FUZZY SET THEORY AND RELATED AREAS

STUDIES ON FUZZY TOPOLOGICAL SEMIGROUPS AND RELATED AREAS

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By

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CERTIFICATE

This is to certify that this thesis is an authentic record of research work carried out by S. Kumari Geetha under my supervision and guidance in the School of Mathematical Sciences, Cochin University of Science and Technology for the Ph.D. degree of the Cochin University of Science and Technology and no part of it has previously formed the basis for the award of any other degree in any other University.

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CHAPTER 0

INTRODUCTION

This thesis is a study of fuzzy topological semigroups.

FUZZY MATHEMATICS

The notion of fuzziness was formally introduced by Lotfi.A.Zadeh in 1965 [77].In a very limited and specific context the concept and the term"fuzzy set" were already used by K.Menger in 1951 [52].It was primarily intended to be applied in areas of pattern classification and information processing, but later it was found to be useful and applicable in various areas of knowledge.

A fuzzy subset A of a set X is defined by a membership function $\mu_A(x)$, which associates with each point x in X, a real number in the interval [0,1] with the value of $\mu_A(x)$ representing the grade of membership of x in A. That is the nearer the value of $\mu_A(x)$ is to 1, the higher is the grade of membership of x in A. No further mathematical meaning is given to the value that a fuzzy set attains at a point x. That is basically a fuzzy set is a class in which there may be a continuum of grades of memberships in comparison to the situation that only two grades of memberships are possible in ordinary set theory,with this dèscription fuzzy sets can be used to model many concepts.

Most of the initial work in the development of the concept of fuzzy set was theoretical in nature.Now a days the theory of fuzzy sets have wider scope of applicability than classical set theory.It has been found applicable in so diverse areas like Linguistics, Robotics, Computer languages. Military control, Artificial intelligence, Law, Psychology, Taxonomy, Economics and Medical and Social sciences.

When Zadeh introduced fuzzy subset he used [0,1] as the membership set. But while developing the theory of fuzzy sets many mathematicians used different lattice structures for the membership set. In 1967, Goguen introduced L-fuzzy set, where L-is an arbitrary lattice with both a minimal and a maximal element, 0 and 1 respectively. Some of the other lattice structures that are used are the following:-

1) Completely distributive lattice with 0 and 1 by T.E.Gantner [23]

2) Complete and completely distributive lattice equipped

with order reversing involution by Bruce Hutton [28]

3) Complete and completely distributive non-atomic Boolean algebra by Mira sarkar [63]

4) Complete Brouwerian lattice with its dual also Brouwerian by Ulrich Hȯhle [27]

5) Completely distributive lattice by S. E. Rodabaugh [59]

General topology is one of the first branches of pure Mathematics to which the notion of fuzzy sets has been applied systematically. In 1968, that is three years after Zadeh's paper had appeared, C.L.Chang [8] first brought together the notion of fuzzy set and General topology. He introduced the notion that we now call Chang fuzzy space and made an attempt to develop basic topological notions for such spaces. This paper was followed by others in which Chang fuzzy space and other topological type structures for fuzzy set systems were considered.

According to Chang, a fuzzy topological space is a pair (X,F), where X is any set and $F \subset I^X$ satisfying the following axioms:-

i) ϕ, $X \in F$

ii) If A, B $\in$ F, then $A \cap B \in F$

iii) If $A_i \in F$ for each $i \in I$, then $\vee_i A_i \in F$

Later R.Lowen in 1976 [37] modified this definition by taking the set of all constant maps instead of ϕ and X in axiom (i) of Chang's definition. Either has disadvantages or advantages over the other; we are following throughout the thesis, definition nearer to Chang's rather than to Lowen's.

Separation axioms in fuzzy topological spaces were studied by R. Lowen [37-41], Pu and Liu [57-58], A.P.Shostak [65],T.E.Gantner [23], S.R.Malghan [46], K.K.Azad [2], S.E.Rodabaugh [59],B.Hutton [29] and R.Srivastava [66]. The important separation property namely Hausdorffness Concept has been defined and studied by many Mathematicians from different view points. At present not less than 10 approaches to the definition of Hausdorff fuzzy topological spaces are known. Some of them differ negligibly; but others do basically. The relation between some of them are avaliable in [54]

Compactness property of a fuzzy topological space is one of the most important notions in Fuzzy topology. The first definition of compactness for fuzzy topological spaces was proposed in 1968 by C. L. Chang. Goguen [24] extended it into the case of L-fuzzy topological spaces.

T.E.Gantner, R.C.Steinlage and R.H.Warren [23] proposed α-Compactness and observed that it is possible to have degrees of compactness. R. Lowen [37], Wang Guojun [69], S.Ganguli and S.Saha [21] and J.T.Chadwick [7] have different definitions for compactness property in fuzzy topological spaces. Fuzzy topology on groups and other algebraic objects was studied by D.H.Foster [20], A.K.Katsaras [30-31], Chun Hai yu [13],Ma Ji liang [43],B.T.Lerner [35],Ulrich Höhle [27],Ahsanullah [1] etc.

In [20] D.H.Foster combined the structure of a fuzzy topological space with that of a fuzzy group defined by Rosenfeld [62] to define a fuzzy topological group. Chun Hai Yu [13] and Ji liang Ma [43] defined fuzzy topological groups in terms of neighbourhoods and Q-neighbourhoods. B.T.Lerner [35] exten ded some of Foster's results on homomorphic images and inverse images to fuzzy right topological semigroups

TOPOLOGICAL SEMIGROUPS

The study of topological semigroups was started in 1953. During these years the subject has developed in many directions and the literature is so vast and it would be difficult to give a brief survey of the developments in this area. Some of the early contributors of this area are A.D.Wallace [On the structure of topological semigroups,Bull.Amer.Math.Soc.61(1955a), 95-112.] K.H.Hofmann and P.S.Mostert [Elements of Compact semigroups, Merrill books, INC, Columbus(1966)], A.B.Palman De Miranda, [Topological semigroups, Math centres Tracts, 11 edition, Mathematische centrum Amstendam 1970], Hewitt [Compact monothetic semigroups, Duke: Math, J.23(1956) 447-457] and R.J. Koch [On Monothetic semigroups, Proce.Amer.Math.Soc. 8(1957a),397-401].

By definition a topological semigroup S is a Hausdorff space with continuous associative multiplication (x,y)--->xy of S x S into S, and if the multiplication is continuous in each variable separately, S is called a semi toplogical semigroup.Some of the major areas of developments in the theory of semitopological semigroups are the theory of compact semi topological semigroups, structure theory of compact semigroups,semigroup compactification and almost periodic and weakly almost periodic compactification. For details regarding these ,one may refer to [3].

SUMMARY OF THE THESIS

chapter 1

In the first chapter we introduce a category of L-fuzzy topological spaces,where L stands for a complete completely distributive lattice with minimal element o and maximal element 1 and also the order reversing involution $a \longrightarrow a^c$ is fixed ($a \leq b$,$a,b \in L \longrightarrow b^c \leq a^c$)

[THROUGH OUT THE THESIS WE USE L IN THE ABOVE SENSE]

In section 1 we define an L-fuzzy topological space (X,μ,F). A subspace of (X,μ,F), an induced fuzzy space of (X,μ,F), product of family of L-fuzzy topological spaces $\left\{(X_i,\mu_i,F_i) \middle| i \in I\right\}$, the quotient of (X,μ,F) are obtained in this section.

In section 2 we define the category FTOP of L-fuzzy topological spaces,Subcategories,subobjects,terminal objects and initial objects of FTOP are also described in this section.

CHAPTER 2

This chapter is a study of L-fuzzy topological semigroups. In section 2 we identify a semigroup object in FTOP as an L-fuzzy topological semigroup. Also we define an induced L-fuzzy topological semigroup and obtain a relation between L-fuzzy topological semigroups and induced L-fuzzy topological semigroups. In section 3 of this chapter we specialize L-fuzzy topological semigroups, for L=[0,1] and $\mu=1_X$ and consider $(X,1_X,F)$ as a fuzzy topological semigroup. We define a fuzzy topological semigroup in terms of neighbourhoods and Q-neighbourhoods. We observe an association between the classes of topological semigroups and fuzzy topological semigroups in this section.

CHAPTER 3

We proceed to study the analogous concept namely L-fuzzy semi topological semigroups in chapter 3. We define an L-fuzzy semi topological semigroup in the categorical point of view. In section 2 we specialize an L-fuzzy semi topological semigroup to a fuzzy topological semigroup and it is defined in terms of neighbourhoods and Q-neighbourhoods .In the last section we give some examples to observe relations between fuzzy topological semigroups and fuzzy semi topological semigroups.

CHAPTER 4

We study fuzzy topology on function spaces in chapter 4. We define point open fuzzy topology and compact open fuzzy topology for a family of fuzzy continuous functions between two fuzzy topological spaces. A few separation properties of compact open fuzzy topological spaces are studied and also we obtain some relations between compact open topology and compact open fuzzy topology for a family of functions between two topological spaces.

CHAPTER 5

In chapter 5 we study homomorphism and Isomorphism between two L-fuzzy (semi) topological semigroups. We define F-morphism and F-isomorphism between two L-fuzzy (semi) topological semigroups. B.T.Lerner studied the homomorphic images and inverse images of fuzzy right topological semigroups. We prove the analogous results for L-fuzzy topological semigroups. In section 2 we prove that the product of a family of L-fuzzy topological semigroups $\left\{X_i . \mu_i, F_i \middle| i \in I\right\}$ is an L-fuzzy topological semigroup and each factor space is an F-morphic image of the product space.

Also we prove that the quotient of an L-fuzzy topological semigroup is also an L-fuzzy topological semigroup. In the last section we consider some categories of fuzzy (semi) topological semigroups.

CHAPTER 6

In chapter 6 we study the problem of fuzzy space compactification on fuzzy topological semigroups. In section 2 we prove that the Bohr fuzzy compactification exists for a fuzzy topological semigroup. We define F-semigroup compactification for a fuzzy topological semigroup and obtain an association with the Bohr fuzzy compactification. In the last section we define an order relation on the set of all F-semigrup compactifications of a fuzzy topological semigroup and prove that it is a complete lattice.

CHAPTER 0.1

PRELIMINARY DEFINITIONS USED IN THE THESIS

Definition 0.1.1

Let X be a set and A,B be two fuzzy sets in X.Then

i) A=B <====> $\mu_A(x) = \mu_B(x)$ for all $x \in X$

ii) $A \subset B$ <====> $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$

iii) $\mathcal{C} = A \cup B$ <====> $\mu_{\mathcal{C}}(x) = \max(\mu_A(x), \mu_B(x))$ for all $x \in X$

iv) $D = A \cap B$ <====> $\mu_D(x) = \min(\mu_A(x), \mu_B(x))$ for all $x \in X$

More generally for a family of fuzzy sets $\mathcal{A} = \{A_i, i \in I\}$, the union $C = \bigcup_{i\in I} A_i$ and the intersection $D = \bigcap_{i\in I} A_i$ are defined by $\mu_C(x) = \sup_{i\in I} \mu_{A_i}(x)$, $x \in X$ and $\mu_D(x) = \inf_{i\in I} \mu_{A_i}(x)$, $x \in X$.

Definition 0.1.2

Let f be a mapping from a set X to a set Y. Let B be a fuzzy set in Y with membership function μ_B. Then the inverse image of B is the fuzzy set in X whose membership function is defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ $\forall x \in X$. Conversely let

A be a fuzzy set in X with membership function μ_A, then the image of A is the fuzzy set in Y whose membership function is defined by $\mu_{f(A)}(y) = \operatorname{Sup}_{z \in f^{-1}(y)} \mu_A(z)$ if $f^{-1}(y)$ is nonempty.

$= 0$ otherwise $\forall\ y \in Y$ where $f^{-1}(y) = \{x\ ; f(x) = y\}$

Definition 0.1.3

Let (X,F), $(Y,\mathcal{U})$ be two fuzzy topological spaces. A mapping f of (X,F) into $(Y,\mathcal{U})$ is fuzzy continuous if and only if for each open fuzzy set v in $\mathcal{U}$, the inverse image $f^{-1}(v)$ is in F.

Definition 0.1.4

A bijective mapping f of a fuzzy topological space (X,F) into a fuzzy topological space $(Y,\mathcal{U})$ is fuzzy homeomorphism if and only if both f and f^{-1} are fuzzy continuous.

Definition 0.1.5

Let (X,T) be a topological space and $I=[0,1]$ equipped with the usual topology, a mapping $\mu:(X,T) \longrightarrow I$ is said to be lower semi continuous (l.s.c) if $\mu^{-1}(\alpha,1]$ is open in X for every $\alpha \in I$.

Let (X,T) be a topological space in the above sense, then $(X,\mathcal{W}(T))$, where $\mathcal{W}(T)=\left\{f:X \longrightarrow [0,1] \;\middle|\; f \text{ is a l.s.c map}\right\}$, is called the associated fuzzy space or topologically generated fuzzy space of (X,T).

