A STUDY OF
TRANSLATES OF FUZZY SUBGROUPS

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CERTIFICATE

Certified that the work reported in this thesis is based on the bona fide work done by **Sri. Souriar Sebastian** under my guidance and supervision in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

Dr.S. BABU SUNDAR
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C O N T E N T S

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INTRODUCTION

Fuzziness is an unavoidable feature of most humanistic systems [22] and it cannot be properly studied within the frame-work of classical set theory and two-valued logic. There has been several attempts to develop the tools required for a proper study of this concept. In 1965, Zadeh [57] introduced the notion of fuzzy subsets as a generalisation of the notion of subsets in ordinary set theory. Goguen's [24] notion of L-fuzzy sets further generalises this concept. In [36] Lou and Pan described fuzziness using four axioms called " fuzzy axioms". The theory of fuzzy sets has been developed through a series of papers by Zadeh[57-62] and hundreds of researchers all over the world. It provides the right tool for studying fuzziness and other human-centered systems [11]. Though there are some controversies regarding the utility and essentiality of fuzzy set theory, it has already developed into a theory which challenges the traditional reliance on two-valued logic and classical set theory as a basis for scientific enquiry [22]. This theory has begun to be applied in multitudes of scientific areas ranging from engineering and computer science to medical diagnosis and social behaviour [22]. The theory of fuzzy sets is, in fact, a step towards a rapprochement between the precision of classical mathematics and the pervasive imprecision of the real world [29].

There were several attempts to fuzzify various mathematical structures. The fuzzification of algebraic structures was initiated by Rosenfeld [45]. He introduced the notions of fuzzy subgroupoids and fuzzy subgroups; and obtained some of their basic properties. Though some other definitions of fuzzy subgroups are available in the literature (see, for example, [2], [7], [8] and [17]), Rosenfeld's definition seems to be the most natural and popular one. Most of the recent works on fuzzy groups follow Rosenfeld's definition. Abu Osman ([1], [2]), Ahsanullah and Khan [3-5], Anthony and Sherwood ([9], [51]), Bhattacharya and Mukherjee ([12-14], [41], [42]), Das [19], Dixit, Kumar and Ajmal [21], Liu [35], Sessa [50], Sidky and Mishref[52] and Wetherilt [53] are some of the names associated with recent developments in the theory of fuzzy groups. The names of Bhattacharya and Mukherjee deserve special mention. In a series of papers, they have developed fuzzy parallels of several concepts in classical group theory, and proved fuzzy generalisations of some important theorems, like Lagrange's and Cayley's theorems. Biswas [16] introduced and studied the dual concept of anti-fuzzy subgroups. Studies on some other fuzzy algebraic structures like fuzzy semi-groups, fuzzy rings, fuzzy vector spaces and fuzzy algebras are also available in the literature (see [18], [28], [30-32], [43-44]). Fuzzy topological groups were studied by Foster [23], Höhle [26], Ji Liang Ma and Chun Hai Yu ([38-39], [54-55]).

In this thesis we study the properties of fuzzy groups and their chains of level subgroups. We also study the effect of group homomorphisms on the chain of level subgroups of a fuzzy group, and its fuzzy translates. Some results available in these areas have been generalised, and many new results obtained. Some of the results have been communicated [46-49].

The thesis is divided into five chapters. Definitions, notations and preliminary results from fuzzy set theory, which are required in the sequel, are included in chapter I.

Generalisations of some results of Das [19] are obtained in chapter II. Examples of fuzzy subgroups are constructed to show the existence of fuzzy groups of any finite level cardinality, and of infinite level cardinality. Existence of fuzzy subgroups of a particular level cardinality is characterised in terms of lengths of chains of subgroups. Groups having fuzzy subgroups of every finite level cardinality are identified as those having non-terminating chains of distinct subgroups.

The effect of group homomorphisms on the chains of level subgroups of fuzzy groups is studied in chapter III. If $f:G \longrightarrow G^*$ is a group homomorphism and $F[F^*]$ is a fuzzy subgroup of $G[G^*]$, then $f(F)$ $[f^{-1}(F^*)]$ is a fuzzy subgroup of $G^*[G]$. Level subgroups of a homomorphic pre-image of a fuzzy group can be obtained as pre-images of level subgroups of the fuzzy group. It is observed that the collection of all pre-images need not be distinct. Surjectiveness of f is shown to be a sufficient condition for the distinctness of pre-images. However, this

condition is not necessary. In the case when the collection of pre-images is countable, a necessary and sufficient condition has been arrived at.
Dixit, Kumar and Ajmal [21] have studied the relationship between the chains of level subgroups of a fuzzy group F and its homomorphic image f(F). They assumed that $|Im(F)| < \infty$. We have studied the general case and obtained a condition for the chain of level subgroups of f(F) to be the same as the image of the chain of level subgroups of F under f. This generalises the corresponding theorem in [21]. Further, we have proved that f-invariance of F is a necessary and sufficient condition for the images of level subgroups of F under f to be distinct.

Fuzzy translation operators $T_{\alpha+}$ and $T_{\alpha-}$ are introduced and studied in chapter IV. Fuzzy subgroups and fuzzy normal subgroups are characterised in terms of their fuzzy translates. It is observed that these operators commute with formations of fuzzy conjugates and fuzzy cosets. However, fuzzy translates of a fuzzy abelian subgroup need not be fuzzy abelian. All constant fuzzy groups and their fuzzy translates are fuzzy abelian, if and only if the group is abelian. For a non constant fuzzy subgroup of a non-abelian group we obtain a condition on the translation parameter α, under which fuzzy abelianness becomes translation invariant.

In the final chapter, the effect of the fuzzy translation operators on the chains of level subgroups of fuzzy groups is analysed. It is proved that, in general, the chains of level subgroups of fuzzy translates of a fuzzy group F are subchains of that of F. A sufficient condition on α is found, for which the above chains coincide. The combined action of the fuzzy translation operators and group homomorphism on a fuzzy group is also studied.

All notations used in this work are either standard or explained in the text. A list of notations is provided at the end for easy reference. We shall follow the usual convention of refering to item m.n in chapter p, by m.n within the same chapter; and by p.m.n elsewhere.

CHAPTER I

PRELIMINARIES

1. INTRODUCTION

In 1965, Zadeh [57] introduced the concept of fuzzy subset as a generalisation of the notion of characteristic function in classical set theory. A fuzzy subset A of a non-empty set X is one which is characterised by its membership function.

This chapter contains some definitions and results in fuzzy set theory which are required in the sequel. We assume familiarity with elementary lattice theory as presented by Birkhoff [15] and Davey and Priestly [20]. Throughout this work X denotes an arbitrary non-empty set and I, the closed unit interval [0,1].

2. FUZZY SUBSETS

2.1. Definition [57]. A *fuzzy subset* A of X is a function $A:X \longrightarrow I$. Fuzzy subsets taking the values 0 and 1 only are said to be *crisp.*

2.2. Notations. The following notations are used throughout this work.

2 : $\{0, 1\}$

I^X : Collection of all fuzzy subsets of X

2^X : Collection of all crisp subsets of X

$\tilde{\alpha}$: Constant fuzzy subset taking the value α.

$\vee$: Supremum or lattice join

$\wedge$: Infemum or lattice meet

$|B|$: Cardinality of B

$\mathrm{Im}(A)$: $\{ A(x) : x \in X \}$

$A_\vee$: $\vee \{ A(x) : x \in X \}$

$A_\wedge$: $\wedge \{ A(x) : x \in X \}$

<u>2.3. Definition</u> [56]. If A and B are fuzzy subsets of X, then,

(a) $A \subseteq B$ iff $A(x) \leqslant B(x)$, $\forall\ x \in X$, and

(b) $A = B$ iff $A(x) = B(x)$, $\forall\ x \in X$.

<u>2.4. Definition</u>. For a fuzzy subset A of X, $|\mathrm{Im}(A)|$ is called the <u>level cardinality</u> of A.

<u>2.5. Definition</u> [56]. Let $\{A_j : j \in J\}$ be any collection of fuzzy subsets of X. Their <u>union</u> and <u>intersection</u> are respectively defined by

$$(\bigcup_{j \in J} A_j)(x) = \bigvee \{A_j(x) : j \in J\}, \text{ and}$$

$$(\bigcap_{j \in J} A_j)(x) = \bigwedge \{A_j(x) : j \in J\}, \forall x \in X.$$

2.6. Remark. The relation $\subseteq$ is a partial order on I^X. Further, $(I^X, \subseteq)$ is a complete lattice with $\tilde{1}$ and $\tilde{0}$ respectively as the lattice infinity and zero. Also 2^X is a complete sublattice of it.

2.7. Definition [37]. The fuzzy complement of a fuzzy subset A of X is defined by $A^c(x) = 1 - A(x), \forall\, x \in X$.

2.8. Definition [45]. A fuzzy subset A of X is said to have $\vee$-property if for every $S \in 2^X$ there exists $x_0 \in S$ such that

$$A(x_0) = \bigvee \{A(x) : x \in S\}$$

$\wedge$-property is defined dually.

2.9. Remark. If A is a fuzzy subset of X with $|Im(A)| < \infty$, then A has both $\vee$ and $\wedge$-properties. In general, A has $\vee$-property does not imply that A has $\wedge$-property, and vice versa.

<u>2.10. Example</u>. Let X be the set N of all natural numbers. Define fuzzy subsets A and B by

$$A(x) = \frac{1}{x}, \quad B(x) = 1 - \frac{1}{x}, \forall\, x \in X.$$

Then A has $\vee$-property, but not $\wedge$; and B has $\wedge$-property, but not $\vee$.

3. IMAGES OF FUZZY SUBSETS.

In this section, we consider images and pre-images of fuzzy subsets, and observe some of their properties. Throughout this section X and Y are non-empty sets; $f:X \longrightarrow Y$ is a function; A, A_j are fuzzy subsets of X; and B, B_j are fuzzy subsets of Y.

<u>3.1. Definition</u> [56]. The <u>image</u> of A under f is the fuzzy subset f(A) of Y, defined by

$$f(A)(y) = \begin{cases} \bigvee\{A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

The <u>pre-image</u> of B under f is the fuzzy subset $f^{-1}(B)$ of X given by

$$f^{-1}(B)(x) = B(f(x)), \forall\, x \in X.$$

3.2. Proposition [56]

(a) $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$

(b) $B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

3.3. Proposition [10]

(a) $f^{-1}\left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} f^{-1}(B_j)$

(b) $f^{-1}\left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} f^{-1}(B_j)$

(c) $f\left(\bigcup_{j \in J} A_j\right) = \bigcup_{j \in J} f(A_j)$, and

(d) $f\left(\bigcap_{j \in J} A_j\right) \subseteq \bigcap_{j \in J} f(A_j)$, J being any index set.

3.4. Proposition

$f(f^{-1}(B)) \subseteq B$. If f is a surjection, then $f(f^{-1}(B))=B$.

Proof: Let $y \in Y$. We have two cases.

Case (i). $y \in f(X)$: Then, we have

$$\begin{aligned} f(f^{-1}(B))(y) &= \vee\{B(f(x)) : f(x)=y\} \\ &= B(y). \end{aligned}$$

Case (ii). $y \notin f(X)$: Then $f^{-1}(y)$ is empty, and hence we have,

$$f(f^{-1}(B))(y) = 0 \leqslant B(y)$$

Hence,

$$f(f^{-1}(B)) \subseteq B.$$

If f is a surjection, then case (ii) is absent, and hence $f(f^{-1}(B)) = B$.

3.5. Proposition

$f^{-1}(f(A)) \supseteq A$. If f is an injection, then $f^{-1}(f(A)) = A$.

Proof: For any $x \in X$,

$$\begin{aligned} f^{-1}(f(A))(x) &= f(A)\,(f(x)) \\ &= \bigvee \{A(z) : f(z) = f(x)\} \quad \cdots\cdots\cdots (i) \\ &\geqslant A(x) \end{aligned}$$

Hence,

$$f^{-1}(f(A)) \supseteq A.$$

If f is an injection, then it follows by (i) that

$$f^{-1}(f(A))(x) = A(x), \forall\, x \in X.$$

We proceed to prove that equality of $f^{-1}(f(A))$ and A can be ensured by a weaker condition than injectiveness of f.

