Some Problems in Topology

ARITHMETIC IN VARIOUS GROWTHS OF TOPOLOGICAL SPACES AND SOME APPLICATIONS TO NUMBER THEORY

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CERTIFICATE

Certified that the work reported in the thesis entitled "Arithmetic in various growths of topological spaces and some applications to number theory" is a bona fide work done by Mrs. Mangalambal N.R., under my guidance and supervision in the Department of Mathematics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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CONTENTS

Chapter 0
INTRODUCTION

This thesis is a study, motivated by the work done by N. Hindman and others, of the extension of the semigroup operations in N, the discrete set of natural numbers; Z x Z, where Z is the discrete set of integers with componentwise operations of addition and multiplication and R, the set of real numbers considered with both the discrete and usual topologies, to their Stone-Čech compactifications βN, $\beta(Z \times Z)$ and βR (in the case when R is discrete) and $\dot{p}R$, the LMC-compactification (when R with usual topology is considered as a semitopological semigroup). Various properties applying the arithmetic on the growth $X^* = \beta X \setminus X$, have been discussed when X is N, Z x Z or R. We have also studied the general situation of E-completely regular spaces X and in particular when E is a topological field we have constructed the maximal E-compactification $\beta_E X$ in a manner analogous to βX.

A compactification of a topological space X is a compact space K together with an embedding $e: X \longrightarrow K$ with e(X) dense in K. We will identify X with e(X) and consider X as a subspace of K. The Stone-Čech compactification is that compactification of X in which X is embedded in such a way that every bounded, real-valued continuous function on X will extend continuously to the compactification and is denoted by βX. In 1930, Tychonoff discovered that those topological spaces which can be embedded in a compact Hausdorff space are precisely the completely regular (Hausdorff) spaces. This was essentially the beginning of the general study of Hausdorff compactifications, since we can obtain a compactification of a space by embedding it in a compact space and then taking its closure. Once one compactification has been obtained, others can generally be constructed as quotient space of it. Tychonoff's original embedding was into a product of closed intervals, using the set of bounded, continuous, real-valued functions as the indexing set of this product. By using an appropriate subset of this indexing set and proceeding in essentially the same way, any given Hausdorff compactification can be obtained. This technique was studied extensively by Čech [1937]. It was this study which apparently established the presently universal notation of βX for the compactification. Using entirely different techniques, Stone [1937] constructed a compactification equivalent to βX and showed that it had the same universal mapping property as that of βX. This construction was simplified by Gelfand and Kolmogoroff[1939] and we use for our purpose this mode of construction of βX. In this construction, βX is taken as the set of all Z-ultrafilters on X with the following topology. Let $\bar{Z} = \{p \in \beta X : Z \in p\}$. Then $\{\bar{Z} : Z$ is a zero-set in $X\}$ is a base for the closed sets in X. In particular, when X is a discrete space, every subset of X is a Zero-set so that $\{\bar{A} : A \subseteq X\}$ is a base for the closed sets (as well as a base for the open sets). (See [G;J] or [RC] for a detailed discussion of βX constructed in this way).

We also have the theory of compact right topological semigroups and in particular, of semigroup compactification. By a semigroup compactification, we mean a compact right topological semigroup which contains a dense continuous homomorphic image of a given semitopological semigroup. The classical example is the Bohr (or almost periodic) compactification (a, AR) of the usual additive real numbers R. Here AR is a compact topological group and $a : R \longrightarrow AR$ is a continuous homomorphism with dense image. An important feature of the Bohr compactification is the following universal mapping property which it enjoys: Given any compact topological group G and any continuous homomorphism $\phi : R \longrightarrow G$, there exists a continuous homomorphism $\psi : AR \longrightarrow G$ such that $\psi = \emptyset . a$.

Compactifications of semigroups can be produced in a variety of ways. We have, for our purpose used the method based on the Gelfand-Naimark theory of commutative C*-algebras. Compactifications of a semitopological semigroup S now appear as the spectra of certain C*-algebras of functions on S. There is the book [BE; JU; MI] which gives a very good account of the whole theory of topological semigroups and their compactifications. When we take S to be a separately continuous, completely regular and Hausdorff topological semigroup, $C_b(S)$, the space of continuous and bounded complex-valued functions on S, then βS, the Stone-Čech compactification of S is the space of continuous, multiplicative linear functionals on $C_b(S)$. βS is compact in the weak * topology and the Gelfand map $f \longmapsto \hat{f}$ defined by $\hat{f}(\mu) = \mu(f)$ is an isometric isomorphism of $C_b(S)$ onto $C(\beta S)$.

We have adopted this technique to study the LMC-compactification pR of R, the set of real numbers under usual topology, considered as a semi-topological semigroup and taking pR as a quotient space of βR. Neil Hindman has considered the unique left continuous extensions of ordinary addition and multiplication to βN, the Stone-Čech compactification of the (discrete) set N of positive integers. It was known previously that there exist associative left continuous operations on βN, but it was Glazer (GL) who observed that these operations can be defined in terms of ultrafilters. (By left continuous we mean that f_x defined by $f_x(y) = x*y$ is continuous). Glazer proved directly that Galvin's almost translation invariant ultrafilters exist and obtained as a corollary the proof of the finite sum theorem. Glazer's observation was that an almost translation invariant ultrafilter is exactly an idempotent with respect to an operation in βN which extends ordinary addition on N.

Hindman has extensively studied the problem of extending an operation on a discrete semigroup S to its Stone-Čech compactification βS and the relationship between these extended operations. He has shown that these extensions and their interrelation have been a useful tool in combinatorial partition theory (Ramsey theory). Hindman prefers to work with ultrafilters. To mention a few of Hindman's work, he has proved that there is a multiplicative idempotent in the topological closure of the set of additive idempotents [HI_1] and that there are no simultaneous additive and multiplicative idempotents. He has presented several results about whether $p+q = r.s$ is possible, where at least one of p,q,r,s is in N and others in $\beta N\setminus N$.

For the elementary definitions and results in topology, reference may be made to [WI]; for theory of ultrafilters, to [GJ], [WA] and [CO; NE].

In chapter 1, we define a new kind of types of ultrafilters on N, called S-types, 'S' standing for semigroup, similar to the types of ultrafilters on N. Types of ultrafilters (on ω) were first defined and considered by W. Rudin [RU]. Frolik [FR_1], [FR_2] uses them in connection with the non-homogeneity of $\beta\omega \setminus \omega$. Hindman and Strauss [H; S] have shown that the only topological and algebraic copies of N* to be found in N* are the trivial ones, namely k.N*, $k \in N$. We have used this fact to define S-types on N* which satisfy many properties analogous to those satisfied by types and relative types. The fundamental properties of the Rudin-Keisler order were studied by M.E. Rudin [RU] and by H.J. Keisler [KE]. We have introduced an order relation among S-types similar to Rudin-Keisler partial order on types of ultrafilters. Though the properties of S-types seem exactly similar to that of types, the members of S-types are different and also the way they occur in the corresponding results is different. Finally, we have shown using the restricted distributive law in βN that the collection of S-types form a semigroup under extended addition in βN.

In chapter II, we consider the space Z x Z, where Z is the discrete set of integers. We have extended as in βN, the componentwise addition and multiplication in Z x Z to $\beta(Z \times Z)$ which makes $(\beta(Z \times Z), +)$ and $(\beta(Z \times Z), .)$ semigroups. We have been able to prove that the natural map from $\beta(Z \times Z)$ to $\beta Z \times \beta Z$ is such that the component-wise addition and multiplication in $\beta Z \times \beta Z$ result from the extended operations of + and . in $\beta(Z \times Z)$. Also the operations in $\beta Z \times \beta Z$ are not distributive. Considering $Z \times Z$ as the Gaussian integers, we have extended the product '**x**' in $Z \times Z$ to $\beta Z \times \beta Z$ and have shown that this extension of the product '**x**' is non-associative. We have also attempted some combinatorial results in $\beta Z \times \beta Z$, analogous to those in βN [HI_1].

In chapter III, we consider the discrete set R of real numbers. Here we have first extended the ordinary addition and multiplication in R to βR which make $(\beta R,+)$ and $(\beta R,.)$ semigroups. We have shown that in contrast to $(\beta N,+)$ and $(\beta N,.)$ [HI_2], βR has solutions to equations of the form $p+q = p.n$, $p+m = p.q$, $p+q = p.q$, where, $p,q \in \beta R \setminus R$ and $m,n \in R$. We have defined the notion of α-remote points for an infinite cardinal α, in a discrete topological field X with $|X| \geqslant \alpha$ and applied the arithmetic defined in βX to the class of α-remote points in βX.

In chapter IV, we have studied the LMC-compactification (p, pR) of R, the set of real numbers with usual topology, considered as a semitopological semigroup. It has been proved [BA; BU] that when R has the usual topology, the ordinary addition and multiplication in R can be extended to βR if and only if LMC(R) = $C_b(R)$. Here pR has been constructed as the quotient space of βR in terms of Z-ultrafilters on R. Also, we have obtained solutions to equations of the form $\rho + \zeta = \beta \cdot \eta$, where at least one of ρ, ζ, β, η is in R and others in $pR \setminus R$. In the case of R, we have the equalities LUC(R) = CK(R) = K(R) = WLUC(R) = LMC(R) [BE;JU;MI]. So the corresponding canonical compactifications are the same so that all the properties that we have studied in pR hold good in these compactifications.

In the fifth chapter, we have shown that remote and non-remote points exist in $pR \setminus R$. We have obtained results analogous to those in chapter IV when we particularly consider the remote and non-remote points.

In chapter VI, we have defined k-uniform Z-ultrafilters in pR, where the definition is analogous to k-uniform ultrafilter [CO; NE]. We have obtained results regarding the ideal structure of the collection of k-uniform Z-ultrafilters in R, analogous to that for a discrete space X.

We have the Appendix A which includes the concept of E-completely regular spaces defined by Engelking and Mrówka [EN; MR]. We have considered E to be a topological field and obtained the maximal E-compactification $\beta_E X$ as the collection of all E-Z-ultrafilters on X with the suitable topology. It turns out that the E-compactification $\beta_E X$ plays a role within a framework that runs parallel to that played by the Stone-Čech compactification βX of a topological space X. We can study situations in $\beta_E X$ analogous to that in βX, as studied in previous chapters. However, we do not embark on it since it involves, among other things, a lot of spade work.

Chapter I

S-TYPES OF ULTRAFILTERS ON N [@]

§ 1.0. Introduction

In [GL] Glazer has defined addition '+' and multiplication '.' in βN, the Stone-Čech compactification of N, the discrete set of natural numbers, in the language of ultrafilters on N. Hindman $[HI_1]$ has proved that these operations + and . are left continuous, associative operations on βN which uniquely extend the ordinary addition and multiplication on N. (By left continuous we mean (in the case of addition) that the function $\lambda_p : \beta N \longrightarrow \beta N$ defined by $\lambda_p(q) = p+q$ is continuous for each $p \in \beta N$. The "topological center" consists of those points for which ρ_x is also continuous, where $\rho_x(p) = p+x$. Similar is the case with multiplication). It is well known that (see [G;J]) any infinite closed subspace of N^* contains a topological copy of all of βN, where $N^* = \beta N \setminus N$. It was then a natural question raised by Van Douwen $[HI_1]$ as to whether there are topological and algebraic copies of $(\beta N, +)$ in N^*. Hindman and Strauss [H;S] have shown that the only topological and algebraic copies of N^* to be found in N^* are the trivial ones, namely $k. N^*$, for $k \in N$.

[@] An earlier version of this chapter has been published in Far East J. Math. Sci. Special Volume (1997), Part I, 75-82.

Types of ultrafilters (on ω) were first defined and considered by W. Rudin [RU]. Frolik [FR_1] [FR_2], uses them in connection with establishing the non-homogeneity of $\beta\omega\setminus\omega$. The fundamental properties of the Rudin-Keisler order were studied by M.E. Rudin [RU] and by H.J. Keisler [KE].

We combine the above two concepts to define a new kind of types of ultrafilters on N called the S-types, 'S' standing for semigroup. In section 1.2 we introduce S-types using the left-continuous extensions in βN, of operations in N and using Strauss's result on the ultrafilters on N. It is known that though there are 2^c types of ultrafilters on N in N* and 2^c N* types [WA]; however we prove that there are only countably infinite number of points in N* of each S-type and 2^c S-types of points in N*. We have also obtained that if p is a P-point, then every member of the S-type of p is a P-point and that there are 2^c S-types of P-points. Similarly for non-P-points.

In section 1.3 we introduce relative S-types analogous to the relative types of ultrafilters on N. We have shown that though the relative S-types have properties analogous to the relative types, any S-type is produced by at most $\aleph_0$ relative S-types and any S-type produces 2^c S-types and as a corollary we get the known result that

N^* is not homogeneous. We have also introduced 'S-orbit' similar to orbit in types of ultrafilters.