Definition 0.1.6

A category consists of

i) A collection $\mathcal{R}$ of objects X,Y,....

ii) A set mor(X,Y) of morphisms associated with every two objects X and Y of $\mathcal{R}$ (written $f:X \longrightarrow Y$ or $X \xrightarrow{f} Y$ if f is an element of mor(X,Y))

iii) a map mor (X,Y) x mor(Y,Z) ——> mor (X,Z) for every three objects X,Y and Z of $\mathcal{R}$ such that 1) if $f:X \longrightarrow Y$; $g:Y \longrightarrow Z$, $h:Z \longrightarrow T$ then $h\circ(g\circ f) = (h\circ g)\circ f$

2) to each X of $\mathcal{R}$ there exists identity morphism $e_X:X \longrightarrow X$ such that $f\circ e_X = f$ and $e_X\circ g = g$ for every $f:X \longrightarrow Y$ & $g: Y \longrightarrow X$.

Definition 0.1.7

A Subcategory $\mathcal{R}$ of a category $\mathcal{E}$ is a collection of some of the objects and some of the morphims of $\mathcal{E}$ such that i) if object X is in $\mathcal{R}$ then so is the identity morphism e_X. ii) if morphism $f: X \to Y$ is in $\mathcal{R}$ so are X and Y iii) if $f:X \to Y$, $g: Y \to Z$ are in $\mathcal{R}$ so is $g\circ f$

Definition 0.1.8

Let $\mathscr{C}$ be a category, A,B,C be objects in $\mathscr{C}$. A morphism $f:A \longrightarrow B$ is a monomorphism in $\mathscr{C}$ whenever for any two morphisms $g_1, g_2:C \longrightarrow A$ the equality $f \circ g_1 = f \circ g_2 \longrightarrow g_1 = g_2$

A morphism $h:A \longrightarrow B$ is a epimorphism in $\mathscr{C}$ when ever for any two morphisms $g_1, g_2 :B \longrightarrow C$ the equality $g_1 \circ h = g_2 \circ h \longrightarrow g_1 = g_2$

Definition 0.1.9

A subobject of an object B is a pair (A,f) where $f:A \longrightarrow B$ is a monomorphism .Subobjects (A,f) & (C,g) are said to be isomorphic if there exits a unique isomorphism $g:A \longrightarrow C$ such that $g \circ h = f$

Definition 0.1.10

An object T is a terminal object in $\mathscr{C}$ if to each object C in $\mathscr{C}$ there exists exactly one morphism $g:C \longrightarrow T$.And an object S is initial in $\mathscr{C}$ if to each object C in $\mathscr{C}$ there exists exactly one morphism $h:S \longrightarrow C$.

Definition 0.1.11

Let $\mathscr{C}$ be a category with finite products. An ordered pair (X,m) is called a semigroup object in $\mathscr{C}$ if i) X is an object of $\mathscr{C}$ .ii) $m : X \times X \longrightarrow X$ is a morphism in $\mathscr{C}$

3) m is associative.

CHAPTER 1

THE CATEGORY FTOP

Introduction

Categorical approach in fuzzy set theory was initiated by J.A.Goguen [25], R.Lowen [42], C.K.Wong [76], S.E.Rodabaugh [59], P.Eklund [17-18] and U.Cerruti [6]. Goguen in [25] gave the first categorical definition of fuzzy sets. He constructed the category Set-L, where the objects are (X,μ) where $\mu: X \longrightarrow L$ is an L-fuzzy subset of X and the morphisms $f:(X,\mu) \longrightarrow (Y,\beta)$ are functions $f:(X,\mu) \longrightarrow (Y,\beta)$ such that for every $x \in X$, $\mu(x) \leq \beta(f(x))$. C.K.Wong [76] defined two categories $\mathcal{R}_1$ and $\mathcal{R}_2$ of fuzzy sub sets as follows:-

Let (X,α) denote the ordered pair of a set X and a fuzzy sub set α of X. Then $\mathcal{R}_1$ is the collection of all $(X,\alpha), (Y,\beta)$......For each pair of objects $\left[(X,\alpha),(Y,\beta)\right]$ of $\mathcal{R}_1$ define a set mor $\left\{(X,\alpha),(Y,\beta)\right\} = \left\{f, f_{\alpha\beta}\right\}$ where f ranges over all possible set mappings from X to Y and $f_{\alpha\beta}$ is the induced mapping from fuzzy set α to fuzzy set β defined by $f_{\alpha\beta}(\mu_\alpha(x)) = \mu_\beta(f(x))$. Also $\mathcal{R}_2$ is the collection of all $(X,\mathscr{A})$ where $\mathscr{A}$ denotes the collection of all fuzzy subsets of X. For any two objects $(X,\mathscr{A}),(Y,\mathscr{B})$ of $\mathcal{R}_2$ define $\text{mor}\left[(X,\mathscr{A}),(Y,\mathscr{B})\right] = \left\{f, f_{\mathscr{A}\mathscr{B}}\right\}$ where f ranges over all possible set mappings from X to Y and $f_{\mathscr{A}\mathscr{B}}$ is the collection of all mappings from any fuzzy set $\mathscr{A}$ in X to any fuzzy set $\mathscr{B}$ in Y induced by f.

In the case of fuzzy topology ,there are various interesting categories of fuzzy topological spaces available in the literature. The collection of all fuzzy topological spaces and fuzzy continuous maps form a category. Since C.L.Chang [8],R.Lowen [37] and J.A.Goguen [24] have defined fuzzy topology in different ways, each of them defines a different category of fuzzy topological spaces.

In 1983 S.E.Rodabaugh [59] defined a new fuzzy topological category FUZZ. It is a significant generalization of all previous approaches to fuzzy topology. The objects of FUZZ are of the form (X,L,T) where (X,T) is an L-fuzzy topological space where L is a complete distributive lattice with universal bounds and order reversing involution. A morphism from (X_1,L_1,T_1) to (X_2,L_2,T_2) is a pair (f,ϕ) satisfying the following conditions.

i) $f:X_1 \longrightarrow X_2$ is a function

ii) $\phi^{-1}:L_2 \longrightarrow L_1$ is a function preserving $\cap$, $\cup$

iii) $V \in T_2 \Longrightarrow \phi^{-1} \circ V \circ f \in T_1$

In this chapter we introduce a new category FTOP, which appears to be the best frame work to define fuzzy topological semigroups. In section 1 of this chapter we define an L-fuzzy topological space (X,μ,F) and obtain some of its basic properties.

In section 2 we define FTOP. The objects of FTOP are "L-fuzzy topological spaces" (X,μ,F) and the morphisms are the "fuzzy continuous maps" between two L-fuzzy topological spaces. The subcategories, subobjects, initial objects and final objects in FTOP are obtained. Also we find relations between FTOP and some other categories of fuzzy topological spaces.

1.1 L-fuzzy topological spaces

Definition 1.1.1

Let X be a set, $\mu:X\text{---->}L$ be an L-fuzzy subset of X and F be a subset of L^X satisfying the following conditions:-

i) $g \in F \Longrightarrow g(x) \leq \mu(x)\ \forall\ x \in X$

ii) $\{g_i, \text{where } i \in I\} \subseteq F \Longrightarrow \cup\{g_i, \text{where } i \in I\} \in F$

iii) $g_1, g_2 \in F \Longrightarrow g_1 \cap g_2 \in F$

iv) $1_\phi, \mu \in F$ where 1_ϕ is a constant map from X to L which takes the value 0 for every point $x \in X$.

The triple (X,μ,F) is called an **L-fuzzy topological** space subordinate to μ (or where there is no chance for confusion just L-fuzzy topological space). The members of F are called L-fuzzy open sets and the complements of members of F are called L-fuzzy closed sets.

If F consists of all L-fuzzy sub sets of X which are less than μ, it is called discrete L-fuzzy topolgy and if it consists of 1_ϕ and μ only, it is called indiscrete L-fuzzy topolgy.

Remark

When $L=[0,1]$ and $\mu=1_X$ ($1_X:X\text{---->}L$ such that $1_X(x)=1$ $\forall\ x \in X$), an L-fuzzy topological space is nothing but a Chang's fuzzy space.

Definition 1.1.2

Let (X_1,μ_1,F_1) and (X_2,μ_2,F_2) be two L-fuzzy topological spaces. A mapping g of (X_1,μ_1,F_1) into (X_2,μ_2,F_2) is fuzzy continuous if :

i. $\mu_1(x) \leq \mu_2(g(x)), \forall\ x \in X,$

ii. $\mu_1 \cap g^{-1}(u) \in F_1, \ \forall\ u \in F_2$

Remark

When $L=[0,1]$, $\mu_1=1_{X_1}$ and $\mu_2=1_{X_2}$ this definition coincides with the definition 0.1.3

Proposition 1.1.3

If $f:(X_1,\mu_1,F_1) \longrightarrow (X_2,\mu_2,F_2)$ and $g:(X_2,\mu_2,F_2) \longrightarrow (X_3,\mu_3,F_3)$ are fuzzy continuous then $g\circ f$ is also fuzzy continuous

Proof

Since f and g are fuzzy continuous ,

$$\left.\begin{array}{l} \mu_1(x) \leq \mu_2(f(x)), \forall\ x \in X_1 \\ \mu_1 \cap f^{-1}(V) \in F_1\ \forall\ V \in F_2 \end{array}\right\} (1)$$

and

$$\left.\begin{array}{l} \mu_2(x) \leq \mu_3(g(x)), \forall\ x \in X_2 \\ \mu_2 \cap g^{-1}(V) \in F_2\ \forall\ V \in F_3 \end{array}\right\} (2)$$

By (1) $\mu_1(x) \leq \mu_2((f(x))\ \forall\ x \in X_1$

$\leq \mu_3(g(f(x))\ \forall\ x \in X_1$ by (2)

$= \mu_3(g\circ f)(x))\ \forall\ x \in X_1$

That is $\mu_1 \cap f^{-1}\left[\mu_2 \cap g^{-1}(V)\right] \in F_1$ by (1) & (2)

That is $\mu_1 \cap f^{-1}(\mu_2) \cap f^{-1}(g^{-1}(V)) \in F_1$

That is $\mu_1 \cap f^{-1}(\ g^{-1}(V)) \in F_1$ $\left\{\mu_1(x) \leq \mu_2((f(x)) \ \forall x \in X_1\right\}$

$\left\{\text{that is } \mu_1(x) \leq f^{-1}(\mu_2)(x) \ \forall x \in X_1\right\}$

That is $\mu_1 \cap (g \circ f)^{-1}(V) \in F_1 \forall \ V \in F_3$

Therefore $g \circ f$ is fuzzy continuous.

Definition 1.1.4

Let(X, μ, F) be an L-fuzzy topological space, $Y \subset X$. Let $\gamma = \mu\big|_Y$ $\left(\text{that is } (\mu \cap 1_Y)\right)$ and $\mathcal{U} = \left\{U\big|_Y, U \in F\right\}$, Then $(Y, \gamma, \mathcal{U})$ is called a subspace of (X, μ, F).

Proposition 1.1.5

If $Y \subset X$ and $f:(X_1, \mu_1, F_1) \dashrightarrow (X_2, \mu_2, F_2)$ is fuzzy continuous, then f restricted to the subspace $(Y, \gamma, \mathcal{U})$ is fuzzy continuous.

<u>Proof</u>:

We have $\gamma = \mu_1\big|_Y$

Let $U \in F_2$, then $f^{-1}(U) \cap \mu_1 \in F_1$

That is $\left\{f^{-1}(U) \cap \mu_1\right\}\Big|_Y \in \mathcal{U}$

That is $(f^{-1}(U) \cap 1_Y) \cap (\mu_1 \cap 1_Y) \in \mathcal{U}$

That is $(f|_Y)^{-1}(U) \cap \gamma \in \mathcal{U}$

Therefore $f|_Y$ is fuzzy continuous.

Definition 1.1.6

Let(X,μ,F) be an L-fuzzy topological space,$\mu' \subset \mu$ be any fuzzy subset of X. Then the induced fuzzy topology on μ' is the family of fuzzy subsets of X which are the intersections with μ of L-fuzzy open subsets of X. The induced L-fuzzy topology is denoted by F_{μ}' and the triple (X,μ',F_{μ}') is called induced fuzzy sub space of (X,μ,F).

Proposition 1.1.7

If:(X_1,μ_1,F_1) ---->(X_2,μ_2,F_2) is fuzzy continuous,and $\mu_i' \subset \mu_i$ for i = 1,2 then f is fuzzy continuous from (X_1,μ_1',F_{μ_1}') to (X_2,μ_2',F_{μ_2}') if $\mu_1'(x) \leq \mu_2'(f(x)) \, \forall x \in X_1$.