<u>3.6. Definition</u> [45]. A is said to be <u>f-invariant</u> if $A(x_1) = A(x_2)$ whenever $f(x_1) = f(x_2)$; $x_1, x_2 \in X$.

<u>3.7. Example</u>. Let $P = \{a,b,c,d\}$ and $Q = N$. Define $g: P \longrightarrow Q$ by $g(a) = g(d) = 5$, $g(b) = 3$, and $g(c) = 4$. Define $F: X \longrightarrow I$ by $F(a) = F(d) = \frac{1}{2}$ and $F(b) = F(c) = \frac{1}{3}$. Then F is g-invariant.

<u>3.8. Remark</u>. If f is an injection, then every fuzzy subset of X is f-invariant. Also, constant fuzzy subsets of X are g-invariant with respect to any function g on X.

<u>3.9. Theorem</u>.

$$f^{-1}(f(A)) = A \iff A \text{ is f-invariant.}$$

<u>Proof</u>: ($\Longrightarrow$): Assume that $f^{-1}(f(A)) = A$. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then,

$$\begin{aligned} A(x_1) &= f^{-1}(f(A))\,(x_1) \\ &= f(A)\,(f(x_1)) \\ &= \bigvee\{A(z) : f(z) = f(x_1)\} \\ &= \bigvee\{A(z) : f(z) = f(x_2)\} \\ &= A(x_2). \end{aligned}$$

($\Leftarrow$): Let A be f-invariant. Then for any $x \in X$, we have,

$$\begin{aligned} f^{-1}(f(A))(x) &= f(A)\,(f(x)) \\ &= \vee\{A(z):f(z) = f(x)\} \\ &= A(x) \end{aligned}$$

<u>3.10. Proposition</u>. Let f be a surjection and $\mathfrak{F}$ be the collection of all f-invariant fuzzy subsets of X. Then $\psi_f : \mathfrak{F} \longrightarrow I^Y$ defined by $\psi_f(A) = f(A), \forall A \in \mathfrak{F}$, is a bijection.

<u>Proof</u>: For $A_1, A_2 \in \mathfrak{F}$,

$$\begin{aligned} \psi_f(A_1) = \psi_f(A_2) &\Longrightarrow f(A_1) = f(A_2) \\ &\Longrightarrow f^{-1}(f(A_1)) = f^{-1}(f(A_2)) \\ &\Longrightarrow A_1 = A_2, \text{ by theorem 3.9.} \end{aligned}$$

Hence ψ_f is an injection. Now, let $B \in I^Y$ and $A = f^{-1}(B)$. Then $A \in I^X$. We want to see that $A \in \mathfrak{F}$. That is, A is f-invariant. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then,

$$A(x_1) = f^{-1}(B)(x_1) = B(f(x_1)) = B(f(x_2)) = A(x_2).$$

Hence $A \in \mathfrak{F}$. Also, by Proposition 3.4, we have,

$$\psi_f(A) = f(A) = f(f^{-1}(B)) = B$$

Thus ψ_f is a surjection also.

<u>3.11. Remark</u>. It follows from the above proposition that for any function $f:X \longrightarrow Y$, there is a set-theoretic isomorphism between the collection $\mathfrak{F}$ of all f-invariant fuzzy subsets of X and $I^{f(X)}$. For any non-empty set X, $| I^{f(X)} | \geqslant c$, where c is the cardinality of the continuum R. Hence, for any function f on X, there are uncountably many f-invariant fuzzy subsets of X.

CHAPTER II

FUZZY SUBGROUPS

1. INTRODUCTION

The study of fuzzy algebraic structures was initiated by Rosenfeld [45]. He defined fuzzy subgroup as an extension of the concept of subgroup in classical group theory. Some other definitions of fuzzy subgroups are available in the literature ([2], [7], [8], [17]). Most of the recent studies on fuzzy groups use Rosenfeld's definition.

In this chapter, we first give some definitions and results from fuzzy group theory which are used in this work. Then we proceed to obtain extensions of some results of Das [19]. We prove that the groups $A(N)$ and $Z(p^{\infty})$ admit fuzzy subgroups of all finite level cardinalities; and give a characterisation of such groups.

The terms and results from classical group theory which are used in this work are as in Herstein [25], Hungerford [27], Lang [34], and Kurosh [33]. Throughout this work G is an arbitrary multiplicative group with e as the identity element.

2. BASIC CONCEPTS

2.1. Definition [45]. A fuzzy subset F of G is said to be a fuzzy subgroup of G if for every $x,y \in G$

(1) $F(xy) \geqslant \wedge \{F(x), F(y)\}$; and

(2) $F(x^{-1}) = F(x)$.

2.2 Remark. All constant fuzzy subsets of G are fuzzy subgroups of G. For any non-empty subset F of G, χ_F is a fuzzy subgroup of G if, and only if, F is a subgroup of G. Hence the above definition extends the concept of subgroup in classical group theory, to the fuzzy context. If F is empty, then $\chi_F = \tilde{0}$ is a fuzzy subgroup of G, but F is not a subgroup of G.

The terms "fuzzy group" and "fuzzy subgroup" are used interchangeably.

Examples of fuzzy groups of small level cardinality are given by various authors. We give below an example of a fuzzy group with infinite level cardinality.

2.3. Example. Let Z be the group of all integers under addition. We observe that if x is a non-zero even integer, then it can be written uniquely as $x = m.2^n$, where m and n are integers, m is old, and $n > 0$. We shall call this the <u>standard form</u> of x. Define $F:Z \longrightarrow I$ by

$$F(x) = \begin{cases} 0 & \text{, if } x \text{ is odd} \\ 1-\frac{1}{n} & \text{, if } x = m.2^n \text{ in the standard form} \\ 1 & \text{, if } x = 0 \end{cases}$$

It follows that $F(-x) = F(x), \forall\ x \in Z$ and $|\mathrm{Im}(F)| = \aleph_0$, where $\aleph_0$ is the cardinality of N. Now, it suffices to verify that

$$F(x+y) \geqslant \wedge \{F(x),F(y)\}\ , \forall\ x,y \in Z.$$

When x or y is zero, the result is obvious. The case when x or y is odd also is trivial. Now let x and y be both non-zero even integers. Let $x = m_1.2^{n_1}$ and $y = m_2.2^{n_2}$ be their standard forms. Let $n_1 < n_2$. Then

$$x+y = (m_1 + m_2.\ 2^{n_2-n_1})\ 2^{n_1}$$

where $m_1+m_2.\ 2^{n_2-n_1}$ is odd. Hence,

$$\begin{aligned} F(x+y) &= 1 - \frac{1}{n_1} \\ &= \wedge \left\{1-\frac{1}{n_1},\ 1-\frac{1}{n_2}\right\}, \text{ since } n_1<n_2 \\ &= \wedge \{F(x),\ F(y)\}. \end{aligned}$$

<u>2.4. Proposition</u> [45]. If F is a fuzzy subgroup of G then,

(a) $F(e) \geqslant F(x), \forall x \in G$; and

(b) $G_F = \{x \in G : F(x) = F(e)\}$ is a subgroup of G.

<u>2.5. Proposition</u> [45]. A fuzzy subset F of G is a fuzzy subgroup if, and only if,

$$F(xy^{-1}) \geqslant \wedge \{F(x), F(y)\}, \forall x, y \in G.$$

<u>2.6. Proposition</u> [45]. The intersection of any collection of fuzzy subgroups of G is a fuzzy subgroup of G.

<u>2.7. Remark</u>. It is known from classical group theory that the union of two subgroups, in general, is not a subgroup. Hence it follows that the union of two fuzzy subgroups need not be a fuzzy subgroup.

Let $\mathfrak{F}(G)$ denote the collection of all fuzzy subgroups of G. Then $\mathfrak{F}(G)$, under fuzzy set inclusion, is a complete lattice with $\tilde{1}$ as the largest and $\tilde{0}$ as the smallest elements. If $F_j \in \mathfrak{F}(G), \forall j \in J$, J being an arbitrary index set, we have,

$$\bigwedge_{j \in J} F_j = \bigcap_{j \in J} F_j; \text{ and}$$
$$\bigvee_{j \in J} F_j = \bigcap \{F \in \mathcal{F}(G) : F \supseteq \bigcup_{j \in J} F_j\}.$$

It may be observed that $\mathcal{F}(G)$ is not a sublattice of I^G.

<u>2.8. Definition</u> [41]. A fuzzy subgroup F of G is said to be <u>fuzzy normal</u> if

$$F(xy) = F(yx), \; \forall \; x,y \in G.$$

We shall denote the collection of all fuzzy normal subgroups of G by $\mathcal{F}\aleph(G)$.

<u>2.9. Proposition</u> [41]. Let F be a non-empty subset of G. Then χ_F is a fuzzy normal subgroup of G iff F is a normal subgroup of G.

<u>2.10. Proposition</u> [41]. A fuzzy subgroup F of G is fuzzy normal iff

$$F(y^{-1}xy) = F(x), \; \forall \; x,y \in G.$$

<u>2.11. Definition</u> [42]. Let F be a fuzzy subgroup of G and $x \in G$. Then the fuzzy subsets Fx and xF of G defined by

$$Fx(g) = F(gx^{-1}), \text{ and}$$

$$xF(g) = F(x^{-1}g), \forall g \in G$$

are respectively called the <u>right</u> and <u>left fuzzy cosets</u> of F determined by x.

If F is an ordinary subgroup of G and $x \in G$, then $(\chi_F)_x = \chi_{Fx}$ and $x(\chi_F) = \chi_{xF}$.

<u>2.12. Definition</u> [42]. If F is a fuzzy subgroup of G and $x \in G$, then the fuzzy subset $Cx(F)$ of G defined by

$$Cx(F)(g) = F(x^{-1}g\,x), \forall g \in G$$

is called the <u>fuzzy conjugate</u> of F determined by x.

<u>2.13. Definition</u> [13]. A fuzzy subgroup F of G is said to be <u>fuzzy abelian</u> if G_F is an abelian subgroup of G.

We shall denote the collection of all fuzzy abelian subgroups of G by $\mathcal{FA}(G)$.

<u>2.14. Proposition</u> [13]. A non-empty subset H of G is an abelian subgroup of G iff χ_H is a fuzzy abelian subgroup of G.

<u>2.15. Proposition</u> [42]. A fuzzy subgroup F of G is fuzzy normal iff $Fx = xF, \forall x \in G$.

<u>2.16. Remark</u>. In recent times, Bhattacharya and Mukherjee ([12-13], [41-42]) have made some significant contributions to the theory of fuzzy groups. They introduced fuzzy parallels of several concepts in classical group theory, and obtained extensions of some important theorems.

Ajmal and Thomas [6] have studied the lattice structure of $\mathfrak{F}(G)$. They proved that, if

$$L_f = \{F \in \mathfrak{F}(G) : |Im(F)| < \infty\}, \text{ and}$$

$$L_t = \{F \in \mathfrak{F}(G) : F(e)=t\}$$

then L_f and L_t, and hence $L_f \cap L_t$ are sublattices of $\mathfrak{F}(G)$. Also, if L_{fnt} denote the collection of all fuzzy normal subgroups of G having finite range and the same support t at e, then L_{fnt} is a modular sublattice of $L_f \cap L_t$; and hence of $\mathfrak{F}(G)$.

We do not go into the details of these as we do not use them in this work.

3. LEVEL SUBGROUPS

Das [19] used Zadeh's notion of level subsets to define level subgroups of a fuzzy group. Many properties of fuzzy groups have been characterised by using their level subgroups; and hence it has become one of the important tools used in the study of fuzzy groups.

