FUZZY MEASURES AND A THEORY OF FUZZY DANIELL INTEGRALS

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CERTIFICATE

Certified that the work reported in this thesis is based on the bona fide work done by Sri. M.S. Samuel, under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Cochin 682 022, and has not been included in any other thesis submitted previously for the award of any degree.

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Chapter 0
INTRODUCTION

L.A. Zadeh [ZA_1] defined fuzzy sets for representing inexact concepts. In 1974 Michio Sugeno [SU] introduced the concept of fuzzy measures and fuzzy integrals. He used fuzzy measures to evaluate the grade of fuzziness of fuzzy subsets of a set X. Since then, without using a probability distribution, it has been widely used to represent available information about an uncertain experiment. Fuzzy set theory finds application in many fields, for example, automata, linguistics, algorithm, pattern recognition, etc. [SU]. In the fuzzy set theory the concept of "fuzziness" is introduced corresponding to randomness in probability theory. Sugeno started with the concept of "grade of fuzziness". He obtained "fuzzy measure" as measuring grade of fuzziness and he compared it with probability measure expressing grade of randomness. He constructed and developed the theory of fuzzy integrals independently of fuzzy set theory. But the concept of fuzzy sets was referred frequently. He used fuzzy measures and fuzzy integrals as a way for expressing human subjectivity and discussed their applications. Sugeno [SU], Ralescu and Adams [RA; AD] and Wang [WA_1] studied the fuzzy measures and fuzzy integrals defined on a classical σ-algebra by deleting the σ-additivity and replacing it by monotonicity and continuity. Fuzzy integrals obtained by Sugeno are analogous to Lebesgue integrals. Lebesgue measures assume additivity whereas fuzzy measures assume only monotonicity. Therefore, human subjective scales can better be approximated by using the conventional one. Sugeno obtained it as an optimisation problem of minimizing the error between the human evaluation and the fuzzy integral output. His fuzzy integral model was applied to examples such as subjective evaluation of female faces and grading similarity of patterns. Sekita and Tabata [SE; TA] used fuzzy integrals for evaluation of human health index. The problem of evaluating the properties of a system was studied by Siegfried Gottwald and Witold Pedrycz [GO; PE] on the basis of the corresponding fuzzy model. They have shown that a grade of satisfaction for a property of the system may be calculated by means of a fuzzy integral with respect to a fuzzy measure. Here the fuzzy measure corresponds to a quantitative representation of the quality of the model constructed. Dubois and Prade [DU; PR] introduced an axiomatic approach to a broad class of fuzzy measures in the sense of Sugeno using the concept of triangular norm (t- norm).

R.R. Yager [YA] and D. Butnario [BU] also studied fuzzy measures in order to measure fuzziness of a fuzzy set. Qiao [QI] and Wang [WA_1] generalised them to a fuzzy σ-algebra of fuzzy sets. Klement [KL] defined a fuzzy σ-algebra axiomatically. Further he established the relationship between a classical σ-algebra and a fuzzy σ-algebra. Suzuki [SUZ_1], [SUZ_2] studied some analytical properties of Sugeno's fuzzy measures especially of atoms of fuzzy measures. He defined a fuzzy integral of Riemann type through atoms. Also he showed that a continuous function is fuzzy integrable in the sense of Riemann and has the same integral as that of Sugeno. Wang [WA_2] and Kruse[KR_1] studied some structural characteristics of fuzzy measures. Eventhough their discussion was limited to a classical σ-algebra, they proved several convergence theorems for a sequence of fuzzy integrals. Qiao [QI] established a theory of fuzzy measure and fuzzy integral in a fuzzy σ-algebra of fuzzy sets by combining Sugeno's theory and Zadeh's fuzzy sets. Qiao [QI] introduced real-valued fuzzy measures and fuzzy integrals for a pseudo complemented infinitely distributive complete lattice L and studied several properties of fuzzy integrals on L-fuzzy sets.

In this thesis we make an attempt to study the theory of fuzzy Daniell integral analogous to the crisp theory [RO]. P.J. Daniell in 1918 published his famous paper [DA_1] in which he introduced the idea of an integral without using the concept of measure. Radon, Young, Riesz and others [DA_1] have extended the idea of integration to a function of bounded variation, based on the fundamental properties of sets of points in a space of finite dimension. E.H. Moore's theory of integration is similar to that of Daniell's; but Moore restricts himself to the use of relatively uniform sequences. Daniell also showed that the Lebesgue integral and the Radon-Young integral were only special cases of the general form of the integral he had obtained.

Frechet in 1915 showed that it is possible to abandon completely the sets with geometric character and integrate functions defined in abstract spaces [FR]. He did this in extending the method of Lebesgue to this general problem. A little later, using the idea of the extension of linear functionals, Daniell also did the same. To generate the theory, a vector lattice L of real-valued functions on a nonempty set X is considered. Here the role of step functions is given to elements of this class so as to reproduce the theory of the Lebesgue integral. Two fundamental facts are needed. If I is a positive linear functional the two fundamental facts are (i) $I(f_n) \longrightarrow 0$ when $f_n(x) \downarrow 0$ a.e., and (ii) if $\{f_n\}$ is an increasing sequence $I(f_n)$ remains bounded implies $f_n(x)$ converges a.e to a finite limit. A triple $(X,L,\int)$ is called a Daniell integral space if the following conditions are satisfied: The family of functions $L \subset R^X$ forms a linear lattice; the functional $\int$ is nonnegative and completely countably additive on $L^+ = \{f \in L: f \geqslant 0\}$. We say that a functional $\int$ is a Daniell integral over the space X if its domain $L \subset R^X$ forms a linear lattice and the triple $(X,L,\int)$ is a Daniell integral space. A Lebesgue integral is a Daniell integral but the following example given in [BO] shows that Daniell integrals need not always be Lebesgue integrals.

Example. Let $X = (0,1]$ and $L = \{ f = re : r \in R \}$, where the function e is the identity map $e(x) = x$ for $x \in X$. Define $\int f = r$, if $f = re$. This triple forms a Daniell integral space but the linear lattice is not closed under the stone operation i.e., $f \rightarrow f \cap 1$. The domain of the integral is one-dimensional.

Daniell showed that it is not necessary to have a measure in the set X on which the integrands are defined for the existence of an integral.

P. Lubcznok [LU], Godfrey C. Muganda [MU] and others have defined a fuzzy vector space. Zadeh [ZA_2] introduced another very useful concept, viz., fuzzy singletons in 1972. Using the notation of fuzzy singleton, Wong [WO] introduced the concept of fuzzy points. He defined it in such a way that a crisp singleton, equivalently, an ordinary point, was not a special case of a fuzzy point. Pu Pao-Ming and Liu Ying-Ming [PU; LI] have redefined it as a crisp singleton, equivalently, an ordinary point, as a special case.

We make use of the earlier definitions of fuzzy vector spaces, fuzzy lattices, fuzzy sets and fuzzy points in order to introduce the concept of fuzzy vector lattices. We use a "fuzzy point approach" throughout the work. This thesis consists of five chapters.

In the first chapter we have defined a fuzzy vector lattice s, a vector lattice of fuzzy points $\tilde{s}$ and introduced some definitions related to sequence of fuzzy points. We have also defined a fuzzy Daniell integral as a positive linear functional τ from $\tilde{s}$ to $\tilde{R}$, where $\tilde{R}$ is the set of fuzzy points in R, and show that τ can be extended to $\tilde{s}_u$ which is the set of the limits of all increasing sequences of fuzzy points in $\tilde{s}$.

The second chapter contains the definitions of upper fuzzy integral, lower fuzzy integral and a study of τ-integrable class of fuzzy points represented by $\tilde{s}_1$. In this chapter we prove that τ can be extended to the class of all fuzzy integrable fuzzy points and that (τ-integrable class of fuzzy points) $\tilde{s}_1$ is a vector lattice.

In the third chapter we have shown that the monotone convergence theorem, Fatou's lemma and the Lebesgue convergence theorem are still true in the fuzzy context under suitable assumptions. The chapter ends with the establishment of the uniqueness of the extension of τ on $\tilde{s}$ to $\tilde{s}_1$.

Fourth chapter begins by quoting the definition of fuzzy σ-algebra, fuzzy measure and fuzzy integral. In this chapter, the definition of measurability of a fuzzy point, measurability of fuzzy set, and integrability of a fuzzy set are given. The main result of this chapter is the fuzzy analogue of Stone's theorem, which says that the fuzzy Daniell integral τ on $\tilde{S}_1$ is equivalent to the fuzzy integral with respect to the fuzzy measure ξ.

The last chapter, viz., chapter five briefly describes the extension of the above theory to fuzzy vector valued integration. Here, we are considering two types of fuzzy vector valued integration: (i) the fuzzy integral of a fuzzy point in the set of all vector valued functions on X with respect to a real valued fuzzy measure, and, (ii) the fuzzy integral of a fuzzy point in the set of all real valued functions on X with respect to a vector valued fuzzy measure.

Chapter I

FUZZY VECTOR LATTICES AND FUZZY DANIELL INTEGRAL*

1.0 INTRODUCTION

Daniell P.J. obtained an extension of elementary Riemann integral to a general form of integral. He starts with a vector lattice L of bounded real valued functions on a set X. Then a nonnegative linear functional I which is continuous under monotone limits, is defined. This functional I is called a Daniell Integral. Then I is extended to a larger class of functions retaining all the properties of L and having additional properties. As per the example cited in Loomis [LO] taking L to be the class of continuous functions on [0,1] and I to be the ordinary Riemann integral the extension of L is then the class of Lebesgue summable functions and the extended I becomes the ordinary Lebesgue integral. Taking the class L_u of limits of monotone increasing sequences of functions in L, I is extended to L_u and proved that L_u is a vector lattice. The fuzzy form of the above part of Daniell's procedure is discussed in this chapter. Here we define a fuzzy vector lattice and fuzzy Daniell functional. P. Lubczonok [LU], Godfrey C. Muganda [MU] and others have already given the definition of fuzzy vector

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* Some of the results given in this chapter have already appeared in J. Fuzzy Maths 3(1995).

space and we make use of this definition in the forthcoming discussion.

1.1. FUZZY VECTOR LATTICES

Let X be any set and L be a vector lattice of extended real valued functions on X.

Notation 1.1.1. A fuzzy set s in L is a map $s : L \longrightarrow [0,1]$. For $f \in L$, $\alpha \in (0,1]$, a fuzzy set f_α is a fuzzy point when

$$\begin{aligned} f_\alpha(h) &= \alpha \quad \text{if} \quad h = f \\ &= 0 \quad \text{if} \quad h \neq f \quad \forall h \in L. \end{aligned}$$

If s is a fuzzy set and f_α a fuzzy point, we say f_α is a fuzzy point of s if $f_\alpha \leqslant s$ i.e., $\alpha \leqslant s(f)$.

Convention. In this thesis, a fuzzy set s in L is always taken to be such that $s(o) = 1$.

Definition 1.1.2 (Def. 2.1 of [LU]). A fuzzy set s in L is a fuzzy vector space if

$$s(af + bg) \geqslant s(f) \wedge s(g) \quad \forall f, g \in L \text{ and } a, b \in R.$$

Definition 1.1.3. s is a fuzzy vector lattice if

(i) $s(af + bg) \geqslant s(f) \wedge s(g)$

(ii) $s(f \vee g) \geqslant s(f) \vee s(g)$

(iii) $s(f \wedge g) \geqslant s(f) \wedge s(g) \quad \forall f,g \in L$ and $a,b \in R$

Notation 1.1.4. $\tilde{s}$ denotes the set of all fuzzy points of s and $\tilde{R}$ the set of all fuzzy points in R.

Definition 1.1.5. $f_\alpha \in \tilde{s}$ is said to be non negative and we write $f_\alpha \geqslant 0$ if $f \geqslant 0$.

Definition 1.1.6. $\forall f_\alpha, g_\beta \in \tilde{s}$, $f_\alpha \leqslant g_\beta$ if $f \leqslant g$ and $\alpha \geqslant \beta$, where $f,g \in L$ and $\alpha,\beta \in (0,1]$. Note that $\leqslant$ is a partial order in $\tilde{s}$.

Theorem 1.1.7. $\forall f_\alpha, g_\beta \in \tilde{s}$,

(i) $f_\alpha \vee g_\beta = (f \vee g)_{\min(\alpha,\beta)}$ and

(ii) $f_\alpha \wedge g_\beta = (f \wedge g)_{\max(\alpha,\beta)}$

Proof. (i) Let $f_\alpha, g_\beta \in \tilde{s}$. We have $f \leqslant f \vee g$ and $\alpha \geqslant \min(\alpha,\beta)$ and so $f_\alpha \leqslant (f \vee g)_{\min(\alpha,\beta)}$; $g \leqslant f \vee g$ and $\beta \geqslant \min(\alpha,\beta)$ and so $g_\beta \leqslant (f \vee g)_{\min(\alpha,\beta)}$. Also if $f_\alpha, g_\beta \leqslant h_\gamma$ then $f \leqslant h$, $g \leqslant h$, $\alpha \geqslant \gamma$, $\beta \geqslant \gamma$ so that

$f \vee g \leqslant h$ and $\min(\alpha,\beta) \geqslant \gamma$. Therefore, $(f \vee g)_{\min(\alpha,\beta)} \leqslant h_\gamma$. Thus $f_\alpha \vee g_\beta = (f \vee g)_{\min(\alpha,\beta)}$.

(ii) Follows similarly.

Remark 1.1.8. From the above theorem we find that whenever $f_\alpha, g_\beta \in \tilde{s}$, $f_\alpha \vee g_\beta \in \tilde{s}$ and $f_\alpha \wedge g_\beta \in \tilde{s}$, i.e., $\tilde{s}$ is having the lattice structure.

Note 1.1.9. (i) For every $f_\alpha, g_\beta \in \tilde{s}$ and $a, b \in R$, we have

$$a\, f_\alpha + b\, g_\beta = (af+bg)_{\min(\alpha,\beta)}$$ from Prop. 3.1 of [MU].

Thus $\tilde{s}$ is a vector space over R.