Proof

We have to show that $\mu_1' \cap f^{-1}(U') \in F_{\mu_1'}$, $\forall\, U' \in F_{\mu_2'}$

Let $U' \in F_{\mu_2'}$

That is $U' = \mu_2' \cap U$ for some $U \in F_2$

Therefore $\mu_1' \cap f^{-1}(U') = \mu'_1 \cap f^{-1}(\mu_2' \cap (U))$

$$= \mu_1' \cap f^{-1}(\mu_2') \cap f^{-1}(U))$$

$$= \mu_1' \cap f^{-1}(U)) \quad \left\{ \text{since } \mu_1' \leq f^{-1}(\mu_2') \right\}$$

$$\in F_{\mu_1'}$$

Definition 1.1.8

Let $\left\{ (X_i, \mu_i, F_i) \,\middle|\, i \in I \right\}$ be a family of L-fuzzy topological spaces. We define their product $\prod_{i \in I} (X_i, \mu_i, F_i)$ to be the L-fuzzy topological space (X, μ, F) ,where $X = \prod_{i \in I} X_i$ is the usual set product, μ the product fuzzy set in X whose membership function is defined by

$\mu(x) = \inf_{i \in I} \left\{ \mu_i(x_i) \,\middle|\, x = (x_i) \in X \right\}$ and F is generated by the sub basis $\beta = \left\{ p_i^{-1}(U_i) \cap \mu \,\middle|\, i \in I \right\}$

Proposition 1.1.9

i) For each $\alpha \in I$ the projection map p_α is fuzzy continuous.

ii) The product L-fuzzy topology is the smallest L-fuzzy topology for X such that i) is true.

iii) Let $(Y,\gamma,\mathcal{U})$ be an L-fuzzy topological space and let f be a function from $(Y,\gamma,\mathcal{U})$ to (X,μ,F), then f is fuzzy continuous if and only if $\forall\ \alpha \in I$, $p_\alpha \circ f$ is fuzzy continuous.

<u>Proof</u>

(i) & (ii) follows from the definition of product L-fuzzy topology.

iii) Suppose $f:(Y,\gamma,\mathcal{U}) \longrightarrow (X,\mu,F)$ is fuzzy continuous since p_α is fuzzy continuous the composition $p_\alpha \circ f$ is fuzzy continuous

conversely suppose $p_\alpha \circ f$ is fuzzy continuous $\forall\ \alpha \in I$

then $\gamma \cap (p_\alpha \circ f)^{-1}(U_\alpha) \in \mathcal{U}\ \forall\ U_\alpha \in F_\alpha\ \forall \alpha$

that is $\gamma \cap f^{-1}(p_\alpha^{-1}(U_\alpha)) \in \mathcal{U}\ \forall\ U_\alpha \in F_\alpha\ \forall \alpha$

that is $\gamma \cap f^{-1}(U) \in \mathcal{U}$ where $U = p_\alpha^{-1}(U_\alpha)$

therefore f is fuzzy continuous. (by using definition 1.1.8)

Definition 1.1.10

Let (X,μ,F) be an L-fuzzy topological space, R be an equivalence relation on X. Let X/R be the usual quotient set and $p:X \longrightarrow X/R$ be the usual quotient map. We define the quotient L-fuzzy topology as follows

let $\nu = p(\mu)$ so that ν is an L-fuzzy set in X/R and $\mathcal{U} = \left\{ U:X/R \dashrightarrow L \;\middle|\; p^{-1}(U) \cap \mu \in F \right\}$. Then $(X\backslash R, \nu, \mathcal{U})$ is the quotient space of (X, μ, F).

Proposition 1.1.11

let (X,μ,F) be an L-fuzzy topological space and $(X\backslash R,\nu,\mathcal{U})$ be the quotient space of (X,μ,F), then :

i) $p:(X,\mu,F) \dashrightarrow (X/R,\nu,\mathcal{U})$ is fuzzy continuous

ii) Let (X_1,μ_1,F_1) be an L-fuzzy topological space and g be a function from the quotient fuzzy space $(X/R,\nu,\mathcal{U})$ to (X_1,μ_1,F_1) then g is fuzzy continuous if and only if $g \circ p$ is fuzzy continuous

<u>Proof</u>

i) It is trivial from the definition of quotient L-fuzzy topology.

ii) Suppose g is fuzzy continuous, then the composition $g \circ p$ is fuzzy continuous

conversely suppose $g \circ p$ is fuzzy continuous

that is $\mu \cap (g \circ p)^{-1}(U) \in F \;\forall\; U \in F_1$

that is $\mu \cap p^{-1}(g^{-1}(U)) \subset F \; \forall \; U \in F_1$

that is $g^{-1}(U) \in \mathcal{U}$ (by the definition of quotient L-fuzzy topology)

hence $\nu \cap (g^{-1}(U))$ is open in X/R.

Therefore g is fuzzy continuous.

1.2 Categories of L-fuzzy topological spaces

Definition 1.2.1

Let $\mathcal{E}$ be the collection of all L-fuzzy topological spaces (X,μ,F)......For each pair of objects (X_1,μ_1,F_1) and (X_2,μ_2,F_2) of $\mathcal{E}$ let mor $\left[(X_1,\mu_1,F_1),(X_2,\mu_2,F_2)\right]$ be the set of all fuzzy continuous mappings from (X_1,μ_1,F_1) to (X_2,μ_2,F_2). Clearly $\mathcal{E}$ constitutes a category; we denote this category by **FTOP**.

Results. 1.2.2

In FTOP the **monomorphisms** are the injections in the usual sense.

For let $f:(X_1,\mu_1,F_1) \dashrightarrow (X_2,\mu_2,F_2)$ be an injection .

consider two morphisms $g_1,g_2:(X_3,\mu_3,F_3) \longrightarrow (X_1,\mu_1,F_1)$

and suppose $f\circ g_1=f\circ g_2$

that is $(f\circ g_1)(x) = (f\circ g_2)(x)\ \forall\ x\in X_3$

that is $f(g_1(x)) = f(g_2(x))\ \forall x \in X_3$

that is $g_1(x) = g_2(x)\ \forall x \in X_3$

that is $g_1= g_2$

Conversely, if f is not an injection ,and let $f\circ g_1=f\circ g_2$

that is $(f\circ g_1)(x)=(f\circ g_2)(x)\ \forall\ x \in X_3$

that is $f(g_1(x)) = f(g_2(x))\ \forall x \in X_3$

that is $g_1(x) \neq g_2(x)\ \forall x \in X_3$

that is $g_1\neq g_2$

Therefore f can not be a monomorphism.

Simil arly we can show that the epimorphisms of FTOP are the surjections .

An object $\left\{\{x\},1_{\{x\}},F\right\}$ where $\{x\}$ denotes any one point set, $1_{\{x\}}$ is an L-fuzzy subset of $\{x\}$ such that $1_{\{x\}}(x)=1$ and $F=\left\{0,1_{\{x\}}\right\}$ is a terminal object of FTOP.

For each object (X,μ,F) of FTOP the subspace $(Y,\gamma,\mathcal{U})$ (cf.definition 1.2.4) and the induced fuzzy subspace $(X,\gamma_\mu,F_{\gamma_\mu})$ (cf.definition 1.2.6) are **subobjects of** (X,μ,F)

$\mathscr{E}_1$ consists of those objects (X,μ,F) for a fixed X together with the morphisms and $\mathscr{E}_2$ consists of those objects (X,μ,F) where μ is fixed for a particular X. Clearly $\mathscr{E}_1$ and $\mathscr{E}_2$ are **sub categories of FTOP.**

Relation between FTOP and other categories

Consider the follwing Functors $\mathscr{F}_1$,$\mathscr{F}_2$

1) where $\mathscr{F}_1,\mathscr{F}_2$:TOP---->FTOP such that $\mathscr{F}_1(X,T)=(X,1_X,F)$

where $F=\left\{g:X\text{---->}[0,1] \;\middle|\; g \text{ is a l.s.c.map}\right\}$

and $\mathscr{F}_1(f) = f$

$\mathscr{F}_2(X,T)= (X,1_X,\mathcal{U})$ where $\mathcal{U} =\left\{ \lambda_U:U \in T \;\middle|\; \lambda_U \text{ denote a characterstic map of } U\right\}$

$\mathscr{F}_2(f)=f$

clearly $\mathscr{F}_1$ embedds TOP into a full subcategory of FTOP

2) Consider $\mathcal{F}_3$:FTOP---->TOP

$\mathcal{F}_3$ $(X,\mu,F) = (X,i(F))$ where i(F) denotes the smallest topology on X such that each member of F becomes a l.s.c.map. and $\mathcal{F}_3(f)=f$

<u>Product objects in FTOP</u>

Let (X_1,μ_1,F_1) and (X_2,μ_2,F_2) be two objects in FTOP. In the categorical sense we define their product (X_1,μ_1,F_1) x $(X_2\ \mu_2,F_2)$ as $(X_1 \times X_2,\mu_1 \times \mu_2,F_1 \times F_2,\pi_1,\pi_2)$ where $F_1 \times F_2$ is generated by the sub basis $S = \left\{ g_1 \times \mu_2 \;\middle|\; g_1 \in F_1 \right\} \cup \left\{ \mu_1 \times g_2 \;\middle|\; g_2 \subset F_2 \right\}$ and π_i: $X_1 \times X_2$---->X_i (i=1,2) are ordinary projections .

CHAPTER 2*

ON L-FUZZY TOPOLOGICAL SEMIGROUPS

Introduction

The study of fuzzy topological groups was started in 1979 by D.H.Foster [20]. He combined the structure of a fuzzy topological space (in the sense of R.Lowen) with that of a fuzzy group defined by Rosenfeld [62] to define a fuzzy topological group. He studied products,quotients and homomorphisms of fuzzy topological groups. Later Chun Hai Yu [13] and Ma Ji Liang [43-44] developed a theory on fuzzy topological groups. In [43] a fuzzy topological group is defined in terms of neighbourhoods and Q-neighbourhoods. Also they discussed about direct products, quotients, Separation properties,Q-compactness and some algebraic properties of fuzzy topological groups. B.T.Lerner [35] defined a fuzzy right topological semigroup and extended some of Foster's results on homomorphic images and inverse images to fuzzy right topological semigroups. In 1991 Ma Ji Liang and Chun Hai Yu [44] defined and studied L-fuzzy topological groups .

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* Some of the results of this chapter has been accepted for publication in the journal of Mathematical Analysis and Applications.

In this chapter we define an L-fuzzy topological semigroup in the categorical point of view. In section 2 we identify a semigroup object of FTOP as an L-fuzzy topological semi group. We specialize the results to the case when L=[0,1] in section 3, in this case an L-fuzzy topological semigroup is called a fuzzy topological semigroup; we study some basic properties of fuzzy topological semigroups.

2.1 Preliminary concepts

Definiton 2.1.1

A fuzzy set U in a fuzzy topological space (X,F) is called a **neighbourhood** of a fuzzy point x_λ ($\lambda > 0$) if there exists a $V \in F \ni x_\lambda \in V \subset U$

Definition 2.1.2

A fuzzy set V is said to be **quasi coincident** with a fuzzy point x_λ ($0 < \lambda \leq 1$) denoted by x_λ q V if $\lambda + V(x) > 1$

A fuzzy set U is said to be a **Q-neighbourhood** of a fuzzy point x_λ if there exists a $V \in F \ni x_\lambda$ q $V \subset U$

Proposition 2.1.3 [58]

Let f be a mapping from a fuzzy topological space (X,F) into a fuzzy topological space $(Y,\mathcal{U})$, then the following are equivalent

i) f is a continuous mapping

ii) f is continuous with respect to neighbourhood at every fuzzy point x_λ.

iii) f is continuous with respect to Q-neighbourhood at every fuzzy point x_λ.

Proof:- See theorem 1.1 [58]

Definition 2.1.4

Let $(X,\circ)$ be a semigroup; an L-fuzzy set $\mu:X \longrightarrow L$ is said to be an L-fuzzy semigroup if

$$\mu(x \circ y) \geq \min\ (\mu(x),\mu(y)),\ \forall\ x,y,\in X.$$

Proposition 2.1.5

The intersection of any set of L-fuzzy semigroups is an L-fuzzy semigroup

Proof:

The proof is same as the proof of proposition 3.1 [62]

2.2 L-fuzzy topological semigroups

Definition 2.2.1

A semigroup object in FTOP is called an L-fuzzy topological semigroup.

That is a quadruple (X,μ,F,m) is an L-fuzzy topological semigroup if

i) (X,μ,F) is an object of FTOP

ii) $m : (X,\mu,F) \times (X,\mu,F) \longrightarrow (X,\mu,F)$ is fuzzy continuous.

that is $m : X \times X \longrightarrow X$ is subjected to the condition

1) $\mu(x) \cap \mu(y) \leq \mu\{m(x,y)\} \; \forall \; x,y \in X$

2) $m^{-1}(g) \cap \mu \times \mu \in F \times F \; \forall \; g \in F$

iii) m is associative

Example

Let $X = (N, \circ)$, $\mu : X \longrightarrow L = [0,1]$ such that

$\mu(x) = 1 - 1/x \,|\, x \in X$ and F, the discrete L-fuzzy topology on X.

Then (X,μ,F) is an L-fuzzy topological semigroup .

Example

Let $X = (R,+)$, $\mu : X \longrightarrow L = [0,1] \ni \mu(x) = \frac{1}{2} \; \forall \; x \in X$

and $F = \{f:X \longrightarrow L=[0,1] \ni f(x)=c \text{ where } c \leq \frac{1}{2}\}$

Then (X,μ,F) is an L-fuzzy topological semigroup .