3.1. Definition [19]. Let A be a fuzzy subset of X and $t \in I$. Then $A_t = \{x \in X : A(x) \geqslant t\}$ is called the level subset of A at t.

3.2. Notation. We shall use the notation F_t for $t \in R \setminus I$ also, in the following sense

$$F_t = \begin{cases} \emptyset \quad , & \text{if } t > 1 \\ X \quad , & \text{if } t < 0 \end{cases}$$

3.3. Proposition [19]. If F is a fuzzy subgroup of G, then

(a) $F_t = \emptyset, \forall\ t > F(e)$, and

(b) F_t is a subgroup of G, for every $0 \leqslant t \leqslant F(e)$.

3.4. Proposition [19]. A fuzzy subset F of G is a fuzzy subgroup of G iff F_t is a subgroup of G, $\forall\, t \in [0,F(e)]$.

3.5. Definition [19]. If F is a fuzzy subgroup of G and $0 \leqslant t \leqslant F(e)$, then F_t is called the level subgroup of F at t.

3.6. Proposition. If F is a fuzzy subgroup of G and $t_1, t_2 \in I$, then

$$t_1 > t_2 \implies F_{t_1} \subseteq F_{t_2} .$$

In particular, if $t_1, t_2 \in Im(F)$, then

$$t_1 > t_2 \iff F_{t_1} \subsetneq F_{t_2} .$$

Proof: Trivial.

The following is the corrected form of theorem 3.1. of Das [19] as given by Mashinchi and Zahedi [40].

3.7. Proposition. Let G be a group, F be a fuzzy subgroup of G and $t_1, t_2 \in I$ with $t_1 < t_2$. Then F_{t_1} and F_{t_2} are equal iff there exists no $x \in G$ such that $t_1 \leqslant F(x) < t_2$.

3.8. Proposition [19]. If F is a fuzzy subgroup of a finite group G with $Im(F) = \{t_i : i=1,2,\ldots,n\}$, then

$\{F_{t_i} : i = 1,2,..,n\}$ contains all level subgroups of F. Further, if $t_1 > t_2 > \dots > t_n$ then the level subgroups of F form a chain $G_F = F_{t_1} \subsetneq F_{t_2} \subsetneq \dots \subsetneq F_{t_n} = G$.

The above proposition considers only fuzzy subgroups having finite level cardinality. We shall extend it to an arbitrary fuzzy subgroup.

<u>3.9. Proposition</u>. Let F be a fuzzy subgroup of G with $\mathrm{Im}(F) = \{t_j : j \in J\}$ and $\mathfrak{F} = \{F_{t_j} : j \in J\}$. Then

(a) there exists a unique $j_o \in J$ such that $t_{j_o} \geq t_j, \forall\, j \in J$.

(b) $G_F = \bigcap_{j \in J} F_{t_j} = F_{t_{j_o}}$

(c) $G = \bigcup_{j \in J} F_{t_j}$, and

(d) the members of $\mathfrak{F}$ form a chain.

<u>Proof</u> (a). Since $F(e) \in \mathrm{Im}(F)$, there exists a unique $j_o \in J$ such that $F(e) = t_{j_o}$. By proposition 2.4(a), we have, $t_{j_o} \geq F(x)$, for every $x \in G$ and hence $t_{j_o} \geq t_j, \forall\, j \in J$.

(b) We have,

$$
\begin{aligned}
F_{t_{j_o}} &= \{x \in G : F(x) \geqslant t_{j_o}\} \\
&= \{x \in G : F(x) = t_{j_o}\}, \text{ by Prop. 2.4(a)} \\
&= G_F
\end{aligned}
$$

Since $t_{j_o} \geqslant t_j, \forall\ j \in J$, by Proposition 3.6., we have

$$F_{t_{j_o}} \subseteq F_{t_j}, \forall\ j \in J.$$

Hence,

$$F_{t_{j_o}} \subseteq \bigcap_{j \in J} F_{t_j}.$$

Also, since $j_o \in J$, we get

$$F_{t_{j_o}} = \bigcap_{j \in J} F_{t_j}.$$

(c) Since $F_{t_j} \subseteq G, \forall\ j \in J$, we have,

$$\bigcup_{j \in J} F_{t_j} \subseteq G.$$

Now let $x \in G$. Since $F(x) \in \mathrm{Im}(F)$, there exists $j_x \in J$ such that $F(x) = t_{j_x}$. Obviously, $x \in F_{t_{j_x}}$, and hence $x \in \bigcup_{j \in J} F_{t_j}$. Hence,

$$G \subseteq \bigcup_{j \in J} F_{t_j}.$$

(d) Since I is a chain, for any $i,j \in J$, either $t_i \geqslant t_j$ or $t_i \leqslant t_j$. Hence, by Proposition 3.6, $F_{t_i} \subseteq F_{t_j}$ or $F_{t_i} \supseteq F_{t_j}$. This proves (d).

<u>3.10. Remark</u>. In the finite case, $\mathfrak{F}$ in the above proposition is the chain of all level subgroups of F. However, in general, $\mathfrak{F}$ need not contain all level subgroups of F. The following example proves this observation.

<u>3.11. Example</u>. Let Q be the set of all rational numbers, Z the set of all integers and p be a prime number. Then

$$Z(p^{\infty}) = \left\{ Z + \frac{a}{b} \in \frac{Q}{Z} : a,b \in Z,\ b = p^{m} \text{ for some integer } m \geqslant 0 \right\},$$

is a Sylow p-subgroup of $\frac{Q}{Z}$, of infinite order. For $n = 0,1,2,\ldots$, let C_n be the cyclic subgroup of $\frac{Q}{Z}$ generated by $Z + \frac{1}{p^n}$. Then C_n is a proper subgroup of $Z(p^{\infty})$ such that $C_n \subsetneq C_{n+1}$, for every n. Further, $Z(p^{\infty}) = \bigcup_{n \in N_o} C_n$, where $N_o = N \cup \{0\}$. [27].

Define F: $Z(p^{\infty}) \longrightarrow I$ by $F(x) = 2^{-n}$, if C_n is the smallest subgroup containing x. Then F is a fuzzy subgroup of $Z(p^{\infty})$ with

$$Im(F) = \{2^{-i} : i \in N_o\}.$$

But,

$$\mathfrak{F} = \{ F_{(2^{-i})} : i \in N_o \}$$

does not contain all level subgroups of F, since $Z(p^{\infty})$ does not belong to $\mathfrak{F}$.

<u>3.12. Theorem</u>. Under the assumptions in Proposition 3.9 $\mathfrak{F}$ contains all level subgroups of F if and only if F attains its infemum on all subgroups of G.

<u>Proof</u>. Necessity. Assume that $\mathfrak{F}$ contains all level subgroups of F. Let H be any subgroup of G. If F is constant on H, there is nothing to prove. Assume that F is not constant on H. We divide the proof into two cases.

<u>Case (i)</u>. H = G: Let $t^* = \bigwedge_{j \in J} t_j$. Then $t^* \leqslant t_j$, for every $j \in J$, and hence

$$F_{t^*} \supseteq F_{t_j}, \forall\ j \in J \qquad .. \qquad (1)$$

Since $\mathfrak{F}$ contains all level subgroups of F and $F_o = G$, we have, $G \in \mathfrak{F}$. Hence, there exists $t_{j^*} \in Im(F)$ such that $G = F_{t_{j^*}}$. Also, by (1), we have, $F_{t^*} \supseteq F_{t_{j^*}} = G$. But all level subgroups of F are subgroups of G, and hence

$$F_{t^*} = F_{t_{j^*}} = G.$$

The proof is complete if we show that $t^* = t_{j^*}$. By the definition of t^*, we have, $t^* \leqslant t_{j^*}$. Suppose $t^* < t_{j^*}$. Then there exists $t_j \in Im(F)$ such that $t^* \leqslant t_j < t_{j^*}$. This implies that $F_{t_j} \supsetneqq F_{t_{j^*}} = G$, which is a contradiction.

<u>Case (ii)</u>. $H \subsetneqq G$: Let F_H denote the restriction of F to H. For any $x,y \in H$, we have $xy^{-1} \in H$, and hence

$$\begin{aligned} F_H(xy^{-1}) &= F(xy^{-1}) \\ &\geqslant \wedge \{F(x), F(y)\} \\ &= \wedge \{F_H(x), F_H(y)\}. \end{aligned}$$

Therefore, by Proposition 2.5, F_H is a fuzzy subgroup of H. Now, let

$$J_H = \{j \in J : F(h) = t_j, \text{ for some } h \in H\}, \text{ and}$$
$$\mathfrak{F}_H = \{(F_H)_{t_j} : j \in J_H\}.$$

Since $\mathfrak{F}$ contains all level subgroups of F, $\mathfrak{F}_H$ contains all level subgroups of F_H; and hence by case (i), there exists $x^* \in H$ such that

$$F_H(x^*) = \bigwedge_{x \in H} F_H(x)$$

By the definition of F_H, this implies that

$$F(x^*) = \bigwedge_{x \in H} F(x).$$

<u>Sufficiency</u>. Assume that the infemum of F on every subgroup of G is attained at some point of it. Let F_t be any level subgroup of F. We want to prove that $F_t \in \mathfrak{F}$. If $t=t_j$, for some $j \in J$, there is nothing to prove. We assume that $t \neq t_j, \forall j \in J$. Then there does not exist $x \in G$ such that $F(x) = t$. Let $H = \{x \in G : F(x) > t\}$. Then,

$$x,y \in H \Rightarrow F(x) > t \text{ and } F(y) > t$$
$$\Rightarrow F(xy^{-1}) \geqslant \wedge\{F(x),F(y)\} > t$$
$$\Rightarrow xy^{-1} \in H.$$

Hence H is a subgroup of G. Therefore, there exists $h^* \in H$ such that

$$F(h^*) = \bigwedge_{h \in H} F(h)$$

Now, $F(h^*) \in Im(F)$ and hence $F(h^*) = t_{j^*}$, for some $j^* \in J$. Then we have,

$$\wedge\{F(x) : F(x) > t\} = t_{j^*}.$$

Obviously, $t_{j^*} \geqslant t$, and hence by assumption, $t_{j^*} > t$. Also there does not exist $x \in G$ such that $t \leqslant F(x) < t_{j^*}$. Hence, by Proposition 3.7, $F_t = F_{t_{j^*}}$ and therefore $F_t \in \mathfrak{F}$

<u>3.13. Remark</u>. It may be observed that in Example 3.11, $\bigwedge_{x \in G} F(x) = 0$. But F does not attain the value zero at any point in G.

If $F \in \mathfrak{F}(G)$ with $|Im(F)| < \infty$, then F attains its infemum on all subgroups of G. Hence the above theorems generalise the corresponding result of Das [19] to the case of an arbitrary fuzzy group.

4. LEVEL CARDINALITY

In this section we present two groups which admit fuzzy subgroups of any finite level cardinality. We also give a characterisation of groups having this property.

Let A(N) be the group of all permutations on N. For any $n \in N$, let

$$A_n = \{f \in A(N) : f(k) = k, \forall\ k = 1,2,\ldots,n\}.$$

Then A_n is a subgroup of A(N) and $A_n \supsetneq A_{n+1}$, $\forall$ n.

4.1. Proposition. A(N) has fuzzy subgroups of level cardinality m, $\forall\ m \in N$.

Proof: The case when m=1 is trivial. Assume that $m \geqslant 2$. We have,

$$A_{m-1} \subsetneq A_{m-2} \subsetneq \cdots \subsetneq A_1 \subsetneq A(N).$$

Choose t_i $(i=1,2,\ldots,m)$ in I such that $t_1 > t_2 > \ldots > t_m$.

Define $F:A(N) \longrightarrow I$ by

$$F(A_{m-1}) = t_1$$

$$F(A_{m-i} \setminus A_{m+1-i}) = t_i, \forall\ i=2,3,\ldots,m-1; \text{ and}$$

$$F(A(N) \setminus A_1) = t_m.$$

Then F is a fuzzy subgroup of $A(N)$ with level cardinality m.

