FUZZY SET THEORY AND FUZZY TOPOLOGY

REFLECTIVE AND COREFLECTIVE SUBCATEGORIES IN FUZZY TOPOLOGY

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SHEELA C.

DEPARTMENT OF MATHEMATICS AND STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
COCHIN – 682 022

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DEPARTMENT OF MATHEMATICS AND STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Phone: 855893

COCHIN - 682 022
KERALA, INDIA

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bona fide work done by Ms. Sheela C., under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

S. BABU SUNDAR
(Research Guide)
Reader

C O N T E N T S

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INTRODUCTION

Fuzzy set theory and fuzzy topology are approached as generalizations of ordinary set theory and ordinary topology. We consider fuzzy subsets as functions from a non empty set to a membership lattice. Through out this work we follow the definition of fuzzy topology given by Chang[3] with membership set as an arbitrary complete and distributive lattice.

Category theory is the branch of mathematics which studies the abstract properties of 'sets with structures' and 'structure preserving functions'. It provides a tool by which many parallel techniques used in several branches of mathematics can be linked and treated in a unified manner.

In this work, we present some applications of category theory in Fuzzy Topology based mainly on two notions 'simple reflection and coreflection'. This thesis is presented in five chapters.

In 1974, C.K. Wong [34] introduced the concept of 'fuzzy point belongs to a fuzzy set'. Later the same concept was defined in different ways by Srivastava, Lal and Srivastava [30]. The definitions of the relation '$\in$' of a fuzzy point belonging to a fuzzy set given independently by these authors seem to be very much alike. But on thorough analysis, they are found to differ in certain aspects. This study is included in chapter I. We arrive at the conclusion that the definition given by Piu and Liu[27] is the most appropriate one for fuzzy set theory. A characterization of fuzzy open set is necessary for the study of fuzzy topology. This leads us to study the fuzzy neighbourhood system of a fuzzy point. Piu and Liu [27], Demitri and Pascali [4] introduced the notion of fuzzy neighbourhood system. Both the definitions do not generalize the corresponding definitions of ordinary topology. To rectify this anomaly we introduce a new definition for fuzzy neighbourhood system by the addition of two more axioms. These axioms are necessary in the fuzzy context. In the case of ordinary topology where $L = \{0,1\}$, these axioms are trivially satisfied. The basics of fuzzy topology is strengthened in chapter I.

Pelham Thomas [26] introduced the concept of associated regular spaces. Later P.M. Mathew [22] introduced associated completely regular spaces. "What is the speciality of these spaces among all subcategories, say reflective, coreflective"? Chapter II provides an answer to this question which holds for all those classes for which interesting characterizations of completely regular spaces and regular spaces are known. As a generalization to this, an associated p-space is constructed and their properties are studied. We formulated these concepts in Category theory and obtained a characterization of the simple reflective subcategories of the category of topological spaces.

In the third chapter a fuzzy parallel of associated completely regular spaces is constructed and their properties studied. Fuzzy completely regular space was introduced and studied by Hutton [10,11]. A different version of fuzzy complete regularity is available in [15]. However, we follow the definition given in [11].

The properties of fuzzy completely regular spaces enable us to construct fuzzy associated p-spaces. We obtain this as a generalization of the concepts that we have introduced in the second chapter. In order to widen the range of application we do this in the language of category theory. The results obtained enable us to treat the known theories in an unified manner. Thus we obtained some characterizations of the simple reflective subcategories of the category of fuzzy topological spaces in the fourth chapter.

In the fifth chapter we present some applications of Category theory in Fuzzy Topology based on the notion 'Coreflection'. The coreflective subcategories of the class of fuzzy topological spaces are considered in the works of Lowenand Wuyts [20]. In this chapter we give an internal description of the coreflection. This was motivated by the work of V. Kannan [13]. The notion of topological coreflections are discussed in the paper by Herrlich and Strecker [8]. V. Kannan [13] characterized the smallest coreflective subcategory

of the category of topological spaces TOP, containing a given subcategory $\mathcal{E}$ of TOP. We introduce the class of induced fuzzy topological spaces $I(\mathfrak{H})$ corresponding an arbitrary family of fuzzy topological spaces $\mathfrak{H}$. The study of induced fuzzy topological spaces coincides with the generation of coreflective subcategories of the category of fuzzy topological spaces. We also characterize coreflection as the lattice meet of all finer fuzzy topologies.

Chapter I

FUZZY TOPOLOGY THROUGH FUZZY NEIGHBOURHOOD SYSTEM

The concepts of fuzzy point, fuzzy point belonging to fuzzy subsets and fuzzy neighbourhood are revisited in this chapter. The various definitions by different authors are analysed. Most appropriate definitions are deduced. A new definition of fuzzy neighbourhood systems is introduced. A characterization of fuzzy topology in terms of fuzzy neighbourhoods is arrived at.

In 1974, C.K. Wong [34] introduced the concept of 'fuzzy point belongs to a fuzzy set'. Later the same concept was defined in different ways by Piu and Liu[27], M. Sarkar [23], Srivastava, Lal and Srivastava [30]. The definitions of the relation '$\in$' of a fuzzy point belonging to a fuzzy set, given independently by these authors seem to be very much alike at a glance. But on thorough analysis, they are found to differ in certain aspects. We arrive at the conclusion that the definition given by Piu and Liu [27] is the most appropriate one for fuzzy set theory.

Piu and Liu [27], Demitri and Pascali [4] introduced the notion of fuzzy neighbourhood system. Both the definitions do not generalize the corresponding definition of ordinary topology [12]. To rectify this anomaly we introduce a new definition for fuzzy neighbourhood system.

1.1 PRELIMINARIES

In this section, some definitions and results that are needed later on, are given. Throughout this chapter, X is taken to be a non empty set. A fuzzy subset of X is considered as a function from X to L, where L is a complete and distributive lattice. The least element and greatest element are denoted by 0 and 1 respectively. The set of all fuzzy subsets of X is denoted as L^X.

1.1.1 Definition

A point x of X with a non zero membership value $\ell \in L$ is a fuzzy point of X, and is denoted by $p(x,\ell)$.

1.1.2 Definition [1]

The fuzzy singleton determined by a fuzzy point $p(x,\ell)$ is a fuzzy subset $s(x,\ell)$ such that for $y \in X$,

$$s(x,\ell)(y) = \begin{cases} 0 & \text{if } y \neq x \\ \ell & \text{if } y = x \end{cases}$$

1.1.3. Definition

A subset R of L is join complete (meet complete) if R is closed for arbitrary join operation (meet operation).

1.1.4. Definition

A lattice L is said to be join complete (meet complete) if every subset of L is join complete (meet complete).

The following remarks are immediate consequences of the definitions.

1.1.5. Remark

(i) A join complete lattice with 0 is complete.

(ii) A meet complete lattice with 1 is complete.

(iii) A join complete (meet complete) lattice is a chain.

(iv) L is a finite chain if and only if it is join complete and meet complete.

1.2. A STUDY ON FUZZY MEMBERSHIP

Different definitions of the relation '$\in$' are given and they are analysed.

1.2.1 Definitions [[23],[26],[29]]

Let a be a fuzzy subset and $p(x,\ell)$ a fuzzy point, of X.

(i) $p(x,\ell) \in a$ if and only if $\ell < a(x)$ -- (A)

(ii) $p(x,\ell) \in a$ if and only if $\ell \leqslant a(x)$ -- (B)

(iii) $p(x,\ell) \in a$ if and only if $\ell = a(x)$ -- (C)

1.2.2 Remark

(1) According to definitions (A) and (B)a fuzzy singleton may contain more than one fuzzy point. However, by definition (C), a fuzzy singleton uniquely contains a fuzzy point.

(2) Ordinary set theory can be considered as a special case of fuzzy set theory, taking $L = \{0,1\}$. But then, by definition (A) $p(x,1) \notin s(x,1)$. Hence definition (A) is not considered further. However, $p(x,1) \in s(x,1)$ according to definition (B) and (C).

In ordinary set theory we have for any two subsets A,B of X, $A \subset B$ if and only if $x \in A \Longrightarrow x \in B$. The existence of corresponding characterization for two fuzzy subsets of X is studied.

1.2.3 Theorem

Let a and b be fuzzy subsets of X. Then the following are equivalent.

(1) $a \leqslant b$ (i.e., $a(x) \leqslant b(x)$ for all x)

(2) $p(x,\ell) \in b$ for all $p(x,\ell) \in a$ (we use definition 1.2.1 (B))

Proof: Let (1) holds and $p(x,\ell) \in a$.
Then $\ell \leqslant a(x)$.

i.e., $\ell \leqslant a(x) \leqslant b(x) \Longrightarrow \ell \leqslant b(x)$
$\Longrightarrow p(x,\ell) \in b$ which is (2)

(2) $\Longrightarrow$ (1) is straight forward.

1.2.4 Remark

If we use definition (C) in (1.2.3) (2) in the above theorem, (1) $\not\Longrightarrow$ (2). It may be noted that if we use definition (C), theorem (1.2.3) holds if and only if $L = \{0,1\}$.

1.2.5 Remark

Owing to the fact that definitions (A) and (C) of (1.2.1) are unable to generalize ordinary set theory, hereafter, only definition 1.2.1 (B) is used.

1.2.6 Theorem

Let $\{a_i : i \in I\}$ be an arbitrary family of fuzzy subsets of X and $a = \bigvee_{i \in I} a_i$. "$p(x, \ell) \in a \Rightarrow p(x, \ell) \in a_i$ for some i", is true if and only if L is join complete.

Proof:

($\Rightarrow$) Suppose L is not join complete. Then there is a subset R of L which is not join complete. Let $R = \{r_j : j \in J\}$ and $r = \bigvee_j r_j$. Then $r \neq r_j$ for every j.

Let $\bar{\ell}$ denotes the constant fuzzy subset such that $\bar{\ell}(x) = \ell$ for all $x \in X$, for $\ell \in L$. Clearly $r \neq 0$. Then $p(x,r) \in \bar{r}$, but $p(x,r) \notin \bar{r}_j$ for every j. Thus $p(x,r) \in \bigvee_j \bar{r}_j \not\Rightarrow p(x,r) \in \bar{r}_j$ for some j. Hence L must be join complete.

($\Leftarrow$) Let $\{a_i : i \in I\}$ be an arbitrary family of fuzzy subsets of X and $a = \bigvee_{i \in I} a_i$, and L be join complete. Let $p(x,\ell) \in a$. Then $\ell \leqslant a(x)$. $a(x) = a_i(x)$ for some i since $\{a_i(x) : i \in I\} \subset L$, and L is join complete.

$\therefore$ $\ell \leqslant a_i(x)$. i.e., $p(x,\ell) \in a_i$ for some $i \in I$.

Hence the theorem.

From the preceding theorem we have immediately the following theorem.

1.2.7. Theorem

Let $\{a_i : i \in I\}$ be an arbitrary family of fuzzy subsets of X and $a = \bigwedge_i a_i$. " $p(x,\ell) \in a_i$ for every i $\Rightarrow p(x,\ell) \in a$" **is true** if and only if L is meet complete.

Proof: Dual of theorem (1.2.6).

1.2.8. Remark

The above theorems (1.2.6) and (1.2.7) hold simultaneously if and only if L is a finite chain (1.1.5 iv).

1.3. FUZZY NEIGHBOURHOOD SYSTEM

In this section fuzzy neighbourhood system of a fuzzy point, is studied.

1.3.1. Definition [3]

An ordered pair (X, δ), where δ, is a family of fuzzy subsets of X is called a fuzzy topological space (fts) if δ satisfies the following conditions.

(1) $\bar{0}, \bar{1} \in \delta$

(2) If $a_1, a_2 \in \delta$ then $a_1 \wedge a_2 \in \delta$

(3) If $\{a_i\}_{i \in I}$ is a family of members of δ, then $\bigvee_i a_i \in \delta$

Every member of δ is called a fuzzy open subset of X.

1.3.2. Definition

Let (X, δ) be a fuzzy topological space. A fuzzy subset 'a' is a fuzzy neighbourhood of a fuzzy point $p(x, \ell)$ if and only if there exists a fuzzy open subset 'g' such that $p(x, \ell) \in g \leqslant a$.

In terms of this concept, we have the following trivial but useful characterization of fuzzy open sets.

1.3.3. Theorem

Let a be a fuzzy subset in a fuzzy topological space (X, δ). 'a' is fuzzy open if and only if for each fuzzy point $p(x, \ell) \in a$, a is a fuzzy neighbourhood of $p(x, \ell)$.

