SOME PROBLEMS IN FLUID MECHANICS

STUDIES ON SOME CONSERVATION LAWS OF NON - BAROTROPIC FLOWS

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CERTIFICATE

This is to certify that the work reported in this thesis is based on the bonafide work done by Sri. George Mathew, under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, Cochin 682 022, and has not been included in any other thesis submitted previously for the award of any degree.

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Chapter 1

INTRODUCTION

1. Variational Principles in Hydrodynamics

The principle of least action put forward by Maupertius in the 18th century was of fundamental and far reaching importance in that it opened up a new field of research leading to invaluable results in many branches of physics. This result was developed into a beautiful theory of dynamics by the ingenious works of Euler, Lagrange, Hamilton and Jacobi. The credit for giving the first exact formulation of the principle of least action goes to Hamilton. The theorem known as Hamilton's variational principle can be stated as follows: "During the motion in a conservative force field the action is stationary". i.e., $\delta \int L \, dt = 0$, where L is the Lagrangian of the dynamical system and δ denotes the variation. In other words Euler-Lagrange conditions for the action integral '$\int L \, dt$' of a dynamical system to be stationary are precisely the equations of motion of the system. In later years this principle found applications in the studies of optics, dynamics of particles and rigid bodies, elasticity, electromagnetism, Einstein's laws of gravitation and many other branches of physics.

It was the great success of variational principles in classical mechanics that stimulated the efforts to formulate the laws of hydrodynamics in a similar way. The two types of description in fluid dynamics-Lagrangian in which field variables are expressed in terms of initial coordinates and time and Eulerian in which they are expressed as functions of current coordinates and time-lead to two forms of variational formulations. The Lagrangian variational formulation is comparatively easy and analogous to that of a dynamical system of discrete particles. When Eulerian description is introduced, this close similarity is lost and it raises mathematical problems.

Attempts for variational formulations of hydrodynamics can be traced back to Bateman (1929), Lichtenstein (1929) and Lamb (1932). Assuming the Clebsch's representation (Clebsch, 1859) of the velocity field apriori, Bateman derived the equations of motion for barotropic flows of an ideal fluid. Eckart (1938) and Taub (1949) tried to extend the variational principles to adiabatic compressible flows. The first successful variational formulation for compressible fluid flows was due to Herivel (1955).

Using the field variables velocity, density and entropy expressed in space-time coordinates, he presented both Lagrangian and Eulerian variational formulations for ideal fluid flows. However, his Eulerian variational principle was incomplete in the sense that for isentropic flows this principle led to irrotational motion of the fluid only. Introducing a new set of constraints- constancy of particle identity- Lin (1963) extended this principle to rotational flows also. The modified version (Herivel-Lin variational formulation) appeared first in an article by Serrin (1959). Serrin showed that every flow of an inviscid fluid corresponds to an extremal of this principle if the Clebsch's potentials are suitably defined. Eckart (1960) used the energy-momentum tensor to derive the equations of motion and some conservation laws in Lagrangian description.

In Herivel-Lin variational formulation, the constraints are taken by means of Lagrangian multipliers (Monge potentials) which lead to a Clebsch's representation of the velocity field involving 8 potentials. It is difficult to assign physical meaning to these potentials.

There have been attempts during the last three decades to find new variational techniques in Eulerian description avoiding the difficulties due to the redundancies and indeterminancies of these Clebsch's potentials.

Hamilton's principle does not exist for the flows of viscous fluids except in some very restricted cases. Discussions regarding the non-existence of variational principle in viscous fluid flows can be found in Finlayson (1972a, 1972b) and Mobbs (1982).

We sketch briefly some more important developments in the studies of variational formulations for ideal fluid flows. Identifying a suitable Lagrangian Penfield (1966a) has derived equations of motion for both relativistic and non-relativistic flows of compressible inviscid fluids using Lin's constraints. Seliger and Whitham (1968) have studied Eulerian and Lagrangian variational principles in continuum mechanics and shown that the number of Clebsch's potentials in Eulerian variational formulation of ideal fluid flows can be reduced to 4 from 8. But Bretherton (1970) has pointed out that though Seliger-Whitham representation is locally valid, in isentropic case the flows determined by such a representation do not include those with non-zero helicity. He has also shown that the relation between the Eulerian and Lagrangian variations of the field variables can be used to derive the equations of motion in Eulerian form from fundamental Lagrangian without using any other constraints. Guderley and Bhutani(1973) have suggested a method to derive the variational principle for three dimensional steady flows of compressible fluids from the Herivel-Lin variational formulation for unsteady flows.

Discussing different variational formulations of Herivel-Lin type, Mobbs (1982) has shown that the most general formulation is that due to Serrin (1959) in which the Clebsch's potentials are identified with initial coordinates and initial velocities. He has also attempted to extend the variational principles to thermally conducting viscous fluids using local potentials. Capriz (1984) has shown that Lin's constraint can be replaced by another one- Euler's expansion formula- in Eulerian formulation of ideal fluid flows. Katz and Lynden-Bell (1982) have suggested a variational principle using a Lagrangian incorporating local conservation of mass, entropy, circulation and potential vorticity. Moreau (1981, 1982, 1985) has introduced a new variational technique- Method of horizontal variations- to derive Euler's equations of motion of inviscid non-barotropic fluids. Some other developments can be found in the works of Benjamin (1984) and Becker (1987).

Arnold (1965a, 1965b), Grinfeld (1981, 1982, 1984) and Lynden-Bell and Katz (1981) have used variational techniques in the study of stability of stationary flows.

Variational formulations suitable to the equations of water waves have been first introduced by Luke (1967). Some other contributions in this field are due to Whitham (1967, 1974), Miles (1977) and Milder (1977). Variational principles applied to other branches of physics are discussed by Lundgren (1963), Penfield (1966b), Mittag, Stephen et al (1968), Lanczos (1970) and Buchdahl (1987).

2. Noether theorems and conservation laws

The study of invariance properties of the action integrals in the calculus of variations was initiated in the early part of this century by Emmy Noether (1918), influenced by the works of Klein (1918) and Lie (1912) on the transformation properties of the differential equations under continuous groups of transformations. Noether proved two fundamental results now known as Noether theorems. Since then a number of papers have appeared in the literature either modifying these theorems or applying the theorems to particular dynamical systems by relating familiar conservation laws to transformation groups. (Trautman (1967), Logan and Blakeslee (1975), Blakeslee and Logan (1976, 1977), Logan (1977), Cantrijn (1982), Benjamin and Olver (1982), Logan and Bdzil (1984) and Olver (1986a, 1986b)).

In the case of hydrodynamics,only few attempts have been made in this direction. Drobot and Rybarski (1959) have introduced a variational principle for barotropic flowsand adapted Noether theorems suitably. Moreau (1977) has obtained the conservation of helicity as a consequence of Noether theorem. Bretherton (1970) and Gouin (1976) have shown that Kelvin's circulation theorem is related to the invariance of the action under certain transformation groups.

3. Conservation laws related to vorticity

Since almost all fluid flows are rotational, the study of rotational motion plays an important role in fluid dynamics. The foundations of the kinematics of vorticity have been developed by the pioneers in fluid dynamics like Euler, Cauchy, Lord Kelvin, Helmholtz and Beltrami. To Truesdell goes the credit for the creation and unification of the discipline of vorticity transport.

The barotropic flows of an inviscid fluid are characterized by the familiar conservation laws: Kelvin's circulation theorem, Helmholtz theorems, conservation laws of helicity and potential vorticity. Mobbs (1981) has generalized these conservation laws to non-barotropic case, replacing vorticity by generalized vorticity and velocity by another suitable vector in some of their occurrences in the relevant equations. Taub (1959) has proved the circulation theorem for relativistic hydrodynamics. Hollmann (1964) has tried to deduce conservation laws from known conserved quantities. Marris and Passman (1968) have studied generalized circulation preserving flows and obtained generalizations of some conservation laws. Thyagaraja (1975) has exploited the concept of Helmholtz fields and proved that some conservation laws in barotropic flows related to vorticity can be generalized using this concept. Mc Donald and Witting (1984) have shown that the local conservation law for a surface velocity variable leads to Kelvin's circulation theorem when the surface is closed and the force field is conservative. Katz (1984) has proved the conservation law of potential vorticity for inviscid flows in general relativity. Moffat (1981) has pointed out the connection between Hopf-invariant in topology and helicity in fluid dynamics.

4. Scope of the thesis

In this thesis we consider a new variational formulation for non-barotropic flows and study some transformation groups leading to conservation laws. The conservation laws involving vorticity both in barotropic and non-barotropic flows are brought under a general framework by using the concept of Helmholtz fields.

It has been shown by Bretherton (1970) that Kelvin's circulation theorem is a consequence of the indistinguishability of fluid elements with equal density, velocity and entropy in Eulerian description. In chapter 2 we give a proof of the Generalized Circulation theorem based on similar arguments.

In chapter 3 we present a variational formulation suitable for non-barotropic flows of an ideal fluid using quadri dimensional field variables. Following Drobot and Rybarski (1959), we restrict the variations of these field variables by explicitly given conditions and term these as hydromechanical variations. The conditions for the extremum of the action under hydromechanical variations of the field variables lead to Euler's equations of motion. This method is an alternative to other Eulerian variational formulations such as those proposed by Seliger and Whitham (1968) and Bretherton(1970).