(ii) $af_\alpha = (af)_\alpha \;\; \forall\, a \in R$ and $f_\alpha \in \tilde{s}$ by Prop. 3.1. of [MU]

(iii) $(|f|)_\alpha = (f^+)_\alpha + (f^-)_\alpha$.

Note that $f_\alpha = (f^+)_\alpha - (f^-)_\alpha$.

Result 1.1.10. If s is a fuzzy vector lattice then $\tilde{s}$ is a vector lattice.

Definition 1.1.11. A sequence $\{(\emptyset_n)_{\alpha_n}\}$ in $\tilde{s}$ decreases means $(\emptyset_{n+1})_{\alpha_{n+1}} \leqslant (\emptyset_n)_{\alpha_n} \; \forall\, n$, i.e., $\emptyset_{n+1} \leqslant \emptyset_n$ and $\alpha_{n+1} \geqslant \alpha_n \; \forall\, n$ and $\{(\emptyset_n)_{\alpha_n}\}$ in $\tilde{s}$ increases means $(\emptyset_n)_{\alpha_n} \leqslant (\emptyset_{n+1})_{\alpha_{n+1}} \; \forall\, n$.

Definition 1.1.12. A sequence $\{(\emptyset_n)_{\alpha_n}\}$ in $\tilde{s}$ increases to $\emptyset_\alpha$ $(\alpha > 0)$ means $\emptyset_n \uparrow \emptyset$ and $\alpha_n \downarrow \alpha$.

Definition 1.1.13. $\lim((\emptyset_n)_{\alpha_n}) = \emptyset_\alpha$ $(\alpha > 0)$ if $\lim \emptyset_n = \emptyset$ and $\lim \alpha_n = \alpha$, i.e., $\emptyset_n(x)$ converges to $\emptyset(x)$ for every $x \in X$ and $\alpha_n \longrightarrow \alpha$ as $n \longrightarrow \infty$.

Remark 1.1.14. In the light of the above definition we get the following:

(i) A sequence $\{(\emptyset_n)_{\alpha_n}\} \in \tilde{s}$ decreases to zero if $\emptyset_n(x) \downarrow 0$ for every $x \in X$ and $\alpha_n \uparrow \alpha > 0$.

(ii) Let $\{(\emptyset_n)_{\alpha_n}\}$ and $\{(\psi_m)_{\beta_m}\}$ be increasing sequences of fuzzy points in $\tilde{s}$. Then $\lim (\emptyset_n)_{\alpha_n} \leqslant \lim (\psi_m)_{\beta_m}$ if $\lim \emptyset_n \leqslant \lim \psi_m$ and $\lim \alpha_n \geqslant \lim \beta_m > 0$.

1.2. FUZZY DANIELL INTEGRAL.

$\tilde{R}$ is a fuzzy vector space by Proposition 3.1. of [MU] having lattice structure and therefore it is a fuzzy vector lattice. Let τ be a map from $\tilde{s}$ to $\tilde{R}$. Then $\tau(f_\alpha) = \lambda_\beta$, where $\lambda \in R$, $\beta \in (0,1]$.

Notation 1.2.1. If $T:L \longrightarrow R$ then the map from $\tilde{s}$ to $\tilde{R}$ defined by $f_\alpha \longrightarrow (T(f))_\alpha$ for every $f_\alpha \in \tilde{s}$ is denoted by τ_T.

Definition 1.2.2. A map τ from $\tilde{s}$ to $\tilde{R}$ is called linear map if $\tau(af_\alpha + bg_\beta) = a\tau(f_\alpha) + b\tau(g_\beta)$ for every $a,b \in R$, and $f_\alpha, g_\beta \in \tilde{s}$ and for each $f_\alpha \in \tilde{s}$, $\tau(f_\alpha) = r_\alpha$ for some $r \in R$.

Remark 1.2.3. If $T:L \longrightarrow R$ is linear then $\tau_T:\tilde{s} \longrightarrow \tilde{R}$ is linear. For,

$$\begin{aligned}
\tau_T(af_\alpha + bg_\beta) &= \tau_T((af+bg)_{\min(\alpha,\beta)}) \\
&= (T(af+bg))_{\min(\alpha,\beta)} \\
&= (aT(f) + bT(g))_{\min(\alpha,\beta)} \\
&= a(T(f))_\alpha + b(T(g))_\beta \\
&= a\tau_T(f_\alpha) + b\tau_T(g_\beta)
\end{aligned}$$

Also $\tau_T(f_\alpha) = (T(f))_\alpha$. Hence τ_T is linear if T is linear.

Definition 1.2.4. $\tau:\tilde{s} \longrightarrow \tilde{R}$ is said to be positive if $\tau(f_\alpha) \geqslant 0$ for every $f_\alpha \geqslant 0$ in $\tilde{s}$. i.e., $\tau(f_\alpha) = \lambda_\beta$ for some $\lambda \geqslant 0$.

Proposition 1.2.5. If τ is positive and linear then it is monotone. i.e., $\tau(f_\alpha) \leqslant \tau(g_\beta)$ whenever $f_\alpha, g_\beta \in \tilde{s}$ and $f_\alpha \leqslant g_\beta$.

Proof: Since $f_\alpha \leqslant g_\beta$, (i.e., $f \leqslant g$ and $\alpha \geqslant \beta$), $g_\beta - f_\alpha = (g-f)_{\min(\alpha,\beta)} \geqslant 0$. i.e., $\tau(g_\beta - f_\alpha) \geqslant 0$.
i.e., $\tau(g_\beta) - \tau(f_\alpha) \geqslant 0$ since τ is linear.
i.e., $s_\beta - r_\alpha \geqslant 0$ where $\tau(f_\alpha) = r_\alpha$, $\tau(g_\beta) = s_\beta$
i.e., $s \geqslant r$ and also we have $\alpha \geqslant \beta$. Therefore $s_\beta \geqslant r_\alpha$. i.e., $\tau(g_\beta) \geqslant \tau(f_\alpha)$.

Note 1.2.6. Clearly, the converse of Proposition 1.2.5 is not true.

Definition 1.2.7. A linear map $\tau : \tilde{s} \to \tilde{R}$ is called fuzzy Daniell functional or fuzzy Daniell integral if for every sequence $\{(\emptyset_n)_{\alpha_n}\} \in \tilde{s}$ and $(\emptyset_n)_{\alpha_n} \downarrow 0$, we have $\lim \tau((\emptyset_n)_{\alpha_n}) = 0$.

Remark 1.2.8. (i) If $\tau(\emptyset_n)_{\alpha_n}) = (r_n)_{\beta_n}$, where $\emptyset_n \in L$, $r_n \in R$, $\beta_n \in (0,1]$ then $\lim \tau((\emptyset_n)_{\alpha_n}) = 0$ implies $\lim r_n = 0$ and $\lim \beta_n = \beta > 0$.

(ii) If $T: L \longrightarrow R$ is linear then $\lim(\tau_T(\emptyset_n)_{\alpha_n}) = 0$ if and only if $\lim (T(\emptyset_n))_{\alpha_n} = 0$ i.e., if and only if $\lim T(\emptyset_n) = 0$ and $\lim \alpha_n = \alpha > 0$.

Lemma 1.2.9. Let τ be a fuzzy Daniell functional. If $\{(f_n)_{\alpha_n}\}$ and $\{(g_m)_{\beta_m}\}$ are increasing sequences from $\tilde{s}$ and if $\lim (f_n)_{\alpha_n} \leqslant \lim (g_m)_{\beta_m}$ then $\lim \tau((f_n)_{\alpha_n}) \leqslant \lim \tau((g_m)_{\beta_m})$.

Proof: Since $\{(f_n)_{\alpha_n}\}$ and $\{(g_m)_{\beta_m}\}$ are both increasing sequences from $\tilde{s}$ and since $\lim (f_n)_{\alpha_n} \leqslant \lim (g_m)_{\beta_m}$, $(f_n)_{\alpha_n} \leqslant (g_m)_{\beta_m} \leqslant \lim (g_m)_{\beta_m}$ for some $\alpha_n, \beta_m \in (0,1]$ and for sufficiently large n and m. Then $\tau((f_n)_{\alpha_n}) \leqslant \tau((g_m)_{\beta_m})$ by Proposition 1.2.5. Therefore $\lim \tau((f_n)_{\alpha_n}) \leqslant \lim \tau((g_m)_{\beta_m})$.

Notation. $\tilde{s}_u$ denotes the set of limits of all increasing sequences of fuzzy points in $\tilde{s}$.

Lemma 1.2.10. A fuzzy Daniell functional τ can be extended as a monotone $\tilde{R}$ valued linear functional on $\tilde{s}_u$, also $\tilde{s}_u$ is a vector lattice.

Proof: Since τ is a monotone $\tilde{R}$ valued linear functional on $\tilde{s}$, $\tau(f_\alpha) \leqslant \tau(g_\beta)$ $\forall f_\alpha \leqslant g_\beta$ and $\tau(af_\alpha + bg_\beta) = a\tau(f_\alpha) + b\tau(g_\beta)$ for every $f_\alpha, g_\beta \in \tilde{s}_u$, $a,b \in R$ by lemma 1.2.9.

Let $(\emptyset_n)_{\alpha_n} \uparrow f_\alpha$ and $(\psi_m)_{\beta_m} \uparrow g_\beta$ then

$(\emptyset_n)_{\alpha_n} \wedge (\psi_m)_{\beta_m} \uparrow f_\alpha \wedge g_\beta \in \tilde{s}_u$ and

$(\emptyset_n)_{\alpha_n} \vee (\psi_m)_{\beta_m} \uparrow f_\alpha \vee g_\beta \in \tilde{s}_u$.

For, $(\emptyset_n)_{\alpha_n} \wedge (\psi_m)_{\beta_m} = (\emptyset_n \wedge \psi_m)_{\max(\alpha_n, \beta_m)}$.

Since $(\emptyset_n)_{\alpha_n} \uparrow f_\alpha$ and $(\psi_m)_{\beta_m} \uparrow g_\beta$, we get $\emptyset_n \uparrow f$, $\alpha_n \downarrow \alpha$ and $\psi_m \uparrow g$, $\beta_m \downarrow \beta$.

Therefore $(\emptyset_n \wedge \psi_m) \uparrow f \wedge g$ and $\max(\alpha_n, \beta_m) \downarrow \max(\alpha, \beta)$.

i.e., $(\emptyset_n)_{\alpha_n} \wedge (\psi_m)_{\beta_m} \uparrow f_\alpha \wedge g_\beta \in \tilde{s}_u$. In the same way it follows that $(\emptyset_n)_{\alpha_n} \vee (\psi_m)_{\beta_m} \uparrow f_\alpha \vee g_\beta \in \tilde{s}_u$.

Hence $\tilde{s}$ is a vector lattice.

Notation 1.2.11. $\sum_{n=1}^{\infty} (\psi_n)_{\alpha_n} = f_\alpha$ means $\sum_{n=1}^{\infty} \psi_n = f$ and $\inf \alpha_n = \alpha$.

Lemma 1.2.12. A non negative fuzzy point f_α belongs to $\tilde{s}_u$ if and only if there is a sequence $\{(\psi_n)_{\alpha_n}\}$ of nonnegative fuzzy points in $\tilde{s}$ such that

$$f_\alpha = \sum_{n=1}^{\infty} (\psi_n)_{\alpha_n}$$

Proof: If $f_\alpha \in \tilde{s}_u$ there exists an increasing sequence of nonnegative fuzzy points $\{(\phi_n)_{\beta_n}\}$ in $\tilde{s}$ whose limit is f_α, and take

$$(\psi_1)_{\alpha_1} = (\phi_1)_{\beta_1}, \quad (\psi_n)_{\alpha_n} = (\phi_n)_{\beta_n} - (\phi_{n-1})_{\beta_{n-1}}$$

$$= (\phi_n - \phi_{n-1})_{\min(\beta_n, \beta_{n-1})} = (\phi_n - \phi_{n-1})_{\beta_n}$$

Therefore $\sum_{n=1}^{\infty} (\psi_n)_{\alpha_n} = \sum_{n=1}^{\infty} (\phi_n - \phi_{n-1})_{\beta_n} = \lim(\phi_n)_{\beta_n}$

i.e., $\quad f_\alpha = \sum_{n=1}^{\infty} (\psi_n)_{\alpha_n}$.

Conversely, let $f_\alpha = \sum_{n=1}^{\infty} (\psi_n)_{\alpha_n}$, we have to show that $f_\alpha \in \tilde{s}_u$. Suppose $(\phi_m)_{\beta_m} = \sum_{n=1}^{m} (\psi_n)_{\alpha_n}$, where $\psi_n \geq 0$. Then $(\phi_m)_{\beta_m} \in \tilde{s}$ and $(\phi_m)_{\beta_m} \uparrow f_\alpha$. But $\tilde{s}_u$ consists of the limit of all increasing sequences of fuzzy points in $\tilde{s}$ so that $f_\alpha \in \tilde{s}_u$.

Result 1.2.13. If $\{(f_n)_{\alpha_n}\}$ is a sequence of nonnegative fuzzy points in $\tilde{s}_u$ such that $\inf \alpha_n = \alpha > 0$, then

$f_\alpha = (\sum_{n=1}^{\infty} f_n)_{\inf \alpha_n}$ is in $\tilde{s}_u$. Also $\tau(f_\alpha) = \sum_{n=1}^{\infty} \tau(f_n)_{\alpha_n}$.

Proof: For each $(f_n)_{\alpha_n}$ of nonnegative fuzzy points in $\tilde{s}_u$ there exists an increasing sequence $\{(g_{n\ell})_{\beta_{n\ell}}\}$ in $\tilde{s}$ such that

$$(f_n)_{\alpha_n} = (\sum_{\ell=1}^{\infty} g_{n\ell})_{\inf \beta_{n\ell}}, \text{ where } \alpha_n = \inf \beta_{n\ell}.$$

Therefore

$$\sum_{n=1}^{\infty} (f_n)_{\alpha_n} = \sum_{n=1}^{\infty} \Big(\sum_{\ell=1}^{\infty} g_{n\ell} \Big)_{\inf \beta_{n\ell}}$$

i.e., $$f_\alpha = \Big(\sum_{n,\ell=1}^{\infty} g_{n\ell} \Big)_{\inf\limits_{n,\ell} \beta_{n\ell}}$$

Thus $f_\alpha \in \tilde{S}_u$.