In section 1.4, as in the case of types and N^*-types of points in βN, we have shown that the producing relation induces total order ' $>_p$' on the set $\mathcal{T}_s[p, N^*]$ of relative S-types of p. We use the family $\{>_p : p \in N^*\}$ to define a partial order '>' on the set T of all S-types in such a way that the restriction of '>' to $\mathcal{T}_s[p,N^*]$ is $>_p$, for each p. Finally, we have shown using the restricted distributive law in βN that the collection of S-types form a semigroup under 'addition' of S-types, where addition here is the extended addition in βN.

§ 1.1. Preliminaries.

We take βN, the Stone-Čech compactification of N, the (discrete) set of natural numbers to be the set of ultrafilters on N with the following topology: Let $\bar{A} = \{p \in \beta N : A \in p\}$. Then, $\{\bar{A} : A \subseteq N\}$ is a base for the closed sets of βN. See [G;J] or [HI_1] for further details.

1.1.1. Definition [HI_1]. Let $A \subseteq N$ and $x \in N$

$$A - x = \{y \in N : y + x \in A\}$$
$$A/_x = \{y \in N : y . x \in A\}$$

Let $p, q \in \beta N$

$$p+q = \{A \subseteq N : \{x \in N : A - x \in p\} \in q\}.$$
$$p.q = \{A \subseteq N : \{x \in N : A/_x \in p\} \in q\}.$$

1.1.2. Theorem $[HI_1]$.

(a) The operation '+' on βN is the unique extension of ordinary addition on N which is left-continuous and has the property that addition on the right by any member of N is continuous. If p or q is in $\beta N \setminus N$, then so is $p+q$.

(b) The operation '.' on βN is the unique extension of ordinary multiplication on N which is left-continuous and has the property that multiplication on the right by any member of N is continuous. If p or q is in $\beta N \setminus N$, then so is $p.q$.

1.1.3. Remark. In βN, the distributive laws fail badly. However, a special case does hold.

1.1.4. Lemma $[HI_3]$. Let $p,q \in \beta N$ and $m \in N$. Then, $(p+q).m = p.m + q.m$.

1.1.5. Lemma [H;S]. Let $\emptyset$ be a continuous one-to-one homomorphism from N^* to N^* and let $e + e = e \in N^* \setminus K(\beta N)$, where $K(\beta N)$ is the smallest two sided ideal in $(\beta N, +)$. There do not exist $m,n \in N$ such that $\emptyset(m+e) = -n + \emptyset(e)$ or $\emptyset(-m+e) = n + \emptyset(e)$.

1.1.6. Theorem [H;S]. Assume that $\emptyset$ is a continuous one-to-one homomorphism from N^* into N^*. There is some $k \in \beta N$ such that for all $p \in N^*$, $\emptyset(p) = k.p$.

1.1.7. Conclusion [H;S]. The only algebraic-topological copies of N^* in N^* are the trivial ones, namely $k.N^*$ for $k \in N$.

1.1.8. Definition (Types and N^*-types) [WA]. Every permutation σ of N extends to a homeomorphism $\beta(\sigma):\beta N \rightarrow \beta N$ and the restriction of $\beta(\sigma)$ to N^*, denoted by σ^* is an automorphism of N^*. For a pair of points p and q of N^*, define $p \sim q$ if $\sigma^*(p) = q$ for some permutation σ. Then '$\sim$' is an equivalence relation. Let T be the set of equivalence classes. Let $\tau: N^* \longrightarrow T$ be the function which assigns to each free ultrafilter on N, its equivalence class. The elements of T are called types of ultrafilters. If $t = \tau(p)$, then t is called the type of p and p is said to be of type t.

Two points of N^* are said to be of the same N^*-type if there is an automorphism of N^* which maps one to the other. Clearly, if p,q are of the same type, they are of the same N^*-type.

§ 1.2. S-types.

1.2.1. Definition. For a pair of points $p,q \in N^*$, define $p \sim q$ if and only if $\sigma^*(p) = q$ for some auto-homeomorphism σ^* of N^*. [The only auto-homeomorphism on N^* are the maps $p \longmapsto m \cdot p$, $m \in N$ so that we may take $q = m \cdot p$ for some $m \in N$]. Evidently, '$\sim$' is an equivalence relation.

Let T be the set of equivalence classes. Let $\tau_s : N^* \longrightarrow T$ be the function which assigns to each member of N^*, its equivalence class. The elements of T are called S-types of ultrafilters. If $t = \tau_s(P)$, then t is called the S-type of p and p is said to be of S-type t.

1.2.2. **Remark.** In the name S-type, 'S' stands for semi-group. Also, for $m \in N$, $\sigma_m^*(p) = m.p$ determines a homeomorphism of N^* onto N^*. So, an S-type is contained in a type and an N^*-type, but not conversely.

1.2.3. Theorem [WA]. There are 2^c types of ultrafilters in N^* and there is a dense set of c ultrafilters of each type.

1.2.4. Theorem [WA]. There are 2^c N^*-types of points in N^* and N^* contains a dense subset of each type.

1.2.5. **Result.** There are 2^c S-types of points in N^* and there are countably infinite points in N^* of each S-type.

Proof: For $p \in N^*$, $\tau_s(p)$ is the equivalence class containing all members of N^* that are equivalent to p under some auto-homeomorphism σ^* on N^* given by $\sigma^*(p) = m.p$ for $m \in N$. So each equivalence class can contain only a countably infinite number of members of N^*.

Thus there are $\aleph_0$ points in N^* of each S-type. But $|N^*| = 2^c$. So there must be 2^c S-types.

1.2.6. Definition. A point of a topological space is called a P-point if every G_δ containing the point is a neighbourhood of the point. Equivalently, a point is a P-point if and only if every zero-set containing the point is a neighbourhood of the point.

1.2.7. Theorem [WA]. N^* has a dense set of 2^c P-points and dense set of 2^c non-P-points.

1.2.8. **Result.** There are 2^c S-types of P-points in N^* and if p is a P-point, then every member of $\mathcal{T}_s(p)$ is a P-point.

Proof: We first show that if p is a P-point of N^*, then so is m.p for every $m \in N$. For this, we prove that every zero-set $Z(\beta(f))$ containing m.p is open in βN. Every zero-set in βN is a countable intersection of closures in βN of zero-sets in N.

So, $Z(\beta(f)) = \bigcap_{n=1}^{\infty} cl_{\beta N} Z_n$, where Z_n's are zero-sets in N.

Hence, $m.p \in \bigcap_{n=1}^{\infty} cl_{\beta N} Z_n \Longrightarrow m.p \in cl_{\beta N} Z_n$ for every n.

Thus, $Z_n \in m.p$ for every n and $Z_{n/m} \in p$ for every n.

Hence $p \in \bigcap_{n=1}^{\infty} cl_{\beta N}(Z_{n/m})$, where $\bigcap_{n=1}^{\infty} cl_{\beta N}(Z_{n/m})$ is open. So, $\bigcap_{n=1}^{\infty} cl_{\beta N} Z_n$ must be open by the homeomorphism $p \longmapsto m.p.$ i.e., $Z(\beta(f))$ is open. Thus $m.p$ is a P-point. Thus all elements of $\tau_s(p)$ are P-points. N* contains a dense set of 2^c P-points. So there are 2^c S-types of P-points.

1.2.9. Result. There are 2^c S-types of non-P-points and if p is a non-P-point, then so is every member of $\tau_s(p)$.

§ 1.3. Relative S-types

1.3.1. Definition. Any iso-homeomorphic copy of N in N* is C*-embedded in βN. By an iso-homeomorphic copy of N, we mean an algebraic-topological copy of N. When the iso-homeomorphism is from X onto X, we call it auto-homeomorphism. Therefore, if X is such a copy of N in N*, then $cl_{\beta N} X \approx \beta N$. If we put $X^* = cl_{\beta X} X \setminus X$, then $X^* \approx N^*$. A point p in X* must then have S-type as a point of X* as well as a point of N*. Let h be an iso-homeomorphism of X* onto N*. Define the S-type of p relative to X to be $\tau_s(h(p))$ and denote this relative S-type by $\tau_s(p,X)$. This definition is independent of the iso-homeomorphism chosen, since for any other iso-homeomorphism g of X* onto N*, $g.h^{-1}$ sends h(p) to g(p) and is an auto-homeomorphism of N* so that g(p) and h(p) are of the same S-type as p.

1.3.2. Convention. By a copy X of N in N* we mean an iso-homeomorphic copy X of N in N*.

1.3.3. Result. Let X and Y be copies of N in N*. Then,

(a) If Y is contained in X and $p \in X^* \cap Y^*$, then $\tau_s(p,X) = \tau_s(p,Y)$.

(b) If p and q belong to X* and Y* respectively, then, $\tau_s(p,X) = \tau_s(q,Y)$ if and only if there is an iso-homeomorphism h of X* onto Y* such that h(p) = q.

(c) If h is an auto-homeomorphism of N* and p belongs to X*, then $\tau_s(p,X) = \tau_s(h(p), h[X])$.

Proof:

(a) Let g be the iso-homeomorphism of Y* onto X* such that g(p) = p. Let h be an iso-homeomorphism of X* onto N*. Then, $\tau_s(p,Y) = \tau_s(h.g(p)) = \tau_s(h(p)) = \tau_s(p,X)$.

(b) Let $\tau_s(p,X) = \tau_s(q,Y)$. Then there exist iso-homeomorphisms f and g of X* and Y* respectively onto N* such that $\tau_s(f(p)) = \tau_s(g(q))$. Then, there exists an auto-homeomorphism k of N* which sends f(p) to g(q). Then, $h = g^{-1}.k.f$ is the required iso-homeomorphism. The converse follows from the definition of the relative S-type.

(c) Follows from (b).

1.3.4. Definition.

(1) For an infinite subset S of N*, define, $\tau_s[p,S] = \{\tau_s(p,X) : X \subseteq S,\ X \approx N\}$ i.e., $\tau_s[p,S]$, is the set of relative S-types of p which occur relative to copies of N contained in S. Then $\tau_s[p, N^*]$ is invariant. Also, $\tau_s[p,N] = \{\tau_s(p)\}$.

(2) If p is a point of N*, we say that a S-type t produces $\tau_s(p)$ or $\tau_s(p)$ is produced by t if $\tau_s(p,X) = t$ for some copy X of N in N*. Thus the set $\tau_s[p, N^*]$ of relative S-types of a point p of N* is the set of S-types which produce the S-type t if there exists a copy X of N in N* and a point p of S-type t in X* such that $\tau_s(p,X) = s$.

1.3.5. Result. Any S-type is produced by atmost $\aleph_0$ S-types and any S-type produces 2^c S-types.

Proof: Consider the countable partition of N* into a union of k. N*, $k \in N$. i.e., $N^* = \bigcup_{k=1}^{\infty} k.\ N^*$. Let $X_k = k.\ N^*$. Then each X_k, $k \in N$ is an iso-homeomorphic copy of N*. Consider X_k. If S_k is countable discrete subspace in X_k, then S_k will be a copy of N in N*, and $cl_{\beta N}\ S_k \approx \beta N$. Since $|X_k| = 2^c$, X_k should contain 2^c such sets S_k. So, if t is any given S-type, say

$t = \tau_s(q)$ for some $q \in N^*$, then $cl_{\beta N} S_k$ should contain a point p_k such that $\tau_s(p_k, S_k) = \tau_s(q) = t$. Thus $\tau_s(p_k)$ is produced by t, by definition 1.3.4. Since only $\aleph_0$ of such points p_k can be of any given S-type, t must produce 2^c S-types.

Now, a given S-type t is produced by a S-type r exactly when there is an ultrafilter p of S-type t and a copy X of N in N* with p in X* and the S-type of p relative to X is r. i.e., $\tau_s(p,X) = r$, where $\tau_s(p)=t$. Let $r = \tau_s(q)$. Now, there are only $\aleph_0$ members in N* having the same S-type t (namely k.p, $k \in N$). For any of these $\aleph_0$ points p_n (where $\tau_s(p_n) = \tau_s(p) = t$) we have $\tau_s(p_n,X) = \tau_s(q) = r$, where $X \approx N$ and $p_n \in X^*$. i.e., $\tau_s(h(p_n)) = \tau_s(q)$, where $h: X^* \longrightarrow N^*$ is an iso-homeomorphism. So, $q = k.p_n$ for some $k \in N$, since there exists some auto-homeomorphism g of N* that sends $h(p_n)$ to q. But h and g being iso-homeomorphisms, we would have $q = k.p_n$, for some $k \in N$. This is true for any $q \in N^*$. So there can be only countably many copies X_n of N in N* and $p_n \in X_n^*$ with $\tau_s(p_n, X_n) = r = \tau_s(q)$. So, t is produced by at atmost $\aleph_0$ relative S-types.

1.3.6. Corollary. N* is not homogeneous.

Proof. Let h be a homeomorphism of N* onto N* and p and q be points of N* such that h(p)=q. If X is any copy of N in N* having p as a limit point, then we have $\tau_s(p,X) = \tau_s(q,h(X))$ from result 1.3.3. Thus the sets $\tau_s[p,N^*]$ and $\tau_s[q,N^*]$ of relative S-types are identical. The family $\{\tau_s[p, N^*]: p \in N^*\}$ of all such sets of relative S-types covers the set T of S-types. However, $|T| = 2^c$ and that of each member of this cover is atmost $\aleph_0$, since each S-type is produced by atmost $\aleph_0$ S-types. So, there must exist points r and s of N* such that $\tau_s[r, N^*] \neq \tau_s[s, N^*]$. But then no homeomorphism on N* can map r to s and so N* is not homogeneous.

