SOME PROBLEMS IN TOPOLOGY

A STUDY OF ČECH CLOSURE SPACES

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By

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CERTIFICATE

Certified that the work reported in this thesis is based on the bonafide work done by Smt.Sunitha T.A., under my guidance in the School of Mathematical Sciences, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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CHAPTER 0

INTRODUCTION

This thesis is concerned with two aspects - One, study of some topological like concepts in closure spaces and two, study of closure semigroups analogous to the theory of topological semigroups.

CLOSURE SPACES

The concept of a topological space is generally introduced in terms of the axioms for the open sets. However alternate methods to describe a topology in the set X are often used - in terms of neighbourhood systems, the family of closed sets, the closure operator, the interior operator etc. Of these, the closure operator was axiomatised by Kuratowski and he associated a topology from a closure space by taking closed sets as sets A such that clA = A, where clA is the toplogical closure of a subset A of X. It is also found that clA is the smallest closed set containing A.

Čech introduced the concept of Čech closure space. (In this thesis we denote Čech closure space as closure space for convenience).

In Čech's approach the condition ccA =cA among Kuratowski axioms need not hold for every subset A of X (Here cA denotes the closure of A in X); when this condition is also true, c is called a topological closure operator. The concept of closure space is thus a generalization of that of topological spaces. Čech closure space is also called A - space by C.Calude - M.Malitza [C-M]. For them a Čech space is obtained by removing $c(A \cup B) = cA \cup cB$ and introducing $A \subset B \Longrightarrow cA \subset cB$ into the axioms of an A-space. However considering universal acceptability we call the former Čech closure spaces and the latter monotone spaces.

In 1978, C.Calude and M.Malitza also mentioned [C-M] the concept of a total Čech space and a total Kuratowski space. A total Čech space [respectively total Kuratoswski space] is a closure space [respectively Kuratoswski closure space] which also satisfies the condition

$$c(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} c(A_i) \text{ [respectively } cl(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} cl(A_i)].$$

Calude and Cazanescu mentioned that [C-C] total Čech spaces are in one-to-one, onto correspondence with reflexive relations and they also studied the category of total Čech spaces and its full subcategory determined by total Kuratowski spaces.

The ideas about the concepts of a continuous mapping and of a set endowed with continuous operations (compositions) play a fundamental role in general mathematical analysis. Analogous to the notion of the continuity, we consider the morphisms throughout this thesis. Čech described continuity in closure spaces by means of neighbourhoods, nets etc. Koutnik studied the convergence in non Hausdorff closure space [KO_1]. He studied more about sequential convergence structure in [KO_2],[KO_3]. Mashour and Ghanim in 1982 defined [MA-G_1] C- almost continuous as a function f:X-->Y, where X and Y are closure spaces and is said to be a C-almost continuous if for each $x\in X$ and each $V \subset Y$ with $f(x) \in V^0$, there is $U \subset X$ such that $x \in U^0$ and $f(U)\subset(c(U))^0$. They also studied some results related to this concept. D.R.Andrew and E.K.Whittlesy [A-W] and James Chew [CHE] studied about closure continuity. D.N.Roth and J.W.Carlson mentioned [RD-C] the degree of a closure operator. They also found that Čech closure spaces of finite degree provide a nontrivial generalisation of topological spaces. It was shown that the category of topological spaces and continuous maps is bi-reflective in the category of Čech closure spaces of finite degree and continuous maps. V.Kannan [K] defined the degree of closure operators in infinite case also and studied more about this.

Separation axioms in closure spaces have different implications than the corresponding axioms in topological spaces. According to Čech, a closure space is said to be seperated [CE_2] if any two distinct points are separated by distinct neighbourhoods. Separation properties in closure spaces have been studied by various authors. D.N.Roth and J.W.Carlson studied [RO-C] a number of seperation properties in closure spaces. They showed that Čech closure operator on a finite set can be represented by a zero-one reflexive matrix. A number of separation properties were studied for finite spaces and characterised in terms of matrix that represents the closure operator. Separation properties that carry over to the underlying topology were also studied. W.J.Thron studied [T] some separation properties in closure spaces. He defined a space as regular if $x \notin c(A)$ (A is closed in X) implies that there exist $D,E \subset X$, $D \cap E = \phi$ such that $x \notin c(X-D)$, $A \cap c(X-E) = \phi$. K.C.Chattopadhyay and W.J.Thron studied [CH-T] some separation properties of extensions and obtained some results on the above. Chattopadhyay mentioned [CH] R_0,R_1 spaces and pointed out that an R_1 space is an R_0 space. Some separation properties on biclosure spaces (X,c_1,c_2) were studied by Chattopadhyay and Hazra in 1990 [CH-H]. According to them a biclosure space (X,c_1,c_2) is called a T_0 - space if for each x,y in X, $\mathcal{G}(c_1,x) \times \mathcal{G}(c_2,x) = \mathcal{G}(c_1,x) \times \mathcal{G}(c_2,y)$

==> x=y where $\mathcal{G}$ is a grill on X and a T_1 space if $c_1(x) \cap c_2(x) = x$ for all x in X . It was proved that the R_1 space (X,c_1,c_2) is T_0 if and only if the space is T_1.

For topological spaces compactness can be expressed in a number of different ways. However for closure spaces some of these statements are not equivalent. Čech defined [CE_2] the term compactness for a closure space (X,c) if every proper filter of sets on X has a cluster point in X. He described the fundamental properties of compact closure spaces. He noted that for a closure space (X,c) to be compact it is necessary and sufficient that every interior cover $\mathcal{V}$ (an analogue of an open cover in topological space) of (X,c) has a finite subcover. Chattopadhyay [CH] defined a compact space as a closure space (X,c) if and only if $[G_c^+(x)=x \in X]$ is a cover of $\Omega(X)$. He denoted by $\Omega(X)$, a set of ultrafilters on X, by $\mathcal{G}$, a grill on X then $\mathcal{G}^+=[\mathcal{U}:\mathcal{U}\in\Omega(X),\ \mathcal{U} \subset \mathcal{G}]$. W.J.Thron mentioned [T] types of compactness. According to him a closure space (X,c) is called linkage (F - linkage) compact if every linked (F - linked) grill on X converges. A grill $\mathcal{G}$ is called linked grill if $A,B \in G \Rightarrow c(A) \cap c(B) = \phi$, F - linked grill if $A_1, A_2, \ldots A_n \in G \Rightarrow \cap[c(A_k)] = \phi$. Some weak forms of compactness like almost c-compactness were introduced and some of its properties were studied [MA-G1] by A.S.Mashour and M.H.Ghanim. Compactness and linkage compactness were defined by K.C.Chattopadhyay [CH].

Čech defined [CE_2] and developed some properties of connected spaces. According to him a subset A of a closure space X is said to be connected in X if A is not the union of two non-empty semi-separated subsets of X, that is $A = A_1 \cup A_2$, $(cA_1 \cap A_2) \cup (A_1 \cap cA_2) = \phi$ implies $A_1 = \phi$ or $A_2 = \phi$. Plastria studied [P] connectedness and local connectedness of simple extensions. The concept of connectedness which was defined by Čech in closure spaces precisely coincides with connectedness in the associated topological spaces.

K.C.Chattopadhyay and W.J.Thron were the first persons who studied [CH-T] the general extension theory of G_0 closure spaces. They studied some special closure operators and considered the case when an extension is topological and also compact. The underlying structure of each nearness space is topological space. The underlying structure of each semi nearness space is a Čech closure space. D.N.Roth and J.W.Carlson showed [RO-C] that finitely generated Čech closure spaces are a natural generalisation of finite Čech closure spaces. K.C.Chattopadhyay developed [CH] an extension theory of arbitarary closure spaces which are in general supposed to satisfy no separation axioms. He introduced the concept of regular extensions of closure spaces and satisfied this concept in detail.

Though much work has been done in topological spaces and in Čech spaces, there are still many problems not attempted. In the first part of this thesis we have made an attempt in this direction.

TOPOLOGICAL SEMIGROUPS

The theory of topological semigroups originated during the fifties. A.D.Wallace has contributed much to this area in its earlier days of development. A topological semigroup is a Hausdorff space S with continuous associative multiplication (x,y) —> xy of S x S into S . After Wallace's introduction, the study of topological semigroups was continued by others. Some of the studies in this direction are due to A.D.Wallace [on the structure of topological semigroups, Bull Amer. Math. Soc. 61 (1955a),95-112] and K.H.Hofmann and P.S.Mostert (Elements of compact semigroups, Merril book, Inc, Colombus (1966), A.B.Paalman De Miranda, [Topological semigroups, Mathematical Centre Tracts, 2nd edition , Mathematiche Centrum Amsterdam 1970].

Topological semigroups which are compact will be called compact semigroups. The theory of compact semigroups is a rich area of research. It is to be noted that a compact semigroup S contains an idempotent. Some standard results in this area can be found in the book " The theory of topological semigroups " J.H.Carruth [CH-A-K_1] etc.

Tietze, Alexandroff, Urysohn etc worked on campactification and introduced the concept of one-point compactification. Tychnoff continued this work and proved that every Tychnoff space can be embedded in a compact Hausdorff space. Later E.Čech and M.H.Stone gave the concept of maximal compactification (βX) and stated its fundamental properties.

Deleeuw, Glicksberg, Hunter etc have studied Bohr compactification of topological semigroups having universal properties analogous to those of βX. J.A.Hildebrant and J.D.Lawson investigated [H-L] the conditions under which, a topological semigroup and a dense ideal have same Bohr compactification. He also stated more results for weak compactification of semitopological semigroups as well as the Bohr compactification. J.W.Baker and R.J.Butcher studied [B-B] about Stone - Čech compactification of a topological semigroup. In 1990 K.S.Kripalini defined semigroup compactification of a topological semigroup and also found some results related to this.

The theory of closure semigroups, the theory of Bohr compactification and other types of closure semigroup compactifications seem to have not been attempted by others. In the second part of this thesis we have made an attempt in this direction.

0.2 SUMMARY OF THE THESIS

CHAPTER 1

In this chapter we introduce the morphisms in the category of closure spaces and study the relation between these morphisms and the continuous functions in the associated topologies. We denote the collection of morphisms from one closure space into another as $S_c(X)$ and find that $S_c(X)$ is a semigroup. We consider an order relation between closure operators and prove some related results. In section 2 we point out the degree of a closure operator and find that the degree is invariant. We also consider the order of a map and study more about this. In the third section we find that a closure space can be associated from a monotone space and mention some preliminary concepts in monotone spaces. We also study the concepts of morphisms and order of map in monotone spaces.

CHAPTER 2

The second chapter is a study of some separation properties. In section 1 of this chapter we consider some pointwise separation properties in closure spaces such as T_0,T_1 and T_2. We also find the relation between the separation properties in (X,c) and those in the associated topological space (X,t). Some higher separation properties in closure spaces have been introduced and studied in section 2. We consider the hereditariness and productivity of some of these properties in closure spaces. In section 3 of this chapter we explain the above separation properties in the case of monotone spaces.

CHAPTER 3

Some properties of compactness are studied in the first section of this chapter. Čech studied some properties of compact closure spaces. We find the relation between compact closure space and the associated compact topological space. Section 2 deals with the concept of connectedness. Though Čech def ined and studied some properties of connectedness in closure spaces, they are more related to associated topological spaces. We define connectedness and find the relation between connected closure spaces and associated topological spaces. It is observed that when a subset A of X is connected cA need not be connected. We consider the image of a connected space under a morphism. In the third section we define and study the concept of local connectedness and path connectedness in closure spaces. We explain the above notions in monotone spaces in section 4.

CHAPTER 4

An attempt has been made to introduce and study closure semigroups in the fourth chapter. We define closure semigroup and find that S is a closure semigroup does neither imply nor is implied by the fact that it is a topological semigroup in the associated topology. We give examples in either direction. In this section we also explain some preliminary concepts of closure semigroups analogous to those in a topological semigroup. The concepts of ideal and product in a closure semigroup have also been studied in section 2. In the third section we consider the notion of homomorphisms and congruences and prove some related results.

CHAPTER 5

The study of closure semigroup is continued in the fifth chapter. We introduce Bohr-type compactification and prove its existence. We also consider the set of all closure semigroup compactifications of a given closure semigroup. We find that this set is an upper complete semilattice.

0.3 PRELIMINARY DEFINITIONS AND RESULTS USED IN THE THESIS

Definition 0.3.1[CE_2]

A function c from a power set of X to itself is called a closure operation for X provided that the following conditions are satisfied.

i) $c\phi = \phi$

ii) $A \subset cA$

iii) $c(A \cup B) = cA \cup cB$

A structure (X,c) where X is a set and c is a closure operation for X will be called closure space or Čech space. Let us consider the following conditions.

iv) $A \subset B \Longrightarrow cA \subset cB$ for every $A,B \subset X$

v) For any family of subsets of X, $\{A_i\}_{i\in I}$

$$c(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} c(A_i)$$

vi) $c(cA) = cA$ for every $A \subset X$

The structure (X,c) where c has the properties (i), (ii) and (iv) is called a monotone space [C-M]. A Čech space which satisfies the condition (vi) is called Kuratowski (topological) space [C-M]. A Čech space (Kuratowski) space is total if the condition (v) holds [C-M].

Definition 0.3.2 [CE_2]

A closure c is said to be coarser than a closure c' on the same set X if $c'(A) \subset c(A)$ for each $A \subset X$. In this case we say $c<c'$.

Definition 0.3.3 [CE_2]

The identity relation on the power set of X is the finest closure for X and it will be called the discrete closure for X. Setting $c\phi=\phi$ and $cA = X$ for every $A \subset X$ we get the coarsest closure for X and it will be called the indiscrete closure for X.

Definition 0.3.4

A subset A of a closure space (X,c) will be called closed if $cA = A$ and open if its complement is closed. That is if $c(X-A) = X-A$.

Note 0.3.5

If (X,c) is a closure space we denote the associated topology on X by t. That is $t=\{A' : cA = A\}$ where A' denotes the complement of A . Members of t are the open sets of (X,c) and their complements the closed sets.

Convention

We consider spaces (X,c) where c denotes a monotone operator on X; the associated closure (See Result 1.3.4) is denoted by c and the associated closure space is (X,c); the associated topology is denoted by t and the associated topological space is (X,t); the closure in (X,t) will be denoted by cl. c, c_1, c_2, c' etc denote closure operators on sets and t, t_1, t_2, t' etc denote the respective associated topologies and cl, cl_1, cl_2, cl' etc the respective topological closures.

