STUDIES IN QUANTUM OSCILLATORS AND q-DEFORMED QUANTUM MECHANICS

Thesis submitted
in partial fulfilment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY

by

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1997

CERTIFICATE

Certified that the thesis presented is based on the original work done by Mr. Vinod G. under my guidance and has not been included in any other thesis submitted previously for the award of any degree.

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27-11-1997.

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(Supervising Guide)

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PREFACE

The mathematical structure called *Quantum Group* has found applications in several areas of theoretical physics. *Quantum group* made its first appearance in the study of integrable quantum systems. Later it was shown that this algebraic structure can be obtained by quantising the Poisson Lie algebras. These *Quantised Universal Enveloping Algebras* are related to the classical q-analysis. The same mathematical structure arises in the theory of knot and link invariants and in non-commutative geometry.

From a physical point of view, *quantum group* includes two basic ideas, namely the deformation of an algebraic structure and the notion of a non-commutative comultiplication. In the algebraic quantisation a deformation parameter q is involved and the undeformed sructure is obtained in the $q \longrightarrow 1$ limit. This deformation parameter is reminiscent of the universal constants c and $\hbar$, which ,in some sense act as deformatin parameters in the transition from Newtonian relativity to Einsteinian relativity and in the transition from classical regime to quantum regime respectively. But unlike c or $\hbar$, q is dimensionless.

The idea of comultiplication is inherent in quantum physics,since the vector addition of angular momentum operators can be thought of as defining a comultiplication in a bialgebra. But it is a *commutative bialgebra.* The ladder operators of angular momentum theory also follow such a comultiplication scheme. The total angular momentum operator and the ladder operators constitute an $su(2)$ algebra. Now, if we deform this algebra using a deformation parameter q,we shall obtain what is called an $su_q(2)$ algebra. The generators of this *quantum group*, are q-deformations of the generators of $su(2)$ algebra and the associated comultiplication is *non-commutative*.

Harmonic oscillator is one of the potential tools that one needs to handle the quantum world. In standard quantum mechanics, it is describable in terms of the creation and annihilation operators which obey a commutation relation $aa^\dagger - a^\dagger a = 1$. These operators can be used for defining the generators of the $su(2)$ algebra. Now if we q-deform commutation relation, we shall obtain the generators of $su_q(2)$ algebra. These q-oscillators or *quantum oscillators*,as we call them, were introduced independently by Biedenharn and Macfarlane. Later several definitions of q-oscillators appeared in the literature. And our investigations are based on the q-oscillator proposed by Greenberg which obeys a q-commutation relation $aa^\dagger - qa^\dagger a = 1$.

The q-deformation of the quantum mechanics is motivated mainly by two things: firstly, if q-deformed harmonic oscillator exists, the q-deformation of more general potentials should also exist and this suggests a general framework in which these quantum oscillators arise naturally. Secondly, the relation between quantum groups and q-analysis suggests the replacement of the differential operators in quantum mechanics by q-deformed operators. This leads to q-deformations of both the Schrödinger equation and the Heisenberg commutation relation.

Now, as a generalisation to systems of infinite degrees of freedom, q-deformed quantum field theories are also relevant. Such a deformed field can be supposed to be made up of q-deformed oscillators. Also,since the q-deformation affects the statistics, it is natural to q-deform the chronological product of fields. We consider these two as the key points in the q-deformation of a field theory. This formalism can easily be constructed for scalar fields.

Extension of these ideas to a curved space-time is interesting and the possibility of particle creation occurs in a q-deformed squeezed vacuum rather than a q-vacuum.

The development of this thesis is as follows: in Chapter 1, we give an introduction to quantum groups. Starting from the historical origin of quantum groups, formal definition of quantum group is given and then it is examined from a physical point of view. In section 2,different approaches to quantum groups are discussed. Section 3 presents the basic tools of q-analysis. Quantum oscillators are introduced in section4 and section 5 serves as a summary of what is to be appeared in the later chapters.

Having developed the basic tools in Chapter1, we study the proprties of quantum oscillators in Chapter2. We follow the commutation relation proposed by Greenberg and construct the eigen value spectrum of the corresponding oscillator. Coherent states and squeezed states of this oscillator are studied in detail and we distinguish between squeezing and q-squeezing.

The squeezing properties of some special Hamiltonians are also studied.

We try to form a generalised framework of *q-quantum mechanics* in Chapter3. Then we introduce a particular deformation of the Schrödinger equation and solve it for a particle confined in an infinite potential. This q-quantum mechanics includes non-observable momentum capable of giving observable kinetic energy.

In Chapter 4, a q-deformed quantum field theory is developed for a scalar field and the renormalisation scheme is presented for a self interacting scalar field quartic interaction.

Chapter5 presents an investigation in to the possibility of gravitationally induced particle creation for a q-deformed scalar field.

The material presented in this thesis is based on the following works:

(1).G Vinod, K.Babu Joseph and V C Kuriakose ,*Pramana.J.of Phys.*, **42**,299(1994).

(2).G Vinod, K.Babu Joseph and K M Valsamma,*Pramana.J.of Phys.*, **45**,311(1995).

(3).G Vinod and K.Babu Joseph (submitted to *Phys.Lett.A*)

(4).G Vinod and K.Babu Joseph (under preparation).

Chapter 1

Introduction

1.1 Historical Development of Quantum Groups

Quantum oscillator is the most widely used paradigm for the realisation of quantum groups,which is one of the most interesting developments in contemporary mathematics and theoretical physics. This mathematical structure appears in diverse areas of mathematical physics. Quantum groups made their first appearance in the physics literature in connection with the quantum inverse scattering method(QISM),a technique for studying integrable quantum systems. Kulish and Reshetikhin[1] showed that the quantum linear problem of the quantum sine-Gordon equation is associated with a deformation of the Lie algebra sl_2 unlike the classical problem, which is associated with sl_2 itself. Later Sklyannin[2,3] showed that deformations of Lie algebraic structures were not bound to this particular equation and they were part of a more general theory. It was realised by Drinfield[4-7] that the algebraic structure associated with QISM can be reproduced by a suitable algebraic quantisation of Poisson Lie algebras. The same relations were obtained independently by Jimbo[8,9] through a somewhat different scheme. These algebraic structures are called quantised universal enveloping algebras (QUEA) and are describable in terms of the Lie (super) algebras and this description making use of the ideas of classical q-analysis[10-18]. These QUEAs or quantum groups, arise topologically in the theory of knot and link invariants[19,20] and geometrically in the study of non-commutative geometries[21-24].

1.1.1 Formal definition of a quantum group

Formally,quantum groups are defined to be Hopf algebras which are in general, non commutative[4,26-30]. Hopf algebra is a bialgebra with an antipode [4,26-30]. Bialgebra is a vector space which is an algebra as well as a coalgebra. As an algebra is a way of multiplying things, a coalgebra is a way of 'unmultiplying' things. Analogous to the notion of product in an algebra, there is the notion of coproduct in a coalgebra. For a bialgebra $\mathcal{A}$ defined over a field k, product is defined as the mapping

$m : A \otimes A \rightarrow A$

whereas coproduct is defined by

$\Delta : A \rightarrow A \otimes A$

unit is defined by

$\eta : k \rightarrow A$

co-unit is defined by

$c : A \rightarrow k$

The antipode ϵ is a linear map $\epsilon : A \rightarrow A$ so that the following conditions are satisfied.

$$m(\epsilon \otimes id)\Delta \;=\; m(id \otimes \epsilon)\Delta \;=\; \eta \odot c \tag{1.1}$$

1.1.2 Quantum group as a key to new physics

Fom the viewpoint of physics,quantum group includes two basic ideas,namely the deformation of an algebraic structure and the notion of a noncommutative comultiplication[27]. The idea of deformation is familiar in physics: the Poincaré group is a deformation of the Galilei group ,which is recovered in the limit $c \rightarrow \infty$. Also quantum mechanics can be considered as a deformation of classical mechanics which is regained in the limit $\hbar \rightarrow 0$. In the deformation of algebraic structure usually a deformation parameter q is introduced and in the limit $q \rightarrow 1$,the original structure is regained. As a result of q-deformation,a commutative algebra becomes a non-commuting one. This is the origin of the term 'quantum' in quantum groups since quantisation is in effect the replacement of commuting things by noncommuting things.

The concept of comultiplication is also inherent in quantum physics. To make it apparent,consider the action of angular momentum operator $\mathbf{J}$ in quantum mechanics. Angular momentum is additive in both classical and quantum mechanics.

$$ie \quad \mathbf{J}_{total} \;=\; \mathbf{J}^{(1)} \;+\; \mathbf{J}^{(2)} \tag{1.2}$$

The action of the total angular momentum operator on the product ket formed from the eigen kets of $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ can be expressed as

$$\mathbf{J}_{total} = \mathbf{J}^{(1)} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{J}^{(2)} \tag{1.3}$$

This is actually a comultiplication Δ:

$$\Delta(\mathbf{J}) = \mathbf{J} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{J} \tag{1.4}$$

Thus the vector addition of angular momentum in quantum mechanics defines a comultiplication in a bialgebra. This is an example of commutative comultiplication. The raising and lowering operators in angular momentum theory also obey such a comultiplication.

The total angular momentum operator $\mathbf{J_0}$ and the ladder operators $\mathbf{J}_\pm$ constitute an su(2) algebra:

$$\begin{aligned} [J_0, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2J_0 \end{aligned} \tag{1.5}$$

A prototype q-deformation of this algebra is[4-7] :

$$\begin{aligned} [J_0, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= [2J_0]_q \end{aligned} \tag{1.6}$$

$[\mathbf{2J}]_\mathbf{q}$ is the q-deformation of **2J** which in general depends on a parameter q. There are more than one definition for q-deformation in q-analysis which we explicitly give in section3. All these deformations are such that when $q \to 1$, the original structure is regained. Thus different $su_q(2)$ are possible and all of them reproduce the su(2) algebra as $q \to 1$.

There are different approaches to quantum groups ; these are compared in section (2). This thesis contains the study of q-oscillators and q-deformed systems. q-oscillators are representations of the quantum group $SU_q(2)$. A general description of them is given in section(4). The study of q-oscillators and q-deformed quantum mechanics are enhanced by the ideas of q-analysis, and section (3) gives the important aspects of q-analysis.

1.2 Different approaches to quantum groups

There are mainly three approaches to quantum groups,namely FRT approach, Lie theoretic approach and non-commutative differential geometric approach.

1.2.1 Lie algebraic method

Drinfeld[4] showed that deformed algebraic structures which were introduced by Faddeev and his colleagues as solutions to the Quantum Yang -Baxter Equation associated with the QISM can be reproduced by the quantisation of Lie groups. Almost at the same time Jimbo[9] showed that the trigonometric case encountered in the work of Kulish and Reshetikhin[1], can be put on the same footing as the rational case by employing the representation theory of the corresponding algebra. Simple Lie algebras do not admit non-trivial deformations in the category of Lie algebras. Hence Drinfeld and Jimbo independently introduced the idea of deforming them in the category of Hopf algebras[29]. The resulting structure called quantum universal enveloping algebra(QUEA) and alternatively,quantum groups though they are not at all groups.

There is no general prescription for defining the mappings for a given algebraic structure so as to make it a Hopf algebra. Consider the sl(2) algebra formed by the generators X^+,X^- and H :

$$\begin{aligned} [H, X^{\pm}] &= \pm X^{\pm} \\ [X^+, X^-] &= 2H \end{aligned} \tag{1.7}$$

The co-product can be defined as

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H \\ \Delta(X^{\pm}) &= X^{\pm} \otimes 1 + 1 \otimes X^{\pm} \end{aligned} \tag{1.8}$$

This is a co-commutative co-product. Now assume that the deformed algebra has the form:

$$\begin{aligned} [H, X^{\pm}] &= \pm X^{\pm} \\ [X^+, X^-] &= f(H) \end{aligned} \tag{1.9}$$

where the form of the function is at present arbitrary. Define co-multiplication in this algebra as:

$$\begin{aligned} \Delta(H) &= H \otimes 1 + 1 \otimes H \\ \Delta(X^{\pm} &= X^{\pm} \otimes f + g \otimes X^{\pm} \end{aligned} \tag{1.10}$$

Then the condition $(id \otimes \Delta)\Delta(X^{\pm}) = (\Delta \otimes id)\Delta(X^{\pm})$ suggests the following definitions for $\Delta(f)$ and $\Delta(g)$:

$$\begin{aligned} \Delta(f) &= f \otimes f \\ \Delta(g) &= g \otimes g \end{aligned} \tag{1.11}$$

These, along with the definition of$\Delta(H)$ suggest the choice $f(H) = e^{\mu H}$ and $g(H) = e^{\nu H}$ $\mu, \nu \epsilon$R If we redefine $X^{\pm}$ by making an apropriate transformation,it is easy to show that [31]

$$\begin{aligned} \Delta(X^{+}) &= X^{+} \otimes e^{\mu H} + 1 \otimes X^{+} \\ \Delta(X^{-}) &= X^{-} \otimes 1 + e^{-\mu H} \otimes X^{-} \end{aligned} \tag{1.12}$$

Then using the relations

$$\begin{aligned} X^{+} e^{\pm\mu H} &= e^{\mp\mu} e^{\pm\mu H} X^{+} \\ X^{-} e^{\pm\mu H} &= e^{\pm\mu} e^{\pm\mu H} X^{-} \end{aligned} \tag{1.13}$$

it can be shown that

$$\begin{aligned} \Delta\,[X^{+},\ X^{-}] &= [\Delta X^{+},\ \Delta X^{-}] \\ &= [X^{+},\ X^{-}] \otimes e^{\mu H} + e^{-\mu H} \otimes [X^{+},\ X^{-}] \end{aligned} \tag{1.14}$$

This relation together with the coproduct for f(H) suggests that $[X^{+},\ X^{-}]$ may be deformed according to

$$[X^{+},\ X^{-}] = \frac{e^{2\mu H} - e^{-2\mu H}}{e^{\mu} - e^{-\mu}} \tag{1.15}$$

If we put $e^{\mu} = q$,the quantised algebra becomes

$$[H, X^{\pm}] = \pm X^{\pm}$$

$$[X^{+}, X^{-}] = \frac{q^{(2H)} - q^{-(2H)}}{q - q^{-1}} \tag{1.16}$$

This is the standard form of $\mathcal{U}_q sl(2)$. Further,if we demand that $H^{\dagger} = H,\ X^{+\dagger} = X^{-},\ X^{-\dagger} = X^{+},\ \ \mathcal{U}_q sl(2) \rightarrow \mathcal{U}_q su(2)$. If we represent the generators by J_0, J_+, J_-,

$\mathcal{U}_q su(2)$ or simply $SU_q(2)$ takes the form

$$[J_0, J_\pm] = \pm J_\pm$$

$$[J_+, J_-] = \frac{q^{(2J_0)} - q^{-(2J_0)}}{q - q^{-1}} = [2J_0]_q \tag{1.17}$$

There are two distinct expressions for the Casimir operator corresponding to integer or half integer values of j in the representation of the algebra. They are $J_- J_+ + [J_0]_q [J_0 + 1]_q$ and $J_- J_+ + ([J_0 + 1/2]_q)^2$ respectively.

To relate the QUEA to the solutions of the QYBE, Drinfeld introduced the notion of a universal Yang-Baxter operator. This is an invertible element $\mathcal{R}$ in $A \otimes A$ satisfying:

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} \tag{1.18}$$

where A is the quantum group associated to the Lie algebra.

Quantisations of other Lie algebras have also appeared in the literature[32-35]. Callegini *etal.*[32] have constructed $H_q(1)$ and $E_q(2)$ by contracting $SU_q(2)$.

The existence of a Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{1.19}$$

is an essential requirement of any Lie algebra . In order to construct the q-anlogue, Chaichian *etal.* defined a q-deformed commutator $[A, B]_q = AB - qBA$. They have observed that the following identity holds for arbitrary values of p and q:

$$[A, [B, C]_p]_q + q\,[B, [C, A]_p]_{q^{-1}} + [C, [A, B]_{pq}\,] = 0 \tag{1.20}$$

The contraction of $su_q(2)$ gives rise to the so called q-oscillator algebra[33,34]. Various aspects of q-oscillator algebra are discussed in the next section.

1.2.2 Non-commutative differential geometry

Manin [25] showed that a system of 2×2 matrices

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with noncommuting matrix elements could be a quantum group provided the bialgebra A generated by a, b, c, d satisfy the following rules for multiplication:

$$ab = qba, ac = qca,$$

$$bd = qdb, cd = qdc,$$

$$bc = cb, ad - da = (q - q^{-1})bc \tag{1.21}$$

$$\Delta T = T \otimes T \tag{1.22}$$

If T_1 and T_2 are two 2×2 matrices with noncommuting elements,and suppose the elements of T_1 and T_2 both satisfy the above relations,but the elements of T_1 commute with those of T_2,then the elements of the matrix product T_1T_2 also satisfy the relations(21). But conditions (21) can be obtained by proposing the idea of a quantum plane, *ie*, if the coordinates satisfy the relation

$$xy = qyx \tag{1.23}$$

If x and y commute with the matrix elements a, b, c, d satisfying the relations (21),then x', y' defined by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

will satisfy $x'y' = qy'x'$,if a, b, c, d satisfies relations(21). In other words,if $\mathcal{A}$ is the algebra generated by x and y with relations (23) and $\mathcal{H}$ that generated by a, b, c, d with relations(21),then the map $\delta : \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ defined by

$$\delta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

is a homomorphism. Thus the relations(21) constitute a sufficient condition on the elements of the matrix $\mathbf{T}$ for the action $\mathbf{x} \to \mathbf{Ax}$ on a column vector $\mathbf{x}$ to preserve the relations (23) between the components of $\mathbf{x}$. The same is true of a row vector $\tilde{\mathbf{x}}$ and the action $\tilde{\mathbf{x}} \to \tilde{\mathbf{x}}$. Conversely, (21) are necessary conditions for the quantum plane to be preserved for both column and row vectors:*ie*, if

$$\begin{aligned} xy = qyx \longrightarrow \quad x'y' &= qy'x' \\ and \quad \tilde{x}'\tilde{y}' &= q\tilde{y}'\tilde{x}' \end{aligned} \tag{1.24}$$

where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{x}' & \tilde{y}' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then a, b, c, d satisfy the relations (21).Thus the relations(21) are consequences of the noncommutativity of space.