1.3.7. Definition. The S-orbit of a point p in N* is the set of points of N* which are images of p under auto-homeomorphisms of N*.

1.3.8. Result. For any point p in N*, there are 2^c points of N* which cannot be mapped to p by auto-homeomorphisms of N*.

Proof. The S-orbits of two points p and q of N* under auto-homeomorphisms of N* are disjoint exactly when no auto-homeomorphisms carries p to q. Thus the set of all such S-orbits decomposes N* into a union of disjoint sets. Since any two points belonging to the same S-orbit have the same set of atmost $\aleph_0$ relative S-types, there must be 2^c distinct S-orbits.

§ 1.4. Order relation in S-types.

1.4.1. Result. If X and Y are copies of N in N*, then the set $Z = (X \cap Y) \cup (X^* \cap Y) \cup (X \cap Y^*)$ is a copy of N in N*, $\text{cl}\, Z = \text{cl}\, X \cap \text{cl}\, Y$ and $Z^* = X^* \cap Y^*$.

Proof. The set Z is discrete since each of the three sets in the union is discrete and no point belonging to any one of the sets can be accumulation point of the other two sets. Also, we have $p.m+q.m=(p+q).m$. The equalities hold because a point belonging to both cl X and cl Y must belong to the closure of one of the three sets making up Z.

1.4.2. Remark. If t_1 and t_2 are both in $\tau_s[p, N^*]$, as a consequence of the preceeding result, either two S-types are equal or one produces the other. Just similar to the situation that the producing relation induces a total order on the set $\tau[p, \beta N]$ of relative types of a point p, here also we can see that the producing relation induces a similar total order on the set $\tau_s[p, N^*]$ of relative S-types of a point p.

The following results are analogous to those by Z. Frolik [FR_3].

1.4.3. Definition. Write $t_1 >_p t_2$ if t_2 produces t_1 and define $t_1 \geqslant_p t_2$ if and only if $t_1=t_2$ or $t_1 >_p t_2$, where t_1, t_2 are relative S-types of p in N*.

1.4.4. Result. The relation $>_p$ is a total order on $\mathcal{T}_s[p, N^*]$. Here two relative S-types of p are either equal or one produces the other.

Proof. The arguments are identical to that for types in βN. Here instead of considering countable discrete subspaces X_1, X_2 of βN, we have taken copies X_1, X_2 of N in N^*.

1.4.5. Remark. As in the case of types, each of the total orders $>_p$ is defined only on the set $\mathcal{T}_s[p, N^*]$ of relative S-types of p. We will use the family $\{>_p : p \in N^*\}$, to define a partial order > on the set T of all S-types in such a way that the restriction of > to $\mathcal{T}_s[p, N^*]$ is $>_p$ for each p.

1.4.6. Definition. For two S-types t_1 and t_2, define $t_1 > t_2$ if $t_1 >_p t_2$ for some p. i.e., $t_1 > t_2$ if and only if t_1 is produced by t_2. Then the relation > is well-defined. i.e., for any two points p and q of N^*, the relation $>_p$ coincides with $>_q$ if and only if $\mathcal{T}_s[p, N^*] = \mathcal{T}_s[q, N^*]$: suppose t_1 and t_2 belong to $\mathcal{T}_s[p,N^*] \cap \mathcal{T}_s[q, N^*]$ and that $t_1 >_p t_2$. Then there are copies of N, say X_1 and X_2 in N^* such that $\mathcal{T}_s[p,X_1]=t_1$ and $\mathcal{T}_s[p,X_2] = t_2$ and X_2 is contained in X_1^*. Since t_1 belongs to $\mathcal{T}_s[q, N^*]$, there is a copy Y of N in N^* such that $\mathcal{T}_s(q,Y) = t_1$. By result 1.3.3, there is an iso-homeomorphism h of X_1^* onto Y^* such that $h(p) = q$.

Then $\tau_s(q,h[X_2]) = t_2$ and since $h(X_2)$ is contained in Y^*, $t_1 >_p t_2$.

1.4.7. Definition. If t_1 and t_2 are S-types, write $t_1 \geqslant t_2$ if and only if $t_1 = t_2$ or $t_1 > t_2$.

1.4.8. Result. The relation $>$ is a partial order on the set of S-types.

Proof: We proceed as in the case of types of βN except for the fact that X_1, X_2, Y_2, Y_3 are copies of N in N^* and the iso-homeomorphism h must send p to q.

1.4.9. Result. If t_1 and t_2 are S-types, then so is t_1+t_2, where + is the extended addition in βN.

Proof. Let $t_1 = [m.p]$, $m \in N$, $t_2 = [m.q]$, $m \in N$, where $p,q \in N^*$. We have the restricted distributive law in βN given by $(p+q).m = p.m + q.m$ by Lemma 1.1.4. Hence,

$$[p.m] + [q.m] = [\,p.m + q.m]$$
$$= [(p+q).m]$$

i.e., $t_1+t_2 = [(p+q).m]$ is also a S-type.

1.4.10. Result. If t_1, t_2, t_3 are S-types, then the addition of S-types as defined in 1.4.9 is associative.

Proof: Let $t_1 = [p.m]$, $m \in N$, $t_2 = [q.m]$, $m \in N$, $t_3 = [r.m]$, $m \in N$, where $p,q,r \in N^*$. Then,

$$
\begin{aligned}
(t_1+t_2)+t_3 &= ([p.m] + [q.m]) + [r.m] \\
&= [(p+q).m] + [r.m] \\
&= [((p+q)+r).m] \\
&= [(p+(q+r)).m], \text{ since addition in } \beta N \text{ is associative } [HI_1] \\
&= [p.m] + [(q+r).m] \\
&= [p.m] + ([q.m] + [r.m]) \\
&= t_1 + (t_2+t_3).
\end{aligned}
$$

Conclusion. The S-types in N^* is a semigroup under the extended addition in βN and form a quotient set in βN.

Chapter-II

ARITHMETIC IN $\beta Z \times \beta Z$ [@]

2.0. Introduction

If γ and δ are cardinals, then Blass [BL] has considered, the map $\emptyset: \beta(\gamma \times \delta) \longrightarrow \beta(\gamma) \times \beta(\delta)$, the Stone-extension of the natural embedding (in fact inclusion) of $\gamma \times \delta$ into $\beta(\gamma) \times \beta(\delta)$ in relation to the products of filters to prove some topological properties. In this chapter, we in particular consider the product $Z \times Z$, where Z is the set of integers with discrete topology. We have component-wise addition + and multiplication '.' in $Z \times Z$ which we have extended to $\beta(Z \times Z)$, the Stone-Čech compactification of $Z \times Z$, making $\beta(Z \times Z)$, semigroups under + and . . The extension of the operations is done in a way that is similar to that of βN (Chapter I).

In section 2.1, we have given the necessary definitions and results extending the componentwise addition '+' and multiplication '.' in $Z \times Z$ to $\beta(Z \times Z)$. Here we have shown that the Stone extension of the natural map $\emptyset: \beta(Z \times Z) \longrightarrow \beta Z \times \beta Z$ accounts for the componentwise addition + and multiplication in $\beta Z \times \beta Z$. Also, considering $Z \times Z$ as the Gaussian integers, we have proceeded to extend the product 'x' in $Z \times Z$ to $\beta Z \times \beta Z$ and have shown that this extension is non-associative.

[@] Some results of this chapter were presented in the National Conference at Pollachi- 'Recent Trends in Topology', March 2-3, 1997.

In section 2.2 we have attempted some combinatorial results similar to that of βN [HI_1]. We have been able to prove that like βN, the distributive law fails in $\beta Z \times \beta Z$ with componentwise addition and multiplication.

§ 2.1. Extension of +, . and x to $\beta Z \times \beta Z$

We know that, βZ, the Stone-Čech compactification of Z is the set of ultrafilters in Z, each point $x \in Z$ being identified with the principal ultrafilter, $\hat{x} = \{A \subseteq Z : x \in A\}$. For $A \subseteq Z$, we let $\bar{A} = \{p \in \beta Z : A \in p\}$. Then the set $\{\bar{A} : A \subseteq Z\}$ forms a basis for the closed sets (as well as a basis for the open sets of βZ). The operations + and . on Z extend uniquely to βZ so that $(\beta Z, +)$ and $(\beta Z, .)$ are left topological monoids with $(Z, +)$ and $(Z, .)$ respectively contained in their topological centres.

We have Z x Z with discrete topology and componentwise addition + and multiplication . and also another product 'x'. i.e., if $(x_1, y_1), (x_2, y_2) \in Z \times Z$, then we have,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2,\ y_1 \cdot y_2)$$
$$(x_1, y_1) \times (x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + y_1 x_2)$$

$\beta(Z \times Z)$ is the set of all ultrafilters on $Z \times Z$. As in βZ, the addition + and multiplication . in $Z \times Z$ can be extended to $\beta(Z \times Z)$ which make $(\beta(Z \times Z), +)$ and $(\beta(Z \times Z), .)$, monoids, with respective identities. We show that the extension of the product 'x' in $Z \times Z$ to $\beta(Z \times Z)$ and thereby to $\beta Z \times \beta Z$ is non-associative.

2.1.1. Definition. Let $C \subseteq Z \times Z$ and $(x,y) \in Z \times Z$. Then,

$$C - (x,y) = \{(a-x,\ b-y) \in Z \times Z : (a,b) \in C\}.$$

$$C/(x,y) = \{(a,b) \in Z \times Z : (a,b).(x,y) \in C\}.$$
$$= \{(a,b) \in Z \times Z : (ax,by) \in C\}.$$

Let $P,Q \in \beta(Z \times Z)$. Define,

$$P+Q = \{C \subseteq Z \times Z : \{(x,y) \in Z \times Z : C-(x,y) \in P\} \in Q\}.$$
$$P.Q = \{C \subseteq Z \times Z : \{(x,y) \in Z \times Z : C/(x,y) \in P\} \in Q\}.$$

2.1.2. Result. The operations + and . are associative left-continuous operations on $\beta(Z \times Z)$. If P or Q is in $\beta(Z \times Z) \setminus Z \times Z$, then so are P+Q and P.Q.

Proof: We shall prove this for '+' only. The proof for '.' is identical.

Let $P,Q \in \beta(Z \times Z)$. We first show that $P+Q \in \beta(Z \times Z)$. Trivially, $\emptyset \notin P+Q$. Let $C,D \in P+Q$. Then,

$$\{(x,y) \in Z \times Z : C-(x,y) \in P\} \in Q \text{ and}$$
$$\{(x,y) \in Z \times Z : D-(x,y) \in P\} \in Q.$$

Therefore,

$\{(x,y) \in Z \times Z : C-(x,y) \in P\} \cap \{(x,y) \in Z \times Z : D-(x,y) \in P\} \in Q.$

But $\{(x,y) \in Z \times Z : C-(x,y) \in P\} \cap \{(x,y) \in Z \times Z : D-(x,y) \in P\}$

$= \{(x,y) \in Z \times Z : (C \cap D) - (x,y) \in P\}$

So $\{(x,y) \in Z \times Z : (C \cap D) - (x,y) \in P\} \in Q.$ Thus $C \cap D \in P+Q.$

Let $C \subseteq Z \times Z$ such that $C \notin P+Q.$ Then,

$\{(x,y) \in Z \times Z : C-(x,y) \in P\} \notin Q.$ Since $Q \in \beta(Z \times Z)$, this means that $[(Z \times Z) \setminus \{(x,y) \in Z \times Z : C-(x,y) \in P\}] \in Q.$

But $(Z \times Z) \setminus \{(x,y) \in Z \times Z : C-(x,y) \in P\}$

$= \{(x,y) \in Z \times Z : ((Z \times Z) \setminus C) - (x,y) \in P\}$

Thus, $\{(x,y) \in Z \times Z : ((Z \times Z) \setminus C) - (x,y) \in P\} \in Q.$

i.e. $(Z \times Z) \setminus C \in P+Q.$ Thus $P+Q \in \beta(Z \times Z)$.

As in βN, we can prove that the function $f_P : \beta(Z \times Z) \longrightarrow \beta(Z \times Z)$ given by $f_P(Q) = P+Q$ is continuous, both the addition as well as multiplication are associative and that if P or Q is in $\beta(Z \times Z)$, then so are P+Q and P.Q. The proofs are identical to those for βN [HI_1]. Also, these operations + and . on $\beta(Z \times Z)$ are the respective unique extensions of componentwise addition and multiplication on Z x Z, which are left-continuous and have the property, component-wise addition (respectively multiplication) on the right by any member of Z x Z is continuous. (Here again proofs are identical to those for βN[HI_1]).