Let $(g_n)_{\beta_n} = \sum_{m=1}^{n} (f_m)_{\alpha_m}$. Then $(g_n)_{\beta_n} \uparrow f_\alpha$ so that $\tau(g_n)_{\beta_n} \uparrow \tau(f_\alpha)$, i.e., $\lim \tau(g_n)_{\beta_n} = \tau(f_\alpha)$ i.e., $\lim \tau\big(\sum_{m=1}^{n} (f_m)_{\alpha_m} \big) = \tau(f_\alpha)$. Hence

$$\tau(f_\alpha) = \lim \sum_{m=1}^{n} \tau(f_m)_{\alpha_m} = \sum_{n=1}^{\infty} \tau(f_n)_{\alpha_n}.$$

Chapter II

UPPER FUZZY INTEGRAL, LOWER FUZZY INTEGRAL AND INTEGRABILITY OF A FUZZY POINT*

2.0. INTRODUCTION

In the previous chapter we have seen how a fuzzy vector lattice s, vector lattice $\tilde{s}$ of fuzzy points, fuzzy Daniell integral τ are defined and obtained the extension of $\tilde{s}$ to $\tilde{s}_u$. Here we continue our development of the fuzzy analogue of Daniell theory. In the crisp case lower and upper integral of an arbitrary function f on X with the assumption that the infimum of the empty set is $+\infty$ are defined and further the integrability of f with respect to I is obtained. The class of all I-integrable functions is denoted by L_1. This class L_1 is a vector lattice of functions containing L and also proves that I is a positive linear functional on L_1 also. In this chapter we are presenting fuzzy analogue of the crisp theory as mentioned above.

2.1. UPPER FUZZY INTEGRAL AND LOWER FUZZY INTEGRAL

Definition 2.1.1. Let f_α be a fuzzy point of the set of all real valued functions on X. Then the upper fuzzy integral $\bar{\tau}(f_\alpha)$ is defined by $\bar{\tau}(f_\alpha) = \inf\limits_{\substack{f_\alpha \leqslant g_\beta \\ g_\beta \in \tilde{s}_u}} \tau(g_\beta)$

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*Some of the results of this chapter have already appeared in Fuzzy Sets and Systems (1995).

Example 2.1.2. Consider remark 1.2.3. Let $\overline{T}:L_u \longrightarrow R$ then $\overline{\tau}_T:\tilde{s}_u \longrightarrow \tilde{R}$ where L_u is the class of all extended real valued functions on X each of which is a limit of a monotone increasing sequence of functions in L. Then

$$\overline{\tau}_T(f_\alpha) = \inf_{\substack{f_\alpha \leq g_\beta \\ g_\beta \in \tilde{s}_u}} \tau_T(g_\beta)$$

$$= \inf_{\substack{f \leq g \\ \alpha \geq \beta \\ g \in L_u}} (T(g))_\beta$$

$$= (\overline{T}(f))_\alpha$$

Definition 2.1.3. The lower fuzzy integral $\underline{\tau}$ is defined by $\underline{\tau}(f_\alpha) = -\overline{\tau}(-f_\alpha)$.

Example 2.1.4. Similar to example 2.1.2, the lower fuzzy integral $\underline{\tau}_T$ is given by

$$\underline{\tau}_T(f_\alpha) = -\overline{\tau}_T(-f_\alpha)$$

$$= -\inf_{\substack{f_\alpha \leq g_\beta \\ g_\beta \in \tilde{s}_u}} \tau_T(-g_\beta)$$

$$= -\inf_{\substack{f \leq g \\ \alpha \geq \beta \\ g \in L_u}} (T(-g))_\beta$$

$$= -(\bar{T}(-f))_\alpha$$
$$= (\underline{I}(f))_\alpha$$

Lemma 2.1.5. Let $f_\alpha, g_\beta, h_\gamma \in \tilde{s}_u$ such that $h_\gamma = f_\alpha + g_\beta$ then $\bar{\tau}(h_\gamma) \leqslant \bar{\tau}(f_\alpha) + \bar{\tau}(g_\beta)$ where right hand side is defined. If $c \geqslant 0$, $\bar{\tau}(cf_\alpha) = c\,\bar{\tau}(f_\alpha)$. If $f_\alpha \leqslant g_\beta$, then $\bar{\tau}(f_\alpha) \leqslant \bar{\tau}(g_\beta)$ and $\underline{\tau}(f_\alpha) \leqslant \underline{\tau}(g_\beta)$.

Proof: Let $(k_1)_{\delta_1}, (k_2)_{\delta_2} \in \tilde{s}$, such that $f_\alpha \leqslant (k_1)_{\delta_1}$ and $g_\beta \leqslant (k_2)_{\delta_2}$. Then $f_\alpha + g_\beta \leqslant (k_1)_{\delta_1} + (k_2)_{\delta_2}$. Let $k_\delta = (k_1)_{\delta_1} + (k_2)_{\delta_2}$. Then $h_\gamma \leqslant k_\delta$. Therefore

$$\begin{aligned}
\bar{\tau}(h_\gamma) &= \inf_{\substack{h_\gamma \leqslant e_\nu \\ e_\nu \in \tilde{s}_u}} \tau(e_\nu) \\
&\leqslant \tau(k_\delta) \\
&= \tau((k_1)_{\delta_1}) + \tau((k_2)_{\delta_2})
\end{aligned}$$

This is true for all $(k_1)_{\delta_1}$ and $(k_2)_{\delta_2}$ such that $f_\alpha \leqslant (k_1)_{\delta_1}$ and $g_\beta \leqslant (k_2)_{\delta_2}$. Therefore

$$\begin{aligned}
\bar{\tau}(h_\gamma) &\leqslant \inf(\tau(k_1)_{\delta_1} + \tau(k_2)_{\delta_2}) \\
&= \inf \tau((k_1)_{\delta_1}) + \inf \tau((k_2)_{\delta_2}) \\
&= \bar{\tau}(f_\alpha) + \bar{\tau}(g_\beta)
\end{aligned}$$

$$\text{If } c > 0,\ \overline{\tau}(c\, f_\alpha) = \inf_{f_\alpha \leqslant g_\beta} \tau(c\, g_\beta)$$

$$= c \inf_{f_\alpha \leqslant g_\beta} \tau(g_\beta)$$

$$= c\, \overline{\tau}(f_\alpha).$$

Let $f_\alpha \leqslant g_\beta$. If $k_\delta \in \tilde{S}$ such that $g_\beta \leqslant k_\delta$ then $f_\alpha \leqslant k_\delta$. Therefore

$$\inf_{f_\alpha \leqslant k_\delta} \tau(k_\delta) \leqslant \inf_{g_\beta \leqslant k_\delta} \tau(k_\delta)$$

i.e., $\overline{\tau}(f_\alpha) \leqslant \overline{\tau}(g_\beta)$. Now $\underline{\tau}(f_\alpha) \leqslant \underline{\tau}(g_\beta)$ follows easily from the definition.

Notation 2.1.6. The set of all real valued functions on X is denoted by F.

Lemma 2.1.7. If f_α is a fuzzy point of F then $\underline{\tau}(f_\alpha) \leqslant \overline{\tau}(f_\alpha)$. If $f_\alpha \in \tilde{S}_u$ then $\underline{\tau}(f_\alpha) = \overline{\tau}(f_\alpha) = \tau(f_\alpha)$.

Proof: For every $g_\beta \in \tilde{S}_u$, $\tau(g_\beta - g_\beta) = \tau(o_\beta) \geqslant 0$

i.e., $\tau(g_\beta) - \tau(g_\beta) \geqslant 0$, i.e., $\tau(g_\beta) \geqslant -\tau(-g_\beta)$;

i.e., $\inf_{f_\alpha \leqslant g_\beta} \tau(g_\beta) \geqslant -\sup_{f_\alpha \leqslant g_\beta} \tau(-g_\beta)$; $\overline{\tau}(f_\alpha) \geqslant \underline{\tau}(f_\alpha)$.

If $f_\alpha \in \tilde{s}_u$, by definition of $\bar{\tau}$, $\bar{\tau}(f_\alpha) \leqslant \tau(f_\alpha)$. If $g_\beta \in \tilde{s}_u$ and $f_\alpha \leqslant g_\beta$, then $\tau(f_\alpha) \leqslant \tau(g_\beta)$ and $\tau(f_\alpha) \leqslant \inf\limits_{f_\alpha \leqslant g_\beta} \tau(g_\beta) = \bar{\tau}(f_\alpha)$. Therefore $\bar{\tau}(f_\alpha) = \tau(f_\alpha)$. Also $\bar{\tau}(-f_\alpha) = \tau(-f_\alpha) = -\tau(f_\alpha)$. Then $-\bar{\tau}(-f_\alpha) = \underline{\tau}(f_\alpha) = \tau(f_\alpha)$. Thus $\underline{\tau}(f_\alpha) = \bar{\tau}(f_\alpha) = \tau(f_\alpha)$.

Lemma 2.1.8. If $\{(f_n)_{\alpha_n}\}$ is a sequence of non negative fuzzy points and if

$$f_\alpha = \left(\sum_{n=1}^{\infty} f_n\right)_{\inf\limits_{n \in N} \alpha_n} \quad \text{then } \bar{\tau}(f_\alpha) \leqslant \sum_{n=1}^{\infty} \bar{\tau}((f_n)_{\alpha_n})$$

Proof: If $\bar{\tau}((f_n)_{\alpha_n}) = \infty$ for some n, then the result follows immediately. If for every n, $\bar{\tau}((f_n)_{\alpha_n}) \neq \infty$ then given $\varepsilon_t \in \tilde{R}$ we can find a fuzzy point $(g_n)_{\beta_n}$ in $\tilde{s}_u$ such that $(f_n)_{\alpha_n} \leqslant (g_n)_{\beta_n}$ and $\tau((g_n)_{\beta_n}) \leqslant \bar{\tau}((f_n)_{\alpha_n}) + 2^{-n} \varepsilon_t$ by Definition 2.1.1. (Here β_n may be taken as sup α_n and t equal to 1). Now $\{(g_n)_{\beta_n}\}$ is a sequence of non negative fuzzy points and therefore by result 1.2.13, $g_\beta = \sum\limits_{n=1}^{\infty} (g_n)_{\beta_n}$ is in $\tilde{s}_u$ and

$$\begin{aligned} \tau(g_\beta) &= \sum_{n=1}^{\infty} \tau((g_n)_{\beta_n}) \\ &\leqslant \sum_{n=1}^{\infty} \bar{\tau}((f_n)_{\alpha_n}) + \varepsilon_t \end{aligned}$$

Now $f_\alpha \leqslant g_\beta$ and $g_\beta \in \tilde{s}_u$ gives $\bar{\tau}(f_\alpha) \leqslant \tau(g_\beta)$

$$\leqslant \sum_{n=1}^{\infty} \bar{\tau}((f_n)_{\alpha_n}) + \varepsilon_t.$$

But ε_t is arbitrary so that $\bar{\tau}(f_\alpha) \leqslant \sum_{n=1}^{\infty} \bar{\tau}((f_n)_{\alpha_n})$.

2.2. INTEGRABILITY OF A FUZZY POINT

Definition 2.2.1. A fuzzy point f_α is τ-integrable if $\bar{\tau}(f_\alpha) = \underline{\tau}(f_\alpha)$ and it is finite.

Notation 2.2.2. The class of all τ-integrable fuzzy points is denoted by $\tilde{s}_1$. For f_α in $\tilde{s}_1$, $\tau(f_\alpha) = \bar{\tau}(f_\alpha)$.

The following proposition shows that the original integral τ on $\tilde{s}$ can be extended to all of $\tilde{s}_1$.

Proposition 2.2.3. The set $\tilde{s}_1$ is a vector lattice of fuzzy points containing $\tilde{s}$ and τ is a positive linear functional on $\tilde{s}_1$ which extends the functional τ on $\tilde{s}$.

Proof: If $f_\alpha \in \tilde{s}_1$ and $c \geqslant 0$ then by lemma 2.1.5,

$$\bar{\tau}(cf_\alpha) = c\,\bar{\tau}(f_\alpha) = c\,\underline{\tau}(f_\alpha) = \underline{\tau}(c\, f_\alpha).$$ Thus $cf_\alpha \in \tilde{s}_1$.

Let $f_\alpha, g_\beta \in \tilde{s}_1$. By lemma 2.1.5,

$$\bar{\tau}(f_\alpha + g_\beta) \leqslant \bar{\tau}(f_\alpha) + \bar{\tau}(g_\beta)$$

Also $\quad -\overline{\tau}(f_\alpha+g_\beta) = \overline{\tau}(-f_\alpha+-g_\beta)$

$$\leqslant \overline{\tau}(-f_\alpha) + \overline{\tau}(-g_\beta)$$

$$= -\underline{\tau}(f_\alpha) - \underline{\tau}(g_\beta)$$

i.e., $\overline{\tau}(f_\alpha+g_\beta) \geqslant \overline{\tau}(f_\alpha) + \overline{\tau}(g_\beta)$.

Therefore

$$\overline{\tau}(f_\beta+g_\beta) = \overline{\tau}(f_\alpha) + \overline{\tau}(g_\beta)$$

$$= \underline{\tau}(f_\alpha) + \underline{\tau}(g_\beta).$$

Similarly,

$$\underline{\tau}(f_\alpha+g_\beta) = \underline{\tau}(f_\alpha) + \underline{\tau}(g_\beta) \text{ follows.}$$

Therefore

$$\overline{\tau}(f_\alpha+g_\beta) = \underline{\tau}(f_\alpha+g_\beta); \text{ i.e., } f_\alpha+g_\beta \in \tilde{s}_1.$$

Thus τ is a linear functional on $\tilde{s}_1$.

We have $\quad f_\alpha \vee g_\alpha = (f_\alpha - g_\alpha) \vee o_\alpha + g_\alpha$.