Relations (21) define the the quantum general linear group $C_qSL(2)$. If we further assume

$$ad - qbc = 1 \tag{1.25}$$

we get the quantum unimodular group $C_qSL(2)$.

The concept of duality is a key point in defining the Lie algebra of a quantum group[36]. Given a Hopf algebra $A(m, \Delta, \eta, k, \epsilon)$,the dual of A, A^* will be endowed with the mappings $m^*, \Delta^*, k^*, \epsilon^*$, such that

$$\begin{aligned}
\langle m^*(f \times g), x\rangle &= \langle(f \times g, \Delta(x))\rangle \\
\langle \Delta^* f, x \times y\rangle &= \langle(f, xy\rangle \\
\langle \eta^*(\alpha), x\rangle &= \alpha\epsilon(x) \\
\epsilon^*(f) &= \langle f, 1\rangle \\
\langle S^* f, x\rangle &= \langle f, S(x)\rangle
\end{aligned} \tag{1.26}$$

$\langle\ ,\rangle$ denotes the pairing between a vector space and its dual.

If we consider the quantisation of $G = GL_(2)$,taking the algebra of functions on G to be the algebra H generated by the noncommuting matrix elements a, b, c, d defined by (21). Further, if we define

$$\mathcal{E} = \frac{\partial}{\partial b}|_I$$

$$\mathcal{F} = \frac{\partial}{\partial c}|_I$$

$$\mathcal{H} = \frac{\partial}{\partial a} - \frac{\partial}{\partial d}|_I$$

$$\mathcal{I} = \frac{\partial}{\partial a} + \frac{\partial}{\partial d} |_I \tag{1.27}$$

where I denotes the identity matrix, we are led to the following relations between $\mathcal{E}, \mathcal{F}, \mathcal{H}, \mathcal{I}$:

$$[\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, \quad [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}$$

$$[\mathcal{I}, .] = 0 \tag{1.28}$$

$$\mathcal{E}\mathcal{F} - q\mathcal{F}\mathcal{E} = \frac{q^{2\mathcal{H}} - q^{-2\mathcal{H}}}{q - q^{-1}} \tag{1.29}$$

The elements E, F, H generate the quantum Lie algebra $U_q sl(2)$. Thus non-commutativity of space leads to quantisation of the Lie algebra.

1.2.3 FRT approach(Faddeev,Reshetikhin,Takhtajan,1987)

This method consists of constructing Lax operators and R-matrices.

This method also is equivalent to the quantisation of Lie algebras and historically it started the study of quantum groups.

Our work is mainly dependent on the algebraic method and it needs some ideas of q-analysis:

1.3 Elements of q-analysis

Classical q-analysis has deep roots down to the beginning of this century. In q-analysis a q-deformation of an integer is given by [14]

$$[n]_q = \frac{q^n - 1}{q - 1} \tag{1.30}$$

Thus,$[1]_q = 1$,and $[0]_q = 0$,independent of the value of q. Also,as $q \to 1$, $[n]_q \to n$ This q-deformation does not have $q \to q^{-1}$ symmetry.The additive inverse of the q-integer is defined by

$$[n]_q + q^n [-n]_q = 0 \tag{1.31}$$

The q-factorial is defined by

$$[n]_q! \; = \; [n]_q [n-1]_q [n-2]_q \ldots\ldots\ldots\ldots [2]_q [1]_q \tag{1.32}$$

The q-exponential function is defined as

$$exp_q X \; = \; \sum_{n=0}^{\infty} \frac{X^n}{[n]_q!} \tag{1.33}$$

For q-exponential functions, $exp_q X exp_q Y \neq exp_q(X + Y)$. But if x and y are in a quaqntum plane,*ie*,if xy = qyx, $exp_q x exp_q y = exp_q(x + y)$[37].

To prove this,let us prove the q-binomial expansion

$$(x + y)^n \; = \; \sum_{r=0}^{n} \frac{[n]_q!}{[r]_q![n-r]_q!} y^r x^{n-r} \tag{1.34}$$

for x and y in a quantum plane. For n=1,equation (34) is true.Now suppose it is true for n=m,some integer.

$$ie \;\; (x + y)^m \; = \; \sum_{r=0}^{m} \frac{[m]_q!}{[r]_q![m-r]_q!} y^r x^{m-r} \tag{1.35}$$

then,

$$(x + y)^{m+1} \; = \; \sum_{r=0}^{m} \frac{[m]_q!}{[r]_q![m-r]_q!} x^{m-r} y^r \times (x + y)$$

$$= \sum_{r=0}^{m} \frac{[m]_q!}{[r]_q![m-r]_q!} y^r x^{m+1-r} + \sum_{r=0}^{m} \frac{[m]_q!}{[r]_q![m-r]_q!} q^{m-r} y^{r+1} x^{m-r}$$

$$= \sum_{r=0}^{m} \left(\frac{[m]_q!}{[r]_q![m-r]_q!} + q^{m+1-r} \frac{[m]_q!}{[r-1]_q![m+1-r]_q!} \right) y^r x^{m+1-r} + y^{m+1}$$

$$= \sum_{r=0}^{m} \frac{[m+1]_q!}{[r]_q![m+1-r]_q!} y^r x^{m+1-r} + y^{m+1}$$

$$= \sum_{r=0}^{m+1} \frac{[m+1]_q!}{[r]_q![m+1-r]_q!} y^r x^{m+1-r} \qquad (1.36)$$

Thus if equation (34) is true for n=m,it is true for n=m+1 also. Since eq.(34) is true for n=1 ,it follows that it is true for all integers. Now,

$$\begin{aligned} exp_q(x+y) &= \sum_{n=0}^{\infty} \frac{(x+y)^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \sum_{r=0}^{n} \frac{[n]_q!}{[r]_q![n-r]_q!} y^r x^{n-r} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{y^r x^{n-r}}{[r]_q![n-r]_q!} \\ &= \sum_{r=0}^{\infty} \frac{y^r}{[r]_q!} \sum_{n=0}^{\infty} \frac{x^{n-r}}{[n-r]_q!} \\ &= \sum_{r=0}^{\infty} \frac{y^r}{[r]_q!} \sum_{s=0}^{\infty} \frac{x^s}{[s]_q!} \\ &= exp_q(y).exp_q(x) \qquad (1.37) \end{aligned}$$

The q-analogues of trigonometric functions are defined as

$$sin_q x = (exp_q(ix) - exp_q(-ix))/2i \tag{1.38}$$

$$cos_q x = (exp_q(ix) + exp_q(-ix))/2 \tag{1.39}$$

The q-difference operator D_x is defined by [13]

$$D_x f(x) = \frac{f(qx) - f(x)}{x(q-1)} \tag{1.40}$$

This operator is defined on a q-lattice in which the lattice points are in a geometric sequence. This q-difference operator does not possess $q \to q^{-1}$ symmetry. As $q \to 1, D_x \to \frac{d}{dx}$,if it exists. It is a linear operator and it satisfies

$$D_x(x^n) = [n]_q x^{n-1} \tag{1.41}$$

Hence the q-exponential function satisfies

$$D_x \; exp_q x = exp_q x \tag{1.42}$$

Also

$$D_x \; sin_q x = cos_q x$$

$$D_x \; cos_q x = sin_q x \tag{1.43}$$

Hence $sin_q(kx)$ and $cos_q(kx)$ are the solutions of the q-difference equation

$$(D_x^2 + k^2) f(x) = 0 \tag{1.44}$$

Successive aplication of D_x on x^n gives

$$D_x^m x^n = \frac{[n]!}{[n-m]!} x^{n-m} \tag{1.45}$$

The q-analogues of Lebnitz rule and quotient rule are found as:

$$D_x \ (u(x)v(x)) = v(x)D_x u(x) + u(qx)D_x v(x)$$

$$D_x \ (u(x)/v(x)) = \frac{v(x)D_x u(x) - u(x)D_x v(x)}{v(qx)v(x)} \tag{1.46}$$

q-integration is defined by

$$\int_a^b f(x)d(qx) = (1-q)\left(b\sum_{r=0}^{\infty} q^r f(q^r b) - a\sum_{r=0}^{\infty} q^r f(q^r a)\right) \tag{1.47}$$

Product rule for this q-integral is:

$$\int g(x)D_x f(x)d(qx) = f(x)g(x) - \int f(qx)D_x g(x)d(qx) \tag{1.48}$$

Alternative definitions of q-basic number and q-difference operator exist in the literature.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{1.49}$$

$$\mathcal{D}_x f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})} \tag{1.50}$$

These definitions have $q \rightarrow q^{-1}$ symmetry. Also,q-integer defned by (49) has the property

$$[n]_q + [-n]_q = 0 \tag{1.51}$$

ie, the additive inverse of the q-deformation of a number is the q-deformation of the additive inverse of the number. This is a special case of a more general property which reads:

$$[m+n]_q = [m]_q + [n]_q \quad m, n \epsilon \mathcal{Z} \tag{1.52}$$

Thus there is an isomorphism $\{[n]_q\} \rightarrow \mathcal{Z}$. The q-difference operator(50) satisfies the following q-analogue of Lebnitz rule:

$$\begin{aligned} \mathcal{D}_x\ (u(x)v(x)) &= \mathcal{D}_x u(x)v(qx) + u(q^{-1}x)\mathcal{D}_x v(x) \\ &= \mathcal{D}_x u(x)v(q^{-1}x) + u(qx)\mathcal{D}_x v(x) \end{aligned} \tag{1.53}$$

q-analogue of quotient rule is

$$\mathcal{D}_x\ (u(x)/v(x)) = \frac{\mathcal{D}_x u(x)v(q^{-1}x) - u(q^{-1}x)\mathcal{D}_x v(x)}{v(qx)v(q^{-1}x)} \tag{1.54}$$

Summation over a q-lattice in which differentiation is defined by (50) suggests the q-analogue of integration as:

$$\int_a^b f(x)d(qx) = b\sum_{r=m}^{n} f(q^{2r}x)(q-q^{-1})q^{2r}x \tag{1.55}$$

where $a = q^m, b = q^n$.

The following identity also holds:

$$\int_{-\infty}^{\infty} f(x)d(qx) = \int_{-\infty}^{\infty} q^n f(q^n x)d(qx) \tag{1.56}$$

It is helpful to introduce a dilation operator $\hat{Q}$:

$$\hat{Q}f(x) = f(qx) \tag{1.57}$$

Let $f(x) = x^m$

$$\begin{aligned} \hat{Q}x^m = q^m x^m &= e^{mlnq}x^m \\ &= \left(1 + \frac{mlnq}{1!} + \frac{(mlnq)^2}{2!} + \ldots\ldots\ldots\ldots\right)x^m \end{aligned}$$

$$= \left(1 + \frac{(lnq)x\partial_x}{1!} + \frac{(lnq)^2(x\partial_x)^2}{1!} + ...\right) x^m$$

$$= e^{(lnq)x\partial_x} x^m$$

$$= q^{x\partial_x} x^m \tag{1.58}$$

Thus $\hat{Q} = q^{x\partial_x}$

We can extend this relation and write

$$q^{\pm ax\partial_x} f(x) = f(q^{\pm a}x) \tag{1.59}$$

With the help of the dilation operator the action of D_x and $\mathcal{D}_x$ can be written explicitly as :

$$D_x f(x) = \frac{q^{x\partial_x} f(x) - f(x)}{x(q-1)} \tag{1.60}$$

$$\mathcal{D}_x f(x) = \frac{q^{x\partial_x} f(x) - q^{-x\partial_x} f(x)}{x(q-q^{-1})} \tag{1.61}$$

Further,we can relate $D_X(q)$ and $\mathcal{D}_x(q^{1/2})$:

$$D_x(q) = q^{1/2x\partial_x}\mathcal{D}_x(q^{1/2}) \tag{1.62}$$

The following q-commutation relations hold:

$$D_x x - qxD_x = 1. \tag{1.63}$$

$$\mathcal{D}_x x - q^{-1}x\mathcal{D}_x = q^{x\partial_x} \tag{1.64}$$

Since our work is based on the definition(40) for D_x, we work out some relations satisfied by it which are applied in the later chapters. Consider an arbitrary function f(x). Then

$$D_x q^{x\partial_x} f(x) = \frac{f(q^2x) - f(qx)}{x(q-1)} \tag{1.65}$$

$$q^{x\partial_x} D_x f(x) = \frac{f(q^2x) - f(qx)}{qx(q-1)} \tag{1.66}$$

Thus

$$D_x q^{x\partial_x} = q q^{x\partial_x} D_x \tag{1.67}$$

Similarly,

$$D_x q^{-x\partial_x} = q^{-1} q^{-x\partial_x} D_x \tag{1.68}$$

We can generalise these results :

$$D_x q^{ax\partial_x} = q^a q^{ax\partial_x} D_x, \quad a \epsilon \mathcal{Q} \tag{1.69}$$

Further,

$$D_x^2 q^{ax\partial_x} = q^a D_x q^{ax\partial_x} D_x = q^{2a} q^{ax\partial_x} D_x^2 \tag{1.70}$$

Generalisation of the above result is straightforward:

$$D_x^n q^{ax\partial_x} = q^{na} q^{ax\partial_x} D_x^n, \quad n \epsilon \mathcal{Z}, a \epsilon \mathcal{Q}. \tag{1.71}$$

An important result that follows from (71) is:

$$\begin{aligned} D_x q^{-1/2x\partial_x} D_x q^{-1/2x\partial_x} &= q^{1/2} D_x^2 q^{-x\partial_x} \\ or, \quad D_x^2 q^{-x\partial_x} &= (q^{-1/4} D_x q^{-1/2x\partial_x})^2 \end{aligned} \tag{1.72}$$

This result is made use of in the q-deformation of quantum mechanics described in Chapter3.

In our work we restrict the domain of q s.t. $q \epsilon [0, 1]$. If q is very close to unity,*ie*, $q = 1 - \epsilon$,where ϵ is a very small positive number,$[n]_q$ can be approximated as:

$$[n]_q = n\{1 - (n-1)\epsilon/2\} \tag{1.73}$$

$[n]_q!$ is approximated as:

$$[n]_q! = n!\{1 - n(n-1)\epsilon/2\} \tag{1.74}$$

Therefore the action of D_x on x^n is approximated as:

$$D_x x^n = n\{1 - n(n-1)\epsilon/2\}x^n \tag{1.75}$$

In addition,

$$D_x^m x^n = \frac{n!}{m!}\{1 - m(2n - m - 1)\epsilon/2\}x^{n-m} \tag{1.76}$$

1.4 Quantum oscillators

The generators of $su(2)$ can be realised by the creation and annihilation operators a and $a^\dagger$ of a single boson in the form

$$J_+ = -1/2a^2, \quad J_- = 1/2a^{\dagger 2}, \quad J_0 = \frac{1}{4}(aa^\dagger + a^\dagger a) \tag{1.77}$$

where the operators $a, a^\dagger$ obey the commutation rule

$$[a^\dagger, a] = aa^\dagger - a^\dagger a = 1. \tag{1.78}$$

In a similar manner, the generators of $su_q(2)$ can be realised if we replace the commutation relation(78) by a q-deformed commutation relation(q-CR) retaining relations (77) with the difference that $J_\pm, J_0$ are now generators of$su_q(2)$ and a and $a^\dagger$ are q-deformed.Biedenharn[38] and Macfarlane[39] independently found two q-CRs for q-deformed operators $\tilde{a}$ and $\tilde{a}^\dagger$. They are

$$\tilde{a}\tilde{a}^\dagger - q^{1/2}\tilde{a}^\dagger\tilde{a} = q^{-\tilde{N}/2}. \tag{1.79}$$

and

$$\tilde{a}\tilde{a}^{\dagger} - q^{-1}\tilde{a}^{\dagger}\tilde{a} = q^{\tilde{N}}. \tag{1.80}$$

where the operator $\tilde{N}$ satisfies the following commutation relations:

$$[\tilde{N}, \tilde{a}] = -\tilde{a}, [\tilde{N}, \tilde{a}^{\dagger}] = \tilde{a}^{\dagger} \tag{1.81}$$

The commutatuion relations(79) and (80) are equivalent and they can be written alternatively as[38-41]:

$$\tilde{a}\tilde{a}^{\dagger} - q\tilde{a}^{\dagger}\tilde{a} = q^{-\tilde{N}}. \tag{1.82}$$

We can construct the representations of(82) in the Fock sopace spanned by the orthonormalized eigenstates $\mid n >$ of the operator $\tilde{N}$:

$$\mid n >= \frac{(\tilde{a}^{\dagger})^n}{\sqrt{[n]!}} \mid 0 >, \tilde{a} \mid 0 >= 0, \tilde{N} \mid n >= n \mid n > \tag{1.83}$$

where $[n]! = [n][n-1]$..............$[2][1]$ and $[n]$ is defined by (1.49). Here onwards,we drop the suffix q for the sake of convenience. The following relations are valid in this Fock space:

$$\tilde{a}\tilde{a}^{\dagger} = [\tilde{N}], \quad \tilde{a}^{\dagger}\tilde{a} = [\tilde{N}+1] \tag{1.84}$$

If the deformation parameter $q = e^{\pm i\pi/m}$,The Fock space breaks up into m-dimensional subspaces not connected by the operators a and $a^{\dagger}$. Each subspace carries an m-dimensional representation of the algebra(82) and can be considered separately. A similar phenomenon occurs whenever $q = e^{ir\pi}$, where r is rational and is different from an integer[41].