Chapter 4 is devoted to a discussion of Noether theorems and conservation laws in connection with the hydromechanical variations. We adapt Noether theorems to our variational principle. The conservation laws of energy, impulse and angular momenta are shown to follow from an application of Noether's first theorem. As an application of the second theorem we consider a particular group of transformations of independent variables defined by the condition that hydromechanical variations of the field variables are vanishing identically. We characterize this group and show that generalized Helmholtz theorems are related to this group.

In chapter 5 we generalize some conservation laws of barotropic (non-barotropic) flows such as conservation laws of helicity, potential vorticity etc. (their analogues in non-barotropic flows) using the concept of Helmholtz fields. This study shows that many of the conservation laws involving vorticity both in barotropic and non-barotropic flows follow from the properties of Helmholtz fields.

In chapter 6 we give a general discussion of the results in this thesis and point out the direction for further research work.

Chapter 2

GENERALIZED CIRCULATION THEOREM AND VARIATIONAL PRINCIPLE

1. Eulerian variational principle

It has been shown by Herivel (1955) and Lin (1963) that the equations of motion for non-barotropic flows can be derived from the variational principle:

$$\delta \int_{t_1}^{t_2} dt \int_V \Big\{ \rho \left[\tfrac{1}{2} |\bar{v}|^2 - E(\rho,S) - U(x) \right] - \rho\alpha \left[\frac{\partial}{\partial t} + \nabla\cdot(\rho\,\bar{v}) \right] - \rho\beta \frac{DS}{Dt} + \rho \sum_{i=1}^{3} \gamma_i \frac{D\lambda_i}{Dt} \Big\} dV = 0 \ , \qquad (2.1)$$

where $\bar{v}$ the velocity, ρ the density, S the specific entropy, E the internal energy and α, β and γ_i the Lagrangian multipliers are considered as functions of current Cartesian coordinates $\bar{x}$ and time t. Here $\frac{D}{Dt}$ denotes the material differentiation and δ denotes the variation.

The results presented in this chapter have been published. (George Mathew and M.J. Vedan (1988)).

The constraints $\frac{D\lambda_i}{Dt} = 0$, $i = 1,2,3$ which correspond to particle identity are due to Lin (1963). In (2.1) V is an arbitrary volume fixed in space and $[t_1, t_2]$ is any given interval of time.

The allowed variations in $\bar{v}, \rho$, S, α, β, γ_i and λ_i, $i = 1,2,3$ are independent, continuously differentiable and vanish for $|x|$ or $|t|$ sufficiently large. The Lagrangian multipliers α and β ensure the conservation of mass and entropy respectively,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0 , \tag{2.2}$$

$$\frac{DS}{Dt} = 0 \ . \tag{2.3}$$

From the Euler-Lagrange equations corresponding to the variation of $\bar{v}$, we get

$$\bar{v} = \nabla \alpha + \beta \nabla S + \sum_{i=1}^{3} \gamma_i \nabla \lambda_i \tag{2.4}$$

so that the Clebsch's representation of $\bar{v}$ involves 8 potentials.

2. Generalized circulation theorem

The famous circulation theorem due to Lord Kelvin can be stated as follows:

'The circulation around a closed curve C consisting of material particles of the fluid lying on a surface of constant entropy is conserved during the motion'.

i.e., $$\frac{D}{Dt} \oint_C \bar{v} . d\bar{l} = 0 . \qquad (2.5)$$

Eckart (1960) has introduced a new quantity called thermodynamic circulation,

$$- \oint (\eta \nabla S) . d\bar{l} \qquad (2.6)$$

where η is the thermasy (Schutz and Sorkin,1977) defined by the relation

$$\left.\begin{aligned} \frac{D\eta}{Dt} &= T, \text{ the temperature} \\ \eta(0) &= 0 \end{aligned}\right\} \qquad (2.7)$$

and proved that in non-barotropic flows, the total circulation- the sum of the kinematic and thermodynamic circulations- around a closed curve C is conserved, as the curve moves with the fluid. Based on the work of Eckart, Mobbs (1981) has defined non-barotropic flows as follows:

Definition (2.8).

The flows in which $\nabla \eta \times \nabla S \neq 0$ are called non-barotropic flows where η is as defined by (2.7) and S is the specific entropy.

Mobbs (1981) has proved that generalizations are possible for almost all other conservation laws of barotropic flows associated with vorticity to non-barotropic flows. He called the quantity,

$$\oint (\bar{v} - \eta \nabla S) . d\bar{l} \qquad (2.9)$$

the 'generalized circulation' and the theorem proved by Eckart as 'generalized circulation theorem'. A detailed discussion of his generalization of other conservation laws is given in chapter 5.

It has been proved by Bretherton (1970) that the equations of motion of an inviscid fluid flow in a conservative force field can be derived from the variation of the action

$$W = \int dt \int_V \rho \left[\frac{1}{2}|\bar{v}|^2 - E(\rho, S) - U(x) \right] dV, \tag{2.10}$$

using the relation between the Eulerian and Lagrangian variations without using any other constraints. He has also proved that the indistinguishability of fluid particles having the same velocity, density and entropy in Eulerian representation corresponds to Kelvin's circulation theorem in isentropic case. In this chapter we prove that the same indistinguishability of fluid particles leads to generalized circulation theorem in non-barotropic case.

3. Derivation of generalized circulation theorem from displacement freedom of particles

We consider an instantaneous displacement $\Delta\bar{X}_1(\bar{x},t)$ under which the material particles in a closed solenoidal filament of infinitesimal cross section A and centre line C_1 are substituted for one another cyclically, the mass Δm passing each cross section of the filament being the same. Particles outside the filament are undisturbed. This change can be viewed as due to localized body forces acting during an infinitesimal interval about time t_1. This displacement $\overline{\Delta X}_1$ must be parallel to the unit tangent vector $\hat{n}$ of the filament and

$$\Delta m = \rho A \, |\overline{\Delta X}_1| = \text{constant}. \tag{2.11}$$

i.e., the density distribution ρ after the displacement is the same as before ($\delta\rho = 0$). We do not restrict that C_1 lies in a surface of constant entropy. An instantaneous displacement at time t_1 implies a particle velocity which is a Dirac delta function of time

$$\delta u = \overline{\Delta X}_1(\overline{x},t) \quad \delta(t-t_1). \tag{2.12}$$

For a small change δS carried along by the fluid particles we have

$$\int dt \int_V \rho \frac{\partial E}{\partial S} \delta S \, dV = \int dt \int_V \rho T \, \delta S,$$

$$\left(\text{since } \left(\frac{\partial E}{\partial S}\right)_\rho = T \right)$$

$$= \int dt \int_V \rho \frac{D\eta}{Dt} \delta S \, dV,$$

(by equation (2.7))

$$= \int dt \frac{D}{Dt} \int \rho \eta \delta S \, dV,$$

(by Reynold's transport theorem (Batchelor, 1968) and the condition $\frac{D}{Dt}(\delta S) = 0$)

$$= \int dt \, \delta(t-t_1) \int_V \rho \eta \, \Delta \overline{X}_1 . \nabla S \, dV, \qquad (2.13)$$

corresponding to the instantaneous displacement around $t = t_1$.

The corresponding change in the variation of the total action W is

$$\delta W_1 = \int dt \int_V [\rho \overline{v} . \delta \overline{v} + (\tfrac{1}{2}|\overline{v}|^2 - E - U - \rho \frac{\partial E}{\partial \rho}) \delta\rho$$

$$- \rho \frac{\partial E}{\partial S} \delta S] \, dV,$$

$$= \int_V (\rho \bar{v} . \Delta \bar{X}_1 - \rho \eta \nabla S . \Delta \bar{X}_1) \, dV,$$

$$= \int_V (\bar{v} - \eta \nabla S).\hat{n} \, \rho \, | \Delta X_1 | A \, dl,$$

$$= \Delta m \oint_{C_1} (\bar{v} - \eta \nabla S).d\bar{l} \, . \qquad (2.14)$$

Here $d\bar{l} = \hat{n}\, dl$ is the line element along C_1 so that the element of volume of the solenoid is $dV = A\, dl$. Previous and subsequent to the infinitesimal interval around $t = t_1$, we suppose that the field ' $\bar{v} - \eta \nabla S$' is unaffected by the variation and except for the contribution (2.14) already calculated the variation of the total action vanishes.

However, this particle substitution does not meet the requirements of a variation under Hamilton's principle. A particle displaced at time $t = t_1$ to a neighbouring point in a physical space will remain on the trajectory through that point. The displacement $\Delta \bar{X}$ changes for $t > t_1$ like the infinitesimal line element separating two material fluid particles and does not vanish as $t \longrightarrow + \infty$.

To restore it to zero a second substitution displacement is necessary at some time t_2 defined by a mass exchange around the solenoid C_2 consisting of the same material particles which were involved in C_1, of magnitude equal to Δm but in the opposite sense. The change in the total action under this second substitution displacement is

$$\delta W_2 = - \Delta m \oint_{C_2} (\bar{v} - \eta \nabla S) . d\bar{l} . \qquad (2.15)$$

Then for times $t > t_2$, the variant trajectories coincide in all respect with the originals and Hamilton's principle applies. It states that,

$$\delta W_1 + \delta W_2 = 0 .$$

i.e., $$\oint_{C_1} (\bar{v} - \eta \nabla S) . d\bar{l} = \oint_{C_2} (\bar{v} - \eta \nabla S) . d\bar{l} . \qquad (2.16)$$

Clearly the same is true at any times t_1, t_2 for any closed material filament C consisting of fluid particles. This is precisely the generalized circulation theorem.