Therefore to prove that $\tilde{s}_1$ is a vector lattice it is sufficient to show that $f_\alpha \vee o_\alpha \in \tilde{s}_1$ for each $f_\alpha \in \tilde{s}_1$. For a given ε_t in $\tilde{R}$ and t equal to 1 it is possible to find two fuzzy points g_β and h_γ in $\tilde{s}_u$ such that

$$-h_\gamma \leqslant f_\alpha \leqslant g_\beta \text{ with } \tau(g_\beta) \leqslant \tau(f_\alpha) + \varepsilon_t \text{ and}$$

$$\tau(h_\gamma) \leqslant -\tau(f_\alpha) + \varepsilon_t,$$

where $\tau(g_\beta)$ and $\tau(h_\gamma)$ are finite.

We have $g_\beta = (g_\beta \vee o_\beta) + (g_\beta \wedge o_\beta)$. Since $g_\beta \wedge o_\beta \in \tilde{s}_u$, $\tau(g_\beta \wedge o_\beta)$ is finite and further since

$$g_\beta \vee o_\beta = g_\beta - (g_\beta \wedge o_\beta).$$

Therefore

$$\tau(g_\beta \vee o_\beta) \leqslant \tau(g_\beta) + \tau(-(g_\beta \wedge o_\beta)) \text{ by lemma 2.1.5.}$$

$$= \tau(g_\beta) - \tau(g_\beta \wedge o_\beta), \text{ which is also}$$

finite since $g_\beta \vee o_\beta \in \tilde{s}_u$. Also $h_\gamma \wedge o_\gamma$ is in $\tilde{s}_u$ and $-(h_\gamma \wedge o_\gamma) \leqslant (f_\alpha \vee o_\alpha) \leqslant g_\beta \vee o_\beta$. We have $g_\beta \geqslant - h_\gamma$ and therefore $g_\beta \vee o_\beta + h_\gamma \wedge o_\gamma \leqslant g_\beta + h_\gamma$. This gives

$$\tau(g_\beta \vee o_\beta) + \tau(h_\gamma \wedge o_\gamma) \leqslant \tau(g_\beta) + \tau(h_\gamma) \leqslant 2\,\varepsilon_t \qquad \text{(i)}$$

Now,

$$-(h_\gamma \wedge o_\gamma) \leqslant f_\alpha \vee o_\alpha \leqslant g_\beta \vee o_\beta \ ;$$

i.e., $-\tau(h_\gamma \wedge o_\gamma) \leqslant \underline{\tau}(f_\alpha \vee o_\alpha) \leqslant \overline{\tau}(f_\alpha \vee o_\alpha) \leqslant \overline{\tau}(g_\beta \vee o_\beta) = \tau(g_\beta \vee o_\beta)$ since $g_\beta \vee o_\beta \in \tilde{s}_u$. We have $\overline{\tau}(f_\alpha \vee o_\alpha) \leqslant \tau(g_\beta \vee o_\beta)$ and

$$-\underline{\tau}(f_\alpha \vee o_\alpha) \leqslant \tau(h_\gamma \wedge o_\gamma) \text{ which gives}$$

$$\overline{\tau}(f_\alpha \vee o_\alpha) - \underline{\tau}(f_\alpha \vee o_\alpha)$$
$$\leqslant \tau(g_\beta \vee o_\beta) + \tau(h_\gamma \wedge o_\gamma)$$
$$\leqslant 2\,\varepsilon_t \text{ from (i).}$$

Therefore $\bar{\tau}(f_\alpha \vee o_\alpha) = \underline{\tau}(f_\alpha \vee o_\alpha)$ since ε_t is arbitrary. Also $\bar{\tau}(f_\alpha \vee o_\alpha) \leqslant \tau(g_\beta \vee o_\beta) < \infty$.

Therefore, $f_\alpha \vee o_\alpha \in \tilde{S}_1$.

Chapter III

THE MONOTONE CONVERGENCE THEOREM, FATOU'S LEMMA, THE LEBESGUE CONVERGENCE THEOREM AND UNIQUENESS OF THE EXTENSION OF THE FUZZY LINEAR FUNCTIONAL

3.0. INTRODUCTION

In the previous chapter we have seen how the fuzzy linear functional τ is extended from $\tilde{s}$ to $\tilde{s}_1$ which is the analogous form of the extension of the non negative linear functional I from L to L_1. In the case of the crisp theory the next step is to show that the non negative linear functional I on L_1 is a Daniell integral on L_1. This is established by means of the analogue of the monotone convergence theorem; then further prove the analogues for the integral I of Fatou's lemma and the Lebesgue convergence theorem which are considered to be very useful in the development of the theory. Here we are establishing that the fuzzy analogues of the monotone convergence theorem, Fatou's lemma and the Lebesgue convergence theorem are true under suitable assumptions. Also similar to the uniqueness of the extension of I to L_1 we show that the extension of τ to $\tilde{s}_1$ is unique.

3.1. THE MONOTONE CONVERGENCE THEOREM, FATOU'S LEMMA AND LEBESGUE CONVERGENCE THEOREM.

Notation 3.1.1. For a sequence $\left\{(f_n)_{\alpha_n}\right\}$ of fuzzy points in $\tilde{s}_1$ $\inf (f_n)_{\alpha_n} = f_\alpha$ when $f = \inf f_n$ and $\alpha = \sup \alpha_n$. What follows is the analogue of monotone convergence theorem.

Proposition 3.1.2. Let $\{(f_n)_{\alpha_n}\}$ be an increasing sequence of fuzzy points in $\tilde{s}_1$ such that $\inf \alpha_n > o$ and let $f_\alpha = \lim (f_n)_{\alpha_n}$. Then $f_\alpha \in \tilde{s}_1$ if and only if $\lim \tau((f_n)_{\alpha_n}) < \infty$. In this case $\tau(f_\alpha) = \lim \tau((f_n)_{\alpha_n})$.

Proof:

Necessity. Let $\{(f_n)_{\alpha_n}\}$ be an increasing sequence and let $f_\alpha = \lim (f_n)_{\alpha_n}$. Then $f_\alpha \geqslant (f_n)_{\alpha_n}$. If $f_\alpha \in \tilde{s}_1$ then $\bar{\tau}(f_\alpha) = \tau(f_\alpha) \geqslant \tau((f_n)_{\alpha_n})$. If $\lim \tau((f_n)_{\alpha_n}) = \infty$, i.e., given k_t there exists n_o such that $\forall\, n \geqslant n_o$, $\tau((f_n)_{\alpha_n}) \geqslant k_t$ with $t=1$ for every $n \geqslant n_o$ then $\bar{\tau}(f_\alpha) = \tau(f_\alpha) = \infty$ which implies $f_\alpha \notin \tilde{s}_1$. Therefore if $f_\alpha \in \tilde{s}_1$ then $\lim \tau(f_n)_{\alpha_n} < \infty$.

Sufficiency. Let $\lim \tau((f_n)_{\alpha_n}) < \infty$. Since $\{(f_n)_{\alpha_n}\}$ is an increasing sequence,

$$\sum_{n=1}^{\infty} ((f_{n+1})_{\alpha_{n+1}} - (f_n)_{\alpha_n}) = \sum_{n=1}^{\infty} (f_{n+1} - f_n)_{\min(\alpha_n, \alpha_{n+1})}$$

$$= \sum_{n=1}^{\infty} (f_{n+1} - f_n)_{\alpha_{n+1}}$$

$$= \left(\sum_{n=1}^{\infty} (f_{n+1}-f_n)\right)_{\inf \alpha_i}, \ i \in N$$

$$= (f-f_1)_{\inf \alpha_i}, \quad i \in N$$

$$= g_\alpha \text{ say where } \alpha = \inf \alpha_i.$$

By lemma 2.1.8,

$$\bar{\tau}(g_\alpha) \leqslant \sum_{n=1}^{\infty} \bar{\tau}((f_{n+1})_{\alpha_{n+1}} - (f_n)_{\alpha_n})$$

$$= \sum_{n=1}^{\infty} (\bar{\tau}(f_{n+1})_{\alpha_{n+1}} - \bar{\tau}(f_n)_{\alpha_n})$$

$$= \sum_{n=1}^{\infty} (\tau(f_{n+1})_{\alpha_{n+1}} - \tau((f_n)_{\alpha_n}))$$

since $(f_n)_{\alpha_n} \in \tilde{S}_1$ for every n.

$$= \lim \tau((f_n)_{\alpha_n}) - \tau(f_1)_{\alpha_1})$$

i.e., $\bar{\tau}(g_\alpha) + \tau((f_1)_{\alpha_1}) \leqslant \lim \tau((f_n)_{\alpha_n})$

Since $\quad g_\alpha + (f_1)_{\alpha_1} = f_\alpha$, $\bar{\tau}(f_\alpha) = \bar{\tau}(g_\alpha + (f_1)_{\alpha_1})$

$$\leqslant \bar{\tau}(g_\alpha) + \bar{\tau}(f_1)_{\alpha_1})$$

$$\leqslant \lim \tau((f_n)_{\alpha_n})$$

We have $(f_n)_{\alpha_n} \leqslant f_\alpha$ so that $\underline{\tau}((f_n)_{\alpha_n}) \leqslant \underline{\tau}(f_\alpha)$

i.e., $\tau((f_n)_{\alpha_n}) = \underline{\tau}((f_n)_{\alpha_n}) \leqslant \underline{\tau}(f_\alpha)$, for $(f_n)_{\alpha_n} \in \tilde{s}_1$.

i.e., $\lim \tau((f_n)_{\alpha_n}) \leqslant \underline{\tau}(f_\alpha)$. But $\lim \tau((f_n)_{\alpha_n}) \leqslant \underline{\tau}(f_\alpha) \leqslant \bar{\tau}(f_\alpha)$.

Since $\bar{\tau}(f_\alpha) \leqslant \lim \tau((f_n)_{\alpha_n})$, $\bar{\tau}(f_\alpha) = \underline{\tau}(f_\alpha) = \lim \tau((f_n)_{\alpha_n})$

i.e., $f_\alpha \in \tilde{s}_1$ and $\tau(f_\alpha) = \lim \tau((f_n)_{\alpha_n})$.

Corollary 3.1.3. The functional τ is a fuzzy Daniell functional on the vector lattice $\tilde{s}_1$.

For, $\tau(f_\alpha) \geqslant 0$ for every $f_\alpha \in \tilde{s}_1$, τ is a positive linear functional on $\tilde{s}_1$ and satisfies the definition of fuzzy Daniell functional.

The next proposition is the analogue of Fatou's lemma.

Proposition 3.1.4. Let $\{(f_n)_{\alpha_n}\}$ be a sequence of nonnegative fuzzy points in $\tilde{s}_1$, where $\sup \alpha_n \geqslant \frac{1}{2}$. Then the fuzzy points $\inf(f_n)_{\alpha_n}$ and $\underline{\lim}\,(f_n)_{\alpha_n}$ are in $\tilde{s}_1$, if $\underline{\lim}\,\tau((f_n)_{\alpha_n}) < \infty$. In this case $\tau(\underline{\lim}\,(f_n)_{\alpha_n}) \leqslant \underline{\lim}\,\tau((f_n)_{\alpha_n})$.

Proof: Define $(g_n)_{\beta_n} = (f_1)_{\alpha_1} \wedge (f_2)_{\alpha_2} \wedge \dots (f_n)_{\alpha_n}$.
Then $\{(g_n)_{\beta_n}\}$ is a sequence of nonnegative fuzzy points in $\tilde{s}_1$ which decrease to $g_\beta = \inf(f_n)_{\alpha_n}$; i.e., $g = \inf f_n$

and $\beta = \sup \alpha_n$. Now $\beta_n \uparrow \beta$ means $(1-\beta_n) \downarrow (1-\beta)$ so that $(-g_n)_{1-\beta_n} \uparrow (-g)_{1-\beta}$. Also $\tau((-g_n)_{1-\beta_n}) \leqslant 0$; i.e., $\lim \tau((-g_n)_{1-\beta_n}) < \infty$. Therefore $(-g)_{1-\beta} \in \tilde{s}_1$ by Proposition 3.1.2.

Now we will show that $g_\beta \in \tilde{s}_1$. Since $(-g)_{1-\beta} = -(g)_{1-\beta} \in \tilde{s}_1$ we have $g_{1-\beta} \in \tilde{s}_1$ so that $\tau(g_{1-\beta}) < \infty$. Since $g \leqslant g$, $\beta \geqslant 1-\beta$ and $\frac{1}{2} \leqslant \beta < 1$, $g_\beta \leqslant g_{1-\beta}$. Therefore $\tau(g_\beta) \leqslant \tau(g_{1-\beta}) < \infty$ i.e., $g_\beta \in \tilde{s}_1$.

Let $(h_\ell)_{\gamma_\ell} = \inf\limits_{n \geqslant \ell} (f_n)_{\alpha_n}$. Then $\{(h_\ell)_{\gamma_\ell}\}$ is a sequence of non negative fuzzy points in $\tilde{s}_1$ which increases to $\underline{\lim}\,(f_n)_{\alpha_n}$ as $\ell \to \infty$. Therefore,

$$(h_\ell)_{\gamma_\ell} \leqslant (f_n)_{\alpha_n} \quad \text{for } \ell \leqslant n ;$$

i.e., $\tau((h_\ell)_{\gamma_\ell}) \leqslant \tau((f_n)_{\alpha_n})$ for $\ell \leqslant n$;

i.e., $\tau((h_\ell)_{\gamma_\ell}) \leqslant \inf\limits_{\ell \leqslant n} \tau((f_n)_{\alpha_n})$

i.e., $\lim \tau((h_\ell)_{\gamma_\ell}) \leqslant \inf\limits_{\ell \leqslant n} \tau((f_n)_{\alpha_n})$

i.e., $\lim \tau((h_\ell)_{\gamma_\ell}) \leqslant \underline{\lim}\, \tau((f_n)_{\alpha_n}) < \infty$

Thus $\underline{\lim}\,(f_n)_{\alpha_n} \in \tilde{s}_1$ by Proposition 3.1.2.

Since $\tau((h_\ell)_{\gamma_\ell}) \leqslant \inf_{\ell \leqslant n} \tau((f_n)_{\alpha_n})$ for every ℓ,

$$\tau(\sup_{\ell} \inf_{\ell \leqslant n} (f_n)_{\alpha_n}) \leqslant \sup_{\ell} \inf_{\ell \leqslant n} \tau((f_n)_{\alpha_n})$$

i.e., $\tau(\underline{\lim}\, (f_n)_{\alpha_n}) \leqslant \underline{\lim}\, \tau((f_n)_{\alpha_n})$

Analogous to Lebesgue convergence theorem we have the following proposition.