2.1.3. Remark. There is a natural map $\emptyset : \beta(Z \times Z) \longrightarrow \beta Z \times \beta Z$ which is the Stone-extension of the natural embedding of $Z \times Z$ into $\beta Z \times \beta Z$ which accounts for the componentwise addition and multiplication in $\beta Z \times \beta Z$, which we can see from the following results.

2.1.4. Construction. We construct a map $\emptyset : \beta(Z \times Z) \longrightarrow \beta Z \times \beta Z$ as given below.

Let $P \in \beta(Z \times Z)$. Then P is an ultrafilter on $Z \times Z$. Consider, $\{A_\alpha \times B_\alpha \in P : A_\alpha, B_\alpha \subseteq Z\}$.
Let $\mathcal{A}_p = \{A_\alpha \subseteq Z: A_\alpha \times B_\alpha \in P\}$. We can show that $\mathcal{A}_p \in \beta Z$.

(a) $\emptyset \notin \mathcal{A}_p$.

(b) Let $A_\alpha, A_\beta \in \mathcal{A}_p$. Then $A_\alpha \times B_\alpha \in P$ and $A_\beta \times B_\beta \in P$. Since $P \in \beta(Z \times Z)$, $(A_\alpha \times B_\alpha) \cap (A_\beta \times B_\beta) \in P$. i.e., $(A_\alpha \cap A_\beta) \times (B_\alpha \cap \beta_\beta) \in P$. So, $A_\alpha \cap A_\beta \in \mathcal{A}_p$.

(c) Suppose that $\emptyset \neq A \subseteq Z$ such that $A \notin \mathcal{A}_p$. Then, $A \times B_\alpha \notin P$ for any $B_\alpha \subseteq Z$. Choose any $B_\alpha \subseteq Z$ so that $A \times B_\alpha \notin P$. Since $P \in \beta(Z \times Z)$, this means that $(Z \times Z) \setminus (A \times B_\alpha) \in P$. Now, $(Z \times Z) \setminus (A \times B_\alpha) = (A \times (Z \setminus B_\alpha)) \cup ((Z \setminus A) \times B_\alpha) \cup ((Z \setminus A) \times (Z \setminus B_\alpha))$. But $A \times (Z \setminus B_\alpha) \notin P$. So, either $(Z \setminus A) \times B_\alpha \in P$ or $(Z \setminus A) \times (Z \setminus B_\alpha) \in P$. In either case, $Z \setminus A \in \mathcal{A}_p$.

In a similar manner, we obtain,

$$\mathcal{B}_p = \left\{ B_\alpha \subseteq Z : A_\alpha \times B_\alpha \in P \right\} \in \beta Z$$

Thus,

$$(\mathcal{A}_p, \mathcal{B}_p) \in \beta Z \times \beta Z .$$

We define $\emptyset : \beta(Z \times Z) \longrightarrow \beta Z \times \beta Z$ to be,

$$\emptyset(P) = (\mathcal{A}_p, \mathcal{B}_p)$$

Evidently $\emptyset$ is well defined.

2.1.5. Result. The map $\emptyset : \beta(Z \times Z) \longrightarrow \beta Z \times \beta Z$ defined by $\emptyset(P) = (\mathcal{A}_p, \mathcal{B}_p)$ is a continuous map of $\beta(Z \times Z)$ onto $\beta Z \times \beta Z$.

Proof: The map $P \longrightarrow (\mathcal{A}_p, \mathcal{B}_p)$ from $\beta(Z \times Z)$ is clearly onto $\beta Z \times \beta Z$, from the definition of $\emptyset$. It is also continuous. To prove this, let $U \times V$ be an open neighbourhood of $(\mathcal{A}_p, \mathcal{B}_p) \in \beta Z \times \beta Z$. Here U and V are open in βZ . So,

$$U \times V = (\beta Z - \cap \overline{A}_i) \times (\beta Z - \cap \overline{B}_j),$$

where $A_i \subseteq Z$, $B_j \subseteq Z$.

So,

$$(\mathcal{A}_p, \mathcal{B}_p) \in (\beta Z - \cap \overline{A}_i) \times (\beta Z - \cap \overline{B}_i).$$

So,

$$A_i \notin \mathcal{A}_p, \ B_j \notin \mathcal{B}_p \text{ for some } i,j, \text{ say for } i_o, j_o.$$

i.e.,

$$A_{i_o} \notin \mathcal{A}_p, \quad B_{j_o} \notin \mathcal{B}_p.$$

Hence by definition of $\mathcal{A}_p$, $\mathcal{B}_p$, $A_{i_o} \times B_{j_o} \notin P$.

Since $P \in \beta(Z \times Z)$, this means that

$$P \notin cl_{\beta(z \times z)} (A_{i_o} \times B_{j_o}).$$

So,

$$P \in \beta(Z \times Z) \setminus cl_{\beta(z \times z)} (A_{i_o} \times B_{j_o}) = W \text{ (say)}$$

Thus W is an open neighbourhood of P in $\beta(Z \times Z)$.

If $Q \in W$, then $Q \notin cl_{\beta(z \times z)}(A_{i_o} \times B_{j_o})$.

i.e., $A_{i_o} \times B_{j_o} \notin Q$. So, by definition, again,

$$A_{i_o} \notin \mathcal{A}_Q, \; B_{j_o} \notin \mathcal{B}_Q.$$

i.e., $\mathcal{A}_Q \in (\beta Z - \cap \overline{A_i})$, $\mathcal{B}_Q \in (\beta Z - \cap \overline{B_j})$

i.e., $(\mathcal{A}_Q, \mathcal{B}_Q) \in (\beta Z - \cap \overline{A}_i) \times (\beta Z - \cap \overline{B}_j) = U \times V$
as desired.

2.1.6. Note. In $\beta Z \times \beta Z$, we have pointwise addition and multiplication. i.e., if (p_1,q_1), $(p_2,q_2) \in \beta Z \times \beta Z$, where p_1,q_1,p_2,q_2 are ultrafilters on Z, then,

$$(p_1,q_1) + (p_2,q_2) = (p_1+p_2, q_1+q_2) \text{ and}$$
$$(p_1,q_1) \cdot (p_2,q_2) = (p_1\cdot p_2, q_1\cdot q_2) \text{ where,}$$

p_1+p_2, q_1+q_2; $p_1\cdot p_2$, $q_1\cdot q_2$ are the respective addition and multiplication in βZ. (Ordinary addition and multiplication in Z have unique left-continuous extensions to addition and multiplication in βZ, just similar to that of βN [HI_1]).

2.1.7. Result. The operations of pointwise addition and multiplication in $\beta Z \times \beta Z$ can be obtained from the corresponding extension of pointwise addition and multiplication in $Z \times Z$ to $\beta(Z \times Z)$ by the map $P \longmapsto (\mathcal{A}_p, \mathcal{B}_p)$.

Proof: For $P, Q \in \beta(Z \times Z)$, we have,

$$P+Q = \left\{C \subseteq Z \times Z : \left\{(x,y) \in Z \times Z : C-(x,y) \in P\right\} \in Q\right\}.$$

Consider $\left\{A_\alpha \times B_\alpha \in P+Q\right\}$. Then,

$$\left\{(x,y) \in Z \times Z : (A_\alpha \times B_\alpha) - (x,y) \in P\right\} \in Q.$$

i.e., $\left\{(x,y) \in Z \times Z : (A_\alpha - x) \times (B_\alpha - y) \in P\right\} \in Q.$

Let $C = \left\{(x,y) \in Z \times Z : (A_\alpha - x) \times (B_\alpha - y) \in P\right\}.$

Then $C \in Q$. Then $C \supseteq C_\alpha \times D_\alpha$, where $C_\alpha, D_\alpha \subseteq Z$ and $C_\alpha \times D_\alpha \in Q$.

We have, for $y \in D_\alpha$,

$$C_\alpha \supseteq \left\{x \in Z : A_\alpha - x \in \mathcal{A}_p\right\} \text{ and } \left\{x \in Z : A_\alpha - x \in \mathcal{A}_p\right\} \in \mathcal{A}_Q.$$

Therefore, $C_\alpha \in \mathcal{A}_Q$. Since $\left\{x \in Z : A_\alpha - x \in \mathcal{A}_p\right\} \in \mathcal{A}_Q$, we have, $A_\alpha \in \mathcal{A}_P + \mathcal{A}_Q$. Thus for each $y \in D_\alpha$, we have $A_\alpha \in \mathcal{A}_P + \mathcal{A}_Q$.

Similarly, for each $x \in C_\alpha$, we get $B_\alpha \in \mathcal{B}_P + \mathcal{B}_Q$.

Thus,

$$\begin{aligned} (\{A_\alpha\}, \{B_\alpha\}) &= (\mathcal{A}_P + \mathcal{A}_Q, \mathcal{B}_P + \mathcal{B}_Q) \\ &= (\mathcal{A}_P, \mathcal{B}_P) + (\mathcal{A}_Q, \mathcal{B}_Q) \end{aligned}$$

Also we have,

$$\mathcal{A}_{P+Q} = \left\{A_\alpha \subseteq Z : A_\alpha \times B_\alpha \in P+Q\right\}$$

$$\mathcal{B}_{P+Q} = \left\{B_\alpha \subseteq Z : A_\alpha \times B_\alpha \in P+Q\right\},$$

where, $\mathcal{A}_{P+Q}, \mathcal{B}_{P+Q} \in \beta Z$.

So, $(\mathcal{A}_{P+Q}, \mathcal{B}_{P+Q}) \in \beta Z \times \beta Z$ and we have,

$$\mathcal{A}_{P+Q} = \mathcal{A}_P + \mathcal{A}_Q, \ \mathcal{B}_{P+Q} = \mathcal{B}_P + \mathcal{B}_Q.$$

Likewise with respect to pointwise multiplication in $Z \times Z$, we have the extended multiplication in $\beta(Z \times Z)$ given by

$$P.Q = \left\{C \subseteq Z \times Z : \left\{(x,y) \in Z \times Z : C \,/\, (x,y) \in P\right\} \in Q\right\}.$$

As in the case of addition, we take $\left\{A_\alpha \times B_\alpha \in P.Q\right\}$. Then we obtain,

$$\begin{aligned}(\left\{A_\alpha\right\}, \left\{B_\alpha\right\}) &= (\mathcal{A}_P . \mathcal{A}_Q, \mathcal{B}_P . \mathcal{B}_Q) \\ &= (\mathcal{A}_P, \mathcal{B}_P).(\mathcal{A}_Q, \mathcal{B}_Q) \text{ in } \beta Z \times \beta Z\end{aligned}$$

i.e., pointwise multiplication in $\beta Z \times \beta Z$ can be obtained from the unique left continuous extension of pointwise multiplication in $Z \times Z$ to $\beta(Z \times Z)$.

2.1.8. Definition. We now define the product '$\times$' in $\beta Z \times \beta Z$. Let $(p_1,q_1), (p_2,q_2) \in \beta Z \times \beta Z$.

Define,

$$(p_1,q_1) \times (p_2,q_2) = (p_1 \cdot p_2 - q_1 \cdot q_2,\ p_1 \cdot q_2 + q_1 \cdot p_2),$$

where,

$$-q_1 \cdot q_2 = -1 \cdot q_1 \cdot q_2 .$$

With respect to this product, $\beta Z \times \beta Z$ is a groupoid, since,

$$(p_1 \cdot p_2 - q_1 \cdot q_2,\ p_1 \cdot q_2 + q_1 \cdot p_2) \in \beta Z \times \beta Z .$$

2.1.9. Result. Let $(p_1,q_1), (p_2,q_2) \in \beta Z \times \beta Z$. Then the product in $\beta Z \times \beta Z$ given by,

$$(p_1,q_1) \times (p_2,q_2) = (p_1 \cdot p_2 - q_1 \cdot q_2,\ p_1 \cdot q_2 + q_1 \cdot p_2)$$

is non-associative.

Proof: We prove this result by an example.
Let $(p,0), (1,1) \in \beta Z \times \beta Z$ where p is a non-principal ultrafilter on Z .

$$(p,0) \times ((1,1) \times (1,1)) = (p,0) \times (1-1,1+1) = (p,0) \times (0,2)$$
$$= (0-0,\ 2p+0) = (0,2p) \qquad (1)$$

$$((p,0) \times (1,1)) \times (1,1) = (p-0,p+0) \times (1,1) = (p,p) \times (1,1)$$
$$= (p-p,\ p+p) \qquad (2)$$

Evidently, $(0,2p) \neq (p-p,\ p+p)$, where $p-p = p + -1 \cdot p$

§ 2.2. Some combinatorial results in $\beta Z \times \beta Z$

2.2.1. Notation. ω represents the set of non-negative integers viewed as ordinals. ω is also the cardinality of countable infinity. Given an infinite set A, we denote by $[A]^{\omega}$, the infinite subsets of A and by $\mathcal{P}_f(A)$ the set of finite non-empty subsets of A.

2.2.2. Definition. Let $A \subseteq Z \times Z$.

Define,

$$FS(A) = \left\{ \Sigma F \; : \; F \in \mathcal{P}_f(A) \right\}$$
$$FP(A) = \left\{ \Pi F \; : \; F \in \mathcal{P}_f(A) \right\},$$

where the addition and multiplication taken here are componentwise.