4. Remarks

The instantaneous localized displacements considered here are not continuous and are not allowed under the rules of variational principles. However, we may approximate them by smooth functions as closely as we wish.

Gouin (1976) has proved that Kelvin's circulation theorem is a consequence of the invariance properties of the action under certain group of transformations. Though it is evident that the circulation theorem can be associated with Noether's second theorem, it has not yet been derived by a direct application of the theorem.

Chapter 3

AN EULERIAN VARIATIONAL PRINCIPLE FOR NON-BAROTROPIC FLOWS OF AN IDEAL FLUID

1. Introduction

As discussed in chapter 1, the indeterminacies and redundancies in the definition of potentials in Eulerian variational principles like Herivel-Lin formulation demand new variational principles avoiding these difficulties. Bretherton's (1970) work was an attempt in this direction. In this chapter, we introduce an Eulerian variational formulation for non-barotropic fluid flows by using suitable field variables. Euler's equations of motion are obtained as Euler-Lagrange equations of variations. It is a generalization of the variational principle for barotropic flows due to Drobot and Rybarski (1959).

2. Matter and entropy flows

We consider the Euclidean four dimensional space X. A point $\bar{x}$ in X has coordinates x^α, $\alpha = 0,1,2,3$, where x^0 is time and x^α, $\alpha = 1,2,3$ are space like coordinates.

Some of the results presented in chapters 3 and 4 have been published. (George Mathew and M.J.Vedan (1989)).

$\overline{p}(x)$ and $\overline{s}(x)$ are four dimensional vector fields (4-vectors) with components p^α and s^α, $\alpha = 0,1,2,3$ defined as follows:

Definition (3.1):

(a) $\overline{p} = (p^0, p^1, p^2, p^3) = (\rho, \rho v^1, \rho v^2, \rho v^3)$,

(b) $\overline{s} = (s^0, s^1, s^2, s^3) = (\rho S, \rho v^1 S, \rho v^2 S, \rho v^3 S)$,

where ρ is the density, v^1, v^2, v^3 are components of the velocity and S is the specific entropy.

Let H denote any three dimensional hypersurface in X and dH_α denote the oriented surface element on H,

$$dH_\alpha = e_{\alpha\beta\gamma\mu}\, dl^\beta\, dl^\gamma\, dl^\mu, \tag{3.2}$$

where $e_{\alpha\beta\gamma\mu} = e^{\alpha\beta\gamma\mu}$ is Levi-Civita tensor and dl^β, dl^γ, dl^μ are three linearly independent vectors lying on H so that dH_α is normal to the hypersurface H.

Definition (3.3):

The mass contained on H represented by the integral $\oint_H dH_\alpha p^\alpha$ is called 'the complete matter flow'.

In particular when the hypersurface H is space-like three dimensional volume V, we have $dH_o = dV$, $dH_1 = dH_2 = dH_3 = 0$ and

$$\int_H dH_\alpha p^\alpha = \int_V p^o dV \tag{3.4}$$

reduces to the usual mass. If H is closed and consists of V_{t_o}, V_{t_1} and moving two dimensional boundary Σ_t of V_t for $t_o < t < t_1$, then

$$\oint_H dH_\alpha \, p^\alpha = \int_{V_{t_1}} p^o \, dV_t - \int_{V_{t_o}} p^o \, dV_t + \int_{t_o}^{t_1} dt \int_{\Sigma_t} p^o \overline{v} \cdot d\Sigma_t \, . \tag{3.5}$$

This is called the matter balance for moving region. In general, the hypersurface may be open. In this case the complete matter flow represents a generalization of the motion of the mass contained on H.

Definition (3.5):

$\oint_H dH_\alpha s^\alpha$ is called 'the complete entropy flow'.

From the definition of $p^\alpha(x)$ and $s^\alpha(x)$ it is clear that

$$\partial_\alpha p^\alpha = \partial_t \rho + \operatorname{div}(\rho \bar{v}) \tag{3.6}$$

and

$$\partial_\alpha s^\alpha = \partial_t(\rho S) + \operatorname{div}(\rho \bar{v} S) \tag{3.7}$$

where $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$, 'div' is the divergence operator in

(x^1, x^2, x^3) space and $\bar{v} = (v^1, v^2, v^3)$.

Definition (3.8):

The action W is the functional

$$W = \int_V dV \; L(\bar{x}, \bar{p}(\bar{x}), \bar{s}(\bar{x})) \tag{3.9}$$

in which the Lagrangian L is any given function depending on $\bar{x}$, $\bar{p}$ and $\bar{s}$ only and V is a 4-dimensional volume.

The usual Lagrangian is given by

$$L = \frac{1}{2p^0}\,[(p^1)^2 + (p^2)^2 + (p^3)^2] - E(p^0,s^0) - p^0\,U(x)\,, \tag{3.10}$$

where E is the internal energy and U(x) is the potential energy.

Let F be a function space of vector valued functions of the 4-vectors $\bar{p}$ and $\bar{s}$ supposed to be regular in X. We consider the following infinitesimal transformations of X and F into themselves:

$$\left.\begin{aligned} \tilde{x}^\alpha &= x^\alpha + \delta x^\alpha(x), \\ \tilde{p}^\alpha &\doteq p^\alpha + \delta p^\alpha(x), \\ \tilde{s}^\alpha &= s^\alpha + \delta s^\alpha(x), \end{aligned}\right\} \tag{3.11}$$

where

$$\left.\begin{aligned} \delta x^\alpha(x) &= e\,\xi^\alpha(x) + o(e), \\ \delta p^\alpha(x) &= e\,\pi^\alpha(x) + o(e), \\ \delta s^\alpha(x) &= e\,\theta^\alpha(x) + o(e). \end{aligned}\right\} \tag{3.12}$$

and

In (3.12) ξ, π and Θ are arbitrary functions belonging to the space F and e is a scalar parameter. The functions $\delta p^\alpha(x)$ and $\delta s^\alpha(x)$ are called local variations of the fields $\bar{p}$ and $\bar{s}$ respectively.

Definition (3.13):

(a) $$\Delta \int_H dH_\alpha p^\alpha = e\left[\frac{d}{de} \int_{\tilde{H}} dH_\alpha \tilde{p}^\alpha(\tilde{x}^\alpha) \right]_{e=0},$$

where $\tilde{H}$ denotes the hypersurface obtained from H by the transformations (3.11) is called the 'total variation of the complete matter flow'.

(b) $$\Delta \int_H dH_\alpha s^\alpha = e\left[\frac{d}{de} \int_{\tilde{H}} dH_\alpha \tilde{s}^\alpha(\tilde{x}^\alpha) \right]_{e=0}$$

is called the 'total variation of the complete entropy flow'.

Following Drobot and Rybarski (1959), we have the following identities:

$$\Delta \int_H dH_\alpha p^\alpha = \int_H dH_\alpha \left[\delta p^\alpha - \partial_\beta (p^\beta \delta x^\alpha - p^\alpha \delta x^\beta) + \partial_\beta p^\beta . \delta x^\alpha \right] \qquad (3.14)$$

and $$\Delta \int_H dH_\alpha s^\alpha = \int_H dH_\alpha \left[\delta s^\alpha - \partial_\beta (s^\beta \delta x^\alpha - s^\alpha \delta x^\beta) + \partial_\beta s^\beta . \delta x^\alpha \right] \qquad (3.15)$$

Definition (3.16):

The functional

$$\Delta W = e \left[\frac{d}{de} \int_{\tilde{V}} dV \, L(\tilde{x}, \tilde{p}(\tilde{x}), \tilde{s}(\tilde{x})) \right]_{e=0} ,$$

where $\tilde{V}$ is obtained from V by the transformations (3.11) is called the total variation of the action W.

Now we impose certain conditions on the variations of $\bar{p}$ and $\bar{s}$.

(i) For every hypersurface H for which $dH_\alpha \delta x^\alpha = 0$ the total variations of the complete matter flow and complete entropy flow should vanish;

i.e., $$\Delta \int_H dH_\alpha p^\alpha = 0 \quad \text{and} \quad \Delta \int_H dH_\alpha s^\alpha = 0, \tag{3.17}$$

and

(ii) the variations δp^α and δs^α satisfy the equations

$$\partial_\alpha(\delta p^\alpha) = 0 \quad \text{and} \quad \partial_\alpha(\delta s^\alpha) = 0 . \tag{3.18}$$

Definition (3.19):

(a) The variations δp^{α} and δs^{α} satisfying conditions (3.17) and (3.18) are called the 'local hydromechanical variations' of the fields $\bar{p}$ and $\bar{s}$ respectively.

We denote the hydromechanical variations of p^{α} and s^{α} by $\delta_o p^{\alpha}$ and $\delta_o s^{\alpha}$ respectively.

(b) The variations given by

$$\Delta_o p^{\alpha} = \delta_o p^{\alpha} + \partial_{\beta} p^{\alpha} \, \delta x^{\beta}$$

and

$$\Delta_o s^{\alpha} = \delta_o s^{\alpha} + \partial_{\beta} s^{\alpha} \, \delta x^{\beta}$$

are called the 'total hydromechanical variations of p^{α} and s^{α}' respectively.

Theorem (3.20):

All local hydromechanical variations of p^{α} and s^{α} are of the form

$$\delta_o p^{\alpha} = \partial_{\beta} (p^{\beta} \delta x^{\alpha} - p^{\alpha} \delta x^{\beta}),$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (3.21)

$$\delta_o s^{\alpha} = \partial_{\beta} (s^{\beta} \delta x^{\alpha} - s^{\alpha} \delta x^{\beta}),$$

where δx^{α} are arbitrary infinitesimal functions belonging to the space F. The proof is similar to that given by Drobot and Rybarski (1959).