For any closure c for a set X there is associated interior operator Int_c.

Definition 0.3.6 [CE_2]

An interior operator Int_c is a function from power set of X to itself such that for each $A \subset X$, $Int_c\, A = X - c(X-A)$. The set $Int_c A$ is called the interior of A in (X,c). Also A is called a neighbourhood of x if $x \in Int\, A$.

Note 0.3.7

A subset X is open if and only if int X = X.

Definition 0.3.8 [CE_2]

Let (X,c) be a closure space and $Y \subset X$. The closure c' on Y is defined as

$c'A = Y \cap cA$ for every $A \subset Y$. The closure space (Y,c') is called the subspace of (X,c).

Result 0.3.9 [CE_2]

Let Y be a subspace of a closure space X. Then (a) If A is closed (open) in X, then $Y \cap A$ is closed (open) in Y.

(b) If Y is closed (open) in X and A is closed (open) in Y then A is closed (open) in X.

Recall : If ccA = cA for every A ⊂ X, then the closure operator c is called topological.

Result 0.3.10 [CE_2]

A closure space (X,c) to be topological, it is necessary and sufficient that for each subspace Y of X every relatively closed (open) set be of the form Y ∩ A with A closed (open) in X.

Definition 0.3.11 [CE_2]

Let $\{(X_a,c_a) : a \in A\}$ be a family of closure spaces, X be the product of the family $\{X_a\}$ of underlying sets and π_a be the projection of X onto X_a for each a. The product closure c is the coarsest closure (which exists) on the product of underlying sets such that all the projections are morphisms.

Result 0.3.12 [CE_2]

If $\{X_a\}$ is a family of closure spaces and Y_a is a subspace of X_a for each a, then $\Pi\{Y_a\}$ is a subspace of $\Pi\{X_a\}$.

Result 0.3.13 [CE_2]

If X is the product of the family $\{X_a\}$ of closure spaces and U is a neigbourhood of x in X, then $\pi_a(U)$ is a neigbourhood of π_a x in X_a; in particular, if U is open then $\pi_a(U)$ is open.

Note 0.3.14

Let $\{(X_a,c_a):a \in A\}$ be a family of closure spaces. The product of the associated topologies is not necessarily the associated topology of the product closure.

Example 0.3.15 (For details, see the appendix)

Let X = {a,b,c}

c be defined on X such that

c{a}={a}, c{b}={b,c}, c{c}={a,c}

c{a,c}={a,c}, c{a,b}=c{b,c}=cX=X, cϕ=ϕ

Then c is a closure operator on X

X x X={(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)}

(X,t)={X,ϕ,{b},{b,c}}

(X x X,t x t)={{(b,b)},{(b,b),(b,c)},{(b,b),(c,b)},{(a,b),(b,b),(c,b)},

{(b,a),(b,b),(b,c)},{(b,b),(b,c),(c,b)},

{(a,b),(b,b),(b,c),(c,b)},{(b,a),(b,b),(b,c),(c,b)},

{(b,b),(b,c),(c,b),(c,c)},{(a,b),(b,b),(b,c),(c,b), (c,c)}

{(a,b),(a,c),(b,b),(b,c),(c,b),(c,c)},

{(a,b),(b,a),(b,b),(b,c),(c,b),(c,c)},

{(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)},

{(a,b),(a,c),(b,a),(b,b),(b,c),(c,b),(c,c)}, {(a,b),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)},

{(a,b),(a,c),(b,a),(b,b),(b,c),(c,b),(c,c)},

{(a,b),(b,a),(b,b),(b,c),(c,b)},XxX,ϕ}

(X,t) x (X,t)= {ϕ,X x X,{(b,b)},{(b,b),(b,c)},{(b,b),(c,b),},

{(b,b),(b,c),(c,b),(c,c)}}

Definition 0.3.16 [CE_2]

Let f: (X,c) --->Y is a surjective on to mapping. Then Y is said to be quotient of X under f if and only if Y is endowed with the closure inductively generated by the mapping f : X --->Y. That is $c'A=f(c(f^{-1}(A)))$, for every $A \subset Y$.

Result 0.3.17

Let g be an onto c-c' morphism from X to Y and for every $U \subset Y$ containing g(x) such that $g^{-1}(U)$ is a neighbourhood of x then U is a neighbourhood of g(x) implies g is quotient.

Proof

We have to show that $c'A=g(c(g^{-1}(A)))$, for every $A \subset Y$

Given that $g^{-1}(U)$ is a neighbourhood of x ==> U is a neighbourhood of g(x)

That is, $x \in X-c(X-g^{-1}(U))$ ==> $g(x) \in Y-c'(Y-U)$

That is, $g(x) \in c'(Y-U)$ ==> $x \in c(X-g^{-1}(U))$

That is, $g(x) \in c'A$==> $x \in cg^{-1}(A)$ by taking A = Y - U

$\Longrightarrow g(x) \in gc(g^{-1}(A))$

Any element in c'A is of the form g(x)

Therefore $c'A \subset gcg^{-1}(A)$

since g is a morphism, $(gc(g^{-1}(A)) \subset c'gg^{-1}(A) \subset c'A$

Therefore $c'A = gcg^{-1}(A)$

so we get g is a quotient map.

CHAPTER 1

MORPHISMS AND ORDER OF A MAP

INTRODUCTION

This chapter begins with the study of c-morphisms from one closure space into another. The order of a map and order of a closure operator are also studied in this chapter.

Čech studied continuous mapping in closure space [CE_2] . He also described this in terms of neighbourhoods and in terms of nets and considered the case when the image set is topological. In [A-W] D.R.Andrew and E.K.Whittlesy studied closure continuity .

In section 1.1 we introduce the morphisms in the category of closure spaces (which are the continuous functions in the terminology of Čech) and study the relation between these morphisms and the continuous functions in (X,t). We consider order between closure operators and prove certain related results like " If X is a set, c_1 and c_2 denote closure operators on X and if f is c_1 morphism and $c_2 \leq c_1$ then f is c_1-c_2 morphism."

David N. Roth and John W.Carlson mentioned the degree of closure operators in [RO-C]. They proved some results like " let (X,c) be a finite Čech closure space of degree k then c^k is the closure operator with respect to underlying topology." In section 1.2 we mention the concept of the degree of a closure operator. V. Kannan defined and studied about order of a map f from a topological space into another set and into another topological space [K]. Based on this we give an analogous study about order of a map f from a closure space into a set and into another closure space in section 1.2

We also study these concepts in monotone space in section 1.3

1.1 MORPHISMS

In this section we study the notion of morphisms from one closure space into another.

Definition 1.1.1

A map f : (X,c) --->(Y,c') is said to be c-c' morphism or just morphism if

$f(cA) \subset c'f(A)$.

Remark 1.1.2

Čech calls a morphism by the term continuous function. However, for us, a function f : (X,c) --->(Y,c') is continuous means f : (X,t) --->(Y,t') is continuous.

Definition 1.1.3 [CE_2]

A neighbourhood of a subset A of a space (X,c) is any subset U of X containing A in its interior where X-c(X-U) is the interior of U. By a neighbourhood of a point x of X we mean a neighbourhood of the one point set {x}.

Note 1.1.4

It is clear that if $A \subset X$ and if W is a neighbourhood of A in (X,t) then W is a neighbourhood of A in (X,c); the converse is not true.

Example 1.1.5

Let X = {a,b,c}, c be defined on X such that

c{a}={a}, c{b}={b,c}, c{c}={a,c}, c{a,c}={a,c}, c{a,b}=c{b,c}=cX=X, cϕ=ϕ

Then c is a Čech closure operator.

Here {a,c} is a neighbourhood of {a} in (X,c), but it is not a neighbourhood of {a} in (X,t).

In order that a mapping f of a closure space (X,c) into another one (Y,c') be c-c' morphism at a point x $\in$ X it is necessary and sufficient that the inverse image $f^{-1}(V)$ of each neighbourhood V of f(x) be a neighbourhood of x, or equivalently that for each neighbourhood V of f(x) there exists a neighbourhood U of x such that f(U) $\subset$ V[CE_2].

Result 1.1.6 [CE_2]

If f is a c-c' morphism of a space (X,c) into a space (Y,c') then the inverse image of each open subset of Y is an open subset of X.

Result 1.1.7

Let (X,c),(Y,c') be two closure spaces. f is a mapping from (X,c) into (Y,c').If f is a c-c' morphism, then f is continuous.

Proof

Let f be a c-c' morphism. Then inverse image of every open set is open. So f is continuous.

Note 1.1.8

f is continuous does not imply that f is c-c' morphism.

Example

X={a,b,c}

Let c be defined on X such that

c{a}={a}, c{b}={b,c}, c{c}={a,c}, c{a,c}={a,c} , c{a,b}=c{b,c}=cX =X, cϕ=ϕ

c is a closure operation on X

c' be defined on X such that

c'{a}={a,b}, c'{b}={b,c}, c'{c}={c}, c'{b,c}={b,c},

c'{a,b}=c'{a,c}=c'X=X, c'ϕ=ϕ

c' is a closure operation on X.

f is a mapping from X-->X defined in such a way that f(a)=c,f(b)=a, f(c)=c

f is continuous. But f is not a morphism. For, fc({b}) $\not\subset$ c'f({b})

Definition 1.1.9 [CE_2]

A cluster point or an accumulation point of a set A in (X,c) is a point x belonging to c(A-(x))

Note 1.1.10

If x is a cluster point of a set A in (X,c) then x is a cluster point of a set A in (X,t). The converse is not true. In Example 1.1.5 a is a cluster point of A = {a,b} in (X,t), a $\in$ cl(A-(a)). Since a $\notin$ c(A-(a)), a is not a cluster point of A={a,b} in (X,c).

Note 1.1.11

The set of all c-morphisms of X is denoted by $S_c(X)$ which is a semigroup under usual composition. The set of all morphisms from (X,c) to (X,c') is denoted by $S_{c\text{-}c'}(X)$. Clearly it is not in general a semigroup under composition.

Note 1.1.12

If $c \leq c'$ then $S_{c\text{-}c'}(X)$ is a semigroup.

For this, f is c-c' morphism and g is c-c' morphism

$(g \circ f)(cA) = g(f(cA))$

$\subset g(c'f(A))$

$\subset g\, c(f(A))$, since $c \leq c'$

$\subset c'g(f(A)) = c'(g \circ f)(A)$

That is g o f is c-c' morphism and this shows that $S_{c\text{-}c'}(X)$ is a semigroup under composition.

Result 1.1.13

Let X be a set, c and c' denote closure operations on X. If f is c-morphism and $c' \leq c$ then f is c-c' morphism.

Proof

$f(cA) \subset cf(A)$, since f is c-morphism

$\subset c'f(A)$, since $c' \leq c$

That is f is c-c' morphism

Similarly if f is c-c' morphism and $c \leq c'$ then f is c-morphism.

Note 1.1.14

If (X,c) is a closure space, cl is the closure operation in the associated topological space. Then $cl \leq c$.

$c(cl(A) = cl(A)$, since cl is closed in the associated topology

$cA \subset c(clA)$, since $A \subset clA$

That is $cA \subset clA$

Equivalently, identity map from (X,c) into (X,cl) is a c-cl morphism.

Proposition 1.1.15

Let X be a set c and c' be closure operations on X and cl and cl' be the closure operations on the associated topological spaces t and t' respectively. If $c' \leq c$, then $cl' \leq cl$

Proof

If $c' \leq c$ then any c' closed set is c- closed. For, if B is any c' closed set, then $B \subset cB \subset c'B = B$. So $B = cB$. It follows that if F is any c' closed set containing A then $F \supset clA$. In particular $cl'A \supset clA$ for any set A. That is $cl' \leq cl$.

Then we get $c' \leq c \Longrightarrow cl' \leq cl$.

The following example shows that the converse of the above proposition is not true.

Example 1.1.16

Let $X = \{a,b,c,d\}$

c be defined on X such that

$c\{a\}=\{a\}$, $c\{b\}=\{b,c\}$, $c\{c\}=\{c,d\}$, $c\{d\}=\{b,d\}$, $c\{a,b\}=\{a,b,c\}$, $c\{a,c\}=\{a,c,d\}$,

$c\{a,d\}=\{a,b,d\}$, $c\{b,c\}=c\{b,d\}=c\{c,d\}=c\{b,c,d\}=\{b,c,d\}$,

$c\{a,b,c\}=c\{a,b,d\}=c\{a,c,d\}=cX=X$, $c\phi=\phi$

Then c is a closure operation on X

c' be defined on X such that

$c'\{a\}=\{a\}$, $c'\{c\}=c'\{d\}=c'\{c,d\}=\{c,d\}$,

$c'\{b\}=c'\{b,c\}=c'\{b,d\}=c'\{b,c,d\}=\{b,c,d\}$,

$c'\{a,c\}=c'\{a,d\}=c'\{a,c,d\}=\{a,c,d\}$,

$c'\{a,b\}=c'\{a,b,c\}=c'\{a,b,d\}=c'X=X$, $c'\phi=\phi$

Then c' is a closure operation on X

$(X,t) = \{X,\phi,\{a\},\{b,c,d\}\}$

$(X,t')=\{X,\phi,\{a\},\{b\},\{a,b\},\{b,c,d\}\}$

Here $cl \leq cl'$. But $c \nleq c'$. For $c'\{b\} \not\subset c\{b\}$.

Result 1.1.17

Let X,Y be sets. c and c' be closure operations on X,Y respectively and cl, cl' be the closure operations on the respective associated topological spaces. If $f : X \longrightarrow Y$ is a c-c' morphism, then it is a c-cl' morphism.

Proof

$f(cA) \subset c'f(A)$ for every $A \subset X$

$\subset cl' f(A)$.

That is f is c-cl' morphism.

In particular when X=Y and c=c', we get , f is c-morphism implies f is continuous (as is seen in Result 1.1.7).

Note 1.1.18

It is clear that if c'<c and f is cl-morphism then f is cl-cl' morphism.

Result 1.1.19

If c'<c, then $S_c(X) \subset S_{c\text{-}c'}(X)$ and is a monoid although $S_{c\text{-}c'}(X)$ is not even a semigroup in general.