Proposition 3.1.5. Let $\{(f_n)_{\alpha_n}\}$ be a sequence of fuzzy points in $\tilde{S}_1$ and let there be a fuzzy point g_β in $\tilde{S}_1$ such that for all n we have $(|f_n|)_{\alpha_n} \leqslant g_\beta$. Then if $f_\alpha = \lim (f_n)_{\alpha_n}$, $\tau(f_\alpha) = \lim \tau((f_n)_{\alpha_n})$.

Proof: $\{(f_n)_{\alpha_n}+g_\beta\}$ is a sequence of non negative fuzzy points in $\tilde{S}_1$ and by Proposition 3.1.4,
$\lim ((f_n)_{\alpha_n}+ g_\beta) = f_\alpha + g_\beta$ is in $\tilde{S}_1$,

since $\tau((f_n)_{\alpha_n}+g_\beta) \leqslant 2\tau(g_\beta)$

Also,

$$\begin{aligned} \tau(f_\alpha+g_\beta) &= \tau(\underline{\lim}\, ((f_n)_{\alpha_n} + g_\beta)) \\ &\leqslant \underline{\lim}\, \tau((f_n)_{\alpha_n} + g_\beta) \\ &= \underline{\lim}\, \tau((f_n)_{\alpha_n}) + \tau(g_\beta). \end{aligned}$$

Since τ is a linear functional on $\tilde{s}_1$,

$$\tau(f_\alpha + g_\beta) = \tau(f_\alpha) + \tau(g_\beta). \text{ Therefore,}$$

$$\tau(f_\alpha) \leqslant \underline{\lim}\, \tau((f_n)_{\alpha_n}).$$

$\{g_\beta - (f_n)_{\alpha_n}\}$ is also a sequence of non negative fuzzy points. Therefore,

$$\begin{aligned}\tau(g_\beta - f_\alpha) &= \tau(\underline{\lim}\,(g_\beta - (f_n)_{\alpha_n}))\\ &\leqslant \underline{\lim}\,\tau(g_\beta - (f_n)_{\alpha_n})\\ &= \tau(g_\beta) - \overline{\lim}\,\tau((f_n)_{\alpha_n});\end{aligned}$$

i.e., $\tau(g_\beta) - \tau(f_\alpha) \leqslant \tau(g_\beta) - \overline{\lim}\,\tau((f_n)_{\alpha_n});$

i.e., $\overline{\lim}\,\tau((f_n)_{\alpha_n}) \leqslant \tau(f_\alpha)$

Thus $\tau(f_\alpha) = \lim \tau((f_n)_{\alpha_n})$

3.2. UNIQUENESS OF THE EXTENSION OF τ ON $\tilde{s}$ TO $\tilde{s}_1$

Let $\tilde{s}_{u\ell}$ be the class of all fuzzy points f_α of F which are the limit of a decreasing sequence of fuzzy points $\{(f_n)_{\alpha_n}\}$ with $\sup \alpha_n \geqslant \frac{1}{2}$ in $\tilde{s}_u$ such that $\tau((f_n)_{\alpha_n}) < \infty$ and $\lim \tau((f_n)_{\alpha_n}) > -\infty$.

Lemma 3.2.1. If f_α is any fuzzy point with $\bar{\tau}(f_\alpha)$ finite then there is a $g_\beta \in \tilde{s}_{u\ell}$ such that $f_\alpha \leqslant g_\beta$ and $\bar{\tau}(f_\alpha) = \tau(g_\beta)$.

Proof: If $\{(f_n)_{\alpha_n}\}$ is a decreasing sequence of fuzzy points with $\sup \alpha_n \geqslant \frac{1}{2}$ in $\tilde{s}_u$ then $\{(-f_n)_{1-\alpha_n}\}$ is an increasing sequence of fuzzy points in $\tilde{s}_u$ with $\lim ((-f_n)_{1-\alpha_n})$ in $\tilde{s}_u$. Then by lemma 2.1.7
$\bar{\tau}(\lim((-f_n)_{1-\alpha_n})) = \underline{\tau}(\lim(-f_n)_{1-\alpha_n}) = \tau(\lim(-f_n)_{1-\alpha_n})$.
Since $\{(-f_n)_{1-\alpha_n}\}$ is an increasing sequence in $\tilde{s}_u$, consequently in $\tilde{s}_1$ by Proposition 3.1.2,
$\lim ((-f_n)_{1-\alpha_n}) \in \tilde{s}_1$. Therefore $\lim (f_n)_{\alpha_n} \in \tilde{s}_1$ as in the proof of Proposition 3.1.5. Thus $\tilde{s}_{u\ell} \subset \tilde{s}_1$.

Let f_α be any fuzzy point in F with $\bar{\tau}(f_\alpha) < \infty$. Then for a given n, there exists $(h_n)_{\gamma_n} \in \tilde{s}_u$ such that

$$\bar{\tau}(f_\alpha) = \inf_{f_\alpha \leqslant (h_n)_{\gamma_n}} \tau((h_n)_{\gamma_n})$$

i.e., $\tau((h_n)_{\gamma_n}) \leqslant \bar{\tau}(f_\alpha) + (\frac{1}{n})_t$, where $(\frac{1}{n})_t \in \tilde{R}$ with t=1

Define

$$(g_n)_{\beta_n} = (h_1)_{\gamma_1} \wedge (h_2)_{\gamma_2} \wedge \ldots \wedge (h_n)_{\gamma_n}$$

$$= (h_1 \wedge h_2 \wedge \ldots \wedge h_n)_{\max(\gamma_1, \gamma_2, \ldots, \gamma_n)}$$

Therefore $f_\alpha \leqslant (g_n)_{\beta_n} \leqslant (h_n)_{\gamma_n}$ and $\{(g_n)_{\beta_n}\}$ is a decreasing sequence of fuzzy points in $\tilde{s}_u$ with

$\overline{\tau}(f_\alpha) \leqslant \tau((g_n)_{\beta_n}) \leqslant \tau((h_n)_{\gamma_n})$;

i.e., $\overline{\tau}(f_\alpha) \leqslant \tau((g_n)_{\beta_n}) \leqslant \overline{\tau}(f_\alpha) + (\frac{1}{n})_t$

Therefore $\lim \tau((g_n)_{\beta_n} = \overline{\tau}(f_\alpha)$; i.e., $\lim \tau(g_n)_{\beta_n}$ exists and $\lim \tau((g_n)_{\beta_n}) = \tau(\lim(g_n)_{\beta_n}) = \tau(g_\beta)$.

Since g_β is the limit of a decreasing sequence $\{(g_n)_{\beta_n}\}$ of fuzzy points in $\tilde{s}_u$, $g_\beta \in \tilde{s}_{u\ell}$. Thus $\overline{\tau}(f_\alpha) = \tau(g_\beta)$.

Definition 3.2.2. A fuzzy point f_α in F is said to be fuzzy null point if $f_\alpha \in \tilde{s}_1$ and $\tau((|f|)_\alpha) = 0$, where $\alpha \in (0,1]$ and 0 means fuzzy singleton 0.

Remark 3.2.3. If f_α is a fuzzy null point and $(|g|)_\beta \leqslant f_\alpha$ then $0 \leqslant \underline{\tau}((|g|)_\beta) \leqslant \overline{\tau}((|g|)_\beta) \leqslant \tau(f_\alpha) = 0$;

i.e., $\underline{\tau}((|g|)_\beta) = \overline{\tau}((|g|)_\beta) = \tau((|g|)_\beta) = 0$

i.e., $g_\beta \in \tilde{s}_1$ and g_β is a fuzzy null point.

Proposition 3.2.4. A fuzzy point f_α in F is in $\tilde{s}_1$ if and only if $f_\alpha = g_\beta - h_\gamma$, where $\alpha = \min(\beta,\gamma)$, $g_\beta \in \tilde{s}_{u\ell}$ and h_γ is a non negative fuzzy null point. A fuzzy point of F, h_γ is a fuzzy null point if and only if there is a fuzzy null point k_δ in $\tilde{s}_{u\ell}$ such that $(|h|)_\gamma \leqslant k_\delta$.

Proof:

Necessity. Let $f_\alpha = g_\beta - h_\gamma$ where $\alpha = \min(\beta,\gamma)$, $g_\beta \in \tilde{s}_{u\ell}$ and h_γ a non negative fuzzy null point. Since $g_\beta \in \tilde{s}_{u\ell}$ and $\tilde{s}_{u\ell} \subset \tilde{s}_1$, $g_\beta \in \tilde{s}_1$. Also since h_γ is a non negative fuzzy null point, $h_\gamma \in \tilde{s}_1$. Now,
$g_\beta - h_\gamma = (g+(-h))_{\min(\beta,\gamma)} = f_\alpha \in \tilde{s}_1$.

Let k_δ be a fuzzy null point in $\tilde{s}_{u\ell}$ and $(|h|)_\gamma \leqslant k_\delta$. Then $\bar{\tau}((|h|)_\gamma) = \tau(k_\delta) = 0$. From the above remark $h_\gamma \in \tilde{s}_1$ and h_γ is a fuzzy null point.

Sufficiency. Let $f_\alpha \in \tilde{s}_1$. Then $\bar{\tau}(f_\alpha) = \tau(f_\alpha)$ is finite and there exists a $g_\beta \in \tilde{s}_{u\ell}$ with $f_\alpha \leqslant g_\beta$ and $\tau(f_\alpha) = \tau(g_\beta)$. Therefore, $h_\gamma = g_\beta - f_\alpha$ is a non negative fuzzy point. Also, $\tau(h_\gamma) = \tau(g_\beta - f_\alpha) = \tau(g_\beta) - \tau(f_\alpha) = 0$. Therefore, h_γ is a fuzzy null point.

Let h_γ be a fuzzy null point. Then $h_\gamma \in \tilde{s}_1$, and $\tau((|h|)_\gamma) = 0$. Therefore by lemma 3.2.1, there is a fuzzy point $k_\delta \in \tilde{s}_{u\ell}$ such that $(|h|)_\gamma < k_\delta$ and $\tau((|h|)_\gamma) = \tau(k_\delta) = 0$.

The following proposition establishes the uniqueness of the extension of τ on $\tilde{s}_1$.

Proposition 3.2.5. Let τ be a fuzzy Daniell integral on a fuzzy vector lattice $\tilde{s}$ and ξ be a fuzzy Daniell integral on a fuzzy vector lattice $\tilde{t} \supset \tilde{s}$.
If $\tau(f_\alpha) = \xi(f_\alpha)$ $\forall f_\alpha \in \tilde{s}$ then $\tilde{t}_1 \supset \tilde{s}_1$ and $\tau(f_\alpha) = \xi(f_\alpha)$ for every $f_\alpha \in \tilde{s}_1$.

Proof: Suppose that $\{(f_n)_{\alpha_n}\}$ be an increasing sequence of fuzzy points in $\tilde{s}_1$ and let $f_\alpha = \lim (f_n)\alpha_n$. By Proposition 3.1.2, $f_\alpha \in \tilde{s}_1$ and $\tau(f_\alpha) = \lim \tau((f_n)_{\alpha_n}) = \lim \xi((f_n)_{\alpha_n}) = \xi(f_\alpha)$. We have shown that the fuzzy Daniell integral τ on $\tilde{s}$ can be extended to $\tilde{s}_1$ by Proposition 2.2.3 and in the same way ξ on $\tilde{t}$ can be extended to $\tilde{t}_1$. From above $f_\alpha \in \tilde{s}_1$ implies $f_\alpha \in \tilde{t}_1$. Hence $\tilde{s}_1 \subset \tilde{t}_1$.

Using proposition 3.1.2. $\tilde{s}_{u\ell} \subset \tilde{s}_1$ and from above $\tilde{s}_1 \subset \tilde{t}_1$ we get $\tilde{s}_{u\ell} \subset \tilde{s}_1 \subset \tilde{t}_1$. Therefore if $f_\alpha \in \tilde{s}_1$ then $f_\alpha \in \tilde{t}_1$ and $\tau(f_\alpha) = \xi(f_\alpha)$ for every $f_\alpha \in \tilde{s}_1$.

Chapter IV

STONE-LIKE THEOREM

4.0. INTRODUCTION

This chapter deals with the fuzzy analogue of following part of the Daniell Integration theory, viz., measurability of a non negative real valued function on X, measurability of a sub set of X and its integrability with respect to I. Daniell established that the measure μ defined over the σ-algebra of measurable subsets of X with respect to I satisfies the property that the integrable sets are the same as the measurable sets of finite measure. The important result of this chapter is the Stone's theorem which says that each function f on X is integrable with respect to I if and only if it is integrable with respect to μ and that $I(f) = \int f \, d\mu$. This chapter ends with analogous result for the establishment of the uniqueness of the measure μ. To establish the parallel theory we define measurability of a fuzzy point and fuzzy measure of a fuzzy set with respect to τ.

4.1. PRELIMINARIES

We adapt definitions 2.1 and 2.2 of [QI] which are respectively quoted below as 4.1.1 and 4.1.2. Klement's [KL] definition requires additionally that every constant belongs to the fuzzy σ- algebra.

Definition 4.1.1. Let X be a non empty set and $\mathcal{F}(X) = \{\widetilde{A};\ \widetilde{A}: X \longrightarrow [0,1]\}$. Then $\widetilde{\mathcal{F}}$ which is a sub-class of $\mathcal{F}(X)$ is a fuzzy σ-algebra if the following conditions are satisfied: (i) $\widetilde{\emptyset}, X \in \widetilde{\mathcal{F}}$ (ii) if $\widetilde{A} \in \widetilde{\mathcal{F}}$, then $\widetilde{A}^c \in \widetilde{\mathcal{F}}$ (iii) if $\{\widetilde{A}_n\} \subset \widetilde{\mathcal{F}}$, then $\bigcup_{n=1}^{\infty} \widetilde{A}_n \in \widetilde{\mathcal{F}}$.