From (81), we get

$$\tilde{a}q^{\pm r\tilde{N}} = q^{\pm r}q^{r\tilde{N}}\tilde{a}$$

$$\tilde{a}^{\dagger}q^{\pm r\tilde{N}} = q^{\pm r}q^{r\tilde{N}}\tilde{a}^{\dagger} \tag{1.85}$$

where r is a rational number. Then defining

$$A = q^{\tilde{N}/2}\tilde{a}, \quad , A^{\dagger} = \tilde{a}^{\dagger} q^{\tilde{N}/2} \tag{1.86}$$

relation (82) can be rewritten as

$$AA^{\dagger} - q^2 A^{\dagger} A = 1 \tag{1.87}$$

We can express(82) in terms of the usual undeformed Bose operators b and $b^{\dagger}$ by means of the change [41,42]

$$\tilde{a} = \left(\frac{[\tilde{N}+1]}{\tilde{N}+1} \right)^{1/2} b, \quad \tilde{a}^{\dagger} = b^{\dagger} \left(\frac{[\tilde{N}+1]}{\tilde{N}+1} \right)^{1/2}, \tag{1.88}$$

Then we obtain the usual bosonic algebra:

$$[b, b^{\dagger}] = 1, [N, b] = -b, [N, b^{\dagger}] = b^{\dagger}, N = b^{\dagger} b \tag{1.89}$$

The q-fermionic algebra is defined as[43,41]:

$$ff^{\dagger} + qf^{\dagger} f = q^{-N}, \ [N, f] = -f \ [N, f^{\dagger}] = f^{\dagger} \tag{1.90}$$

where N is the q-fermion number operator. The orthonormalised eigenstates of N are defined by

$$\mid n >= \frac{(f^{\dagger})^n}{\sqrt{[n]^f!}} \mid 0 >, \ \ f \mid 0 > \ \ N \mid n >= n \mid n > \tag{1.91}$$

where

$$[n]^f = \frac{q^{-n} - (-1)q^n}{q + q^{-1}} \tag{1.92}$$

For generic values of q,this representation is infinite dimensional. But for some special values of q,the Fock space breaks up into disjoint subspaces each carrying a finite dimensional representation of (88). Also for $q = 1$,the Fock space breaks up into two-dimensional subspaces,and the Pauli exclusion principle follows from $f^2 = f^{\dagger 2} = 0$. In the case of q-fermion oscillators defined by (88),there does not exist a change of operators analogous to(84) in order that these oscillators be expressed in terms of the undeformed fermion operators[41]. If we make the substitution $\tilde{a} = q^{(1/4)\tilde{N}}a, \tilde{a}^\dagger = a^\dagger q^{(1/4)\tilde{N}}$, (79) can be written as

$$aa^\dagger - qa^\dagger a = 1 \tag{1.93}$$

whose multimode generalisation is the quon algebra proposed by Greenberg[44,45]. A number operator corresponding to(93) and satisfying the conditions

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger \tag{1.94}$$

is[46]

$$\begin{aligned} N &= \sum_{n=1}^{\infty} \frac{(1-q)}{(1+q^n)} a^{\dagger n} a^n \\ &= a^\dagger a + \left(\frac{1-q}{1+q}\right) a^{\dagger 2} a^2 + \ldots\ldots. \\ &+ \frac{(1-q)^n}{1+q+\ldots..+q^{n-1}} a^{\dagger n} a^n + \ldots.. \end{aligned} \tag{1.95}$$

This algebra interpolates between fermions and bosons as q goes from -1 to +1 on the real axis. The realisation given in (93) is better adapted to discussions of the quantum plane (finite transformations),while the algebra (79) is adapted to infinitesmal transformations acting on q-analogue state vectors[47].

We can construct the representation of (93) in the Fock space spanned by the orthonormalised eigenstates $| n >$ of N.

$$| n >= \frac{a^{\dagger n}}{\sqrt{[n]!}} | 0 >, a | 0 >= 0, N | n >= n | n > \tag{1.96}$$

where $[n]! = [n][n-1]$.............$[2][1]$ and $[n]$ is defined by (30).For $q = 0$,from (99),we obtain

$$N = \sum_{n=1}^{\infty} a^{\dagger^n} a^n \tag{1.97}$$

which corresponds to the single mode Greenberg oscillator obeying $aa^{\dagger} = \mathbf{I}$. The normalised states in the Fock space are $| 0 >, a^{\dagger} | 0 >, a^{\dagger^2} | 0 >, \ldots.$

Making use of definition (30) for q-deformed number,(93) can be had with the following relations:

$$aa^{\dagger} = [N], \quad a^{\dagger}a = [N+1] \tag{1.98}$$

The q-commutator (93) can be written as[46]:

$$[a, a^{\dagger}] = f(N). \tag{1.99}$$

where

$$\begin{aligned} f(N) &= q^N \ for \ \ q \neq 0 \\ f(N) &= \theta(1-N) \ \ for \ \ q = 0 \end{aligned} \tag{1.100}$$

where $\theta(x)$=1 for $x > 0$ and 0 for $x \leq 0$. Multimode generalisations of q-bosonic and q-fermionic algebras are[41] :

$$\tilde{a}_i \tilde{a}_j^{\dagger} - ((q-1)\delta_{ij} + 1)\tilde{a}_j^{\dagger} \tilde{a}_i = \delta_{ij} q^{-\tilde{N}_i}.$$

$$f_i f^{\dagger}{}_j + ((q-1)\delta_{ij} + 1) f^{\dagger}{}_j f_i = \delta_{ij} q^{-\hat{N}_i}$$

(1.101)

and the multimode quon algebra is[40,44,45]

$$a_i a^{\dagger}{}_j - q a^{\dagger}{}_j a_i = \delta_{ij} \tag{1.102}$$

There is a clear-cut distinction between the q-oscillator algebras defined by (101) and quons defined by (102). In the case of bosonic or fermionic oscillators, different modes commute or anti-commute,whereas for quons,different modes q-mute. Besides,no commutation relation can be impossed on the quonic operators aa and $a^{\dagger}a^{\dagger}$. Relations like $a_i a_j - q a_j a_i = 0$ or $a^{\dagger}{}_i a^{\dagger}{}_j - q a^{\dagger}{}_j a^{\dagger}{}_i = 0$ are valid only when $q^2 = 1$

The fact that the quon algebra can be used to interpolate between bosons and fermions, shows some similarity between quons and anyons. But there is an essential difference between these two:anyons arise only in two dimensions,while q-oscillators can be defined in any space dimension. Also q-oscillators can be interpreted as the Fourier components of local field operators,whereas anyons are non-local objects as a consequence of their braiding properties and the essential difference in their commutation relations[41]. Since our work is based on (93),we restrict our discussion to the properties of (93).

1.4.1 q-analogue of harmonic oscillator

Let us define the q-momentum operator and q-position operator in terms of the q-deformed creation and annihilation operators $a^\dagger$ and a:

$$X_q = (\frac{\hbar}{2m\omega})^{1/2}(a + a^\dagger) \tag{1.103}$$

$$P_q = -i(\frac{m\hbar\omega}{2})^{1/2}(a - a^\dagger) \tag{1.104}$$

Then the Hamiltonian for the q-oscillator takes the form:

$$H_q = \frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) \tag{1.105}$$

This q-Hamiltonian is diagonal on the eigenstates $\mid n >$ and has the eigen values

$$E_q(n) = \frac{\hbar\omega}{2}([n] + [n+1]) \tag{1.106}$$

where $[n]$ is defined by (30). Similar eigen value spectrum exists for the q-oscillator corresponding to (82) except with the difference that $[n]$ is defined by(49). In either case, the energy level spacings are non-uniform. Also,the uncertainty product of q-position and q-momentum operators is minimal and independent of the eigen value n only when $q = 1$. For generic values of q,the uncertainty increases with n which we will explicitly show in the next chapter.

1.5 General q-deformed systems

The deformation of harmonic oscillators can be extended to more general systems. The stationary states of a quantum system are determined by the Schrödinger equation. Therefore the spectrum of a general q-deformed system will be governed by a q-deformed Schrödinger equation. The concepts of quantum physics should be unaffected and only the mathematical structure be affected by this q-deformation. Chapter 3 presents such an investigation on q-deformed quantum mechanics. We suggest a deformation of the Schrodinger equation and give the solutions for a simple potential.

Now q-deformation is applied to systems with infinite degrees of freedom also. As in the standard case,the field can be Fourier decomposed,but the second quantised operators are supposed to follow a deformed commutation relation,which we take as the quon algebra. The chronological product can also be q-deformed and such a deformation of a real scalar field is discussed in Chapter4. Normalisation scheme for a self interacting (ϕ_q^4) field is given.

In the dynamical evolution of a q-deformed field through a curved space-time,it has been established that particle creation is impossible in the vacuum state[48]. But if the field is assumed to be made up of coherent states,there is a nonzero possibility of particle creation. In the q-squeezed vacuum also, there is possibility of particle creation. These aspects are discussed in Chapter5.

The real physical meaning of q-deformation is yet to be explored. Theoretically, it allows some variations from the results predicted by standard($q = 1$) quantum mechanics and offers measurement in which the uncertainty can be reduced below the value predicted by Heisenberg's relation. But the feasibility of this and the other results of q-deformed systems depends on the physical sense of the deformation parameter q, which unlike $\hbar$ or c is dimensionless.

Chapter 2

Coherent States and Squeezed States of q-deformed Oscillators

2.1 Coherent states of a harmonic oscillator

Coherent states of a harmonic oscillator were introduced by Schrödinger [48] as minimum uncertainty states which exhibit in some sense the classical behaviour of the oscillator. These states which have arbitrary energy and momentum are represented by Gaussian wave packets with the width of the ground state and the wave packets follow the classical motion of the classical oscillator and preserve their shape . On the basis of Schrödinger's work,coherent states have the following properties[49]:

(1) They are the subset of three-parameter family of minimum uncertainty states $\psi(x) = (\pi\hbar\lambda)^{-1/4} exp[ixp_0/\hbar - (x-x_0)^2/(2\hbar\lambda)]$, the subset being fixed by the condition $\lambda = (m\omega)^{-1}$ where m is the mass and ω is the frequency of the oscillator and x_0 and p_0 denoting the classical values of position and momentum at $t = 0$.

(2)The time evolution of these states is such that at time t, these states are represented by $\psi(x)$ with the difference that x_0 and p_0 are replaced by the classical values of position and momentum at time t. (3) The uncertainties in position and momentum are independent of time and the uncertainty product is always the minimum value fixed by Heisenberg's relation. The study of these states was renewed in 1960s by the works of Klauder, Sudarsan and Glauber[50-58]. The coherent states had been defined using some displacement operator or as the eigen states of the annihilation operator. The displacement operator method is based on group-theoretic techniques. Consider the oscillator algebra generated by $a, a^\dagger, a^\dagger a$,and I. The displacement operator is the unitary exponentiation of the elements of the factor algebra, spanned by a and $a^\dagger$:

$$\begin{aligned} D(\alpha) &= exp(\alpha a^\dagger - \alpha^* a) \\ &= exp(\frac{-1}{2} \mid \alpha \mid^2) exp(\alpha a^\dagger) exp(-\alpha^* a) \end{aligned} \tag{2.1}$$

(using the Baker-Campbell-Hausdorff relation and the CR, $[a, a^\dagger] = 1$.) The displacement operator acting on the ground state gives the coherent state,

$$\begin{aligned} D(\alpha) \mid 0> &= exp\left(\alpha a^\dagger - \alpha^* a\right) \mid 0> \\ &= exp\left(-\frac{1}{2} \mid \alpha \mid \alpha^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n > = \mid \alpha > \end{aligned} \tag{2.2}$$

where $\mid n >$ are the number states. Definitions (1) and (2) are the same if we make the identification $\alpha = \sqrt{m\omega/2\hbar} x_0 + \sqrt{2m\hbar\omega} p_0$ [59]. Coherent states are also defined as eigen states of the annihilation operator a:

$$a \mid \alpha > = \alpha \mid \alpha > \tag{2.3}$$

Definitions (2) and (3) are equivalent for the harmonic oscillator as a consequence of the Heisenberg-Weyl algebra satisfied by the operators $a, a^\dagger, a^\dagger a$. and I. Thus, for the harmonic oscillator, the above three definitions are equivalent. Coherent states are not orthogonal.But they are linearly independent and provide a continous basis for the Hilbert space of the oscillator. In a coherent state the two quadrature components have equal variances. Also, sum of the variances is minimum in a CS. We can define two self-adjoint operators a_1 and a_2 callee the quadrature components such that $a = a_1 + ia_2$. then

$$\begin{aligned} Var a_1 &= \frac{1}{4} \\ Var a_2 &= \frac{1}{4} \\ (\Delta a_1)(\Delta a_2) &= \frac{1}{4} \end{aligned} \tag{2.4}$$

where variance of an operator is defined by Variance of an operator A is defined by

$$VarA = (\Delta A)^2 =< A^2 > - < A >^2 \tag{2.5}$$

Thus uncertainties of both the components are equal in a CS which is same as their uncertainties in the ground state. The probability of measuring n quanta in the CS is

$$\begin{aligned} P_n &= \left[\exp(\mid \alpha \mid^2)\right]^{-1} \left(\frac{\mid \alpha \mid^{2n}}{[n]!}\right) \\ &= [\exp(< n >)]^{-1} \left(\frac{< n >^n}{n!}\right) \end{aligned} \tag{2.6}$$

where we have made use of the fact that

$$\mid \alpha \mid^2 =< \alpha \mid a^\dagger a \mid \alpha >=< \alpha \mid N \mid \alpha >=< n > \tag{2.7}$$

This represents a Poissonian distribution. Coherent states can be used for dealing with the coherent properties of electromagnetic fields. The optical field Hamiltonian is expressed as a superposition of oscillator type Hamiltonians. Although the number states desribe the state of the field, they are unsuitable for calculations when the description of the field involves phase and amplitude variables. A radiation mode such as a plane propogating wave is best described by a CS that is a particular linear combination of the number states expressing the cooperative behaviour of the photons. The quantum fluctuations in a CS are equal to zero point fluctuations and are randomly distibuted in phase. These zero point fluctuations represent the standard quantum limit to the reduction of noise in a signal.

2.2 Squeezed states

Squeezed states are also minimum uncertainty states but the two quadratures have different uncertainties; one of the components has uncertainty below that fixed by Heisenberg's relation but the other has uncertainty above the limit, the product being equal to the minimum value(uncertainty in the vacuum state)[62]. The component whose uncertainty goes below 1/2 is said to be squeezed at the expense of the other. Squeezed states were first introduced by Yuen [60] as two photon coherent states. His method is simple:for a single mode field, mix a part of the field with its phase conjugate to produce a new mode b such that

$$b = \mu a + \nu a^{\dagger}, \quad |\, \mu \,|^2 - |\, \nu \,|^2 = 1 \tag{2.8}$$

For a mode a in a coherent state the mode b will be in a squeezed state. Or, let the eigen states of b represent squeezed states:

$$b \mid \beta >= \beta \mid \beta > \tag{2.9}$$

We can expand $\mid \beta >$ in terms of the number states:

$$\mid \beta >= \sum_{n=0}^{\infty} C_n \mid n > \tag{2.10}$$

The coefficients follow the recursion relation

$$C_n = \frac{\beta C_{n-1} - \nu\sqrt{n-1}C_{n-2}}{\mu\sqrt{n}} \tag{2.11}$$

Yuen [60] also introduced the concept of generalised number states. Analogous to constructing number states from the vacuum,the generalised number states can be constructed by successive application of $b^\dagger$ on a generalised vacuum which is an eigen state of b with the eigen value 0. A generalised number state is constructed from its adjacent lower state according to the relation

$$\mid m+1 >= \frac{b^\dagger}{\sqrt{m+1}} \mid m > \tag{2.12}$$

Expectation value of $b^\dagger b$,which is the number operator for the generalised number states in $\mid \beta >$ is $\mid \beta \mid^2$. But the ordinary number operator $a^\dagger a$ has the expectation $\mid \beta \mid^2 (\mu - \nu)^2 + \nu^2$ in the squeezed state. $\mid \beta >$'s are also called $SU(1,1)$ squeezed states because of the condition $\mid \mu \mid^2 - \mid \nu \mid^2 = 1$. If we make the identifications $K_+ = \frac{1}{2}a^\dagger a^\dagger, K_- = \frac{1}{2}aa, K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2})$, the $su(1,1)$ algebra is obtained:

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0 \tag{2.13}$$

The $SU(1,1)$ squeezed states have the form $exp(\frac{1}{2}\alpha(a^\dagger)^2 \mid 0 >$. An alternative but inequivalent characterisation of the squeezed state can be found in [63].

2.3 Coherent states and squeezed states of q-deformed oscillators

Coherent states and squeezed states of q-oscillators have been sudied extensively[64-67]. Majority of the work in this area make use of the algebra(1.82). Our interest is in studying the coherent and squeezed states of a q-oscillator whose defining algebra is (1.93). A normalised q-coherent state(q-CS) is defined as

$$a \mid \alpha >_q = \alpha \mid \alpha >_q \tag{2.14}$$

$$\mid \alpha >_q = \left[\exp_q \mid \alpha \mid^2\right]^{-1/2} \mid \alpha >_q \tag{2.15}$$

where the q-exponential is defined by (1.33).

The q-CS can also be obtained from the q-vacuum by the action of a q-displacement operator $D_q(\alpha)$:

$$\mid \alpha >_q = D_q(\alpha) \mid 0 >_q \tag{2.16}$$

where $D_q(\alpha)$ has the form

$$D_q(\alpha) = \left[\exp_q \mid \alpha \mid^2\right]^{-1/2} \exp_q(\alpha a^\dagger) \tag{2.17}$$

In $q = 1$ theory displacement operator has the form $\exp(\alpha a^\dagger - \alpha^* a)$. But the q-analogue of this,namely $\exp_q(\alpha a^\dagger - \alpha^* a)$ is different from $D_q(\alpha)$ due to the property of q-exponential functions,which reads $\exp_q A . \exp_q B \neq \exp_q(A + B)$ for generic values of q. The probability of measuring n quanta in the q-CS is

$$\begin{aligned} P_n &= \left[\exp_q(\mid \alpha \mid^2)\right]^{-1} \left(\frac{\mid \alpha \mid^{2n}}{[n]!}\right) \\ &= \left[\exp_q(< [n] >)\right]^{-1} \left(\frac{< [n] >^n}{[n]!}\right) \end{aligned} \tag{2.18}$$

where we have made use of the fact that

$$| \alpha |^2 = < \alpha | a^\dagger a | \alpha > = < \alpha | [N] | \alpha > = < [n] > \tag{2.19}$$

Equation(18) represents a q-Poisson distribution.