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Proof: ($\Longrightarrow$) obvious.

($\Longleftarrow$): Let $p(x,\ell)$ be an arbitrary fuzzy point of a. Then a is a fuzzy neighbourhood of $p(x,\ell)$. Then $g_{p(x,\ell)}$ be fuzzy open such that $p(x,\ell) \in g_{p(x,\ell)} \leq a$.

$\bigvee_{p(x,\ell)\in a} g_{p(x,\ell)} \leq a$. i.e., for every $y \in X$, $\bigvee g_{p(x,\ell)}(y) \leq a(y)$

We claim that $\bigvee g_{p(x,\ell)}(y) = a(y)$ for every $y \in Y$.

Suppose not:

i.e., there exists $a, y \in Y$ such that $\bigvee g_{p(x,\ell)}(y) < a(y)$.

i.e., $g_{p(x,\ell)}(y) < a(y)$ for every $p(x,\ell) \in a$

Then $p(y,a(y)) \in a$ and $p(y,a(y)) \notin g_{p(y,a(y))}$ which is a contradiction. Hence the claim.

1.3.4 Definition

Let (X,δ) be a fuzzy topological space and $p(x,\ell)$ be a fuzzy point. Let $\mathcal{N}_{p(x,\ell)}$ be the set of all fuzzy neighbourhoods of $p(x,\ell)$. The family $\mathcal{N}_{p(x,\ell)}$ is called the fuzzy neighbourhood system at $p(x,\ell)$.

The next theorem is similar to theorem [12(2.5)] about neighbourhood system in ordinary topology [12]. The theorem lists properties of the fuzzy neighbourhood systems which can be used to generate fuzzy neighbourhood system without invoking a fuzzy topology.

1.3.5. Theorem

Let (X, δ) be a fuzzy topological space and $p(x,\ell)$ be a fuzzy point. Let $\mathcal{N}_{p(x,\ell)}$ be the fuzzy neighbourhood system at $p(x,\ell)$. Then

(1) If $a \in \mathcal{N}_{p(x,\ell)}$ then $p(x,\ell) \in a$

(2) For any $a, b \in \mathcal{N}_{p(x,\ell)}$, $a \wedge b \in \mathcal{N}_{p(x,\ell)}$

(3) If $a \in \mathcal{N}_{p(x,\ell)}$ and $b \geqslant a$ then $b \in \mathcal{N}_{p(x,\ell)}$

(4) If $a \in \mathcal{N}_{p(x,\ell)}$ then there exists $b \in \mathcal{N}_{p(x,\ell)}$ such that $b \leqslant a$ and $b \in \mathcal{N}_{p(y,m)}$ for every $p(y,m) \in b$.

Proof:

(1) Let $a \in \mathcal{N}_{p(x,\ell)}$. Then a is a fuzzy neighbourhood of $p(x,\ell)$. i.e., there exists a fuzzy open subset g such that $p(x,\ell) \in g \leqslant a$. By theorem (1.2.3) $p(x,\ell) \in a$.

(2) Let $a, b \in \mathcal{N}_{p(x,\ell)}$. Then there exists fuzzy open sets g, h such that $p(x,\ell) \in g \leqslant a$ and $p(x,\ell) \in h \leqslant b$. Then $p(x,\ell) \in g \wedge h \leqslant a \wedge b$.

$$\Rightarrow a \wedge b \in \mathcal{N}_{p(x,\ell)} \text{ since } g \wedge h \in \delta.$$

(3) Let $a \in \mathcal{N}_{p(x,\ell)}$. Then there exists fuzzy open set g such that $p(x,\ell) \in g \leqslant a$. But $b \geqslant a$. Then $p(x,\ell) \in g \leqslant a \leqslant b$.

$$\Rightarrow b \in \mathcal{N}_{p(x,\ell)} .$$

(4) Let $a \in \mathcal{N}_{p(x,\ell)}$. Then there exists fuzzy open subset b such that $p(x,\ell) \in b \leqslant a$. This $b \in \mathcal{N}_{p(y,m)}$ for every $p(y,m) \in b$ by (1.3.3).

We now introduce fuzzy neighbourhood systems independent of a fuzzy topology.

1.3.6. Theorem

Let X be an arbitrary set, L a join complete lattice and suppose for each fuzzy point $p(x,\ell)$, a non empty family $\mathcal{N}_{p(x,\ell)}$ of fuzzy subsets of X is given satisfying conditions (1) to (4) of theorem (1.3.5). Then there exists a unique fuzzy topology δ on X such that for each fuzzy point $p(x,\ell)$, $\mathcal{N}_{p(x,\ell)}$

coincides with the family of all fuzzy neighbourhoods of $p(x,\ell)$ with respect to δ.

Proof:

Let $\delta = \{a \in L^X : a \in \mathcal{N}_{p(x,\ell)} \text{ for all } p(x,\ell) \in a\}$

We shall now prove that δ is a fuzzy topology on X.

Obviously, $\bar{0}$ belongs to δ. From condition (3) it follows that $\bar{1} \in \mathcal{N}_{p(x,\ell)}$ for all $p(x,\ell) \in \bar{1}$ and then $\bar{1} \in \delta$. Now let $a_1, a_2 \in \delta$ and $p(x,\ell) \in a_1 \wedge a_2$. Then $p(x,\ell) \in a_1$ and $p(x,\ell) \in a_2$ so that $a_1 \in \mathcal{N}_{p(x,\ell)}$ and $a_2 \in \mathcal{N}_{p(x,\ell)}$. By condition (2), $a_1 \wedge a_2 \in \mathcal{N}_{p(x,\ell)}$. Since this holds for every $p(x,\ell) \in a_1 \wedge a_2$, $a_1 \wedge a_2 \in \delta$. Now let $\{a_i : i \in I\}$ be an arbitrary family of members of δ. If $p(x,\ell) \in \bigvee_i a_i$, by theorem (1.2.6), there exists $i_o \in I$ such that $p(x,\ell) \in a_{i_o}$. Since $a_{i_o} \in \delta$, we have $a_{i_o} \in \mathcal{N}_{p(x,\ell)}$ and therefore by condition (3), $\bigvee_i a_i \in \mathcal{N}_{p(x,\ell)}$. Since this holds for every $p(x,\ell) \in \bigvee_i a_i$, it follows that $\bigvee_i a_i \in \delta$. Thus the family δ is fuzzy topology on X.

Condition (4) means that for every fuzzy point $p(x,\ell)$ and $a \in \mathcal{N}_{p(x,\ell)}$, there exists a fuzzy open subset 'b' such that $b \in \mathcal{N}_{p(x,\ell)}$ and $b \leqslant a$. From condition (1) it follows that $p(x,\ell) \in b$. Hence every member of $\mathcal{N}_{p(x,\ell)}$ is a fuzzy neighbourhood of $p(x,\ell)$ with respect to δ.

Conversely, let 'a' be a fuzzy neighbourhood of $p(x,\ell)$ with respect to δ. There exists 'b' fuzzy open such that $p(x,\ell) \in b \leqslant a$. Since $b \in \delta$, we have $b \in \mathcal{N}_{p(x,\ell)}$ and thus by condition (3) $a \in \mathcal{N}_{p(x,\ell)}$. Thus the fuzzy neighbourhoods of $p(x,\ell)$ with respect to δ are precisely the members of $\mathcal{N}_{p(x,\ell)}$ for each $p(x,\ell)$.

If δ' is a fuzzy topology for X, where $\mathcal{N}_{p(x,\ell)}$ is again the fuzzy neighbourhood system at $p(x,\ell)$ for each fuzzy point $p(x,\ell)$, then $\delta = \delta'$ (by 1.3.3).

To show that the condition: 'L is join complete' is necessary, consider the following example.

1.3.7. Example

Let $L = [0,1]$

Define $\forall x,\ \forall \ell < \frac{1}{2},\ \mathcal{N}_{p(x,\ell)} = \left\{ a \in L^X :\ a \geqslant \bar{\ell} \right\}$

$$\forall x,\ \forall \ell \geqslant \tfrac{1}{2},\ \mathcal{N}_{p(x,\ell)} = \left\{ \bar{1} \right\}$$

$\mathcal{N}_{p(x,\ell)}$ satisfies all the conditions in the above theorem.

Define fuzzy subsets a_n such that $a_n(x) = \frac{1}{2} - \frac{1}{n}$ for $n \in \mathbb{N}$, $n > 2$, for every x. Now

$p(x,\ell) \in a_n \Rightarrow \ell \leq \frac{1}{2} - \frac{1}{n}$. i.e., $\ell \leq a_n \Rightarrow a_n \in \mathcal{N}_{p(x,\ell)}$

$\therefore$ a_n's are fuzzy open. But $\bigvee a_n = \overline{\frac{1}{2}}$ is not fuzzy open since $p(x, \frac{1}{2}) \in \overline{\frac{1}{2}}$ and $\overline{\frac{1}{2}} \notin \mathcal{N}_p(x, \frac{1}{2})$.

A modified version of theorem (1.3.5) by introducing two more properties of the fuzzy neighbourhood system is given below.

1.3.8. **Theorem**

Let (X,δ) be a fuzzy topological space and $p(x,\ell)$ be a fuzzy point. Let $\mathcal{N}_{p(x,\ell)}$ be the fuzzy neighbourhood system at $p(x,\ell)$. Then the following hold.

(1) If $a \in \mathcal{N}_{p(x,\ell)}$ then $p(x,\ell) \in a$

(2) For any $a,b \in \mathcal{N}_{p(x,\ell)}$, $a \wedge b \in \mathcal{N}_{p(x,\ell)}$.

(3) If $a \in \mathcal{N}_{p(x,\ell)}$ and $b \geq a$ then $b \in \mathcal{N}_{p(x,\ell)}$

(4) If $a \in \mathcal{N}_{p(x,\ell)}$ then there exists $b \in \mathcal{N}_{p(x,\ell)}$ such that $b \leqslant a$ and $b \in \mathcal{N}_{p(y,m)}$ for all $p(y,m) \in b$.

(5) If $\ell, m \in L \setminus \{0\}$ and $\ell < m$ then $\mathcal{N}_{p(x,m)} \subset \mathcal{N}_{p(x,\ell)}$

(6) If $\ell = \bigvee_{\alpha \in J} \ell_\alpha$ and $a \in \mathcal{N}_{p(x,\ell_\alpha)}$ for every α, then $a \in \mathcal{N}_{p(x,\ell)}$.

Proof:

(1) to (4) are proved in theorem (1.3.5). Hence we need only to prove (5) and (6).

(5): Let $\ell < m$ and $a \in \mathcal{N}_{p(x,m)}$. Then there exists a fuzzy open subset g such that $p(x,m) \in g \leqslant a$. i.e., $m \leqslant g(x) \leqslant a(x)$ for all x.

But $\ell < m$. Then $\ell < m \leqslant g(x) \leqslant a(x)$ for all x.

i.e., $p(x,\ell) \in g \leqslant a$.

i.e., $a \in \mathcal{N}_{p(x,\ell)}$. Therefore $\mathcal{N}_{p(x,m)} \subset \mathcal{N}_{p(x,\ell)}$.

(6): $a \in \mathcal{N}_{p(x,\ell_\alpha)}$ for α and let $\ell = \bigvee_\alpha \ell_\alpha$. Then there exist fuzzy open sets g_α's such that $p(x,\ell_\alpha) \in g_\alpha \leqslant a$ for every α.

i.e., $\ell_\alpha \leq g_\alpha(x) \leq a(x)$ for every α and for every x.

i.e., $\vee \ell_\alpha \leq \bigvee_\alpha g_\alpha(x) \leq a(x)$

i.e., $p(x, \bigvee_\alpha \ell_\alpha) \in \bigvee_\alpha g_\alpha \leq a$.

Since $\bigvee_\alpha g_\alpha$ is fuzzy open, $a \in \mathcal{N}_{p(x,\ell)}$.

Now we partially generalise theorem (1.3.6) by omitting the condition of L being join complete.

1.3.9. Theorem.

Let L be a complete chain. If with each fuzzy point $p(x,\ell)$ is associated a family $\mathcal{N}_{p(x,\ell)}$ of fuzzy subsets of X satisfying conditions (1) to (6) of theorem (1.3.8) are satisfied, then there exists a unique fuzzy topology on X with $\mathcal{N}_{p(x,\ell)}$ as the fuzzy neighbourhood system at $p(x,\ell)$.

Proof:

Let $\delta = \{a \in L^X : a \in \mathcal{N}_{p(x,\ell)}$ for all $p(x,\ell) \in a\}$
We shall now prove that δ is a fuzzy topology on X.