Thus equations (3.21) define an infinitesimal group of transformations of the vector fields $\bar{p}(\bar{x})$ and and $\bar{s}(\bar{x})$ depending on arbitrary functions δx^{α}.

The conditions defining the hydromechanical variations are appropriate counterparts of the necessary constraints for fluid flows.

3. <u>Generalized Variational Principle</u>

We now state our variational principle as follows:

' For all δx^{α} vanishing on the boundary of the region V, the total variation of the action vanishes, i.e., $\triangle_o W = \triangle \int_V dV\ L(\bar{x}, \bar{p}(\bar{x}), \bar{s}(\bar{x}))$,

$$= \int_V dV \left[\frac{\partial L}{\partial p^\alpha} \delta p^\alpha + \frac{\partial L}{\partial s^\alpha} \delta s^\alpha + \partial_\alpha (L \, \delta x^\alpha) \right],$$

$$= 0, \tag{3.22}$$

provided that δp^α, δs^α are hydromechanical variations'.

Using the expressions for $\delta_o p^\alpha$ and $\delta_o s^\alpha$ given by (3.21), we have

$$\triangle_o W = \int_V dV \left[\frac{\partial L}{\partial p^\alpha} \partial_\beta (p^\beta \delta x^\alpha - p^\alpha \delta x^\beta) \right.$$

$$+ \frac{\partial L}{\partial s^\alpha} \partial_\beta (s^\beta \delta x^\alpha - s^\alpha \delta x^\beta)$$

$$\left. + \partial_\alpha (L \ \delta x^\alpha) \right] , \tag{3.23}$$

$$= \int_V dV \ \partial_\beta \ (T^\beta_\alpha \delta x^\alpha) - \int_V dV \, \psi_\alpha \, \delta x^\alpha , \tag{3.24}$$

where

$$T^\beta_\alpha = p^\beta \frac{\partial L}{\partial p^\alpha} + s^\beta \frac{\partial L}{\partial s^\alpha} + \delta^\beta_\alpha (L - p^\gamma \frac{\partial L}{\partial p^\gamma} - s^\gamma \frac{\partial L}{\partial s^\gamma}) \tag{3.25}$$

and

$$\mathcal{E}_\alpha = p^\beta \left[\partial_\beta \left(\frac{\partial L}{\partial p^\alpha} \right) - \partial_\alpha \left(\frac{\partial L}{\partial p^\beta} \right) \right]$$

$$+ s^\beta \left[\partial_\beta \left(\frac{\partial L}{\partial s^\alpha} \right) - \partial_\alpha \left(\frac{\partial L}{\partial s^\beta} \right) \right], \qquad (3.26)$$

δ^β_α being the Kronecker delta.

Definition (3.27):

The expressions $\mathcal{E}_\alpha$, $\alpha = 0,1,2,3$ are called 'hydromechanical Euler-Lagrange expressions'.

Since δx^α vanish on the boundary of V, the first integral on the right hand side of (3.24) vanishes. Thus we obtain from our variational principle (3.22)

$$\int_V dV \; \mathcal{E}_\alpha \; \delta x^\alpha = 0 \; . \qquad (3.28)$$

Since δx^α are arbitrary functions in the interior of V, we get the equations of motion as

$$\mathcal{E}_\alpha = 0, \; \alpha = 0,1,2,3. \qquad (3.29)$$

Since $p^\alpha \mathcal{E}_\alpha = 0$, (3.30) the four equations (3.29) are linearly dependent. When the Lagrangian takes the usual form (3.10) we get the following equations:

$\alpha = 0$:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\bar{v}|^2 \right) + \bar{v} \cdot \nabla\left(\frac{1}{2}|\bar{v}|^2\right)$$

$$+ \bar{v}\cdot\left(\frac{\nabla P}{p^o} + \nabla U \right) = 0, \tag{3.31}$$

, $\alpha = 1,2,3$:

$$\frac{\partial}{\partial t}(\bar{v}) + (\bar{v}.\nabla)\bar{v} = - \frac{\nabla P}{p^o} - \nabla U \ , \tag{3.32}$$

where $P = p^o \dfrac{\partial E}{\partial p^o} + s^o \dfrac{\partial E}{\partial s^o} - E$ is the pressure.

Note that the equation (3.31) is a generalized form of Bernoulli's equation and the equations (3.32) are Euler's equations of motion.

All these equations are deduced under the assumption that the functions p^α and s^α are continuous in the whole region.

Thus we have extended the variational principle of Drobot and Rybarski to non-barotropic adiabatic inviscid fluid flows and obtained Euler's equations of motion as Euler-Lagrange equations of variations.

Chapter 4

NOETHER THEOREMS AND CONSERVATION LAWS

1. Introduction

In the calculus of variations the theorems of Noether (1918) describe a relationship between the invariance of the action integral with respect to given groups of transformations and some identities satisfied by Euler-Lagrange expressions. There are two types of Noether theorems. The first theorem deals with transformations depending on scalar parameters and the second theorem deals with transformations depending on functions. They can be roughly stated as follows (Logan, 1977):

(i) If the action W is div-invariant under an r-parameter continuous group of transformations of the variables, then there result r identities between Euler-Lagrange expressions $\mathcal{L}_k$ and quantities which can be written as divergences.

(ii) If the action is div-invariant under a group of transformations which depend upon r arbitrary functions and their derivatives upto some order q, there exist r identities between the Euler-Lagrange expressions $\mathcal{L}_k$ and their derivatives upto order q.

In this chapter we adapt these theorems to the variational principle discussed in chapter 3 and examine some applications.

2. Noether's first theorem and Galilean group of transformations:

We consider a class of transformations depending on scalar parameters.

Let $$W = \int_V dV\, L\,(\bar{x}, \bar{p}(\bar{x}), \bar{s}(\bar{x})) \qquad (4.1)$$

be defined on a suitable function space. We consider the infinitesimal transformations defined by

$$\tilde{x}^\alpha = x^\alpha + \Delta x^\alpha, \ \alpha = 0,1,2,3, \qquad (4.2)$$

where Δx^α are functions of x^β, $p^\beta(x)$, $s^\beta(x)$ and their derivatives.

Then the functional W is transformed to

$$\tilde{W} = W + \Delta_0 W, \qquad (4.3)$$

$$= \int_{\tilde{V}} dV \; L(\tilde{x}, \tilde{p}(\tilde{x}), \tilde{s}(\tilde{x})), \qquad (4.4)$$

where $\tilde{V}$ is the transformed region of V.

Now we have,

$$\triangle_o W = \int_V dV \; [\frac{\partial L}{\partial p^\alpha} \delta_o p^\alpha + \frac{\partial L}{\partial s^\alpha} \delta_o s^\alpha + \partial_\alpha (L \triangle x^\alpha)], \qquad (4.5)$$

where

$$\delta_o p^\alpha = \partial_\beta (p^\beta \triangle x^\alpha - p^\alpha \triangle x^\beta), \qquad (4.6)$$

and

$$\delta_o s^\alpha = \partial_\beta (s^\beta \triangle x^\alpha - s^\alpha \triangle x^\beta). \qquad (4.7)$$

Definition (4.8):

(a) The functional W is said to be'hydromechanically invariant upto a divergence (div-invariant)' with respect to the transformations (4.2) if there exists a vector (C^α) such that $\triangle_o W = \partial_\alpha C^\alpha$ identically in V.

(b) If $C^\alpha = 0$ in (a) so that $\triangle_o W = 0$, then W is said to be 'absolutely invariant' with respect to the transformations (4.2).

Noether's first theorem can be adapted to our variational principle as follows:

Theorem (4.9):

If the functional W is div-invariant with respect to the transformations (4.2), depending on r arbitrary parameters, then exactly r linearly independent linear forms of the Euler-Lagrange expressions $\mathcal{L}_\alpha$ are divergences, provided the variations of the field variables are restricted to hydromechanical variations.

Proof:

By the hypothesis of the theorem, within the infinitesimals of first order,

$$\Delta x^\alpha = g^\alpha_m \, \varepsilon^m \tag{4.10}$$

and

$$C^\alpha = C^\alpha_m \, \varepsilon^m, \; m = 1,2,3, \ldots, r, \tag{4.11}$$

where $\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^r$ are infinitesimal scalar parameters and g^α_m, C^α_m are given functions.

From equation (3.24) we have

$$\Delta_o W = \int_V dV \; [\; \partial_\beta (T^\beta_\alpha \Delta x^\alpha) - \psi_\alpha \Delta x^\alpha \;]. \tag{4.12}$$

Substituting from the equations (4.10) and (4.11) in the equation (4.12), we get

$$\varepsilon^m \int_V dV \; \partial_\beta \; C^\beta_m = \varepsilon^m \int dV \; [\partial_\beta (T^\beta_\alpha \, g^\alpha_m) - \psi_\alpha \, g^\alpha_m \,], \tag{4.13}$$

identically in ε^m and V. Hence

$$\psi_\alpha \, g^\alpha_m = \partial_\beta \; (T^\beta_\alpha \, g^\alpha_m - C^\beta_\alpha \,), \tag{4.14}$$

which completes the proof.