Using (1.103),

$$\begin{aligned} X_q^{\,2} &= \frac{\hbar}{2m\omega}(a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a) \\ &= \frac{\hbar}{2m\omega}(a^2 + a^{\dagger 2} + (1+q)a^\dagger a + 1) \end{aligned} \tag{2.20}$$

And using(1.104),

$$\begin{aligned} P_q^{\,2} &= \frac{m\hbar\omega}{2}(-a^2 - a^{\dagger 2} + aa^\dagger + a^\dagger a) \\ &= \frac{\hbar}{2m\omega}(-a^2 - a^{\dagger 2} + (1+q)a^\dagger a + 1) \end{aligned} \tag{2.21}$$

In the q-CS $| \alpha >_q$, the expectation values are calculated as

$$\begin{aligned} < X_q > &= (\frac{\hbar}{2m\omega})^{1/2} (\alpha + \alpha^*) \\ < X_q^{\,2} > &= \frac{\hbar}{2m\omega}(\alpha^2 + \alpha^{*2} + (1+q)| \alpha |^2 + 1) \\ < P_q > &= -i(\frac{m\hbar\omega}{2})^{1/2} (\alpha - \alpha^*) \\ < P_q^{\,2} > &= -\frac{m\hbar\omega}{2}(\alpha^2 + \alpha^{*2} - (1+q)| \alpha |^2 - 1) \end{aligned} \tag{2.22}$$

Hence variances of X_q and P_q are calculated:

$$\begin{aligned} VarX_q &= \frac{\hbar}{2m\omega}\left(| \alpha |^2(q-1) + 1\right) \\ VarP_q &= \frac{m\hbar\omega}{2}\left(| \alpha |^2(q-1) + 1\right) \end{aligned} \tag{2.23}$$

Since variance should be positive, equation(23) fixes an upper limit for α:

$$| \alpha |^2 \leq (1-q)^{-1} \tag{2.24}$$

But the energy of corresponding to a q-CS $\mid \alpha >_q$ is given by

$$\begin{aligned} E_\alpha &= {}_q< \alpha \mid H \mid \alpha >_q \\ &= {}_q< \alpha \mid (aa^\dagger + a^\dagger a) \mid \alpha >_q \\ &= \frac{\hbar\omega}{2}\left((1+q)\mid \alpha \mid^2 + 1\right) \end{aligned} \tag{2.25}$$

Thus the maximum energy that a q-CS can possess is $\hbar\omega/(1-q)$. Variances of X_q and P_q in the number states are calculated as:

$$\begin{aligned} VarX_q &= \frac{\hbar}{2m\omega}\left([n] + [n+1]\right) \\ VarP_q &= \frac{m\hbar\omega}{2}\left([n] + [n+1]\right) \end{aligned} \tag{2.26}$$

Hence uncertainty product for the number states are:

$$(\Delta X_q)_n(\Delta P_q)_n = \frac{\hbar}{2}\left([n] + [n+1]\right) \tag{2.27}$$

In particular, the uncertainty product for the q-vacuum state is :

$$(\Delta X_q)_0(\Delta P_q)_0 = \frac{\hbar}{2} \tag{2.28}$$

which coincides with that of the ordinary vacuum state. In $q = 1$ theory, position-momentum uncertainty is the same for vacuum as well as coherent state, and it is the minimum . But for $q \neq 1$,uncertainty for q-CS is different from that of q-vacuum and is α- dependent.

Let us introduce two self adjoint operators A_1 and A_2,which are the quadrature components. such that

$$\begin{aligned} a &= A_1 + iA_2 \\ a^\dagger &= A_1 - iA_2 \end{aligned} \tag{2.29}$$

then it can be seen that

$$\begin{aligned} VarA_1 &= \frac{1}{4}\left((q-1)\mid \alpha \mid^2 + 1\right) \\ VarA_2 &= \frac{1}{4}\left((q-1)\mid \alpha \mid^2 + 1\right) \\ (\Delta A_1)(\Delta A_2) &= \frac{1}{4}\left((q-1)\mid \alpha \mid^2 + 1\right) \end{aligned} \tag{2.30}$$

The quantum noise energy[60] is zero for q-CS also:

$$_q < \alpha \mid (\Delta a)^\dagger (\Delta a) \mid \alpha >_q = 0 \tag{2.31}$$

Let us now define two new operators B and $B^\dagger$:

$$\begin{pmatrix} B \\ B^\dagger \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \nu^* & \mu^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

with

$$bb^\dagger - b^\dagger b = q^N \tag{2.32}$$

The SU(1,1) squeezed states of the q-oscillator are defined as follows:

$$b \mid \beta >_q = \beta \mid \beta >_q \tag{2.33}$$

We can express $\mid \beta >_q$ as a linear combination of the number states $\mid n >_q$ [61]:

$$\mid \beta >_q = \sum_{n=0}^{\infty} C_n \mid n >_q \tag{2.34}$$

$$b \mid \beta >_q = (\mu a + \nu a^\dagger) \sum_{n=0}^{\infty} C_n \mid n >_q = \beta \sum_{n=0}^{\infty} C_n \mid n >_q \tag{2.35}$$

It follows that

$$C_1 = \beta C_0/\mu, C_2 = (\beta C_1 - \nu C_0)/\sqrt{[2]}\mu \tag{2.36}$$

Or,in general,

$$C_n = \frac{\beta C_{n-1} - \nu\sqrt{[n-1]}C_{n-2}}{\mu\sqrt{[n]}} \tag{2.37}$$

Although, in general μ and ν are arbitrary, we consider the evolution of a state which is initially a q-coherent state ($\mu(0) = 1, \nu(0) = 0$) into a q-squeezed state $\mid \beta >_q$ at t. This procedure is motivated by similar considerations made in [68] in the context of ordinary coherent states and squeezed states. We have

$$\begin{aligned} a &= \mu^* B - \nu B^\dagger \\ a^\dagger &= \nu^* B + \mu B^\dagger \end{aligned}$$

Hence we can write the Hamiltonian as

$$H = \frac{\hbar\omega}{2}\left(2\mu^*\nu^* B^2 - 2\mu\nu B^{\dagger 2} + BB^\dagger + B^\dagger B\right) \tag{2.38}$$

It can easily been shown that

$$\begin{aligned} < A_1 >_\beta &= \frac{1}{2}\left(\beta(\mu^* - \nu^*) + \beta^*(\mu - \nu)\right) \\ < A_2 >_\beta &= \frac{1}{2}\left(\beta^*(\mu + \nu) + \beta(\mu^* + \nu^*)\right) && (2.39) \\ Var A_1 &= \frac{1}{4}\mid \mu - \nu \mid^2 \sigma(t) \\ Var A_2 &= \frac{1}{4}\mid \mu + \nu \mid^2 \sigma(t) && (2.40) \\ (\Delta A_1)(\Delta A_2) &= \frac{1}{4}\mid \mu^2 - \nu^2 \mid \sigma(t) && (2.41) \end{aligned}$$

where

$$\sigma(t) = \sum_{n=o}^{\infty} q^n \mid C_n \mid^2 \tag{2.42}$$

Normalisation of $\mid \beta >_q$ demands

$$\sum_{n=0}^{\infty} \mid C_n \mid^2 = 1 \tag{2.43}$$

This implies the inequality $\sigma(t) \leq 1$. At this point we wish to draw a distinction between squeezing and q-squeezing. By squeezing we mean that the variance goes below the value of the uncertainty product in the q-vacuum state,which is the same as that for the ordinary vacuum state. For an undeformed oscillator the uncertainty product is the same for vacuum as well as for CS. But when $q \neq 1$,the uncertainty product has different values corresponding to q-vacuum and q-CS.

From(37),

$$C_n(0) = \frac{\beta C_{n-1}}{\sqrt{[n]}} = \frac{\beta^n C_0}{\sqrt{[n]!}} \tag{2.44}$$

Hence

$$\sum_{n=0}^{\infty} | C_n(0) |^2 = C_0^2 \exp_q | \beta |^2 \tag{2.45}$$

and

$$\begin{aligned} \sigma(0) &= \sum_{n=0}^{\infty} q^n | C_n(0) |^2 \\ &= | C_0 |^2 \exp_q(q| \beta |^2) \\ &= \frac{\exp_q(q| \beta |^2)}{\exp_q(| \beta |^2)} \end{aligned} \tag{2.46}$$

We assumed that $| \beta >_q$ is prepared intially as a q-CS. Hence

$$\begin{aligned} \frac{1}{4}\sigma(0) &= \frac{1}{4}\left((q-1)| \alpha |^2 + 1\right) \\ ie \quad \frac{\exp_q(q| \beta |^2)}{\exp_q(| \beta |^2)} &= \left((q-1)| \alpha |^2 + 1\right) \end{aligned} \tag{2.47}$$

Thus in the q-deformed case, α and β are interrelated. But for $q = 1$ they are independent of each other.

Squeezing means (ΔA_1) or $(\Delta A_2) \leq \frac{1}{4}$ and q-squeezing means (ΔA_1) or $(\Delta A_2) \leq \frac{1}{4}\sigma(0)$. We may define the degree of q-squeezing and the degree of q-squeezing respectively as

$$S^{(A)} = \frac{2 < (\Delta A)^2 > -1/4}{1/4} \quad (2.48)$$

$$S_q^{(A)} = \frac{2 < (\Delta A)^2 > -\sigma(0)/4}{\sigma(0)/4} \quad (2.49)$$

Squeezing corresponds to $S^{(A_1)} < 0$ or $S^{(A_2)} < 0$,while q-squeezing implies $S_q^{(A_1)} < 0$ or $S_q^{(A_2)} < 0$.

2.4 $SU(1,1)$ squeezed states of some q-deformed systems

2.4.1 q-deformed Batemann Hamiltonian

Baseia *etal.* [68] studied the appearance of squeezed states for the Batemann Hamiltonian where the mass of the oscillator changes suddenly. Here we discuss the squeezing and q-squeezing properties associated with $SU(1,1)$ squezed states of a real q-deformed Batemann Hamiltonian, defined by the relation

$$H(t) = \frac{P^2}{2M(t)} + \frac{1}{2}M(t)\omega^2 X^2 \quad (2.50)$$

where $M(t) = M_0 e^{\lambda t}$. We take the corresponding q-deformed Hamiltonian in the form

$$H_q(t) = \frac{P_q^2}{2M(t)} + \frac{1}{2}M(t)\omega^2 X_q^2 \quad (2.51)$$

Setting

$$\mu(t) = \cosh(\lambda t/2) = \frac{1}{2}\left(\sqrt{\frac{M(t)}{M_0}} + \sqrt{\frac{M_0}{M(t)}}\right)$$

$$\nu(t) = \sinh(\lambda t/2) = \frac{1}{2}\left(\sqrt{\frac{M(t)}{M_0}} - \sqrt{\frac{M_0}{M(t)}}\right) \quad (2.52)$$

the condition $| \mu |^2 - | \nu |^2 = 1$ is easily satisfied.

We wish to consider the case where mass changes suddenly as $M_0 \to M_1$ at $t = t_1$:

$$M(t) = M_0\theta(t_1 - t) + M_1\theta(t - t_1) \tag{2.53}$$

where θ is the Heaviside step-function. The quadrature components A_1 and A_2 have equal variances:

$$(\Delta A_1)^2{}_\beta(\Delta A_2)^2{}_\beta = \frac{1}{4}\sigma(t) = (\Delta A_1)^2_\beta(\Delta A_2)^2_\beta \tag{2.54}$$

Thus $| \beta >_q$ behaves as a q-CS. For $t > t_1$,

$$\begin{aligned} M(t) &= M_0 + \Delta M = M_0(1+\delta) \\ \delta &= \frac{\Delta M}{M_0}, \Delta M = M_1 - M_0 \end{aligned} \tag{2.55}$$

Then

$$\begin{aligned} (\Delta A_1)^2 &= \frac{\sigma(t)}{4(1+\delta)} \\ (\Delta A_2)^2 &= \frac{\sigma(t)(1+\delta)}{4} \\ (\Delta A_1)(\Delta A_2) &= \frac{\sigma(t)}{4} \end{aligned} \tag{2.56}$$

This suggests that the uncertainty product can go below $\frac{1}{4}$. For $(\sigma(t) - 1) < \delta < (\frac{1}{\sigma(t)} - 1)$ both $(\Delta A_1)^2$ and $(\Delta A_2)^2$ fall below this value. In this region,one can in principle measure both A_1 and A_2 with uncertainties less than that predicted by Heisenberg's uncertainty principle. Also in the region $(\frac{\sigma(t)}{\sigma(0)} - 1) < \delta < (\frac{\sigma(0)}{\sigma(t)} - 1)$, both A_1 and A_2 have variances below that in q-CS. The squeezing pattern is as follows:

A_1 is squeezedif $\delta > (\sigma(t) - 1)$

and A_2 is squeezed if $\delta > (\frac{\sigma(t)}{\sigma(0)} - 1)$

Similar remarks apply to squeezing and q-squeezing of the A_2 component.

2.4.2 System with constant mass and time dependent frequency

Consider the q-deformed Hamiltonian

$$H_q = \frac{P_q^2}{2M} + \frac{1}{2}M{\omega_0}^2[1 + \xi\cos{(2\omega_0 + \epsilon)t}]{X_q}^2 \tag{2.57}$$

If we put

$$\begin{aligned}
\mu(t) &= \frac{1}{2}[\left(\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi}\right)^{1/2} + \left(\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi}\right)^{-1/2}] \\
\nu(t) &= \frac{1}{2}\{\left(\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi}\right)^{1/2} - \left(\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi}\right)^{-1/2}\}
\end{aligned} \tag{2.58}$$

the condition $\mid \mu \mid^2 - \mid \nu \mid^2 = 1$ is satisfied. Then

$$\begin{aligned}
(\Delta A_1)^2 &= \frac{1}{4}\{\left(\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi}\right)^{-1}\}\sigma(t) \\
(\Delta A_2)^2 &= \frac{1}{4}\{\left(\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi}\right)\}\sigma(t) \\
(\Delta A_1)(\Delta A_2) &= \frac{1}{4}\sigma(t)
\end{aligned} \tag{2.59}$$

A_1 is squeezed if $\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi} > \sigma(t)$

and A_1 is q-squeezed if $\frac{1+\cos{(2\omega_0+\epsilon)t}}{1+\xi} > \frac{\sigma(t)}{\sigma(0)}$

2.4.3 Harmonic oscillator with time dependent mass and frequency

Consider the q-deformed harmonic oscillator whose mass and frequency change exponentially with time. The Hamiltonian has the form

$$H_q(t) = \frac{P_q^2}{2M(t)} + \frac{1}{2}M(t)\omega(t)^2 X_q^2 \tag{2.60}$$

where

$$M(t) = M_0 e^{\lambda t}, \quad \omega(t) = \omega_0 e^{-\rho t} \tag{2.61}$$

The coefficients $\mu(t)$ and $\nu(t)$ are taken as

$$\begin{aligned} \mu(t) &= \cosh\left(\frac{\lambda - \rho}{2}\right) t \\ \nu(t) &= \sinh\left(\frac{\lambda - \rho}{2}\right) t \end{aligned} \tag{2.62}$$

The variances of quadrature components are calculated as :

$$\begin{aligned} (\Delta A_1)^2 &= \frac{1}{4} e^{-(\lambda-\rho)t}\sigma(t) \\ (\Delta A_2)^2 &= \frac{1}{4} e^{(\lambda-\rho)t}\sigma(t) \\ (\Delta A_1)(\Delta A_2) &= \frac{1}{4}\sigma(t) \end{aligned} \tag{2.63}$$

Here also the uncertainty product may decrease below $\frac{1}{4}$. A_1 is squeezed if $(\lambda-\rho) < \frac{-1}{t}\ln(\sigma(t))$ and q-squeezed if $(\lambda - \rho) < \frac{1}{t}\ln\left(\frac{\sigma(0)}{\sigma(t)}\right)$. Similar conditions can be obtained for A_2 also.

2.5 Conclusion

We have constructed the coherent states and squeezed states corresponding to a q-deformed harmonic oscillator. The q-CS is found to obey a q-deformed Poissonian distribution. Although the variances of both the quadrature components are same in q-CS,they differ from the variance of the q-vacuum. Also the variances are dependent on the parameter α which is the eigen value of q-deformed annihilation operator.

Since the variances of q-vacuum and q-CS are different, we can for the generic values of q, define squeezing and q-squeezing. We considered the SU(1,1) squeezed states which were prepared initially as q-CS. The parameters α and β are found to be inter-related in this case, which is not observed in the $q = 1$ theory. If we could achieve a situation in which $q < 1$,we could measure X_q or P_q (or both) with variances below $\hbar^2/4$. Squeezing and q-squeezing have been illustrated using three model Hamiltonians.

We have calculated the probability distribution P_n for various values of q keeping α fixed. In figure 1,$log_{10}P_n(q)$ is plotted against n for different values q,namely,$1, 0.9, 0.8$ and 0 for $\alpha = 0.5$. Figure 2 gives the corresponding graphs for $\alpha = 0.9$. The graphs corresponding to the same q but different α are similar .

In figure 3 ,$log_{10}P_0(q)/P_0(1)$ is plotted against q where $P_0(q)$ is the probability of the ground state for a generic value of q and $P_0(1)$ is for the special case $q = 1$. These graphs plotted for two different values of α namely,$\alpha = 0.5$ and $\alpha = 0.9$ show marked variation in their behaviour except near the standard value,namely,$q = 1$,where they converge.