Obviously $\bar{0} \in \delta$. From condition (3) it follows that $\bar{I} \in \delta$. Let $a_1, a_2 \in \delta$ and $p(x,\ell) \in a_1 \wedge a_2$. Then $p(x,\ell) \in a_1$ and $p(x,\ell) \in a_2$ so that $a_1 \in \mathcal{N}_{p(x,\ell)}$ and $a_2 \in \mathcal{N}_{p(x,\ell)}$. By condition (2) $a_1 \wedge a_2 \in \mathcal{N}_{p(x,\ell)}$. Since this holds for every $p(x,\ell) \in a_1 \wedge a_2$, $a_1 \wedge a_2 \in \delta$. Let $\{a_\alpha\}_{\alpha \in I}$ be an arbitrary family of members of δ and $\bigvee a_\alpha = a$. We shall now show that $a \in \delta$. Let $p(x,\ell) \in a$. Then $\ell \leqslant a(x)$. Then we have two cases:

(i) $\ell < a(x)$, (ii) $\ell = a(x)$.

Case (i) $\ell < a(x)$.

Since L is a chain, there exists $\alpha_0 \in I$ such that $\ell \leqslant a_{\alpha_0}(x) < a(x)$. Since $p(x,a_{\alpha_0}(x)) \in a_{\alpha_0}$, $a_{\alpha_0} \in \mathcal{N}_{p(x,a_{\alpha_0}(x))}$. By (5) $a_{\alpha_0} \in \mathcal{N}_{p(x,\ell)}$. But $a_{\alpha_0} \leqslant a$ and by (3) $a \in \mathcal{N}_{p(x,\ell)}$.

Case (ii) $\ell = a(x)$.

We have $a_\alpha \in \mathcal{N}_{p(x,a_\alpha(x))}$ for every α and $a_\alpha \leqslant a$. By (3) $a \in \mathcal{N}_{p(x,a_\alpha(x))}$ for every α. By (6) $a \in \mathcal{N}_{p(x,a(x))}$.

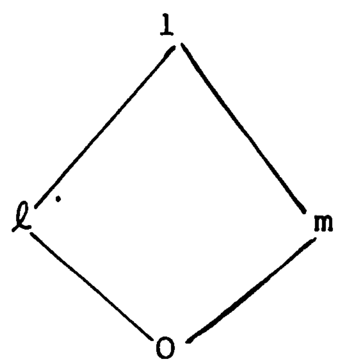
i.e., $a \in \mathcal{N}_{p(x,\ell)}$. Thus in both cases $p(x,\ell) \in a \Longrightarrow a \in \mathcal{N}_{p(x,\ell)}$. i.e., $a \in \delta$. Thus the family δ defines a fuzzy topology on X. Last part of the theorem follows from theorem (1.3.6).

The following examples illustrate that the condition 'L is a complete chain' in the above theorem is not necessary.

1.3.10. Example.

Let the membership lattice be

L =



The theorem (1.3.9) holds in this case because, in this case also if $k < \bigvee_{i \in I} \ell_i \Rightarrow k \leq \ell_i$ for some $i \in I$ since

Case (i) if some ℓ_i' is 1 then the proof is trivial.

Case (ii) none of the ℓ_i's, is exclusively from $\{0,\ell,1\}$ or $\{0,m,1\}$ then, being completeness the proof trivial.

Case (iii) Otherwise some of the ℓ_i's are ℓ and others are either m or 0, then $\vee \ell_i = 1$ and $k < \vee \ell_i$ $\Rightarrow k = \ell$ or m or 0 and the result holds.

The following examples show that the conditions (5) and (6) are independent from the others and is necessary in the fuzzy context.

1.3.11. I Example

Let L = [0,1].

Define $\forall x, \forall \ell < \frac{1}{2}, \mathcal{N}_{p(x,\ell)} = \{a \in L^X : a \geq \bar{\ell}\}$

$\forall x, \forall \ell \geq \frac{1}{2}, \mathcal{N}_{p(x,\ell)} = \{\bar{1}\}$

Here $\mathcal{N}_{p(x,\ell)}$ satisfies (1) — (5) but does not satisfy property (6).

Define fuzzy subsets a_n such that

$a_n(x) = \frac{1}{2} - \frac{1}{n}$ for every x and for every $n > 2$,

$n \in \mathbb{N}$

$\bigvee a_n = \bar{\frac{1}{2}}$

Let $\delta = \{a \in L^X : a \in \mathcal{N}_{p(x,\ell)}$ for all $p(x,\ell) \in a\}$.

Now $a_n \in \delta$, for every n.

In particular $p(x, \frac{1}{2}) \in \bigvee_n a_n = \bar{\frac{1}{2}}$

But $\bar{\frac{1}{2}} \notin \mathcal{N}_{p(x, \frac{1}{2})}$.

$\therefore$ $\bigvee a_n \notin \delta$. Hence δ is not a fuzzy topology.

1.3.12. II Example

Let L = [0,1]. Define for every x, for every $\ell < \frac{1}{2}$, $\mathcal{N}_{p(x,\ell)} = \{\bar{1}\}$, $\forall x, \forall \ell \geq \frac{1}{2}$, $\mathcal{N}_{p(x,\ell)} = \{a \in L^X : a \geq \bar{\ell}\}$

We can easily observe that this family satisfies properties (1) —— (4) and (6) but does not satisfy (5).

Define $\delta = \{a \in L^X : a \in \mathcal{N}_{p(x,\ell)},$ for all $p(x,\ell) \in a\}$ and let $a_n \in \delta$ where $a_n(x) = \frac{1}{2} - \frac{1}{n}$ for every x, $n > 2$. $p(x, \frac{1}{2} - \frac{1}{n}) \in \bar{\frac{1}{2}}$. But $\bar{\frac{1}{2}} \notin \mathcal{N}_{p(x, \frac{1}{2} - \frac{1}{n})}$

1.3.13. Remark

Conditions (5) and (6) of theorem (1.3.9) do not have parallels in ordinary topology. However, they trivially true when $L = \{0,1\}$. It may also be noted that L, then is a complete chain as well as a join complete lattice. Theorem (1.3.9) shows that fuzzy neighbourhood systems is indeed a generalization of neighbourhood system in ordinary topology. Thus the characterization theorem generalizes the corresponding theorem in ordinary topology.

Chapter II

ASSOCIATED p-SPACE AND SIMPLE REFLECTION

J. Pelham Thomas [26] studied associated regular spaces and proved that to every topology T on a set X, there exists a regular topology $T^* \subset T$ on X such that the continuous functions from X to a regular space Y are the same for T and T*. P.M. Mathew [22] introduced the concept of associated completely regular spaces as follows:

A closed subset A of X is * closed if there exists a continuous map $f:X \longrightarrow [0,1]$ such that $f(x)=0$ and $f(a)=1$ for each $a \in A$ and $x \notin A$. Complement of * closed sets are * open. Then $T^* = \{G \subset X : G \text{ is * open}\}$ is a completely regular topology coarser than T.

In this chapter, these concepts are generalized to the case of any initial property p, and associated p-space are defined. The above concepts are reformulated using the language of category theory and obtained a characterization of the simple reflective subcategories of the category of topological spaces.

For any two topological space X and Y, C(X,Y) denotes the set of all continuous functions from X to Y.

2.1. PRELIMINARIES

2.1.1 Definition

Let $\mathcal{C}$ be an arbitrary class of topological spaces and let (X,T) be a topological space. A subset U of X is called $\mathcal{C}^*$ open if there is a continuous function $f:X \longrightarrow Y$, $Y \in \mathcal{C}$ and an open subset V of Y such that $U = f^{-1}(V)$. Complements of $\mathcal{C}^*$ open subsets are called $\mathcal{C}^*$ closed.

2.1.2 Remark

Every $\mathcal{C}^*$ open set ($\mathcal{C}^*$ closed set) is open (closed).

2.1.3 Lemma

Let X,Y be topological spaces and $f:X \longrightarrow Y$ be continuous. If V is $\mathcal{C}^*$ open ($\mathcal{C}^*$ closed) in Y with respect to a class $\mathcal{C}$ of topological spaces then $f^{-1}(V)$ is $\mathcal{C}^*$ open ($\mathcal{C}^*$ closed) in X.

Proof:

Let V be $\mathcal{C}^*$ open in Y. Clearly $f^{-1}(V)$ is open. Since V is $\mathcal{C}^*$ open, there exists $g: Y \longrightarrow Z$ continuous, $Z \in \mathcal{C}$ and $H \subset Z$, open such that $V = g^{-1}(H)$.

i.e., $$f^{-1}(V) = f^{-1}\left(g^{-1}(H)\right) = (g \circ f)^{-1}(H)$$

Now $g \circ f : X \longrightarrow Z$ is continuous such that

$$f^{-1}(V) = (g \circ f)^{-1}(H)$$

$\Rightarrow$ $f^{-1}(V)$ is $\mathcal{C}^*$ open in X.

2.1.4 Definition [25]

Let $\{f_i : X \longrightarrow Y_i\}_{i \in I}$ be a family of functions from a common domain X to topological spaces Y_i. Then the topology generated by the subbasis $\{f_i^{-1}(V_i) : V_i \subset Y_i\}_{i \in I}$ is called the initial topology of the family $\{f_i\}$.

2.1.5 Remark

When the family of functions is a singleton $\{f\}$ the initial topology is simply the preimage topology by f.

2.1.6. Definition

Let p be a topological property. p is said to be an initial property if for every family of functions $\{f_i : X \longrightarrow Y_i\}_{i \in I}$, whenever each Y_i has p, the initial space of $\{f_i\}$ also has p.

2.1.7. Theorem

If $\mathcal{C}$ is a class of topological spaces which satisfies an initial property-p, then

$T^* = \{U \subset X : U \text{ is } \mathcal{C}^* \text{ open}\}$ is a topology on X and $(X,T^*) \in \mathcal{C}$.

Proof:

$\emptyset \in T^*$ and $X \in T^*$ (Trivial). Let $U_1, U_2 \in T^*$ then there exists continuous functions $f_1 : X \longrightarrow Y_1$, $Y_1 \in \mathcal{C}$ and $f_2 : X \longrightarrow Y_2$, $Y_2 \in \mathcal{C}$ and $V_1 \subset Y_1$, $V_2 \subset Y_2$ such that $U_1 = f_1^{-1}(V_1)$ and $U_2 = f_2^{-1}(V_2)$. Let $Y = Y_1 \times Y_2$. Since $\mathcal{C}$ satisfies initial property, $\mathcal{C}$

is productive. Therefore $Y \in \mathcal{P}$. Let $f: X \longrightarrow Y$ be the evaluation map.

Then
$$U_1 \cap U_2 = f_1^{-1}(V_1) \cap f_2^{-1}(V_2)$$
$$= \bigcap_{i=1}^{2} (\pi_i \circ f)^{-1}(v_i)$$
$$= \bigcap_{i=1}^{2} [f^{-1}(\pi_i^{-1}(V_i))]$$
$$= f^{-1}[\bigcap_{i=1}^{2} \pi_i^{-1}(V_i)]$$
$$\Longrightarrow U_1 \cap U_2 \in T^*$$

Let $\{U_i\}_{i \in I}$ be an arbitrary family of elements of T^*. Let f_i, Y_i, U_i be the corresponding functions, spaces (elements in $\mathcal{P}$) and open subsets. Take $Y = \pi_i Y_i (Y \in \mathcal{P})$ and let f be the evaluation map of the family $\{f_i\}$. Let $H_i \subset Y$ be such that $H_i = Y_1 \times Y_2 \times \ldots \times U_i \times Y_{i+1} \times \ldots$ then $\bigcup_i U_i = f^{-1}(\bigcup_i H_i) \in T^*$. Hence T^* is a topology on X. Clearly $T^* \subset T$ and T^* is the weak topology induced by all continuous functions in $C((X,T), \mathcal{P})$. Therefore $(X,T^*) \in \mathcal{P}$.

2.1.8. Corollary

$T = T^*$ if and only if $(X,T) \in \mathcal{L}$.

2.1.9. Theorem.

Let (X,T) be a topological space. Then the set of all continuous functions from (X,T) to $(Y,\mathcal{U})$ for any Y in $\mathcal{L}$ ($\mathcal{L}$ as in theorem 2.1.7) is the same as that from (X,T^*) to Y. Further T^* is characterized by this property.