During the motion we have $\psi_\alpha = 0$, and equation (4.14) becomes

$$\partial_\beta \; (T^\beta_\alpha \, g^\alpha_m - C^\beta_\alpha \,) = 0. \tag{4.15}$$

We apply the formula (4.15) to the case when (4.2) is the group of Galilean transformations. For this, we suppose that the action is absolutely invariant with respect to the transformations defined by

$$\Delta x^{\alpha} = a^{\alpha} + a^{\alpha}_{\beta} x^{\beta}; \quad \alpha,\beta = 0,1,2,3, \tag{4.16}$$

in which a^{α} and a^{α}_{β} are scalar parameters satisfying the conditions

$$\left.\begin{aligned} a^{o}_{\beta} &= a^{\alpha}_{o} = 0, \quad \text{for } \alpha,\beta = 0,1,2,3 \\ \text{and} \quad a^{\alpha}_{\beta} &+ a^{\beta}_{\alpha} = 0, \quad \text{for } \alpha,\beta = 1,2,3. \end{aligned}\right\} \tag{4.17}$$

In this case $C^{\alpha} = 0$ and by equation (4.15) we have the following conservation laws:

$$\partial_{\alpha} (T^{\beta}_{\alpha}) = 0; \quad \alpha,\beta = 0,1,2,3. \tag{4.18}$$

and

$$\partial_{\alpha}(M^{\alpha}_{\beta\gamma}) = 0; \quad \alpha = 0,1,2,3 \quad \beta,\gamma = 1,2,3. \tag{4.19}$$

where

$$M^{\alpha}_{\beta\gamma} = T^{\alpha}_{\beta}\, x^{\gamma} - T^{\alpha}_{\gamma}\, x^{\beta}. \tag{4.20}$$

Substituting in equations (4.18) and (4.19) the expressions for T_α^β given by (3.26), we get the conservation laws of energy, impulse and angular momenta respectively. If L takes the usual form given by (3.10) these laws become the familiar ones:

$$\text{(i)} \qquad \rho \frac{D}{Dt} \left(\frac{1}{2} |\bar{v}|^2 + U + \frac{1}{\rho} (P+E) \right) = \frac{\partial}{\partial t} P, \qquad (4.21)$$

$$\text{(ii)} \qquad \rho \frac{D}{Dt} (\bar{v}) = - \nabla P, \qquad (4.22)$$

and

$$\text{(iii)} \qquad \rho \frac{D}{Dt} (\bar{x} \times \bar{v}) = - \bar{x} \times \nabla P. \qquad (4.23)$$

3. Noether's second theorem and generalized Helmholtz theorems

Now we state Noether's second theorem adapted to hydromechanical variations.

Theorem (4.24):

If the functional W is div-invariant with respect to the transformations

$$\Delta x^{\alpha} = \Lambda^{\alpha}_{s} (\phi^{s}),$$

$$= A^{\alpha}_{s} \phi^{s} + A^{\alpha\lambda_1}_{s} \partial_{\lambda_1} \phi^{s} + A^{\alpha\lambda_1\lambda_2}_{s} \partial_{\lambda_1\lambda_2} \phi^{s} + \ldots + A^{\alpha\lambda_1\lambda_2\ldots\lambda_q}_{s} \partial_{\lambda_1\lambda_2\ldots\lambda_q} \phi^{s}, \tag{4.25}$$

depending essentially on r arbitrary functions ϕ^{s}, $s = 1,2,\ldots,r$ and their derivatives upto a given order q, the coefficients A^{α}_{s}, $A^{\alpha\lambda_1}_{s}$, ... etc. being given functions, there exist exactly r linearly independent identities between the Euler-Lagrange expressions $\mathcal{L}_{\alpha}$ and their derivatives, provided the variations of the field variables $\bar{p}$ and $\bar{s}$ are restricted to hydromechanical variations.

Proof:

By the hypothesis, W is div-invariant under the transformations (4.2).

In definition (4.8a) we take

$$C^{\alpha} = \Gamma^{\alpha}_{s} (\phi^{s})$$

$$= C^{\alpha}_{s} \phi^{s} + C^{\alpha\lambda_1}_{s} \partial_{\lambda_1} \phi^{s} + C^{\alpha\lambda_1\lambda_2}_{s} \partial_{\lambda_1\lambda_2} \phi^{s} + \ldots + C^{\alpha\lambda_1\lambda_2\cdots\lambda_q}_{s} \partial_{\lambda_1\cdots\lambda_q} \phi^{s}, \qquad (4.26)$$

where C^{α}_{s}, $C^{\alpha\lambda_1}_{s}$, ... are given functions.

Substituting from equations (4.25) and (4.26) in equation (4.12), we have

$$\int_V dV\, \partial_{\alpha} (\Gamma^{\alpha}_{s} (\phi^{s}))$$

$$= \int_V dV \left[\partial_{\beta}(T^{\beta}_{\alpha} \Lambda^{\alpha}_{s} (\phi^{s})) - \mathcal{F}_{\alpha} \Lambda^{\alpha}_{s}(\phi^{s}) \right], \qquad (4.27)$$

identically in the functions ϕ^{s} and in the region V.

Let $\tilde{\Lambda}^{\alpha}_{s}(.)$ be the operator adjoint to the operator $\Lambda^{\alpha}_{s}(.)$

i.e., $$\tilde{\Lambda}^{\alpha}_{s}(\phi^{s}) = A^{\alpha}_{s} \phi^{s} - \partial_{\lambda_1}(A^{\alpha\lambda_1}_{s} \phi^{s}) + \ldots + (-1)^{q} \partial_{\lambda_1\lambda_2\cdots\lambda_q}(A^{\alpha\lambda_1\lambda_2\cdots\lambda_q}_{s} \phi^{s}). \qquad (4.28)$$

Integration by parts gives

$$\int dV \tilde{\Lambda}^{\alpha}_{s}(\mathcal{L}_{\alpha})\, \phi^{s} = \oint_{\partial V} dV_{\beta}\, [\Gamma^{\beta}_{\alpha}(\phi^{s}) + T^{\beta}_{\alpha}\Lambda^{\alpha}_{s}(\phi^{s})\,]$$

$$+ \oint_{\partial V} dV_{\lambda_1} \mathcal{L}_{\alpha}\Lambda^{\alpha}_{s}(\phi^{s}). \qquad (4.29)$$

As the functions ϕ^{s} are arbitrary, we select them so as to vanish along with their derivatives upto order q-1 on the boundary ∂V of V. Therefore, from the identity (4.29), it follows that

$$\tilde{\Lambda}^{\alpha}_{s}(\mathcal{L}_{\alpha}) = 0, \qquad (4.30)$$

and these are the identities for the Euler-Lagrange expressions $\mathcal{L}_{\alpha}$. This completes the proof.

We apply this theorem to the group of transformations (4.2), consisting of those Δx^{α} for which $\delta_{o}p^{\alpha} = 0$ and $\delta_{o}s^{\alpha} = 0$ (i.e., hydromechanical variations of the field variables are vanishing identically). The following theorem characterize this group of transformations.

Theorem (4.31):

Hydromechanical variations of p^α and s^α vanish;

i.e., $\delta_o p^\alpha = \partial_\beta (p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta) = 0$ (4.32)

and

$$\delta_o s^\alpha = \partial_\beta (s^\beta \Delta x^\alpha - s^\alpha \Delta x^\beta) = 0, \tag{4.33}$$

if and only if

$$\Delta x^\alpha = p^\alpha \phi + \frac{u_\beta e^{\alpha\beta\lambda\mu}}{p^\nu u_\nu} \partial_\lambda \phi_\mu, \tag{4.34}$$

where ϕ is an arbitrary scalar function, u_β is an arbitrary four vector and ϕ_μ is any vector satisfying the conditions

$$p^\lambda (\partial_\lambda \phi_\mu - \partial_\mu \phi_\lambda) = 0 \tag{4.35}$$

and

$$\partial_\beta S \, e^{\alpha\beta\lambda\mu} \partial_\lambda \phi_\mu = 0, \tag{4.36}$$

S being the specific entropy.

Following Drobot and Rybarski (1959, theorem 4), we have the following results.

Lemma (4.37):

$\delta_o p^\alpha = 0$ if and only if

$$\Delta x^\alpha = p^\alpha \phi + \frac{u_\beta e^{\alpha\beta\lambda\mu}}{p^\nu u_\nu} \partial_\lambda \phi_\mu , \tag{4.38}$$

where ϕ is an arbitrary scalar, (u_β) an arbitrary 4-vector and (ϕ_μ) is any 4-vector satisfying the conditions (4.35).

Lemma (4.39):

If $\Delta x^\alpha = p^\alpha \phi + \frac{u_\beta e^{\alpha\beta\lambda\mu}}{p^\nu u_\nu} \partial_\lambda \phi_\mu$,

where ϕ, (u_β) and (ϕ_μ) are chosen as in lemma (4.37), then

$$p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta = e^{\alpha\beta\lambda\mu} \partial_\lambda \phi_\mu . \tag{4.40}$$

Proof of the theorem (4.31):

Let $f^{\alpha\beta} = p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta$. (4.41)

Since $s^\alpha = p^\alpha S$, (4.42)

$$\delta_o s^\alpha = \partial_\beta(s^\beta \Delta x^\alpha - s^\alpha \Delta x^\beta),$$

$$= \partial_\beta[S(p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta)],$$

$$= \partial_\beta S\ f^{\alpha\beta} + S\ \delta_o p^\alpha. \tag{4.43}$$

Necessary part:

Let $\delta_o p^\alpha = 0$ and $\delta_o s^\alpha = 0$. Then from equation (4.43) we have

$$\partial_\beta S\ f^{\alpha\beta} = 0\ . \tag{4.44}$$

Since $\delta_o p^\alpha = 0$, by lemma (4.37), Δx^α takes the form (4.34) with the conditions (4.35). Then lemma (4.39) and equation (4.44) ensure the condition (4.36). This completes the proof.