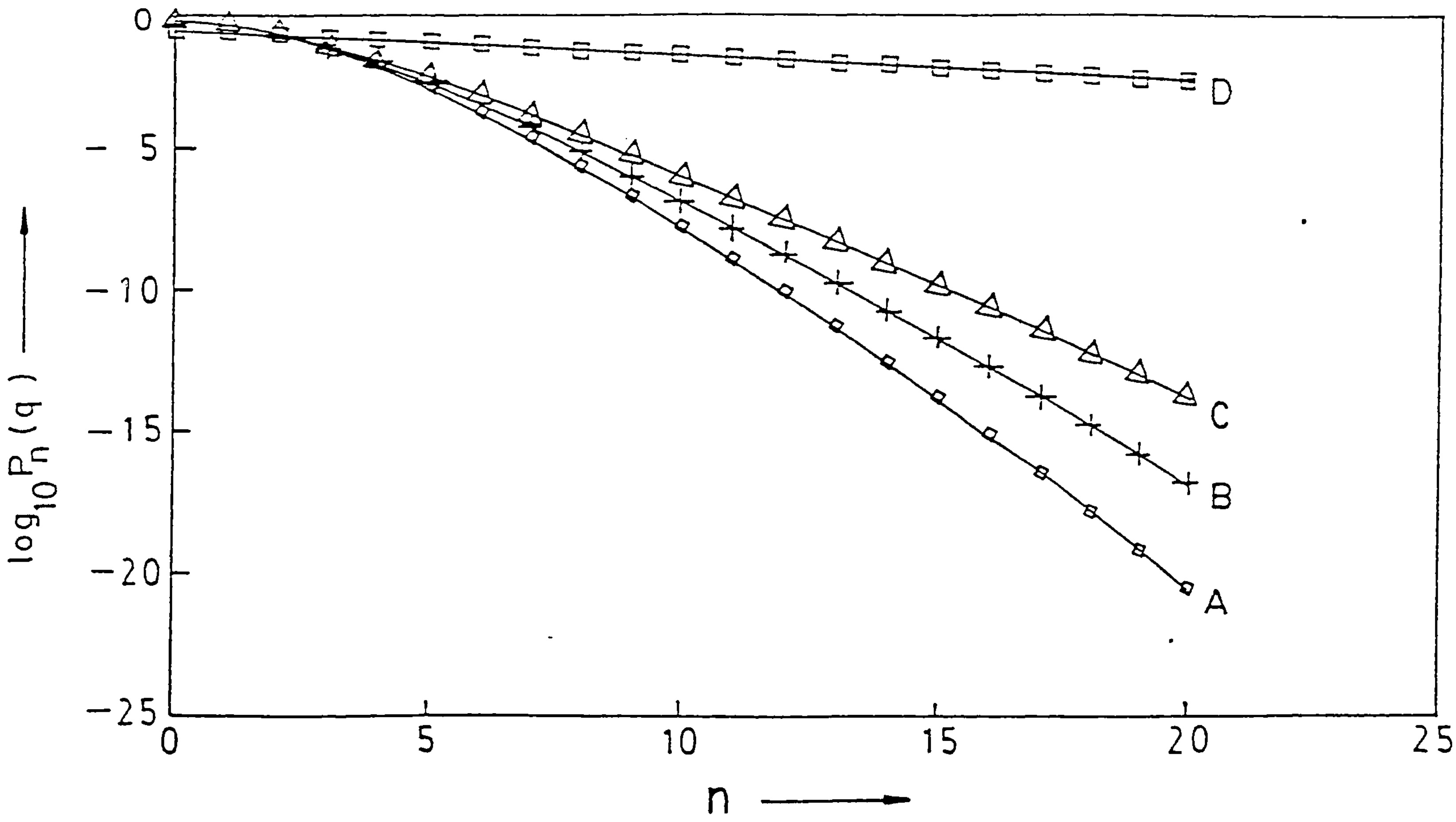

Fig 1.1 $\log_{10} P_n(q)$ versus n for $(\alpha = 0.5)$:

(A). $q = 1$; (B) $q = 0.9$; (C) $q = 0.8$; (D) $q = 0$;

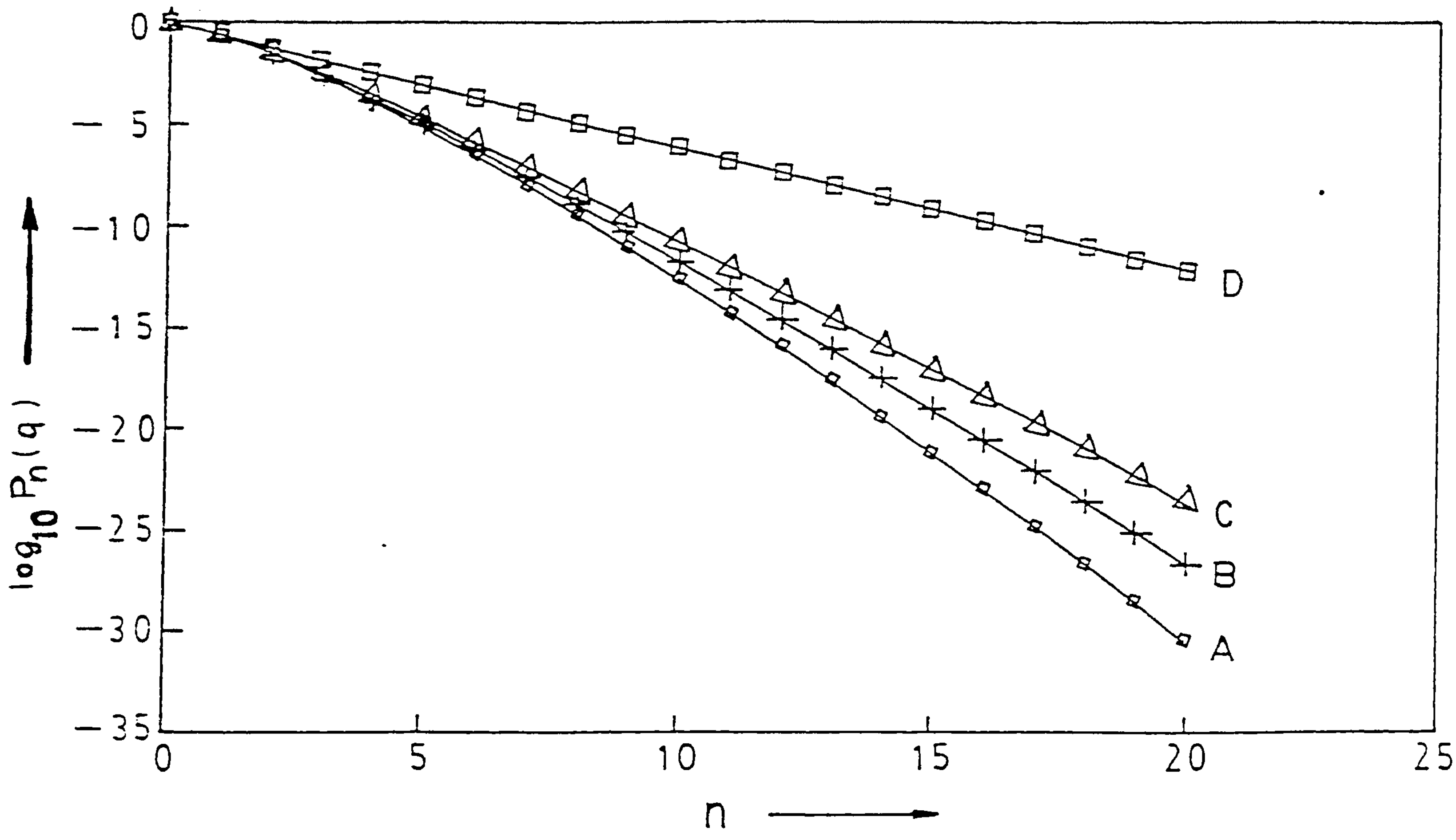

Fig 1.2 $\log_{10} P_n(q)$ versus n for$(\alpha = 0.9)$:

(A). $q = 1$; (B)$q = 0.9$; (C)$q = 0.8$; (D)$q = 0$;

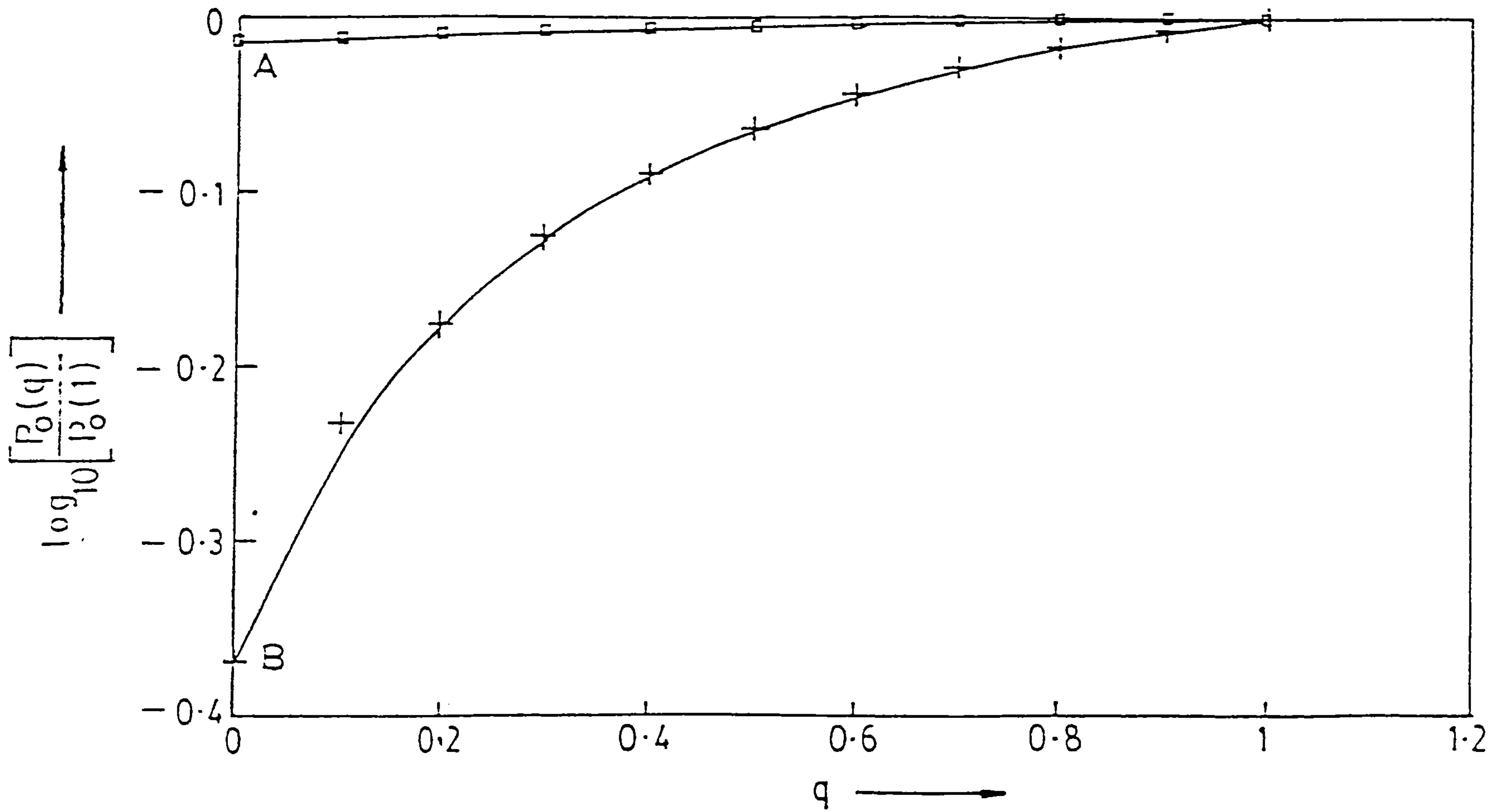

Fig 1.3 Variation of $\log_{10} \frac{P_0(q)}{P_0(1)}$ with q:

(A) $\alpha = 0$; (B) $\alpha = 0.9$

Chapter 3

q-Deformed quantum mechanics

The q-deformed harmonic oscillators[38,39] inspire the search for a q-deformed quantum mechanics which produce the results of standard quantum mechanics when the deformation parameter approaches a particular value. The q-deformed calculus,when applied to quantum mechanics with its fundamental postulates preserved ,gives rise to q-quantum mechanics. Several years ago Janussis et al. [69,70] discussed q-quantisation and the eigenvalue problem of q -differential operators. Quantum mechanics is usually deformed in two different ways: either one may replace the canonical commutation relation by a q-commutation relation or may replace the momentum operator in the Schrödinger equation (SE) by a q-deformed one. Minahan [71] has considered a q-extension of SE (q-SE) and obtained the spectrum of q-harmonic oscillator. The energy spectrum of a q-analog of hydrogen atom has been obtained by Yang and Xu[72]. Usually the Schrödinger equation is deformed by replacing the momentum operator by its q-extension. The form of the q-momentum operator depends on the definition of the q-difference operator used. Wess and Zumino[73] have constructed a q-deformed momentum operator which is Hermitian when q is a root of unity. Li and Sheng [74] have presented a q-deformation of quantum mechanics and have solved the q-SE for the harmonic oscillator potential.

Here we propose a general formalism for q-quantum mechanics and develop a q-deformed Schrödinger equation . Further the q-hermiticity (explained in section 1) of q-momentum operators and its powers is studied. It is obsereved that the momentum itself (along with its odd powers) is not q-Hermitian,but the square of momentum (and all its powers) is q-Hermitian.

3.1 General formulation of q-quantum mechanics

In the linear space $\mathcal{E}$,we define a norm mapping

$\mathcal{E} \rightarrow \mathcal{R}^{+}$: $v \rightarrow \| v \|$such that

$\| \alpha v \| = | \alpha | \| v \|$, $\| v + u \| \leq \| v \| + \| u \|$,

and $\| v \| = 0 \rightarrow v = 0$.

$u\ , v\ \epsilon\ \mathcal{E}$, $\alpha\ \epsilon\ \mathcal{C}$. For square integrable functions,the norm is defined as,

$$\| v \| = \left[\int v^{*}(x) v(x) d(qx) \right]^{\frac{1}{2}} \tag{3.1}$$

where the integral will be defined shortly. Also we define the scalar product of two vectors u and v in $\mathcal{E}$ as

$$\begin{aligned} \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{C} : (u, v) \rightarrow \ \langle u \mid v \rangle &= \frac{1}{4} \Big\{ \| u + v \|^{2} - \ \| u - v \|^{2} \\ &-i \ \ \| u + iv \|^{2} + i \| u + v \|^{2} \Big\} \end{aligned} \tag{3.2}$$

Using(1),

$$\langle u \mid v \rangle = \int u^*(x)v(x)d(qx) \tag{3.3}$$

which has the properties

$\| v \|^2 = \langle v \mid v \rangle \, , \langle v \mid u \rangle = \langle u \mid v \rangle^* \, , \langle v \mid \alpha u \rangle = \alpha \, \langle v \mid u \rangle,$

$\langle u \mid v + w \rangle = \langle u \mid v \rangle + \langle u \mid w \rangle$ and $\langle v \mid v \rangle = 0 \quad iff \;\; v = 0.$

The integral should satisfy the above conditions as well as the fundamental theorem of integral calculus,namely

$$\int_a^b [D_x f(x)] \, d(qx) = f(b) \; - \; f(a) \tag{3.4}$$

A linear vector space is said to be complete if there exists a set of vectors in it such that every vector v in the space can be expressed as a convergent sum of these vectors.

A complete linear space, with a norm defined by (1) and (2) is called a q-Banach space.

The linear space $\mathcal{E}'$ of the continous linear functionals of a q-Banach space is called its q-dual space.

Let $\alpha(\mathcal{E}, \mathcal{F})$ denotes the space of continous linear mappings of the q-Banach space $\mathcal{E}$ into the q-Banach space $\mathcal{F}$. Then $A \;\; \epsilon \;\;\; \alpha(\mathcal{E}, \mathcal{F})$ induces a mapping $A^\dagger : \;\; \mathcal{F}' \rightarrow \;\; \mathcal{E}'$ known as the q-adjoint operator. The q-adjoint operator is unique and is defined by

$$\int u^*(x)Av(x)d(qx) \;\; = \;\; \int \Big[A^\dagger u(x) \Big]^* v(x)d(qx) \tag{3.5}$$

If $A = A^{\dagger}$, A is said to be q-Hermitian.

$$ie,\ \int u^{*}(x)Av(x)d(qx) = \int [Au(x)]^{*}\, v(x)d(qx) \tag{3.6}$$

The eigen values of a q-Hermitian operator are real:

Let $Au(x) = \alpha u(x)$

$$(\alpha - \alpha^{*}) \int u^{*}(x)u(x)d(qx) = 0 \tag{3.7}$$

which implies $\alpha = \alpha^{*}$

Also eigen vectors of a q-Hermitian operator belonging to different eigen values are q-orthogonal:

Let $Au(x) = \alpha u(x)$ and $Av(x) = \beta v(x)$

$$\begin{aligned} \int u^{*}(x)Av(x)d(qx) &= \int [Au(x)]^{*}\, v(x)d(qx) \\ ie\ \ (\alpha - \beta) \int u(x)^{*}v(x)d(qx) &= 0 \end{aligned} \tag{3.8}$$

3.2 q-Deformation of the Schrödinger equation

The q-deformed Schrödinger equation is

$$H_q \psi_q = E_q \psi_q \tag{3.9}$$

$$H_q = \mathbf{p}^2/2m + V(x) \tag{3.10}$$

where $\mathbf{p}$ is the q-deformed momentum operator.