<u>4.2. Proposition</u>. $Z(p^\infty)$ has fuzzy subgroups of level cardinality $m, \forall\ m \in N$; and of infinite level cardinality.

<u>Proof</u>: The case when $m=1$ is trivial. Assume that $m \geq 2$. Then we have,

$$C_o \subsetneq C_1 \subsetneq C_2 \subsetneq \ldots\ C_{m-2} \subsetneq Z(p^\infty)$$

Fix $t_i (i=1,2,\ldots,m)$ in I such that $t_1 > t_2 > \ldots > t_m$ and define $F:Z(p^\infty) \longrightarrow I$ by

$$F(C_o) = t_1$$

$$F(C_i \setminus C_{i-1}) = t_{i+1}, \forall\ i=1,2,3,\ldots,m-2; \text{ and}$$

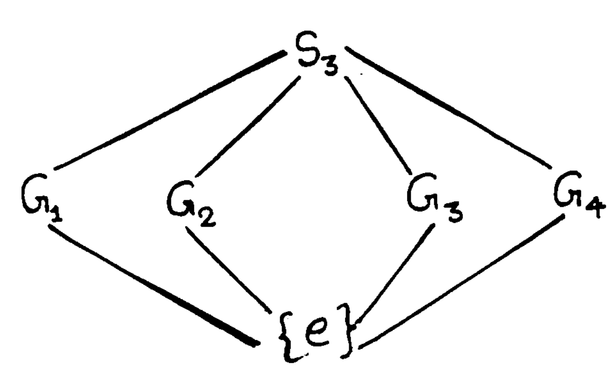
$$F(Z(p^\infty) \setminus C_{m-2}) = t_m$$

It is straight forward to verify that F is a fuzzy subgroup of $Z(p^{\infty})$ with level cardinality m.

Fuzzy subgroup of infinite level cardinality is constructed in example 3.11.

<u>4.3. Remark</u>. If G has a fuzzy subgroup of level cardinality m, then ${}_{o}(G) \geqslant m$. But the converse is not true, as evident from the example below.

<u>4.4. Example</u>. Let $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and $m = 4$. Then ${}_{o}(G) > m$. But G does not have a fuzzy subgroup of level cardinality m. For: If G has a fuzzy subgroup F of level cardinality 4, let $Im(F) = \{t_i : i=1,\ldots,4\}$, $t_1 > t_2 > t_3 > t_4$. Then by Proposition 3.8, $F_{t_1} \subsetneq F_{t_2} \subsetneq F_{t_3} \subsetneq F_{t_4}$ must be a chain of distinct subgroups of G. But we have the following lattice diagram for subgroups of S_3.



where $G_1 = \{e, (123), (132)\}$, $G_2 = \{e, (12)\}$,

$G_3 = \{e, (13)\}$ and $G_4 = \{e, (23)\}$.

Hence G cannot have a chain of subgroups with more than three components.

<u>4.5. Theorem</u>. G has a fuzzy subgroup of level cardinality m if and only if G admits a chain of distinct subgroups of length m.

<u>Proof</u>. <u>Necessity</u>. Assume that G has a fuzzy subgroup of level cardinality m. Let $Im(F) = \{s_i : i=1,2,\ldots,m\}$ where $s_1 > s_2 > \ldots > s_m$. Then by Proposition 3.8, we have the chain

$$F_{s_1} \subsetneq F_{s_2} \subsetneq \ldots \subsetneq F_{s_m}$$

of length m.

<u>Sufficiency</u>. Assume that G admits a chain of distinct subgroups of length m, say, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$. Without loss of generality, we assume that $G_m = G$. Fix $t_i (i=1,2,..m)$ such that $t_1 > t_2 > \ldots > t_m$. Define $F: G \longrightarrow I$ by

$$F(G_1) = t_1, \ F(G_i \setminus G_{i-1}) = t_i, \ \forall \ i=2,3,\ldots,m$$

Then F is a fuzzy subgroup of G with level cardinality m.

<u>4.6. Corollary</u>. G has a fuzzy subgroup of level cardinality m if and only if G has fuzzy subgroups of level cardinality n, $\forall \ n \leqslant m$

<u>4.7. Corollary</u>. If G has a fuzzy subgroup of level cardinality m, then it has uncountably many fuzzy subgroups of level cardinality m.

We now proceed to give a characterisation of groups like A(N) and $Z(p^{\infty})$ which admit fuzzy subgroups of every finite level cardinality. Its proof is an extension of the proof of theorem 4.5, and hence is omitted.

<u>4.8. Theorem</u>. G has fuzzy subgroups of every finite level cardinality iff G has a non-terminating chain of distinct subgroups.

CHAPTER III

FUZZY GROUPS AND GROUP HOMOMORPHISMS

1. INTRODUCTION.

The effect of group homomorphisms on fuzzy groups was studied by Rosenfeld [45], Anthony and Sherwood [8], Sidky and Mishref [52] and Akgul [7]. Rosenfeld [45] proved that if f is a group homomorphism on G, then

$$F \in \mathfrak{F}(G) \Rightarrow f(F) \in \mathfrak{F}(f(G))$$

provided F has V-property; and

$$F^* \in \mathfrak{F}(f(G)) \Rightarrow f^{-1}(F^*) \in \mathfrak{F}(G).$$

Later, Anthony and Sherwood [8] observed that in the first case, the restriction "F has V-property" is redundant. In [52], Sidky and Mishref proved that if $f:G \longrightarrow G^*$ is a group homomorphism and F is a fuzzy subgroup of G "with respect to a continuous t-norm T" then f(F) is a fuzzy subgroup of G* with respect to T. Since $\wedge$ is a continuous t-norm [8], it follows that, $f(F) \in \mathfrak{F}(G^*)$ whenever $F \in \mathfrak{F}(G)$.

It was proved by Akgul [7] that

$$F^* \in \mathcal{F}(G^*) \Rightarrow f^{-1}(F^*) \in \mathcal{F}(G).$$

We, in this chapter, study the effect of group homomorphisms on the chains of level subgroups of fuzzy groups. Throughout this chapter $f:G \longrightarrow G^*$ is a group homomorphism.

2. HOMOMORPHIC PRE-IMAGES OF FUZZY GROUPS.

2.1. Notation. For any fuzzy group F, C(F) denotes the chain of all its level subgroups. Distinctness of the components of C(F) is assumed throughout. We shall denote by $f(C(F))$ $[f^{-1}(C(F))]$, the chain consisting of ima es [inverse images] under f of members of C(F).

2.2. Proposition. Let $F^* \in \mathcal{F}(G^*)$ and $\{F^*_{t_j} : j \in J\}$ be the collection of all level subgroups of F^*. Then $\{f^{-1}(F^*_{t_j}) : j \in J\}$ is the collection of all level subgroups of $f^{-1}(F^*)$.

Proof: Let $F = f^{-1}(F^*)$ and $t \in I$. Then,

$$x \in F_t \iff f^{-1}(F^*)(x) \geqslant t$$

$$\Longleftrightarrow F^*(f(x)) \geqslant t$$

$$\Longleftrightarrow f(x) \in F^*_t$$

$$\Longleftrightarrow x \in f^{-1}(F^*_t)$$

Hence,

$$F_t = f^{-1}(F_t{}^*), \forall\, t \in I \tag{1}$$

In particular, we have,

$$F_{t_j} = f^{-1}(F^*_{t_j}), \ \forall\, j \in J.$$

If F has a level subgroup F_t which does not belong to $\{f^{-1}(F^*_{t_j}):j \in J\}$, then F* must have a level subgroup F^*_t which does not belong to $\{F^*_{t_j} :j \in J\}$ such that (1) holds. This is a contradiction. Hence $\{f^{-1}(F^*_{t_j}):j \in J\}$ is the collection of all level subgroups of F .

We observe that some of the $f^{-1}(F^*_{t_j})$'s may be equal, so that $C(f^{-1}(F^*))$ has fewer components than $C(F^*)$, as evident from the following example.

<u>2.3. Example</u>. Let $G = \{1,-1,i,-i\}$ and $G^* = S_3$. Define $f:G \longrightarrow G^*$ by $f(x) = e$, for every $x \in G$.

Then f is a group homomorphism. Define $F^*: G^* \longrightarrow I$ by

$$e \longmapsto 1,\ (12) \longmapsto 0.5,\ G^* \setminus \{e, (12)\} \longmapsto 0.3$$

Then $F^* \in \mathcal{F}(G^*)$ with level subgroups

$$F_1^* = \{e\},\ F_{0.5}^* = \{e, (12)\},\ F_{0.3}^* = G^*.$$

And $F = f^{-1}(F^*)$ is defined by $F(x)=1, \forall\ x \in G$. Hence,

$$F_1 = F_{0.5} = F_{0.3} = G$$

2.4. Proposition. If $F_{t_j}^*$ $(j \in J)$ of Proposition 2.2 are distinct and f is a surjection, then $f^{-1}(F_{t_j}^*)$ are all distinct.

Proof. Suppose $f^{-1}(F_{t_j}^*)$ are not all distinct. Then there exist $p,q \in J$ with $t_p \neq t_q$ and $f^{-1}(F_{t_p}^*) = f^{-1}(F_{t_q}^*)$. Hence, $f(f^{-1}(F_{t_p}^*)) = f(f^{-1}(F_{t_q}^*))$. Since f is a surjection, this implies that $F_{t_p}^* = F_{t_q}^*$. This contradicts the assumption that $F_{t_j}^*$'s are all distinct.

2.5. Corollary. If f is a surjection and $F^* \in \mathcal{F}(G^*)$, then

$$C(f^{-1}(F^*)) \equiv f^{-1}(C(F^*)) .$$

The following example shows that surjectiveness of f is not a necessary condition for the distinctness of all $f^{-1}(F^*_{t_j})$.

2.6. Example. Let $G = \{1,-1\}$ and $G^* = \{1,-1,i,-i\}$. Define $f: G \longrightarrow G^*$ by $f(x)=x$, for every $x \in G$. Then f is a group homomorphism which is not surjective. Define $F^* : G^* \longrightarrow I$ by $F^*(1) = 0.8$ and $F^*(x) = 0.5$ for every $x \neq 1$. Then F^* is a fuzzy subgroup of G^* with level subgroups

$$F^*_{0.8} = \{1\}, \quad F^*_{0.5} = G^*$$

Now, $f^{-1}(F^*)$ is given by

$$1 \longmapsto 0.8, \; -1 \longmapsto 0.5.$$

Its level subgroups are

$$f^{-1}(F^*_{0.8}) = \{1\}, \; f^{-1}(F^*_{0.5}) = G$$

which are distinct.

We proceed to derive a necessary and sufficient condition for distinctness of all $f^{-1}(F^*_{t_j})$, in the case when J is countable. For $t \in Im(F^*)$,

$$Fib_{F^*}(t) = \{x \in G^* : F^*(x) = t\}$$

is called the fiber of F^* at t.

<u>2.7. Theorem</u>. Let $F^* \in \mathcal{F}(G^*)$ with $Im(F^*)=\{t_j : j \in J\}$ where J is a countable index set. Then $f^{-1}(F^*_{t_j})$ are all distinct iff

$$f(G) \cap Fib_{F^*}(t_j) \neq \emptyset, \forall\, j \in J.$$

<u>Proof</u>: Let $J = \{j_i : i \in K\}$, where $K \subseteq N$. Without loss of generality, we assume that either $K = N$ or $K = \{1,2,3,\ldots,k\}$ for some $k \in N$; and $t_{j_i} > t_{j_{i+1}}$, for every i, $i+1 \in K$.