Sufficient part:

Let Δx^α take the form given by (4.34), together with conditions (4.35) and (4.36). Then $\delta_o p^\alpha = 0$ by lemma (4.37) and $f^{\alpha\beta} = e^{\alpha\beta\lambda\mu}\,\partial_\lambda \phi_\mu$, by lemma (4.39). Thus $\delta_o p^\alpha = 0$ and $\partial_\beta S\, f^{\alpha\beta} = 0$, which imply $\delta_o s^\alpha = 0$ by the relation (4.43). This completes the proof.

As the differential operator defining Δx^α in (4.34) has to satisfy the side conditions (4.35) and (4.36) we cannot apply the theorem (4.24) directly to get identities like (4.30). We use Lagrangian multipliers to incorporate the side conditions and derive identities corresponding to (4.30) in an indirect way.

Since $\delta_o p^\alpha = 0$ and $\delta_o s^\alpha = 0$, from equations (4.12) we get

$$\int dV\, \psi_\alpha \Delta x^\alpha = \int dV\ \partial_\beta[(T^\beta_\alpha - \delta^\beta_\alpha L)\Delta x^\alpha]\ . \tag{4.45}$$

We transform the right hand side of the equation (4.45) by Gauss formula into a hypersurface integral over the boundary ∂V of the region V. On ∂V we take $\phi = \phi_\mu = 0$ and the vector u_β normal to it. Then $\Delta x^\alpha = 0$ on ∂V and equation (4.45) becomes

$$\int dV\, \psi_\alpha \Delta x^\alpha = \int dV\, \frac{\psi_\alpha u_\beta e^{\alpha\beta\lambda\mu}}{p^\nu u_\nu}\ \partial_\lambda \phi_\mu = 0, \tag{4.46}$$

provided that the side conditions (4.35) and (4.36) hold. (Here we have used the relation (3.30)). These side conditions are taken by means of the Lagrangian multipliers Ω^{μ} and ζ^{α}.

Thus we have

$$\int_V dV \Big[\frac{\psi_\alpha \, u_\beta \, e^{\alpha\beta\lambda\mu}}{p^\nu u_\nu} + \Omega^\mu p^\lambda - \Omega^\lambda p^\mu$$

$$+ \zeta^\alpha \, \partial_\beta \, S \, e^{\alpha\beta\lambda\mu} \Big] \partial_\lambda \phi_\mu = 0, \qquad (4.47)$$

where the functions ϕ_μ are arbitrary functions provided that they vanish on the boundary ∂V. Integrating by parts, the last identity leads to the following conclusion: there exist vectors Ω^{μ}, ζ^{α} such that

$$\partial_\lambda \left(\frac{u_\beta}{p^\nu u_\nu} e^{\alpha\beta\lambda\mu} \psi_\alpha \right) = \partial_\lambda \left(\Omega^\lambda p^\mu - \Omega^\mu p^\lambda \right)$$

$$-\partial_\lambda \left(\zeta^\alpha \partial_\beta S \, e^{\alpha\beta\lambda\mu} \right). \qquad (4.48)$$

These are the identities corresponding to (4.30). As Ω^{μ}, ζ^{μ} do not depend on the particular choice of u_β, we can choose u_β arbitrarily. As the side conditions (4.35) are linearly dependent, one of the

Lagrangian multipliers can be chosen arbitrarily.

Putting $u_o = 1$, $u_1 = u_2 = u_3 = 0$, $\Omega^o = 0$ in the equation (4.48), we get the following identities:

$\mu = 0$:

$$\mathrm{div}\,(\bar{\Omega} + \bar{\zeta} \times \nabla S) = 0\,, \tag{4.49}$$

where

$$\bar{\Omega} = (p^o\Omega^1\,,\; p\,\Omega^2\,,\; p\,\Omega^3\,),$$

$$\bar{\zeta} = (\,\zeta^1,\,\zeta^2,\,\zeta^3\,),$$

and 'div' is the divergence operator and three dimensional vector notations are used for convenience.

$\mu = 1,2,3$:

$$\begin{aligned} \nabla \times \left(\frac{\bar{\xi}}{p^o} \right) = &- \frac{\partial}{\partial t}\,(\bar{\Omega} + \bar{\zeta} \times \nabla S\,) \\ &- \nabla \times ((\bar{\Omega} + \bar{\zeta} \times \nabla S) \times \bar{v}\,) \\ &+ \nabla\,(\,\zeta^o + \bar{v} \cdot \bar{\zeta}\,) \times \nabla S, \end{aligned} \tag{4.50}$$

using the relation $\frac{DS}{Dt} = 0$.

Since $\bar{\psi} = 0$ during the motion, equations (4.50) reduce to

$$\frac{\partial}{\partial t}(\bar{\Omega}) + \nabla \times (\bar{\Omega} \times \bar{v}) + \frac{\partial}{\partial t}(\bar{\zeta} \times \nabla S)$$
$$+ \nabla \times \left((\bar{\zeta} \times \nabla S) \times \bar{v} \right)$$
$$= \nabla(\zeta^{0} + \bar{v}.\bar{\zeta}) \times \nabla S . \qquad (4.51)$$

We shall show that these identities lead to the conditions equivalent to generalized Helmholtz theorems, when the Lagrangian takes the usual form given by (3.10).

Since the usual Lagrangian does not depend on s^1, s^2 and s^3 the last three conditions in (4.36) can be deleted. Then the equations (4.51) reduce to

$$\frac{\partial}{\partial t}(\bar{\Omega}) + \nabla \times (\bar{\Omega} \times \bar{v}) = \nabla \zeta^{0} \times \nabla S . \qquad (4.52)$$

If we define the quantity τ as the time integral of ζ^{0},

i.e., $$\tau = \int_0^t \zeta^{0}(x,t)dt, \qquad (4.53)$$

using the condition $\frac{DS}{Dt} = 0$, we can write the equation (4.52) in the form

$$\frac{\partial}{\partial t}(\bar{\Omega} - \nabla\tau \times \nabla S) + \nabla \times \left((\bar{\Omega} - \nabla\tau \times \nabla S) \times \bar{v}\right) = 0, \quad (4.54)$$

which is the 'Helmholtz-Zorawski criterion' (Truesdell (1954), Truesdell and Toupin (1960)) for the conservation of the vector lines and the strength of the vector tubes of the vector field '$\bar{\Omega} - \nabla\tau \times \nabla S$'.

Note that when L takes the usual form the Euler-Lagrange equations $\bar{\psi} = 0$ become the vector equation (3.32).

Using the thermodynamic relation

$$\frac{\nabla P}{\rho} = \nabla I - T \nabla S,$$

where I is the specific enthalpy and taking curl of the vector equation (3.32) we get

$$\frac{\partial}{\partial t}(\bar{\omega}) + \nabla \times (\bar{\omega} \times \bar{v}) = \nabla T \times \nabla S . \qquad (4.55)$$

Comparing the equations (4.52) and (4.55) we can easily see that

$\bar{\Omega} = \bar{\omega}$ and $\zeta^{o} = T$ is a solution of the equation (4.52).

In this case, $\mathcal{T} = \int_0^t T\, dt = \eta$, the thermasy defined by (2.7) and the equation (4.52) becomes the 'Helmholtz-Zorawski criterion' for the generalized vorticity vector ' $\bar{\omega} - \nabla\eta \times \nabla S$ ' . Thus we have the following results:

(i) The vector lines of ' $\bar{\omega} - \nabla\eta \times \nabla S$ ' are material lines.

(ii) The strengths of the generalized vortex tubes are conserved during the motion.

These are precisely generalised Helmholtz theorems. Thus the identity (4.52) obtained as the Noether identity of the transformation group defined by (4.34, 4.35 and 4.36) corresponds to generalized Helmholtz theorems.

Chapter 5

HELMHOLTZ FIELDS AND GENERALIZED CONSERVATION LAWS

1. Introduction

In barotropic flows of an ideal fluid we have the following conservation laws related to vorticity field $\bar{\omega}$.

Kelvin's circulation theorem (5.1):

If C is a closed curve consisted of material particles lying on an isentropic surface, then

$$\frac{D}{Dt} \oint \bar{v} . \bar{dl} = 0 , \tag{5.2}$$

where $\bar{v}$ is the velocity vector.

Helmholtz vorticity theorems (5.3):

(i) If C_1 and C_2 are any two circuits encircling a vortex tube in the same direction then the circulation around C_1 is equal to the circulation around C_2

i.e., $$\oint_{C_1} \bar{v}.\bar{dl} = \oint_{C_2} \bar{v}. \bar{dl} . \tag{5.4}$$

Some of the results presented in this chapter will appear in the Journal of Mathematical and Physical Sciences. (Thomas Joseph and George Mathew).

(ii) Vortex lines are material lines.

(iii) The strength of vortex tube defined as the circulation around any closed circuit encircling the tube remains constant as the tube moves with the fluid.