Proof:

Let Y be any element in $\mathcal{L}$ and $f:(X,T^*) \longrightarrow Y$ be continuous. Since $T^* \subset T$, $f:(X,T) \longrightarrow Y$ is also continuous. On the other hand, let $f:(X,T) \longrightarrow Y$ be continuous. Let U be open in Y. Since $Y \in \mathcal{L}$ by (2.1.8) U is $\mathcal{L}^*$ open in Y. Then $f^{-1}(U)$ is $\mathcal{L}^*$ open in X (2.1.3). i.e., $f^{-1}(U)$ is open in (X,T^*).

Hence $f : (X,T^*) \longrightarrow Y$ is continuous. Thus for any $Y \in \mathcal{L}$, the continuous functions $X \longrightarrow Y$ are the same for T and T^*.

Let T' be any topology on X and $(X,T') \in \mathcal{L}$ such that $T' \subset T$. Since $T^* \subset T$, the identity map

$i : (X,T) \longrightarrow (X,T^*)$ is continuous. Since $(X,T^*) \in \mathcal{L}$, by assumption the identity map $i: (X,T') \longrightarrow (X,T^*)$ is continuous. i.e., $T^* \subset T'$. Similarly we have $T' \subset T^*$. Hence $T' = T^*$.

We call T^* as the associated topology and (X,T^*) as the associated p-space.

2.1.10. Definition [12]

A space X is said to be completely regular if for any point $x \in X$ and closed set A not containing x, there exists a continuous function $f:X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in A$, where the continuity is with respect to the usual topology on the unit interval [0,1].

2.1.11. Definition [25]

A topological space (X,T) is called an R_0 space if it satisfies the condition: $x \in G \in T \Rightarrow \bar{x} \subset G$

2.1.12. Definition [25]

A topological space X is called an R_1 space if it satisfies the condition: $\bar{x} \neq \bar{y} \Rightarrow$ x and y have disjoint neighbourhoods.

2.1.13. **Definition**

A space X is zero dimensional if and only if each point of X has a neighbourhood base consisting of open-closed sets.

2.1.14. **Remark**

Regularity, complete regularity, R_0, R_1, zero dimensionality are initial properties. Hence it may be noted that the theorem (2.1.7) proved in the general setting of associated p-spaces is equivalent to several theorems, one for each of these particular classes.

2.1.15. **Theorem**

Let $\mathcal{E}$ be the collection of all topological spaces having an initial property p.

For a space (X,T), T* is the lattice join of all topologies on X weaker then T and which belong to $\mathcal{E}$.

Proof:

Let $\{T_\alpha\}_{\alpha \in J}$ be the collection of all topologies on X which belong to $\mathcal{C}$ and weaker than T. Since $(X,T^*) \in \mathcal{C}$ and weaker than T,

$$T^* \subset \bigvee_{\alpha} T_\alpha \qquad .. \qquad (1)$$

For every $\alpha \in J$, $T_\alpha \subset T$. Then the identity map $i : (X,T) \longrightarrow (X, T_\alpha)$ is continuous. By theorem [2.1.9] $i : (X,T^*) \longrightarrow (X,T_\alpha)$ is continuous. Hence $T_\alpha \subset T^*$ for every $\alpha \in J$.

$$\Longrightarrow \bigvee_{\alpha} T_\alpha \subset T^* \qquad .. \qquad (2)$$

From (1) and (2)

$$T^* = \bigvee_{\alpha} T_\alpha .$$

2.1.16. Theorem.

If T_1 and T_2 are topologies on X such that $T_1 \subset T_2$, then $T_1^* \subset T_2^*$.

Proof:

We have $T_1 \subset T_2$. Then $T_1^* \subset T_2$. Since T_1^* is coarser than T_2, by (2.1.15) $T_1^* \subset T_2^*$.

2.2 A DESCRIPTION OF SIMPLE REFLECTION

In this section, we express some of the results of the earlier section in the language of categories. We assume that all considered categories are full and replete subcategories of the category of topological spaces: TOP.

2.2.1. Definition

Let $\mathcal{A}$ be a class of objects A,B,C ... $\in obj(\mathcal{A})$ together with

(1) a family of mutually disjoint sets $\{Mor(A,B)\}$ for all objects $A,B \in Obj(\mathcal{A})$ whose elements $f,g,h, \ldots \in Mor(A,B)$ are called morphisms and

(2) a family of maps

$$\{Mor(A,B) \times Mor(B,C) \ni (f,g) \longmapsto g \circ f \in Mor(A,C)\}$$

for all $A,B,C \in obj(\mathcal{A})$, called compositions. $\mathcal{A}$ is called a category if it satisfies the following axioms:

(i) **Associativity:** For all $A,B,C,D \in obj(\mathcal{A})$ and all $f \in Mor(A,B)$, $g \in Mor(B,C)$ and $h \in Mor(C,D)$, we have $h\circ(g\circ f) = (h\circ g)\circ f$

(ii) **Identity:** For each object $A \in obj(\mathcal{A})$ there is a morphism $1_A \in Mor(A,A)$, called the identity such that we have

$f\circ 1_A = f$ and $1_A\circ g = g$ for all $B,C \in obj(\mathcal{A})$ and all $f \in Mor(A,B)$ and $g \in Mor(C,A)$.

2.2.2. **Definition**

A category $\mathcal{B}$ is called a subcategory of a category $\mathcal{A}$ if

(i) $obj\ (\mathcal{B}) \subset obj(\mathcal{A})$

(ii) $Mor\ \mathcal{B} \subset Mor\ (\mathcal{A})$

(iii) **The composition** of morphisms in $\mathcal{B}$ coincides with the composition of the same morphisms in $\mathcal{A}$.

(iv) For every object B of $\mathcal{B}$, the identity morphism on B coincides with that in $\mathcal{A}$.

2.2.3 **Definition**

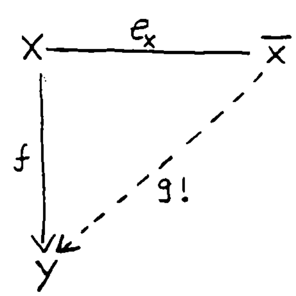
A category is said to be replete if it contains all isomorphic images of its members.

2.2.4 **Definition**

A subcategory $\mathcal{R}$ of a category $\mathcal{A}$ is said to be a full subcategory if $Mor_{\mathcal{R}}(A,B) = Mor_{\mathcal{A}}(A,B)$ for each pair of objects A,B in $\mathcal{R}$ where $Mor_{\mathcal{R}}(A,B)$ denotes the set of all morphisms in $\mathcal{R}$ which have domain A and range B.

2.2.5. **Definition**

A subcategory $\mathcal{R}$ of a category $\mathcal{A}$ is reflective in $\mathcal{A}$, if for each object **X in** $\mathcal{A}$ there exists an object $\bar{X}$ in $\mathcal{R}$ and a morphism $e_X: X \longrightarrow \bar{X}$ such that given any $Y \in \mathcal{R}$ and a morphism $f: X \longrightarrow Y$, there exists a unique morphism $g: \bar{X} \longrightarrow Y$ such that the diagram commutes.



Here $\bar{X}$ is called the reflection of X in $\mathcal{R}$.

2.2.6 Definition

A reflective subcategory $\mathcal{R}$ of TOP is said to be simple reflective subcategory of TOP if in the above definition each e_X is an one-one onto map i.e. a bijection.

For any initial property p, the collection of p-spaces forms a category with continuous functions as morphisms. The category is denoted as $\mathcal{P}$. This category is shown to be simple reflective in the category of topological spaces.

2.2.7 Theorem

The subcategory $\mathcal{P}$ of p-spaces (the class of topological spaces which satisfies initial property-p) is simple reflective in TOP. Here the reflection is the associated p-space.

Proof:

By theorem (2.1.7 and 2.1.9) for each $(X,T) \in$ TOP, we have (X,T^*) in $\mathcal{P}$ such that the continuous functions X to Y, $Y \in \mathcal{P}$ are the same for T and T*. If we take e_X

as the identity function from (X,T) to (X,T^*), then we have for every continuous $f:X \longrightarrow Y$, $Y \in \mathcal{P}$, there is a unique $\overline{f} : (X,T^*) \longrightarrow Y$ ($\overline{f} = f$ itself) such that $f = \overline{f} \circ e_X$. By definition (2.2.5) $\mathcal{P}$ is a simple reflective subcategory of TOP and that (X,T^*) is the simple reflection of (X,T) in $\mathcal{P}$.

The following theorem establishes that p-spaces are the only simple reflective subcategories of TOP.

2.2.8. Theorem

Let $\mathcal{S}$ be a simple reflective subcategory of TOP. Then $\mathcal{S}$ is the category of p-spaces. i.e., $\mathcal{S} = \mathcal{P}$

Proof:

If $\mathcal{S}$ is a simple reflective subcategory, then the reflection morphism is a bijection. The reflection of any space may be taken to have the same set with a weaker topology so that the reflection morphisms are identity maps. If $(X,\overline{T})$ is the reflection of (X,T) in $\mathcal{S}$, then we have $C((X,T),\mathcal{S}) = C((X,\overline{T}),\mathcal{S})$. i.e., $\overline{T}$ is the weak topology induced by all continuous maps from $(X,T) \rightarrow \mathcal{S}$

such that $(X,\overline{T}) \in \mathcal{G}$. Hence $\mathcal{G}$ is the class of p-spaces. Simple reflection of an object in TOP is characterized in the following theorem.

2.2.10. **Theorem.**

Let $\mathcal{L}$ be a simple reflective subcategory of TOP and $(X,T) \in$ TOP. $(X,\overline{T})$ is the simple reflection of (X,T) if and only if $\overline{T}$ is the lattice join of all topologies on X weaker than T which belong to $\mathcal{L}$.

Proof:

$(\Rightarrow)$ Since $(X,\overline{T})$ is the simple reflection of (X,T), $(X,\overline{T}) \in \mathcal{L}$. But $\overline{T} = T^*$ (2.2.9). Also by (2.1.15) T^* is the join of all weaker topologies on X which belong to $\mathcal{L}$. Hence $\overline{T}$ do so.

$(\Leftarrow)$ Given $\overline{T} = \bigvee_{\alpha} T_\alpha$, $T_\alpha \subset T$ and $(X,T_\alpha) \in \mathcal{L}$.

Since $\mathcal{L}$ is simple reflective, by (2.1.6) and (2.2.5) and $(X, \vee T_\alpha)$ can be got as the subspace of the product of the spaces (X,T_α).

i.e., $(X, \vee T_\alpha) \in \mathcal{L}$

i.e., $(X, \overline{T}) \in \mathcal{L}$.

Let (X,T') be the simple reflection of (X,T) in $\mathcal{L}$. Then $(X,\overline{T}) = \overline{X}$ is finer than the reflection of X in $\mathcal{L}$. i.e., $T' \subset \overline{T}$. But $i : (X,T) \longrightarrow (X,\overline{T})$ has to split through the reflection of X. Hence $\overline{T} = T'$.

Remark:

We note that the simple reflection $(X,\overline{T})$ of (X,T) in a simple reflective subcategory induces a map $T \longrightarrow \overline{T}$ from the lattice of topologies onto itself. The map taking T to $\overline{T}$ is order preserving in the lattice of topologies, since, if T_1, T_2 are two topologies on X such that $T_1 \subset T_2$. Then by (2.1.16). $T_1^* \subset T_2^*$

i.e. $\overline{T}_1 \subset \overline{T}_2$

Chapter III

ASSOCIATED FUZZY COMPLETELY REGULAR SPACES

A study on associated completely regular spaces was made by P.M. Mathew [22a]. The study was further carried out in [22b]. A fuzzy analogue of the results are made in this chapter. Fuzzy completely regular spaces was introduced and studied by Hutton [10,11]. A different version of fuzzy complete regularity is available [15]. However, we follow the definition given in [11].

Through out this chapter 'L' denotes a complete and distributive lattice with order reversing involution : ^{o} .

3.1. DEFINITIONS

Concepts that are required for the study in this chapter, are defined below.

3.1.1. Definition [11]

A fuzzy unit interval [0,1] (L) is the set of all monotonic decreasing maps $\lambda : \mathbb{R} \longrightarrow L$ for which

$$\lambda(t) = \begin{cases} 1 & \text{for } t < 0 \\ 0 & \text{for } t > 1, \end{cases}$$ $t \in \mathbb{R}$, after the identi-

fication of $\lambda : \mathbb{R} \longrightarrow L$ and $\mu : \mathbb{R} \longrightarrow L$

if $\lambda(t-) = \mu(t-)$ for $t \in \mathbb{R}$

$\lambda(t+) = \mu(t+)$ for $t \in \mathbb{R}$, where

$$\lambda(t+) = \bigvee_{s>t} \lambda(s); \quad \lambda(s-) = \bigwedge_{s<t} \lambda(s)$$

We define fuzzy topology on $[0,1](L)$ as the topology generated by the subbase

$\{L_t, R_t \mid t \in \mathbb{R}\}$, where $L_t : [0,1](L) \longrightarrow L$

and $R_t : [0,1](L) \longrightarrow L$ defined by

$$L_t(\lambda) = \lambda(t-)^0$$

$$R_t(\lambda) = \lambda(t+)$$

This topology is called the usual topology for $[0,1](L)$. $[0,1](L)$ and its topology reduces to $[0,1]$ and its usual topology for $L = \{0,1\}$.