2.2.3. Result. There exist $(p_1,q_1),(p_2,q_2) \in \beta Z \times \beta Z \setminus (Z \times Z)$ such that $(p_1,q_1)+(p_1,q_1) = (p_1,q_1)$ and $(p_2,q_2).(p_2,q_2)=(p_2,q_2)$.

Proof. + and . are associative left-continuous operations on $(\beta Z \times \beta Z) \setminus (Z \times Z)$ which is compact.

2.2.4. Remark. As in βN [HI_1] we have the following results and the proofs are somewhat identical in some results to those results for βN.

2.2.5. Result. Let $(p,q), (p',q') \in (\beta Z \times \beta Z) \setminus (Z \times Z)$ such that $(p,q)+(p,q) = (p,q)$ and $(p',q').(p',q') = (p',q')$.

If $A \times B \in (p,q)$, $C \times D \in (p',q')$, then there exist $E \in [A\times B]^\omega$, $F \in [C\times D]^\omega$ such that $FS(E) \subseteq A\times B$ and $FP(F) \subseteq C\times D$.

2.2.6. Corollary. Let $Z \times Z = \bigcup_{i<r} A_i$. Then there exist $i<r$, $j<r$, $A \in [A_i]^\omega$, $B \in [A_j]^\omega$ such that $FS(A) \subseteq A_i$, $FP(B) \subseteq A_j$.

2.2.7. Definition.

$$\Gamma = \{A \times B \subseteq Z \times Z : \text{there exist } C \in [A\times B]^\omega \text{ such that } FS(C) \subseteq A\times B\}.$$

$$\overline{\Gamma} = \{(p,q) \in \beta Z \times \beta Z : (p,q) \subset \Gamma\}.$$

2.2.8. Result. $\overline{\Gamma}$ is a closed nonempty subset of $\beta Z \times \beta Z$ and $\cdot : \overline{\Gamma} \times \overline{\Gamma} \longrightarrow \overline{\Gamma}$ and $\times : \overline{\Gamma} \times \overline{\Gamma} \longrightarrow \overline{\Gamma}$.

Proof. That $\overline{\Gamma} \neq \emptyset$ is a consequence of the extension of the finite sum theorem. To see that $\overline{\Gamma}$ is closed, let $(p,q) \in (\beta Z \times \beta Z) \setminus \overline{\Gamma}$. Pick $A\times B \in (p,q) \setminus \Gamma$. Then $cl_{(\beta Z \times \beta Z)}(A\times B) \cap \overline{\Gamma} = \emptyset$. That $\overline{\Gamma} \subseteq (\beta Z \times BZ) \setminus (Z \times Z)$ follows from the fact that every member of Γ is infinite.

We prove that $\times : \overline{\Gamma} \times \overline{\Gamma} \longrightarrow \overline{\Gamma}$.

Let $(p_1,q_1), (p_2,q_2) \in \overline{\Gamma}$, and $A\times B \in (p_1,q_1) \times (p_2,q_2)$.

i.e., $A\times B \in (p_1 \cdot p_2 - q_1 \cdot q_2,\ p_1 \cdot q_2 + q_1 \cdot p_2)$

i.e., $(\{x \in Z : A-x \in p_1 \cdot p_2\} \in^{-} q_1 \cdot q_2,$
$\{y \in Z : B-y \in p_1 \cdot q_2\} \in q_1 \cdot p_2)$.

Pick $(x,y) \neq (0,0)$ such that $A-x \in p_1 \cdot p_2$, $B-y \in p_1 \cdot q_2$.
[We have $\cdot : \overline{\Gamma} \times \overline{\Gamma} \longrightarrow \overline{\Gamma}$, the proof of which is identical to that for βN]. So there exist $C \in [A-x]^{\omega}$ such that $FS(C) \subseteq A-x$; $D \in [B-y]^{\omega}$ such that $FS(D) \subseteq B-y$.
Let $E \times F = \left\{ (x,y) + (z_i,z_j) : (z_i,z_j) \in C \times D \right\}$.
Then, $FS(E) \times FS(F) \subseteq A \times B \in (p_1,q_1) \times (p_2,q_2)$. Thus $(p_1,q_1) \times (p_2,q_2) \in \overline{\Gamma}$.

2.2.9. Result. Let $(p,q) \in (\beta Z \times \beta Z) \setminus (Z \times Z)$; $(m,n) \in Z \times Z$. If $(p,q) + (p,q) = (p,q)$, then, $mZ \times nZ \in (p,q)$.

Proof: We have,

$$(Z \times Z) \setminus (\{m\} \times \{n\}) = ((Z \setminus \{m\}) \times \{n\}) \cup (\{m\} \times (Z \setminus \{n\})) \cup (Z \setminus \{m\} \times Z \setminus \{n\})$$

Since $\{m\} \times \{n\} \notin (p,q)$, $(Z \times Z) \setminus (\{m\} \times \{n\}) \in (p,q)$.

i.e., $((Z \setminus \{m\}) \times \{n\}) \cup (\{m\} \times (Z \setminus \{n\})) \cup (Z \setminus \{m\} \times Z \setminus \{n\}) \in (p,q)$

Here the only possibility is $Z \setminus \{m\} \times Z \setminus \{n\} \in (p,q)$.
i.e., $(\{x \in Z : (Z \setminus \{m\}) - x \in p\}, \{y \in Z : (Z \setminus \{n\}) - y \in q\}) \in (p,q)$, because, $(p,q) + (p,q) = (p,q)$.

Pick $(x_1,y_1) \in Z \setminus \{m\} \times Z \setminus \{n\}$ such that,

$$(Z \setminus \{m\} - x_1) \times (Z \setminus \{n\} - y_1) \in (p,q)$$

Pick $(x_2,y_2) \in (Z \setminus \{m\} \times Z \setminus \{n\}) \cap ((Z \setminus \{m\} - x_1) \times (Z \setminus \{n\} - y_1))$

$$= (Z \setminus \{m\} \cap (Z \setminus \{m\} - x_1)) \times (Z \setminus \{n\} \cap (Z \setminus \{n\} - y_1))$$

We have,

$$(Z \times Z)\setminus(\{m\} \times \{n\}) = \bigcup_{t<n} \bigcup_{t<m} (mZ + t) \times (nZ + t)$$

Pick $(a_1,b_1), (a_2,b_2) \in Z \times Z$ such that

$$(x_1,y_1) = (a_1m+t,\ b_1n+t)$$

$$(x_2,y_2) = (a_2m+t,\ b_2n+t)$$

Then $(x_2,y_2) + (x_1,y_1) = ((a_1+a_2)m+2t,\ (b_1+b_2)m+2t)$

while $(x_2,y_2)+ (x_1,y_1) \in (mZ +t) \times (nZ +t)$, a contradiction.

Thus $t = 0$. So, $mZ \times nZ \in (p,q)$.

2.2.10. Result. Let $\{z_n\}= \{(x_n,y_n)\}_{n<\omega}$ be an increasing sequence in $Z \times Z$. Define

$$\tau: Z \times Z \longrightarrow FS(\{z_n : n<\omega\}) \text{ by}$$

$$\tau\left(\sum_{n\in F} 2^{f(n)}, \sum_{n\in F} 2^{f(n)}\right) = \sum_{n\in F} z_n, \text{ where}$$

$F \in \mathcal{P}_f(\omega)$ and $f: N \longrightarrow Z$ such that

$$f(n) = n/2 \text{ if } n \text{ is even}$$

$$= -\frac{(n-1)}{2}, \text{ if } n \text{ is odd.}$$

Let $(p,q) \in (\beta Z \times \beta Z)\setminus(Z \times Z)$ such that $(p,q)+(p,q)=(p,q)$.

Let $(r,s) = \{A \times B \subseteq Z \times Z :$ there exists $C\times D \in (p,q)$ with $\tau(C\times D) \subseteq A\times B\}$.

Then $(r,s) \in \beta Z \times \beta Z\setminus(Z \times Z)$ and $(r,s)+(r,s) = (r,s)$.

Proof:

Since (r,s) and (r,s)+(r,s) are both ultrafilters, it suffices to show that $(r,s) \subseteq (r,s)+(r,s)$. Let $A \times B \in (r,s)$. Pick $C \times D \in (p,q)$ such that $\tau(C \times D) \subseteq A \times B$.

Let $E = \{(x,y) \in Z \times Z : C \times D - (x,y) \in (p,q)\}$. We claim that $\tau(E) \subseteq \{(x,y) \in Z \times Z : A \times B - (x,y) \in (r,s)\}$. Let $(x,y) \in \tau(E)$. Pick $(x_o,y_o) \in E$ such that $\tau(x_o,y_o) = (x,y)$. Pick $F \in \mathcal{P}_f(\omega)$ with $(x_o,y_o) = (\sum_{n \in F} 2^{f(n)}, \sum_{n \in F} 2^{f(n)})$.

Let m = max F. Since $(x_o,y_o) \in E$, $(C \times D) - (x_o,y_o) \in (p,q)$. i.e., $(C-x_o) \times (D-y_o) \in (p,q)$.
Also, $2^{m+1}Z \times 2^{m+1}Z \in (p,q)$ by Result 2.2.8.

So, $\tau[(C-x_o) \times (D-y_o) \cap (2^{m+1}Z \times 2^{m+1}Z)] \subseteq (A-x) \times (B-y)$.

To prove this, let

$$(z_1,z_2) \in [(C-x_o) \times (D-y_o) \cap (2^{m+1}Z \times 2^{m+1}Z)]$$

Pick

$$G \in \mathcal{P}_f(\omega) \text{ with } (z_1,z_2) = (\sum_{n \in G} 2^{f(n)}, \sum_{n \in G} 2^{f(n)})$$

Then min G > m, since $(z_1,z_2) \in 2^{m+1}Z \times 2^{m+1}Z$.

$$\text{Thus, } \tau((x_o,y_o)+(z_1,z_2)) = (\sum_{n \in F \cup G} 2^{f(n)}, \sum_{n \in F \cup G} 2^{f(n)})$$

$$= \sum_{n \in F \cup G} (x_n,y_n) = \sum_{n \in F} (x_n,y_n) + \sum_{n \in G} (x_n,y_n).$$

$$= \tau(x_o,y_o) + \tau(z_1,z_2)$$

$$= (x,y) + \tau(z_1,z_2).$$

Since $(x_o, y_o) + (z_1, z_2) \in C \times D$, we have,

$$(x, y) + \tau(z_1, z_2) \in \tau(C \times D) \subseteq A \times B.$$

Thus $\tau(z_1, z_2) \subseteq A \times B - (x, y)$, as desired.

To see that (r,s) is an ultrafilter on $Z \times Z$, let, $\mathcal{J} \in \mathcal{P}_f(r,s)$ and pick $\mathcal{G} \in \mathcal{P}_f(p,q)$ such that for each $A \times B \in \mathcal{J}$, there is a $C \times D \in \mathcal{G}$ with $\tau(C \times D) \subseteq A \times B$. Then $\bigcap \mathcal{G} \in (p,q)$ and $\tau(\bigcap \mathcal{G}) \subseteq \bigcap \mathcal{J}$, so $\bigcap \mathcal{J} \in (r,s)$. Since $\emptyset \notin (r,s)$, it suffices to prove that $A \times B \in (r,s)$ or $(Z \times Z) \setminus (A \times B) \in (r,s)$ whenever $A \times B \subseteq Z \times Z$. Let $A \times B \subseteq Z \times Z$ and $C \times D = \tau^{-1}(A \times B)$. If $C \times D \in (p,q)$, then, $\tau(C \times D) = \tau(\tau^{-1}(A \times B)) \subseteq A \times B$ so that $A \times B \in (r,s)$. Otherwise, $(Z \times Z) \setminus (C \times D) \in (p,q)$ in which case,

$\tau((Z \times Z) \setminus (C \times D)) \subseteq (Z \times Z) \setminus (A \times B)$ so that $(Z \times Z) \setminus (A \times B) \in (r,s)$. That (r,s) is non-principal follows from the fact that τ is finite to one.

We now establish that the distributive laws fail on βZ and hence on $\beta Z \times \beta Z$.

2.2.11. Result. Let $\{A_n : n \in N\} \cup \{B_n : n \in N\} \subseteq [Z]^{\omega}$ with $|A_n \cap B_m| < \omega$, whenever $m, n \in N$. Then,

$$cl_{\beta Z \setminus Z}\left(\bigcup_{n \in N} (\overline{A}_n \setminus Z)\right) \cap cl_{\beta Z \setminus Z}\left(\bigcup_{n \in N} (\overline{B}_n \setminus Z)\right) = \emptyset.$$

Proof. Let $C = \bigcup_{n \in N} (A_n \setminus \bigcup_{k<n} B_k)$. For $n \in N$, we have,

$A_n \setminus C = A_n \setminus (\bigcup_{k<n} B_k)$ so that $|A_n \setminus C| < \omega$. Thus for $n \in N$, we have, $\overline{A_n} \setminus Z \subseteq \overline{C} \setminus Z$. Also, for $n \in N$, $(\overline{C} \setminus Z) \cap (\overline{B}_n \setminus Z) = \emptyset$ because, $C \cap B_n \subseteq \bigcup_{k<n} (A_k \cap B_n)$ so that $|C \cap B_n| < \omega$. Therefore, $\overline{C} \setminus Z$ is an open and closed subset of $\beta Z \setminus Z$ containing $\bigcup_{n \in N} (\overline{A_n} \setminus Z)$ and missing $\bigcup_{n \in N} (\overline{B_n} \setminus Z)$.