Cauchy's vorticity formula (5.5):

$$\frac{D}{Dt}\left(\frac{\bar{\omega}}{\rho}\right) = \left(\frac{\bar{\omega}}{\rho}\cdot\nabla\right)\bar{v}\,. \tag{5.6}$$

Conservation of helicity (5.7):

If V is a volume of fluid bounded by a closed vortex tube then

$$\frac{DH}{Dt} = 0, \tag{5.8}$$

where H is the total helicity of the region V defined by

$$H = \int_V \bar{v}\cdot\bar{\omega}\, dV\,. \tag{5.9}$$

This conservation law was discovered by Moreau (1961) and Moffat (1969, 1978, 1986) independently.

Conservation of potential vorticity (5.10):

If λ is a fluid property satisfying $\frac{D\lambda}{Dt} = 0$, then

$$\frac{D}{Dt}\left(\frac{\omega . \nabla\lambda}{\rho} \right) = 0 \quad (5.11)$$

The quantity $\frac{\bar{\omega} . \nabla\lambda}{\rho}$ is called potential vorticity. This conservation law is due to Ertel (1942) where he uses S, the specific entropy, in the place of .

In non-barotropic flows none of these conservation laws hold. But Mobbs (1981) has shown that all these conservation laws can be extended to non-barotropic case by replacing the velocity $\bar{v}$ by $\bar{v}-\eta\nabla S$ and vorticity field $\bar{\omega}$ by $(\bar{\omega} - \nabla\eta \times \nabla S)$ in some of their ocurrences in the relevant equations. Mobbs calls the quantity $(\bar{\omega} - \nabla\eta \times \nabla S)$, the generalized vorticity. Thus in non-barotropic flows we have the following results:

Generalized circulation theorem (5.12):

$$\frac{D}{Dt} \oint_C (\bar{v}-\eta\nabla S).d\bar{l} = 0 , \quad (5.13)$$

where C is any closed curve moving with the fluid but need not be on an isentropic surface.

Generalized Helmholtz theorems (5.14):

(i) If C_1 and C_2 are any two circuits encircling a generalized vortex tube in the same direction then the generalized circulation around C_1 and is equal to that around C_2.

(ii) Generalized vortex lines are material lines.

(iii) The strength of a generalized vortex tube, defined as the generalized circulation around any circuit encircling the tube, remains constant as the tube moves with the fluid.

Generalized vorticity formula (5.15):

$$\frac{D}{Dt}\left(\frac{\bar{\omega} - \nabla\eta \times \nabla S}{\rho} \right) = \left(\left(\frac{\omega - \nabla\eta \times \nabla S}{\rho} \right) . \nabla \right) \bar{v} . \quad (5.16)$$

Conservation law of generalized helicity (5.17):

For any fluid region bounded by a closed generalized vortex tube the total generalized helicity is conserved.

i.e., $$\frac{D}{Dt} \int_V (\bar{v} - \eta \nabla S) \cdot (\bar{\omega} - \nabla \eta \times \nabla S)\, dV = 0 \; . \qquad (5.18)$$

Conservation of generalized potential vorticity (5.19):

If λ is a fluid quantity such that $\lambda = \lambda(S,T)$ and $\frac{D\lambda}{Dt} = 0$, then

$$\frac{D}{Dt}\left(\frac{(\omega - \nabla\eta \times \nabla S) \cdot \nabla\lambda}{\rho}\right) = 0 \; . \qquad (5.20)$$

In this chapter we show that the properties of any smooth solenoidal vector field $\bar{g}(x,t)$ satisfying the differential equation of the type (5.6) lead to conservation laws similar to those discussed above. Using this concept we generalize some conservation laws related to generalized vorticity field. We borrow the terminology 'Helmholtz fields' for such fields from Thyagaraja (1975), who has studied some of its properties in barotropic flows.

We use the following equations describing non-barotropic flows of perfect fluids:

Conservation of momentum,

$$\frac{D\bar{v}}{Dt} = -\nabla(I+U) + T\nabla S \, , \qquad (5.21)$$

The continuity equation,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{v} = 0, \tag{5.22}$$

and conservation of entropy,

$$\frac{DS}{Dt} = 0, \tag{5.23}$$

where $\bar{v}$ is the velocity vector, ρ the density, S the specific entropy, I the specific enthalpy and U is the potential energy due to any conservative body forces. It is assumed that the equation of state can be written in the form

$$E = E(\rho, S), \tag{5.24}$$

where E is the internal energy.

2. Helmholtz fields and conservation laws

Definition (5.25):

A Helmholtz field is a vector field $\bar{g}(x,t)$ satisfying the equations

$$\nabla \cdot \bar{g} = 0 \tag{5.26}$$

and $$\frac{D}{Dt}\left(\frac{\bar{g}}{\rho}\right) = \left(\frac{\bar{g}}{\rho} \cdot \nabla\right) \bar{v} .$$

From the equations (5.6) and (5.16) it follows that $\bar{\omega}$ and '$\bar{\omega} - \nabla\eta \times \nabla S$' are Helmholtz fields in barotropic and non-barotropic flows respectively.

The general solution of the equations (5.26) are discussed in Truesdell (1954), Lamb (1932), Serrin (1959) and Marris and Passman (1968).

Definition (5.27):

(a) A $\bar{g}$- tube in a fluid is a material structure formed by closed field lines of a Helmholtz field $\bar{g}$.

(b) A $\bar{g}$- filament is a $\bar{g}$- tube of infinitesimal cross section.

Theorem (5.28):

In a non-barotropic flow

$$\frac{D}{Dt} \int_V \bar{g} \cdot (\bar{v} - \eta \nabla S)\, dV = 0, \tag{5.29}$$

where V is any volume of fluid bounded by a $\bar{g}$- tube Σ.

Proof:

$$\frac{D}{Dt}\int_V \bar{g}.(v-\eta\nabla S)dV = \frac{D}{Dt}\int_V \left(\frac{\bar{g}}{\rho}\right).(\bar{v}-\eta\nabla S)\rho\, dV,$$

$$= \int_V \frac{D}{Dt}\left(\frac{\bar{g}}{\rho}\right).(\bar{v}-\eta\nabla S)\rho\, dV$$

$$+ \int_V \left(\frac{\bar{g}}{\rho}\right).\frac{D}{Dt}(\bar{v}-\eta\nabla S)\rho\, dV,$$

(by Reynold's transport theorem)

$$= \int_V \left(\frac{\bar{g}}{\rho}.\nabla\right)\bar{v}.(\bar{v}-\eta\nabla S)\rho\, dV$$

$$+ \int_V \bar{g}.[-\nabla(I+U) + T\nabla S - \frac{D}{Dt}(\eta\nabla S)]dV, \tag{5.30}$$

(by equations (5.21) and (5.26))

$$= \int_V [-(\bar{g}.\nabla)\bar{v}.(\eta\nabla S)+\bar{g}.(\eta(\nabla S.\nabla)\bar{v})$$

$$+ \bar{g}.(\nabla S \times \bar{\omega})\eta]\, dV$$

$$+ \int_V ((\bar{g}.\nabla)\bar{v}).\bar{v} - \int_V \bar{g}.\nabla(I+U)dV, \tag{5.31}$$

where we have used

$$\frac{D}{Dt}(\eta \nabla S) = T\nabla S + \eta[-(\nabla S.\nabla)\bar{v} - \nabla S \times \bar{\omega}] , \qquad (5.32)$$

which follows from the equations (2.7), (5.23) and the identity

$$\frac{D}{Dt}(\nabla\lambda) = \nabla\left(\frac{D\lambda}{Dt}\right) + \omega \times \nabla\lambda - (\nabla\lambda . \nabla)\bar{v} . \qquad (5.33)$$

By straight forward simplifications we can show that the integrand of the first integral in the right hand side of the equation (5.31) vanishes identically.

Now,

$$\int_V ((\bar{g}.\nabla)\bar{v}).\bar{v}\, dV = \int_V \nabla\cdot\left(\frac{1}{2}|\bar{v}|^2\,\bar{g}\right) dV,$$

$$= \int_\Sigma \frac{1}{2}\,|\bar{v}|^2\,\bar{g}.\hat{n}\, d\Sigma = 0 , \qquad (5.34)$$

since $g.\hat{n} = 0$ on the $\bar{g}$-tube bounding the volume V.

Similarly,

$$-\int_V \bar{g}.\nabla(I+U)dV = -\int_V \nabla\cdot((I+U)\bar{g})\, dV,$$

$$= -\int_\Sigma (I+U)\,\bar{g}.\hat{n}\, d\Sigma,$$

$$= 0, \qquad (5.35)$$

since $\bar{g}.\hat{n} = 0$ on Σ.

Hence the theorem.

Theorem (5.36):

Let C be a curve defined by a closed $\bar{g}$- filament and let the volume of the filament be V_C. Then

$$\int_{V_C} \bar{g}.(\bar{v}-\eta\nabla S)dV = |\bar{g}|\ \delta\sigma \oint(\bar{v}-\eta\nabla S).\bar{dl}\ ,$$

where ' $\delta\sigma$' is the infinitesimal cross sectional area of the filament.

Proof:

The result follows from the solenoidal nature of the $\bar{g}$-field which makes the strength of the $\bar{g}$-filament $|\bar{g}|\ \delta\sigma$ constant along the tube.

Theorem (5.37):

$$\frac{D}{Dt}\int_C (\bar{v}-\eta\nabla S).\bar{dl} = 0\ ,$$

for any closed curve C defined by a $\bar{g}$-filament.

Proof:

The proof follows from theorems (5.28) and (5.36).

Theorem (5.38):

In a non-barotropic flow of an inviscid fluid, the quantity $\frac{\bar{q}.\nabla\lambda}{\rho}$ is constant in time for each fluid element if λ is a fluid property such that $\frac{D\lambda}{Dt} = 0$.