Proof

If $f \in S_c(X)$, then $f(cA) \subset cf(A)$

$$\subset \; c'f(A), \text{ since } c' \leq c$$

This is true for every $A \subset X$. Therefore $f \in S_{c\text{-}c'}(X)$. That is $S_c(X)$ is a semigroup under the induced operation is trivial.

Definition 1.1.20

A homeomorphism is a bijective (one - one - onto) mapping f such that both f and f^{-1} are morphisms. That is $f(cA) = cf(A)$ for every $A \subset X$.

Note 1.1.21

If a function f from (X,c) onto (Y,c') is a homeomorphism then f is a homeomorphism from (X,t) onto (Y,t'). But the converse is not true.

Example 1.1.22

$X = \{a,b,c\}$

Let c be defined on X such that

$c\{a\} = \{a\}$, $c\{b\}=\{b,c\}$, $c\{c\}=\{a,c\}$, $c\{a,c\}=\{a,c\}$,

$c\{a,b\}=c\{b,c\}=cX=X, c\phi=\phi$

c is a closure operation on X.

c' be defined on X such that

$c'\{b\}=\{b,c\}$, $c'\{c\}=\{c\}$, $c'\{b,c\}=\{b,c\}$,

$c'\{a\}=c'\{a,b\}=c'\{a,c\}=c'X=X$, $c'\phi=\phi$

Then c' is a closure operation on X.

$(X,t) = \{X,\phi,\{b\},\{b,c\}\}$ and $(X,t') = \{X,\phi,\{a\},\{a,b\}\}$

f is a mapping from (X,c) onto (X,c') defined in such a way that $f(a)=c$, $f(b)=a$, $f(c)=b$. f is a homeomorphism from (X,t) onto (Y,t'). But f is not a homeomorphism from (X,c) onto (X,c') because $f\,c(\{b\}) \neq c'\,f(\{b\})$.

1.2 DEGREE OF CLOSURE OPERATOR AND ORDER OF A MAP

Definition 1.2.1

Let (X,c) be a Čech closure space. We define for each ordinal α, the operator c^{α} as $c^{\alpha}(A)=\bigcup_{\beta<\alpha} c^{\beta}(A)$ if α is limit ordinal and $c(c^{\beta}(A))$ if $\alpha=\beta+1$. Then c^{α} is a closure operator. The degree of c is defined to be k if k is the smallest ordinal number for which $c^{k}=c^{k+1}$ (It is called the order of the closure space in [K]).

Example 1.2.2

(1) Let $X=\{a,b,c\}$

c be defined on x such that

$c\{a\}=\{a\}$, $c\{b\}=\{b,c\}$, $c\{c\}=\{a,b,c\}$,

$c\{a,b\}=c\{b,c\}=c\{a,c\}=cX=X$, $c\phi=\phi$

Then c is a closure operation on X.

Here the degree of closure operator is two.

(2) Let X=N

c be defined on X such that

$cA = A \cup \{x+1: x \in A\}$ for every $A \subset X$

Then c is a closure operation on X

Here the degree of closure operator is ω

(3) Let $X=[1,\Omega]$ ordinal space.

c be defined on X such that

$cA = A \cup (A+1), (A+1)=\{x+1 : x \in A\}$

Then c is a closure operator and degree of c is Ω .

Result 1.2.3 [RO-C]

Let (X,c) be a finite Čech closure space. Then

1. For every natural number n, and $A \subset X$, $c^n(A) \subset clA$

2. For each $A \subset X$ there exists a smallest $m \in N$ such that $c^m(A)= clA$.

Result 1.2.4 [RO-C]

Let (X,c) be a finite Čech closure space of degree k. Then (1) c^k is a Kuratowski closure opearator on X

(2) c^k is the closure operator with respect to the associated topology (X,t).

Result 1.2.5

The order of a closure operator is invariant.

Proof

Let (X,c) and (X,c') be closure spaces which are homeomorphic. Let the degree of c be k. Then clearly the degree of c' is also k.

Now we consider the order of a map from a closure space X to another set Y and from a closure space X to another closure space (analogous to the study made in [K]).

Let X be a closure space and Y be any set. Let f be a function from X onto Y.

Take any subset A of Y. Define

$A^0_f = A$

$A^1_f = f(c(f^{-1}(A)))$

$A^2_f = f(c(f^{-1}(f(c(f^{-1}(A))))))$ etc.

$A^\alpha_f = \bigcup_{\beta<\alpha} A^\beta_f$ if α is a limit ordinal

$A^\alpha_f = (A^\beta_f)'_f$, if $\alpha=\beta+1$

We define $A^{\sim}_f = \cup\{ A^\alpha_f : \alpha$ is an ordinal number $\}$

Definition 1.2.6

Let X be a closure space and Y be a set f be a map from X onto Y. The order of the map f denoted by $\sigma(f)$ is the least ordinal number α such that $A^{\sim}_{f} = A^{\alpha}_{f}$ for every subset A of Y.

Example 1.2.7

Let X = {a,b,c}; Y = {1,2,3}

c be defined on X such that

c{a}={a}, c{b}={b,c}, c{c}={a,c},

c{a,b}=c{b,c}=cX=X, c{a,c}={a,c}, $c\phi=\phi$

Then c is a closure operation on X

Let f be a map from X onto Y such that f(a)=1 f(b)=3 f(c)=2

The order of the map is 2.

Result 1.2.8

Let (X,c) be a closure space, (X,t) the associated topological space and Y be a set. If $\sigma(f)$ is the order of a map defined from X to Y and $\sigma_*(f)$ is the order of the map in the sense of [K] from (X,t) to Y then $\sigma_*(f) \leq \sigma(f)$.

Proof

Let * denote the corresponding notions with respect to (X,t). We know that

$cA \subset clA$ for every $A \subset X$

$A^0_f = A^0_{f*}$

$A^1_f = f(c(f^{-1}(A))) \subset f(cl(f^{-1}(A))) = A^1_{f*}$ etc.

$A^\alpha_f \subset A^\alpha_{f*}$

for every ordinal α.

Then $A^{\sim}_f \subset A^{\sim}_{f*}$

Since $\sigma_*(f)$ is the least ordinal number α such that $A^{\sim}_{f*} = A^\alpha_{f*}$ for every subset A of Y,

$\sigma_*(f)$ is less than or equal to $\sigma(f)$ ·

Example 1.2.9

Let X=Y=N

c be defined on X such that

$c(A) = A \cup (A+1)$ for every subset A of X.

Then c is a closure operation on X

Let f be an identity map from X to Y

Here $\sigma_*(f)$ is 2. But $\sigma(f)$ is ω.

Note 1.2.10

The following are analogous to the corresponding results in topological spaces [K] :

(1) As α increases, A^{α}_{f} also increases. That is if $\alpha<\beta$ then $A^{\alpha}_{f} \subset A^{\beta}_{f}$ for each $A \subset Y$

(2) If for some α, we have $A^{\alpha}_{f} = A^{(\alpha+1)}_{f}$, then $A^{\alpha}_{f} = A^{\beta}_{f}$ for every $\beta>\alpha$.

(3) $(A^{\sim}_{f})'_{f} = A^{\sim}_{f}$ for every $A \subset Y$

(4) For a subset B of Y, the following are equivalent

(a) $B = A^{\sim}_{f}$ for some $A \subset Y$

(b) $B = B^{\sim}_{f}$

(c) $B = B'_{f}$

(d) $f^{-1}(B)$ is closed.

(5) If $A \subset B \subset Y$, then $A^{\alpha}_{f} \subset A^{\beta}_{f}$ for every ordinal number α.

Now we consider that Y is also provided with a closure operator. Take two closure spaces (X,c) and (Y,c'). f is c-c' morphism.

Note 1.2.11

1. If f is c-c' morphism, then $A^{1}_{f} \subset c'A$ for every $A \subset Y$

$A^{1}_{f} = f(c(f^{-1}(A)))$

$cf^{-1}(A) \subset f^{-1}(c^{1}A)$, since f is c-c^{1} morphism.

$f(c(f^{-1}(A))) \subset ff^{-1}(c^{1}A) \subset c^{1}(A)$

That is $A^{1}_{f} \subset c^{1}A$

2. Conversely if $A'_f \subset c'A$ for every $A \subset Y$, then f is c-c' morphism

$f(c(f^{-1}(A))) \subset c'A$

So $f^{-1} f(c(f^{-1}(A))) \subset f^{-1}(c'A) \subset ...$

Therefore $cf^{-1}(A) \subset f^{-1}(c'A)$

Hence f is c-c' morphism.

3. If f is a closed c-c' morphism, then $c'A = A'_f$ for every $A \subset Y$.

4. If f is a closed c-c' morphism then $\sigma(f) < 1$ ·

5. If f and g are two maps from a closure space onto a set Y and if $A'_f = A'_g$ for each subset A of Y, then $\sigma(f)=\sigma(g)$.

1.3 C_*-MORPHISMS AND ORDER OF MAP IN MONOTONE SPACES

To each monotone space, we can associate uniquely a Čech closure space and thereby a topological space. The interelations are discussed in this section.

Definition 1.3.1

A monotone operator c_* is said to be coarser than a monotone operator c'_* if $c_*A \supset c'_*A$ for each $A \subset X$. In this case we say $c_* \leq c'_*$

Definition 1.3.2

A subset A of a monotone space (X,c_*) will be called closed if $c_*A = A$, open if its complement is closed.

Result 1.3.3

For each monotone operator there is a uniquely associated closure operator.

Proof

Let X be a set and c_* be a monotone operator on X. Take the collection

$\{c_\alpha : c_\alpha$ is a closure operator coarser than $c_*\}$. Take $cA = \cap c_\alpha A$ for all A

Then c is a closure operator.

For, $c\phi = \cap c_\alpha(\phi) = \phi$

$c(A \cup B) = \cap (c_\alpha(A \cup B)) = \cap(c_\alpha A \cup c_\alpha B) = (\cap c_\alpha A) \cup (\cap c_\alpha B) = cA \cup cB$

$A \subset c_\alpha A$ for every α

Therefore $A \subset \cap c_\alpha A$

Thus $A \subset cA$.

Example 1.3.4

$X=\{a,b,c\}$

c_* be defined on X such that

$c_*\{a\}=\{a\}$, $c_*\{b\}=\{b,c\}$, $c_*\{c\}=\{c\}$, $c_*\{a,b\}=c_*\{b,c\}=c_*\{a,c\}=c_*X=X$, $c_*\phi=\phi$

Then c_* is a monotone operator.

Associated closure operator is

$c\phi=\phi$, $c\{a\}=\{a,b\}$, $c\{b\}=\{b,c\}$, $c\{c\}=\{a,c\}$, $c\{a,b\}=c\{b,c\}=c\{a,c\}=cX=X$.

The associated topology is indiscrete.

Remark 1.3.5

From (X,c) we can associate a topology (X,t) in the usual manner. From c_* we can associate in a similar way a Kuratowski closure operator directly. It is clear that both the above two Kuratowski operators are the same.

Definition 1.3.6

A neighbourhood of a subset A of a space (X,c_*) is any subset U of X such that $A\subset X-c_*(X-U)$. By a neighbourhood of a point x of X we mean a neighbourhood of the one point set $\{x\}$.

Note 1.3.7

It is clear that if W is a neighbourhood of $A \subset (X,c)$, then it is a neighbourhood of A in (X,c_*). But the converse is not true. In the Example 1.3.2, $\{a,b\}$ is a neighbourhood of $\{a\}$ in (X,c_*). But it is not a neighbourhood of $\{a\}$ in (X,c).

Definition 1.3.8

Let (X,c_*) be a monotone space and let $Y \subset X$. The monotone operator c'_* on Y is defined as $c'_*A = Y \cap c_*A$ for every $A \subset Y$. Then $c'_*$is called the relativisation of c_* to Y and the space (Y,c'_*) is called the subspace of (X,c_*).

Definition 1.3.9

Let X be a set and c_* denote a monotone operator on X. A map f : X—>X is said to be a c_*-morphism if $f(c_*A) \subset c_*f(A)$ for every $A \subset X$.

Definition 1.3.10

A map f : (X,c_*) —> (Y,c'_*) is said to be c_*-c'_* morphism if $f(c_*A) \subset c'_*f(A)$ for each $A \subset X$.

Note 1.3.11

f is c_*-c'_* morphism need not imply that it is c-c' morphism and vice-versa.

The following example shows thatf:(X,c_*) —> (X,c'_*) is not a c_*-c'_* morphism but it is a c-c' morphism.

Example 1.3.12

X = {a,b,c}

c_* be defined on X such that

$c_*\{a\}=\{a\}$, $c_*\{b\}=\{b,c\}$, $c_*\{c\}=\{c\}$, $c_*\{a,b\}=c_*\{b,c\}=c_*\{a,c\}=c_*X=X$, $c_*\phi=\phi$.

c_* is a monotone operator on X.

c is given by

$c\{a\}=\{a,b\}$, $c\{b\}=\{b,c\}$, $c\{c\}=\{a,c\}$, $c\{a,b\}=c\{b,c\}= c\{a,c\}=cX=X$, $c\phi=\phi$

Let f be a map from (X,c_*) into (X,c_*) defined in such a way that

$f(a) = b$, $f(b) = c$, $f(c) = a$

$f : (X,c) \longrightarrow (X,c)$ is c-c morphism. But it is not c_*-c_* morphism because

$f\,c_*(\{b\}) \not\subset c_*f(\{b\})$.

Definition 1.3.13

Let (X,c_*) be a monotone space and Y be a set. f be a map from X onto Y. The order of the map f denoted by $S(f)$ is the least ordinal number α such that $A^{\sim}_f = A^{\alpha}_f$ ($A^{\sim}_f$, A^{α}_f defined in monotone space similar to that in closure spaces) for every subset A of Y.

Note 1.3.14

It is clear that $S(f) \leq \sigma(f)$.

CHAPTER 2

SOME SEPARATION PROPERTIES IN CLOSURE SPACES

INTRODUCTION

This chapter is devoted to the study of some separation properties of closure spaces and of monotone spaces analogous to the separation properties of topological spaces.

Tietze, Kolmogoroff, Frechet, Riez, Hausdorff and others studied separation properties in topological spaces. The separation properties in closure spaces were defined and discussed by E.Čech [CE_2]. According to him, any two points can be separated by distinct neighbourhoods in a separated space. Any point x and closed set not containing x can be separated by distinct neighbourhoods in a regular space. David.N.Roth and J.W.Carlson studied a number of separation properties like T_0 ,T_1 , R_0 , R_1 etc [RO-C] W.J.Thron also discuss separation properties [T].