<u>Necessity</u>. Assume that $f^{-1}(F^*_{t_j})$, $j \in J$, are all distinct. Let e^* denote the identity element in G^*. Since f is a homomorphism, $e^* \in f(G)$. Also, $t_{j_1} \geqslant t_{j_i}$, for every $i \in K$ and hence $F^*(e^*) = t_{j_1}$. Hence $e^* \in Fib_{F^*}(t_{j_1})$. Therefore,

$$f(G) \cap Fib_{F^*}(t_{j_1}) \neq \emptyset.$$

Suppose $f(G) \cap Fib_{F^*}(t_{j_p})$ is empty, for some $p > 1$. Since $t_{j_{p-1}} > t_{j_p}$, by Proposition 2.3.6,

$$F^*_{t_{j_{p-1}}} \subsetneq F^*_{t_{j_p}}$$

and hence

$$f^{-1}(F^*_{t_{j_{p-1}}}) \subseteq f^{-1}(F^*_{t_{j_p}}).$$

Now,

$$x \in f^{-1}(F^*_{t_{j_p}}) \Longrightarrow f(x) \in F^*_{t_{j_p}} = F^*_{t_{j_{p-1}}} \cup \mathrm{Fib}_{F*}(t_{j_p})$$

$$\Longrightarrow f(x) \in F^*_{t_{j_{p-1}}}, \text{ by assumption}$$

$$\Longrightarrow x \in f^{-1}(F^*_{t_{j_{p-1}}})$$

Hence,

$$f^{-1}(F^*_{t_{j_p}}) \subseteq f^{-1}(F^*_{t_{j_{p-1}}})$$

Therefore,

$$f^{-1}(F^*_{t_{j_p}}) = f^{-1}(F^*_{t_{j_{p-1}}}).$$

This contradicts the assumption that $f^{-1}(F^*_{t_j})$ are all distinct. Hence,

$$f(G) \cap \mathrm{Fib}_{F*}(t_j) \neq \emptyset, \forall\, j \in J.$$

<u>Sufficiency</u>. Assume that $f^{-1}(F^*_{t_j})$ are not distinct. Then we can find $p,q \in J$ such that $t_p \neq t_q$ and

$$f^{-1}(F^*_{t_p}) = f^{-1}(F^*_{t_q}) \tag{1}$$

Without loss of generality, we assume that $t_p < t_q$. Since,

$$f(G) \cap \mathrm{Fib}_{F*}(t_p) \neq \emptyset$$

there exists $x \in G$ such that

$$f(x) \in \mathrm{Fib}_{F*}(t_p).$$

This implies that $F^*(f(x)) = t_p$. Since $t_p < t_q$, we have, $f(x) \in F^*_{t_p}$ and $f(x) \notin F^*_{t_q}$. Therefore,

$$x \in f^{-1}(F^*_{t_p}) \text{ and } x \notin f^{-1}(F^*_{t_q}) \tag{2}$$

(1) and (2) contradict. Therefore, $f^{-1}(F^*_{t_j})$ are all distinct.

<u>2.8. Remark</u>. It can be observed from the proof that the sufficiency of the condition in the above theorem hold even when J is uncountable.

If f is a surjection, then,

$$f(G) \cap \mathrm{Fib}_{F*}(t_j) \neq \emptyset, \ \forall\ j \in J$$

and hence Proposition 2.4 can be derived as a particular case of theorem 2.7.

3. HOMOMORPHIC IMAGES OF FUZZY GROUPS

Dixit, Kumar and Ajmal [21] have briefly studied the relationship between C(F) and C(f(F)). We have the following result from [21].

3.1. Proposition. Let G be a finite group, $f:G \longrightarrow G^*$ be a surjective group homomorphism, and F be a fuzzy subgroup of G with $\mathrm{Im}(F) = \{t_i: i=1,2,\dots,n\}$ where $t_1 > t_2 > \dots > t_n$. If the chain of level subgroups of F is

$$F_{t_1} \subseteq F_{t_2} \subseteq \dots \subseteq F_{t_n}$$

then the chain of level subgroups of f(F) is

$$f(F_{t_1}) \subseteq f(F_{t_2}) \subseteq \dots \subseteq f(F_{t_n})$$

In the following proposition we remove the restriction on the finiteness of both G and Im(F).

<u>3.2. Proposition</u>. Let $f: G \longrightarrow G^*$ be a surjective group homomorphism and F be a fuzzy subgroup of G having V-property. If $\{F_{t_j} : j \in J\}$ is the collection of all level subgroups of F, then $\{f(F_{t_j}) : j \in J\}$ is the collection of all level subgroups of f(F).

<u>Proof</u>. Let $f^* = f(F)$. For any $t \in I$,

$$\begin{aligned} u \in F_t^* &\Longrightarrow F^*(u) \geqslant t \\ &\Longrightarrow \bigvee \{F(x) : x \in f^{-1}(u)\} \geqslant t, \text{ since } f \text{ is surjective.} \end{aligned}$$

Since F has V-property, this implies that, $F(x_o) \geqslant t$, for some $x_o \in f^{-1}(u)$, Hence, $x_o \in F_t$. Therefore, $f(x_o) \in f(F_t)$, and hence, $u \in f(F_t)$. Thus we have,

$$F_t^* \subseteq f(F_t)$$

Now, if $u \in f(F_t)$, then $u = f(x)$ for some $x \in F_t$; and hence,

$$\begin{aligned} F^*(u) &= \bigvee \{F(z) : z \in f^{-1}(u)\} \\ &= \bigvee \{F(z) : f(z) = f(x)\} \\ &\geqslant F(x) \\ &\geqslant t, \text{ since } x \in F_t. \end{aligned}$$

Therefore, $u \in F_t^*$ and hence $f(F_t) \subseteq F_t^*$. Thus we have,

$$F_t^* = f(F_t), \forall\, t \in I \qquad (1)$$

In particular,

$$F_{t_j}^* = f(F_{t_j}), \forall\, j \in J.$$

Hence all $f(F_{t_j})$'s are level subgroups of F^*. Also, it follows from (1) and the assumption that these are the only level subgroups of F^*.

<u>3.3. Remark</u>. Proposition 3.1 can be obtained as a corollary to Proposition 3.2.

The following example shows that surjectiveness of f is essential in the above proposition.

<u>3.4. Example</u>. Let $G = \{1,-1\}$ and $G^* = \{1,-1,i,-i\}$. Define $f : G \longrightarrow G^*$ by $f(x) = x$, $\forall\, x \in G$. Then f is a group homomorphism which is not surjective. Define $F{:}G \longrightarrow I$ by $F(1) = 0.3$ and $F(-1) = 0.1$. Then F is a fuzzy subgroup of G having V-property. The level subgroups of F are $F_{0.3} = \{1\}$ and $F_{0.1} = G$. Now. $F^* = f(F)$ is defined by

$$1 \longmapsto 0.3, \quad -1 \longmapsto 0.1, \quad \{i,-i\} \longmapsto 0$$

Hence the level subgroups of F* are

$$F^*_{0.3} = f(F_{0.3}) = \{1\}$$

$$F^*_{0.1} = f(F_{0.1}) = \{1,-1\}, \text{ and}$$

$$F^*_{0} = G^*$$

Therefore, $\{f(F_{0.3}),\ f(F_{0.1})\}$ does not contain all level subgroups of F*

We observe from the following example that surjectiveness of f does not guarantee the distinctness of all $f(F_{t_j})$.

<u>3.5. Example</u>. Let $G = S_3$ and G* be the subgroup $\{e,\ (12)\}$ of S_3. Define $f:G \longrightarrow G^*$ by

$$\{e,\ (123),\ (132)\} \longmapsto e,\ \{(12),\ (13),(23)\} \longmapsto (12)$$

Then f is a surjective group homomorphism. Fix $t_1 > t_2 > t_3$ in I and define $F:G \longrightarrow I$ by

$$e \longmapsto t_1,\quad (12) \longmapsto t_2,\ G \setminus \{e,(12)\} \longmapsto t_3$$

Then F is a fuzzy subgroup of G having V-property. The level subgroups of F are:

$$F_{t_1} = \{e\},\ F_{t_2} = \{e,(12)\},\ F_{t_3} = G$$

Now, $f(F)$ is given by

$$e \longmapsto t_1, \quad (12) \longmapsto t_2$$

and hence its level subgroups

$$f(F_{t_1}) = \{e\}, \quad f(F_{t_2}) = G^*, \; f(F_{t_3}) = G^*$$

are not distinct.

The following theorem shows that f-invariance of F is a necessary and sufficient condition for the distinctness of the images of the level subgroups.

__3.6. Theorem.__ Let F of Proposition 3.2. be such that $\mathrm{Im}(F) = \{t_j : j \in J\}$ where J is a countable index set. Then $f(F_{t_j})$, $j \in J$ are all distinct iff F is f-invariant.

__Proof:__ Let J and K be as in the proof of Theorem 2.7.

__Necessity.__ Suppose $f(F_{t_j})$'s are all distinct. Since $t_{j_i} > t_{j_{i+1}}$, $\forall\, i, i+1 \in K$, by assumption, we have

$$F_{t_{j_i}} \subsetneqq F_{t_{j_{i+1}}}$$

and hence

$$f(F_{t_{j_i}}) \subsetneqq f(F_{t_{j_{i+1}}}), \forall i, \; i+1 \in K.$$

Let $x,y \in G$ such that $f(x) = f(y)$. Let $f(Ft_{j_p})$ be the smallest $f(Ft_j)$ which contains $f(x)$.

Case (i): $j_p=j_1$. Then $f(x), f(y) \in Ft_{j_1} = G_F$, and hence $F(x) = F(y) = F(e)$.

Case (ii): $j_p > j_1$. Then $f(x), f(y) \in f(Ft_{j_p})$ and $f(x), f(y) \notin f(Ft_{j_{p-1}})$. Hence $x,y \in Ft_{j_p}$ and $x,y \notin Ft_{j_{p-1}}$. Therefore, $F(x) = F(y) = t_{j_p}$.

Thus in both cases we have $F(x) = F(y)$, and hence F is f-invariant.

Sufficiency. Assume that F is f-invariant. Then for any $z \in G^*$,

$$f(F)(z) = F(x), \forall\, x \in f^{-1}(z) \qquad (1)$$

If $f(Ft_j)$'s are not distinct, then there exists $t_p, t_q \in \mathrm{Im}(F)$ such that $t_p \neq t_q$ and $f(Ft_p) = f(Ft_q)$. Since $t_p, t_q \in \mathrm{Im}(F)$, $F(x) = t_p$ and $F(y) = t_q$ for some $x,y \in G$. Hence, by (1), we have, $f(F)\,(f(x)) = t_p$, and $f(F)(f(y)) = t_q$. Hence, $t_p, t_q \in \mathrm{Im}(f(F))$, and it follows by Proposition 2.3.6

that $f(F_{t_p}) \neq f(F_{t_q})$. This is a contradiction. Hence $f(F_{t_j})$, $j \in J$, are all distinct.

We observe that the proof of the sufficiency part does not require the countability of J. Hence, we have,

<u>3.7. Corollary</u>. If $f: G \longrightarrow G^*$ is a surjective group homomorphism and F is an f-invariant fuzzy subgroup of G having V-property, then

$$C(f(F)) \equiv f(C(F))$$

<u>3.8. Corollary</u>. Let $f: G \longrightarrow G^*$ be a surjective group homomorphism and F be a fuzzy subgroup of G with $Im(F) = \{t_i : i=1,2,\ldots,n\}$ where $t_1 > t_2 > \ldots > t_n$. Then,

(a) $\{f(F_{t_i}) : i=1,2,\ldots,n\}$ contains all level subgroups of $f(F)$.

(b) $f(F_{t_i})$, $i=1,\cdots,n$, are all distinct iff F is f-invariant.

(c) If F is f-invariant, then $Im(f(F)) = Im(F)$. and

$$C(f(F)) \equiv f(F_{t_1}) \subsetneq f(F_{t_2}) \subsetneq \ldots \subsetneq f(F_{t_n})$$

CHAPTER IV

TRANSLATES OF FUZZY GROUPS

1. INTRODUCTION.

As an abstraction of the geometric notion of translation, we introduce two operators $T_{\alpha+}$ and $T_{\alpha-}$ called the fuzzy translation operators. First we define the operators on fuzzy sets and derive some of their properties. Then we investigate their action on fuzzy groups. We prove that all translates of fuzzy subgroups [fuzzy normal subgroups] are fuzzy subgroups [fuzzy normal subgroups]. We study the interaction of these operators with fuzzy coset formation and fuzzy conjugation; and we prove that the operators commute with both. We observe that fuzzy abelianness is not translation invariant. We prove that, with suitable restrictions on α, fuzzy abelianness can be made invariant under the action of $T_{\alpha+}$ and $T_{\alpha-}$.