2.2.12. Result. Let $H = \{ p \in \beta Z \setminus Z$: for all q and r in $\beta Z \setminus Z$, $p.(q+r) \neq p.q + p.r$ and $(p+q).r \neq p.r + q.r \}$. Then the interior in $\beta Z \setminus Z$ of H is dense in $\beta Z \setminus Z$.

Proof: A basis for the open sets in $\beta Z \setminus Z$ is, $\{ \overline{A} \setminus Z : A \in [Z]^\omega \}$. Let $A \in [Z]^\omega$ and define a monotonically increasing sequence $\{x_n\}_{n \in N}$ such that $x_n \in A$, whenever $n \in N$. Let $B = \{ x_n : n \in N \}$. Then $B \in [A]^\omega$. We show that $\overline{B} \setminus Z \subseteq H$. To this end, let $p \in \overline{B} \setminus Z$. Let,

$C = cl_{\beta z \setminus z} \bigcup \{ \overline{Bm} : m \in Z \}$, $D = cl_{\beta z \setminus z} \bigcup \{ \overline{Bm+n},\ m,n \in Z\ ,\ m > n \}$,

$E = cl_{\beta z \setminus z} \bigcup \{ \overline{Bm+n} : m,n \in Z,\ m \leqslant n \}$.

We now establish the following.

(1) If $m,m',n,n' \in Z$, and $(m,n) \neq (m',n')$, then $|(Bm+n) \cap (Bm'+n')| < \omega$.

Let $m,m',n,n' \in Z$ with $(m,n) \neq (m',n')$. Let $a = m+n+m'+n'$. We show that if $k>a$, $t>a$, then $x_k m+n \neq x_t m'+n'$ ('a' can be negative, zero or positive) and hence $|(Bm+n)\cap(Bm'+n')| < 2a$. Let $k > a$, $t > a$. Assume first that $k=t$. If $m=m'$, then $n\neq n'$. So, $x_k m+n \neq x_k m'+n'$. Then we assume that $m > m'$. Then,

$$\begin{aligned}(x_k m+n) - (x_t m'+n') &= x_k(m-m') + n-n' \\ &\neq 0 \text{ except in the case when } \\ &\quad x_k=0 \text{ and } n=n'\text{, in which case}\end{aligned}$$

the difference between the $(k+1)^{th}$ term onwards is different from zero.

Now assume that $k > t$. Then,

$$(x_k m+n) - (x_t m'+n') = x_k m - x_t m' + n-n' \neq 0$$

even when $n=n'$ because $x_k m > x_t m'$.

(2) C,D,E are pairwise disjoint.

We show that $D\cap E = \emptyset$, the other two proofs being similar. If $m,m',n,n' \in Z$ such that $(m,n) \neq (m',n')$, then by (1), $|(Bm+n)\cap(Bm'+n')| < \omega$. So by the previous result, $D\cap E = \emptyset$.

(3) For any $q,r \in \beta Z\setminus Z$.

(a) $p.q \in C$, (b) $p.q+r \in D$, (c) $(p+q).r \in E$.

Let $q,r \in \beta Z\setminus Z$. To see that $p.q \in C$, let $G \in p.q$. Then, $\{x \in Z : G/_x \in p\} \in q$. So pick $m \in Z$ such that $G|_m \in p$.

Then $|G/_m \cap B| = \omega$. So $|G \cap Bm| = \omega$. Then, $\bar{G} \cap C \neq \emptyset$. So $p.q \in C$. To see that $p.q+r \in D$, let $G \in p.q+r$. Then $\{x \in Z : G-x \in p.q\} \in r$. So, pick $n \in Z$ with $G-n \in p.q$. Then $\{x \in Z : (G-n)/_x \in p\} \in q$ and $q \in \beta Z \setminus Z$, so pick $m > n$ such that $(G-n)|_m \in p$. Then $|(G-n)/_m \cap B| = \omega$. So, $|G \cap (Bm+n)| = \omega$. So, $\bar{G} \cap D \neq \emptyset$. Thus $p.q+r \in D$. To see that $(p+q).r \in D$, let $G \in (p+q).r$. Then, $\{x \in Z: G/x \in p+q\} \in r$. So pick $m \in Z$ with $G/_m \in p+q$. Then $\{x \in Z : G|_m - x \in p\} \in q$, so pick $x \in Z$ with $G|_m - x \in p$. Let $n = mx$. Then $|[(G|m)-x] \cap B| = \omega$, so, $|G \cap (Bm+n)| = \omega$. So $\bar{G} \cap E = \emptyset$ and hence $(p+q).r \in E$. Let $q, r \in \beta Z \setminus Z$. Then $p.(q+r) \in C$ and $p.q+p.r \in D$ so that $p.(q+r) \neq p.q + p.r$. Also, $(p+q).r \in E$ and $p.r \in D$ so that $(p+q).r \neq p.r+q.r$.

2.2.13. Result. The distributive law fails in $\beta Z \times \beta Z$ with componentwise addition and multiplication.

Proof follows from 2.2.12.

Chapter III

ARITHMETIC IN βR FOR DISCRETE R

3.0. Introduction

In this chapter we consider the set R of real numbers with discrete topology. As in βN and βZ it can be seen that the ordinary addition and multiplication in R can be uniquely extended to βR, making $(\beta R,+)$ and $(\beta R,.)$ semigroups with identities 0 and 1 respectively. The extended operations are left continuous and associative and the topological centers of $(\beta R,+)$ and $(\beta R,.)$ are $(R,+)$ and $(R,.)$ respectively. Though all these properties are analogous to that in βN and βZ, here we have obtained situations that are in contrast to those in βN mainly because, R is algebraically a field.

In section 3.1, we define addition and multiplication in βR analogous to that in βN and discuss properties characterizing sums and products in βR.

In section 3.2, using the characterizations discussed in section 3.1, we obtain solutions to equations such as $p+q = r.s$ when at least one of p,q,r,s is in R and others in $\beta R\setminus R$. We have shown that solutions exist in the case when the members of $\beta R\setminus R$ are strongly summable ultrafilters on R.

In section 3.3 we introduce the concept of α-remote points in βX for a discrete topological field X where $\omega \leqslant \alpha \leqslant |X|$. We have obtained several results using the arithmetic in βX using the α-remote points.

§ 3.1. + and . in βR.

Taking R , the set of real numbers with discrete topology, as mentioned earlier, βR is the collection of all ultrafilters on R with the following topology. Let $\bar{A} = \{p \in \beta R : A \in p\}$. Then $\{\bar{A} : A \subseteq R\}$ is a base for the closed sets in βR . The points of R are identified with the principal ultrafilters.

Definition 3.1.1. Let $p, q \in \beta R$. We define + and . in βR as follows:

$$p+q = \{A \subseteq R : \{x \in R : A-x \in p\} \in q\}.$$

$$p.q = \{A \subseteq R : \{x \in R : A/_x \in p\} \in q\}, \quad \text{where}$$

for $A \subseteq R$, and $x \in R$,

$$A-x = \{y \in R : x+y \in A\} = \{z-x : z \in A\}.$$

$$A/_x = \{y \in R : xy \in A\} = \{z/_x : z \in A\}, \text{ when } x \neq 0$$

$$= R, \text{ when } x = 0 \in A$$

$$= \emptyset \text{ when } x = 0 \notin A .$$

Result 3.1.2. The operations + and . are associative left continuous operations on βR . If p or q is in $\beta R \setminus R$, then so are p+q and p.q.

Proof: We shall prove this for + only. The proof for . can be similarly obtained.

Let $p,q \in \beta R$. First of all $p+q \neq \emptyset$, because, $R \in p$, $R \in q$ and $R + R \in p+q$, since for each $x \in R$, $R-x = R$.

$\emptyset \notin p+q$, for

$$\emptyset \in p+q \Rightarrow \{x \in R : \emptyset - x \in p\} \in q$$
$$\Rightarrow \{x \in R : \emptyset \in p\} \in q \Rightarrow \emptyset \in q, \text{ not possible.}$$

Let $A,B \in p+q$. Then,

$$\{x \in R : A-x \in p\} \in q \text{ and } \{x \in R : B-x \in p\} \in q.$$

So $\{x \in R : A-x \in p\} \cap \{x \in R : B-x \in p\} \in q$.

But $\{x \in R : A-x \in p\} \cap \{x \in R : B-x \in p\} = \{x \in R : (A \cap B)-x \in p\}$.

Thus $\{x \in R : (A \cap B)-x \in p\} \in q$. Therefore $A \cap B \in p+q$.

Let $A \subseteq R$ and assume that $A \notin p+q$. Then,

$$\{x \in R : A-x \in p\} \notin q. \text{ So, } R \setminus \{x \in R : A-x \in p\} \in q.$$

But $R \setminus \{x \in R : A-x \in p\} = \{x \in R : (R \setminus A)-x \in p\}$.

Thus $\{x \in R : (R \setminus A)-x \in p\} \in q$. Therefore, $R \setminus A \in p+q$.

Thus $p+q \in \beta R$.

Let $p \in \beta R$ and define $f_p : \beta R \longrightarrow \beta R$ by $f_p(q) = p+q$. We show that f_p is continuous. Let $q \in \beta R$ and U be an open neighbourhood of $p+q$. Then $V = \beta R \setminus U$ is closed set in βR. So, $V = \bigcap \overline{Z}$, where Z's are some subsets in R.

Now, $p+q \notin V$. So, there exists $\overline{Z}$ such that $p+q \notin \overline{Z}$.
i.e., $Z \notin p+q$. i.e., $\{x \in R : Z-x \in p\} \notin q$.
Let $B = \{x \in R : Z-x \in p\}$. Then, $B \notin q$. So, $q \notin \overline{B}$.
Now $C = \beta R \setminus \overline{B}$ is an open set in βR containing q.
Let $r \in C$. Then $r \notin \overline{B}$. i.e., $B \notin r$.
i.e., $\{x \in R : Z-x \in p\} \notin r$. i.e., $Z \notin p+r$. So, $p+r \notin \overline{Z}$.
Hence, $p+r \notin \bigcap \overline{Z} = V$. Hence $p+r \in U$. i.e., $f_p(r) \in U$.
i.e., $f_p(C) \subset U$. Thus $q \longmapsto p+q$ is a left-continuous operation on βR. To see that + is associative, let $p, q, r, \in \beta R$ and $A \subseteq R$.

$$
\begin{aligned}
A \in p + (q+r) &\leftrightarrow \{x \in R : A-x \in p\} \in q+r \\
&\leftrightarrow \{y \in R : \{x \in R : A-x \in p\} - y \in q\} \in r \\
&\leftrightarrow \{y \in R : \{x \in R : (A-y)-x \in p\} \in q\} \in r \\
&\leftrightarrow \{y \in R : A-y \in p+q\} \in r \\
&\leftrightarrow A \in (p+q) + r.
\end{aligned}
$$

Let $p, q \in \beta R$ and assume $p+q \notin \beta R \setminus R$. Pick $x \in R$ such that $p+q = \{A \subseteq R : x \in A\}$. Then $\{x\} \in p+q$. So, $\{y \in R : \{x\} - y \in p\} \in q$. Since $y \neq z \Rightarrow (\{x\} - y) \cap (\{x\} - z) = \emptyset$, there must be a unique y such that $\{x\} - y \in p$. Then, $\{x-y\} \in p$ and $\{y\} \in q$, so both p and q are principal ultrafilters.

Result 3.1.3. The operations + and . are the unique extensions of + and . respectively on R, which are left-continuous.

Proof: We prove the statement for + only, the proof of . is essentially identical. Let $e: R \longrightarrow \beta R$ be the embedding, where for $x \in R$, $e(x) = \{A \subseteq R : x \in A\}$. Let $x, y \in R$. We show that $e(x)+e(y) = e(x+y)$. For this it suffices to prove that $\{x+y\} \in e(x) + e(y)$.

i.e., $\{z \in R : \{x+y\} - z \in e(x)\} \in e(y)$. But for $z \in R$,

$\{x+y\} - z \in e(x)$ if and only if $x+z \in \{x+y\}$. Thus,

$\{z \in R : \{x+y\} - z \in e(x)\} = \{y\}$.

Result 3.1.4. The centers of the monoids $(\beta R, +)$ and $(\beta R, .)$ contain R.

Proof: Let $x \in R$, $p \in \beta R \setminus R$. We shall show that $p+e(x) = e(x)+p$, the proof for . being essentially the same. Let $A \in p+e(x)$. Then, $\{z \in R : A-z \in p\} \in e(x)$, so that $A-x \in p$.

$$\begin{aligned} \text{But } A-x &= \{z \in R : x+z \in A\} \\ &= \{z \in R : x \in A-z\} \\ &= \{z \in R : A-z \in e(x)\}. \end{aligned}$$

Thus, $A \in e(x)+p$. So, $p+e(x) \subseteq e(x)+p$. Both being ultrafilters, equality holds.