Definition 2.2.2

Let (X,μ,F,m) be an L-fuzzy topological semigroup and $\mu' \subset \mu$ be an L-fuzzy semigroup of (X,m). Then the induced L-fuzzy topological space $(X,\mu',F_{\mu'})$ is said to be an induced L-fuzzy topological semigroup if the mapping $m:(x,y) \longrightarrow xy$ of $(X,\mu',F_{\mu'}) \times (X,\mu',F_{\mu'}) \longrightarrow (X,\mu',F_{\mu'})$ is fuzzy continuous .

Proposition 2.2.3

If (X,μ,F,m) is an L-fuzzy topological semigroup and $\gamma :X \longrightarrow L$ is an L-fuzzy semigroup of (X,m) and $\gamma \leq \mu$ then (X,γ,F_{γ},m) is an induced L-fuzzy topological semigroup of (X,μ,F,m).

Proof

Let $g \in F_{\gamma}$ then there exists an $f \in F \ni f \cap \gamma = g$

consider $(m^{-1}(g) \cap \gamma \times \gamma)(x,y) = \{m^{-1}(f \cap \gamma) \cap (\gamma \times \gamma)\}(x,y)$

$$= (f \cap \gamma)\bullet m(x,y) \cap (\gamma \cap \gamma)(x,y)$$

$$= (f \cap \gamma)(xy) \cap (\gamma \cap \gamma)(x,y)$$

$$= f(xy) \cap \gamma(xy) \cap \gamma(x) \cap \gamma(y)$$

$$= f(xy) \cap \gamma(x) \cap \gamma(y) \quad \left[\gamma(xy) \geq \gamma(x) \cap \gamma(y) \right]$$

$$= m^{-1}(f)\ (x,y) \cap \gamma(x) \cap \gamma(y)$$

$$= \{m^{-1}(f) \cap (\gamma \times \gamma)\}(x,y)$$

$$\therefore m^{-1}(g) \cap \gamma \times \gamma = m^{-1}(f) \cap \gamma \times \gamma$$

$\in F_{\gamma} \times F_{\gamma}$ (by the definition of induced L-fuzzy topological space)

Therefore (X,γ,F_γ,m) is an induced L-fuzzy topological semigroup.

Proposition 2.2.4

Let (X,μ,F) be an L-fuzzy topological semigroup. $Y \subset X$ be a subsemigroup of X, then the subspace $(Y,\gamma,\mathcal{U})$ of (X,μ,F) is also an L-fuzzy topological semigroup.

Proof

We have $\gamma = \mu\big|_Y$ and $\mathcal{U} = \left\{U\big|_Y, U \in F\right\}$

Let $x,y \in X$, f be the mapping such that $f:(x,y) \longrightarrow xy$. We have to show that $f\big|_Y$ is fuzzy continuous.

For let $U' \in \mathcal{U}$

that is there exists a $U \in F$ such that $U\big|_Y = U'$

we have $f^{-1}(U) \cap \mu \cap \mu \in F \times F$

That is $(f^{-1}(U) \cap \mu \cap \mu)\big|_{YxY} \in \mathcal{U} \times \mathcal{U}$

that is $f^{-1}(U\big|_{YxY}) \cap (\mu \cap \mu)\big|_{YxY} \in \mathcal{U} \times \mathcal{U}$

that is $f^{-1}(U') \cap \gamma \times \gamma \in \mathcal{U} \times \mathcal{U}$

Therefore $f\big|_Y$ is fuzzy continuous

Therefore $(Y,\gamma,\mathcal{U})$ is an L-fuzzy topological semigroup.

2.3 Fuzzy topological semigroups

Proposition 2.3.1

Let X be a semigroup and F a fuzzy topology on X.Then the following conditions are equivalent.

1) For all $x,y \in X$ the mapping $f:(x,y) \longrightarrow xy$ of $(X,F) \times (X,F)$ into (X,F) is fuzzy continuous.

2) For all $x,y \in X$ and any neighbourhood W of fuzzy point $(xy)_\lambda (0 < \lambda \leq 1)$ there are neighbourhoods U of x_λ and V of y_λ such that $U \circ V \subset W$.

3) For all $x,y \in X$ and any Q-neighbourhood W of fuzzy point $(xy)_\lambda (0 < \lambda \leq 1)$ there are Q-neighbourhoods U of x_λ and V of y_λ such that $U \circ V \subset W$.

Proof:Analogous to proposition 2.2 [43]

Definition 2.3.2

When L=[0,1] and $\mu=1_X$ an L-fuzzy topological semigroup (cf.definition 2.2.1) is called a **fuzzy topological semigroup.**

That is if X be a semigroup and F, a fuzzy topology on X, then the fuzzy topological space (X,F) is said to be a fuzzy topological semigroup if it satisfy any one of the 3 equivalent conditions in proposition 2.3.1

Remark

1) A semigroup with discrete fuzzy topology is a fuzzy topological semigroup.

2) The set of all constant fuzzy sets on a semigroup is a fuzzy topology on it and with respect to this fuzzy topology it is a fuzzy topological semigroup.

The following two propositions gives an association between fuzzy topological semigroups and topological semigroups.

Proposition 2.3.3

A topological semigroup X is a fuzzy topological semigroup with fuzzy topology $F = \{ g : X \text{----}> [0,1] \mid g$ is a lower semi continuous function $\}$

Proof

We claim that the semigroup operation $\nu : (x,y) \text{----}> xy$ of (X,F) x (X,F) into (X,F) is fuzzy continuous.

For let $U \in F$, then $\nu^{-1}(U) = U \circ \nu$ (by definition 0.1.2)

consider $(U \circ \nu)^{-1}(\alpha,1] = \nu^{-1}(U^{-1}(\alpha,1]), 0 < \alpha \leq 1$

$= \nu^{-1}(W)$ where $W = U^{-1}(\alpha,1]$ is open X

Since X is a topological semigroup $\nu^{-1}(W)$ is open in X x X

Therefore $U \circ \nu$ is lower semicontinuous

Therefore $\nu^{-1}(U)$ is open fuzzy set in X x X

Therefore ν is fuzzy continuous .

Proposition 2.3.4

If (X,F) is a fuzzy topological semigroup ,then (X,i(F)) is a topological semigroup,where i(F) denotes the smallest topology on X such that each member of F is a lower semi continuous function.

Proof

We have to show that for $x,y \in X$, the mapping $f:(x,y) \longrightarrow xy$ of $(X,i(F)) \times (X,i(F))$ into $(X,i(F))$ is continuous.

For, let $W \in i(F)$ be a subbasic open set in X,then there exists a $\mu \in F$ such tht $W = \mu^{-1}(\alpha,1]$ for $\alpha>0$

Now $\mu \in F$ implies $f^{-1}(\mu) \in F \times F$ (since (X,F) is a fuzzy topological semigroup).

That is $\mu \circ f \in F \times F$

That is $(\mu \circ f)^{-1}(\alpha,1] \in i\ (F x F)$

That is $f^{-1}(\mu^{1}\ (\alpha,1] \in i(F x F)$

That is $f^{-1}(W) \in i(F \times F)$

Therefore f is continuous.

Note:

We do not insist that a topological semigroup should have Hausdorff separation property, However if F is Hausdroff,then i(F) is Hausdorff.

Proposition 2.3.5

Let (X,F) be a fuzzy topological semigroup, then a sub semigroup of X with subspace topology is a fuzzy topological semigroup and a fuzzy semigroup of X with induced fuzzy topology is an induced fuzzy topological semigroup of (X,F).

Proof

Let $Y \subset X$ be a subsemigroup of X,then the fuzzy topology on Y with respect to F is

$F_Y = \left\{ U|_Y \text{ that is } (U \cap 1_Y), U \in F \right\}$. clearly (Y, F_Y) is a fuzzy topological semigroup.

Let μ be a fuzzy semigroup of X,then by proposition 2.2.4 (μ, F_μ) is an induced fuzzy topological semigroup of (X,F).

CHAPTER 3*

ON L-FUZZY SEMITOPOLOGICAL SEMIGROUPS

Introduction

In this chapter we define an L-fuzzy semi topological semigroup from the categorical point of view, and study some properties of an L-fuzzy semi topological semigroup.

In section 2 we specialize an L-fuzzy semi topological semi group to a fuzzy semi topological semigroup. Also we observe an association between the classes of semi topological semigroups and fuzzy semi topological semigroups. In the last section we give some examples to illustrate the relations between fuzzy topological semigroups and fuzzy semi topological semigroups.

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*.This chapter will appear as a research paper in the Journal of Mathematical Analysis and Applications.

3.1 L- Fuzzy semi topological semigroups

Definition 3.1.1

An object (X,μ,F) in FTOP is an L-fuzzy semi topological semigroup if

i) X is a semigroup

ii) μ is an L-fuzzy semigroup

iii) [1]. $\rho_t : x \longrightarrow xt$ [2], $\lambda_t : x \longrightarrow tx$ from (X,μ,F) to (X,μ,F) are fuzzy continuous.

Note

If (X,μ,F) satisfies all the above conditions except condition iii(2) it is called an **L- fuzzy right topological semigroup** and if (X,μ,F) satisfies all the above conditions except condition iii(1) it is called an **L-fuzzy left topological semigroup.**

Definition 3.1.2

Let (X,μ,F) be an L-fuzzy semi topological semigroup, γ be an L-Fuzzy semigroup of X.Then (X,γ,F_γ) is said to be an **induced L-fuzzy semi topologicalsemigroup** if $\rho_t : x \longrightarrow xt$ and $\lambda_t : x \longrightarrow tx$ of (X,γ,F_γ) into (X,γ,F_γ) are fuzzy continuous.

Proposition 3.1.3

Let (X,μ,F) be an L-fuzzy semi topological semigroup, γ be an L-fuzzy semigroup of X. Then (X,γ, F_γ) is an induced L-fuzzy semi topological semigroup if γ satisfies the following conditions:

i) $\gamma(xt) \geq \gamma(x)$ ii) $\gamma(tx) \geq \gamma(x)$

Proof

We have to show that $\rho_t : x \longrightarrow xt$ and $\lambda_t : x \longrightarrow tx$ of (X,γ,F_γ) into (X,γ,F_γ) are fuzzy continuous.

For let $U \in F_\gamma$

that is $U = \gamma \cap f \mid f \in F$

$$\text{Consider } \left\{\gamma \cap \rho_t^{-1}(U)\right\}(x) = \gamma(x) \cap U \circ \rho_t(x)$$

$$= \gamma(x) \cap (\gamma \cap f) \circ \rho_t(x)$$

$$= \gamma(x) \cap \gamma \circ \rho_t(x) \cap f \circ \rho_t(x)$$

$$= \gamma(x) \cap \gamma(xt) \cap f(xt)$$

$$= \gamma(x) \cap f(xt) \quad (\text{ since } \gamma(xt) \geq \gamma(x))$$

$$= \gamma \cap \rho_t^{-1}(f)(x)$$

$$\therefore \gamma \cap \rho_t^{-1}(U) = \gamma \cap \rho_t^{-1}(f)$$

$$\in F_\gamma$$

Therefore ρ_t is fuzzy continuous. Similarly λt

Proposition 3.1.4

Let (X,μ,F) be an L-fuzzy semitopological semigroup. $Y \subset X$ be a subsemigroup of X. Then the subspace $(Y,\gamma,\mathcal{U})$ is also an L-fuzzy semitopological semigroup.

Proof : is obvious

3.2 Fuzzy semi topological semigroups

Definition 3.2.1

Let $(X.\circ)$ be a semigroup. U,V be two fuzzy sets in X. We define UV,Uy and xV as follows:

$$UV(z) = \sup_{x \circ y = z} \min (U(x), V(y))$$

$$Uy(z) = \sup_{x \circ y = z} U(x) \quad \text{if there exists such } x$$
$$= 0 \text{ otherwise}$$

$$xV(z) = \sup_{x \circ y = z} V(y) \text{ if there exists such } y$$
$$= 0 \text{ otherwise}$$

Proposition 3.2.2

For the same semigroup X and for the same fuzzy topology F, the following conditions are equivalent.

1) For $x, y \in X$ the mapping $g_1 : x \longrightarrow xy$ and $g_2 : x \longrightarrow yx$ for every fixed y are fuzzy continuous.

2) For each $x, y \in X$ and any neighbourhood W of $(xy)_\lambda (0 < \lambda \leq 1)$ there exists neighbourhoods U of x_λ and V of y_λ such that $g_1(U) \subset W$ and $g_2(V) \subset W$

3) For each $x, y \in X$ and any neighbourhood W of $(xy)_\lambda$ $(0 < \lambda \leq 1)$ there exists neighbourhoods U of x_λ and V of y_λ such that $Uy \subset W$ and $xV \subset W$

4) For each $x, y \in X$ and any Q-neighbourhood W of $(xy)_\lambda$ $(0 < \lambda \leq 1)$ there exists Q-neighbourhoods U of x_λ and V of y_λ such that $g_1(U) \subset W$ and $g_2(V) \subset W$.