2. THE OPERATORS $T_{\alpha+}$ and $T_{\alpha-}$

2.1. Definition. Let X be a non-empty set, $\alpha \in I$ and A be a fuzzy subset of X. We define

$$T_{\alpha+}(A)(x) = \wedge \{A(x) + \alpha, 1\} \text{ , and}$$
$$T_{\alpha-}(A)(x) = \vee \{A(x)-\alpha, 0\}, \forall x \in X$$

$T_{\alpha+}(A)$ and $T_{\alpha-}(A)$ are respectively called the <u>α-up</u> and <u>α-down fuzzy translates</u> of A. We shall refer to $T_{\alpha+}$ and $T_{\alpha-}$ as the <u>fuzzy translation operators</u>.

<u>2.2. Proposition.</u> For any $A \in I^X$ and $\alpha \in I$,

(a) $|Im(T_{\alpha+}(A))| \leqslant |Im(A)|$, and

(b) $|Im(T_{\alpha-}(A))| \leqslant |Im(A)|$

<u>Proof</u>: Trivial.

<u>2.3. Proposition.</u> For any $A \in I^X$ and $\alpha \in I$,

(a) $T_{\alpha+}(A^c) = (T_{\alpha-}(A))^c$, and

(b) $T_{\alpha-}(A^c) = (T_{\alpha+}(A))^c$

<u>Proof</u>. For any $x \in X$, we have,

$$\begin{aligned} T_{\alpha+}(A^c)(x) &= \wedge \{A^c(x) + \alpha, 1\} \\ &= \wedge \{1-A(x) + \alpha, 1\} \\ &= 1 - \vee\{A(x)-\alpha, 0\} \\ &= 1 - T_{\alpha-}(A)(x) \\ &= (T_{\alpha-}(A))^c(x) \end{aligned}$$

The proof of (b) is similar.

We observe that, in general, for a given α, $T_{\alpha+}$ and $T_{\alpha-}$ are not inverse operators. That is, $T_{\alpha+}(T_{\alpha-}(A))$ and $T_{\alpha-}(T_{\alpha+}(A))$ may be different from A.

<u>2.4. Example</u>. Let $X = \{a,b,c\}$ and define $A:X \longrightarrow I$ by $A(a) = 1$, $A(b) = \frac{1}{2}$ and $A(c) = 0$. Put $\alpha = \frac{1}{3}$. Then the actions of $T_{\alpha+}$ and $T_{\alpha-}$ are as follows:

$$T_{\alpha+}(A) : a \longmapsto 1, \quad b \longmapsto \tfrac{5}{6}, \quad c \longmapsto \tfrac{1}{3}$$

$$T_{\alpha-}(A) : a \longmapsto \tfrac{2}{3}, \quad b \longmapsto \tfrac{1}{6}, \quad c \longmapsto 0$$

Therefore, we have,

$$T_{\alpha-}(T_{\alpha+}(A)) : a \longmapsto \tfrac{2}{3}, \; b \longmapsto \tfrac{1}{2}, \; c \longmapsto 0$$

$$T_{\alpha+}(T_{\alpha-}(A)) : a \longmapsto 1, \; b \longmapsto \tfrac{1}{2}, \; c \longmapsto \tfrac{1}{3}$$

Hence,

$$T_{\alpha-}(T_{\alpha+}(A)) \neq A \text{ and } T_{\alpha+}(T_{\alpha-}(A)) \neq A.$$

We further notice that

$$T_{\alpha-}(T_{\alpha+}(A)) \neq T_{\alpha+}(T_{\alpha-}(A))$$

<u>2.5. Theorem</u>. For any $A \in I^X$,

(a) $T_{\alpha-}(T_{\alpha+}(A)) = A \iff \alpha \leqslant 1 - A_{\vee}$

(b) $T_{\alpha+}(T_{\alpha-}(A)) = A \iff \alpha \leqslant A_{\wedge}$

Proof (a)($\Longrightarrow$): An easy computation shows that, for any $x \in X$,

$$T_{\alpha-}(T_{\alpha+}(A))(x) = \begin{cases} A(x), & \text{if } A(x) + \alpha < 1 \\ 1-\alpha, & \text{if } A(x) + \alpha \geqslant 1 \end{cases} \tag{1}$$

Case (i). $A(x) + \alpha < 1, \forall x \in X$: Then $A_{\vee} + \alpha \leqslant 1$; and hence $\alpha \leqslant 1 - A_{\vee}$.

Case (ii). $A(x)+\alpha \geqslant 1$, for some $x \in X$:

By assumption, we have,

$$T_{\alpha-}(T_{\alpha+}(A))(x) = A(x), \forall x \in X.$$

By (1), this implies that $A(x) = 1-\alpha$, for every $x \in X$ for which $A(x)+\alpha \geqslant 1$. Therefore, $A_{\vee} = 1-\alpha$; and hence $\alpha = 1 - A_{\vee}$.

($\Longleftarrow$): Assume that $\alpha \leqslant 1-A_{\vee}$. Then $A_{\vee} + \alpha \leqslant 1$. If $A_{\vee} = 1$, we have, $\alpha = 0$; and hence $T_{\alpha-}(T_{\alpha+}(A)) = A$. Now, let $A_{\vee} < 1$. Since $A(x) \leqslant A_{\vee}, \forall x \in X$, we have

$$A(x)+\alpha \leqslant A_{\vee} + \alpha < 1, \forall\, x \in X.$$

Therefore,

$$T_{\alpha+}(A)(x) \quad = \quad A(x) + \alpha$$

and hence

$$T_{\alpha-}(T_{\alpha+}(A))(x) = A(x), \forall\, x \in X$$

The proof of (b) is similar.

<u>2.6. Proposition</u>. For any $A \in I^X$ and $\alpha, \beta \in I$,

(a) $$T_{\alpha+}(T_{\beta+}(A)) \quad = \quad T_{\beta+}(T_{\alpha+}(A))$$

$$= \begin{cases} T_{(\alpha+\beta)+}(A), & \text{if } \alpha+\beta < 1 \\ \tilde{1} & , \text{if } \alpha+\beta \geqslant 1 \end{cases}$$

(b) $$T_{\alpha-}(T_{\beta-}(A)) \quad = \quad T_{\beta-}(T_{\alpha-}(A))$$

$$= \begin{cases} T_{(\alpha+\beta)-}(A), & \text{if } \alpha+\beta < 1 \\ \tilde{0} & , \text{if } \alpha+\beta \geqslant 1 \end{cases}$$

<u>Proof</u>. Straight forward.

<u>2.7. Remark</u>. For $A, B \in I^X$ with $A \subseteq B$, we have,

$T_{\alpha+}(A) \subseteq T_{\alpha+}(B)$ and $T_{\alpha-}(A) \subseteq T_{\alpha-}(B), \forall\ \alpha \in I$.

Hence the fuzzy translation operators are isotones. Further, they are lattice homomorphisms which are, in general, not isomorphisms.

3. TRANSLATION OF FUZZY GROUPS

In this section we study the action of $T_{\alpha+}$ and $T_{\alpha-}$ on fuzzy groups. We prove that these operators take a fuzzy group to a fuzzy group, and preserve some properties of fuzzy groups.

<u>3.1. Theorem.</u> The following are equivalent:

(a) $F \in \mathfrak{F}(G)$

(b) $T_{\alpha+}(F) \in \mathfrak{F}(G), \forall\ \alpha \in I$, and

(c) $T_{\alpha-}(F) \in \mathfrak{F}(G), \forall\ \alpha \in I$.

<u>Proof</u>: (a) $\Longleftrightarrow$ (b): Let $F \in \mathfrak{F}(G)$ and $\alpha \in I$. For any $x, y \in G$, we have,

$$T_{\alpha+}(F)(xy^{-1}) = \wedge \{F(xy^{-1})+\alpha,\ 1\}$$

$$\geqslant \wedge \{\wedge\{F(x), F(y)\} + \alpha, 1\}, \text{ by Prop.2.2.5}$$

$$= \wedge \{\wedge\{F(x) + \alpha, 1\}, \wedge\{F(y) + \alpha, 1\}\}$$
$$= \wedge \{T_{\alpha+}(F)(x), T_{\alpha+}(F)(y)\}$$

Hence, by Proposition 2.2.5, $T_{\alpha+}(F) \in \mathfrak{F}(G)$.
The converse follows from $T_{0+}(F) = F$.

The Proof of the equivalence of (a) and (c) is similar

If $T_{\alpha+}(F)$ [$T_{\alpha-}(F)$] is a fuzzy subgroup of G for a particular $\alpha \in I$, then it cannot be deduced that F is a fuzzy subgroup of G.

3.2. Example. Let G be the Klein 4-group $\{e,a,b,ab\}$ where $a^2 = b^2 = (ab)^2 = e$. Define $F:G \longrightarrow I$ by

$$e \longmapsto \frac{1}{2}, \quad a \longmapsto \frac{3}{4}, \{b, ab\} \longmapsto \frac{1}{4}.$$

Put $\alpha = \frac{3}{5}$. Then $T_{\alpha+}(F)$ is given by

$$\{e,a\} \longmapsto 1, \quad \{b, ab\} \longmapsto \frac{17}{20}$$

It is easy to verify that $T_{\alpha+}(F) \in \mathfrak{F}(G)$ and $F \notin \mathfrak{F}(G)$

The following results improve theorem 3.1.

<u>3.3. Proposition</u>. If $T_{\alpha+}(F) \in \mathfrak{F}(G)$ for some $\alpha \in I$ with $\alpha < 1 - \bigvee\{F(x) : x \in G \setminus G_F\}$, then $F \in \mathfrak{F}(G)$.

<u>Proof</u>: Let $\alpha < 1 - \bigvee\{F(x) : x \in G \setminus G_F\}$ and $T_{\alpha+}(F) \in \mathfrak{F}(G)$. Then by Proposition 2.2.5, for any $x, y \in G$, we have,

$$T_{\alpha+}(F)(xy^{-1}) \geqslant \bigwedge\{T_{\alpha+}(F)(x),\ T_{\alpha+}(F)(y)\} \qquad (1)$$

<u>Case (i)</u>. $T_{\alpha+}(F)(x) = 1$ and $T_{\alpha+}(F)(y) = 1$: Then $F(x) + \alpha \geqslant 1$ and $F(y) + \alpha \geqslant 1$. By the choice of α, this implies that $x, y \in G_F$. Since G_F is a subgroup of G, we have, $xy^{-1} \in G_F$; and hence $F(xy^{-1}) = F(e) = \bigwedge\{F(x),\ F(y)\}$.

<u>Case (ii)</u>. $T_{\alpha+}(F)(x) = 1$ and $T_{\alpha+}(F)(y) < 1$: Then $F(x) + \alpha \geqslant 1$ and $F(y) + \alpha < 1$. Hence, from (1), we have,

$$\bigwedge\{F(xy^{-1}) + \alpha, 1\} \geqslant \bigwedge\{1, F(y) + \alpha\}$$

$$\Longrightarrow \quad F(xy^{-1}) + \alpha \geqslant F(y) + \alpha$$

$$\Longrightarrow \quad F(xy^{-1}) \geqslant F(y) = \bigwedge\{F(x),\ F(y)\}.$$

<u>Case (iii)</u>. $T_{\alpha+}(F)(x) < 1$ and $T_{\alpha+}(F)(y) < 1$:
Then (1) implies that,

$$\wedge\{F(xy^{-1}) + \alpha,\ 1\} \geq \wedge\{F(x) + \alpha,\ F(y) + \alpha\}$$

$$\Longrightarrow \quad F(xy^{-1}) + \alpha \geq \wedge\{F(x),\ F(y)\} + \alpha$$

$$\Longrightarrow \quad F(xy^{-1}) \geq \wedge\{F(x),\ F(y)\}$$

Hence, by Proposition 2.2.5, $F \in \mathfrak{F}(G)$

<u>3.4. Proposition</u>. If $T_{\alpha-}(F) \in \mathfrak{F}(G)$ for some $\alpha \in I$ with $\alpha < \wedge\{F(x) : F(x) > F_v\}$, then $F \in \mathfrak{F}(G)$.