In the coordinate representation of quantum mechanics, $-i\hbar\frac{d}{dx}$ serves as the momentum operator. So the q-momentum operator should contain a q-difference operator.There are several realisations of q-difference operator. In quantum mechanics the harmonic oscillator serves as a key to more general systems since any general potential can be approximated to harmonic oscillator potential near equilibrium points. In standard quantum mechanics,the creation and annihilation operators corresponding to the harmonic oscillator obey the algebra $[a\ , a^{\dagger}] = 1$. The differential operator $\frac{\mathrm{d}}{\mathrm{dx}}$ satisfies the commutation relation $[\frac{\mathrm{d}}{\mathrm{dx}},\ \mathbf{x}] = 1$. There is a homomorphism between $\mathbf{a}$ and $\frac{\mathrm{d}}{\mathrm{dx}}$ as well as between $\mathbf{a}^{\dagger}$ and $\mathbf{x}$. So the q-difference operator is expected to be homomorphic to the q-annihilation operator. Although there are several versions of q-oscillator algebra,we prefer the algebra proposed by Greenberg[44,45], which reads

$$aa^{\dagger} - qa^{\dagger}a = 1 \tag{3.11}$$

The q-difference operator is assumed to obey the q-commutation relation

$$D_x x - qxD_x = 1 \tag{3.12}$$

We further demand that D_x should obey the q-product rule

$$D_x f(x)g(x) = f(qx)D_x g(x) + [D_x f(x)]g(x) \tag{3.13}$$

and

$$D_x(x) = 1 \tag{3.14}$$

These conditions lead to a q-difference operator of the form

$$D_x f(x) = \frac{f(qx) - f(x)}{x(q-1)} \tag{3.15}$$

which is same as (1.40). The q-integration,which is the inverse operation of q-differentiation is defined as:

$$\int_{x_i}^{x_f} f(x)d(qx) = \sum_{n=i}^{f} q^n x_0(q-1)f(q^n x_0) \tag{3.16}$$

where $x_i = q^i, x_f = q^f x_0$ and x_0 is an arbitrary positive constant. Note that the q-difference operator and q-integral operator introduced in this article are defined on a lattice in which the lattice spacings vary in a geometric fashion. When the integration extends over the entire lattice,

$$\int f(x)d(qx) = \sum_{n=-\infty}^{\infty} q^n x_0(q-1)f(q^n x_0) \tag{3.17}$$

The q-integral defined above satisfies all the requirements pointed out in the previous section. The prouct rule for this q-integral reads:

$$\int f(x)g(x)d(qx) = f(x)g(x) - \int g(qx)\{D_x f(x)\}d(qx) \tag{3.18}$$

If we define the q-momentum operator as $-i\hbar D_x$ and if we use equation(12) to obtain a q-commutation relation for position and momentum operators some inconsistencies will arise on taking the adjoint of the q-commutator. This arises because of the assumption that **p** is real,which can not be true for real values of q. In fact, $-iD_x$ is not a self-adjoint operator . It is clear from eq.(5) that

$$(D_x)^\dagger = -D_x q^{-x\partial_x} \tag{3.19}$$

But $-D_x^2 q^{-x\partial_x}$ will be q-Hermitian which can be proved explicitly if we use the definition of q-adjointness as well as the product-rule for q-integration(1.48):

$$\begin{aligned}\int u^*(x)D_x{}^2 q^{-x\partial_x} v(x)d(qx) &= \int u^*(x)D_x(D_x v(q^{-1}x))d(qx) \\ &= \mid u^*(x)D_x v(q^{-1}x) \mid - \int D_x u^*(x) q^{1/2} D_x v(x)d(qx) \\ &= -q^{1/2}\int D_x u^*(x)D_x v(x)d(qx) \end{aligned} \tag{3.20}$$

Similarly it can be shown that

$$\int (D_x^2 q^{-x\partial_x} u(x))^* v(x)d(qx) = -q^{1/2}\int D_x u^*(x)D_x v(x)d(qx) \tag{3.21}$$

Hence we can take

$$\mathbf{p}^2 \;=\; -\hbar^2 D_x^2 q^{-x\partial_x} \tag{3.22}$$

Hence the Hamiltonian becomes

$$\mathbf{H_q} \;=\; \frac{-\hbar^2}{2m} D_x^2 q^{-x\partial_x} \;+\; \mathbf{V(x)} \tag{3.23}$$

Thus the q-Schrödinger(q-SE) equation takes the form

$$\left(\frac{-\hbar^2}{2m} D_x^2 q^{-x\partial_x} \;+\; \mathbf{V(x)}\right)\psi_q \;=\; E_q\psi_q \tag{3.24}$$

3.3 Physical implications of q-deformation

It is easy to show that

$$-\,D_x^2 q^{-x\partial_x} \;=\; \left(-iq^{-1/4} D_x q^{-1/2x\partial_x}\right)^2 \tag{3.25}$$

So $-i\hbar q^{-1/4} D_x q^{-1/2x\partial_x}$ is defined as the q-momentum operator. But surprisingly, **p** is not q-Hermitian. It can be shown that for real q, **x** and **p** can not be simultaneously q-Hermitian if the q-commutator is a c-number function. Assume a q-commutation relation of the form:

$$\mathbf{xp} \;-\; f(q)\mathbf{px} \;=\; i\hbar g(q) \tag{3.26}$$

where f and g are real valued functions of q and $f(q) \to 1$ and $g(q) \to 1$ as $q \to 1$. If **x** and **p** are q-Hermitian, taking the adjoint of the above equation gives

$$\mathbf{px} \;-\; f(q)\mathbf{xp} \;=\; -i\hbar g(q) \tag{3.27}$$

This leads to a contradictory result **xp** = -**px** for $q \neq 1$. But if the c-number function g(q) is replaced by an operator, the contradiction is solved. For example if we replace g(x) by $q^{x\partial_x+1}$, the q-commutation relation will be consistent with q-Hermitian **x** and **p**.

Thus **p** defined by (22) is not a q-Hermitian operator ;but its square is q-Hermitian. This is not possible in standard(q=1) quantum mechanics where only Hermitian or skew-Hermitian operators can yield a Hermitian operator on squaring. In q-deformed case,there may exist operators which themselves are neither hermitian nor anti-Hermitian,but their squares are Hermitian. This suggests the following q-symmetrised form for any operator **F** in a q-Hilbert space:

$$\mathbf{F} = \mathbf{A} + \xi(q)\mathbf{B} \tag{3.28}$$

where **A** and **B** are two anti commuting q-Hermitian operators and $\xi(q) \to 0$ as $q \to 1$.

For a q-lattice, the Taylor expansion for a function is

$$f(x+a) = exp_q(aD_x)f(x) \tag{3.29}$$

where

$$exp_q x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{3.30}$$

with [n]!=[n][n-1][n-2]......[2][1]. and $[n] = \frac{q^n-1}{q-1}$. The q-translation operator is not unitary since p is not q-Hermitian. Thus translational invariance is lost as a result of q-deformation. But space inversion symmetry is not affected by q-deformation since $D_x q^{-x\partial_x}$ is unchanged by the replacement $x \to -x$.

It is natural to enquire about the hermitian nature of the powers of **p**. We can easily show that the even powers of **p** are q-Hermitian while the odd powers are not q-Hermitian: For n=1,2,3,.....

$$\begin{aligned}\int\phi^*(x)\mathbf{p}^{2n}\psi(x)d(qx) &= \int\phi^*(qx)\mathbf{p}^2\mathbf{p}^{2n-2}\psi(x)d(qx)\\ &= q^{n^2-5n+4}\int\mathbf{p}^2\phi^*(qx)\mathbf{p}^{2(n-1)}\psi(qx)d(qx)\\ &= q^{n^2-9n+9}\int\mathbf{p}^4\phi^*(q^2x)\mathbf{p}^{2(n-2)}\psi(q^2x)d(qx)\\ &= q^{n^2-13n+16}\int\mathbf{p}^6\phi^*(q^3x)\mathbf{p}^{2(n-3)}\psi(q^3x)d(qx)\end{aligned} \tag{3.31}$$

In general,for $m\epsilon Z$, m $\leq$ n,

$$\int\phi^*(x)\mathbf{p}^{2n}\psi(x)d(qx) = q^{n^2-(4m+1)n+(m+1)^2}\int\mathbf{p}^{2m}\phi^*(q^mx)\mathbf{p}^{2(n-m)}\psi(q^mx)d(qx) \tag{3.32}$$

If we put m=n/2 in (32),

$$\begin{aligned}\int\phi^*(x)\mathbf{p}^{2n}\psi(x)d(qx) &= q^{-3n^2/4+1}\int\mathbf{p}^n\phi^*(q^{n/2}x)\mathbf{p}^n\psi(q^{n/2}x)d(qx)\\ &= \int\left[\mathbf{p}^{2n}\phi^*(x)\right]^*\psi(x)d(qx)\end{aligned} \tag{3.33}$$

Thus even powers of **p** are q-Hermitian.

For odd powers of **p**,(n= 0,1,2,3,....)

$$\int\phi^*(x)\mathbf{p}^{2n+1}\psi(x)d(qx) = const.\int\mathbf{p}^{2m}\phi^*(q^mx)\mathbf{p}^{2(n-m)+1}\psi(q^mx)d(qx) \tag{3.34}$$

Since the momentum operator contains $q^{-1/2x\partial_x}$,which multiplies the arguments of the operands by $q^{-1/2}$,it follows that the odd powers of **p** are not q-Hermitian.

3.4 Representation dependence of q-quantum mechanics

Standard (q =1) quantum mechanics is representation independent. But q-deformed quantum mechanics is found to be representation dependent. The entire nature of the q-deformed theory determined by the q-momentum operator, which depends on the choice of the q-difference operator. Here we have preferred the definition (1.40) for the q-difference operator and the corresponding q-momentum operator is not q-Hermitian. But if one chooses the definition(1.50), one can have a self adjoint q-momentum operator[71,74]. But we prefer this particular definition due to two main reasons: firstly, it is more suited for the description of non-commutative geometry and is closely related to infinite statistics, secondly,the possibility of non-Hermitian momentum giving Hermitian energy may be helpful in circumventing the difficulties associated with the theories of the Early Universe.

3.5 The particle in a box

The infinite potential well is characterised by a constant potential in a finite region outside which the potential is infinite. Particles subject to such a potential are trapped inside the constant potential region.This model potential has been used in the free electron model of metals. Study of any problem in q-quantum mechanics is of interest not only for pedagogic reasons but also for comprehending the uniqueness of standard (q=1) quantum mechanics. One cannot also rule out the possible existence of q-systems in nature.

In this problem,

$$\begin{aligned} V(x) &= 0 \quad for \quad 0 \le x \le a, \\ &= \infty \quad otherwise. \end{aligned} \tag{3.35}$$

Solution of the q-SE for this potential is

$$\psi_q(x) = N \sum_{r=0}^{\infty} \frac{(-1)^2 q^{r(r+2)} (kx)^{2r+1}}{[2r+1]!} \tag{3.36}$$

N is the normalisation constant. Since $\psi_q(0) = \psi_q(a) = 0$,the admissible solutions of ka are those which satisfy

$$\sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r+2)} (ka)^{2r+1}}{[2r+1]!} = 0 \tag{3.37}$$

Only numerical solutions are possible. The q-normalisation constant N can be evaluated from

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\ d(qx) = 1 \tag{3.38}$$

Let $\psi^*\psi$ be expressed as

$$\psi^*(x)\psi(x) = \sum_{r=0}^{\infty} b_r (kr)^2 r \tag{3.39}$$

$$b_r = \begin{cases} -2\sum_{s=0}^{(r-2)/2} \frac{q^{(r-s-1)(r-s+1)+s(s+2)}}{[2s+1]!\,[2r-2s-1]!} & \text{for even r} \\ 2\sum_{s=0}^{(r-1)/2} \frac{q^{(r-s-1)(r-s+1)+s(s+2)}}{[2s+1]!\,[2r-2s-1]!} - \frac{q^{(r-1)(r+3)/2}}{[r]![r]!} & \text{for odd r} \end{cases}$$

Since the series expansion of $\psi^*(kx)\psi(kx)$ is convergent,we may q-integrate each term in the expansion (36) employing the identity

$$\int x^n d(qx) = \frac{x^{n+1}}{[n+1]} \tag{3.40}$$

which gives

$$\begin{aligned} N^2 \sum_{r=0}^{\infty} b_r \frac{(k)^2 r a^{2r+1}}{[2r+1]} &= 1 \\ or, \quad N &= \left(a \sum_{r=0}^{\infty} \frac{b_r C_n^{2r}}{[2r+1]} \right)^{\frac{-1}{2}} \end{aligned} \tag{3.41}$$

where C_n are the solutions of ka given by (37).

3.6 Analytic solutions for q $\approx$ 1

Analytic solutions exist when q is close to unity. Let us take $q = 1 - \delta$, δ being a very small quantity.. The following approximations are valid.

$$[n] = n\{1 - (n-1)\delta/2\} \tag{3.42}$$

$$[n]! = n!\{1 - n(n-1)\delta/2\} \tag{3.43}$$

The wavefunction is approximated by

$$\psi(kx) = N\sin kx + N\sum_{r=0}^{\infty} \frac{(-1)^{r+1}(kx)^{r+1}}{(2r+1)!} \tag{3.44}$$

The coefficients b_r take the form

$$b_r = \begin{cases} -2\sum_{s=0}^{(r-2)/2} \Delta_{r,s} & \text{for even r} \\ 2\sum_{s=0}^{(r-1)/2} \Delta_{r,s} - \frac{1}{r!^2}\left(\frac{1-(r-1)(2r+3)\delta}{1+r(r-1)\delta}\right) & \text{for odd r} \end{cases}$$

where

$$\Delta_{r,s} = \frac{1}{(2s+1)!(2r-2s-1)!}\left(1 + \frac{(r-1)(2r+3)\delta}{1+r(r-1)\delta}\right) \tag{3.45}$$

3.7 Conclusion

A general formalism for q-deformed quantum mechanics is proposed and the representation dependence is pointed out. A particular representation in which energy is Hermitian while momentum is non Hermitian is given. q-SE is constructed and numerical solutions are obtained for a particle confined in a one dimensional infinite box for values of q ranging from 0 to 1. It is observed that for lower values of q,higher energy levels are forbidden. This is due to the rapidly converging nature of the wavefunction for lower values of q. The numerical solutions are tabulated and the solution corresponding to the ground state of the system is plotted against q. The numerical solutions show that for $q \neq 1$,the energy eigenvalues have an upperbound even in systems which possess an infinite number of energy eigenvalues when q =1.

Table 1

Allowed values of ka

q	n=1	n=2	n=3	n=4
0.1	35.11869			
0.2	13.91709	69.87712		
0.3	8.564421	28.97956		
0.4	6.335347	16.46896	41.17533	
0.5	5.319289	10.92827	22.82981	45.21165
0.6	4.822798	8.039686	15.5551	24.57137
0.7	4.044871	7.795353	11.70577	15.03165
0.8	3.770396	7.872589	13.08165	14.74317
0.9	3.394796	6.803214	8.21286	

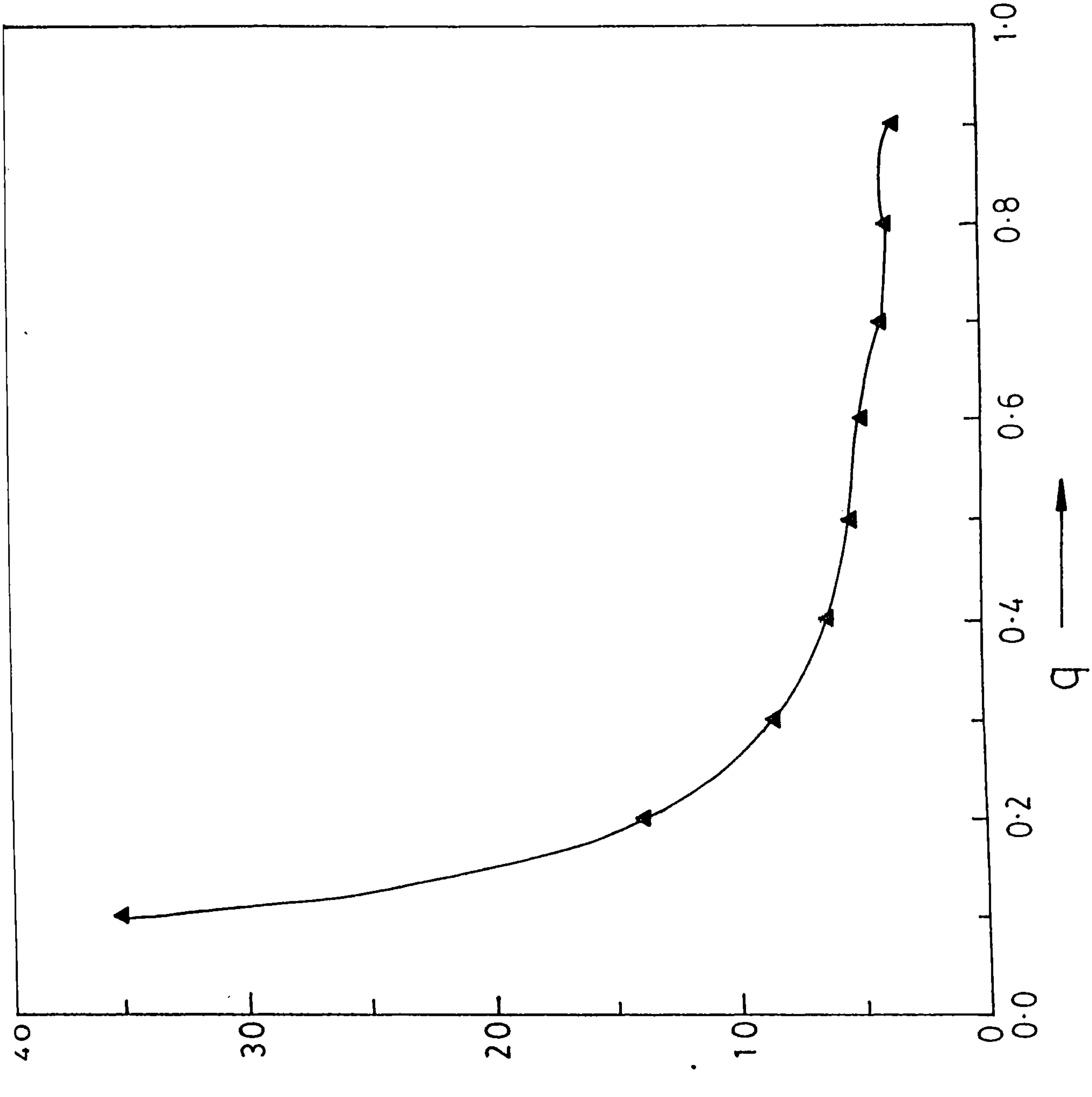

Fig 2.1 Variation of $| k_q a |$ (corresponding to the ground state) with q

Chapter 4

q-Deformed quantum field theory

Having applied the idea of q-deformation to the single particle quantum mechanics,it is natural to search for a q-deformed quantum field theory. The quon algebra introduced in Chapter1 can be made use in the formulation of a q-deformed field theory. Chaichian *etal.*[41] have constructed the single particle temperature Green function for non-relativistic quons. Goodison and Tom [75] have obtained the canonical partition function for quons. But quons corresponding to different modes do not commute;they actually qummute. Also for quons no commutation relation can be impossed on aa and $a^\dagger a^\dagger$. Relations such as $a_i a_j - q a_j a_i = 0$ and $a_i^\dagger a_j^\dagger - q a_j^\dagger a_i^\dagger = 0$ are meaningful only when $q^2 = 1$, since for generic q,interchange of i and j in those relations will lead to incosistencies. As a consequence, a many particle system composed of quons has no definite symmetry properties under particle interchange.