Section 2.1 is mainly focused to define and study some point separation properties like T_0,T_1 and T_2 in closure spaces.

We introduce and study some higher separation properties in section 2.2. We also find the relation between c-separation properties and t-separation properties.

Some results in the same area related to monotone space have also been studied in section 2.3.

2.1. POINT SEPARATION PROPERTIES IN CLOSURE SPACES

In this section we introduce and study some point separation properties in closure spaces.

Definition 2.1.1

A closure space (X,c) is said to be T_0 if for every $x \neq y$ in X either $x \notin c\{y\}$ or $y \notin c\{x\}$ (Čech termed this as feebly semiseparated).

Result 2.1.2

If (X,t) is T_0 then (X,c) is T_0.

Proof

Let (X,t) be T_0. If $x \neq y$ in X, then either $x \notin cl\{y\}$ or $y \notin cl\{x\}$.

But $cA \subset cl\,A$ for every $A \subset X$.

So we get $x \notin c\{y\}$ or $y \notin c\{x\}$

The following example shows that the converse of the above Result 2.1.2 is not true.

Example 2.1.3

$X = \{a,b,c\}$

Let c be defined on X such that

c {a}=a,b}, c{b}={b,c}, c{c}= {a,c}, c {a,b}= c {b,c}= c{a,c}= cX = X, cϕ=ϕ

c is a closure operation on X.

Here (X,c) is T_0. But (X,t) is the indiscrete topology which is not T_0.

Definition 2.1.4

A closure space (X,c) is said to be T_1 if for $x \neq y$ we have $x \notin c\{y\}$ and $y \notin c\{x\}$ (Čech termed this as semiseparated space).

Result 2.1.5

For a closure space (X,c) the following are equivalent.

(1) The space (X,c) is T_1.

(2) For any $x \in X$, the singleton set {x} is closed .

(3) Every finite subset of X is closed.

Proof

(1) ==> (2)

Let (X,c) be T_1 . If possible, suppose x is not closed. That is $c\{x\} \neq \{x\}$. So there exists $y \neq x$, $y \in c\{x\}$. But this contradicts the fact that (X,c) is T_1. Therefore {x} is closed.

(2) ==> (3)

For any $x \in X$, the singleton set $\{x\}$ is closed. Since finite union of closed sets is closed, every finite subset of X is closed.

(3) ==> (2)

Trivial

(2) ==> (1)

Singleton sets are closed. Therefore $c\{x\}=x$, $c\{y\}=y$ and so $x \notin c\{y\}$ and $y \notin c\{x\}$. Therefore (X,c) is T_1.

Corollary 2.1.6

(X,c) is T_1, if and only if (X,t) is T_1.

Note 2.1.7

It is clear that every T_1 space is also T_0. (X,c) is T_0 need not imply that (X,c) is T_1. In Example 2.1.3 (X,c) is T_0 but it is not T_1.

Definition 2.1.8

A closure space (X,c) is said to be semi-Hausdorff if for $x \neq y$ either there exists an open set U such that $x \in U$ and $y \notin cU$ or there exists an open set V such that $y \in V$ and $x \notin cV$. If both conditions hold, then (X,c) is said to be pseudo- Hausdorff.

Result 2.1.9

If (X,t) is Hausdorff, then (X,c) is pseudo-Hausdorff.

Proof

Let (X,t) be Hausdorff. Then $x \neq y$ implies there exists disjoint open sets U,V such that $x \in U$, $y \in V$. That is there exists an open set U such that $x \in U$ and $y \notin cl\, U$ and also there exists an open set V such that $y \in V$ and $x \notin cl V$. But $cA \subset cl\, A$ for each $A \subset X$. Therefore (X,c) is pseudo Hausdorff.

Note 2.1.10

The converse of the above result is not true.

Example 2.1.11

$X = N \times N \cup \{x,y\} \cup \{a_i : i \in N\} \cup \{b_j : j \in N\}$, a_i's ,b_j's, x,y are all distinct and do not belong to N x N.

Let c be defined on X such that

$cA = A \cup \{b_i$: there exists an infinite number of j such that $(i,j) \in A\}$

$\cup \{a_j$: there exists an infinite number of i such that $(i,j) \in A\}$, if $A \subset N\times N$

$cA = c(A \cap N \times N) \cup A \cup \{x\}$, if A contains an infinite number of a_i's and at most finitely many b_j's.

$cA = c(A \cap N \times N) \cup A \cup \{y\}$, if A contains an infinite number of b_j's and at most finitely many a_i's.

$cA = c(A \cap N \times N) \cup A \cup \{x,y\}$ if A contains an infinite number of a_i's and an infinte number of b_j's.

Here (X,c) is pseudo-Hausdorff. In (X,t) x and y cannot be separated by disjoint open sets and so (X,t) is not Hausdorff.

Definition 2.1.12

A closure space (X,c) is said to be Hausdorff (Čech termed it as separated) if for any two distinct points, there exists neighbourhoods U of x and V of y such that $U \cap V = \phi$.

Result 2.1.13

If (X,t) is Hausdorff then (X,c) is Hausdorff.

Proof

Let (X,t) be Hausdorff, then for any two distinct points x and y, there exist disjoint open sets U and V containing x and y respectively. Since an open set is a neighbourhood of each of its points and a neighbourhood in (X,t) is also a neighbourhood in (X,c) , U and V are disjoint neighbourhoods for x and y in (X,c). Hence (X,c) is Hausdorff.

Note 2.1.14

The converse of the above result is not true.

Result 2.1.15

(X,c) is pseudo-Hausdorff implies (X,c) is T_1 .

Proof

If (X,c) is not T_1, then there exists $x \in X$ such that $c\{x\} = \{x\}$. Let $y \in c\{x\}$. Thus if U is an open set containing x, $cU \supset c\{x\} \ni y$. So (X,c) is not pseudo-Hausdorff.

Note 2.1.16

(X,c) is semi-Hausdorff need not imply that (X,c) is pseudo-Hausdorff.

Note 2.1.17

(X,c) is pseudo-Hausdorff need not imply that (X,c) is Hausdorff and vice versa.

Result 2.1.18

Every subspace of a pseudo-Hausdorff space is pseudo-Hausdorff.

Proof

Let (X,c) be a pseudo-Hausdorff space and $A \subset X$. Since X is pseudo-Hausdorff there exists points $x \neq y$ and open U,V such that $X \in U$, $y \notin cU$ and $y \in V$, $x \notin cV$. Then $A \cap U$ and $A \cap V$ are open sets in A, such that $x \in A \cap U$, $y \in A \cap V$ and $x \notin [A \cap c(A \cap V)]$, $y \notin [A \cap c(A \cap U)]$ which shows that A is pseudo-Hausdorff.

It is clear that if a closure space is T_1 (respectively T_0), then every subspace is T_1 (respectively T_0).

Result 2.1.19

A nonempty product space is pseudo- Hausdorff if and only if each factor space is pseudo- Hausdorff.

Proof

Suppose X_α is pseudo-Hausdorff for each $\alpha \in A$. Let $x \neq y$ in ΠX_α. Then for some co-ordinate α, $x_\alpha \neq y_\alpha$. Since each X_α is pseudo-Hausdorff, for $x_\alpha \neq y_\alpha$ there exists open sets U_α , V_α such that $x_\alpha \in U_\alpha$, $y_\alpha \in V_\alpha$ and $x_\alpha \notin c_\alpha V_\alpha$, $y_\alpha \notin c_\alpha U_\alpha$. Since the projection maps are c - c_α morphisms [CE_2] and the inverse image of an open set under a c- morphism is open, $\pi_\alpha^{-1} U_\alpha$ and $\pi_\alpha^{-1} V_\alpha$ are open in ΠX_α and $x \in \pi_\alpha^{-1} U_\alpha$, $y \in \pi_\alpha^{-1} U_\alpha$. $x \notin \pi_\alpha^{-1} c_\alpha V_\alpha)$ and $y \notin \pi_\alpha^{-1}(c_\alpha U_\alpha)$. But $c(\pi_\alpha^{-1} U_\alpha) \subset \pi_\alpha^{-1}(c_\alpha U_\alpha)$ and $c(\pi_\alpha^{-1} V_\alpha) \subset \pi_\alpha^{-1}(c_\alpha V_\alpha)$ Therefore, $x \notin c(\pi_\alpha^{-1}(V_\alpha))$and $y \notin c(\pi_\alpha^{-1}(U_\alpha))$.

Conversely, suppose that ΠX_α is nonempty pseudo-Hausdorff. Take a fixed point $b=(b_\alpha)$ where $b_\alpha \in X_\alpha$ for each $\alpha \in A$. Then the subspace $B_\alpha = \{x \in \Pi X_\beta : x_\beta = b_\beta$ except for $\beta = \alpha\}$ is pseudo-Hausdorff. B_α is homeomorphic to X_α. Therefore X_α is pseudo-Hausdorff.

Definiiton 2.1.20

A closure space (X,c) is said to be Urysohn space if for $x \neq y$, there exists open sets U,V such that $x \in U$, $y \in V$ and $cU \cap cV = \phi$.

Result 2.1.21

(X,t) is Urysohn space implies (X,c) is Urysohn space.

The proof is similar to the Proof of 2.1.13.

Note

The converse is not true.

Remarks 2.1.22

It is clear that every Urysohn space is pseudo-Hausdorff space. (X,c) is a pseudo-Hausdorff space need not imply that (X,c) is Urysohn Space. In Example 2.1.11 (X,c) is pseudo-Hausdorff but it is not Urysohn.

Result 2.1.23

Every subspace of a Urysohn space is Urysohn. The proof is similar to the Proof of 2.1.18.

Definition 2.1.24

A closure space (X,c) is said to be functionally Hausdorff, if for every pair of distinct points x,y, there exists a c-cl_1, morphism f : x —>[0,1] such that f(x)=0 and f(y)=1 where cl_1 is the usual closure operation in [0,1].

Result 2.1.25

If (X,t) is functionally Hausdorff, then (X,c) is functionally Hausdorff.

Proof

Let (X,t) be functionally Hausdorff space. Therefore for every pair of distinct points x,y there exists a cl-cl_1 morphism f : X —>[0,1] such that f(x)=0, f(y)=1 where cl is the closure operation in the associated topological space(X,t). Now $f(clA) \subset cl_1 f(A)$

for every $A \subset X$. Since $cA \subset clA$ for every $A \subset X$, $f(cA) \subset f(clA) \subset cl_1 f(A)$ That is f is c-cl_1 morphism. That is (X,c) is functionally Hausdorff.

Note 2.1.26

The converse of the above result is not true.

Result 2.1.27

Every subspace of a functionally Hausdorff space is functionally Hausdorff.

Proof

Let (X,c) be functionally Hausdorff space and $Y \subset X$ and $x,y \in Y$. Since X is functionally Hausdorff, there exists a c-cl_1 morphism from X ->[0,1] such that f(x)=0 and f(y)=1. Let c' be the induced closure operation on Y. Since $c'A=cA \cap Y \subset cA$ for every $A \subset Y$, f restricted to Y is c'-cl_1 morphism. Thus Y is functionally Hausdorff.

2.2 <u>HIGHER SEPARATION PROPERTIES IN CLOSURE SPACES.</u>

In this section we study regularity, normality, complete regularity etc, in closure spaces.

Definition 2.2.1

A closure space(X,c) is said to be quasi-regular if for every point x and a closed set A not containing x, there exists an open set U such that $x \in U$ and $cU \cap A = \phi$. X is said to be semi-regular if for every point x and closed set A not containing x, there exists an open set V such that $A \subset V$ and $x \notin cV$. If both conditions hold, then X is said to be pseudo-regular.

Definition 2.2.2 [CE_2]

A closure space (X,c) is said to be regular, if for each point x of X and each subset A of X such that $x \notin cA$ and there exists neighbourhoods U of x and V of y such that $U \cap V = \phi$.

Result 2.2.3

If (X,t) is regular then (X,c) is pseudo-regular. The proof is similar to the Proof of 2.1.9

Note 2.2.4

The converse of the above result is not true. In Example 2.1.11 (X,c) is pseudo-regular but (X,t) is not regular.

The following example shows that (X,c) is quasi-regular does not imply that it is semi-regular and pseudo-regular.

Example 2.2.5

Let X be a set of real numbers and let $A = \{1/n : n=1,2,3...\}$.

Define the Smirnov's deleted sequence topology τ on X by letting $G \in \tau$ if $G=U-B$ where $B \subset A$ and U is an open set in the Euclidean topology on X. Let c be the closure in this topology. Then (X,c) is quasi-regular but not semi-regular.

Note 2.2.6

(X,c) is semi-regular does not imply that (X,c) is pseudo-regular.

Result 2.2.7

If (X,t) is regular, then (X,c) is regular. The proof is similar to the Proof of 2.1.13.

Note 2.2.8

The converse of the above result is not true.

Note 2.2.9

(X,c) is regular does not imply that (X,c) is pseudo-regular and vice-versa.

Result 2.2.10 [CE_2]

Every subspace of a regular space is regular.

Result 2.2.11

Every closed subspace of a pseudo-regular space is pseudo-regular.

Proof

Let (X,c) be pseudo-regular. Y be closed in X. Let A be closed in Y. Then A is closed in X [CE_2]. y be a point in Y not in A. Since (X,c) is pseudo-regular, there exists open sets U of A and V of y such that $y \notin cU$ and $cV \cap A = \phi$. Then $Y \cap U$ and $Y \cap V$ are open in Y containing A and y respectively. $A \cap [Y \cap c(Y \cap V)] = \phi$ and $y \notin [Y \cap c(Y \cap U)]$ which shows that Y is pseudo-regular.

Result 2.2.12

X is a closure space. X is quasi-regular if and only if for each $x \in X$ and an open neighbourhood U of x there exists a neighbourhood V of x with $x \in V \subset cV \subset U$.

Proof

Let X be quasi-regular. x in X and U is an open neighbourhood of x. Then U' is closed in X. Since X is quasi-regular,for the point x and the closed set U' there exists open V such that $x \in V$ and $U' \cap cV = \phi$. Then $cV \subset U$, that is $x \in V \subset cV \subset U$.