<u>Proof</u>: Similar to the proof of proposition 3.3

Some results on the translation of fuzzy normal subgroups are given below.

<u>3.5. Theorem</u>. The following are equivalent:

(a) $F \in \mathfrak{FN}(G)$

(b) $T_{\alpha+}(F) \in \mathfrak{FN}(G), \forall\ \alpha \in I$, and

(c) $T_{\alpha-}(F) \in \mathfrak{FN}(G), \forall\ \alpha \in I$

<u>Proof</u>: (a) $\Longleftrightarrow$ (b): Let $F \in \mathfrak{FN}(G)$ and $\alpha \in I$.
By theorem 3.1, $T_{\alpha+}(F) \in \mathfrak{F}(G)$. Now, for $x,y \in G$,

$$
\begin{aligned}
T_{\alpha+}(F)(xy) &= \bigwedge \{F(xy) + \alpha, 1\} \\
&= \bigwedge \{F(yx) + \alpha, 1\}, \text{ since } F \in \mathfrak{F}\mathfrak{N}(G) \\
&= T_{\alpha+}(F)\ (yx)
\end{aligned}
$$

Hence $T_{\alpha+}(F) \in \mathfrak{F}\mathfrak{N}(G)$. The converse follows from $T_{0+}(F) = F$.

The proof of the equivalence of (a) and (c) is similar

Conditions (b) and (c) in the above theorem can be relaxed. We state the results without proof.

<u>3.6. Proposition</u>. If $T_{\alpha+}(F) \in \mathfrak{F}\mathfrak{N}(G)$ for some $\alpha \in I$ with $\alpha < 1 - \bigvee \{F(x) : x \in G \setminus G_F\}$, then $F \in \mathfrak{F}\mathfrak{N}(G)$

<u>3.7. Proposition</u>. If $T_{\alpha-}(F) \in \mathfrak{F}\mathfrak{N}(G)$ for some $\alpha \in I$ with $\alpha < \bigwedge \{F(x) : F(x) > F_\wedge\}$, then $F \in \mathfrak{F}\mathfrak{N}(G)$

<u>3.8. Proposition</u>. For any $F \in \mathfrak{F}(G)$, $x \in G$ and $\alpha \in I$,

$$(T_{\alpha+}(F))_x = T_{\alpha+}(F_x).$$

Proof: For any $g \in G$, we have,

$$(T_{\alpha+}(F))_x\,(g) = T_{\alpha+}(F)\,(gx^{-1})$$

$$= \wedge\left\{F(gx^{-1}) + \alpha, 1\right\}$$

$$= \wedge\left\{Fx(g) + \alpha, 1\right\}$$

$$= T_{\alpha+}(Fx)(g)$$

3.9. Remark. Similar to proposition 3.8, we can prove that for any $F \in \mathfrak{F}(G)$, $x \in G$ and $\alpha \in I$,

(1) $(T_{\alpha-}(F))_x = T_{\alpha-}(Fx)$

(2) $x(T_{\alpha+}(F)) = T_{\alpha+}(xF)$, and

(3) $x(T_{\alpha-}(F)) = T_{\alpha-}(xF)$.

That is, the fuzzy translation operators commute with fuzzy coset formation.

In the next proposition we prove that the fuzzy translation operators and formation of fuzzy conjugates also commute.

3.10. Proposition. For any $F \in \mathfrak{F}(G)$, $x \in G$ and $\alpha \in I$,

(a) $C_x(T_{\alpha+}(F)) = T_{\alpha+}(C_x(F))$, and

(b) $C_x(T_{\alpha-}(F)) = T_{\alpha-}(C_x(F))$.

<u>Proof</u>: (a) For any $g \in G$, we have,

$$\begin{aligned} C_x(T_{\alpha+}(F))(g) &= T_{\alpha+}(F)\ (x^{-1}g\ x) \\ &= \wedge \{F(x^{-1}g\ x) + \alpha, 1\} \\ &= \wedge \{C_x(F)(g) + \alpha,\ 1\} \\ &= T_{\alpha+}(C_x(F))(g) \end{aligned}$$

The proof of (b) is similar

4. TRANSLATION OF FUZZY ABELIAN GROUPS.

In the previous section we observed that some fuzzy group theoretic concepts are well-behaved with respect the fuzzy translation operators, in the sense that, they either remain invariant under the action of the operators or commute with them. But all properties of fuzzy groups are not so. The following example shows that fuzzy translates of a fuzzy abelian subgroup need not be fuzzy abelian.

4.1. Example. Let $G = S_3$. Define $F : G \longrightarrow I$ by

$e \longmapsto \frac{4}{5}$, $\{(123),\ (132)\} \longmapsto \frac{3}{4}$, $\{(12),(13),(23)\} \longmapsto \frac{2}{3}$

Then $F \in \mathcal{F}(G)$. Also $G_F = \{e\}$ is an abelian subgroup of G; and hence $F \in \mathcal{F}\mathcal{A}(G)$. Now, put $\alpha = \frac{1}{2}$ and $\beta = \frac{5}{6}$. Then,

$$T_{\alpha+}(F) = \tilde{1}\ ,\quad T_{\beta-}(F) = \tilde{0}\ .$$

Hence,

$$G_{T_{\alpha+}(F)} = G_{T_{\beta-}(F)} = S_3$$

which is not abelian. Therefore, $T_{\alpha+}(F)$ and $T_{\beta-}(F)$ are not fuzzy abelian

4.2. Proposition. For any $F \in \mathcal{F}(G)$ and $\alpha \in I$, $G_F \subseteq G_{T_{\alpha+}(F)}$ and $G_F \subseteq G_{T_{\alpha-}(F)}$.

Proof. Trivial

4.3. Remark. If F is a constant fuzzy subgroup of G, then for every $\alpha \in I$,

$$G_{T_{\alpha+}(F)} = G_{T_{\alpha-}(F)} = G_F = G;$$

and hence the following statements are equivalent:

(1) $F \in \mathfrak{F}\mathcal{A}(G)$

(2) $T_{\alpha+}(F) \in \mathfrak{F}\mathcal{A}(G), \forall\ \alpha \in I$

(3) $T_{\alpha-}(F) \in \mathfrak{F}\mathcal{A}(G), \forall\ \alpha \in I$; and

(4) G is abelian.

<u>4.4. Theorem</u>. Let F be a non-constant fuzzy subgroup of G. Then,

$$\alpha < 1 - \bigvee\{F(x) : x \in G \setminus G_F\} \Longrightarrow G_{T_{\alpha+}(F)} = G_F$$

The converse holds if F has V-property.

<u>Proof</u>. Let $\alpha < 1 - \bigvee\{F(x) : x \in G \setminus G_F\}$ (1)

By proposition 4.2.,

$$G_F \subseteq G_{T_{\alpha+}(F)} \tag{2}$$

Let $y \in G$ such that $y \notin G_F$. Then

$$F(y) \neq F(e)$$

$\Longrightarrow$ $F(y) < F(e)$, by Prop. 2.2.4.(a)

$\Longrightarrow$ $F(y) \leqslant \bigvee\{F(x) : x \in G \setminus G_F\}$

$\Longrightarrow$ $F(y) + \alpha < 1$, by (1) (3)

Also,

$$F(y) + \alpha < F(e) + \alpha \tag{4}$$

By (3) and (4), we have,

$$F(y) + \alpha < \wedge \{F(e)+\alpha, 1\} = T_{\alpha+}(F)(e)$$

This implies that

$$F(y) < T_{\alpha+}(F)(e)$$

Hence,

$$y \notin G_{T_{\alpha+}(F)}$$

This together with (2) proves that

$$G_{T_{\alpha+}(F)} = G_F.$$

Conversely, let

$$G_{T_{\alpha+}(F)} = G_F \tag{5}$$

Also assume that F has $\vee$-property. If possible, let

$$\alpha \geqslant 1 - \vee \{F(x) : x \in G \setminus G_F\} \tag{6}$$

Since F has $\vee$-property, there exists $x_o \in G \setminus G_F$ such that

$$F(x_o) = \vee \{F(x) : x \in G \setminus G_F\} .$$

By (6), we have,

$$\alpha \geqslant 1 - F(x_o)$$

and hence,

$$F(x_o) + \alpha \geqslant 1$$

Therefore,

$$\begin{aligned} T_{\alpha+}(F)(x_o) &= \bigwedge \{F(x_o) + \alpha, 1\} \\ &= 1 \\ &\geqslant T_{\alpha+}(F)(e) \end{aligned}$$

By proposition 2.2.4(a), this implies that

$$T_{\alpha+}(F)(x_o) = T_{\alpha+}(F)(e)$$

Hence $x_o \in G_{T_{\alpha+}(F)}$. Since $x_o \in G \setminus G_F$, this contradicts (5). Therefore,

$$\alpha < 1 - \bigvee\{F(x) : x \in G \setminus G_F\} .$$

If $|\mathrm{Im}(F)| < \infty$, and in particular if G is a finite group, then F has V-property, and hence we have,

<u>4.5. Corollary</u>. Let $F \in \mathfrak{F}(G)$ with $1 < |\mathrm{Im}(F)| < \infty$. Then,

$$G_{T_{\alpha+}(F)} = G_F \iff \alpha < 1 - \bigvee\{F(x) : x \in G \setminus G_F\} .$$

<u>4.6. Theorem.</u> Let F be a non-constant fuzzy subgroup of G and $\alpha < 1 - \bigvee\{F(x) : x \in G \setminus G_F\}$. Then

$$T_{\alpha+}(F) \in \mathfrak{FA}(G) \Longleftrightarrow F \in \mathfrak{FA}(G)$$

<u>Proof:</u>

$$T_{\alpha+}(F) \in \mathfrak{FA}(G)$$

$\Longleftrightarrow$ $G_{T_{\alpha+}(F)}$ is abelian and $T_{\alpha+}(F) \in \mathfrak{F}(G)$

$\Longleftrightarrow$ G_F is abelian and $F \in \mathfrak{F}(G)$, by theorem 4.4 and prop. 3.3.

$\Longleftrightarrow$ $F \in \mathfrak{FA}(G)$

We now state the corresponding results for $T_{\alpha-}(F)$ without proof.

<u>4.7. Theorem.</u> Let F be a non-constant fuzzy subgroup of G. If $\alpha < F(e)$, then $G_{T_{\alpha-}(F)} = G_F$. The converse holds if F has $\bigvee$-property

<u>4.8. Corollary.</u> Let $F \in \mathfrak{F}(G)$ with $1 < |\mathrm{Im}(F)| < \infty$. Then

$$G_{T_{\alpha-}(F)} = G_F \Longleftrightarrow \alpha < F(e)$$

<u>4.9. Theorem.</u> Let F be a non-constant fuzzy subgroup of G and $\alpha < \bigwedge \{F(x) : F(x) > F_{\wedge}\}$ Then,

$$T_{\alpha-}(F) \in \mathfrak{F}\mathcal{A}(G) \iff F \in \mathfrak{F}\mathcal{A}(G)$$

CHAPTER V

LEVEL SUBGROUPS OF THE TRANSLATES

1. INTRODUCTION.

In this chapter we investigate how the operators $T_{\alpha+}$ and $T_{\alpha-}$ affect the chain of level subgroups of a fuzzy group F. We prove that for any $\alpha \in I$, $C(T_{\alpha+}(F))$ and $C(T_{\alpha-}(F))$ are formed with members of $C(F)$; and with some restriction on α,

$$C(T_{\alpha+}(F)) \equiv C(T_{\alpha-}(F)) \equiv C(F).$$

We proceed to study the interaction between the fuzzy translation operators and group homomorphisms, by observing the change produced in $C(F)$ by their combined action on F. We prove that if f is a group homomorphism, then for any $\alpha \in I$,

$$C(f(T_{\alpha+}(F))) \equiv C(T_{\alpha+}(f(F))) \equiv f(C(T_{\alpha+}(F)))$$

and for some particular choice of α, we have,

$$C(f(T_{\alpha+}(F))) \equiv f(C(F))$$

We give similar results for $T_{\alpha-}(F)$ also.