Proof

1====>2

Assume that the mapping $g_1 : x \longrightarrow xy$ and $g_2 : x \longrightarrow yx$ for every fixed y are fuzzy continuous.

Let W be an open neighbourhood of $(xy)_\lambda (0 < \lambda \leq 1)$.

That is $\lambda \leq W(xy)$

Also $g_1^{-1}(W)$ and $g_2^{-1}(W)$ are open in X.

consider $\mu_{g_1^{-1}(W)}(x) = \mu_W \circ g_1(x)$

$$= \mu_W (xy)$$

$$\geq \lambda$$

$\therefore$ $g_1^{-1}(W)$ is a neighbourhood of x_λ

Let $U = g_1^{-1}(W)$

$\therefore g_1(U) \subset W$

similarly $g_2(V) \subset W$.

2====>3

We have $g_1(U) \subset W$ and $g_2(V) \subset W$

Consider $Uy(z) = \left\{ \text{Sup } U(x) \mid x \in g_1^{-1}(z) \right\}$, if $g_1^{-1}(z) = \phi$

= 0 otherwise

$= g_1(U)(z)$

$\leq W(x)$

Similarly $Xv \subset W$.

3====>4

Let W be a Q-neighbourhood of $(xy)_\lambda$.

That is $\lambda + W(xy > 1$

$1-\lambda < W(xy)$---->(i)

Choose λ_1 such that $1-\lambda < \lambda_1 < W(xy)$

Then W is a neighbourhood of $(xy)_{\lambda_1}$

By(3) there exists neighbourhoods U of x_{λ_1} and V of y_{λ_1} such that $Uy \subset W$ and $xV \subset W$-----> (ii)

That is U and V are Q-neighbourhoods of x_λ and y_λ respectively.

Now we have to show that $g_1(U) \subset W$ and $g_2(V) \subset W$

Consider

$$g_1(U)(z) = \left\{ \text{Sup } U(x) \;\middle|\; x \in g_1^{-1}(z) \right\}, \text{ if } g_1^{-1}(z) = \phi$$

$$= 0 \text{ otherwise , where } g_1^{-1}(z) = \left\{ x \ni g_1(x) = z \right\}$$

$$= Uy(z)$$

$$\leq W(z)$$

Similarly $g_2(V) \subset W$.

4====>1

Let $x, y \in X$ and W be any Q-neighbourhood of $(xy)_\lambda$. Then there exists Q-neighbourhoods U of x_λ, V of y_λ such that $g_1(U) \subset W$ and $g_2(V) \subset W$.

Assume that $U,V \in F$ then $x_\lambda q\, U$ and $y_\lambda q\, V$.

Now $g_1(U) \subset \omega$ implies that g_1 is fuzzy continuous at x_λ.

Therefore g_1 is fuzzy continuous.(cf proposition 2.1.3)

Similarly g_2 is also fuzzy continuous.

Definition 3.2.3

Let X be a Semigroup,F a fuzzy topology on X.Then the fuzzy topological space (X,F) is a **fuzzy semitopological semigroup** if it satisfies any one of the conditions in the above proposition

Proposition 3.2.4

A semi topological semigroup (X,T) is a fuzzy semi topological semigroup with fuzzy topology F Where

$$F=\left\{ U:X \longrightarrow [0,1] \;\middle|\; U \text{ is a l.s.c map} \right\}$$

Proof

Similar to the proof of proposition 2.3.3

Proposition 3.2.5

If (X,F) is a fuzzy semi topological semigroup,then (X,i(F)) is a semi topological semigroup.

Proof

Simil ar to the proof of proposition 2.3.4

Proposition 3.2.6

If (X,F) be a fuzzy semi topological semigroup and X has group structure then for a fixed element 'a' the mappings ν_a:x---->xa and ψ_a:x---->ax of (X,F) onto (X,F) are fuzzy homeomorphisms.

Proof

It is clear that ν_a is one one and onto.We have to show that ν_a and ν_a^{-1} are fuzzy continuous.Since (X,F) is a fuzzy semi topological semigroup there exists neighbourhoods V of $(xa)_\lambda (0 < \lambda \leq 1)$ and U of x_λ such that Ua $\subset$ V

$\therefore \nu_a$ is fuzzy continuous.

Now ν_a^{-1} is the mapping such that ν_a^{-1}:x---->xa^{-1} which is also fuzzy continuous by the same arguement

$\therefore \nu_a$ is a fuzzy homeomorphism.

Corollary 3.2.7

If (X,F) is a fuzzy semi topological semigroup and X has group structure ,then for any $x_1,x_2 \in X$ there exists a fuzzy homeomorphism f of (X,F) such that $f(x_1)=x_2$

Proof

Let $x_1^{-1}x_2 = a \in X$.

consider the mapping f:x---->xa, then f is fuzzy homeomorphism by the proposition 3.2.6.

Note

A fuzzy topological space satisfies the above corollary is called a homogeneous fuzzy space.

Remark

A fuzzy topological semigroup is a fuzzy semi topological semigroup ,but not conversely.

Example

Consider $R \cup \{\infty\}$ of the additive group of real numbers with the operation extended by $x + \infty = \infty + x = \infty$. Define a fuzzy topology F on $R \cup \{\infty\}$ as follows:

$$F = \left\{ g : R \cup \{\infty\} \longrightarrow [0,1] \text{ where } g \text{ is a l.s.c map} \right\}$$

Then $R \cup \{\infty\}$ is a fuzzy semi topological semigroup ,but it is not a fuzzy topological semigroup.

Remark

A fuzzy right topological semigroup is not necessarily a fuzzy left topological semigroup..

Example

Let $Y=(X^X, \circ, T)$ where X be any topological space, T,the Tychonoff product topology on X^X and $\circ$ is the composition of functions. Then Y is a fuzzy right topological semigroup but it is not a fuzzy left topological semigroup.

These examples are adapted from [3]

Proposition 3.2.8

A fuzzy topological space can be made a fuzzy topological semigroup.

Proof:

Let (X,F) be a fuzzy topological space. Define a semigroup operation α on X such that for $x,y \in X$, $\alpha(x,y)=x$. We have to show that α is fuzzy continuous. It is equivalent to show that for any open neighbourhood W of a fuzzy point $(x)_\lambda$ $(0 < \lambda \leq 1)$ there exists open neighbourhoods U of $(x,y)_\lambda$ such that $\alpha(U) \subset W$

Let W be an open neighbourhood of a fuzzy point $(x)_\lambda$ $(0 < \lambda \leq 1)$.

$\lambda \leq W(x)$

Consider $\alpha^{-1}(W)(x,y) = W \circ \alpha(x,y)$

$= W(x)$

$\geq \lambda$

$\therefore \alpha^{-1}(W)$ is an open neighbourhood of $(x,y)_\lambda$.

Let $U = \alpha^{-1}(W)$

$\therefore \alpha(U) \subset W$

$\therefore \alpha$ is fuzzy continuous.

$\therefore$ (X,F) is a fuzzy topological semigroup.

CHAPTER 4

COMPACT OPEN FUZZY TOPOLOGICAL SPACES

Introduction

Fuzzy topology on Function spaces were studied by Peng,Yu wei [55].For given fuzzy topological spaces (X,F), (Y,$\mathcal{U}$) and $F=\{ g:X\text{------}>Y \}$,they defined point wise convergent fuzzy topology and compact open fuzzy topology on F based on crisp subsets of X and Y. Some separation properties of this fuzzy topological spaces were also discussed in [55]

In this chapter we define fuzzy topology for a family of functions between two fuzzy topological spaces. We define point open fuzzy topology and Compact open fuzzy topology depends on fuzzy subsets of X and Y. Also we obtain some relations between compact open fuzzy topology and some other fuzzy topologies on Function spaces.

4.1 Preliminary concepts

In this chapter by a Hausdorff fuzzy topological space we mean that the definition introduced in [23] and by compactness of fuzzy topological space we mean N-compactness introduced by Wang Guojun [69].

Definition 4.1.1 [23]

Let (X,F) be a fuzzy toplogical space. Then it is called a T_2 fuzzy space if $x,y \in X$ with $x \neq y$ imply that there exists a and b in F with $a(x)=b(y)=1$ and $a \cap b=0$

Definition 4.1.2

Let (X,F) be a fuzzy topological space, x_λ, $0<\lambda\leq 1$ be a fuzzy point, and P a closed fuzzy set in X. Then P is called a remoted neighbourhood(R-nbd) of x_λ if $x_\lambda \notin P$

Definition 4.1.3

A fuzzy net S is a function $S:\mathcal{D}\longrightarrow\mathcal{T}$ where $\mathcal{D}$ is a directed set with order relation $\geq$ and $\mathcal{T}$ the collection of all fuzzypoints in X. Then for each $n\in\mathcal{D}$, S(n) is a fuzzy point belonging to $\mathcal{T}$ let its value in (0,1] be denoted by λ_n. Thus we get a crisp net $V(S)=\{\lambda_n, n\in\mathcal{D}\}$ in the half open interval (0,1]. If V(S) converges to $\alpha\in(0,1]$ we say that V(S) is a α net.

Definition 4.1.4

A fuzzy point x_λ is called a cluster point of a fuzzy net $S=\{S(n); n\in \mathcal{D}\}$ if for each R-nbd P of x_λ we have frequently $S(n) \not\leq P$

Definition 4.1.5

A fuzzy set μ in a fuzzy topological space is called N-compact if each α-net $(0< \alpha\leq 1)$ contained in μ has atleast a cluster point x_α with value α .

A fuzzy topological space (X,F) is N-compact if the constant fuzzy set 1 is N-compact.

Note :

All the usual important properties of N-compactness are valid simultaneously for Chang Spaces as well as Lowen spaces.

4.2 Compact open fuzzy topological spaces

Definition 4.2.1

Let X be any set and Y be a fuzzy topological space and C(X,Y) be any set of functions from X into Y; for a fuzzy point a_λ in X and an open fuzzy set U in Y Consider $\theta: C(X,Y) \longrightarrow [0,1]$ such that $\theta(f)= 1-\sup_{y\in Y} \left[(f(a_\lambda) \cup U)-U\right](y)$, now the collection of all such θ's for fuzzy points a_λ in X and open fuzzy sets U in Y, generate a fuzzy topology on C(X,Y) called the point open fuzzy topology

It is to be noted that $\theta(f)= \begin{cases} 1 \text{ if } \lambda < U(f(a)) \\ 1-\lambda+U(f(a)), \text{otherwise} \end{cases}$

Definition 4.2.2

Let X,Y be two fuzzy topological spaces and let

$C_F(X,Y)=\left\{f:X\text{----->}Y \middle| f \text{ is a fuzzy continuous map}\right\}$, Consider

$\theta:C(X,Y)\text{----->}[0,1]$ such that $\theta(f)=1-\operatorname{Sup}_{y\in Y}\left[(f(A)y \cup B(y)-B(y)\right]$

for a compact fuzzy set A of X and an open fuzzy set B of Y. Now the collection of all such θ's for fuzzy sets A in X and open fuzzy sets B in Y, generate a fuzzy topology on $C_F(X,Y)$ called the compact open fuzzy topology $F_{\mathcal{E}}$

Remark

1) When X is a discre te fuzzy topological space, the point op en fuzzy topology is equal to the compact open fuzzy topology.

2) When X is any fuzzy topological space, the point op en fuzzy topology is smaller than the compact open fuzzy topology.