3.1.2. **Definition** [11]

A fuzzy topological space (X,δ) is fuzzy completely regular if for each $a \in \delta$ there is a family of fuzzy sets $\{u_i : i \in I\}$ and a family of maps $\{f_i : X \longrightarrow [0,1](L)\}$ such that $\bigvee_i u_i = a$ and $u_i(x) \leqslant f_i(x)(1-) \leqslant f_i(x)(0+) \leqslant a(x)$ $\forall x, \forall i$.

3.1.3. **Definition**

If (Y,γ) is a fuzzy topological space and $f: X \longrightarrow Y$ is a function, then $\{f^{-1}(b) : b \in \gamma\}$ is a fuzzy topology on X and is called the preimage fuzzy topology of γ by f.

3.1.4. **Definition** [3]

A map $f: X \longrightarrow Y$ between fuzzy topological spaces (X,δ) and (Y,γ) is said to be fuzzy continuous if $f^{-1}(v) \in \delta$ for each $v \in \gamma$.

3.1.5. **Definition**

A property p is said to be preimage invariant property if whenever a fuzzy topological space Y has P, the preimage space also has p.

3.2. FUZZY COMPLETELY REGULAR SPACES

Properties of fuzzy completely regular spaces are studied in this section. The lattice **theoretic** properties of the class of fuzzy completely regular spaces are also studied.

3.2.1. Theorem

If (Y,γ) is a fuzzy completely regular space and $f:X \longrightarrow Y$ is a function, then $\delta^* = \{f^{-1}(b) : b \in \gamma\}$ is a fuzzy completely regular topology on X.

i.e. Fuzzy complete regularity is pre-image invariant.

Proof:

$\bar{0}, \bar{1} \in \delta^*$

Let $a_1, a_2 \in \delta^*$. Then $a_1 = f^{-1}(b_1)$, $a_2 = f^{-1}(b_2)$ for some $b_1, b_2 \in \gamma$.

$$a_1 \wedge a_2 = f^{-1}(b_1) \wedge f^{-1}(b_2)$$

$$= f^{-1}(b_1 \wedge b_2)$$

$$\Longrightarrow a_1 \wedge a_2 \in \delta^* \text{ since } b_1 \wedge b_2 \in \gamma .$$

Let $\{a_i : i \in I\}$ be an arbitrary family of members of δ. Then $a_i = f^{-1}(b_i)$ for some $b_i \in \gamma$.

$$\bigvee_i a_i = \bigvee_i f^{-1}(b_i) = f^{-1}(\bigvee_i b_i)$$

Since $\bigvee_i b_i \in \gamma$, $\bigvee_i a_i \in \delta^*$.

Hence δ^* is a fuzzy topology on X.

Let $a \in \delta^*$. Then $a = f^{-1}(u)$ for some $u \in \gamma$. Since (Y,γ) is fuzzy completely regular, there exists fuzzy sets k_i such that $u = \bigvee_i k_i$ and fuzzy continuous functions $g_i : (Y,\gamma) \longrightarrow [0,1]\ (L)$ such that

$$k_i(y) \leqslant g_i(y)(1-) \leqslant g_i(y)\ (0\ +) \leqslant u(y) \cdots\cdots (1)$$

for every $y \in Y$
for every i

$$\text{Then} \quad a = f^{-1}(u) = f^{-1}(\bigvee_i k_i) = \bigvee_i f^{-1}(k_i) \qquad (2)$$

Let $h_i = g_i \circ f$

$\therefore h_i(x) = (g_i \circ f)(x) = g_i(f(x))$, where

$f(x) \in Y$.

$$h_i(x)(1-) = g_i(f(x))(1-) \leq g_i(f(x))(0+) \leqslant uf(x)$$

By (1) $k_i f(x) \leqslant g_i f(x)(1-) \leqslant g_i f(x)(0+) \leqslant uf(x)$,

since $f(x) \in Y$.

i.e. $f^{-1}(k_i)(x) \leqslant h(x)(1-) \leqslant h(x)(0+) \leqslant f^{-1}(u)(x)$

$= a(x)$ (3)

i.e. for $a \in \delta^*$ there is a family of fuzzy sets $\{f^{-1}(k_i)\}$ and a family of maps $\{h_i : (X, \delta^*) \rightarrow [0,1](L)\}$ such that $\bigvee_i f^{-1}(k_i) = a$ and

$$f^{-1}(k_i)(x) \leqslant h_i(x)(1-) \leqslant h_i(x)(0+) \leqslant a(x)$$

for every x,
for every i.

i.e. δ^* is fuzzy completely regular.

3.2.2 Theorem [15]

Fuzzy complete regularity is productive.

A unique fuzzy topological space associated with a given fuzzy topological space is constructed and proved that it is fuzzy completely regular.

3.2.3. Theorem

If δ is a fuzzy topology on X, then there is a unique fuzzy completely regular topology δ^*, coarser than δ such that if Y is any fuzzy completely regular space, the fuzzy continuous maps $(X,\delta) \longrightarrow Y$ are the fuzzy continuous maps $(X, \delta^*) \longrightarrow Y$.

Proof:

Define δ^* to be the family of all $a \in \delta$ for which there exists a fuzzy completely regular space Y, a fuzzy continuous map $f: (X, \delta) \longrightarrow Y$ and a fuzzy open subset b of Y for which $a = f^{-1}(b)$

Claim: δ^* is a fuzzy topology.

$\bar{0}, \bar{1} \in \delta^*$ (trivial)

Let $a_1, a_2 \in \delta^*$. Let $Y = Y_1 \times Y_2$ (Y is fuzzy completely regular) and $f:(X,\delta) \longrightarrow Y$ be such that $f(x) = (f_1(x), f_2(x))$

$$a_1 \wedge a_2 = f_1^{-1}(b_1) \wedge f_2^{-1}(b_2), \text{ where } a_1 = f_1^{-1}(b_1), \; a_2 = f_2^{-1}(b_2)$$

$$= (\pi_1 \circ f)^{-1}(b_1) \wedge (\pi_2 \circ f)^{-1}(b_2)$$

$$= f^{-1}\Big((\pi_1^{-1}(b_1) \wedge \pi_2^{-1}(b_2)\Big)$$

$$\Longrightarrow a_1 \wedge a_2 \in \delta^*$$

Let $\{a_i\}_{i \in I}$ be an arbitrary family of elements of δ^*. Let f_i, Y_i, b_i be the corresponding fuzzy maps, fuzzy completely regular spaces and fuzzy open sets.

Take $Y = \pi_i Y_i$ and let $f: (X, \delta) \longrightarrow Y$ be such that $f(x) = (f_i(x))_i$. Then f is fuzzy continuous. Let h_i be a fuzzy subset of Y such that $h_i = Y_1 x Y_2 \ x \ldots x \ b_i \ x \ldots$

Then $\bigvee_i a_i = f^{-1}(\bigvee_i h_i) \in \delta^*$

δ^* is a fuzzy topology on X.

Clearly $\delta^* \subset \delta$.

Let $a \in \delta^*$. Then there exists a fuzzy continuous function $f: X \longrightarrow (Y, \gamma)$, (Y, γ) fuzzy completely regular and a fuzzy open subset b of Y such that $a = f^{-1}(b)$.

Since (Y,γ) is fuzzy completely regular and $b \in \gamma$ implies that there exist a family of fuzzy subsets $\{u_i : i \in I\}$ and a family of maps $\{g_i : (Y,\gamma) \rightarrow [0,1](L)\}$ such that $\bigvee_i u_i = b$ and

$$u_i(y) \leq g_i(y)\,(1-) \leqslant g_i(y)(0+) \leqslant b(y) \qquad \text{(I)}$$

for every $i \in I$
for every $y \in Y$

But $a = f^{-1}(b) = f^{-1}(\bigvee_i u_i) = \bigvee_i f^{-1}(u_i)$

Since $f(x) \in Y$ and (I) is true for all $y \in Y$,

$$u_i\, f(x) \leqslant g_i\, f(x)(1-) \leqslant g_i f(x)(0+) \leqslant bf(x)$$

for every $i \in I$

i.e. $f^{-1}(u_i)(x) \leqslant (g_i \circ f)(x)\,(1-) \leqslant (g_i \circ f)(x)(0+) \leqslant f^{-1}(b)(x)$

i.e. $f^{-1}(u_i)(x) \leqslant h_i(x)(1-) \leqslant h_i(x)\,(0+) \leqslant a(x)$, (II)

for every $x \in X$
for every $i \in I$

i.e. $a = \bigvee_i f^{-1}(u_i)$ and (II) shows that δ^* is fuzzy completely regular.

Let Y be a fuzzy completely regular space and $f: (X,\delta) \longrightarrow Y$ be fuzzy continuous. Let a be fuzzy open in Y, then by definition of δ^*, $f^{-1}(a)$ is fuzzy open in δ^*.

i.e. $f : (X, \delta^*) \longrightarrow Y$ is fuzzy continuous. On the other hand, if $g: (X, \delta^*) \longrightarrow Y$ is fuzzy continuous. Since $\delta \supset \delta^*$, $g : (X, \delta) \longrightarrow Y$ is also fuzzy continuous. Thus for any fuzzy completely regular space Y, fuzzy continuous maps from X to Y are the same for δ and δ^*

Let δ' be a fuzzy completely regular topology on X, weaker than δ. Since $\delta^* \subset \delta$, the identity map $i : (X, \delta) \longrightarrow (X, \delta^*)$ is fuzzy continuous. Since (X, δ^*) is fuzzy completely regular, by assumption, the identity map $i : (X, \delta') \longrightarrow (X, \delta^*)$ is also fuzzy continuous.

i.e. $$\delta^* \subset \delta' \qquad (1)$$

Conversely $\delta' \subset \delta$, $i : (X, \delta) \longrightarrow (X, \delta')$ is fuzzy continuous. Since (X, δ') is fuzzy completely regular and by assumption, $i : (X, \delta^*) \longrightarrow (X, \delta')$ is also fuzzy continuous.

i.e., $$\delta' \subset \delta^* \qquad (2)$$

Hence from (1) and (2) $\delta' = \delta^*$.

We call (X, δ^*) as the fuzzy completely regular space associated with (X, δ).

3.2.4 Corollary

$\delta = \delta^*$ if and only if δ is fuzzy completely regular.

Two characterizations of the associated fuzzy completely regular topology is given below.

3.2.5 Theorem

For a fuzzy topological space (X, δ), δ^* is the weak topology induced by all fuzzy continuous maps from $(X, \delta) \longrightarrow [0,1]$ (L).

Proof:

The proof is straight forward from the definition.

3.2.6 Theorem

For a fuzzy topological space (X, δ), δ^* is the least upper bound of all fuzzy completely regular topologies on X weaker than δ.

Proof:

Let $\{\delta_\alpha\}_{\alpha \in I}$ be the collection of all fuzzy completely regular topologies on X weaker than δ. Since δ * is fuzzy completely regular and weaker than δ,

$$\delta^* \subset \bigvee_\alpha \delta_\alpha \tag{1}$$

For every $\alpha \in I$, $\delta_\alpha \subset \delta$. Then the identity map i: $(X, \delta) \longrightarrow (X, \delta_\alpha)$ is fuzzy continuous.

By theorem (3.2.3) i: $(X, \delta^*) \longrightarrow (X, \delta_\alpha)$ is fuzzy continuous

$$\Rightarrow \delta_\alpha \subset \delta^* \quad \text{for every } \alpha \in I$$

$$\Rightarrow \bigvee_\alpha \delta_\alpha \subset \delta^* \tag{2}$$

From (1) and (2), $\delta^* = \bigvee_\alpha \delta_\alpha$.

3.2.7 Remark

If δ_1 and δ_2 are fuzzy topologies on X such that $\delta_1 \subset \delta_2$, then $\delta_1^* \subset \delta_2^*$.