Conversely, suppose the condition holds. Let x in X and A a closed set in X. Then for x and its neighbourhood A' (complement of A), there exists an open set V such that $x \in V \subset cV \subset A$. That is $x \in V$ and $cV \cap A' = \phi$. That is X is quasi-regular.

Definition 2.2.13

A closure space (X,c) is said to be semi-normal, if for each pair of disjoint closed sets A and B either there exists an open set U such that $A \subset U$ and $cU \cap B = \phi$ or there exists an open set V such that $B \subset V$ and $A \cap cV = \phi$. If both conditions hold X is said to be pseudo-normal.

Definition 2.2.14 [CE_2]

A closure space (X,c) is said to be normal, if for any pair of disjoint closed sets A and B there exists disjoint neighbourhoods U and V containing A and B respectively.

Result 2.2.15

(X,t) is normal ==> (X,c) is pseudo-normal. The proof is similar to the Proof of 2.1.9

Result 2.2.16

(X,t) is normal ==> (X,c) is normal. The proof is similar to the Proof of 2.1.13

Note 2.2.17

The converse of the above result is not true.

Note 2.2.18

(X,c) is pseudo-normal does not imply that (X,t) is normal. In Example 2.1.11 (X,c) is pseudo-normal but (X,t) is not normal.

Note 2.2.19

(X,c) is semi-normal does not imply that (X,c) is pseudo-normal.

The following example shows that (X,c) is pseudo-normal does not imply that (X,c) is pseudo-regular.

Example 2.2.20

Let X = N

c is defined on X such that

$cA = A \cup \{x+1 : x \in A\}$

Vacuously (X,c) is pseudo-normal. But it is not pseudo-regular.

Note 2.2.21

(X,c) is normal does not imply that (X,c) is pseudo-normal and (X,c) is pseudo-normal does not imply that (X,c) is normal.

Result 2.2.22 [CE_2]

In a closure space every closed subspace of a normal space is normal.

Result 2.2.23

In a closure space every closed subspace of a pseudo-normal space is pseudo-normal.

The proof is similar to the Proof of 2.2.11

Definition 2.2.24

A closure space (X,c) is said to be completely normal if for any two disjoint closed sets A and B in X there exists open sets U,V such that $A \subset U$, $B \subset V$ and $cU \cap cV = \phi$.

Note 2.2.25

If (X,t) is completely normal, then (X,c) is completely normal and the converse is not true.

Remark 2.2.26

It is clear that every completely normal space is pseudo-normal space. In Example 2.1.11 (X,c) is pseudo-normal but not completely normal.

Definition 2.2.27

A closure space (X,c) is said to be completely regular, if for every point x and a closed set A not containing x, there exists a c-cl_1 morphism f : x —> [0,1] such that f(x)=0 and f(y)=1 for every y $\in$ A .

Result 2.2.28

If (X,t) is completely regular, then (X,c) is completely regular and the converse is not true.

Result 2.2.29

Every subspace of a completely regular space is completely regular.

2.3 SEPARATION PROPERTIES IN MONOTONE SPACES

In this section we introduce and study some separation properties in monotone spaces.

Definition 2.3.1

A monotone space (X,c_*) is said to be T_0 , if $x \neq y$ implies either $x \notin c_*\{y\}$ or $y \notin c_*\{x\}$ and T_1 if $x \neq y$ implies $x \notin c_*\{y\}$ and $y \notin c_*\{x\}$.

Note 2.3.2

It is clear that if (X,c) is T_0 space, then (X,c_*) is T_0 and the converse is not true.

Example 2.3.3

X = N

Let c_* be defined on X such that

$c_*A = A \cup (A+1)$, if A is a one point set, that is $c_*\{x\}=\{x, x+1\}$

$c_*A = N$, when A is not a singleton.

Here (X,c_*) is T_0. The associated closure space (X,c) is not T_0.

Note 2.3.4

If (X,c) is T_1 , then (X,c_*) is T_1. The converse is not true.

Example 2.3.5

X = N

Let c_* be defined on X such that

$c_*A = A$, if A is a singleton set

$c_*A = N$, when A is not a singleton.

Then c_* is a monotone operator on X. Here (X,c_*) is T_1. But (X,c) is not T_1.

Remark 2.3.6

It is clear that if (X,c_*) is T_1, then it is T_0. The following example shows that the converse is not true.

Example 2.3.7

Let $X = \{a,b,c\}$

c_* be defined on X such that

$c_*\{a\} = \{a,b\}$, $c_*\{b\} = \{b\}$, $c_*\{c\} = \{a,c\}$,

$c_*\{a,b\} = c_*\{b,c\} = c_*\{a,c\} = c_*X = X$, $c_*\phi = \phi$.

c_* is a monotone operator on X.

It is T_0 but not T_1.

Definition 2.3.8

A monotone space (X,c_*) is said to be semi-Hausdorff if $x \neq y$, then either there exists an open set U containing x and $y \notin c_*U$ or there exists an open set V containing y and $x \notin c_*V$. If the both conditions hold, then (X,c_*) is said to be pseudo-Hausdorff.

Definition 2.3.9

A monotone space (X,c_*) is said to be Hausdorff, if for every pair of distinct points x and y there exist disjoint neighbourhoods U and V containing x and y respectively.

Definition 2.3.10

A monotone space (X,c_*) is said to be quasi-regular, if for every point x and closed set A not containing x, there exists an open set U such that $x \in U$ and $c_*U \cap A = \phi$. X is said to be semi-regular, if for every point x and closed set A not containing x, there exists an open set V such that $A \subset V$ and $x \notin c_*V$. If both conditions hold ,X is said to be pseudo-regular.

Definition 2.3.11

A monotone space (X,c_*) is said to be regular, if for every point x and closed set A not containing x, there exist disjoint neighbourhoods U and V containing x and A respectively.

Definition 2.3.12

A monotone space (X,c_*) is said to be semi-normal, if for each pair of closed sets A and B, either there exists an open set U such that $A \subset U$ and $c_*U \cap B = \phi$ or there exists an open set V such that $B \subset V$ and $c_*V \cap A = \phi$. If both conditions hold, X is said to be pseudonormal.

Definition 2.3.13

A monotone space (X,c_*) is said to be normal, if for every pair of distinct closed sets A and B, there exists disjoint neighbourhoods U and V of A and B respectively.

Note 2.3.14

It is clear that (X,c) is pseudo-Hausdorff, pseudo-regular, pseudo-normal implies (X,c_*) is pseudo-Hausdorff, pseudo-regular and pseudo-normal respectively. The converse is not true.

Example 2.3.15

$X = N \cup \{0\}$

Let c_* be defined on X such that

$c_*A = A$, if A is finite

$c_*B = B \cup \{0\}$ where $B \subset N$ and N-B is finite

$c_*A = X$, if A = N is infinite and not in the form B.

$c_*N = X$

Then c_* is a monotone operator.

Here (X,c_*) is pseudo-Hausdorff, pseudo-regular and pseudo-normal. Here we get cA = A, if A is finite and cA = X, if A is not finite. Thus (X,c) is not pseudo-Hausdorff, not pseudo-regular and not pseudo-normal.

CHAPTER 3

SOME PROPERTIES OF COMPACTNESS AND CONNECTEDNESS IN CLOSURE SPACES

INTRODCUTION

In this chapter we firstly describe the fundamental properties of compactness. We define compact closure spaces and study some properties of compactness.

Čech defined closure space X to be compact if the intersection of the closures of sets belonging to any proper filter in X is nonempty. He proved some properties of compactness in closure spaces [CE_2]. In section 1 of this chapter, we find the relation between compactness in (X,c) and (X,t) and prove some related results.

Čech described the concept of connectedness in [CE_2] as "a subset A of a closure space X is said to be connected in X if A is not the union of two nonempty semi-separated subsets of X. That is $A = A_1 \cup A_2$,$(cA_1 \cap A_2) \cup (A_1 \cap cA_2)=\phi$ implies that $A_1=\phi$ or $A_2=\phi$" . It can be easily seen that this is precisely the connectedness of the associated topological space. Plastria, F obtained certain conditions which imply the connectedness of simple extensions [P]; it has been proved that local connectedness of certain subspaces implies the local connectedness of simple extensions.

We define the concept of connectedness in section 3.2 in a slightly different and perhaps more reasonable way and prove some results in connectedness. We note that the image of a connected space under a c -c_1 morphism need not be connected.

In section 3.3 we introduce the concepts of local connectedness and path connectedness. We also define compactness and connectedness in monotone spaces in section 3.4.

3.1 SOME PROPERTIES OF COMPACTNESS

The following definitions and results are due to E.Čech.

Definitions 3.1.1

(i) Let (X,c) be a closure space , $\mathcal{F}$ be a proper filter on X and x be an element of X. We shall say that x is a cluster point of F in (X,c) if x belongs to $\cap$ {cF: F$\in\mathcal{F}$}, that is if each neighbourhood of x intersects each F$\in$ $\mathcal{F}$.

(ii) A closure space (X,c) is said to be compact, if every proper filter of sets on X has a cluster point in X.

Results 3.1.2

(i) For a closure space (X,c) to be compact, it is necessary and sufficient that every interior cover $\mathcal{V}$ of (X,c) has a finite subcover.

(ii) Any image under a c-morphism of a compact space (X,c) is compact.

(iii) If (Y,c) is a compact subspace of a Hausdorff closure space (X,c), then Y is closed in (X,c).

(iv) Every closed subspace of a compact space (X,c) is compact.

Result 3.1.3

If (X,c) is compact, then (X,t) is compact.

Proof

Let (X,c) be compact. Then every proper filter of sets on X has a cluster point in X. Let $\mathcal{F}$ be a proper filter of sets on X and x be a cluster point. Then $x \in \cap (cF)$, $F \in \mathcal{F}$. That is $\cap (cF) \neq \phi$ but $cF \subset clF$ for every $F \in \mathcal{F}$. Then $\cap (clF) \neq \phi$. So (X,t) is compact.

Note 3.1.4

The converse of the above result is not true.

Example

Consider $X = N \times N \cup \{x,y\} \cup \{ a_i : i \in N\} \cup \{b_j : j \in N\}$,

a_i's, b_j's, x,y are all distinct and do not belong to N x N.

Let c be defined on X as in Example 2.1.11

Let $A_k = \{(m, m) : m \geq k \}$ for $k \in N$.

The family $\mathcal{F} = \{A_k : k \in N \}$ is a filter base.

$cA_k = A_k$, for every $A_k \in \mathcal{F}$ but $\cap^{\infty}_{k=1} cA_k = \phi$.

So (X,c) is not compact. But (X,t) is compact as can be proved easily.

Result 3.1.5

Any image under a c -c' morphism of a compact closure space (X,c) is compact in the associated topology of c'.

Using the Result 3.1.2 (ii) and the Result 3.1.3, we get this result.

Note 3.1.6

If (X,cl) is compact and f : (X,cl) ––> (Y,c') is a surjective c- c' morphism, then (Y,c') need not necessarily be compact.

Result 3.1.7

The associated space (Y,t') of a compact closure space (Y,c') is closed as a subspace of the Hausdorff space (X,c)

Using the Result 3.1.2 (iii) and cA=X ==>clA=X, we get the above result.

Result 3.1.8

Every closed closure subspace of an associated topological space (X,t) of a compact closure space (X,c) is compact.

Proof

Let (Y,c') be a closed subspace of a compact space (X,t). Let $\mathcal{F}$ be a proper filter on (Y,c'). Let us consider the smallest filter $\mathcal{G}$ on X containing $\mathcal{F}$. $\mathcal{F}$ is a filter base for $\mathcal{G}$. Since clY=Y, we have c'A=clA for each A⊂Y and hence ∩(c'F)=∩(cl F). Therefore ∩ (cl F)=∩(cl G). Since (X,t) is compact ∩ (cl G)≠ϕ. That is ∩(c' F)≠ϕ.

Corollary 3.1.9

Closed subspace (Y,t') of compact space (X,c) is compact.

Result 3.1.10

(X,c) is compact. $Y \subset X$. Then cY is compact.

Proof

Let c' be the closure on cY induced by c. Let $\mathcal{F}$ be a filter on cY. We have to prove that $\cap(c' F)$, $F \in \mathcal{F}$ is nonempty. $\{cF \cap cY : F \in \mathcal{F}\}$ is a filter base on X. Since X is compact, $\cap(cF \cap cY)$ is nonempty. So $\cap\, c' F = \cap(cF \cap cY) \neq \phi$.

Definition 3.1.11

A closure space (X,c) is locally compact if and only if each point in X has a neighbourhood base consisting of compact sets.

Note 3.1.12

(X,c) is locally compact does not imply that (X,t) is locally compact and vice-versa.

Result 3.1.13

Let (X,c) be locally comapct. If f is an open c-c' morphism from (X,c) onto (Y,c'), then Y is locally compact.

Proof

Suppose $y \in Y$. Let V be a neighbourhood of y. Take $x \in f^{-1}(y)$. Since f is c-c' morphism and X is locally compact, we can find a compact neighbourhood U such that $f(U) \subset V$. $x \in Int_x U$ so $y \in f(Int_x U) \subset f(U)$. Since f is open, $f(Int_x U)$ is a neighbourhood of g. Hence f(U) is a compact neighbourhood of y contained in V.

3.2 CONNECTEDNESS IN CLOSURE SPACES.

In this section we introduce and study connectedness.

Definition 3.2.1

(X,c) is said to be disconnected if it can be written as two disjoint nonempty subsets A and B such that $cA \cup cB=X$, $cA \cap cB = \phi$ and cA and cB are nonempty. A space which is not disconnected is said to be connected.

Example 3.2.2

$X = \{a,b,c\}$

c can be defined on X such that

$c\{a\}=\{a,b\}$, $c\{b\}=c\{c\}=c\{b,c\}=\{b,c\}$, $c\{a,b\}=c\{a,c\}=cX=X$, $c\phi=\phi$

Then c is a closure operation on X.

Here (X,c) is connected because we can not find nonempty subsets A and B such that $cA \cup cB=X$ and $cA \cap cB=\phi$.

Definition 3.2.3

(X,c) is said to be feebly disconnected if it can be written as two disjoint nonempty subsets A and B such that $A \cup cB=cA \cup B=X$ and $cA \cap B=\phi=A \cap cB$.

Note 3.2.4

It is clear that (X,c) is disconnected implies (X,c) is feebly disconnected. The following example shows that the converse is not true.