Note 4.2.3

Relation between compact open topologies and some fuzzy topologies on Function spaces.

case(i)

Let (X,W(T) and (Y,W($\mathcal{U}$)) be associated fuzzy topological spaces of topological spaces (X,T) and (Y,$\mathcal{U}$) respectively. consider

$C_T(X,Y) = \left\{ f:X \text{----}> Y \,\middle|\, f \text{ is a continuous map} \right\}$ and

$C_F(X,Y) = \left\{ g:X \text{----}> Y \,\middle|\, g \text{ is a fuzzy continuous map with respect to the associated fuzzy topological spaces} \right\}$.

then $C_T(X,Y) = C_F(X,Y)$ (cf:proposition 3.1 [37])

Consider the compact open topologies (crisp) $\mathcal{E}$ and $\mathcal{E}'$ on $C_T(X,Y)$ generated by the following sub basis. Let $\mathcal{E}$ be the compact open topology generated by $N(K,U) = \left\{ f \in C(X,Y) \text{ such that } f(K) \subset U \right\}$, for a compact subset K of X and an open set U of Y, let $\mathcal{E}'$ be the compact open topology generated by $N(K',U') = \left\{ f \in C_F(X,Y) \text{ such that } f(K') \subset U' \right\}$, for a compact fuzzy set K' of X and an open fuzzy set U' of Y.

clearly $\mathcal{E} \subset \mathcal{E}'$

$W(\mathcal{E})$ and $W(\mathcal{E}')$ are the associated fuzzy topologies with respect to $\mathcal{E}$ and $\mathcal{E}'$ respectively.

clearly $W(\mathcal{E}) \subset W(\mathcal{E}')$ -------→(1)

Result

$W(\mathcal{E}) \subset W(\mathcal{E}') \subset F_{\mathcal{E}}$

For let $\mathcal{O} \in W(\mathcal{E}')$

that is $\mathcal{O}$ is l.s.c with respect to $\mathcal{E}'$

that is $\mathcal{O}^{-1}(\alpha,1] \in \mathcal{E}'$, $\forall \alpha \in (0,1]$

that is $\left\{ f: \mathcal{O}(f) > \alpha, \alpha \in (0,1] \right\} \in \mathcal{E}'$ ------→(2)

Each f in (2) satisfies the condition $f(A) \subset B$, for a compact fuzzy set A of X and an open fuzzy set B of Y

By our definition for $F_{\mathcal{E}}$, $\mathcal{O}(f)=1$ for each f in (2)

$\therefore \mathcal{O} \in F_{\mathcal{E}}$

that is $W(\mathcal{E}') \subset F_{\mathcal{E}}$

$$W(\mathcal{E}) \subset W(\mathcal{E}') \subset F_{\mathcal{E}}$$

case(ii)

Let (X,T) and $(Y,\mathcal{U})$ be two discrete topological spaces, (X,F_1) and (Y,F_2) be the corresponding discrete fuzzy topological spaces. Then $C_{\mathcal{T}}(X,Y) = C_F(X,Y)$

Let $\mathscr{E}$ and $\mathscr{E}'$ are the compact open topologies on $C(X,Y)$ generated by the sub basis $N(K,U)$ and $N(K',U')$ respectively as above. clearly $\mathscr{E} \subset \mathscr{E}'$

If $W(\mathscr{E})$ and $W(\mathscr{E}')$ are the associated fuzzy topologies with respect to $\mathscr{E}$ and $\mathscr{E}'$, then $W(\mathscr{E}) \subset W(\mathscr{E}')$

If $F_{\mathscr{E}}$ is the compact open fuzzy topology of $C_F(X,Y)$ then $W(\mathscr{E}) \subset W(\mathscr{E}') \subset F_{\mathscr{E}}$

case(iii)

Let (X,T) and $(Y,\mathcal{U})$ be two topological spaces, F_1 and F_2 be fuzzy topologies on X and Y so that $F_1 =$

$$\left. \begin{array}{l} \mu_A : X \longrightarrow [0,1] \ni \mu_A(x) = 1 \text{ if } x \in A, \text{for } A \in T \\ \qquad\qquad\qquad\qquad\quad = 0 \text{ if } x \notin A \end{array} \right\}$$

$$\text{and } F_2 = \left\{ \begin{array}{l} \mu_A : Y \longrightarrow [0,1] \ni \mu_B(x) = 1 \text{ if } x \in B, \text{for } B \in \mathcal{U} \\ \qquad\qquad\qquad\qquad\qquad\qquad = 0, \text{if } x \notin B \end{array} \right\}$$

Then $C_T(X,Y) = C_F(X,Y)$

for let $f \in C_T(X,Y)$

that is $f:(X,T) \longrightarrow (Y,\mathcal{U})$ is continuous.

for every $U \in \mathcal{U}$, then there exists a $\mu : Y \longrightarrow [0,1]$

$$\ni \mu_U(y) = 1 \quad \text{if } y \in U$$
$$= 0 \quad \text{if } y \notin U$$

that is $\mu_U \in F_2$

Consider $f^{-1}(\mu_U)(x) = \mu_U \circ f(x)$

$=1$ if $f(x) \in U$

$= 0$ otherwise

$=\mu_U(f(x))$

$f^{-1}(\mu_U) \in F_2$

$\therefore f \in C_F(X,Y)$

$\therefore C_{\mathcal{T}}(X,Y) \subset C_F(X,Y)$ ||ly $C_{\mathcal{F}}(X,Y) \subset C_{\mathcal{T}}(X,Y)$.

Let $\mathcal{E}$ and $\mathcal{E}'$ are the compact open topologies generated by the subbasis $N(K,U)$ and $N(K',U')$ respectively as above then $\mathcal{E} \subset \mathcal{E}'$

If $W(\mathcal{E})$ and $W(\mathcal{E}')$ are the associated fuzzy topologies with respect to $\mathcal{E}$ and $\mathcal{E}'$, then $W(\mathcal{E}) \subset W(\mathcal{E}')$

If $F_{\mathcal{E}}$ is the compact open fuzzy topology of $C_F(X,Y)$ then $W(\mathcal{E}) \subset W(\mathcal{E}') \subset F_{\mathcal{E}}$

Proposition 4.2.4

If Y is a Hausdorff topological space, then C(X,Y) is also a Hausdorff topological space in the compact open topology $\mathcal{E}'$, where $\mathcal{E}'$ is generated by $N(K',U')$ (cf:note 4.2.3)

Proof

Let $f \neq g$ in C(X,Y). That is there exists atleast an $a \in X$ such that $f(a) \neq g(a)$. Since Y is a Hausdorff topologial space it is a Hausdorff fuzzy topologial space in the associated fuzy topology.

$\therefore$ There exists disjoint open fuzzy sets U and v such that

$U(f(x))=V(g(x))=1$ and $U \cap V = 0$

That is $f \in N(x,U)$ and $g \in N(x,V)$ and

$N(x,U) \cap g\in N(x,V)=\phi$

$\therefore$ C(X,Y) is a Hasudorff topological space

Proposition 4.2.5

If Y is a Hausdorff topological space,then C(X,Y) is also a Hausdorff fuzzy topological space in the associated fuzzy topology $W(\mathcal{E}')$.

Proof

We have C(X,Y) is a Hausdorff topological space in the compact open topology $\mathcal{E}$.A Hausdorff topological space is a fuzzy Hausdorff space in the the associated fuzzy topology $W(\mathcal{E}')$.

Proposition 4.2.6

If Y is a Hausdorff fuzzy topological space,then $C_F(X,Y)$ is also a fuzzy Hausdorff space in the compact open fuzzy topology $F_{\mathcal{C}}$

Proof

Let f≠g in $C_F(X,Y)$. That is there exists atleast an $a \in X$ such that f(a)≠g(a), given that Y is a Hausdorff fuzzy topologial space

∴ There exists disjoint open fuzzy sets U and V such that U(f(x))=V(g(x))=1and $U\cap V=\phi$

That is $f\in N(x,U)$ and $g\in N(x,V)$

$N(x,U) \cap N(x,V)=\phi$

Consider θ_1 and θ_2 from C(X,Y)---->[0,1] such that

$$\theta_1(k)= 1-\sup_{y\in Y} \left[k(a_\lambda) \cup U-U\right] (y), \quad \text{and}$$

$$\theta_2(k)= 1-\sup_{y\in Y} \left[k(a_\lambda) \cup V-V\right] (y),$$

Then $\theta_1(f)= 1$ and $\theta_2(g)=1$

We have to show that $\theta_1 \cap \theta_2 = 0$

for $h \in C_F(X,Y)$ $\quad (\theta_1 \cap \theta_2)(h) = \min\left\{\theta_1(h), \theta_2(h)\right\}$

Since $U\cap V=0$, if U(y) >0 then V(y)=0 and if V(y)>0 then U(y)=0

$\therefore$ if $\theta_1(h)>0$ then $\theta_2(h)=0$

&

if $\theta_2(h)>0$ then $\theta_1(h)=0$

$(\theta_1 \cap \theta_2)(h)= 0$

$\therefore$ $C_F(X,Y)$ is a Hausdorff fuzzy topological space.

Proposition 4.2.7

If X is a locally compact Hausdorff topological space, then $C_F(X,X) = \{ g:X \dashrightarrow X \mid g$ is a fuzzy continuous map with respect to the associated fuzzy topology on X $\}$ is a fuzzy topological semigroup with the associated fuzzy topology $W(\mathcal{E})$

Proof

We have $C_F(X,X)$ is a topological semigroup under composition of functions and with compact open topology $\mathcal{E}$. Then C(X,X) is a fuzzy topological semigroup with fuzzy topology $W(\mathcal{E})=\{ g:C(X,X) \dashrightarrow [0,1] \mid g$ is a l.s.c.map$\}$ by proposition 2.3.3

Note

We believe that the above proposition is true if we take compact open fuzzy topology on C(X,X).

CHAPTER 5 *

FUZZY HOMOMORPHISM AND FUZZY ISOMORPHISM

Introduction

Homomorphism and Isomorphism between two fuzzy (semi) topological semigroups are defined in this chapter. B.T.Lerner [35] studied the homomorphic and inverse images of fuzzy right topological semigroups. We prove the analogous results for induced L-fuzzy (semi) topological semigroups. In section 2 we prove that the product of a family of L-fuzzy topological semigroups is an L-fuzzy topological semigroup and the quotient space of an L-fuzzy topological semigroup is also an L-fuzzy topological semigroup. In the last section we give a brief discription about the categories of L-fuzzy topological semigroups.

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* Some of the results of this chapter has been communicated for publication in the Journal of Mathematical Analysis and Application

5.1 Preliminary concepts

Definition 5.1.1

Let X be a semigroup. An L-fuzzy set B of X is an L-fuzzy left ideal if $\mu_B(xy) \geq \mu_B(y)$; an L-fuzzy right ideal if $\mu_B(xy) \geq \mu_B(x)$; and an L-fuzzy ideal if

$$\mu_B(xy) \geq \max\left\{\mu_B(x), \mu_B(y)\right\} \ \forall\ x, y \in X$$

Remark 5.1.2

i) An L- fuzzy ideal of X is an L- fuzzy semigroup of X.

ii) Every constant L-fuzzy set in a semigroup X is an L-fuzzy ideal of X.

iii) When X is an abelian semigroup, an L-fuzzy left (or right) ideal becomes an L-fuzzy ideal.

Definition 5.1.3

Let X be a semigroup, a relation R on X is said to be left (or right) compatible with the operation on X if

$$x, y \in X \implies (ax, ay) \in R \ \left[\text{or } (xa, ya) \in R\right] \ \forall\ x, y, a \in X$$ and

compatible if R is both left and right compatible.

Definition 5.1.4

A compatible equivalence on a semigroup X is called a congruence.

If R is a congruence on a topological semigroup X, then R is called a closed congruence if R is a closed subset of XxX

Note

The following propositions are analogous to the propositions 3.3,4.1,4.2 of [62], the difference being only that we have L in place of I=[0,1]. For completeness sake we indicate the proof of proposition 5.1.7

Proposition 5.1.5

The intersection or union of any set of L-fuzzy left ideals,right ideals or two sided ideals respectively is an L-fuzzy left,right or two sided ideal.

Proof :See proposition 3.3 [62]

Proposition 5.1.6

Let X and Y be two semigroups, f a homomorphism of X into Y.If A is an L-fuzzy semigroup (L-fuzzy ideal) of Y, then $f^{-1}(A)$ is an L-fuzzy semigroup (L-fuzzy ideal) of X.

Proof:See propositin 4.1 [62]

Note

Rosenfeld has defined, A fuzzy set μ in X to have the sup property if for any subset $T \subseteq X$ there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$

Proposition 5.1.7

Let X and Y be semigroups and a homomorphism f of X into Y. Let G be an L-fuzzy semigroup (L-fuzzy ideal) of X that has the sup property, then the image f(G) of G is an L-fuzzy semigroup (L-fuzzy ideal) of Y.

Proof

Let $U, V \subset Y$. If either $f^{-1}(U)$ or $f^{-1}(V)$ is empty then the proposition is trivially satisfied.

Suppose neither $f^{-1}(U)$ nor $f^{-1}(V)$ is nonempty.

Let $r_0 \in f^{-1}(U)$, $s_0 \in f^{-1}(V)$ be such that

$\mu_G(r_0) = \text{Sup } \mu_G(t)$ where $t \in f^{-1}(U)$ and

$\mu_G(s_0) = \text{Sup } \mu_G(t)$ where $t \in f^{-1}(V)$

then $\mu_{f(G)}(UV) = \text{Sup } \mu_G(w)$ where $w \in f^{-1}(UV)$

$$\geq \text{Min}\left\{\mu_G(r_0), \mu_G(s_0)\right\}$$

$$= \text{Min}\left\{\mu_{f(G)}(U), \mu_{f(G)}(V)\right\}$$

$\therefore$ f(G) is an L-fuzzy semigroup (L-fuzzy ideal) of Y.

5.2 Fuzzy Homomorphism (F-morphism)

Definition 5.2.1

Let $(X_1, \mu_1, F_1), (X_2, \mu_2, F_2)$ be two L-fuzzy (semi) topological semigroups. A mapping g of (X_1, μ_1, F_1) to (X_2, μ_2, F_2) is an F-morphism if

1) g is an algebraic homomorphism of X into Y

ii) g is a fuzzy continuous mapping of (X_1, μ_1, F_1) to (X_2, μ_2, F_2).

And g is an F-isomorphism if

i) g is algebraically an isomorphism of X into Y

ii) g is fuzzy homeomorphism of (X_1, μ_1, F_1) to (X_2, μ_2, F_2).