Proof:

Since $\delta_1^* \subset \delta_1$, δ_1^* is a fuzzy completely regular topology weaker than δ_2. But by (3.2.6) δ_2^* is the join of all fuzzy completely regular topologies weaker than it, therefore, $\delta_1^* \subset \delta_2^*$.

Now we shall show that fuzzy complete regularity is sup invariant.

3.2.8. Result

Join of an arbitrary collection of fuzzy completely regular topologies on a set X is fuzzy completely regular.

Proof:

Let $\{\delta_\alpha\}_{\alpha \in I}$ be a collection of fuzzy completely regular topologies on X. Let $\delta = \bigvee_\alpha \delta_\alpha$. By (3.2.6) δ^* is the join of all fuzzy completely regular topologies weaker than δ. Since $\delta_\alpha \subset \delta$ and δ_α's are fuzzy completely regular $\delta_\alpha \subset \delta^*$. i.e., $\delta \subset \delta^*$. But always $\delta^* \subset \delta$. Hence $\delta = \delta^*$, a fuzzy completely regular topology.

3.2.9. Remark

The collection of all fuzzy completely regular topologies on a set forms a complete lattice under the usual ordering.

Chapter IV

A DESCRIPTION OF SIMPLE REFLECTION IN THE CATEGORY OF FUZZY TOPOLOGICAL SPACES

This chapter is devoted to the study of simple reflective subcategories of the category of Fuzzy topological spaces FTOP. The properties of fuzzy completely regular spaces enable us to construct fuzzy associated p- spaces. We obtain this as a generalization of the concepts that we have introduced in the second chapter. Some characterizations of the simple reflective sub categories of FTOP are obtained and their properties are studied.

4.1 $\mathcal{C}_*$- FUZZY OPEN SETS

A method to construct an associated fuzzy topology is presented. For this $\mathcal{C}_*$ fuzzy open subsets are introduced corresponding to a given class $\mathcal{C}$ of fuzzy topological spaces.

4.1.1 Definition

Let (X, δ) be a fuzzy topological space and $\mathcal{C}$ be an arbitrary class of fuzzy topological spaces. A fuzzy subset 'a' of X is called $\mathcal{C}_*$ fuzzy open if there is a fuzzy open subset 'b' of some Y in $\mathcal{C}$ and a fuzzy continuous function $f:X \longrightarrow Y$ such that $a = f^{-1}(b)$.

4.1.2 **Remark**

Every $\mathcal{C}_*$ fuzzy open subset is fuzzy open.

Now we shall show that inverse image of a $\mathcal{C}_*$ fuzzy open subset is $\mathcal{C}_*$ fuzzy open.

4.1.3 **Theorem**

Let (X,δ), (Y,γ) be fuzzy topological spaces and $f:X \longrightarrow Y$ be fuzzy continuous. If 'a' is $\mathcal{C}_*$ fuzzy open subset of Y then $f^{-1}(a)$ is a $\mathcal{C}_*$ fuzzy open subset of X.

4.1.4 **Definition** [17]

Let X be a set, $\{(Y_i, \gamma_i)\}_i$ be fuzzy topological spaces. The weakest fuzzy topology on X making all the functions $f_i: X \longrightarrow Y_i$, $i \in I$ fuzzy continuous is called the initial fuzzy topology of the family of functions.

We now introduce the fuzzy analogue of initial property.

4.1.5. **Definition**

A fuzzy topological property 'p' is said to be a fuzzy initial property if for every family of functions $f_i : X \longrightarrow Y_i$, whenever each fuzzy topological space Y_i has p, the initial space of $\{f_i\}$ also has p.

4.1.6. **Remark**

Fuzzy initial property $\Rightarrow$ productive.

The class of fuzzy topological spaces which satisfies a fuzzy initial property p is denoted as fip(p). An example is given below.

4.1.7. **Theorem**

Fuzzy completely regularity is a fuzzy initial property.

Proof:

Fuzzy complete regularity is preimage invariant and sup invariant [(3.2.1) and (3.2.8)]. The initial fuzzy topology induced by $\{f_i\}$ is also the supremum of the preimage fuzzy topologies by f_i.

Corresponding to any fuzzy topological space, and an initial property, a unique fuzzy initial property topology can be associated with it. $F\mathcal{C}[(X,\delta), (Y,\beta)]$ denotes the collection of fuzzy continuous functions with domain X and codomain Y.

4.1.8. **Theorem**

Let (X,δ) be a fuzzy topological space and $\mathcal{L}$ be the class of fip(p), determined by an initial property 'p'. Then there is a unique fuzzy topology δ^* weaker than δ such that $(X,\delta^*) \in \mathcal{L}$ and for any (Y,β) in $\mathcal{L}$, $F\mathcal{C}[(X,\delta),(Y,\beta)] = F\mathcal{C}[(X,\delta^*),(Y,\beta)]$.

Proof:

Define $\delta^* = \{a \in L^X : a \text{ is } \mathcal{L}_* \text{ fuzzy open}\}$

$\overline{0}, \overline{1} \in \delta^*$ choosing X with indiscrete fuzzy topology and f as the identity function.

Let $a_1, a_2 \in \delta^*$. Then there exist fuzzy open subsets b_1, b_2 of some Y_1, Y_2 in $\mathcal{L}$ and fuzzy continuous functions f_1, f_2 on X such that $a_1 = f_1^{-1}(b_1)$ and

$a_2 = f_2^{-1}(b_2)$. Then the fuzzy topological product $Y_1 \times Y_2 \in \mathcal{C}$ [4.1.6]. Let $f = (f_1, f_2)$ be the fuzzy evaluation map defined in (X, δ).

$$
\begin{aligned}
a_1 \wedge a_2 &= f_1^{-1}(b_1) \wedge f_2^{-1}(b_2) \\
&= (\pi_1 \circ f)^{-1}(b_1) \wedge (\pi_2 \circ f)^{-1}(b_2) \\
&= f^{-1}(\pi_1^{-1}(b_1)) \wedge f^{-1}(\pi_2^{-1}(b_2)) \\
&= f^{-1}[\pi_1^{-1}(b_1) \wedge \pi_2^{-1}(b_2)]
\end{aligned}
$$

$$\Rightarrow a_1 \wedge a_2 \in \delta^*$$

Let $\{a_i\}_{i \in I}$ be an arbitrary family of elements of δ^*. Let f_i, Y_i, b_i be the corresponding fuzzy continuous functions, elements in $\mathcal{C}$ and fuzzy open subsets such that $a_i = f_i^{-1}(b_i)$. Let Y be the product space $\prod_i Y_i$ and f be the evaluation map $(f_i(x))_i$ defined on (X, δ), $Y \in \mathcal{C}$ [4.1.6].

$$\text{Let } k_j = \prod_i u_i, \text{ where } u_i = \begin{cases} b_j & \text{for } i = j \\ Y_i & \text{for } i \neq j \end{cases}$$

Clearly k_j's belong to Y.

Then $\bigvee_i a_i = f^{-1}(\bigvee_i k_i) \in \delta^*$

Hence δ^* is a fuzzy topology on X. Further $\delta^* \subset \delta$, follows from remark (4.1.2).

Clearly δ^* is the weak topology induced by all fuzzy continuous maps in $F\mathcal{C}[(X,\delta),\mathcal{C}]$.

$\therefore$ $(X, \delta^*) \in \mathcal{C}$.

Now we shall show that

$$F\mathcal{C}[(X, \delta),Y)] = F\mathcal{C}[(X,\delta^*),Y] \text{ for any } Y \in \mathcal{C}$$

Let $f: (X, \delta^*) \longrightarrow Y$ be a fuzzy continuous function. Since $\delta^* \subset \delta$, $f: (X, \delta) \longrightarrow Y$ is also fuzzy continuous.

Let $g: (X, \delta) \longrightarrow Y$ be a fuzzy continuous function and b a fuzzy open subset of Y. Then by (4.1.3) $g^{-1}(b)$ is $\mathcal{C}_*$ fuzzy open in (X, δ). i.e., $g^{-1}(b)$ is fuzzy open in (X, δ^*).

i.e., $g: (X, \delta^*) \longrightarrow Y$ is fuzzy continuous.

Thus for any $Y \in \mathcal{L}$, the fuzzy continuous function $X \longrightarrow Y$ are the same for δ and δ^*. It remains to show that δ^* is unique.

Let δ' be any fuzzy topology weaker than δ with $(X, \delta') \in \mathcal{L}$ and $F\mathcal{C}[(X, \delta'), Y] = F\mathcal{C}[(X, \delta), Y]$ for every $Y \in \mathcal{L}$. Since $\delta^* \subset \delta$, the identity function $i : (X, \delta) \longrightarrow (X, \delta^*)$ is fuzzy continuous. Since $(X, \delta^*) \in \mathcal{L}$, by assumption the identity function $i : (X, \delta') \longrightarrow (X, \delta^*)$ is also fuzzy continuous.

i.e., $\delta^* \subset \delta'$ (1)

Since $\delta' \subset \delta$, $i : (X, \delta) \longrightarrow (X, \delta')$ is fuzzy continuous. Since $(X, \delta') \in \mathcal{L}$, by first part of theorem, $i : (X, \delta^*) \longrightarrow (X, \delta')$ is also fuzzy continuous. i.e., $\delta' \subset \delta^*$ (2)

Hence $\delta' = \delta^*$

i.e., δ^* is unique.

4.1.9 Definition

For any fuzzy topological space (X, δ) and a fuzzy initial property p, the δ^* obtained in the theorem (4.1.8) is called the associated p-fuzzy topology of δ and (X, δ^*) is called the associated fuzzy p-space.

4.1.10. Remark

For a fuzzy topological space (X, δ), δ^* is the lattice join of all weaker fuzzy topologies on X which belong to $\mathcal{P}$.

4.1.11. Remark

If δ_1, δ_2 are fuzzy topologies on X such that $\delta_1 \subset \delta_2$, then $\delta_1^* \subset \delta_2^*$.

4.2 SIMPLE REFLECTION IN FTOP

The class of fuzzy topological spaces with fuzzy continuous functions as morphisms form a category and is denoted as FTOP. In this section, simple reflective subcategories of FTOP is characterized in terms of associated fuzzy topological spaces.

4.2.1. **Theorem**

Let $\mathcal{C}$ be a class of fip(p), then $\mathcal{C}$ is simple reflective in FTOP and the simple reflection of any fuzzy topological space (X, δ) is the associated fuzzy topological space. Conversely, any simple reflective subcategory of FTOP must arise only in this way.

Proof:

For each $(X, \delta) \in$ FTOP, we have

$$(X, \delta^*) \in \mathcal{C} \text{ with } F\mathcal{C}[(X, \delta), Y] = F\mathcal{C}[(X, \delta^*), Y]$$

for all $Y \in \mathcal{C}$ by (4.1.8). If we take e_X as the identity map from (X, δ) to (X, δ^*), then we can see that for every $f: X \longrightarrow Y$, $Y \in \mathcal{C}$, there is a unique $\bar{f} : (X, \delta^*) \longrightarrow Y$ ($\bar{f} = f$ itself) such that $f = \bar{f} \circ e_X$. Since e_X is a bijection it follows that $\mathcal{C}$ is a simple reflective subcategory of FTOP and that (X, δ^*) is the simple reflection of (X, δ) in $\mathcal{C}$.

Conversely, let $\mathcal{C}$ be a simple reflective subcategory of FTOP. Then the reflection morphism e_X is a bijection. Let $(X, \bar{\delta})$ be the simple reflection of (X, δ) in $\mathcal{C}$. Then

$(X, \bar{\delta})$ is homeomorphic to a weaker fuzzy topology on X. By the definition of reflection then, any fuzzy continuous function f: $X \longrightarrow Y$, $Y \in \mathcal{C}$ splits uniquely through this e_X and hence $f:(X, \bar{\delta}) \longrightarrow Y$ is also fuzzy continuous. Thus $\bar{\delta}$ is a weaker fuzzy topology having the same family $F\mathcal{C}[(X, \delta), \mathcal{C}]$ of fuzzy continuous functions. Since $\bar{\delta}$ is the weak topology induced by the family of fuzzy continuous maps in $F\mathcal{C}[(X, \bar{\delta}), \mathcal{C}]$ and $(X, \bar{\delta}) \in \mathcal{C}$, $\mathcal{C}$ is the class of fip(p). Moreover $\bar{\delta} = \delta^*$.

It is shown that for the case of simple reflective subcategory remark (4.1.10) can be further strengthened.

4.2.2. Theorem

Let $\mathcal{C}$ be a simple reflective subcategory of FTOP. Let $(X, \delta) \in$ FTOP and $\bar{\delta} \subset \delta$. $(X, \bar{\delta})$ is the simple reflection of (X, δ) if and only if $\bar{\delta}$ is the lattice join of all weaker fuzzy topologies on X which belong to $\mathcal{C}$.