Definition 4.1.2. A mapping $\xi : \widetilde{\mathcal{F}} \longrightarrow [0,\infty]$ is said to be a fuzzy measure on $\widetilde{\mathcal{F}}$ if and only if (i) $\xi(\emptyset) = 0$, (ii) for any $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{F}}$, if $\widetilde{A} \subset \widetilde{B}$ then $\xi(\widetilde{A}) \leqslant \xi(\widetilde{B})$, (iii) whenever $\{\widetilde{A}_n\} \subset \widetilde{\mathcal{F}}$, $\widetilde{A}_n \subset \widetilde{A}_{n+1}$, n=1,2,...,

$\xi(\bigcup_{n=1}^{\infty} \widetilde{A}_n) = \lim_{n\to\infty} \xi(\widetilde{A}_n)$ (continuity from below),

(iv) whenever $\{\widetilde{A}_n\} \subset \widetilde{\mathcal{F}}$, $\widetilde{A}_n \supset \widetilde{A}_{n+1}$, n=1,2,..., and there exists n_o such that $\xi(\widetilde{A}_{n_o}) < \infty$, then

$\xi(\bigcap_{n=1}^{\infty} \widetilde{A}_n) = \lim_{n\to\infty} \xi(\widetilde{A}_n)$ (continuity from above).

4.2. MEASURABILITY AND FUZZY MEASURE OF A FUZZY POINT

Definition 4.2.1. A non negative fuzzy point f_α in F is said to be measurable with respect to τ if $g_\beta \wedge f_\alpha$ is in $\tilde{s}_1$ for each g_β in $\tilde{s}_1$.

Note 4.2.2. If $(f_1)_{\alpha_1}$ and $(f_2)_{\alpha_2}$ are two non negative measurable fuzzy points, then $(f_1)_{\alpha_1} + (f_2)_{\alpha_2}$ is measurable.

Proof:

We have for g_β in $\tilde{S}_1$,

$$((f_1)_{\alpha_1} + (f_2)_{\alpha_2}) \wedge g_\beta = (f_1+f_2)_{\min(\alpha_1,\alpha_2)} \wedge g_\beta$$

$$= ((f_1+f_2) \wedge g)_{\max(\min(\alpha_1,\alpha_2),\beta)}$$

$$= ((f_1 \wedge g)+(f_2 \wedge g))_{\max(\min(\alpha_1,\alpha_2),\beta)}$$

$$= ((f_1 \wedge g)+(f_2 \wedge g))_{\min(\max(\alpha_1,\beta),\max(\alpha_2,\beta))}$$

$$= (f_1 \wedge g)_{\max(\alpha_1,\beta)}+(f_2 \wedge g)_{\max(\alpha_2,\beta)}$$

$$= (f_1)_{\alpha_1} \wedge g_\beta + (f_2)_{\alpha_2} \wedge g_\beta \in \tilde{S}_1.$$

Therefore $(f_1)_{\alpha_1} + (f_2)_{\alpha_2}$ is measurable.

Lemma 4.2.3. If f_α and g_β are non negative measurable. fuzzy points then $f_\alpha \wedge g_\beta$ and $f_\alpha \vee g_\beta$ are measurable. If $\{(f_n)_{\alpha_n}\}$ is a sequence of non negative measurable fuzzy points which converge pointwise to a fuzzy point f_α, then f_α is measurable.

Proof: Let f_α, g_β be non negative measurable fuzzy points and let h_γ be in $\tilde{S}_1$. Then $h_\gamma \wedge f_\alpha$ and $h_\gamma \wedge g_\beta$ are in $\tilde{S}_1$.

Also,

$$h_\gamma \wedge (f_\alpha \wedge g_\beta) = h_\gamma \wedge (f \wedge g)_{\max(\alpha,\beta)}$$

$$= (h \wedge (f \wedge g))_{\max(\gamma,\ \max(\alpha,\beta))}$$

$$= ((h \wedge f) \wedge (h \wedge g))_{\max(\max(\gamma,\alpha),\ \max(\gamma,\beta))}$$

$$= (h \wedge f)_{\max(\gamma,\alpha)} \wedge (h \wedge g)_{\max(\gamma,\beta)}$$

$$= (h_\gamma \wedge f_\alpha) \wedge (h_\gamma \wedge g_\beta) \in \tilde{s}_1.$$

To prove $h_\gamma \wedge (f_\alpha \vee g_\beta) \in \tilde{s}_1$ we can see that following relationship is true in all the three cases, (1) $\alpha<\beta<\gamma$, (2) $\alpha < \gamma < \beta$ and (3) $\gamma < \alpha < \beta$ and by symmetry the other three relationships connecting α,β,γ are satisfied. We have,

$$h_\gamma \wedge (f_\alpha \vee g_\beta) = h_\gamma \wedge (f \vee g)_{\min(\alpha,\beta)}$$

$$= (h \wedge (f \vee g))_{\max(\gamma,\ \min(\alpha,\beta))}$$

$$= ((h \wedge f) \vee (h \wedge g))_{\min(\max(\gamma,\alpha),\ \max(\gamma,\beta))}$$

$$= (h \wedge f)_{\max(\gamma,\alpha)} \vee (h \wedge g)_{\max(\gamma,\beta)}$$

$$= (h_\gamma \wedge f_\alpha) \vee (h_\gamma \wedge g_\beta) \in \tilde{s}_1$$

Thus $f_\alpha \wedge g_\beta$ and $f_\alpha \vee g_\beta$ are non negative measurable fuzzy points.

Suppose that $\{(f_n)_{\alpha_n}\}$ is a sequence of non negative measurable fuzzy points converging to f_α and g_β be a fuzzy point in $\tilde{S}_1$. Then $\{(f_n)_{\alpha_n} \wedge g_\beta\}$ is a sequence of fuzzy points in $\tilde{S}_1$ converging to $f_\alpha \wedge g_\beta$. Now $(|f_n \wedge g|)_{\max(\alpha,\beta)} \leqslant g_\beta$. Therefore $\lim \tau((f_n)_{\alpha_n}) \wedge g_\beta) = \tau(f_\alpha \wedge g_\beta)$ by Proposition 3.1.5. Since $\lim \tau((f_n)_{\alpha_n} \wedge g_\beta) < \infty$, by Proposition 3.1.2, $f_\alpha \wedge g_\beta$ is in $\tilde{S}_1$. Hence f_α is measurable.

Lemma 4.2.4. A non negative fuzzy point f_α in F is measurable with respect to τ if $\emptyset_\gamma \wedge f_\alpha$ is in $\tilde{S}_1$ for each $\emptyset_\gamma$ in $\tilde{S}$.

Proof: Let $\{(g_n)_{\beta_n}\}$ be an increasing sequence of fuzzy points in $\tilde{S}$, whose limit g_β is in $\tilde{S}_u$. Then $\{(g_n)_{\beta_n} \wedge f_\alpha\}$ is an increasing sequence in $\tilde{S}$ converging to $g_\beta \wedge f_\alpha$ and $\lim \tau((g_n)_{\beta_n} \wedge f_\alpha) < \infty$. Therefore $g_\beta \wedge f_\alpha \in \tilde{S}_1$ for $g_\beta \in \tilde{S}_u$ and $\tau(g_\beta) < \infty$ by Proposition 3.1.2.

Suppose that $\{(g_n)_{\beta_n}\}$ is a decreasing sequence of fuzzy points in $\tilde{S}_u$, whose limit $g_\beta \in \tilde{S}_{u\ell}$ with $\lim \tau((g_n)_{\beta_n}) < \infty$, and $\lim \tau((g_n)_{\beta_n}) > -\infty$. Then

$\{(g_n)_{\beta_n} \wedge f_\alpha\}$ is a sequence of fuzzy points in $\tilde{s}_1$ and $(|g_n \wedge f|)_{\max(\beta_n,\alpha)} < g_\beta$ for every n. Using proposition 3.1.5, $\tau(g_\beta \wedge f_\alpha) = \lim \tau((g_n)_{\beta_n} \wedge f_\alpha)$, i.e. $g_\beta \wedge f_\alpha \in \tilde{s}_1$ for every $g_\beta \in \tilde{s}_{ul}$. By proposition 3.2.4, if h_γ is any fuzzy point in $\tilde{s}_1$ then $h_\gamma = g_\beta - k_\delta$ where k_δ is a non negative fuzzy null point and $g_\beta \in \tilde{s}_{ul}$. Therefore $k_\delta = g_\beta - h_\gamma$, i.e., $g_\beta \wedge f_\alpha - h_\gamma \wedge f_\alpha = k_\delta \wedge f_\alpha \leqslant k_\delta$. Since k_δ is a fuzzy null point, $g_\beta \wedge f_\alpha - h_\gamma \wedge f_\alpha$ is a fuzzy null point. This shows that $g_\beta \wedge f_\alpha - h_\gamma \wedge f_\alpha$ is in $\tilde{s}_1$. But $g_\beta \wedge f_\alpha$ is in $\tilde{s}_1$ and therefore $h_\gamma \wedge f_\alpha$ must be in $\tilde{s}_1$, i.e., f_α is measurable.

Definition 4.2.5. A fuzzy set μ_α in X is measurable with respect to τ if $\mu_\alpha \wedge g_\beta \in \tilde{s}_1$ for every $g_\beta \in \tilde{s}_1$. A fuzzy set μ_α is τ-integrable if $\mu_\alpha \in \tilde{s}_1$.

Remark 4.2.6. (i) $\mu: X \longrightarrow [0,1]$, $g: X \longrightarrow R$. In the previous definition f_α is measurable if $f_\alpha \wedge g_\beta \in \tilde{s}_1$ for every $g_\beta \in \tilde{s}_1$. Here $f: X \rightarrow R$. In the risp case A is measurable if $\chi_A \wedge g$ is integrable. Therefore here $(\chi_A)_\alpha \wedge g_\beta \in \tilde{s}_1$. This $(\chi_A)_\alpha$ is taken to be μ_α. Thus $\mu_\alpha \wedge g_\beta = (\mu \wedge g)_{\max(\alpha,\beta)}$. In particular take $\alpha=1$ then $\mu \wedge g_\beta = (\mu \wedge g)_1$.

(ii) If a fuzzy set μ_α is measurable then $(1-\mu)_t$ is also measurable for every $t \geqslant \alpha$ and $\mu \geqslant \frac{1}{2}$.

For, we have $1-\mu \leqslant \mu$ and $t \geqslant \alpha$ so that $(1-\mu)_t \leqslant \mu_\alpha$, i.e., $(1-\mu)_t \wedge g_\beta \leqslant \mu_\alpha \wedge g_\beta$.
i.e., $\tau((1-\mu)_t \wedge g_\beta) \leqslant \tau(\mu_\alpha \wedge g_\beta) < \infty$ by proposition 3.1.4.
Therefore $(1-\mu)_t$ is measurable for every $t \geqslant \alpha$ and $\mu \geqslant \frac{1}{2}$.

Lemma 4.2.7. If μ_α is a measurable fuzzy set with $\mu \geqslant \frac{1}{2}$ then $(1-\mu)_t$ is measurable where $t \geqslant \alpha$. If $\{(\mu_n)_{\alpha_n}\}$ is a sequence of measurable fuzzy sets then $\sup\limits_{n \in N} (\mu_n)_{\alpha_n}$ is measurable. If 1 and 0 are measurable, then the class $\tilde{\sigma}$ of measurable fuzzy sets is a fuzzy σ - algebra.

Proof: If μ_α is a measurable fuzzy set then $\mu_\alpha \wedge g_\beta \in \tilde{s}_1$ for every $g_\beta \in \tilde{s}_1$. Now $(1-\mu)_t \leqslant \mu_\alpha$, i.e. $(1-\mu)_t \wedge g_\beta \leqslant \mu_\alpha \wedge g_\beta$.
i.e., $\tau((1-\mu)_t \wedge g_\beta) \leqslant \tau(\mu_\alpha \wedge g_\beta) < \infty$ by proposition 3.1.4.
Thus $(1-\mu)_t$ is measurable for every $t \geqslant \alpha$ and $\mu \geqslant \frac{1}{2}$.

If $\{(\mu_n)_{\alpha_n}\}$ is a sequence of measurable fuzzy sets then $(\mu_n)_{\alpha_n} \wedge g_\beta \in \tilde{s}_1$ for every $g_\beta \in \tilde{s}_1$ and for every $(\mu_n)_{\alpha_n}$ and $\sup\limits_n \tau((\mu_n)_{\alpha_n} \wedge g_\beta) < \infty$.

For $g_\beta \in \tilde{S}_1$, $g_\beta \wedge (\sup(\mu_n)_{\alpha_n}) = \sup_n \ (g_\beta \wedge (\mu_n)_{\alpha_n})$.

Since $\{(\mu_n)_{\alpha_n}\}$ is a sequence of measurable sets, $(\mu_n)_{\alpha_n} \wedge g_\beta \in \tilde{S}_1$. Using proposition 3.1.2, $g_\beta \wedge (\sup\ (\mu_n)_{\alpha_n}) \in \tilde{S}_1$, i.e., $\sup_n \ (\mu_n)_{\alpha_n}$ is measurable.

If 1 and 0 are measurable, then $1 \wedge g_\beta \in \tilde{S}_1$ for every $g_\beta \in \tilde{S}_1$ and $0 \wedge g_\beta \in \tilde{S}_1$ are measurable. Hence by definition 4.1.1, σ will form a fuzzy σ - algebra.

4.3. STONE-LIKE THEOREM

Let $f: X \longrightarrow R$ and $\mu_\delta: R \longrightarrow [0,1]$. Then

$$f^{-1}(\mu_\delta)(x) \ = \delta \text{ if } f(x) = \mu$$
$$= 0 \text{ otherwise}$$

i.e., $$f^{-1}(\mu_\delta)(x) \ = \delta \text{ if } x \in f^{-1}(\mu)$$
$$= 0 \quad \text{otherwise}$$

Let f_α be a fuzzy point in F. Then

$f_\alpha : F \longrightarrow [0,1]$ and $f_\alpha^{-1}(\mu_\delta)(x) = \delta \wedge \alpha$ if $x \in f^{-1}(\mu)$
$= 0$ otherwise

Let $E_{\lambda_\beta} = \{ f_\alpha^{-1}(\mu_\delta) : \mu_\delta > \lambda_\beta, \lambda_\beta \in \tilde{R} \}$. Then E_{λ_β} is a set of fuzzy points in X.