2. CHAINS OF LEVEL SUBGROUPS OF TRANSLATES

2.1. Proposition. For any $F \in \mathcal{F}(G)$ and $\alpha \in I$, let $F^{\alpha+} = T_{\alpha+}(F)$. Then

$$F_t^{\alpha+} = F_{t-\alpha}, \ \forall\, t \in I$$

Proof:

$$x \in F_t^{\alpha+} \iff F^{\alpha+}(x) \geqslant t$$

$$\iff \wedge\{F(x) + \alpha,\ 1\} \geqslant t$$

$$\iff F(x) + \alpha \geqslant t, \text{ since } t \leqslant 1$$

$$\iff F(x) \geqslant t-\alpha$$

$$\iff x \in F_{t-\alpha}$$

We now investigate the chain of level subgroups of $F^{\alpha+}$ more closely.

2.2. Theorem. Let $F \in \mathcal{F}(G)$ with $\mathrm{Im}(F) = \{t_j : j \in J\}$, $\alpha \in I$, $F^{\alpha+} = T_{\alpha+}(F)$, $\mathrm{Im}\,(F^{\alpha+}) = \{s_m : m \in M\}$, where J and M are suitable indexing sets, $t_{j_0} = F(e)$ and $s_{m_0} = F^{\alpha+}(e)$. Then,

(a) $|M| \leqslant |J|$

(b) If $\alpha \leqslant 1-t_{j_0}$, then $|M| = |J|$ and for every $m \in M$, $s_m = t_j+\alpha$, for some $j \in J$.

(c) If $\alpha > 1-t_{j_o}$, then $s_{m_o} = 1$ and for every $m\ (\neq m_o) \in M$, $s_m = t_j+\alpha$, for some $j \in J$.

(d) For every $m \in M$,

$$F_{s_m}^{\alpha+} = F_{t_j}, \text{ for some } j \in J.$$

(e) If $\alpha < 1- \bigvee\{t_j\colon j \neq j_o\}$, then for every $j \in J$, $F_{t_j} = F_{s_m}^{\alpha+}$, for some $m \in M$.

<u>Proof</u>.

(a) Follows from proposition 4.2.2.

(b) Let $\alpha \leqslant 1-t_{j_o}$. Then $t_j+\alpha \leqslant 1, \forall_j \in J$ and hence, for any $m \in M$,

$$\begin{aligned} s_m &= F^{\alpha+}(x), \text{ for some } x \in G \\ &= \wedge\{F(x) + \alpha, 1\} \\ &= \wedge\{t_j+\alpha, 1\}, \text{ for some } j \in J \\ &= t_j + \alpha \end{aligned}$$

(c) Let $\alpha > 1-t_{j_o}$. Then $t_{j_o} + \alpha > 1$ and hence $s_{m_o} = F^{\alpha+}(e) = \wedge\{t_{j_o}+\alpha, 1\} = 1$.

Now, for $m\ (\neq m_o) \in M$,

$$s_m = F^{\alpha+}(x), \text{ for some } x \in G \text{ with } F^{\alpha+}(x)<1$$

$= \bigwedge\{F(x) + \alpha, 1\}$

$= \quad F(x) + \alpha,$

$= \quad t_j+\alpha$, for some $j \in J$

(d) Follows from Proposition 2.1, (b) and (c).

(e) Let $\alpha < 1 - \bigvee\{t_j : j \neq j_o\}$. Consider any $F_{t_j}, j \in J$. If $j = j_o$, then we have, $F_{t_{j_o}} = F^{\alpha+}_{s_{m_o}}$. If $j \neq j_o$, then $t_j+\alpha < 1$. Hence $t_j+\alpha = s_m$ for some $m \in M$. Therefore, by Proposition 2.1, $F^{\alpha+}_{s_m} = F_{t_j}$.

We now state the corresponding results for $T_{\alpha-}(F)$ without proof.

<u>2.3. Proposition</u>. For $F \in \mathcal{F}(G)$ and $\alpha \in I$, if $F^{\alpha-} = T_{\alpha-}(F)$, then

$$F^{\alpha-}_t = F_{t+\alpha}, \quad \forall\, t \in I.$$

<u>2.4. Theorem.</u> Let $F \in \mathcal{F}(G)$ with $\mathrm{Im}(F) = \{t_j : j \in J\}$, $\alpha \in I$, $F^{\alpha-} = T_{\alpha-}(F)$, $\mathrm{Im}(F^{\alpha-}) = \{s_m : m \in M\}$, where J and M are suitable indexing sets, $t_{j_1} = \bigwedge\{t_j : j \in J\}$, and

$s_{m_1} = \bigwedge\{s_m : m \in M\}$. Then,

(a) $|M| \leqslant |J|$

(b) If $\alpha \leqslant t_{j_1}$ then $|M| = |J|$ and for every $m \in M$, $s_m = t_j - \alpha$, for some $j \in J$.

(c) If $\alpha > t_{j_1}$, then $s_{m_1} = 0$ and for every $m\ (\neq m_1) \in M$, $s_m = t_j - \alpha$, for some $j \in J$.

(d) For every $m \in M$,

$F^{\alpha-}_{s_m} = F_{t_j}$, for some $j \in J$.

(e) If $\alpha < \bigwedge\{t_j\colon j \neq j_1\}$, then for any $j \in J$, $F_{t_j} = F^{\alpha-}_{s_m}$, for some $m \in M$.

<u>2.5. Remark</u>. The above results completely describe the relationship of $C(F)$ with $C(F^{\alpha+})$ and $C(F^{\alpha-})$. For any $\alpha \in I$, we have, $C(F^{\alpha+}) \subseteq C(F)$ and $C(F^{\alpha-}) \subseteq C(F)$. But if $\alpha < 1 - \bigvee\{t_j\colon j \neq j_o\}$, then $C(F^{\alpha+}) \equiv C(F)$ and if $\alpha < \bigwedge\{t_j\colon j \neq j_1\}$, then $C(F^{\alpha-}) \equiv C(F)$. In particular, if

$$\alpha < \bigwedge\{1 - \bigvee\{t_j : j \neq j_o\}, \bigwedge\{t_j\colon j \neq j_o\}\}$$

we have,

$$C(F^{\alpha+}) \equiv C(F) \equiv C(F^{\alpha-}).$$

3. TRANSLATION AND GROUP HOMOMORPHISMS

In this section we study the combined action of the fuzzy translation operators and group homomorphisms on fuzzy groups. We shall continue to use the notations in the last section.

3.1. Theorem. Let $f:G \longrightarrow G^*$ be a surjective group homomorphism and $F^* \in \mathcal{F}(G^*)$. Then for any $\alpha \in I$,

$$C(f^{-1}(T_{\alpha+}(F^*))) \equiv C(T_{\alpha+}(f^{-1}(F^*))) \equiv f^{-1}(C(T_{\alpha+}(F^*)))$$

In particular, if $\alpha < 1-\bigvee\{t_j: j \neq j_o\}$, then

$$C(f^{-1}(T_{\alpha+}(F^*))) \equiv f^{-1}(C(F^*))$$

Proof: Follows from Corollary 3.2.5 and Remark 2.5.

3.2. Remark. In view of Theorem 3.2.7, surjectiveness of f in the above theorem may be replaced by the weaker condition

$$F(G) \cap \mathrm{Fib}_{T_{\alpha+}(F^*)}(s_m) = \emptyset,\ \forall\, m \in M.$$

We now give some similar results without proof.

3.3. Theorem. If $f: G \longrightarrow G^*$ is a surjective group homomorphism, and $F^* \in \mathcal{F}(G^*)$, then for any $\alpha \in I$,

$$C(f^{-1}(T_{\alpha-}(F^*))) \equiv C(T_{\alpha-}(f^{-1}(F^*))) \equiv f^{-1}(C(T_{\alpha-}(F^*)))$$

In particular, if $\alpha < \bigwedge\{t_j : j \neq j_1\}$, then

$$C(f^{-1}(T_{\alpha-}(F^*))) \equiv f^{-1}(C(F^*))$$

<u>3.4. Theorem</u>. Let $f:G \longrightarrow G^*$ be a surjective group homomorphism and F be an f-invariant fuzzy subgroup of G having $\bigvee$-property. Then for any $\alpha \in I$,

$$C(f(T_{\alpha+}(F))) \equiv C(T_{\alpha+}(f(F))) \equiv f(C(T_{\alpha+}(F)))$$

and

$$C(f(T_{\alpha-}(F))) \equiv C(T_{\alpha-}(f(F))) \equiv f(C(T_{\alpha-}(F))).$$

If

$$\alpha < 1-\bigvee\{t_j : j \neq j_o\}, \text{ then}$$

$$C(f(T_{\alpha+}(F))) \equiv f(C(F))$$

and if $\alpha < \bigwedge\{t_j : j \neq j_1\}$, then

$$C(f(T_{\alpha-}(F))) \equiv f(C(F))$$

CONCLUSION

In the thesis, we have made an attempt to study more about fuzzy groups. Several existing results have been extended. However, there remains a lot more to be explored. For example, the membership set has been taken as [0,1]. Taking the inspiration from Goguen [24] and many other researchers the study could be extended, replacing [0,1] by a chain, and later by an arbitrary lattice. Apparently several results of the thesis could be generalised to the case where the membership lattice is a chain.

A study on the structure of the collection of fuzzy subgroups will be a propserous venture. We have made a humble beginning in this direction. We have studied the action of some operators on certain sub-lattices of the complete lattice of all fuzzy subgroups of a fixed group. Lattice properties of the collection of fuzzy groups is a problem to be tackled further.

LIST OF NOTATIONS

2	:	$\{0, 1\}$
I	:	$[0, 1]$
2^X	:	Collection of all crisp subsets of X.
I^X	:	Collection of all fuzzy subsets of X.
$\tilde{\alpha}$	:	Constant fuzzy subset taking the value α
$\emptyset$	:	The empty set
χ_A	:	Characteristic function of A
$A \setminus B$	:	Set complement of B in A.
$\lvert A \rvert$	:	Cardinality of A.
G	:	An arbitrary multiplicative group
$o(G)$	:	Order of G
e	:	Identity element in G.
F_t	:	$\{x \in G: F(x) \geq t\}$
G_F	:	$\{x \in G: F(x) = F(e)\}$
$\mathfrak{F}(G)$	:	Collection of fuzzy subgroups of G.
$\mathfrak{FN}(G)$	:	Collection of all fuzzy normal subgroups of G.
$\mathfrak{FA}(G)$	:	Collection of all fuzzy abelian subgroups of G.

$T_{\alpha+}(A)$	:	α-up translate of A.
$T_{\alpha-}(A)$	:	α-down translate of A.
$Im(A)$	:	Range of A
$\vee$	:	Supremum or lattice join
$\wedge$	:	Infemum or lattice meet
$A_{\vee}$	:	$\vee\{A(x):x \in X\}$, if $A \in I^X$
A	:	$\wedge\{A(x):x \in X\}$, if $A \in I^X$
$C(F)$	:	Chain of level subgroups of F.
N	:	The set of all natural numbers
Z	:	The set of all integers
Q	:	The set of all rational numbers
$A(N)$	:	Group of all permutation on N
$Z(p^{\infty})$	:	Sylow p-subgroup of Q/Z
S_3	:	Symmetric group of degree three.

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