Proof:

($\Rightarrow$) If $\mathcal{C}$ is simple reflective and $(X, \bar{\delta})$ is the simple reflection of (X, δ) then $\bar{\delta} = \delta^*$ (4.2.1). By (4.1.10) δ^* is the lattice join of all weaker fuzzy topologies on X which belong to $\mathcal{C}$.

($\Leftarrow$) Since $\mathcal{C}$ is simple reflective subcategory it is the class of fip(p) (4.2.1). Let $(X, \delta) \in$ FTOP and $\bar{\delta} = \bigvee_\alpha \delta_\alpha$, $\delta_\alpha \subset \delta$ and $(X, \delta_\alpha) \in \mathcal{C}$. Since $\mathcal{C}$ is the class of fip(p), $(X, \bigvee_\alpha \delta_\alpha) \in \mathcal{C}$. i.e. $(X, \bar{\delta}) \in \mathcal{C}$.

Let (X, δ') be the simple reflection of (X, δ) in $\mathcal{C}$. Then $\delta' \subset \bar{\delta}$. By the definition of simple reflection, the identity function

$i : (X, \delta) \longrightarrow (X, \bar{\delta})$ has to split through the reflection of (X, δ). Hence $\bar{\delta} = \delta'$.

Remark

The simple reflection $(X, \bar{\delta})$ of (X, δ) in a simple reflective subcategory induces a map $\delta \longmapsto \bar{\delta}$ from the lattice of fuzzy topologies onto itself. We note that this map is order preserving **and idempotent.**Also the class of fuzzy topological spaces on a set X which satisfies a fuzzy initial property-p forms a complete lattice under the usual ordering.

Chapter V

COREFLECTIVE SUBCATEGORIES IN FTOP

In this chapter we present some applications of category theory in Fuzzy topology based mainly on the notion 'coreflection'. The coreflective subcategories of the class of fuzzy topological spaces are considered in the works of Lowen and Wuyts [20]. Their results do not serve for our purpose. We give an internal description of the coreflection in this chapter. This was motivated by the work of V. Kannan [13].

The notion of topological coreflections are discussed in the papers of Herrlich and Strecker ([8], [9]). V. Kannan [13] characterized the smallest coreflective subcategory of the category of topological spaces TOP, containing a given subcategory $\mathcal{E}$ of TOP.

We introduce the class of induced fuzzy topological spaces I($\mathfrak{F}$), where $\mathfrak{F}$ is an arbitrary class of fuzzy topological spaces. The study of induced fuzzy topological spaces provides the ground work for investigating properties of coreflective subcategories of the category of fuzzy topological spaces FTOP. We obtain coreflective hull of $\mathfrak{H}$ and some results related to fuzzy topological properties. Also we discuss the behaviour of coreflection of fuzzy topological space in the lattice of fuzzy topologies.

5.1 INDUCED FUZZY TOPOLOGICAL SPACE

Corresponding to any class of fuzzy topological spaces, we introduce a class of induced fuzzy topological spaces.

Let $F\mathcal{C}[\mathfrak{H},X]$ denotes the family of all fuzzy continuous functions with domain in $\mathfrak{H}$ and codomain, (X,δ). $F\mathcal{C}[\mathfrak{H},X]$ induces a fuzzy topology on X.

5.1.1. Theorem

Let $\mathfrak{H}$ be a class of fuzzy topological spaces and let (X,δ) be a fuzzy topological space. Then the family

$$\bar{\delta} = \{a \in L^X : f^{-1}(a) \text{ is fuzzy open in the domain of } f \text{ for each } f \text{ in } F\mathcal{C}[\mathfrak{H},X]\}$$

is a fuzzy topology on X such that $\delta \subset \bar{\delta}$.

Proof:

$\overline{0}, \overline{1} \in \overline{\delta}$ (Trivial)

Let $a_1, a_2 \in \overline{\delta}$. Since $f_\beta^{-1}(\bigwedge_{i=1}^{2} a_i) = \bigwedge_{i=1}^{2} f_\beta^{-1}(a_i)$

fuzzy open in X_β, $\bigwedge_{i=1}^{2} a_i \in \overline{\delta}$. That is $a_1 \wedge a_2 \in \overline{\delta}$.

Let $\{a_\alpha\}_{\alpha \in J} \subset \overline{\delta}$. We have $f_\beta^{-1}(\bigvee_\alpha a_\alpha) = \bigvee_\alpha f_\beta^{-1}(a_\alpha)$,

fuzzy open in X_β. Therefore $\bigvee_\alpha a_\alpha \in \overline{\delta}$.

Hence $\overline{\delta}$ defines a fuzzy topology on X. Clearly

$\delta \subset \overline{\delta}$.

5.1.2. Remark

The members of $\overline{\delta}$ are called induced fuzzy open subsets of X.

5.1.3. **Definition**

A fuzzy topological space (X, δ) is said to be induced by $\mathfrak{F}$ if every induced fuzzy open subset of X is fuzzy open.

5.1.4. Notation

$I(\mathfrak{H})$ denotes the family of all fuzzy topological spaces induced by $\mathfrak{H}$.

5.1.5. Example

Let $\mathfrak{H}$ be the family of all fuzzy discrete spaces. Then a fuzzy topological space X is induced by $\mathfrak{H}$ if and only if X is fuzzy discrete.

5.1.6. Theorem

A fuzzy topological space (X, δ) is induced by $\mathfrak{H}$ if and only if it has the strongest topology having the same family of X-valued fuzzy continuous functions from members of $\mathfrak{H}$.

Proof:

$(\Rightarrow)$ Let (X, δ) be induced by $\mathfrak{H}$. Let δ_1 be a fuzzy topology such that $\delta \subset \delta_1$ and

$FC[\mathfrak{H}, (X, \delta_1)] = FC[\mathfrak{H}, (X, \delta)]$

We have to show that $\delta_1 = \delta$.

Let $a \in \delta_1$. Then $f^{-1}(a)$ is fuzzy open in the domain of f for each f in $FC[\mathfrak{H}, (X, \delta_1)] = FC[\mathfrak{H}, (X, \delta)]$

Since (X, δ) is induced by $\mathfrak{H}$, a is fuzzy open in X.

($\Leftarrow$) Consider $\bar{\delta} = \{a \in L^X : f^{-1}(a)$ is fuzzy open in the domain of f for each f in $FC[\mathcal{H}, (X, \delta)]\}$

This family forms a fuzzy topology by [5.1.1] and $\delta \subset \bar{\delta}$. Let $a \in \bar{\delta}$. Then $f^{-1}(a)$ is fuzzy open in the domain of f for each f in $FC[\mathcal{H}, (X, \delta)]$. But $FC[\mathcal{H}, (X, \delta)] = FC[\mathcal{H}, (X, \bar{\delta})]$. This implies that f is fuzzy continuous even when X is given a finer fuzzy topology. Hence by the given condition a is fuzzy open in X. Thus $\delta = \bar{\delta}$. i.e., (X, δ) is induced by $\mathcal{H}$.

5.1.7 **Theorem**

If $(X, \delta) \in \mathcal{H}$ then (X, δ) is induced by $\mathcal{H}$. i.e., $(X, \delta) \in I(\mathcal{H})$.

Proof:

Let $(X, \delta) \in \mathcal{H}$ and let 'a' be an induced fuzzy open subset of X. Now the identity map

$$i : (X, \delta) \longrightarrow (X, \delta) \in FC(\mathcal{H}, X)$$

Since a is induced fuzzy open, $i^{-1}(a)$ is fuzzy open in X. That is, a is fuzzy open in X. Thus any induced fuzzy open subset of X is fuzzy open in X. Hence X is induced by $\mathcal{H}$.

Concepts that are required for the study are defined below.

5.1.8 Definition: Quotient Fuzzy Topology [36]

Let X be a space of points. Let R be an equivalence relation defined on X. Let X/R be the usual quotient set, and let p be the usual projection from X onto X/R. If (X, δ) is a fuzzy topological space, the quotient fuzzy topology is the largest fuzzy topology such that p is fuzzy continuous.

5.1.9 Definition

Let $(X_\alpha, \delta_\alpha)$ be fuzzy topological spaces. Then the coproduct or fuzzy topological sum is denoted by $\bigoplus X_\alpha$ and is defined as

$$\bigoplus X_\alpha = \left\{ a : a \wedge X_\alpha \text{ is fuzzy open in } X_\alpha \right\} .$$

i.e. A fuzzy set is open in the coproduct if and only if it's restriction to each component is fuzzy open in that component.

Now we will try to present some results on the basis of the above definitions.

5.1.10. Theorem

Let $(X, \delta) \in I(\mathfrak{H})$ and $p: (X, \delta) \longrightarrow (Y, \gamma)$ be a fuzzy quotient function. Then $(Y, \gamma) \in I(\mathfrak{H})$.

Proof:

Let 'a' be an induced fuzzy open subset of Y. Then $f^{-1}(a)$ is fuzzy open in the domain of f for each f in $F\mathcal{C}[\mathfrak{H}, Y]$. We want to show that a is fuzzy open in Y. Since $p: X \longrightarrow Y$ is a fuzzy quotient function, we need only to show that $p^{-1}(a)$ is fuzzy open in X. Let $g \in F\mathcal{C}(\mathfrak{H}, X)$. Then $p \circ g \in F\mathcal{C}(\mathfrak{H}, Y)$. Since a is induced fuzzy open in Y, $(p \circ g)^{-1}(a)$ is fuzzy open in the domain of $p \circ g$. i.e., $g^{-1}(p^{-1}(a))$ is fuzzy open for every $g \in F\mathcal{C}(\mathfrak{H}, X)$. Thus $p^{-1}(a)$ is induced fuzzy open in X. Since X is induced by $\mathfrak{H}$ and p is a quotient function, a is fuzzy open on Y. Thus Y is induced by $\mathfrak{H}$.

Now we prove that $I(\mathfrak{H})$ is closed under the formation of 'sums'.

5.1.11. Theorem

Let $\{X_\alpha\}_{\alpha \in J}$ be a set of fuzzy topological spaces such that $X_\alpha \in I(\mathfrak{H})$ for every α. Then

$X = \oplus X_\alpha$ also belongs to $I(\mathfrak{H})$

Proof:

A fuzzy subset of X is induced fuzzy open if and only if $a \wedge X_\alpha$ is induced fuzzy open in X_α for every $\alpha \in J$. $a \wedge X_\alpha$ is induced fuzzy open if and only if $a \wedge X_\alpha$ is fuzzy open in X_α for every $\alpha \in J$ (since each $X_\alpha \in I(\mathfrak{H})$). That is, a is fuzzy open in X. Thus $X \in I(\mathfrak{H})$.

5.1.2. Theorem

A fuzzy topological space (X, δ) is induced by $\mathfrak{H}$ if and only if X is a quotient of a sum of members of $\mathfrak{H}$.

Proof:

($\Rightarrow$) Let X be induced by $\mathfrak{H}$ and a be an induced fuzzy open subset of X. Then there exists a fuzzy continuous function $f_a \in F\mathcal{C}(\mathfrak{H}, X)$ such that $f_a^{-1}(a)$ is open in the domain D_a of f_a. Thus to each induced fuzzy open a, there is a fuzzy topological space $D_a \in \mathfrak{H}$ and a fuzzy continuous function $f_a : D_a \longrightarrow X$ such that $f_a^{-1}(a)$ is open. Let $\sum_a f_a = p$ with domain

$\sum_a D_a = D$. Also $p|_{D_a} = f_a$ for every a. Since each f_a is fuzzy continuous, p is fuzzy continuous. Now we can show that $p: D \longrightarrow X$ is a fuzzy quotient function.

Let 'b' be fuzzy subset of X, such that $p^{-1}(b)$ is fuzzy open in D. i.e., $p^{-1}(b) \wedge D_b$ is fuzzy open in D_b. But $p^{-1}(b) \wedge D_b = f_b^{-1}(b)$, is fuzzy open in D_b. Since X is induced by $\mathcal{H}$, b is fuzzy open in X. Hence p is a quotient map and X is a quotient of a sum, of members of $\mathcal{H}$.

($\Leftarrow$) Let X be a quotient of sum of members of $\mathcal{H}$. Then by (5.1.10 and 5.1.11) X is induced by $\mathcal{H}$.