Definition 4.3.1. $f_\alpha^{-1}(\mu_\delta) = (f^{-1}(\mu))_{\alpha \wedge \delta}$

Definition 4.3.2. Let $\mathcal{A}$ be a fuzzy σ-algebra of fuzzy subsets of X, f_α be a measurable fuzzy point in F and let $\tilde{E}$ be a fuzzy subset of X belonging to $\mathcal{A}$ then the fuzzy integral of f_α with respect to ξ is

$$\int_{\tilde{E}} f_\alpha \, d\xi = \sup_{\lambda_\beta \in \tilde{R}} \xi\,(E_{\lambda_\beta} \cap \tilde{E})$$

where $E_{\lambda_\beta} = \left\{ f_\alpha^{-1}(\mu_\delta) \;:\; \mu_\delta > \lambda_\beta,\ \lambda_\beta \in \tilde{R} \right\}$.

Lemma 4.3.3. Let l be measurable and define a set function ξ on the class $\mathcal{A}$ of measurable fuzzy sets by

$$\begin{aligned} \xi(\tilde{E}) &= \tau(\mu_{\tilde{E}}) \quad \text{if } \mu_{\tilde{E}} \in \tilde{s}_1 \\ &= \infty \quad\quad\ \ \text{if } \mu_{\tilde{E}} \notin \tilde{s}_1 . \end{aligned}$$

Then ξ is a fuzzy measure.

Proof: $\xi(\tilde{\emptyset}) = \tau(o) = 0$. Let $\tilde{E}_1$ and $\tilde{E}_2$ be two measurable fuzzy sets such that $\tilde{E}_1 \subset \tilde{E}_2$ so that

$\xi(\tilde{E}_1) = \tau(\mu_{\tilde{E}_1}) \leqslant \tau(\mu_{\tilde{E}_2}) = \xi(\tilde{E}_2)$ for any $\tilde{E}_1, \tilde{E}_2 \in \tilde{\mathcal{F}}$.

Whenever $\{\tilde{E}_n\} \subset \tilde{\mathcal{F}}$, $\tilde{E}_n \subset \tilde{E}_{n+1}$, $n = 1,2,\ldots,$

$$\xi\left(\bigcup_{n=1}^{\infty}\tilde{E}_n\right) = \tau\left(\sup_n \mu_{\tilde{E}_n}\right) = \tau\left(\lim \mu_{\tilde{E}_n}\right) = \lim \tau(\mu_{\tilde{E}_n})$$
$$= \lim \xi(\tilde{E}_n)$$

If $\{\tilde{E}_n\} \subset \tilde{\mathcal{J}}$, $\tilde{E}_n \supset \tilde{E}_{n+1}$, $n=1,2,\ldots,$ and if there exists n_o such that $\xi(\tilde{A}_n) < \infty$, then

$$\xi\left(\bigcap_{n=1}^{\infty}\tilde{E}_n\right) = \tau\left(\inf_n \mu_{\tilde{E}_n}\right) = \tau\left(\lim \mu_{\tilde{E}_n}\right) = \lim \tau(\mu_{\tilde{E}_n})$$
$$= \lim \xi(\tilde{E}_n)$$

Hence ξ is a fuzzy measure.

Lemma 4.3.4. If f_α is a non negative τ-integrable fuzzy point, then for each fuzzy real number λ_β the set $E_{\lambda_\beta} = \left\{ f_\alpha^{-1}(\mu_\delta) : \mu_\delta > \lambda_\beta \right\}$ is measurable.

Proof: We have to show that E_{λ_β} is measurable. Define

$$g_u = (\lambda^{-1}f)_{\beta\wedge\alpha} - (\lambda^{-1}f)_{\beta\wedge\alpha} \wedge 1_\gamma \quad \text{if } \lambda_\beta \neq 0$$

$$= f_\alpha \text{ if } \lambda_\beta = 0.$$

Since g_u is the difference of two fuzzy points in $\tilde{s}_1$, g_u is in $\tilde{s}_1$ and g_u is positive for $f_\alpha^{-1}(\mu_\delta) \in E_{\lambda_\beta}$ and $g_u = 0$ for $f_\alpha^{-1}(\mu_\delta) \notin E_{\lambda_\beta}$. Define $(\emptyset_n)_{u_n} = 1_\gamma \wedge (n\, g_u)$. Then $(\emptyset_n)_{u_n} \in \tilde{s}_1$ and $(\emptyset_n)_{u_n} \uparrow \chi_{E_{\lambda_\beta}}$. Applying lemma 4.2.3 $\chi_{E_{\lambda_\beta}}$ is measurable and hence E_{λ_β} is measurable.

Following is the fuzzy version of the Stone's theorem.

Theorem 4.3.5. Let $\tilde{s}$ be a vector lattice of fuzzy points f_α in a fuzzy vector lattice s with the property that if $f_\alpha \in \tilde{s}$ then $1_\gamma \wedge f_\alpha \in \tilde{s}$ and let τ be a fuzzy Daniell integral on $\tilde{s}$. Then there is a fuzzy σ-algebra $\mathcal{A}$ of measurable fuzzy sets and a fuzzy measure ξ on $\mathcal{A}$ such that each f_α is τ-integrable if and only if it is τ-integrable with respect to ξ. Also $\tau(f_\alpha) = \int_{\tilde{E}} f_\alpha \, d\xi$.

Proof: Let $\mathcal{A}$ be the class of fuzzy subsets of X which are measurable with respect to τ. By lemma 4.2.4 singleton 1 is measurable. Using lemma 4.2.7, $\mathcal{A}$ is a fuzzy σ-algebra. Here we are considering only non negative τ-integrable fuzzy points which are measurable with respect to $\mathcal{A}$. By lemma 4.3.4, each non negative τ-integrable fuzzy point is measurable with respect to $\mathcal{A}$. Since each τ-integrable fuzzy point is the difference of two non negative τ-integrable fuzzy points every τ-integrable fuzzy point must be measurable with respect to $\mathcal{A}$.

Let ξ be a fuzzy measure as defined in lemma 4.3.3 and let f_α be a non negative fuzzy point which is integrable with respect to τ. For each non negative $\lambda_\beta \in \tilde{R}$, let $E_{\lambda_\beta} = \left\{ f_\alpha^{-1}(\mu_\delta) : \mu_\delta > \lambda_\beta \right\}$. Now E_{λ_β} is measurable for each $\lambda_\beta \in \tilde{R}$.

Define $(\phi_n)_{\alpha_n} = \sup_{\lambda_{n_{\beta_n}} \in \tilde{R}} \mu_{E_{\lambda_{n_{\beta_n}}}} \cap \tilde{E}$, where $\tilde{E}$ is any fuzzy subset of X. Then $(\phi_n)_{\alpha_n} \in \tilde{S}_1$ and $(\phi_n)_{\alpha_n} \uparrow f_\alpha$. Therefore $\tau(f_\alpha) = \lim \tau((\phi_n)_{\alpha_n})$. We have

$$\tau((\phi_n)_{\alpha_n} = \tau(\sup_{\lambda_{n_{\beta_n}} \in \tilde{R}} \mu_{E_{\lambda_{n_{\beta_n}}}} \cap \tilde{E})$$

$$= \sup_{\lambda_{n_{\beta_n}} \in \tilde{R}} \tau(\mu_{E_{\lambda_{n_{\beta_n}}}} \cap \tilde{E})$$

$$= \sup_{\lambda_{n_{\beta_n}} \in \tilde{R}} \xi(E_{\lambda_{n_{\beta_n}}} \cap \tilde{E})$$

$$= \int_{\tilde{E}} (\phi_n)_{\alpha_n} \, d\xi$$

But $\int f_\alpha \, d\xi = \lim \int (\phi_n)_{\alpha_n} \, d\xi$. By monotone convergence theorem $\tau(f_\alpha) = \lim \tau((\phi_n)_{\alpha_n}) = \lim \int (\phi_n)_{\alpha_n} d\xi = \int f_\alpha \, d\xi$. This shows that f_α is integrable with respect to ξ. Suppose f_α be any τ - integrable fuzzy point. Then it can be written as the difference of two non negative

τ -integrable fuzzy points and therefore every such f_α must also be integrable with respect to ξ and hence $\tau(f_\alpha) = \int f_\alpha d\xi$.

Let f_α be a non negative fuzzy point in F and let it be integrable with respect to ξ . Let $E_{\lambda_\beta} = \{ f_\alpha^{-1}(\mu_\delta) : \mu_\delta > \lambda_\beta \}$ and $(\emptyset_n)_{\alpha_n} = \sup_{\lambda_{n_{\beta_n}} \in \tilde{R}} \mu_{E_{\lambda_{n_{\beta_n}}}} \cap \tilde{E}$, where $\tilde{E}$ is a fuzzy subset of X. Since f_α is integrable with respect to ξ , $\int_{\tilde{E}} f_\alpha d\xi < \infty$. Fuzzy measure of each $E_{\lambda_{n_{\beta_n}}}$ is finite and therefore $\mu_{E_{\lambda_{n_{\beta_n}}}}$ and $(\emptyset_n)_{\alpha_n} \in \tilde{s}_1$.

Now $(\emptyset_n)_{\alpha_n} \uparrow f_\alpha$ and $\lim \tau((\emptyset_n)_{\alpha_n}) = \tau(\lim(\emptyset_n)_{\alpha_n}) =$ $\tau(f_\alpha) = \int f_\alpha d\xi < \infty$. Therefore by monotone convergence theorem $f_\alpha \in \tilde{s}_1$. Thus each f_α which is integrable with respect to ξ is also integrable with respect to τ .

The following proposition establishes the uniqueness of the fuzzy measure ξ .

Proposition 4.3.6. Let $\tilde{s}$ be a vector lattice of fuzzy points and suppose that $1_\gamma \in \tilde{s}$. Let $\mathcal{B}$ be the smallest fuzzy σ-algebra of fuzzy subsets of X and let each

fuzzy point in $\tilde{s}$ be measurable with respect to $\mathcal{B}$.
Then for each fuzzy Daniell integral τ there is a unique fuzzy measure ξ on $\mathcal{B}$ such that for every $f_\alpha \in \tilde{s}$, $\tau(f_\alpha) = \int f_\alpha \, d\xi$.

Proof: By theorem 4.3.5 the fuzzy measure ξ exists and we have to show that ξ is unique on $\mathcal{B}$. Let $\mathcal{A}$ be the fuzzy σ-algebra of measurable fuzzy sets as in lemma 4.2.7. By lemma 4.2.4, each f_α in $\tilde{s}$ is measurable with respect to $\mathcal{A}$ and therefore $\mathcal{B} \subset \mathcal{A}$. $\mu_B \in \tilde{s}_1$ for each fuzzy set $B \in \mathcal{A}$ and hence for each $B \in \mathcal{B}$. To prove uniqueness we have to show that $\xi(B) = \tau(\mu_B)$ for each $B \in \mathcal{B}$.

Let $\tilde{t}$ be the set of fuzzy points in F which are measurable with respect to $\mathcal{B}$ and are integrable with respect to ξ and let us take $\tau'(f_\alpha) = \int f_\alpha d\xi$ for $f_\alpha \in \tilde{t}$. Then by proposition 3.2.5, $\tau'(f_\alpha) = \tau(f_\alpha)$ for $f_\alpha \leqslant \tilde{s}_1 \wedge \tilde{t}$. But if $B \in \mathcal{B}$ then $\mu_B \leqslant \tilde{s}_1 \wedge \tilde{t}$ and so $\xi(B) = \tau'(\mu_B) = \tau(\mu_B)$. Hence τ determines ξ uniquely on $\mathcal{B}$.

Chapter V

FUZZY VECTOR VALUED INTEGRATION

5.0 INTRODUCTION

Here we are going to show that the theory of fuzzy Daniell integrals is true for (i) the fuzzy integral of a fuzzy point of the set of all vector valued functions with respect to a scalar valued fuzzy measure, and (ii) the fuzzy integral of a scalar valued fuzzy point with respect a vector valued fuzzy measure. It can be seen that almost all the results of the preceding chapters are true and in some cases with slight modifications. Therefore, we are giving only those definitions and results which differ from the earlier situation.

5.1. FUZZY INTEGRAL OF A FUZZY POINT OF THE SET OF ALL VECTOR VALUED FUNCTIONS WITH RESPECT A SCALAR VALUED MEASURE

5.1.0. Preliminaries. Let X be any set and L be an extended vector lattice of functions f from X to R^n. Then L is an extended vector lattice of vector valued functions of X. Let s be a fuzzy set in L such that

$s(o) = 1$ and f_α be a fuzzy point of s which means $f_\alpha \leqslant s$ i.e, $\alpha \leqslant s(f)$. Here $\tilde{s}$ represents the set of all fuzzy points of s and $\tilde{R}$ be the set of fuzzy points in R.

Definition 5.1.1. A fuzzy point $f_\alpha \in \tilde{s}$ is non negative if $f \geqslant 0$, i.e., $f^i \geqslant 0$ for every i, i=1,2,...,n, where $f = (f^1, f^2, \ldots, f^n)$.

Definition 5.1.2. For every $f_\alpha, g_\beta \in \tilde{s}$, $f_\alpha \leqslant g_\beta$ means $f \leqslant g$ and $\alpha \geqslant \beta$, where $f, g \in L$ and $\alpha, \beta \in (0,1]$.
$f \leqslant g$ means $f(x) \leqslant g(x)$.
i.e., $(f^1(x), f^2(x), \ldots, f^n(x)) \leqslant (g^1(x), g^2(x), \ldots, g^n(x))$
i.e., $f^i(x) \leqslant g^i(x)$, for every i, i=1,2,...,n.