Example 3.2.5

$X=\{a,b,c\}$

$c\{a\}=\{a,c\},c\{b\}=c\{c\}=c\{b,c\}=\{b,c\},c\{a,b\}=c\{a,c\}=cX=X,c\phi=\phi$

c is a closure operation on X.

Here (X,c) is feebly disconnected, but not disconnected.

Result 3.2.6

(x,t) is disconnected ==> (X,c) is disconnected.

Proof

(X,t) is disconnected implies that it is the union of two disjoint nonempty subsets A and B such that $clA \cup clB=X$, $clA \cap clB=\phi$ and clA, clB are nonempty. $clA \cap clB=\phi$. So $cA \cap cB=\phi$. That is (X,c) is disconnected.

Note 3.2.7

(X,t) is connected need not imply that (X,c) is connected.

Example

X={a,b,c} . Let c be a closure operation defined on X in such a way that

c{a}={a},c{b}={b,c},c{c}=c{a.b}=c{b,c}=cX=X, cϕ=ϕ

(X,t)={X,ϕ,{b,c}}

Here (X,c) is disconnected, but (X,t) is connected.

Remark

Connectedness of a subspace Y of (X,c) can be defined in the same manner.

Note 3.2.8

Let (X,c) be a closure space and Y be a connected subset of (X,c). Then cY need not be connected.

Example 3.2.9

X={a,b,c,d,e}

Let c be defined on X such that

c{a}={a},c{b}={a,b,c},c{c}={b,c},c{d}={b,c,d},

c{a,b}=c{a,c}=c{b,c}=c{a,b,c}={a,b,c},

c{c,d}={b,c,d}, c{a,d}=c{b,d}=c{a,b,d}=c{a,c,d}=c{b,c,d}=c{a,b,c,d}={a,b,c,d},

c{e}=c{a,e}=c{b,e}=c{c,e}=c{d,e}=c{a,b,e}=c{a,c,e}=c{a,d,e}=c{b,c,e}=c{c,d,e}

=c{b,d,e}=c{a,b,d,e}=c{a,c,d,e}=c{b,c,d,e}=cX=X,cϕ=ϕ

Here Y={b,c} is connected.

$cY=\{a,b,c\}$; if c' isthe induced closure operation on cY, then

$c'\{a\}=\{a\}, c'\{c\}=\{b,c\}, c'\{b\}=c'\{a,b\}=c'\{b,c\}=c'\{a,c\}=c'cY=cY$.

cY is disconnected.

Note 3.2.10

If cA and cB form a separation of X and if Y is a connected subset of X, then Y need not be entirely within either cA or cB.

Example 3.2.11

$X=\{a,b,c\}$

Let c be a closure operation defined on X such that

$c\{a\}=\{a\}, c\{b\}=\{b,c\}$ $c\{c\}=\{a,c\}, c\{a,b\}=c\{b,c\}=cX=X, c\{a,c\}=\{a,c\}$.

$Y=\{a,c\}$ is connected ·

Note 3.2.12

The image of a connected space under a c -c' morphism need not be connected.

Example

Let $X=\{a,b,c,d,e\}$. A closure operation c is defined on X as in Example 3.2.9

Let $Y=\{a,b,c\}$

c' be defined on Y such that

$c'\{a\}=\{a\}, c'\{b\}=\{b,c\}, c'\{c\}=c'\{a,b\}=c'\{b,c\}=c'\{a,c\}=c'X=X$ $c'\phi=\phi$.

Let f be a map from (X,c) into (Y,c') defined in such a way that f(a)=a, f(b)=c, f(c)=b, f(d)=c, f(e)=c

Here f is a c-c' morphism. But f(X) is disconnected.

Result 3.2.13

Suppose c_1 is a closure operator on Y with degree k and f is a c-c_1 morphism from (X,c) to (Y,c_1). If $c_1^k(A)$ and $c_1^k(B)$ form a separation of Y, then $c(f^{-1}(c_1^k(A))$ and $c(f^{-1}(c_1^k(B))$ form a separation on X.

Proof

Let $c_1^k(A) \cup c_1^k(B)=Y$ and $c_1^k(A) \cap c_1^k(B)=\phi$.

Then $f^{-1}(c_1^k(A)) \cup f^{-1}(c_1^k(B))=X$

That is $c(f^{-1}(c_1^k(A)) \cup c(f^{-1}(c_1^k(B))=X$, since $f^{-1}(c_1^k(A)) \subset c(f^{-1}(c_1^k(A))$

$f^{-1}(c_1^k(A)) \cap f^{-1}(c_1^k(B))=\phi$.

But $c(f^{-1}(c_1^k(A)) \subset f^{-1}(c_1(c_1^k(A))=f^{-1}c_1^{k+1}(A)=f^{-1}c_1^k(A)$

In similar manner

$c(f^{-1}(c_1^k(B)) \subset f^{-1}(c_1^k(B))$. Therefore, $c(f^{-1}(c_1^k(A)) \cap c(f^{-1}(c_1^k(B))= \phi$

Hence $c(f^{-1}(c_1^k(A))$ and $c(f^{-1}(c_1^k(B))$ form a separation on X ·

Result 3.2.14

Let (X,c) be connected and f is a c-c_1 morphism from (X,c) on to (Y,c_1). Then (Y,t_1) is connected.

Proof

Since $f(cA) \subset c_1 f(A) \subset cl_1 f(A)$, f being c -c_1 morphism and we get f is c - cl_1 morphism. Suppose cl_1 A and cl_1 B form a separation on Y . Then $cl_1 A \cup cl_1 B=Y$ and $cl_1 A \cap cl_1 B=\phi$. $f^{-1}(cl_1 A) \cup f^{-1}(cl_1 B)=X$ and $f^{-1}(cl_1 A) \cap f^{-1}(cl_1(B)=\phi$. By the above result $c(f^{-1}(cl_1(A))$ and $c(f^{-1}(cl_1 B))$ form a separation on X. This is a contradiction . Hence (Y,t) is connected.

3.3 PATHWISE AND LOCAL CONNECTEDNESS

In this section we define and study pathwise connectedness and local connnectedness.

Definition 3.3.1

A space (X,c) is pathwise connected if and only if for any two points x and y in X, there is a cl_1 -c morphism f : I —>X such that f(o) =x and f(1)=y where cl_1 is the usual closure on I, f is called a path from x to y.

Result 3.3.2

(X,c) is pathwise connected implies (X,t) is pathwise connected.

Proof

If (X,c) is pathwise connected, then for any two points x and y in X there is a cl_1-c morphism f : I—> X such that f(0)=x and f(1)=y. If f is cl_1-c morphism,then f is cl_1-cl morphism. Therefore (X,t) is pathwise connected.

Note 3.3.3

The converse of the above result is not true.

Note 3.3.4

Pathwise connected space need not be a connected space .

Definition 3.3.5

A space X is said to be locally connected at x if for every neighbourhood U of x, there is a connected neighbourhood V of x contained in U. If X is locally connected at each of its points, then X is said to be locally connected.

Definintion 3.3.6

A space X is said to be locally path connected at x if for every neighbourhood U of x, there is a path connected neighbourhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be locally path connected.

Note 3.3.7

A space (X,c) is locally connected need not imply that (X,t) is locally connected and vice-versa.

A parallel study of the above concepts in the set up of closure spaces is interesting; however we are not attempting it in this thesis.

3.4. COMPACTNESS AND CONNECTEDNESS IN MONOTONE SPACES

Definition 3.4.1

Let (X,c_*) be a monotone space. $\mathcal{F}$ be a proper filter on X and x be an element of X. We shall say that x is a cluster point of $\mathcal{F}$ in (X,c_*) if x belongs to $\cap\{c_*F : F\in\mathcal{F}\}$. That is each neighbourhood of x intersects each $F\in\mathcal{F}$.

Definition 3.4.2

A monotone space (X,c_*) is said to be compact, if every proper filter of sets on X has a cluster point in X.

Remark 3.4.3

It is clear that if (X,c_*) is compact, then (X,c) is compact but the converse is not true.

Result 3.4.4

Any image under a c-c_* morphism of a compact monotone space (X,c_*) onto a monotone space (Y,c_*') is compact.

The proof is similar to the Proof of 41 A-15 in [CE_2].

Result 3.4.5

Every closed subspace of a compact monotone space is compact.

The proof is similar to the Proof of 41 A-10 in [CE_2].

Result 3.4.6

If (Y,c') is a compact subspace of a Hausdorff monotone space (X,c_*),then Y is closed in X.

The proof is similar to the Proof of 41 A-11 in [CE_2] .

Definition 3.4.7

A monotone space (X,c_*) is said to be disconnected if it can be written as two disjoint nonempty subsets A and B such that $c_*A \cup c_*B = X$, $c_*A \cap c_*B=\phi$. A space which is not disconnected is said to be connected.

Remark 3.4.8

(X,c) is disconnected implies (X,c_*) is disconnected,and the converse is not true.

Example 3.4. 9

$X=\{a,b,c\}$

c_* be defined on X such that

$c_*\{a\}=\{a\}, c_*\{b\}=\{b,c\}$, $c_*\{c\}=\{b,c\}$, $c_*\{a,b\}=c_*\{b,c\}=c\{a,c\}=c_*X=X, c_*\phi=\phi$

c_* is a monotone operator.

(X,c_*) is disconnected. But (X,c) is connected.

CHAPTER 4

CLOSURE SEMIGROUPS

INTRODUCTION

In this chapter we introduce closure semigroups and study some of their properties. The concepts like homomorphism,congruences and products in the context of closure semigroups are discussed.

A topological semigroup is a Hausdorff space S with continuous associative multiplication (x,y) -> xy of S x S into S. The study of topological semigroups was initiated perhaps by A.D. Wallace during fifties; it was continued by others like K.H.Hofmann, P.S. Mostert, A.B. Paalman De Miranda and Hewitt and during these years the subject has developed in many directions.

The notion of a topologized algebraic structure was studied [CE_2] by Čech. According to Čech, a topological group is a triple (G, σ, u) where G is a set, u is a closure on G and σ is a mapping satisfying the following conditions :

(1) σ : ((G, u) x (G,u)) -> (G,u) is "continuous"

(2) The mapping x -> x^{-1} : (G,u) -> (G,u) is "continuous"

In the terminology that we are going to introduce here, Čech's topological group should have been called a closure group. He proved that every "topological group" is a topological space. That is the underlying space of a "topological group" is a topological space. He also studied topological rings and fields.

In section 1 of this chapter we define closure semigroups and prove the non-relation between a closure semigroup and the associated topological semigroup. A preliminary study of closure semigroups is also attemped here.

The concepts of homomorphisms and congruences in the context of closure semigroups are studied in section 3. The study of closure semigroups that we do here is somewhat on the same lines as the development of the theory of topological semigroups as is available in [CA-H-K_1].

4.1.BASIC CONCEPTS

In this section we define and study some properties of closure semigroups.

Definitions 4.1.1

A closure semigroup is a nonempty set S together with an associative multiplication (x,y)—>xy from ((SxS),c x c) into (S,c) which is a c x c-c morphism.

Example 4.1.2

Let X=N.

Let c be defined on X such that $cA = A \cup \{x+1 : x \in A\}$

Then c is a closure operation on X. Here X is a semigroup under the operation(x,y)—>max{x,y}. Since the map(x,y) —>max{x,y} is c x c-c morphism,(X,c) is a closure semigroup.

Note 4.1.3

If A and B are subsets of a closure semigroup (S,c), we use the notation $AB=\{ab:a \in A \text{ and } b\in B\}$.

Remark

A closure semigroup need not be a topological semigroup in the associated topology.

Example 4.1.4

Consider $X=\{a,b,c\}$.

Let c be defined on X such that

$c\{a\}=\{a\}$, $c\{b\}=\{b,c\}$, $c\{c\}=\{a,c\}$, $c\{a,c\}=\{a,c\}$, $c\{a,b\}=c\{b,c\}=cX=X$, $c\phi=\phi$

Then c is a closure operator on X

$X \times X=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

$(X,t)=\{X,\phi,\{b\},\{b,c\}\}$

$(X,t) \times (X,t)=\{\phi,X\times X,\{(b,b)\},\{(b,b),(b,c)\},\{(b,b),(c,b)\},\{(b,b),(b,c),(c,b),(c,c)\}\}$

(See Appendix for details)

It is a closure semigroup under the binary composition $(x,y) \longrightarrow x$, but not a topological semigroup. Since the inverse image of $\{b\}$ is not open in $(X,t) \times (X,t)$.

Remark

A semigroup which is also a closure space and is a topological semigroup under the associated topology need not be a closure semigroup.

Example 4.1.5

Let X=N

c be defined on X such that $cA=A \cup \{x+1 : x \in A\}$

Then c is a closure operator on X.

X is a semigroup under (x,y) --->x+y. It is a topological semigroup. But it is not a closure semigroup.

For, (4,2)--->6

Let W={5,6} be a neighbourhood of 6.

We cannot find neighbourhoods U,V for 4 and 2 such that $f(U \times V) = U+V \not\subset W$.

Definition 4.1.6

A subgroup G of a closure semigroup is a closure group if the map x --->x^{-1} sending x to its inverse is c-c morphism on G.

In this context, it will be interesting to note the following :-

Result 4.1.7 [CE_2]

Every closure group is a topological space; more precisely the underlying closure space of a closure group is topological.

Theorem 4.1.8

Let $(X,c_1),(Y,c_2),(Z,c_3)$ be closure spaces. Let A be a compact subset of X,B a compact subset of Y, $f : X \times Y \to Z$ a $c_1 \times c_2$-c_3 morphism and W a neighbourhood of f(A x B) in Z. Then there exist neighbourhoods U of A in X and V of B in Y such that $f(U \times V) \subset W$.