Notation 3.1.5. The principal ultrafilter $e(m)$, $m \in R$ represents m of R and so we denote it by m rather than $e(m)$.

Result 3.1.6. $(R,+)$ is a sub group of the monoid $(\beta R,+,0)$ and $(R-\{0\},.)$ is a subgroup of the monoid $(\beta R,.,1)$.

Proof: The principal ultrafilter $-x$ is the additive inverse of the principal ultrafilter x and for $x \neq 0$, the principal ultrafilter $1/x$ is the multiplicative inverse of the principal ultrafilter x.

Corollary 3.1.7. For $p,q \in \beta R$, $m \in R$,

$$m+p = m+q \Rightarrow p=q$$
$$m.p = m.q \Rightarrow p=q \text{ if } m \neq 0.$$

Result 3.1.8. For $p \in \beta R \setminus R$, $m \in R$, $p+m = \{A+m : A \in p\}$.

Proof: $\emptyset \notin p+m$ because, $\emptyset \notin p$. Let $A+m$, $B+m \in p+m$. Then $(A+m) \cap (B+m) = (A \cap B)+m \in p+m$, since $A \cap B \in p$. Let $B \subsetneq R$ be such that $B \notin p+m$. Then, $B-m \notin p$. So, $R \setminus (B-m) \in p$. But $R \setminus (B-m) = (R \setminus B)-m$ so that $(R \setminus B)-m \in p$. So, $R \setminus B \in p+m$. i.e., $p+m \in \beta R$. By the definition of addition in βR,

$$p+m = \{A \subsetneq R : \{x \in R : A-x \in p\} \in m.$$

Let $B \in p+m$. Then,

$$\{x \in R : B-x \in p\} \in m. \quad \text{So, } B-m \in p.$$

Hence $(B-m)+m \in p+m$. i.e., $B \in p+m$, defined as above.

Conversely, let $C \in p+m = \{A+m : A \in p\}$. Then $C-m \in p$. So, $m \in \{x \in R : C-x \in p\}$. So, $\{x \in R : C-x \in p\} \in m$. i.e., $C \in p+m$.

We now give the characterisation of sums and products in βR. The proofs are omitted being similar to those in βN [HI_2].

Result 3.1.9. Let $p, q \in \beta R$ and $A \subseteq R$.

(1) $A \in p+q$ if and only if there exist $C \in q$ and a family $\{B_n : n \in C\} \subset p$ such that $A \supset \bigcup_{n \in C} (B_n + n)$.

(2) $A \in p.q$ if and only if there exist $C \in q$ and a family $\{B_n : n \in C\} \subset p$ such that $A \supset \bigcup_{n \in C} (B_n . n)$.

Result 3.1.10. Let $p \in \beta R \setminus R$, $m \in R$ where $m \neq 0$. Then there exists $q \in \beta R \setminus R$ such that $q+m = p$.

Result 3.1.11. Let $p \in \beta R \setminus R$, $m \in R$, where $m \neq 0$. Then there exists $q \in \beta R \setminus R$ such that $q.m = p$.

Result 3.1.12. Let $p \in \beta R$ and $n \in R$. Then there exists $q \in \beta R$ such that $p+q = p.n$ if and only if for each $A \in p$ and each function $f: R \longrightarrow p$, there exists $m \in R$ such that $(f(m)+m) \cap (A.n) \neq \emptyset$.

Result 3.1.13. Let $p \in \beta R \setminus R$, $m \in R$. Then there exists $q \in \beta R$ such that $p+m = p.q$ if and only if for each $A \in p$ and each function $f : R \longrightarrow p$, there exists $n \in R$ such that $(A+m) \cap (f(n).n) \neq \emptyset$.

Result 3.1.14. Let $p, q \in \beta R \setminus R$. Then $p+q \neq p.q$ if and only if there exists $B \in q$ and a family $\{A_n : n \in B\} \subset P$ such that $(n.A_n) \cap (m+A_m) = \emptyset$, whenever $m, n \in B$.

Corollary 3.1.15. Let $p, q \in \beta R \setminus R$. Then $p+q = p.q$ if and only if whenever $B \in q$ and a family $\{A_n : n \in B\} \subset p$, there exists $m, n \in B$ such that $(n.A_n) \cap (m+A_m) \neq \emptyset$.

§ 3.2. Solutions to some equations in βR .

Result 3.2.1. Let $p \in \beta R \setminus R$, $n \in R$, $n \neq 1$. Then there exists $q \in \beta R \setminus R$ such that $p+q = p.n$.

Proof: Given $p \in \beta R \setminus R$, let $A \in p$. Also, $n \in R$, where $n \neq 1$ is given. For each $a \in A$, consider $A.n - a$. Define, $B_A = \bigcup_{a \in A} (A.n-a)$. Let $\mathcal{B} = \{B_A : A \in p\}$. Then $\mathcal{B} \neq \emptyset$ and $B_{A_1} \cap B_{A_2} \supseteq B_{A_1 \cap A_2}$ for $A_1, A_2 \in p$, so that $\mathcal{B}$ is a filter base. Let q be an ultrafilter generated by $\mathcal{B}$. For this q we claim that $p+q = p.n$. For this, let $A \in p$. Then $A.n \in p.n$. Also $B_A \in q$. We claim that for at least one $m \in B_A$, $A.n-m \in p$. Otherwise,

$A.n-m \notin p$ for every $m \in B_A$. So, $R\setminus(A.n-m) \in p$ for every $m \in B_A$. i.e., $(R\setminus A.n)-m \in p$ for every $m \in B_A$. Since $A \in p$, we get $A \cap (R\setminus A.n-m) \in p$ for every $m \in B_A$. Let $x \in A \cap (R\setminus A.n - m_i)$ for $m_i \in B_A$. Then $x \in A$ and $x \in (R\setminus A.n)-m_i$. i.e., $x \in A$ and $x+m_i \in R\setminus A.n = (R\setminus A).n$.

So, $\quad x + m_i = y.n$, where $y \in R\setminus A$.

i.e., $\quad m_i = y.n-x$, where $y \in R\setminus A$.

i.e., $\quad m_i \in (R\setminus A).n-x$, where $x \in A$.

In a similar manner every $m \in B_A$ belongs to $(R\setminus A.n)-a$ for some $a \in A$. So, we have, $B_A \subseteq \bigcup_{a \in A} (R\setminus A.n - a)$.

But by definition $B_A = \bigcup_{a \in A} (A.n-a)$. So, we have, $\bigcup_{a \in A} (A.n-a) \subseteq \bigcup_{a \in A} (R\setminus A.n-a)$ which is not possible.

Hence for at least one $m \in B_A$, where,

$m \in \bigcup_{a \in A} (A.n-a) \setminus \bigcup_{a \in A} (R\setminus A.n-a)$, we should have,

$A.n - m \in p$. Suppose that for $m_j \in B_A$, $A.n - m_j \in p$, where $m_j \in \bigcup_{a \in A}(A.n-a) \setminus \bigcup_{a \in A}(R\setminus A.n-a)$. Using 3.1.12, for any function $f: R \longrightarrow p$, $f(m_j) \in p$.

So, $f(m_j) \cap (A.n - m_j) \in p$. If $z \in f(m_j) \cap (A.n-m_j)$, then

$$z + m_j \in f(m_j)+m_j \text{ and } z + m_j \in A.n$$

i.e., $\quad z + m_j \in (f(m_j)+m_j) \cap A.n$ so that,

$(f(m_j)+m_j) \cap (A.n) \neq \emptyset$. Thus $p+q = p.n$.

This is contrast to what we have in the case of N.

Remark 3.2.2. Let $p,q \in \beta N \setminus N$ and $n \in N \setminus \{1\}$. Then, $p+q \neq p \circ n$ [HI_2].

Result 3.2.3. Let $p \in \beta R \setminus R$, $m \in R$, $m \neq 0$. Then there exists $q \in \beta R \setminus R$ such that $p+m = p.q$.

Proof: Given $p \in \beta R \setminus R$, let $A \in p$. For each $a \in A$, $a \neq 0$, we take the set $(A+m)/a$, where $m \in R$, $m \neq 0$ is given.

Define $B_A = \bigcup_{\substack{a \in A \\ a \neq 0}} \left(\frac{A+m}{a}\right)$. Then $\mathcal{B} = \left\{B_A : A \in p\right\}$ is a filter base. If q is any ultrafilter generated by $\mathcal{B}$, then we can show that for this q, $p+m = p \circ q$.

As in 3.2.1, we can prove that for at least one $n \in B_A$,

$$n \in \bigcup_{\substack{a \in A \\ a \neq 0}} \left(\frac{A+m}{a}\right) \setminus \bigcup_{\substack{a \in A \\ a \neq 0}} \left(R \setminus \left(\frac{A+m}{a}\right)\right),$$ say n_k, $\frac{A+m}{n_k} \in p$.

Using 3.1.13, if $f: R \longrightarrow p$ is any function, then $f(n_k) \in p$. So, $f(n_k) \cap \left(\frac{A+m}{n_k}\right) \in p$. If $x \in f(n_k) \cap \frac{(A+m)}{n_k}$, then,

$x.n_k \in \left(f(n_k).n_k \text{ and } A+m\right)$. So, $(f(n_k).n_k) \cap (A+m) \neq \emptyset$.

Hence, $p+m = p.q$.

Remark 3.2.4. We do not know whether the equation $p+m = p.q$ has solutions with $m \in N$ and $p \in \beta N \setminus N$ [HI_2].

Result 3.2.5. Given $p \in \beta R \setminus R$, there exists $q \in \beta R \setminus R$ such that $p+q = p \circ q$.

Proof: Given $p \in \beta R \setminus R$. Let $A \in p$.

Define $B_A = \left\{ \frac{a}{b-1} , b \neq 1, a,b \in A \right\}$.

Let $\mathcal{B} = \left\{ B_A : A \in p \right\}$. Then $\mathcal{B}$ is a filter-base. Let q be any ultrafilter generated by $\mathcal{B}$. For this q, we claim that $p+q = p \circ q$, for,

Let $C \in q$ and $\left\{ A_n : n \in C \right\} \subset p$. Since $\mathcal{B}$ generates q, $C = B_A$, for some $A \in p$. Then $\left\{ A \cap A_n : n \in C \right\} \subset p$. The number of elements in B_A is the same as the number of sets $A \cap A_n$, which belong to p. Also, for each $m \in B_A$, $A \cap A_m \subset A$ so that $B_{A \cap A_m} \subseteq B_A$ and the number of members in B_A equals the number of sets $B_{A \cap A_m}$, $m \in B_A$. So we have $B_A = \bigcup_{j \in B_A} (B_{A \cap A_j})$. Let the members in B_A be such that whenever $m_j \in B_A$, then m_j is a member of $B_{A \cap A_{m_j}}$. Consider, any $n_k \in B_A$. Then by definition, $n_k \in B_{(A \cap A_{n_k})}$. So $n_k = \frac{a}{b-1}$, $b \neq 1$, $a,b \in A \cap A_{n_k}$. So $(b-1)n_k = a$. i.e., $b.n_k = a+n_k$, where $b.n_k \in (A \cap A_{n_k}).n_k$ and $a+n_k \in (A \cap A_{n_k})+n_k$. Thus $((A \cap A_{n_k})+n_k) \cap ((A \cap A_{n_k}).n_k) \neq \emptyset$.

Hence, $(A_{n_k}+n_k) \cap (A_{n_k}.n_k) \neq \emptyset$. Hence, using the corollary 3.1.5, we have $p+q = p.q$.

Definition 3.2.6. An ultrafilter $p \in \beta R \setminus R$ is strongly summable if and only if for each $A \in p$, there exists $B \in [A]^{\omega}$ such that $FS(B) \subseteq A$ and $FS(B) \in p$, where, $FS(B) = \{\Sigma F: F \in \mathcal{P}_f(B)\}$. Here $[A]^{\omega}$ means infinite subsets of A and $\mathcal{P}_f(B)$ denotes finite subsets of B.

Result 3.2.7. If p is a strongly summable ultrafilter in $\beta R \setminus R$, then so is q in each of the following equations.

(1) $p+q = p.n$, given $p \in \beta R \setminus R$ and $n \in R$, $n \neq 1$.

(2) $p+m = p.q$, given $p \in \beta R \setminus R$ and $m \in R$, $m \neq 0$.

(3) $p+q = p.q$, given $p \in \beta R \setminus R$.

Proof: We prove only the first one, the proofs of the second and third being essentially identical. The existence of $q \in \beta R \setminus R$ in each of the above equations has been proved in the previous results. In (1), given $p \in \beta R \setminus R$, $n \in R$, $n \neq 1$, we have obtained q to be any ultrafilter generated by the filter base $\mathcal{B} = \{B_A : A \in p\}$, where, $B_A = \bigcup_{a \in A} (A.n-a)$. Since p is strongly summable, for each $A \in p$, there exists $C \in [A]^{\omega}$ such that $FS(C) \subseteq A$ and $FS(C) \in p$, by definition. For each $A \in p$, we have $B_A \in q$ and $FS(C) \in p$ means $B_{FS(C)} \in q$, by the definition of q.