Proof:

$$\frac{D}{Dt}\left(\frac{q.\nabla\lambda}{\rho} \right) = \frac{D}{Dt}\left(\frac{q}{\rho} \right).\nabla\lambda + \frac{\bar{q}}{\rho} \cdot \frac{D}{Dt}(\nabla\lambda),$$

$$= \nabla\lambda\cdot\left(\frac{\bar{q}}{\rho}.\nabla \right)\bar{v} + \frac{\bar{q}}{\rho}\left[\nabla \left(\frac{D\lambda}{Dt}\right) + \omega \times \nabla\lambda - (\nabla\lambda\cdot\nabla)\,\bar{v} \right]. \qquad (5.39)$$

(By equations (5.26) and (5.33))

By straight forward simplifications we can show that the right hand side of the equation (5.39) is identically zero if $\frac{D\lambda}{Dt} = 0$. This completes the proof.

Theorem (5.40):

In the class of non-barotropic flows in which $\bar{g}\cdot\nabla S$ vanishing identically,

$$\frac{D}{Dt}\int_V \bar{v}.\bar{g}\; dV = 0\;,$$

where V is any fluid region bounded by a closed $\bar{g}$-surface Σ.

Proof:

$$\frac{D}{Dt}\int_V \bar{v}.\bar{g}\; dV = \frac{D}{Dt}\int_V \rho\,\bar{v}.\left(\frac{\bar{g}}{\rho}\right) dV,$$

$$= \int_V \rho\,\frac{D}{Dt}\left(\bar{v}.\frac{\bar{g}}{\rho}\right) dV,$$

(by Reynold's transport theorem)

$$= \int_V \left[\rho\,\frac{D\bar{v}}{Dt}\cdot\frac{\bar{g}}{\rho} + \rho\,\bar{v}.\left(\frac{\bar{g}}{\rho}.\nabla\right)\bar{v}\right] dV,$$

$$= \int_V (-\nabla(I+U) + T\nabla S).\bar{g}\; dV$$

$$+ \int \bar{v}.\,(\bar{g}.\nabla)\bar{v}\; dV\;,$$

$$= \int_V \nabla\cdot\left(\left(\tfrac{1}{2}|\bar{v}|^2 - I - U\right)\bar{g}\right) dV,$$

(since $\bar{g}.\nabla S = 0$)

$$= \int_\Sigma \left(\tfrac{1}{2}|\bar{v}|^2 - I - U\right)\bar{g}.\,\hat{n}\; d\Sigma \;=\; 0,$$

since $\bar{g}.\hat{n} = 0$ on the $\bar{g}$- surface Σ.

3. Remarks

From the present discussion we find that most of the conservation laws involving vorticity fields both in barotropic and non-barotropic flows follow from the properties of Helmholtz fields.

Since for non-barotropic flows ' $\bar{\omega} - \nabla\eta \times \nabla S$ ' is a Helmholtz field, replacing g by '$\omega - \nabla\eta \times \nabla S$' in theorem (5.28) we get the conservation law of generalized helicity. Note that the theorem (5.37) is a particular case of generalized circulation theorem. Theorem (5.38) is a generalization of the conservation law (5.20) and theorem (5.40) is a generalization of the result proved by Gaffet (1985): The total helicity in fluid region bounded by a closed vortex tube is conserved during motion of non-barotropic flows of an ideal fluid in which potential vorticity is indentically zero every where.

Chapter 6

DISCUSSION

Kinematics of rotational motion contains the essence of fluid dynamics. In particular classical hydrodynamics may be characterized by the kinematical statement of Kelvin's circulation theorem and in this way all the general properties of barotropic flows of inviscid fluids subject to extraneous forces will appear as certain purely kinematical theorems valid for arbitrary medium. As Mobbs (1981) has generalized almost all the conservation laws related to vorticity in barotropic flows to non-barotropic cases, the kinematical theorems valid in barotropic flows can be studied as special cases of more general ones. This is the basis of our studies contained in this thesis.

In the second chapter we have used the familiar Eulerian variational principle to derive generalized circulation theorem. In a homogeneous fluid, particles with the same velocity, density and entropy are indistinguishable and they may be interchanged without affecting the physically interesting properties of the system at all. Associated with the invariance of the action integral under a reshuffling of particles which leaves the velocity, density and entropy unaltered, we expect some fundamental invariants of motion and these are shown to be those implied by generalized circulation theorem. It is to be noted that unlike in Bretherton's (1970) treatment we are allowing the variations of entropy also.

In the third chapter we have suggested an Eulerian variational principle in a quadridimensional formalism using the field variables $\bar{p}$ and $\bar{s}$ defined by

$$\bar{p} = (p^0, p^1, p^2, p^3) = (\rho, \rho v^1, \rho v^2, \rho v^3)$$

and

$$\bar{s} = (s^0, s^1, s^2, s^3) = (\rho S, \rho v^1 S, \rho v^2 S, \rho v^3 S),$$

as functions of space-time coordinates. Note that in this vector notation the familiar conservation laws of mass and entropy take the simple forms,

$$\partial_\alpha p^\alpha = 0 \quad \text{and} \quad \partial_\alpha s^\alpha = 0.$$

The variations of these field variables are restricted by conditions conformable to the familiar constraints of fluid flows. We call the variations of these field variables as hydromechanical variations. It may be noted that the conditions used for defining the hydromechanical variations of p^α and s^α have the following implications: Condition (3.17) requires that for the variations δx^α tangential to the hypersurface H, the total variation of the complete matter and entropy flows contained on H shall vanish and this corresponds to the conservations of particle identity and particle entropy, as pointed out by Finlayson (1972a). Condition (3.18) is equivalent to $\delta(\partial_\alpha p^\alpha) = 0$ and $\delta(\partial_\alpha s^\alpha) = 0$, which require that no local variations of the sources of mass flux and entropy flux. We have shown that the hydromechanical variations of the action leads to Euler-Lagrange equations of motion and a generalization of Bernoulli equation. In this variational formulation we do not use any Lagrangian multipliers as in other Eulerian variational formulations like Herivel-Lin or Seliger-Whitham methods. It may be noted that though Zaslovskii and Perfil̆év (1969), Bretherton (1970), Wilhelm (1977) and Bampi and Morro (1984) have obtained equations of motion of ideal fluid flows without using Lagrangian multipliers, they had to apply hybrid methods-mixing both Lagrangian and Eulerian variations.

In the third chapter we have disscussed the invariance properties of the action (3.9) in connection with the hydromechanical variational principle. We have adapted Noether theorems to our variational principle suitably. Noether's first theorem is applied to the Galilean group of transformations to obtain the conservation laws of energy, impulse and angular momenta. To apply Noether's second theorem it is necessary to find transformation groups depending on arbitrary functions under which the action is div-invariant. This in general depends on the form of the Lagrangian. But whatever be the form of the Lagrangian, there is a transformation group leaving the action div-invariant; viz. those variations of the independent variables for which $\delta_0 p^\alpha = 0$ and $\delta_0 s^\alpha = 0$. We have characterized this group and obtained the associated conservation laws. As the differential operator defining Δx^α in theorem (4.31) needs to satisfy the side conditions (4.35) and (4.36), we have to use Lagrangian multipliers to derive identities equivalent to Noether identities. We have shown that when the Lagrangian takes the usual form, these identities correspond to 'Helmholtz-Zorawski criterion' (Truesdell, 1954) leading to generalized Helmholtz theorems. As generalized circulation theorem is implied by generalized Helmholtz theorem, it can also be considered to follow from Noether's theorem. The direct application of Noether's second theorem, without using Lagrangian multipliers, to derive generalized Helmholtz theorems is an open problem. The problem of identifying suitable transformation groups associated with the conservation laws of generalized helicity and generalized potential vorticity can also be investigated.

In chapter 5 we have used Cauchy's formula to define a solenoidal vector field $\bar{g}$ (Helmholtz field) which is a generalization of vorticity in barotropic flows and generalized vorticity in non-barotropic flows respectively. Using this mathematical concept we are able to prove some conservation laws involving $\bar{g}$-field from which the familiar results related to vorticity and generalized vorticity can be recovered by giving particular values to $\bar{g}$. We have obtained these conservation laws involving $\bar{g}$-fields with the help of the differential equation (5.26), Euler's equations of motion, equation of continuity and conservation of particle entropy.

This shows the importance of Cauchy's vorticity formula both in barotropic and non-barotropic flows. Truesdell's remark (1954): 'although both Stokes and Kirchoff appreciated the central importance of the Cauchy's vorticity formula in classical hydrodynamics, it is rarely given the prominence it deserves' is valid even today. The work presented in chapter 5 is an attempt to shed some more light on this topic.

It seems that the $\bar{g}$-fields are related to the Clebsch's representation of the velocity field in an Eulerian variational principle. We can see that the conservation laws associated with generalized vorticity are closely related to generalized Weber transformations (Serrin (1959), Mobbs (1981)),

$$\bar{v} = \nabla\varphi + \eta\nabla S + \bar{v}_o . \nabla\bar{\alpha},$$

where $\bar{v}_o$ and $\bar{\alpha}$ are Lagrangian variables.

It follows that $\bar{g} = \nabla A \times \nabla B$, where A and B are Lagrangian variables defines a class of Helmholtz fields. The exact relation between the $\bar{g}$-fields and Clebsch's potentials remains to be investigated.

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