Here a q-deformed quantum field theory is considered in which different modes commute,but the single particle behaviour is quonic. Propagators are calculated for a scalar field and the single particle propagator is the same as that for

$q = 1$ theory except for a multiplication factor. Renormalisation of a self interacting field is discussed in this frame work.

4.1 q-deformed scalar field

The field operator of the q-deformed scalar field is expressed as

$$\phi_q(x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}(2\omega_k)^{\frac{1}{2}}} \left\{ a_q(k)e^{ikx} + a_q^\dagger(k)e^{-ikx} \right\} \tag{4.1}$$

The following CRs are postulated for the q-bosonic Fock space:

$$\begin{aligned} [a_q(k),\ a_q(k')] &= 0 = [a_q^\dagger(k),\ a_q^\dagger(k')] \\ \left[a_q(k),\ a_q^\dagger(k')\right] &= q^{N(k)}\ \delta(k - k') \end{aligned} \tag{4.2}$$

where $[N(k)] = a(k)^\dagger a(k)$. Using the above CRs and the plane wave expansion ,we obtain

$$[\ \phi_q(x)\ ,\ \phi_q(x')\ ] = i\ \Delta_q\ (x - x') \tag{4.3}$$

where the q-deformed function $\Delta_q\ (x - x')$ is

$$i\Delta_q\ (x - x') = \frac{-i}{(2\pi)^3} \int \frac{d^3k}{\omega_k}\ q^{N(k)}\ \sin\omega_k(x - x_0) \tag{4.4}$$

which is manifestly Lorentz invariant. Also ,the equal-time commutator of two field ampitudes vanishes as is evident from equations (3) and (4).It then follows that

$$[\ \phi_q(x)\ ,\ \phi_q(x')\ ] = 0 \quad for\ (x - x')^2 < 0 \tag{4.5}$$

Thus,q-deformations are consistent with the principle of microscopic causality. Hence at two points which cannot be causally connected, measurements of the field strengths are independent of each other. Thus without any harm we can interpret ϕ_q as a physical observable. The q-deformed chronological product is defined as[41]:

$$\begin{aligned} T\left(\phi_q\,(x)\phi_q\,(x')\right) &= \phi_q\,(x)\phi_q\,(x')\theta\,(x_0 - x_0') \\ &+\; q\,\phi_q\,(x')\phi_q\,(x)\;\theta\,(x_0' - x_0) \end{aligned} \tag{4.6}$$

where θ is the Heaviside step function. We define the q-Feynman propagator as

$$i\,\Delta_{F,q}\,(x - x') \;=\; {}_q\langle 0 \mid T\left(\phi_q\,(x)\phi_q\,(x')\,\right) \mid 0\,\rangle_q \tag{4.7}$$

It is easy to show that

$$i\,\Delta_{F},q\,(x - x') \;=\; \frac{i}{(2\pi)^4}\int\frac{d^4k e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon}\,\{\theta(x_0 - x_0') + q\theta(x_0' - x_0)\} \tag{4.8}$$

In momentum space,

$$\begin{aligned} i\Delta_{F,q}\,(k) &= i\int d^4x\; e^{ik'x}\;\Delta_{F,q}\,(x) \\ &= \int\frac{d^4k}{k^2 - m^2 + i\epsilon}\;\;\delta^3(k - k\,') \\ &\quad\times\int dx_0\;e^{-i(k_0 - k_0'\,)x_0} \\ &\quad\times\{\theta(x_0) + q\;\theta(-x_0)\;\} \end{aligned} \tag{4.9}$$

The x_0 integration on the right hand side can be evaluated as:

$$\begin{aligned} \int dx_0\;e^{-i(k_0 - k_0'\,)x_0}\;\{\;\theta\,(x_0) +\;q\,\theta\,(-x_0)\;\} \;&=\; \int_0^\infty dx_0\;e^{-i(k_0 - k_0'\,)x_0} \\ &+\;q\int_{-\infty}^0 dx_0\;e^{-i(k_0 - k_0'\,)x_0} \end{aligned} \tag{4.10}$$

Consider the first integral appearing on the right hand side of (10). Here the integrand has oscillating nature. It can be evaluated by multiplying with an exponentially decaying quantity and then taking the apropriate limit.

$$\int dx_0 \; e^{-i(k_0 - k_0')x_0} = \lim_{\eta \to 0} \int_0^\infty dx_0 \; e^{-[\, i\,(k_0 - k_0')\, + \, \eta \,]x_0}$$
$$= \frac{-i}{k_0 - k_0'} \tag{4.11}$$

Also,

$$\int_{-\infty}^0 dx_0 \; e^{-i(k_0 - k_0')x_0} = \int_{-\infty}^\infty dx_0 \; e^{-i(k_0 - k_0')x_0}$$
$$- \int_0^\infty dx_0 \; e^{-i(k_0 - k_0')x_0}$$
$$= \delta(k_0 - k_0') + \frac{i}{k_0 - k_0'} \tag{4.12}$$

Hence

$$\int dx_0 \; e^{-i(k_0 - k_0')x_0} \; \{ \, \theta\,(x_0) + \; q\,\theta\,(-x_0) \, \} = \delta(k_0 - k_0') + (q-1)\frac{i}{k_0 - k_0'} \tag{4.13}$$

From equations (9), and (12),

$$i\Delta_{F,q}\,(k) = \frac{iq}{k^2 - m^2 + i\epsilon}$$
$$+ \; i(q-1)\int \frac{d^4k\delta^3(k - k')}{(k^2 - m^2 + i\epsilon)(k_0 - k_0')} \tag{4.14}$$

We can take

$$\int_{-\infty}^\infty \frac{dk_0}{(k^2 - m^2 + i\epsilon)(k_0 - k_0')} = \lim_{\alpha \to \infty} \int_{-\alpha}^\alpha \frac{dk_0}{(k^2 - m^2 + i\epsilon)(k_0 - k_0')} \tag{4.15}$$

But it can be shown that

$$\int_{-\alpha}^\alpha \frac{dx}{(x^2 - a^2)(x - b)} = \frac{b}{a(a^2 - b^2)} \{\ln(\alpha + a) - \ln(\alpha - a)\}$$
$$+ \frac{1}{(b^2 - a^2)} \{\ln(\alpha - b) - \ln(\alpha + b)\} \tag{4.16}$$

Using equations (14), (15), and (16), we get

$$i\Delta_{F,q}(k) = q\left\{\frac{i}{k^2 - m^2 + i\epsilon}\right\} \tag{4.17}$$

The temperature Green's function for quons obtained by Chaichian *etal.* [41] has the form $\frac{1}{ik_4-\epsilon_4}$ which is the same as standard propagator with the exception that the frequency k_4 is given by $\frac{2n\pi}{\beta} - \frac{1}{\beta}lnq$. But the zero temperature limit of this q-propagator is q-independent. Thus at zero temperaturem, quon field theory is trivial. Effects of q-deformation appears only at finite temperatures. This is due to the particular choice of the field Hamiltonian in [41] ,which is proportional to the number operator which assumes that the quons are free. If the Hamiltonian is chosen as (1.105), partition function cannot be calculated in a closed form for generic values of q. Here we have calculated the q-Feynman propagator for q-oscillators, but no linear relationship is assumed between the field Hamiltonian and the number operator.

4.2 Renormalisation of ${\phi_q}^4$ theory

To exhibit the non-trivial consequences of multiplication of the propagator by q, let us renormalise the q-deformed real scalar field theory with ϕ_q^4 interaction. Though the ϕ_q^4 theory is purely pedagogical, since it is free from complications like charge and spin, it can effectively illustrate the properties of renormalisation. The Feynman rules remain the same except that propagators are now having the form $q\left\{\frac{i}{k^2-m^2+i\epsilon}\right\}$.

The renormalisation can be easily done via the method of dimensional regularisation

79]. In this scheme, integrations are performed in a D -dimensional momentum space rather than the usual 4 -dimensional space. D is a variable which is allowed to vary continously and need not be an integer. The sense in which the integral to be regarded as a D -dimensional one is in the sense in which the integral may be defined as a function of D by the process of analytic continuation in D. The integrals in ϕ_q^4 theory are perfectly finite if D is slightly less than 4. So we prefer to perform the calculations in $(4 - \epsilon)$ dimensions,with $\epsilon > 0$. Thus we can separate the integrals into those parts which will be infinite and those which will be finite at the value $D = 4$. Infinities appearing in different diagrams are cancelled by the corresponding counter terms.

We set $\hbar = c = 1$ so that the action integral must be dimensionless while length and mass are measured in inverse units:$[x]^{-1} = [m]$. The canonical dimensions of the field is determined by the condition that the kinetic term in the action, which has the form $\int d^n x \phi_q \partial^2 \phi_q$ is dimensionless. It demands that $[\phi] = [m]^{(n-2)/2}$. In the same manner the dimensions of the couplng constant follows from the requirement that the corresponding term in the action is dimensionless. Thus $\lambda\phi_q^4$ should be dimensionless which implies $[\lambda] = [m]^{4-n}$. Since the coupling constant λ becomes dimensionful we write it as $\lambda\mu^\epsilon$ where μ is some arbitrary mass scale.

Our task is to calculate the 1PI Green's functions for the three divergent diagrams in Fig.1 [79].

Fig.1(b) has a symmetry factor of $\frac{1}{2}$ and the Feynman integral in four dimensional momentum space is

$$\mathcal{I}_1 = \frac{1}{(2\pi)^4}\frac{\lambda^2 q^2}{2}\int d^4k \frac{1}{[k^2 - m^2 + i\epsilon][(k - p_1 - p_2)^2 - m^2 + i\epsilon]} \tag{4.18}$$

The denominators in the integrand can be combined using the Feynman formula

$$\frac{1}{A_1 A_2} = \int_0^1 \frac{d\alpha}{[\alpha A_1 + (1 - \alpha A_2)]^2} \tag{4.19}$$

which is a corollary of the identity

$$\frac{1}{A_1.A_2........A_n} = \Gamma(n)\int_0^1 \frac{d\alpha_1......d\alpha_n \delta(\sum_i \alpha_i - 1)}{[\alpha_1 A_1 + ...\alpha_n A_n]^2} \tag{4.20}$$

Changing the variable from k to $k' = k + (1 + \alpha)s$,where $s = p_1 + p_2$, the integral becomes

$$\mathcal{I}_1 = \frac{1}{(2\pi)^4}\frac{\lambda^2 q^2}{2}\int \frac{dk\ d\alpha}{[k'^2 - m^2 + (1 - \alpha)\alpha s]^2}$$

Without any harm we can relabel k' as k, so that

$$\mathcal{I}_1 = \frac{1}{(2\pi)^4}\frac{\lambda^2 q^2}{2}\int \frac{d^4k\ d\alpha}{[k^2 - m^2 + (1 - \alpha)\alpha s]^2} \tag{4.21}$$

But

$$\int \frac{d^n k}{[k^2 + b^2]^p} = i\pi^{n/2}\frac{\Gamma(p - n/2)}{\Gamma(p)}\frac{1}{(b^2)^{p-n/2}} \tag{4.22}$$

Since in eq.(22) n has the value 2, and since the gamma function has poles at zero and negative integers,the gamma function in the numerator of (22) becomes infinite. Thus in four dimensions, $\mathcal{I}_1$ diverges(logarithmically). However, if we perform the integration in $4 - \epsilon$ dimensions the result is finite.

As mentioned previously, dimensional considerations demand the multiplication of λ by μ^ϵ in $4 - \epsilon$ dimensions. Formula(22) is defined,by analytic continuation of the RHS,to have a value even when n is not an integer. Thus (22) can be used when $n = D = 4 - \epsilon$. Now

$$\mathcal{I}_1 = \frac{\lambda^2 q^2 \mu^{2\epsilon}}{2} \int d\alpha \int \frac{d^n k}{(2\pi)^n [k^2 - m^2 + (1-\alpha)\alpha s]^2} \tag{4.23}$$

After a Wick rotation, the integral takes the form

$$\mathcal{I}_1 = \frac{\lambda^2 q^2 \mu^{2\epsilon}}{2} \int d\alpha \int \frac{d^n k}{(2\pi)^n [k^2 + m^2 - (1-\alpha)\alpha s]^2} \tag{4.24}$$

where k is a vector in n-dimensional Euclidian space. Thus the $(4 - \epsilon)$ dimensional integral takes the form

$$\frac{i}{(4\pi)^n/2} \lambda \mu^\epsilon q^2 \lambda \Gamma(\epsilon/2) \int_0^1 d\alpha \left(\frac{m^2 - s\alpha(1-\alpha)}{\mu^2} \right)^{-\epsilon/2} \tag{4.25}$$

Using the approximations

$$\Gamma(x) = \frac{1}{x} - \gamma, \tag{4.26}$$

$$\frac{1}{(b^2)^{2-n/2}} = 1 - (2 - n/2) ln b^2 + ..., \tag{4.27}$$

where

$$\gamma = \lim_{n \to \infty} \left\{ 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n \right\} = 0.577$$

is the Euler-Mascheroni constant, and taking the limit $\epsilon = (4 - n) \to 0$

$$\mathcal{I}_1 = i \frac{\lambda^2 q^2}{16\pi^2} \frac{1}{\epsilon} - i \frac{\lambda^2 q^2}{2(4\pi)^2} [\int_0^1 d\alpha ln(\frac{m^2 - s\alpha(1-\alpha)}{\mu^2}) + \gamma - ln 4\pi] \tag{4.28}$$

The diverging contribution of Fig.1(b) can be cancelled by the vertex counter term $-i\lambda\,(Z_1 - 1)^{(n)}$(n denotes the order of λ to which Z_1 is approximated). Since Fig.1(b) has three sub diagrams having equal contributions,

$$(Z_1 - 1)^{(1)} = 3\, q^2\, \frac{1}{\epsilon}\, \frac{\lambda}{16\pi^2} \tag{4.29}$$

Diagrams (1.a) and (1.b) need both mass and wavefunction renormalizations. Fig.1(a) has a diverging contribution $q\,\frac{1}{2}\,\left\{i\;2\;\lambda\;\;m^2/(4\pi)^2\epsilon\right\}$. This is cancelled by the mass counter term $(-i)(Z_0-1)^{(n)}m^2$. (n denotes the order of λ to which Z_0 is approximated.) This gives

$$(Z_0-1)^{(1)} = q\,\frac{\lambda}{16\;\pi^2}\frac{1}{\epsilon} \tag{4.30}$$

$$(Z_3-1)^{(1)} = 0 \tag{4.31}$$

Fig.1(c) has a diverging contribution [80]

$$\frac{i.\lambda^2\;q^3}{(16\pi^2)^2}\left\{\frac{-m^2}{\epsilon^2}+\frac{1}{\epsilon}(m^2\ln\frac{m^2}{4\pi\mu^2}+\frac{p^2}{12}+(\gamma-\frac{3}{2})m^2)\right\}$$

This term needs both mass and wavefunction renormalisations. Since Fig.1(c) is of order λ^2, we have to consider the other diagrams of the same order which are shown in Fig.2 [79]. Diverging parts of their amplitudes are

$$iq\;\frac{\lambda^2 m^2}{(16\pi^2)^2}\;\frac{3}{2}\;\left\{\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{\epsilon}-\frac{\gamma}{\epsilon}\right\},$$

$$iF_0{}^2q^2\;\frac{\lambda^2 m^2}{(16\pi^2)^2}\;\frac{1}{2}\left\{\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\;\ln\frac{m^2}{4\pi\mu^2}-\frac{\gamma}{\epsilon}\right\},$$

$$\textit{and}\qquad -iq^2\frac{\lambda^2 m^2}{(16\pi^2)^2}.\frac{1}{2}\left\{\frac{2}{\epsilon^2}-\frac{2}{\epsilon}.\ln\frac{m^2}{4\pi\mu^2}+\frac{1}{\epsilon}-\frac{2\gamma}{\epsilon}\right\}$$

respectively. These divergences are to be cancelled by the diverging parts of the contributions from the counter terms

$-i(Z_0-1)^{(2)}m^2$ and $i(Z_3-1)^{(2)}$.

It then follows that

$$\begin{aligned}(Z_3-1)^{(2)} &= \frac{-\lambda^2 q i^3}{12(16\pi^2)^2}\cdot\frac{1}{\epsilon} \\ and\ \ (Z_0-1)^{(2)} &= \frac{\lambda^2}{(16\pi^2)^2}\cdot\left\{\frac{1}{\epsilon}(q^3+\frac{1}{2}\xi^2 F_0{}^2-\frac{3}{2}q)(\ln\frac{m^2}{4\pi\mu^2}+\gamma)\right. \\ &+ \left.\frac{1}{\epsilon^2}(3q-q^3)-\frac{1}{\epsilon}\frac{1}{2}\,(3q^3+q^2-3q)\right\} \end{aligned} \tag{4.32}$$

So we finally get the renormalisation parameters as (up to order λ^2).

$$\begin{aligned} Z_0 &= 1+q\frac{\lambda}{16\pi^2}\frac{1}{\epsilon} \\ &+ \frac{\lambda^2}{(16\pi^2)^2}\left\{\frac{1}{\epsilon}(q^3+\frac{1}{2}\xi^2 F_0^{\,2}-\frac{3}{2}q)(\ln\frac{m^2}{4\pi\mu^2}+\gamma)\right. \\ &+ \left.\frac{1}{\epsilon^2}(3q-q^3)-\frac{1}{2}\frac{1}{\epsilon}(3q^3+q^2-3q)\right\} \end{aligned} \tag{4.33}$$

$$Z_1 = 1+q^2\frac{3\lambda}{16\pi^2}\frac{1}{\epsilon} \tag{4.34}$$

$$Z_3 = 1-q^3\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon} \tag{4.35}$$

4.3 Conclusion

Here we have developed a q-deformed quantum field theory whose algera coincides with the quon algebra in the single particle limit. We have introduced a q-deformed chronological product which treats particles and antiparticles on different footings.Assuming a nonlinear relationship between the number operator and the field Hamiltonian,we calculated the propagator for a scalar field whose field quanta obey a quon algebra in the single mode case, but different modes of the field commute. The propagator so obtained is nothing but the standard propagator multiplied by the deformation parameter q. Thus q-deformed field theory has no remarkable variation from the standard theory. But in perturbation calculations this deformation has nonlinear impact. As an illustration of the nontrivial consequences renormalisation calculations are done for q-deformed real scalar field with quartic interaction. The divergences are found to have direct dependence on the deformation parameter q. This imposes some physical significance on the q-deformation of the field since it is closely related to the divergence of observable quantities. This implies another possibility: if we could achieve a situation in which q vanishes, the divergences encountered in the measurement of the field quantities disappear spontaneously.