Proof

f is $c_1 \times c_2$-c_3 morphism and W is neighbourhood of f(A x B). Then $f^1(W)$ is a neighbourhood of A x B in X x Y. For each (x,y) in A x B there exist neighbourhoods M of x and N of y such that $M \times N \subset f^{-1}(W)$. Since B is compact, for fixed $x \in A$, there is finite interior cover $M_1, M_2 \ldots M_n$ in X containing x and correspondingly $N_1, N_2 \ldots N_n$ in Y such that $B \subset N_1 \cup \ldots \cup N_n = Q$. Let $P = M_1 \cap \ldots \cap M_n$. Then P is a neighbourhood of B in Y and $P \times Q \subset f^{-1}(W)$. Since A is compact there exists a finite interior cover $P_1 \ldots P_m$ in X and correspondingly $Q_1 \ldots Q_m$ in Y such that $B \subset Q_1 \cap \ldots Q_m$ and $A \subset P_1 \cup \ldots \cup P_m$. Let $U = Q_1 \cap \ldots \cap Q_m$ and $V = P_1 \cup \ldots \cup P_m$. U and V are neighbourhoods of A and B in X and Y and $U \times V \subset f^{-1}(W)$. $f(U \times V) \subset W$.

Result 4.1.9

Let A and B be subsets of a closure semigroup. If A and B are compact, then AB is compact.

Proof

AB = {ab: aA,b$\in$B}, AB = f(A x B). Since S is a closure semigroup and f : S x S--> S is a c x c-c morphism, then f(A,B) =AB is compact.

Note 4.1.10

If A and B are connected subsets of closure semigroup S, then AB need not be connected.

Definition 4.1.11

An element e of a semigroup S is called an idempotent if e^2=e and the set of idempotents of S is denoted E (S) .

Theorem 4.1.12 [CE_2]

A closure space (X,c) is Hausdorff if and only if the diagonal $\triangle$ ={(x,x):x$\in$X} of X x X is closed in the product space X x X.

Result 4.1.13

Let (X,c) be a Hausdorff space and f :X-->X a c-morphism. Then the set of fixed points of f is closed in X.

Proof

Let f : (X,c) -->(X,c) be a c - morphism. Define g : X--> X x X by g(x)=(f(x),x). Then g is c-cl morphism. For this,

Let g(A) =(f(A),A) where (f(A),A)= {(f(x),x) : x $\in$ A}

$$g(cA) = (f(cA),cA) \subset (c\, f(A),cA) \subset (cl\, f(A),cl\, A) = cl(f(A),A) = clg(A)$$

Therefore g is c-cl morphism.

Let D = {x : f(x)=x}. We have to prove that D is closed. That is cD=D.

Let y ϵcD. Then $g(cD)=g\, c(g^{-1} \triangle)$ where $\triangle$ is the diagonal elements in XxX

$$\subset cl(gg^{-1} \triangle) \subset cl\triangle = \triangle$$

$$g(cD) \subset \triangle$$

$$g(y) = (f(y),y) \in \triangle$$

That is, f(y) =y, yϵD

Therefore cD=D. D is closed. Hence the set of all fixed points is closed .

Results 4.1.14

If S is a closure semigroup, then E (S) is a closed subset of S.

Proof

E(S) is the set of fixed points of the c -morphisms $x \longrightarrow x^2$. By the above result E(S) is closed.

Result 4.1.15

Let S be a closure semigroup. For e, fϵE(S), define $e \leq f$ if ef=fe=e. Then $\leq$ is a partial order on E and is a closed subspace of S x S.

The proof is similar to the Proof of 1.6 as in [CA-H-K_1]

Definition 4.1.16

If S is a semigroup and a$\in$S, then the function x-->xa is called right translation by a and is denoted by ρ_a and x -->ax is called left translation by a and is denoted by λ_a.

Definition 4.1.17

An element e of a semigroup S is called a left identity for S if ex=x for all x$\in$S, a right identity for S if xe=x for all x$\in$S and identity for S if e is both left and right identity. A semigroup which has an identity is called a monoid.

Definition 4.1.18

A subsemigroup of a closure semigroup S is a nonempty subset T of S such that $T^2 \subset T$ (That is $TT \subset T$).

Result 4.1.19

Let (S,c) be a closure semigroup . T be a subspace of S whcih is also a subsemigroup. Then T is also a closure semigroup.

Proof

Let (S,c) be a closure semigroup. Then the multiplication f : (S,c) x (S,c) is a c x c-c morphism.

$T \subset S$, $c'A = T \cap cA$ for each $A \subset T$

c' is closure operation on T

Then the multiplication (T,c') x (T,c')--> (T,c') is a c' x c'-c' morphism.

Example 4.1.20

Consider X=N

c be defined on X such that $cA=A \cup \{x+1 : x \in A\}$

Y be the set of all even natural numbers

$c'A=Y \cap cA = A$, for every $A \subset Y$.

Thus c' is the discrete closure operation on Y and (Y,c') is a closure subsemigroup.

4.2. IDEALS OF A CLOSURE SEMIGROUP

In this section we define and study ideals in closure semigroups.

Definition 4.2.1

A nonempty subset L of a semigroup S is a left ideal of S if $SL \subset L$.

Note 4.2.2

Here we consider only the case of left ideals. In a similar manner we can consider the case of right ideals and ideals.

Result 4.2.3

Let L be a left ideal of a closure semigroup S. Then cL is a left ideal of S.

Proof

Let L be a left ideal of S. We have to show that $ScL \subset cL$

$ScL = cScL$, Since $cS = S$.

$\subset c(SL)$, since multiplication is a morphism and $f((cA) \subset cf(A)$

$\subset cL$, since L is left ideal and $SL \subset L$

Therefore $ScL \subset cL$.

Note 4.2.4

If A is a subset of a semigroup S, then we denote $L(A)=S'A=A \cup SA$.

Result 4.2.5

Let S be a compact semigroup and let A be a compact subspace of S. Then $L(A)=A \cup SA$ is compact.

Corollary 4.2.6

Let S be a compact semigroup and let $a \in S$. Then $L(a)$ is compact.

Definition 4.2.7

A left ideal of a semigroup S is called a minimal left ideal if it properly contains no other left ideal.

Result 4.2.8

If S is a compact semigroup, then each minimal left ideal is compact.

Proof

Let L be a minimal left ideal of S and $x \in L$. Then Sx is a left ideal of S and $Sx \subset L$. Since L is minimal,we get $L = Sx$.

But $Sx = x \cup Sx$, since $x \cup Sx$ is left ideal and $Sx = L$ is minimal.

Thus $L = x \cup Sx = L(x)$ and by corollary 4.2.6, we get L is compact.

As in the case of a topological semigroup, we can find the cartesian product of a collection of closure semigroups.

Definition 4.2.9

Let $\{S_i\}_{i \in I}$ be a collection of closure semigroups. Then coordinatewise multiplication on $\Pi\{S_i\}_{i \in I}$ is given by $(fg)(j) = f(j)g(j)$.

Result 4.2.10

Let $\{S_i\}_{i \in I}$ be a collection of closure semigroups and $S = \Pi\{S_i\}_{i \in I}$. Then S with coordinatewise multiplication and product closure is a closure semigroup and each projection $\pi_j : S \longrightarrow S_j$ is an onto c-c_j morphism.

Proof

Let $\{S_i\}_{i \in I}$ be a collection of closure semigroups. In each S_i, the multiplication is associative. Therefore the multiplication is associative in S.

$\pi_j(xy) = (xy)_j = x(j)\, y(j) = \pi_j(x)\pi_j(y)$. Thus each π_j is a homomorphism.

That is multiplication on S is a morphism follows from the fact that its composition with each projection is a morphism.

Result 4.2.11

Let $\{S_i\}_{i \in I}$ be a collection of semigroups and let $S = \Pi\{S_i\}_{i \in I}$. Then,

(1) If $e_i \in Si$ for each $i \in I$ and $e \in S$ is defined by $e(i)=e_i$ for each $i \in I$, then $e \in E(S)$ if and only if $e_i \in E(S_i)$ for each $i \in I$.

(2) If $A_i \subset S_i$ for each $i \in I$, then $\Pi\{A_i\}_{i \in I}$ is a left ideal of S if and only if A_i is a left ideal of S_i for each $i \in I$.

The proof is analogous to the Proof of theorem 2.2 of [CA-H-K_1].

4.3. CONGRUENCES IN CLOSURE SEMIGROUPS

In this section we introduce the concept of congruences in closure semigroups analogous to that in topological semigroups. Recall that if S and T are semigroups, a function Φ:S—>T is called a homomorphism if $\Phi(xy)=\Phi(x)\,\Phi(y)$ for each $x,y \in S$. If Φ is surjective (onto) then Φ is called a surmorphism, If Φ is also injective then Φ is called an algebraic isomorphism and S and T are said to be algebraically isomorphic.

Definition 4.3.1

If S and T are closure semigroups and Φ: S—> T is both an algebraic isomorphism and a c- homomorphism, then Φ is called a closure isomorphism and S and T are said to be closurewise isomorphic.

Definition 4.3.2

A relation R on a semigroup S is said to be left compatible, if $(a,b) \in R$ and $x \in S$ implies that $(xa,xb) \in R$.

In a similar manner, we can define right compatible and compatible.

Definition 4.3.3

A compatible equivalence on a semigroup S is called a congruence on S. An equivalence R on a semigroup S is a congruence if and only if $(a,b)\in R$ and $(c,d)\in R$ imply $(a.c,b.d)\in R$.

Definition 4.3.4

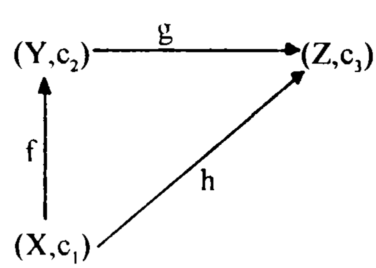
If R is an equivalence on a set X and $x \in X$ then, $\{y \in X: (x,y) \in R\}$ is called the R -class of X containing x. The set X/R of R- classes is called the quotient of X mod R and the function π: X-->X/R which assigns to each x in X the R-class containing x is called the natural map. π is onto map. The set $\pi^{-1}(\pi(x))$ is the R class of x.

Definition 4.3.5

If S and T are semigroups and Φ : S—> T is a homomorphism, we denote by $k(\Phi)$ the relation $\{(x,y) \in S\times S : \Phi(x)=\Phi(y)\}$.

Result 4.3.6

Let X,Y and Z be spaces and let f,g and h be functions such that the following diagram commutes:



(a) If f is quotient and h is c_1-c_3 morphism, then g is c_2-c_3 morphism and

(b) If both f and h are quotient, then g is quotient.

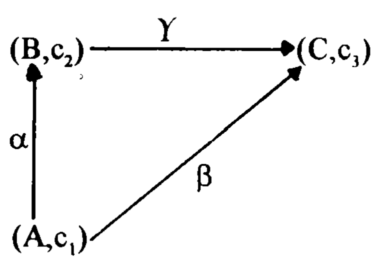
Proof

(a) Let $x \in Y$, $y \in Y$ and $y \in g^{-1}h(x)$ and $x \in f^{-1}(y)$. Let U be a c_3- neighbourhood of h(x). Then $h^{-1}(U) = f^{-1}(g^{-1}(U))$ is a c_1 neighbourhood of $x \in X$. Since f is quotient $g^{-1}(U)$ is a c_2- neighbourhood of $y \in Y$. Then g is c_2- c_3 morphism.

(b) Let $y \in Y$. U be a subset of Z containing g(y) such that $g^{-1}(U) = f^{-1}(g^{-1}(U))$ is a c_1- neighbourhood of $x \in X$, since f is a c_1 -c_2 morphism. Since h is quotient, U is a c_1 - neighbourhood of g(y). By (a), g is a c_1 - c_3 morphism. Then g is quotient by the result 0.3.17

Result 4.3.7

Let (A, c_1), (B, c_2) and (C, c_3) be closure semigroups, $\alpha : A \dashrightarrow B$ a quotient surmorphism and $\beta : A \dashrightarrow C$ a c_1 - c_3 homomorphism such that $k(\alpha) \subset k(\beta)$. Then there exists a unique c_2- c_3 homomorphism $\gamma : B \dashrightarrow C$ such that the following diagram commutes.

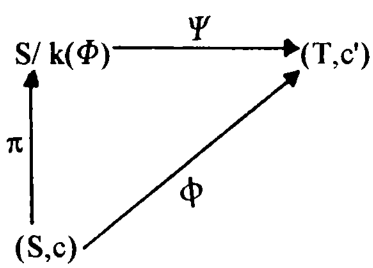


Proof

Define $\gamma(x) = \beta(\alpha^{-1}(x))$ for each $x \in B$. γ is a homomorphism. By the above result, γ is c_2-c_3 morphism.

Result 4.3.8 [CA -H -K$_1$]

Let S and T be semigroups and let $\Phi : S \rightarrow T$ be a surmophism. Then $k(\Phi)$ is a congruence on S and there exists a unique algebraic isomorphism $\Psi : S/ k(\Phi) \rightarrow T$ such that the following diagram commutes.



Result 4.3.9

If (S,c_1) and (T,c_2) are Hausdorff closure semigroups and $\Phi : S \rightarrow T$ is a c_1- c_2 morphism, then $k(\Phi)$ is a closed congruence on S and the following are equivalent :

(a) Ψ^{-1} is a closure morphism

(b) Ψ is a closure isomorphism

(c) Φ is a quotient.

Finally if any one of these conditions is satisfied, then $S/ k(\Phi)$ is a closure semigroup.

Proof

By the above result, $k(\Phi)$ is a congruence on S.

$\Phi : S \dashrightarrow T$ is a c_1 -c_2 morphism. $\pi : S \dashrightarrow S / k(\Phi)$ is a natural map.

$(k(\Phi) = \{ (x,y) \in S \times S : \Phi(x) = \Phi(y) \}$, $k(\pi) = \{ (x,y) \in S \times S : \pi(x) = \pi(y) \}$. It is clear that $k(\pi) = k(\Phi)$. Using the Result 4.3.6 and $k(\pi) = k(\Phi)$, we obtain a c_1-c_2 morphism such that the above diagram commutes.

To prove $k(\Phi)$ is a closed congruence

$k(\Phi) = (\Phi \times \Phi)^{-1}(\triangle(T))$, $\triangle T$ is closed in T x T.

Then $k(\Phi)$ is closed.

(a) ==> (b)

Ψ^{-1} is a closure isomorphism, π is onto and is a quotient map.

We know that Ψ is algebraic isomorphism.

By 4.3.6 Ψ is a closure isomorphism.

(b) ==> (a)

Ψ is closure isomorphism, π is quotient map. By Result 4.3.6 (b), Φ is quotient.

(c) ==> (a)

Φ is quotient and π is also quotient. Then by Result 4.3.6 (a) Ψ^{-1} is closure morphism.