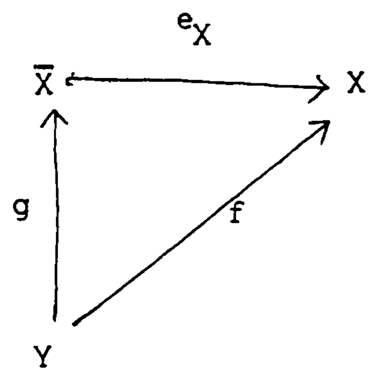
5.2. COREFLECTION

Here we translate the study of the earlier section into categorical language.

5.2.1 Definition

A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is said to be coreflective in $\mathcal{A}$ if, for each object X in $\mathcal{A}$

there exists an object $\overline{X}$ in $\mathbb{B}$ and a morphism $e_X: \overline{X} \longrightarrow X$ such that given any Y in $\mathbb{B}$ and a morphism f: $Y \longrightarrow X$, there exists a unique morphism $g: Y \longrightarrow \overline{X}$ such that the diagram commutes.



In the course of our investigation the following results are obtained.

5.2.2. **Theorem.**

Let $\mathfrak{H}$ be any family of fuzzy topological spaces. Then I($\mathfrak{H}$), the family of induced fuzzy topological spaces is coreflective in FTOP.

Proof:

Let $(X, \delta) \in$ FTOP and $\overline{\delta}$ be the family of all induced open subsets of X. Then

$$\delta \subset \overline{\delta} \tag{5.1.1}$$

Let i: $(X,\bar{\delta}) \longrightarrow (X,\delta)$ be the identity map.

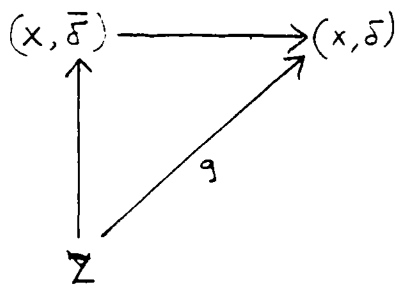
By (5.1.6) $F\mathcal{C}[(Y,\gamma),(X,\delta)] = F\mathcal{C}[(Y,\gamma),(X,\bar{\delta})]$

Let a be induced fuzzy open in $(X,\bar{\delta})$. Then $f^{-1}(a)$ is fuzzy open in (Y,γ). i.e., a is induced fuzzy open in (X,δ).

i.e., a is fuzzy open in $(X,\bar{\delta})$.

Hence $(X,\bar{\delta}) \in I(\mathscr{H})$.

Now let $(Z,\beta) \in I(\mathscr{H})$ and let g: $(Z,\beta) \longrightarrow (X,\delta)$ be fuzzy continuous. Let $a \in \bar{\delta}$. i.e., a is induced fuzzy open in (X,δ). Since $(X,\bar{\delta}) \in I(\mathscr{H})$, a is fuzzy open in (X,δ).



i.e., $g^{-1}(a)$ is fuzzy open in (Z,β).

g: $(Z,\beta) \longrightarrow (X,\bar{\delta})$ is fuzzy continuous.

Thus the diagram commutes. Uniqueness condition is trivially satisfied.

Hence $I(\mathscr{H})$ is coreflective in FTOP.

5.2.3. **Theorem.**

$\mathcal{H}$ is coreflective if and only if $\mathcal{H} = I(\mathcal{H})$.

Proof:

($\Rightarrow$) Let $(X, \delta) \in I(\mathcal{H})$ and let $(X, \bar{\delta})$ be its coreflection in $\mathcal{H}$. Consider the identity map $i:(X, \bar{\delta}) \longrightarrow (X, \delta)$ as the coreflection map. Then by the definition of coreflection [5.2.1], $f:(Y,\gamma) \longrightarrow (X,\delta)$ with $(Y,\gamma) \in \mathcal{H}$ splits uniquely through i and hence $f:(Y,\gamma) \longrightarrow (X, \bar{\delta})$ is also fuzzy continuous. Thus $\bar{\delta}$ is a finer fuzzy topology in X having the same family of fuzzy continuous functions. Hence by (5.1.6) $\bar{\delta} = \delta$. But $(X, \bar{\delta}) \in \mathcal{H}$. So $(X, \delta) \in \mathcal{H}$. Thus $I(\mathcal{H}) \subset \mathcal{H}$. But by (5.1.7) $\mathcal{H} \subset I(\mathcal{H})$. Hence $\mathcal{H} = I(\mathcal{H})$.

($\Leftarrow$) Let $\mathcal{H} = I(\mathcal{H})$. By (5.2.2) $I(\mathcal{H})$ is coreflective. This implies that $\mathcal{H}$ is coreflective.

5.2.4. **Theorem**

$I(\mathcal{H})$ is the smallest coreflective subcategory of FTOP containing $\mathcal{H}$.

Proof:

Let G be a coreflective subcategory of FTOP containing $\mathfrak{F}$ such that $G \subset I(\mathfrak{F})$ (1)

We shall show that $I(\mathfrak{F}) = G$.

Since G is coreflective $I(G) = G$.

But $F \subset G \Longrightarrow I(F) \subset I(G) = G$.

i.e., $I(F) \subset G$ (2)

Hence from (1) and (2) $I(\mathfrak{F}) = G$.

Now we characterize coreflective subcategory of FTOP using fuzzy topological properties.

5.2.5. Theorem

$\mathfrak{F}$ is coreflective in FTOP if and only if $\mathfrak{F}$ is closed under the formation of sums and quotients.

$(\Longrightarrow)$ $\mathfrak{F}$ is coreflective if and only if $\mathfrak{F} = I(\mathfrak{F})$ by (5.2.3). But $I(\mathfrak{F})$ is closed under the formation of fuzzy sums and fuzzy quotients [5.1.10 and 5.1.11]. This implies that $\mathfrak{F}$ is closed under the formation of fuzzy sums and fuzzy quotients.

Conversely, $\mathcal{H}$ is closed under the formation of sums and quotients. But, by (5.1.7) $\mathcal{H} \subset I(\mathcal{H})$. Also $I(\mathcal{H})$ is the smallest family having above properties. Then $\mathcal{H}$ coincides with $I(\mathcal{H})$. Hence $\mathcal{H}$ is coreflective in FTOP.

Here we discuss the behaviour of coreflection in the lattice of fuzzy topological spaces. The coreflective subcategory $\mathcal{H}$ induces a function $\delta \longmapsto \overline{\delta}$ from the lattice of fuzzy topologies on X onto itself.

5.2.6. **Theorem**

The induced map $\delta \longmapsto \overline{\delta}$ is order preserving, and idempotent on the lattice of fuzzy topologies on X.

Proof:

Let $\delta_1 \subset \delta_2$. We want to show that $\overline{\delta}_1 \subset \overline{\delta}_2$.

$i : (X, \delta_2) \longrightarrow (X, \delta_1)$ is fuzzy continuous since $\delta_2 \supset \delta_1$. Since $\delta_1 \subset \overline{\delta_2}$,

$i : (X, \overline{\delta}_2) \longrightarrow (X, \delta_1)$ is fuzzy continuous.

But $(X, \overline{\delta}_2) \in \mathcal{H}$. Therefore by definition of coreflection

$i : (X, \overline{\delta}_2) \longrightarrow (X, \overline{\delta}_1)$ is fuzzy continuous.

Thus $\overline{\delta}_1 \subset \overline{\delta}_2$.

Since $\delta \subset \overline{\delta}$, $\overline{\delta} \subset \overline{\overline{\delta}}$ (1)

Since $\delta \subset \overline{\overline{\delta}}$, $i : (X, \overline{\overline{\delta}}) \longrightarrow (X, \delta)$ is fuzzy continuous. Since $(X, \overline{\delta})$ is the coreflection of (X, δ), $i: (X, \overline{\delta}) \longrightarrow (X, \delta)$ is the same as $i : (X, \overline{\delta}) \longrightarrow (X, \overline{\overline{\delta}})$.

Then $i : (X, \overline{\delta}) \longrightarrow (X, \overline{\overline{\delta}})$ is fuzzy continuous.

i.e., $\overline{\overline{\delta}} \subset \overline{\delta}$ (2)

From (1) and (2) $\overline{\overline{\delta}} = \overline{\delta}$ (idempotent)

5.2.7. Theorem

Let $\mathcal{H}$ be a coreflective subcategory of FTOP. $(X, \overline{\delta})$ be the coreflection of (X, δ) in $\mathcal{H}$ if and only if $(X, \overline{\delta})$ is the lattice meet of all finer fuzzy topologies on X which belong to $\mathcal{H}$.

Proof:

$(\Longrightarrow)$ $(X, \overline{\delta}) \in \mathcal{H}$ since $(X, \overline{\delta})$ is the coreflection of X. Let δ' be any fuzzy topology finer than δ with $(X,\delta') \in \mathcal{H}$. Since $\delta \subset \bar{\delta}$, $i:(X,\bar{\delta}) \longrightarrow (X,\delta)$ is fuzzy continuous. Since $(X,\bar{\delta}) \in \mathcal{H}$, and by definition of coreflection $i: (X,\delta') \longrightarrow (X,\bar{\delta})$ is fuzzy continuous. i.e., $\bar{\delta} \subset \delta'$.

i.e., any fuzzy topology δ' finer than δ which belongs to $\mathcal{H}$ is finer than $\bar{\delta}$ also. This implies that $\bar{\delta}$ is the lattice meet of all finer fuzzy topologies on X which belong to $\mathcal{H}$.

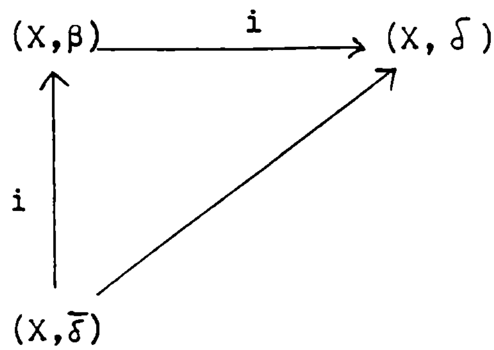
($\Longleftarrow$) Let $\{(X,\delta_\alpha)\}_{\alpha \in J}$ be members of $\mathcal{H}$ such that $\delta_\alpha \supset \delta$. Since $\mathcal{H}$ is coreflective, $\mathcal{H} = I(\mathcal{H})$ and therefore $(X,\delta_\alpha) \in I(\mathcal{H})$. Also $(X, \wedge\delta_\alpha)$ can be got as the quotient of the sum of the fuzzy topological spaces (X,δ_α). Hence by (5·1·2), $(X, \wedge\delta_\alpha) \in I(\mathcal{H})$. But $I(\mathcal{H}) = \mathcal{H}$. Hence $(X,\wedge\delta_\alpha) \in \mathcal{H}$ i.e., $(X,\bar{\delta}) = (X, \wedge\delta_\alpha) \in \mathcal{H}$

Now we want to show that $(X,\bar{\delta})$ coincides with the coreflection of X in $\mathcal{H}$.

Let (X,β) be the coreflection of (X,δ) in $\mathcal{H}$ Then,

$$\bar{\delta} \subset \beta \qquad (1)$$

Consider the identity map $i:(X,\bar{\delta}) \longrightarrow (X,\delta)$ and the diagram,



From the diagram,

$i : (X,\bar{\delta}) \longrightarrow (X,\beta)$ is **fuzzy continuous**

i.e., $\beta \subset \bar{\delta}$ (2)

From (1) and (2) $\beta = \bar{\delta}$

i.e., $(X,\bar{\delta})$ coincides with the coreflection of (X,δ) in $\mathcal{H}$.

5.2.8. **Remark**

Let $\mathcal{H}$ be a family of fuzzy topological spaces. Then there exists a smallest coreflective subcategory of FTOP containing $\mathcal{H}$ (5.2.4). This we shall call the coreflective subcategory generated by $\mathcal{H}$ or the coreflective hull of $\mathcal{H}$ in FTOP. For example, $I(\mathcal{H})$ is the coreflective hull of $\mathcal{H}$ in FTOP.

5.2.9. Remark

We can observe that, if $\mathcal{H}$ be a family of fuzzy topologies in X and each $(X, \delta_\alpha) \in I(\mathcal{H})$ then $(X, \wedge\delta_\alpha) \in I(\mathcal{H})$. In this case $I(\mathcal{H})$ forms a complete lattice under the usual ordering.

Conclusion

The investigation made in Chapter III to V is a humble beginning in the direction of the study of the class of fuzzy topology. There remains a lot of research work to be done. Several lattice theoretic properties are to be investigated, especially by fixing the underlying set. In the category theoretic study, more general subcategories deserve a closer look. We are attempting some of them.

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