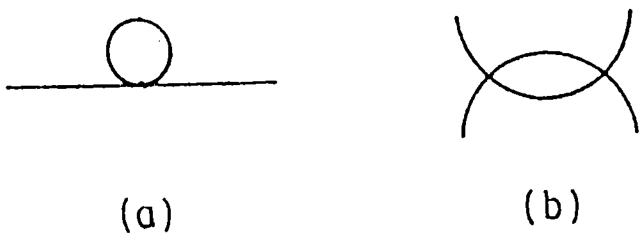

Fig 3.1

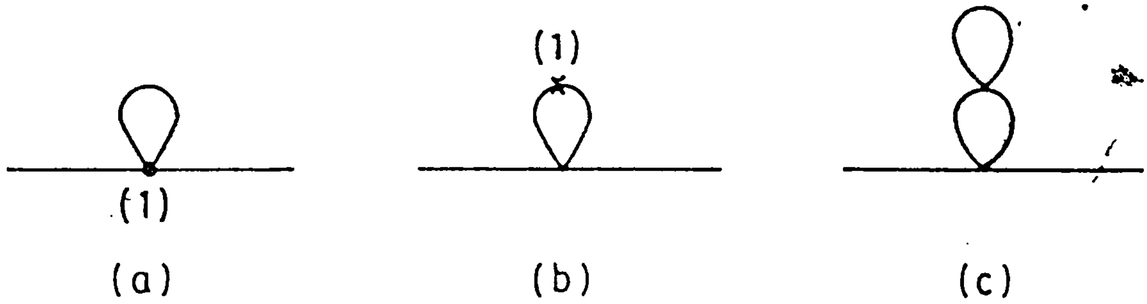

Fig 3.2

Chapter 5

Gravitationally induced particle creation

The study of quantum field theory in curved spacetime is important as it is an essential key to the knowledge of the scenario in the Early Universe. The behavior of the classical scalar field near the initial singularity is best approximated quantum mechanically by constituting a complete set of coherent states for each mode of the scalar field[81]. The quantum state of the scalar field near the initial singularity is inaccessible to an observer at the present time, and Hawking[82] proposed that this inaccessible nature can be expressed by taking a random superposition of all allowed states in the inaccessible region. Berger[83] realised this by superposing coherent states in a random manner. Parker[84] studied the particle creation in an expanding universe,with gravitational metric treated as an unquantised external field. Considering the evolution of a scalar field in an expanding Universe, Goodison and Tom[85] established that if the field quanta obey the quantum statistics, then particle creation is impossible in the vacuum state. Their result dismisses the possibility of quon statistics for the Early Universe in the ordinary vacuum but not for a coherent state or squeezed state.

Here we study the evolution of a scalar field whose quanta obey q-oscillator algebra in the asymptotic region,which is assumed to be flat. By constructing a superposition of coherent states in both in- and out-regions,we show that there is a nonzero probability for particle creation. We calculate the expectation values of the stress energy tensor in the coherent state of each mode of the field. Section 2 contains the calculation of the different components of the stress energy tensor. In section 3,the coherent state representation of the field and coherent state expectations are discussed. In section 4, we discuss the situation where the field is in a squeezed state. Section 5 is the concluding section.

5.1 Stress energy tensor of the scalar field in curved spacetime.

We assume a spatially flat Robertson-Walker spacetime with the metric

$$ds^2 = -dt^2 + \sum_i R^2(t)(dx^i)^2 \tag{5.1}$$

where$R(t) = R_1$ for $t \leq t_1$ and $R(t) = R_2$ for $t \geq t_2$. We call the portion of spacetime with $t < t_1$ the in-region and that with $t > t_2$ the out region. In the in-region,the scalar field is expanded as

$$o_q(x) = \sum_k \frac{1}{(2\pi)^{3/2}} \frac{1}{(2\omega_k)^{1/2}} \left[a_k F_k(x) + a_k^\dagger F_k^*(x)\right] \tag{5.2}$$

where ω_k is given by

$$\omega_k^2 = g(\sum_i \frac{k_i^2}{R^2} + m) \tag{5.3}$$

with

$$g = \mid g_{\mu\nu} \mid \tag{5.4}$$

We take $\phi_q(x)$ as a real scalar field in the Heisenberg picture and $F_k(x)$s are assumed to form a complete set of positive frequency solutions to the Klein-Gordon equation:

$$(F_k \, . F_{k'}) = (2\pi)^3(2\omega_k)\delta_{kk'} \quad , \quad (F_k \, , F_{k'}^*) = 0 \tag{5.5}$$

In the in-region, the field statistics is assumed to be given by

$$a_k a_{k'}^\dagger - a_{k'}^\dagger a_k = q^{N_k}\delta_{kk'} \tag{5.6}$$

with $\mid q \mid \leq 1$ and $[N_k] = a_k a_k^\dagger$. Also operators corresponding to different modes commute:

$$[a_k, \, a_{k'}] = 0 = [a_k^\dagger, \, a_{k'}^\dagger] \tag{5.7}$$

In the out-region, the field is expanded as

$$\phi_q(x) = \sum_k \frac{1}{2\pi^{3/2}} \frac{1}{(2\omega_k)^{1/2}} \left[b_k G_k(x) + b_k^\dagger G_k^*(x) \right] \tag{5.8}$$

$G_k(x)$s also are asuumed to form a complete set of positive frequency solutions. In general, b_k differ from a_k if there is particle creation due to the expansion of the universe. In the out-region,the statistics is assumed as

$$b_k b_{k'}^\dagger - \prime b_{k'}^\dagger b_k = q'^{\mathcal{N}_k}\delta_{kk'} \tag{5.9}$$

with $[\mathcal{N}_k] = b_k^\dagger b_k$ and $\mid q' \mid \leq 1$. Generally q' may not be equal to q. Here also

$$[b_k, \, b_{k'}] = 0 = [b_k^\dagger, \, b_{k'}^\dagger] \tag{5.10}$$

Since the mode solutions in both the regions are complete, one set of solutions can be expressed in terms of the other:

$$\begin{aligned} G_k(x) &= \sum_{k'} (\alpha_{kk'} F_{k'}(x) \ + \ \beta_{kk'} F^*_{k'}(x)) \\ F_{k'}(x) &= \sum_{k} (\alpha_{kk'} G_k(x) \ - \ \beta_{kk'} G^*_k(x)) \end{aligned} \tag{5.11}$$

Above relations are called the Bogolubov transformations and the coefficients $\alpha_{kk'}$ and $\beta_{kk'}$ are called Bogolubov coefficients[86,87]. Using the Bogolubov transformations and equating the field expansions in the in- and out- regions,we can obtain the following relation:

$$a_k = \sum_{k'} (\alpha_{kk'} F_{k'}(x) + \beta_{kk'} F^*_{k'}(x)) \tag{5.12}$$

For a spacetime whose metric is given by (1), the Bogolubov coefficients are diagonal. Hence we can write

$$a_k = \alpha_k b_k + \beta_k b^\dagger_k \tag{5.13}$$

Expansions given by (2) and (8) are special cases of the expansion given by Parker[84]. The stress energy tensor for the scalar field is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + m^2 \phi^2) \tag{5.14}$$

If we transform the coordinates as

$$g^{1/2} d\tau = dt \tag{5.15}$$

then

$$T_{00} = \frac{1}{2g} \left[\left(\frac{\partial \phi}{\partial \tau}\right)^2 + g \left[\sum_i \frac{1}{R^2} (\partial_i \phi)^2 + m^2 \phi^2 \right] \right] \tag{5.16}$$

$$T_{ii} = (\partial_i \phi)^2 + \frac{1}{2g} R^2 \left(\frac{\partial \phi}{\partial \tau}\right)^2 - \frac{1}{2} R^2 \left[\sum_j \frac{1}{R^2} (\partial_j \phi)^2 + m^2 \phi^2 \right] \tag{5.17}$$

The spatial average of $T_{\mu\nu}$ is defined by[83],

$$\bar{T}_{\mu\nu} = \frac{1}{(2\pi)^3}\int d^3x T_{\mu\nu} \quad . \tag{5.18}$$

Taking $a_{-k} = a_k^\dagger$ and $b_{-k} = b_k^\dagger$,the spatial averages of T_{00} and T_{ii} in the in-region are

$$(\bar{T}_{00})in = \frac{1}{16\pi^3 g}\sum_k \omega_k^2 (a_k a_k^\dagger + a_k^\dagger a_k) \tag{5.19}$$

$$(\bar{T}_{ii})in = \frac{R_i^2}{16\pi^3 g}\sum_k \frac{1}{\omega_k}(\frac{2k_i^2}{R^2}g - \omega_k^2)(a_k a_k^\dagger + a_k^\dagger a_k) \tag{5.20}$$

Similarly for the out-region

$$(\bar{T}_{00})out = \frac{1}{16\pi^3 g}\sum_k \omega_k^2 (b_k b_k^\dagger + b_k^\dagger b_k) \tag{5.21}$$

$$(\bar{T}_{ii})out = \frac{R_i^2}{16\pi^3 g}\sum_k \frac{1}{\omega_k}(\frac{2k_i^2}{R^2}g - \omega_k^2)(b_k b_k^\dagger + b_k^\dagger b_k) \tag{5.22}$$

5.2 Coherent state representation

Berger[83] represented the field as a superposition of coherent states because it is a natural method for relating the quasi-classical behavior to singularity parameters. Coherent states are the eigenstates of the annihilation operator and are normalisable to unity but are not orthogonal. In a coherent state,the fluctuations in the two quadratures are equal and minimize the uncertainty product given by Heisenberg's uncertainty relation.

We represent the field as a superposition of q-coherent states,which are eigenstates of q-annihilation operators a_k and b_k.

$$a_k \mid \lambda_k\rangle = \lambda_k \mid \lambda_k\rangle \tag{5.23}$$

$$b_k \mid \chi_k\rangle = \chi_k \mid \chi_k\rangle \tag{5.24}$$

The states $\mid \lambda_k\rangle$ and $\mid \chi_k\rangle$ represent coherent states for the k^{th} mode. The set $\lambda_k = 0$ or $\chi_k = 0$ for all **k** refers to vacuum in in- and out-regions respectively.

$$\left\langle \lambda_k \mid \bar{T}^k_{00} \mid \lambda_k \right\rangle = \frac{1}{16\pi^3 g} g\omega_k^2 \left((1+q) \mid \lambda_k \mid^2 + 1\right) \tag{5.25}$$

$$\left\langle \lambda_k \mid \bar{T}^k_{ii} \mid \lambda_k \right\rangle = \frac{R_i^2}{16\pi^3 g\omega_k}\left(\frac{2k_i^2}{R^2}g - \omega_k^2\right)\left[(1+q) \mid \lambda_k \mid^2 + 1\right] \tag{5.26}$$

The vacuum energy density in the in-region is

$$\begin{aligned} \rho_0(in) &= -\left\langle 0 \mid \bar{T}^0_0 \mid 0 \right\rangle \\ &= \frac{1}{16\pi^3 g}\sum_k \omega_k^2 \end{aligned} \tag{5.27}$$

The anisotropic pressure in this region is

$$\begin{aligned} P_{i0}(in) &= \left\langle 0 \mid \bar{T}_i^i \mid 0 \right\rangle \\ &= \frac{1}{16\pi^3 g}\sum_k \frac{1}{\omega_k}\left(\frac{2k_i^2}{R^2}g - \omega_k^2\right) \end{aligned} \tag{5.28}$$

In the in-region, the field is represented by a coherent state $\prod_k \mid \lambda_k\rangle$. The expectation value of the stress energy tensor in that region

$$\begin{aligned} \rho(in) &= \rho_0(in) + \rho^{cl}(in) \\ P_i(in) &= P_{i0}(in) + \rho_i^{cl}(in) \end{aligned} \tag{5.29}$$

where

$$\rho^{cl}(in) = \frac{1}{16\pi^3 g}\sum_k {\omega_k}^2 \left[(1+q) \mid \lambda_k \mid^2\right]$$

$$and \;\; P_i^{cl}(in) = \frac{1}{16\pi^3 g}\sum_k \frac{1}{\omega_k}|\lambda_k|^2(\frac{2k_i^2}{R^2}g - \omega_k^2) \quad (5.30)$$

For the out-region

$$\rho(out) = \rho_0(out) + \rho^{cl}(out)$$
$$P_i(out) = P_{i0}(out) + \rho_i^{cl}(out) \quad (5.31)$$

where

$$\rho^{cl}(out) = \frac{1}{16\pi^3 g}\sum_k {\omega_k}^2\left[(1+q')|\chi_k|^2\right]$$
$$and \;\; P_i^{cl}(out) = \frac{1}{16\pi^3 g}\sum_k \frac{1}{\omega_k}|\chi_k|^2(\frac{2k_i^2}{R^2}g - \omega_k^2) \quad (5.32)$$

Thus the vacuum expectation values of energy density and anisotropic pressure are the same for both in and out-regions. Neverthless,for a general coherent state.the expectation values for energy density and anisotropic pressure differ for the in and out regions.

The change in energy density for each mode depends on the value of $|\chi_k|^2 - |\lambda_k|^2$. Similarly the probability for observing a nonzero number of quanta in the k^{th} mode in the in-region is

$$1 - |\langle\lambda_k \mid 0\rangle|^2 = \frac{exp_q|\lambda_k|^2 - 1}{exp_q|\lambda_k|^2} \quad (5.33)$$

where the q-exponential $exp_q z$ is defined by

$$exp_q z = \sum_{n=0}^{\infty}\frac{z^n}{[n]!} \quad (5.34)$$

with [n]!= [n].[n-1].[n-2].......[2].[1]. and $[n] = \frac{q^n-1}{q-1}$.

In the out-region, such a probability is given by

$$1 - \mid \langle \chi_k \mid 0 \rangle \mid^2 = \frac{exp_{q'} \mid \chi_k \mid^2 - 1}{exp_{q'} \mid \chi_k \mid^2} \tag{5.35}$$

So if the Bogoliubov coefficients α_k and β_k are nonzero, there is a nonzero probability of particle creation as the field evolves from the in-region to the out-region.

5.3 Squeezed states

Like coherent states, squeezed states are minimum uncertainty states. In squeezed state the variance of one of the quadrature components goes below the minimum value allowed by Heisenberg's uncertainty principle,while that of the other component goes above it,keeping the product at the minimum value. Grishchuk and Sidorov [88] introduced the language of squeezed states ,a well known concept in quantum optics, in to cosmology, and showed that gravitons and other primordial perturbations created from zero point quantum fluctuations in the process of cosmological evolution should now be in a strongly squeezed state. Gasperini and Giovannini[89] established the dependence of the entropy growth in the cosmological process of pair creation on the associated squeezing parameter. Albericht *etal.*[90] have found that the squeezed state formalism provides a framework for studying the amplifying process during the cosmological inflation. Making use of the squeezed state formalism, Hu et al.[91] arrived at a systematic description of the dependence of particle creation on the initial state. Suresh *etal.*[92] have calculated the the expectation values of the stress-energy tensor of the scalar field in curved spacetime as well as the quantum fluctuations of density and anisotropic pressure by making use of the squeezed state formalism. Motivated by these works,we cousruct squeezed states corresponding to the in and out-regions and calculate the expectation values of the stress-energy tensor in these states. We define two new operators A_k and B_k with their adjoint operators by

$$
\begin{aligned}
A_k &= \mu a_k + \nu a_k^\dagger && (5.36)\\
B_k &= \eta b_k + \theta b_k^\dagger && (5.37)\\
with \quad |\mu|^2 - |\nu|^2 &= 1 && (5.38)\\
and \quad |\eta|^2 - |\theta|^2 &= 1 && (5.39)
\end{aligned}
$$

The eigenstates of A_k and B_k are defined as q-squeezed states.

$$
\begin{aligned}
A_k \mid \beta_k\rangle &= \beta_k \mid \beta_k\rangle && (5.40)\\
B_k \mid \gamma_k\rangle &= \gamma_k \mid \gamma_k\rangle && (5.41)
\end{aligned}
$$

The squeezed vacuum energy density in the in-region defined as $\lim_{\beta_k \to 0} \sum_k \left\langle \beta_k \mid \tilde{T}^k_{00} \mid \beta_k \right\rangle$ is given by

$$
\rho_0(\beta_k)(in) = \frac{1}{16\pi^3 g[1 - |\nu|^2(q-1)]} \sum_k \omega_k^2 \left(|\mu|^2 + |\nu|^2 \right) \qquad (5.42)
$$

Note that as $\mu \to 1$ and $\nu \to 0$, the above expectation value reduces to the vacuum energy density given by (27) ,as it should. In the out-region, squeezed vacuum energy density is given by

$$
\rho_0(\gamma_k)(out) = \frac{1}{16\pi^3 g[1 - |\theta|^2(q-1)]} \sum_k \omega_k^2 \left(|\eta|^2 + |\theta|^2 \right) \qquad (5.43)
$$

which also reduces to (27) in the apropriate limit. Thus squeezed vacuum energy density values in the in and out-regions are different and both differ from the vacuum energy density given by (27).As the field evolves, the squeezed vacuum energy density changes depending on $|\eta|^2 + |\theta|^2 - |\mu|^2 - |\nu|^2$.

Since in general, $|\eta|^2 + |\theta|^2 \neq |\mu|^2 + |\nu|^2$, there can be particle creation in the interacting region.

5.4 Conclusion

Goodison and Tom[85] showed that for $q \neq 1$,particle creation is impossible in the ordinary vacuum state for the dynamical evolution of the field through a curved spacetime. We have considered the evolution of a q-deformed field in a curved spacetime and observed that if the field is in a coherent state,there is a non-trivial difference between the energy values in the in-and out-regions, indicating the possibility of particle production. Also the expectation value of energy density in the squeezed vacuum differs from the vacuum expectation value. These results show that q-oscillator algebra can not be ruled out completely for the Early Universe, because if it was in a coherent state or a squeezed state then scalar particles obeying q-oscillator algebra would be produced.

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