Finally if these conditions are satisfied,then clearly S/k(Φ) is a semigroup. That is a closure semigroup follows from the fact that (ak(Φ), bk(Φ)) --> abk(Φ) is a morphism.

CHAPTER 5

CLOSURE SEMIGROUP COMPACTIFICATIONS

INTRODUCTION

E. Čech and M.H.Stone gave the concept of maximal compactification βX and stated its fundamental properties. Deleeuw,Glicksberg and Hunter have studied Bohr compactification of topological semigroups having universal properties analogous to those of βX. The theory of Bohr compactification and other types of closure semigroup compactifications seems to have not been attempted by others.

In 1990 K.S.Kripalini defined the semigroup compactification of a topological semigroup. In [KR_1], [KR_2],[KR_3], she proved that if (β,B) is a Bohr compactification of a topological semigroup S and R is any closed congruence on B, then the quotient space B/R is a semigroup compactification of S and conversely any semigroup compactification (α,A) of S is topologically isomorphic to B/R for some closed congruence on B. The lattice structure of the collection of all semigroup compactifications of a topological semigroup are also studied.

In section 1 of this chapter we define in the closure space context the semigroup compactification and Bohr-type compactification and prove the existence of this analogous to Bohr compactification of a topological semigroup. We find also the relation between Bohr-type compactification and other semigroup compactifications. In section 2, we define an order between closure semigroup compactification and find that it is a partial order. We also prove that $k_1(S)$, the set of all closure semigroup compactifications is an upper complete semilattice.

5.1. SEMIGROUP COMPACTIFICATIONS

Definition 5.1.1

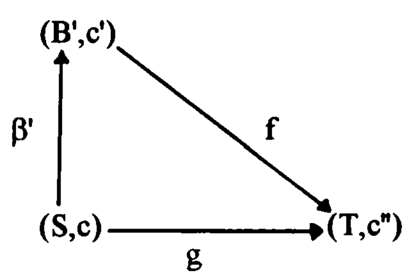
A semigroup compactification of a closure semigroup (S,c) is an ordered pair (g,(T,c')) where (T,c') is a compact semigroup and g : S—>T is a dense c-c' homomorphism of S into T.

Definition 5.1.2

A subset A of (X,c) is said to be dense in (X,c) if cA=X.

Definitiom 5.1.3

If (S,c) is a closure semigroup,then a Bohr-type compactification is a pair (β',(B',c')) such that (B',c') is compact, β': S —> B' is a dense c-c' homomorphism and if g:(S,c)—>(T,c") is a c-c" homomorphism of S into a compact semigroup (T,c"), then there exists a unique c'-c" homomorphism f : (B'-c") —> (T,c") such that the following diagram commutes.



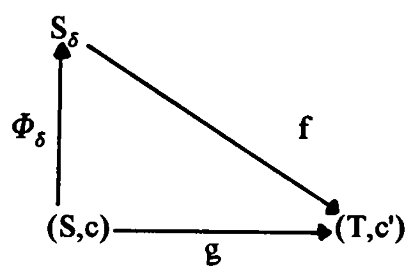
Result 5.1.4

Let D be a dense subset of a Hausdorff space X. Then card X $\leq 2^{2^{D}}$.

The proof is similar to the Proof of 2.42 in [CA - H -K_1].

Result 5. 1.5

Let (S,c) be a closure semigroup. Then there exists a collection $\{(\Phi_\alpha, S_\alpha)\}:\alpha \in A\}$ such that S_α is a compact semigroup and $\Phi_\alpha : S \longrightarrow S_\alpha$ is a dense c - c_2 homomorphism for each $\alpha \in$ A, and if g : S —> T is dense c - c' homomorphism of S into compact semigroup (T,c'), then there exists $\delta \in$ A and a closure isomorphism f : $S_\sigma \longrightarrow$ T such that the following diagram commutes.



The proof is similar to the Proof of 2.43 in [CA -H-K_1].

Result 5.1.6

If (S,c) is a closure semigroup, then there exists a Bohr-type compactification (β',(B',c')) of S.

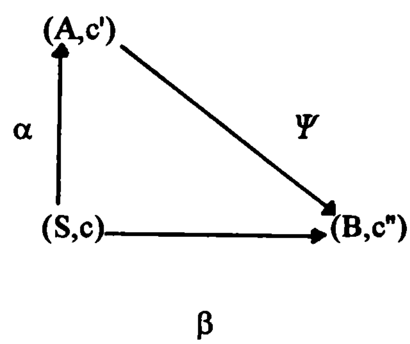
Proof

Let $\{S_\alpha\}_{\alpha \in A}$ be the collection of Result 5.1.5

Define $\sigma : S \longrightarrow \Pi \{S_\alpha\}_{\alpha \in A}$ such that $\sigma(x)(\alpha) = \Phi_\alpha(x)$. σ is c -Πc_α homomorphism. (cf. Theorem 17.c .10 of [CE_2]). Let $B' = c\sigma(S)$. Define $\beta' : S \longrightarrow B'$ so that $\beta'(x) = \sigma(x)$ for each $x \in S$. Then β' is a dense morphism. Suppose $g:(S,c) \longrightarrow (T,c'')$ is a c - c'' homomorphism. With no loss of generality, we can assume that g is dense. Then by 5.1.5, there exists $\delta \in A$ and a closure isomorphism $h : S_\delta \longrightarrow T$ such that $h \circ \Phi = g$. Define $f : B' \longrightarrow T$ by $f = h \circ \pi_\delta$. Then $f \circ \beta = g$ and f is a c'-c'' homomorphism. Since $\beta' : S \longrightarrow B'$ is dense, f is unique.

Result 5.1.7

Let (S,c) be a closure semigroup and let $(\alpha,(A,c'))$ and $(\beta',(B',c'))$ be Bohr-type compactifications of S. Then there exists a c'-c'' isomorphism $\Psi : A \longrightarrow B$ such that the following diagram commutes.



Proof

Since $(\alpha,(A,c'))$ is a Bohr-type compactification of (S,c), there exists a c' - c" homomorphism Ψ: $(A,c') \longrightarrow (B,c")$ such that the diagram commutes. That is $\beta=\Psi \circ \alpha$. Similarly since $(\beta',(B',c"))$ is a Bohr-type compactification of (S,c), there exists a c"-c' homomorphism $\Phi : (B,c") \longrightarrow (A,c')$ such that the diagram commutes. $\alpha = \Phi \circ \beta$

$\Phi \circ \Psi$ is unique . We know that $\Phi \circ \Psi = I_A$ and similarly $\Psi \circ \Phi = I_B$. Then Ψ is c'-c" isomorphism.

Result 5.1.8

Let (S,c) be a closure semigroup with Bohr-type compactification $(\beta',(B',c'))$. If $(\alpha,(A,c"))$ is any semigroup compactification of S, then

(a) there exists a c'-c" homomorphism $\gamma :(B,c') \longrightarrow (A,c")$ such that $\gamma\beta=\alpha$.

(b) The equivalence defined by γ on B is a closed congruence.

Proof

a) By definition of $(\beta,(B,c'))$ there exits a c' -c" homomorphism $\gamma : B \longrightarrow A$ such that $\gamma\beta=\alpha$.

(b) Let R be a relation defined on B by γ , $R = \{(x,y)\in B \times B : \gamma(x) = \gamma(y)\}$ is an equivalence relation. R is a congruence on B. To prove this ,

Let $(x,y), (x',y') \in R$

$(x,y) \in R \Longrightarrow \gamma(x) = \gamma(y)$

$(x',y') \epsilon R \Longrightarrow \gamma(x') = \gamma(y')$

$\gamma(xx') = \gamma(x)\gamma(x')$ since γ is homomorphism.

$= \gamma(y)\gamma(y') = \gamma(yy')$

That is $(xx',yy') \epsilon R$. Hence R is a congruence on B.

Note

When A is Hausdorff then R is closed congruence. For this, γ is a c-c' morphism. Then $\gamma \times \gamma$ is a c-c' morphism.

$R = (\gamma \times \gamma)^{-1}(\triangle(A))$

$\triangle A$ is closed in A x A. Hence R is closed in B x B.

Result 5.1.9

Let (S,c) be a Hausdorff closure semigroup with Bohr-type compactification (β,(B,c')). If R is a closed congruence on (B,c'), then there exists a semigroup compactification (α,A) of S so that the congruence defined by this compactification is R.

Proof

Let R be a closed congruence on B. B is a compact semigroup. Then B/R is a compact semigroup. Define $\gamma : B \longrightarrow B/R$, the natural map. Take A=B/R with the quotient map γ. Define $\alpha : S \longrightarrow A$ such that $\alpha = \gamma \circ \beta$, α is well defined and α is a c-c" morphism. Since γ is c' - c" morphism and β is c -c" morphism. α is dense. For,

$c(\alpha(S)) = c((\gamma \circ \beta)(S))$

$=\gamma(c(\beta(S)))$, [γ is closed].

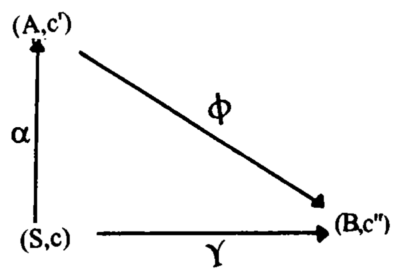
$= \gamma(B) = A$, [γ is surjective].

Thus we have $\alpha : S \longrightarrow A$ is a dense c -c" homomorphism. Therefore (α, A) is a semigroup compactification.

5.2. LATTICE STRUCTURE OF SEMIGROUP COMPACTIFICATIONS

Definition 5.2.1

Two semigroup compactifications (α,(A,c')),(γ,(B,c")) are said to be equivalent if there exists a one-one onto homomorphism $\Phi : A \longrightarrow B$ such that Φ is c' -c" and c"-c' morphisms and commute the following diagram.



Note

(α,(A,c'))$\geq$(γ,(B,c")) if there exists a one-one onto c' -c" homomorphism $f : A \longrightarrow B$ such that $f\alpha = \gamma$.

Result 5.2.2.

Two semigroup compactifications (α,(A,c')) and (γ,(B,c")) are equivalent if and only if (α,(A,c'))$\geq$(γ,(B,c")) and (γ,(B,c"))$\geq$(α,(A,c')).

Proof

Suppose $(\alpha,(A,c'))$ and $(\gamma,(B,c''))$ are equivalent. By the definition of equivalence, there exists a one-one onto homomorphism $\Phi : A \longrightarrow B$ such that Φ is c'-c" and c"-c' morphisms and $\Phi\alpha = \gamma$. Therefore, $(\alpha,(A,c')) \geq (\gamma,(B,c''))$, $\alpha = \Phi^{-1}\gamma$

$\Phi^{-1} : (B,c'') \longrightarrow (A,c')$ is one-one onto c"- c' morphism. Then $(\gamma,(B,c'')) \geq (\alpha,(A,c'))$.

Conversely, suppose that $(\alpha,(A,c')) \geq (\gamma,(B,c''))$ and $(\gamma,(B,c'')) \geq (\alpha,(A,c'))$. Since $(\alpha,(A,c')) \geq (\gamma,(B,c''))$, there exists a one-one onto c'-c" homomorphism $f_1 : A \longrightarrow B$ such that $f_1\alpha = \gamma$. Since $(\gamma,(B,c'')) \geq (\alpha,(A,c'))$ there exists a one-one onto c" -c' homomorphism $f_2 : B \longrightarrow A$ such that $f_2\gamma = \alpha$.

$f_2 \circ f_1$ is a c'-c' morphism. For,

$$f_2 \circ f_1 (c_1A) = f_2 (f_1 (c_1A)) \subset f_2 (c_2(f_1A)) \subset c_1f_2f_1 (A) = c_1 (f_2 \circ f_1 (A))$$

$$f_2 \circ f_1 \circ \alpha = f_2 \circ \gamma = \alpha$$

$f_2 \circ f_1 = I_A$, where I_A is identity on X. Similarly $f_1 \circ f_2 = I_B$.

Note 5.2.3

The collection $k_1(S)$ of all semigroup compactifications of (S,c) is a partially ordered set.

Result 5.2.3

k_1 (S) is an upper complete semilattice.

Proof

To prove k_1 (S) is a upper complete semilattice, we have to prove that the set $\{\alpha_i S\}_{i\in I}$ has a least upper bound with respect to the partial order relation $\geq$. Define $\alpha: S \to \Pi_{i\in I} \{\alpha_i S\}$ by $(\alpha(x))_i = \alpha_i(x)$. Since $\{\alpha_i S\}$ is a subset of $k_1(S)$, each $\alpha_i S$ is a c-c_i morphism. It follows that α is a c-Πc_i morphism. Since product of compact closure spaces are compact [C E_2], $\Pi_{i\in I} \{\alpha_i S\}$ is a compact semigroup. Let $A=c(\alpha(S))$ is a compact semigroup. Therefore $\alpha : S \to A$ is a dense morphism and (α, A) is a semigroup compactification of S.

For each $i\in I$, let $\alpha : A \to \alpha_i S$ be the restriction to A of the projection map.

$(\pi_i \circ \alpha)(x) = (\alpha(x))_i = \alpha_i(x)$ so that $\pi_i \alpha = \alpha_i$

Thus $(\alpha, A) \geq (\alpha_i, \alpha_i S)$ for each $i\in I$

Let $(\alpha_0, \alpha_0 S) \geq (\alpha_i, \alpha_i S)$ for every $i\in I$

$g_i : \alpha_0 S \to \alpha_i S$ is defined by $g_i \alpha_0 = \alpha_i$

Define $f: \alpha_0 S \to, \Pi_{i\in I} \{\alpha_i S\}$ by $(f(y))_i = g_i(y)$

$\pi_i \circ f = g_i$ so that f is c -c' morphism

$f(\alpha_0 x)_i = g_i(\alpha_0(x)) = \alpha_i(x) = (\alpha(x))_i$

From this we get $f\alpha_0 = \alpha$

$$(f\alpha_0)(S) = \alpha(S) = A$$

$$f(\alpha_0 S) \supset f(\alpha_0(S)) = (f\alpha_0)(S) = \alpha(S)$$

Therefore $f(\alpha_0 S) \supset \alpha(S)$

$$c\, f(\alpha_0 S) \supset c\alpha(S) = A$$

$(\alpha_0, \alpha_0 S) \geq (\alpha, A)$

Therefore (α, A) is the least upperbound.

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