Example 5.2.2

Let X=(N,+)

F_X=the discrete L-fuzzy topology on X

Let Y=(N,+) and $F_Y = \left\{ g : Y \longrightarrow [0,1] \;\middle|\; g \text{ is a constant map} \right\}$

Define $\Theta : X \longrightarrow Y \ni \Theta(x) = 2x$

clearly Θ is an F-morphism of $(X, 1_X, F)$ into (Y, I_Y, F_Y).

Example 5.2.3

Let $X = (\mathcal{R}, +)$ and $Y = (\mathcal{R} - \{0\}, \bullet)$

Let F_X and F_Y be the associated fuzzy topologies on X and Y respectively for their usual topology.

Define $\Theta : X \longrightarrow Y \ni \Theta(x) = 2^x$

Clearly Θ is an F-morphism of $(X, 1_X, F)$ to $(Y, 1_Y, F_Y)$

Proposition 5.2.4

Let (X_1, μ_1, F_1) and (X_2, μ_2, F_2) be two L-fuzzy topological semigroups and $f : (X_1, \mu_1, F_1) \longrightarrow (X_2, \mu_2, F_2)$ be an F-morphism. If $\mu_2' \subset \mu_2$ be an L-fuzzy semigroup of X_2, then $(X_1, f^{-1}(\mu_2'), F_{1f^{-1}(\mu_2')}$ $\left[\text{where } F_{1f^{-1}(\mu_2')} \text{ is the induced L-fuzzy topology with respect to } f^{-1}(\mu_2') \text{ on } (X_1, \mu_1, F_1)\right]$ is an induced L-fuzzy topological semigroup of (X_1, μ_1, F_1).

Proof

By proposition (5.1.4) $f^{-1}(\mu'_2)$ is an L-fuzzy semigroup of X_1, then by proposition (2.2.3) $(X_1, f^{-1}(\mu_2'), F_1 f^{-1}(\mu_2'))$ is an induced L-fuzzy topological semigroup of (X_1, μ_1, F_1).

Proposition. 5.2.5

Let (X_1, μ_1, F_1) and (X_2, μ_2, F_2) be two L-fuzzy topological semigroups and $f : (X_1, \mu_1, F_1) \longrightarrow (X_2, \mu_2, F_2)$ be an F-morphism. If $\mu_1' \subset \mu_1$ be an L-fuzzy semigroup of X_1 satifies the sup property then $(X_2, f(\mu_1'), F_{2f(\mu_1')}$ is an induced L-fuzzy topological semigroup of $((X_2, \mu_2, F_2)$

Proof

It is clear from the propositin (5.2.5 and 5.2.3.)

Proposition 5.2.6

Given semigroups X,Y and a homomorphism f of X onto Y and an L-fuzzy topology $\mathcal{U}$ on Y, let X have L-fuzzy topology T where T is the inverse image under f of $\mathcal{U}$ and let $(Y,G,\mathcal{U}_G)$ be an induced L-fuzzy topological semigroup of $(Y,1_Y,\mathcal{U})$,then the inverse image $(X,f^{-1}(G),T_{f^{-1}(G)})$ is an induced L-fuzzy topological semigroup of $(X,1_X,T)$

Proof

The proof is analogous to the proof of proposition 6.1 [20] .

Note .

Let X,Y be two semigroups, f:X———>Y be a homomorphism of X into Y .Let G be a fuzzy semigroup of X ,then the membership function μ_G of G is f-invariant if for all $x_1,x_2 \in X$ such that $f(x_1)=f(x_2)$ we have $\mu_G(x_1) = \mu_G(x_2)$

Proposition 5.2.7

Given semigroups X,Y and a homomorphism f of X into Y,and an L-fuzzy topology F on X, let Y have L-fuzzy topology $\mathcal{U}$ where $\mathcal{U}$ is the image under f of F and let (X,G,F_G) be an induced L-fuzzy topological semigroup of $(X,1_X,F)$. If the membership function of G is f-invariant,the image $(Y.f(G),\mathcal{U}_{f(G)})$ is an induced L-fuzzy topological semigroup of $(Y,1_Y,\mathcal{U})$.

Proof

The proof is analogous to the proof of proposition 6.2 [20]

Theorem 5.2.8

Let $\left\{X_i, \mu_i, F_i \middle| i\in I\right\}$ be a collection of L-fuzzy topological semigroups, then their product $\prod_{i\in I} (X_i, \mu_i, F_i) = (X, \mu, F)$ is also an L-fuzzy topological semigroup, where $X = \prod_{i\in I} (X_i)$, the usual set product and $\mu = \prod_{i\in I} \mu_i$ be the product L-fuzzy set in X whose membership function is defined by $\mu(x) = \inf_{i\in I} \left\{\mu_i(x_i) \middle| x = x_i \in X\right\}$ and F is the smallest L-fuzzy topology on X for which each projection mapping p_j of X onto X_j is fuzzy continuous.

Proof

We have X is a semigroup under coordinate wise multiplication, where the associativity of multiplication follows from that of the semigroups in the collection

$\{X_i, i\in I\}$ and $x_i, y_i \in X_i$ the mapping $g_i : (x_i, y_i) \longrightarrow x_i y_i$ of $(X_i, \mu_i, F_i) \times (X_i, \mu_i, F_i)$ into (X_i, μ_i, F_i) is fuzzy continuous.

[each (X_i, μ_i, F_i) is an L-fuzzy topological semigroup]

let g be the mapping such that $g(x,y) = xy$, and

$p_i \circ g = g_i$

Then g is fuzzy continuous by proposition 1.2.10

$\therefore \prod_{i \in I} (X_i, \mu_i, F_i)$ is an L-fuzzy topological semigroup.

Remark

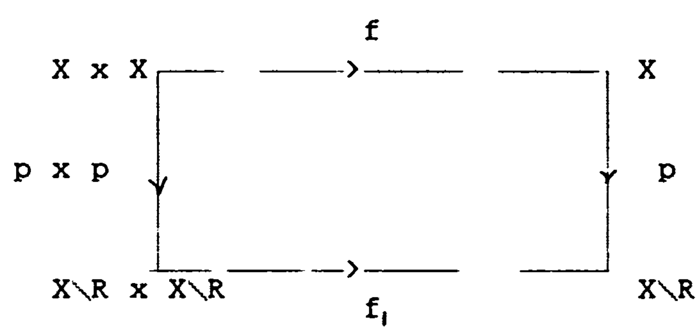
From the definition of multiplication on X each p_j is an F-morphism of X onto X_j

Definition 5.2.9

let (X, μ, F) be an L-fuzzy topological semigroup, R be a closed congruence on X, then the quotient space $(X \setminus R, \gamma, \mathcal{U})$ of (X, μ, F) is an L-fuzzy topological semigroup.

Proof

Consider the diagram



where $f:(x,y) \longrightarrow xy$ of $(X, \mu, F) \times (X, \mu, F)$ into (X, μ, F) is fuzzy continuous.

We have to show that $f_1 : (X/R,\gamma,\mathcal{U}) \times (X/R,\gamma,\mathcal{U})$ into $(X/R,\gamma,\mathcal{U})$ is fuzzy continuous.

let $B \in \mathcal{U}$, then $\mu \cap p^{-1}(B) \in F$ (since p is fuzzy continuous)

That is $(\mu \times \mu) \cap f^{-1}(\mu \cap p^{-1}(B)) \in F \times F$

That is $(\mu \times \mu) \cap f^{-1}(\mu) \cap f^{-1}(p^{-1}(B)) \in F \times F$

That is $(\mu \times \mu) \cap (p \circ f)^{-1}(B) \in F \times F$ (since $\mu \times \mu \subseteq f^{-1}(\mu)$)

That is $(p \times p)\left\{(\mu \times \mu) \cap (p \circ f)^{-1}(B)\right\} \in \mathcal{U} \times \mathcal{U}$ (since p is fuzzy open so is p x p)

That is $(p \times p)(\mu \times \mu) \cap (p \times p)\ (p \circ f)^{-1}(B) \in \mathcal{U} \times \mathcal{U}$

That is $(p x p)(\mu x \mu) \cap (p x p)(f_1 \circ p x p)^{-1}(B) \in \mathcal{U} \times \mathcal{U}$ $(p \circ f = f_1 \circ p x p)$

That is $(p \times p)(\mu \times \mu) \cap (p \times p)\ (p \times p)^{-1} f_1^{-1}(B) \subset \mathcal{U} \times \mathcal{U}$.

That is $\gamma \times \gamma \cap f_1^{-1}(B) \in \mathcal{U} \times \mathcal{U}$

$\therefore f_1$ is fuzzy continuous

Corollary

The quotient L-fuzzy topological semigroup $(X/R,\gamma,\mathcal{U})$ of (X,μ,F) is an F-morphic image of (X,μ,F).

Definition 5.2.10

Let X be a semigroup. A fuzzy relation R is said to be a fuzzy equivalence relation if:

i. $R(x,x)=1, \forall x \in X$

ii $R(x,y) = R(y,x), \forall x,y \in X$

iii. $R(x,z) \geq R(x,y) \cap R(y,z) \; \forall x,y,z \in X$

A fuzzy equivalence relation of X which is compatible with the semigroup structure of X is called a fuzzy congruence of X.

Example.

i) If X is a semigroup, every fuzzy ideal in X is a fuzzy congruence

ii) The constant fuzzy set in a semigroup is a fuzzy congruence of X.

Proposition 5.2.11

Let R_1, R_2 be two fuzzy congruences of X, then $R_1 \cap R_2$ is also a fuzzy congruence.

Proof

The intersection of two fuzzy equivalence relations is a fuzzy equivalence relation and the intersection of two fuzzy semigroups is a fuzzy semigroup.

$\therefore R_1 \cap R_2$ is also a fuzzy congruence.

Remark

$R_1 \cup R_2$ need not be a fuzzy congruence.

Theorem 5.2.12

Let (X,F) be a Fuzzy topological semigroup, R be a fuzzy congruence of X. Then R with induced fuzzy topology is an induced fuzzy topological semigroup of (X,F) x (X,F)

Proof: See proposition 2.2.3

5.3 Categories of fuzzy topological semigroups

Definition 5.3.1

Let $\mathcal{R}_1$ be the collection of all L-fuzzy topological semigroups with F-morphisms and $\mathcal{R}_2$ be the collection of all L-fuzzy semi topological semigroups with F-morphisms, Clearly $\mathcal{R}_1$ and $\mathcal{R}_2$ constitues sub categories of FTOP (not full sub categories).

Results 5.3.2

Monomorphisms in the above categories are precisely injective F-morphisms and the epimorphisms are F-surmorphisms.(this follows from the fact that the same is true in FTOP.(Result 1.2.2)

Let $\mathcal{E}_1$ be the collection of all L-fuzzy topological semigroups (X,μ,F) for a fixed semigroup X together with the F-morphisms and let $\mathcal{E}_2$ be the collection of all L-fuzzy topological semigroups (X,μ,F) for a fixed fuzzy semigroup μ together with the F-morphisms. Clearly $\mathscr{E}_1$ and $\mathscr{E}_2$ are subcategories of $\mathscr{R}_1$

Remark :

Using the following Functors we get an association between the category of topological semigroups $\mathscr{E}$ and the category of fuzzy topological semigroups $\mathscr{D}$

Define $\mathscr{F}_1$ and $\mathscr{F}_2$ such that $\mathscr{F}_1 : \mathscr{E} \longrightarrow \mathscr{D}$

$$\mathscr{F}_1(X,m,T) = \left\{ (X,m,U) \mid U \text{ is a l.s.c map} \right\}$$

$$\mathscr{F}_1(f) = f$$

Similarly $\mathscr{F}_2 : \mathscr{D} \longrightarrow \mathscr{E}$ such that

$$F_2(X,m,F) = (X,m,i(F)) \text{ where } i(F) = \left\{ V^{-1}(\alpha,1] ; V \in F, \alpha \in (0,1] \right\}$$

$$\mathscr{F}_2(f) = f$$

clearly $\mathscr{F}_2$ embedds $\mathscr{E}$ into $\mathscr{D}$

CHAPTER 6*

F-SEMIGROUP COMPACTIFICATION

Introduction

The study of fuzzy space compactification was started in 1978 by T.E.Gantner, R.C.Steinlage and R.H.Warren in [23]. They considered the fuzzy situation of the idea of Alexandorff one point compactification ; and constructed one point α-compactification of a fuzzy topological space. Another attempt was by H.W.Martin [48]. He constructed ultra fuzzy compactification of a fuzzy topological space (X,F), if (X,i(F)) is completely regular, where i(F) is the smallest topology on X such that all functions in F are lower semi continuous. The third approach was due to Cerruti [6]. It is based on the ideas and techniques of Categorical topology and mainly applicable to weakly induced fuzzy spaces. Later many others studied and constructed various types of compactifications of a fuzzy topological space. Wang Guojun [69] has investigated N- compactification of a fuzzy topological space.