Theorem 5.1.3. For every $f_\alpha, g_\beta \in \tilde{s}$,

(i) $f_\alpha \vee g_\beta = (f \vee g)_{\min(\alpha,\beta)}$

(ii) $f_\alpha \wedge g_\beta = (f \wedge g)_{\max(\alpha,\beta)}$

Here, $f_\alpha \vee g_\beta = (f^1, f^2, \ldots, f^n)_\alpha \vee (g^1, \ldots, g^n)_\beta$

$$= (f^1 \vee g^1, f^2 \vee g^2, \ldots, f^n \vee g^n)_{\min(\alpha,\beta)}$$

$$f_\alpha \wedge g_\beta = (f^1 \wedge g^1, f^2 \wedge g^2, \ldots, f^n \wedge g^n)_{\max(\alpha,\beta)}$$

Note 5.1.4. (i) For every $f_\alpha, g_\beta \in \tilde{s}$ and $a,b \in R$ we have

$$af_\alpha + bg_\beta = (af+bg)_{\min(\alpha,\beta)}$$

(ii) $af_\alpha = (af)_\alpha$ for every $a \in R$ and $f_\alpha \in \tilde{s}$

(iii) $(|f|)_\alpha = (f^+)_\alpha + (f^-)_\alpha$

We have $f(x) = (f^1(x), f^2(x), \ldots, f^n(x))$

$$|f|(x) = (|f^1|(x), |f^2|(x), \ldots, |f^n|(x))$$

$$= ((f^1)^+(x) + (f^1)^-(x), \ldots, (f^n)^+(x) + (f^n)^-(x))$$

$$= ((f^1)^+(x), (f^2)^+(x), \ldots (f^n)^+(x)) + ((f^1)^-(x), \ldots, (f^n)^-(x))$$

Therefore,

$$(|f|)_\alpha = (f^+)_\alpha + (f^-)_\alpha \text{ where}$$

$$(f^+)_\alpha = ((f^1)^+, (f^2)^+, \ldots (f^n)^+)_\alpha$$

and $$(f^-)_\alpha = ((f^1)^-, (f^2)^-, \ldots (f^n)^-)_\alpha$$

Definition 5.1.5. A sequence $\{(\emptyset_m)_{\alpha_m}\} \in \tilde{s}$ decreases means $(\emptyset_{m+1})_{\alpha_{m+1}} \leqslant (\emptyset_m)_{\alpha_m}$ for every m, i.e., $\emptyset_{m+1} \leqslant \emptyset_m$ and $\alpha_{m+1} \geqslant \alpha_m$ for every m.

i.e., $\emptyset^i_{m+1} \leq \emptyset^i_m$ for every i, i=1,2,...,m and

$\alpha_{m+1} \geq \alpha_m$. Similarly for increasing case.

Definition 5.1.6. A sequence $\{(\emptyset_m)_{\alpha_m}\} \in \tilde{s}$ increases to $\emptyset_\alpha$ ($\alpha > 0$) means $\emptyset_m \uparrow \emptyset$ and $\alpha_m \downarrow \alpha$.
i.e., $(\emptyset^1_m, \emptyset^2_m, \ldots, \emptyset^n_m) \uparrow (\emptyset^1, \emptyset^2, \ldots, \emptyset^n)$ and $\alpha_m \downarrow \alpha$.

Definition 5.1.7. $\lim ((\emptyset_m)_{\alpha_m}) = \emptyset_\alpha$
if $\lim (\emptyset^1_m, \emptyset^2_m, \ldots, \emptyset^n_m) = (\emptyset^1, \emptyset^2, \ldots, \emptyset^n)$ and $\lim \alpha_m = \alpha$.

Remark 5.1.8. $\{(\emptyset_m)_{\alpha_m}\} \in \tilde{s}$ decreases to zero if $\emptyset_m(x) \downarrow 0$ for every $x \in X$ and $\alpha_m \uparrow \alpha > 0$. $\emptyset_m(x) \downarrow 0$ means $(\emptyset^1_m(x), \emptyset^2_m(x), \ldots, \emptyset^n_m(x)) \downarrow (0,0,\ldots,0)$ i.e., $\emptyset^i_m(x) \downarrow 0$ for every i=1,2,...,n.

Let $\tau : \tilde{s} \rightarrow \tilde{R}$. Then $\tau(f_\alpha) = \lambda_\beta$, where $\lambda \in R$ and $\beta \in (0,1]$. For this τ, definition of linearity, positive linear functional, fuzzy Daniell integral and almost all propositions hold.

Theorem 5.1.9. f_α is measurable with respect to τ if and only if each of its coordinate points are measurable.

Proof: Let $f_\alpha = (f^1_{\alpha_1}, f^2_{\alpha_2}, \ldots, f^n_{\alpha_n})$

$$= (f^1, f^2, \ldots, f^n)_\alpha, \text{ where } \alpha = \min(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

and f_α be a fuzzy point in F. If f_α is measurable with respect to τ then there exists $g_\beta \in \tilde{s}_1$ such that

$$f_\alpha \wedge g_\beta = (f^1, f^2, \ldots, f^n)_\alpha \wedge (g^1, g^2, \ldots, g^n)_\beta$$

$$= (f^1 \wedge g^1, f^2 \wedge g^2, \ldots, f^n \wedge g^n)_{\alpha \vee \beta}$$

$$= ((f^1 \wedge g^1)_{\alpha_1 \vee \beta_1}, (f^2 \wedge g^2)_{\alpha_2 \vee \beta_2}, \ldots, (f^n \wedge g^n)_{\alpha_n \vee \beta_n}),$$

where $g_\beta = ((g^1)_{\beta_1}, (g^2)_{\beta_2}, \ldots, (g^n)_{\beta_n})$

$f^i : X \to R^1$, $g^i : X \longrightarrow R^1$ so that $f^i \wedge g^i$ is the i^{th} coordinate of $(f^1, f^2, \ldots, f^n) \wedge (g^1, g^2, \ldots, g^n), i = 1, 2, \ldots, n$. Therefore if f_α is measurable with respect to τ, $(f^i)_{\alpha_i}$ is measurable with respect to τ for each i. Also $(f^1)_{\alpha_1} = (f^1, o, \ldots, o)_{\alpha_1}, (f^2)_{\alpha_2} = (o, f^2, o, \ldots, o)_{\alpha_2}$ etc.

Conversely, if $(f^i)_{\alpha_i}$ is measurable with respect to τ then there exists $(g^i)_{\beta_i}$ such that

$(f^i)_{\alpha_i} \wedge (g^i)_{\beta_i} = (f^i \wedge g^i)_{\alpha_i \vee \beta_i} \in \tilde{s}_1.$

Since $(g^i)_{\beta_i} = (o,o,\ldots,g^i,\ldots,o)_{\beta_i}$ and

$(f^i)_{\alpha_i} = (o,o,\ldots,f^i,\ldots,o)_{\alpha_i}$,

$(f^1_{\alpha_1} \wedge g^1_{\beta_1},\ f^2_{\alpha_2} \wedge g^2_{\beta_2},\ \ldots\ f^n_{\alpha_n} \wedge g^n{}_{\beta_n}) \in \tilde{s}_1$

i.e., $f_\alpha \wedge g_\beta \in \tilde{s}_1$ for every $g_\beta \in \tilde{s}_1$.

Hence if each coordinate of f_α is measurable with respect to τ then f_α is measurable.

Theorem 5.1.10. f_α is integrable with respect to τ if and only if each of its coordinate points are integrable.

Proof: Similar to the above.

Lemma 5.1.11. If f_α is a non-negative τ-integrable fuzzy point of the set of all vector valued functions then for each fuzzy point $\lambda_\beta \in \tilde{R}^n$, the set $E_{\lambda_\beta} = \{ f_\alpha^{-1}(\nu_\delta) : \nu_\delta > \lambda_\beta \}$ is measurable.

Proof: Let g_u be a fuzzy point in F which is of the form $g_u = (g^1, g^2, \ldots, g^n)_u = ((g^1)_{u_1}, (g^2)_{u_2}, \ldots, (g^n)_{u_n})$, where

$u = \min(u_1, u_2, \ldots, u_n)$. f_α is a fuzzy point in $\tilde{s}_1$ and it is of the form $f_\alpha = ((f^1)_{\alpha_1}, \ldots, (f^n)_{\alpha_n})$ where $\alpha = \min(\alpha_1, \ldots, \alpha_n)$. $\lambda_\beta \in R^n$ so that $\lambda_\beta = ((\lambda^1)_{\beta_1}, \ldots (\lambda^n)_{\beta_n})$, where $\beta = \min(\beta_1, \beta_2, \ldots, \beta_n)$.

Now define

$$g^i_{u_i} = ((\lambda^i)^{-1} f^i)_{\beta_i \wedge \alpha_i} - ((\lambda^i)^{-1} f^i)_{\beta_i \wedge \alpha_i} \wedge 1_\gamma, \quad \text{if } \lambda^i_{\beta_i} \neq 0.$$

$$= (f^i)_{\alpha_i} \text{ if } (\lambda^i)_{\beta_i} = 0, \; i=1,2,\ldots,n$$

We see that $g^i_{u_i}$ is the difference of two fuzzy points in $\tilde{s}_1$ so that g_u is in $\tilde{s}_1$. Also g_u is positive for $f_\alpha^{-1}(\nu_\delta) \in E_{\lambda_\beta}$ and $g_u = 0$ for $f_\alpha^{-1}(\nu_\delta) \notin E_{\lambda_\beta}$. Suppose that $(\emptyset_m)_{u_m} = 1_\gamma \wedge (m\, g_u)$. Then $(\emptyset_m)_{u_m} \in \tilde{s}_1$.

Also $(\emptyset^i_m)_{u_m} \uparrow 1$ if $f_\alpha^{-1}(\nu_\delta) \in E_{\lambda_\beta}$ and $(\emptyset^i_m)_{u_m} \downarrow 0$ if $f_\alpha^{-1}(\nu_\delta) \notin E_{\lambda_\beta}$. Therefore E_{λ_β} is measurable.

Theorem 5.1.12 (Stone-Like). Let $\tilde{s}$ be a vector lattice of fuzzy points f_α in a fuzzy vector space s satisfying

the property that if $f_\alpha \in \tilde{s}$ then $1_\gamma \wedge f_\alpha \in \tilde{s}$ and let τ be a fuzzy Daniell integral on $\tilde{s}$. Then there is a fuzzy σ-algebra $\mathcal{A}$ of measurable fuzzy sets and a fuzzy measure ξ on $\mathcal{A}$ such that f_α is τ-integrable if and only if it is τ-integrable with respect to ξ.
Also $\tau(f_\alpha) = \int_{\tilde{E}} f_\alpha \, d\xi$.

Proof: By replacing $\tilde{R}$ by $\tilde{R}^n$ in the proof described in the previous chapter the result is established easily.

Remark 5.1.13. Uniqueness of the fuzzy measure ξ follows in this situation also.

5.2. FUZZY INTEGRAL OF A SCALAR VALUED FUZZY POINT WITH RESPECT TO A VECTOR VALUED FUZZY MEASURE

Preliminaries 5.2.1. Let X be any set and L be a vector lattice of extended real valued functions f on X. Let s be a fuzzy set in L such that $s(o)=1$ and f_α be a fuzzy point of s which means $\alpha \leqslant s(f)$. Suppose $\tilde{s}$ represent the set of fuzzy points of s and $\tilde{R}^n$ be the set of fuzzy points in R^n. Let $\tau : \tilde{s} \longrightarrow \tilde{R}^n$

Replacing R by R^n and $\tilde{R}$ by $\tilde{R}^n$ almost all the results of the preceding chapters will follow. For example we give below the following definitions and results.

Definition 5.2.2. A map τ from $\tilde{s}$ to $\tilde{R}^n$ is called linear map if $\tau(af_\alpha+bg_\beta) = a\tau(f_\alpha)+b\tau(g_\beta)$ $\forall a,b \in R$, and $f_\alpha, g_\beta \in \tilde{s}$ and for each $f_\alpha \in \tilde{s}$, $\tau(f_\alpha) = r_\alpha$ for some $r \in R^n$.

Definition 5.2.3. A linear map $\tau:\tilde{s} \to \tilde{R}^n$ is called fuzzy Daniell functional or fuzzy Daniell integral if for every sequence $\{(\emptyset_n)_{\alpha_n}\} \in \tilde{s}$ and $(\emptyset_n)_{\alpha_n} \downarrow 0$, we have $\lim \tau((\emptyset_n)_{\alpha_n}) = 0$.

Definition 5.2.4. A mapping $\xi: \tilde{\mathcal{F}} \to [o,\infty]^n$ is said to be a fuzzy measure on $\tilde{\mathcal{F}}$ if and only if (i) $\xi(\tilde{\emptyset}) = 0$, (ii) for any $\tilde{A},\tilde{B} \in \tilde{\mathcal{F}}$, $\xi(\tilde{A}) \leqslant \xi(\tilde{B})$ if $A \subset B$,

(iii) $\xi(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \xi(\tilde{A}_n)$ whenever $\{\tilde{A}_n\} \subset \tilde{\mathcal{F}}$, $\tilde{A}_n \subset \tilde{A}_{n+1}$, $n=1,2,\ldots$ (continuity from below), (iv) if $\{\tilde{A}_n\} \subset \tilde{\mathcal{F}}$, $\tilde{A}_n \supset \tilde{A}_{n+1}$, $n=1,2,\ldots$, and if $\exists$ n_o such that $\xi(\tilde{A}_{n_o}) < \infty$, then

$$\xi(\bigcap_{n=1}^{\infty} \tilde{A}_n) = \lim_{n\to\infty} \xi(\tilde{A}_n).$$

We conclude this chapter by stating the Stone-like theorem when ξ is a vector valued fuzzy measure.

Theorem 5.2.5. Let $\tilde{s}$ be a vector lattice of fuzzy points f_α in a fuzzy vector space s satisfying the property that if $f_\alpha \in \tilde{s}$ then $1_\gamma \wedge f_\alpha \in \tilde{s}$ and let τ be a fuzzy Daniell integral on $\tilde{s}$. Then there is a fuzzy σ-algebra $\mathcal{A}$ of measurable fuzzy sets and a vector valued fuzzy measure ξ on $\mathcal{A}$ such that f_α is τ-integrable if and only if it is τ-integrable with respect to ξ. Also $\tau(f_\alpha) = \int_{\widetilde{E}} f_\alpha \, d\xi$.

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