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In this chapter we study compactification problem on fuzzy topological semigroups. In the theory of topological semigroups the study of compactification is not so developed as in general topological spaces. Bohr (almost periodic) compactification of a topological semigroup has been studied by J.H.Carruth, J.A.Hildebrant and R.J.Koch [5]. Also they discussed about the concept of Group compactification and One point compactification of a topological semigroup.

Semigroup compactification of a topological semigroup was defined by K.S.Kripalini in 1990 [34]. In her Ph.D thesis it was proved that the semigroup compactifications of a topological semigroup S are precisely the quotients of the Bohr compactification of S under closed congruences. The lattice structure of the family of semigroup compactifications of a topological semigroup was also investigated. In this chapter our study is in some sense analogous to that in [34].

In section 2 of this chapter we prove that Bohr fuzzy compactification exists for a fuzzy topological semigroup, and it is unique determined upto F-isomorphism. In section 3 we introduce F-semigroup compactification analogous to the notion of Semigroup compactification of topological semi groups and obtain a relation with Bohr fuzzy compactification. We define an order relation on F-semigroup compactifications and prove that the set of all F-semigroup compactifications of a fuzzy topological semigroup is an upper complete semi lattice.

6.1 Preliminary concepts

In this section we recall some definitions and results from [48].

Definition 6.1.1

Let (X,F) be a fuzzy topological space. If $S \subset X$, let μ_S denote the characteristic function of S. That is μ_S is 1 on S and 0 on X-S.

If g is a fuzzy set on X, let $cl_F(g)$ denotes the fuzzy closure of g, that is the intersection of all fuzzy closed sets which contain g. A fuzzy set g on X is dense in (X,F) provided that $cl_F(g) = \mu_X$. If (Y,H) is a fuzzy sub space of (X,F), then (Y,H) is dense in (X,F) if μ_Y is dense in (X,F).

A fuzzy set g is ultra dense in (X,F) provided that whenever($0 < \alpha \leq 1$),then the set $g^{-1}(\alpha,1]$ is dense in the topological space (X.i(F)).

A fuzzy subspace (Y,H) of (X,F) is ultra dense in (X,F) if μ_Y is ultra dense.

Theorem 6.1.2.[48]

A fuzzy set g is ultra dense in (X,F) if and only if g is dense in the induced space $(X,\mathcal{W}(i(F))$

Corollary

If the fuzzy set g is ultradense in (X,F), then g is dense in (X,F)

Theorem 6.1.3

Let $(X,\mathcal{W}(T))$ be the fuzzy topological space topologically generated by a crisp topological space (X,T), then $(X,\mathcal{W}(T))$ is N-compact if and only if (X,T) is N-compact.

Proof:

See theorem 3.6 [69]

Theorem 6.1.4

Let (X,F) and $(Y,\mathcal{U})$ be fuzzy topological spaces, A be an N-compact fuzzy set in X and f: $(X,F) \longrightarrow (Y,\mathcal{U})$, is fuzzy continuous mapping; then f(A) is an N.compact fuzzy set in Y.

Proof :See theorem 3.8 [69].

6.2 Bohr fuzzy Compactification

Definition 6.2.1

If S is a topological semigroup, then a Bohr Compactification of S is a pair (β,B) where B is a compact topological semigroup β:S ———> B is a continuous homomorphism of S into a compact topological semigroup B,and if g:S———>T is a continuous homomorphism of S into a compact toplogical semigroup T, then there exists a unique continuous homomorphism f :B ———>T such that f$\circ\beta$=g.

Analogous to the above definition we can define the Bohr fuzzy compactification of a fuzzy topological semigroup as follows

Definition 6.2.2

Let X be a fuzzy topological semigroup. The Bohr fuzzy compactification of X is a pair (α,Y) such that Y is a compact fuzzy topological semigroup, α:X———>Y is an F-morphism provided whenever β:X———>Z is an F-morphism of X into a Compact fuzzy topological semigroup Z, then there exists a unique F-morphism f:Y —- —> Z such that f$\circ\alpha$=β

The existence of Bohr fuzzy compactification of a fuzzy topological semigroup can be proved by using the following lemmae

Lemma 6.2.3

Let (X,F) be a topologically generated fuzzy topological semigroup. Then (X,i(F)) is a topological semigroup.

Proof: See proposition 2.3.4

Lemma 6.2.4

Let (β,B) be the Bohr Compactification of (X,i(F)), where B is a compact topological semigroup and β a continuous homomorphism of X into B. Let $F_B = \left\{ g \mid g \text{ is l.s.c from B into I such that } g|_X \in F \right\}$. Then (B,F_B) is a compact fuzzy topological semigroup, and (X,F) is dense in (B,F_B).

Proof

(B,F_B) is a compact fuzzy topological semigroup by theorem 6.1.3. and proposition 2.3.4

Now we can show that β is an F-morphism of (X,F) into (B,F_B).

For let $\mu \in F_B$, then there exist a $V \in i(F_B)$ such that

$\mu^{-1}(\alpha,1] = V$ for $\alpha \geq 0$.

We have $\beta^{-1}(V)$ is open in X (since β is continous)

That is $\beta^{-1}(\mu^{-1}(\alpha,1]$ is open in X

That is $\mu o \beta$ is l.s c

That is $\beta^{-1}(\mu) \in F$

$\therefore \beta$ is fuzzy continuous

$\therefore \beta$ is an F-morphism of (X,F) into (B,F_B).

By theorem 6.1.2 we get (X,F) is dense in (B,F_B).

<u>Lemma</u> 6 2.5

Let X be a fuzzy topological semigroup and let (f,Y), (g,Z) be Bohr fuzzy compactifications of X. Then there exists an F-isomorphism h:Y——>Z such that hof=g

<u>Proof</u>

Since (f,Y) is a Bohr fuzzy compactification of X, there exists an F-morphism h:Y———>Z such that hof = g similarly there exists an F-morphism π:Z-———>Y such that πog = f

It is clear that $\pi oh = I_Y$ and $ho\pi = I_Z$

That is $\pi o h$ and $h o \pi$ are identity morphisms of Y and Z respectively

∴ h is a fuzzy homeomorphism and hence we conclude that h is an F-isomorphism of Y onto Z

Lemma 6.2.6

Let (X,F) be a fuzzy topological semigroup.Then there exists a Bohr fuzzy compactification of (X,F),in fact a Bohr fuzzy compactification of (X,F) is unique determined up to F-Isomorphism .

6.3 F-Semigroup Compactification

Definition 6.3.1

Let X denote a fuzzy topological semigroup. By an F-semigroup compactification of X we mean an ordered pair (α,A),where A is a compact fuzzy topological semigroup and $\alpha:X\longrightarrow A$ is a dense F-morphism

Remark

The Bohr fuzzy compactification of a fuzzy topological semigroup (X,F) is an F-semigroup compactification of (X,F).

Lemma 6.3.2

The quotient fuzzy topological semigroup $(X/R, \mathcal{U})$ of a compact fuzzy topological semigroup (X,F) is also compact.

Proof

The quotient map $\pi:(X,F) \longrightarrow (X/R,\mathcal{U})$ is fuzzy continuous, and the fuzzy continuous image of a compact fuzzy space is compact (ref: theorem 6.1.4)

Theorem 6.3.3

Let (X,F) be a fuzzy topological semigroup and (B,β) the Bohr fuzzy compactification of (X,F). If R is a closed congruence on B, then $(B/R,\mathcal{U})$ is an F-semigroup compactification of (X,F).

Proof

Since $(B/R,\mathcal{U})$ is compact the only thing we have to show is that there exists an F-morphism $\emptyset:(X,F) \longrightarrow (B/R,\mathcal{U})$ such that $\emptyset(X)$ is dense in $(B/R,\mathcal{U})$,

Consider F-morphisms $\beta:(X,F) \longrightarrow (B,F_B)$ and

$\alpha:(B,F_B) \longrightarrow (X/R,\mathcal{U})$

Define $\emptyset:(X,F) \longrightarrow (B/R,\mathcal{U})$ such that $\emptyset=\alpha\circ\beta$

Now $\emptyset$ is fuzzy continuous

For let $U \in \mathcal{U}$, then $\alpha^{-1}(U) \in F_B$ (since α is fuzzy continuos)

Since β is fuzzy continuous, $\beta^{-1}(\alpha^{-1}(U)) \in F$

That is $(\alpha \cdot \beta)^{-1}(U) \in F$

$\emptyset^{-1}(U) \in F$

$\emptyset$ is fuzzy continuous

It is obvious that $\emptyset$ is an algebraically a homomorphism.

$\therefore$ $\emptyset$ is an F-morphism

By theorem 6.1.2 $\emptyset(X)$ is dense in B/R.

$(B/R, \mathcal{U})$ is an F-semigroup compactification of (X,F).

Theorem 6.3.4

Let X be a fuzzy topological semigroup with Bohr fuzzy compactification (B,β). If R is a closed congruence on B, then there exists an F-semigroup compactification (α,A) of X, so that the congruences defined by this compactification is R.

<u>Proof</u>

Analogous to the proof of result 1.2.4 [34]

Definition 6.3.5

Two F-Semigroup compactifications (α,A) and (β,B) of a fuzzy topological semigroup X are said to be equivalent if there exits an F-isomorphism $\nu: A \longrightarrow B$ such that $\nu o \alpha = \beta$

Definition 6.3.6

If (α,A) and (β,B) are F-Semigroup compactifications of a fuzzy topological semigroup X then we write $(\alpha,A)\geq(\beta,B)$ if and only if there exits an F-surmorphism (onto F-morphism) $\nu:A\longrightarrow B$ such that $\nu o\alpha=\beta$.

Proposition 6.3.7

Two F-Semigroup compactifications of a fuzzy topological semigroup X are equivalent if and only if $(\alpha,A)\geq(\beta,B)$ and $(\beta,B)\geq(\alpha,A)$.

Proof:

The proof is straight forward and so can be omitted

Notation

F(X) denotes the set of equivalence classes of F-Semigroup compactifications of X

Lemma 6.3.8

F(X) under the ordering $\geq$ is a partially ordered set

Proof: obvious

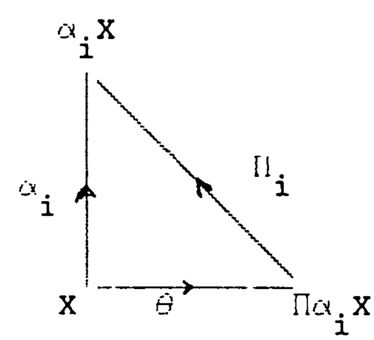
Theorem 6.3.9

$(F(X),\geq)$ is an upper complete semilattice

Proof:Analogous to the proof of 2.1.5 [34].For completeness sake, we indicate below the line of arguement.

Let $\{\alpha_i X \mid i \in I\}$ be a subset of F(X). We have to show that this set has a least upperbound with respect to $\geq$. Consider the product $\prod_{i\in I}\{\alpha_i X\}$, since each $\alpha_i X$ is a compact fuzzy topological semigroup, $\prod_{i\in I}\{\alpha_i X\}$, is also a compact fuzzy topological semigroup.{cf: theorem 4.2 [69] and theorem 5.2.8}

Define a mapping $\theta : X \longrightarrow \prod_{i\in I}\{\alpha_i X\}$ by $(\theta(x))_i = \alpha_i X$



from the diagram $\pi_i \circ \theta = \alpha_i$

then θ is fuzzy continuos (by proposition 1.1.9)

Consider $\theta\,(xy)_i = \alpha_i(xy)$

$$= \alpha_i(x), \alpha_i(y)$$

$$= (\theta(x))_i\ (\theta(y))_i$$

That is, θ is algebraicaly a homomorphism .

$\therefore$ θ is an F-morphism of X into $\prod_{i\in I} \alpha_i X$

Let θX denote $\overline{\theta(X)}$, then θX is a compact fuzzy topological semigroup (by proposition 2.2.5)

Also $\theta : X \longrightarrow \theta X$ is a dense F-Morphism.

$(\theta X, \theta)$ is an F-semigroup compactification of X.

For each $i \in I$, let $f_i : \theta X \longrightarrow \alpha_i X$ be the restriction of the projection to θX.

Also $(f_i \circ \theta)(x) = \alpha_i(x)$

$$f_i \circ \theta = \alpha_i$$

$$\therefore (\theta X, \theta) \geq (\alpha_i X, \alpha_i) \ \forall \ i \in I$$

Suppose $(\theta_0 X, \theta_0) \geq (\alpha_i X, \alpha_i) \ \forall \ i \in I$

Define $f: \theta_0 X \longrightarrow \prod_{i \in I} (\alpha_i X)$ by $(f(y))_i = g_i(y)$

that is $\prod_i \circ f = g$ and $\therefore$ f is also an F-morphism

consider $f(\theta_0(x))_i = (\theta(x))_i$

$\therefore f \circ \theta_0 = \theta$

and $\overline{f(\theta_0 X)} = \theta X$

$\therefore$ f is a dense F-morphism and $f \circ \theta_0 = \theta$

$(\theta_0 X, \theta_0) \geq (\theta X, \theta)$

$(\theta X, \theta)$ is the least upperbound of $\left\{ \alpha_i X \right\}_{i \in I}$

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