Also, $C \in [A]^{\omega} \Longrightarrow B_C \in [B_A]^{\omega}$ and we have $B_{FS(C)} = FS(B_C)$,

for,

Let $z \in FS(B_C)$. Then, $z = \sum_{k=1}^{m} x_k$, where $x_k \in B_C$.

i.e., $z = \sum_{k=1}^{m} (c_k.n-b_k)$, where $c_k.n-b_k$ with $c_k, b_k \in C$, belongs to B_C by the definition of B_C.

$= (\sum_{k=1}^{m} c_k).n - (\sum_{k=1}^{m} b_k)$, where $\sum_{k=1}^{m} b_k = b \in FS(C)$

$\sum_{k=1}^{m} c_k = c \in FS(C)$

$= b.n-C$

$\in B_{FS(C)}$, by definition.

$\therefore \quad FS(B_C) \subseteq B_{FS(C)}$.

Conversely, let $y \in B_{FS(C)}$. Then, $y = a.n-b$, where $a,b \in FS(C)$.

i.e., $y = (\sum_{k=1}^{m} a_k).n - (\sum_{k=1}^{p} b_k)$, where,

$a = \sum_{k=1}^{m} a_k$ and $b = \sum_{k=1}^{p} b_k$, a,b being members of FS(C), $a_k, b_k \in C$.

So, $y = \sum_{k=1}^{m} (a_k.n-b_k)$ if $m>p$, where we take $b_{p+1} = \ldots = b_m = 0$

or $y = \sum_{k=1}^{p} (a_k.n-b_k)$ if $p>m$, where we take $a_{m+1} = \ldots = a_m = 0$.

Therefore, $y \in FS(B_C)$, since $a_k.n-b_k \in B_C$, for $k=1,\ldots,m$ or $k=1,\ldots,p$.

Thus $B_{FS(C)} \subseteq FS(B_C)$.

Therefore, $B_{FS(C)} = FS(B_C)$ so that we have the desired requirement.

§ 3.3. α- remote points in βX, for a discrete topological field X.

Convention 3.3.1. X is a discrete topological field.

Definition 3.3.2. [CO; NE]. Let $p \in \beta X$. The norm of p denoted by $\| p \|$ is defined by $\|p\| = \min \{|Z| : Z \in p\}$. p is said to be k-uniform if $\|p\| > k$. The space $U_k(S) = \{p \in \beta X : \|p\| > k\}$ is closed in βX.

Definition 3.3.3. Let $\omega \leqslant \alpha \leqslant |X|$. A point $p \in \beta X$ is said to be α-remote if $p \notin cl_{\beta X} D$, where $D \subset X$ such that $|D| \leqslant \alpha$. For $k > \alpha > \omega$, every k-uniform ultrafilter in βX is an α-remote point. Also, if p is α-remote in βX, then p is β-remote for $\beta < \alpha$. So, p will be an α-non-remote point if $p \in cl_{\beta X} D$ for some $D \subset X$ such that $|D| \ngtr \alpha$.

Result 3.3.4. Let p be an α-remote point in βX. If $m \in X$, then, p+m is an α-remote point.

Proof: We have, $p+m = \{A+m : A \in p\}$ and $p \longmapsto p+m$ is a homeomorphism. So, if $p+m \in cl_{\beta X} D$, where $D \subset X$ such that $|D| \ngtr \alpha$, then $D \in p+m$. i.e., $D-m \in p$. i.e., $p \in cl_{\beta X}(D-m)$, where $D-m \subset X$ and $|D-m| \ngtr \alpha$, which means that p is α-non-remote. So we must have p+m α-remote in βX.

Result 3.3.5. Let p be α-non-remote. Then for any $m \in X$, p+m is α-non-remote.

Result 3.3.6. Let p,q be α-remote. Then p+q is α-remote.

Proof: Suppose that $p+q \in cl_{\beta X}D$, where $D \subset X$ such that $|D| \not> \alpha$. Then $D \in p+q$. So, $B = \{x \in X : D-x \in p\} \in q$. For any $m \in B$, $D-m \in p$. i.e., $D \in p+m$. So, $p+m \in cl_{\beta X}D$, which is a contradiction by Result 3.3.4. So p+q must be α-remote.

Result 3.3.7. Let p be α-remote and q α-non-remote in βX. Then p+q is α-remote and q+p is α-non-remote.

Proof: Suppose that p+q is not α-remote. Let $p+q \in cl_{\beta X}D$, where $D \subset X$ is such that $|D| \not> \alpha$. Then $D \in p+q$. So, $A = \{x \in X : D-x \in p\} \in q$. Then for every $m \in A$, $D-m \in p$. i.e., $D \in p+m$. So, $p+m \in cl_{\beta X}D$, which is a contradiction, because of Result 3.3.4. Thus p+q is α-remote.

Suppose that $q+p \notin cl_{\beta X}D$ for any $D \subset X$ with $|D| \not> \alpha$. Then $D \notin q+p \Rightarrow X \setminus D \in q+p$.

$$\Rightarrow B = \{x \in X : (X \setminus D)-x \in q\} \in p.$$

So for any $m \in B$, $X \setminus D-m \in q$. i.e., $X \setminus D \in q+m$. Hence $D \notin q+m$. i.e., $q+m \notin cl_{\beta X}D$ for any $D \subset X$ with $|D| \not> \alpha$, which is a contradiction because of Result 3.3.5. Thus q+p is α-non-remote in βX.

Result 3.3.8. If p is α- non remote and q is β- non remote for $\beta<\alpha$, then p+q is α-non-remote and q+p is β-non-remote.

Proof: Suppose that $p+q \notin cl_{\beta X} D$ for any $D \subset X$ such that $|D| \ngtr \alpha$. Then, as in Result 3.3.6, we obtain a contradiction. So p+q is α- non remote. Similarly, we may prove that q+p is β-non remote.

Result 3.3.9. If p is α-remote and q is β- remote then p+q is α-remote and q+p is β-remote.

Result 3.3.10. We have the following similar situations. Let $\omega \leqslant \beta \leqslant \alpha$.

(1) p is α-remote, q is β-non remote $\Rightarrow$ p+q is α-remote, q+p is β-non remote.

(2) p is α-non remote, q is β-remote $\Rightarrow$ p+q is α-non remote, q+p is β-remote.

Result 3.3.11. Given that p is α-remote in βX, where $\omega \leqslant \alpha \leqslant |X|$, $n \in X$, $n \neq 1$, there exists q α-remote in βX such that $p+q = p.n$.

Proof: As in βR, it can be shown that given $p \in \beta X$, $n \in X$, $n \neq 1$, there exists $q \in \beta X$ such that $p+q = p.n$, where we obtain q as the ultrafilter generated by the filter base $\mathcal{B} = \{B_A : A \in p\}$, where $B_A = \bigcup_{a \in A} (A.n-a)$ [Result 3.2.1].

(Here $A.n = \{a.n : a \in A\}$ and $-a$ is the additive inverse of a). Now, given that p is α-remote, we have $|A| > \alpha$ for every $A \in p$ so that $|B_A| > \alpha$ for every $B_A \in \mathcal{B}$ and so q generated by $\mathcal{B}$ is also α-remote.

Result 3.3.12. Given p α-non-remote in βX where $\omega \leqslant \alpha \leqslant |X|$, $n \in X$, $n \neq 1$, there exists q α-non-remote in βX such that $p+q = p.n$.

Proof: As mentioned in Result 3.3.11, we have for a given $p \in \beta X$, $n \in X$, $n \neq 1$, a $q \in \beta X$ generated by the filter base $\mathcal{B} = \{B_A : A \in p\}$, where $B_A = \bigcup_{a \in A} (A.n-a)$ (Result 3.2.1). Given that p is α-non-remote we have, $p \in cl_{\beta X} D$ with $|D| \not> \alpha$. For this D, we get $B_D = \bigcup_{a \in D} (D.n-a)$ where, $|D.n-a| \not> \alpha$ for each $a \in D$ so that $|B_D| \not> \alpha$. Thus $q \in cl_{\beta X} B_D$, where $|B_D| \not> \alpha$. Thus q is α-non-remote.

Result 3.3.13. Given $p \in \beta X$, α- remote for $\omega \leqslant \alpha \leqslant |X|$, $m \in X$, $m \neq 0$, there exists $q \in \beta X$, where q is α-remote such that $p+m = p.q$. When p given is α-non-remote, then so is the solution q.

Proof: We can proceed as in Results 3.3.11 and 3.3.12 once we know that given $p \in \beta X$, $m \in X$, $m \neq 0$, q can be obtained as an ultrafilter generated by the filter base

$\mathcal{B} = \left\{B_A : A \in p\right\}$, where, $B_A = \bigcup_{\substack{a \in A \\ a \neq 0}} (A+m).a^{-1}$.

(where a^{-1} denotes the multiplicative inverse of a).
[Result 3.2.3].

Result 3.3.14. Given $p \in \beta X$, α-remote, $\omega \leqslant \alpha \leqslant |X|$, there exists $q \in \beta X$, α-remote such that $p+q = p.q$. When p is α- non remote, then so is q.

Proof: Proceed as in Results 3.3.11 and 3.3.12, where q is an ultrafilter generated by the filter base $\mathcal{B} = \left\{B_A : A \in p\right\}$, where, $B_A = \bigcup_{\substack{a \in A \\ a \neq 1}} A.(a-1)^{-1}$ (Result 3.2.5).

Remark: In the case of βR, the above situations do not hold, once we assume the continuum hypothesis. But in the background where we assume negation of continuum hypothesis, then the above definitions and results hold in βR.

Chapter IV

ARITHMETIC IN THE LMC-COMPACTIFICATION OF R

§ 4.0. Introduction

We shall take R to be the set of real numbers considered as semitopological semigroup. Let $C_b(R)$ be the C^*-algebra of continuous and bounded complex-valued functions on R and βR, the Stone-Čech compactifications of R. Then βR is the space of continuous, multiplicative linear functionals on $C_b(R)$ and βR is compact in the weak $*$ topology and the Gelfand map $f \longmapsto \hat{f}$ defined by $\hat{f}(\mu) = \mu(f)$, $\mu \in \beta R$ is an isometric isomorphism of $C_b(R)$ onto $C_b(\beta R)$.

In [BA; BU] it has been proved that for a large class of semigroups S, it is impossible to introduce an Arens type product onto βS, in particular, this is so if S is a closed subsemigroup of a locally compact group which is neither compact nor discrete.

In [MI], Mitchell has introduced the space LMC(S) (a C^*-subalgebra of $C_b(S)$). It is easy to see that there is an Arens type product on βS, if and only if $LMC(S)=C_b(S)$. So, if (p,pS) denotes the canonical LMC(S)-compactification of S, then it is of interest to study pS as a semigroup, particularly when S = R.

In section 4.1 we include the preliminaries required for the construction of (p, pR), in order that pR can be studied as a quotient space of βR .

In section 4.2 we construct pR as the family of equivalence classes of z-ultrafilters on R so that pR is the quotient space of βR. We then extend the operations addition and multiplication on R to pR which makes pR left-continuous semigroup with respect to + and with respect to . .

In section 4.3 we characterize the sums and products in pR .

In section 4.4, using the characterization of sums and products in pR , we have shown that there exist solutions for equations in pR of the form $\rho + \eta = \alpha . \beta$, where at least one of $\rho, \eta, \alpha, \beta$ belong to R. This behaviour of pR is in contrast to that of βN [HI_2].

§ 4.1. Preliminaries

This section is devoted to a review of the main definitions and results which are needed for the later results. These are not original and they are included here for the sake of completeness; in the treatment of the topics which relate to them, we follow [BE; JU; MI], [RU_3], [WI].

4.1.1. Convention. S is a separately continuous, completely regular, Hausdorff topological semigroup.

4.1.2. Definitions

(1) Let $s \in S$ and f be a function on S. By the left (resp. right) multiplication by s in S, is meant the mapping λ_s(resp. ρ_s) defined from S into itself by $\lambda_s(t) = st$ (resp. $\rho_s(t) = ts$) for all $t \in S$.

(2) The left (resp. right) translate of f by s is the function $f \cdot \lambda_s$ (resp. $f \cdot \rho_s$) which we will denote by $L_s f$ (resp. $R_s f$).

(3) A set F of functions on S is said to be left (resp. right) translation invariant if $L_s f$ (resp. $R_s f$) belongs to F whenever $f \in F$ and $s \in S$. F is said to be translation invariant if it is both left and right translation invariant.

(4) A subalgebra of $C_b(S)$ is said to be perfect if it is a translation invariant C^*-subalgebra of $C_b(S)$ containing the constant functions.

(5) Let F be a left translation invariant normed vector subspace of $C_b(S)$. For each $\mu \in F^*$, the topological dual of F, a bounded linear mapping T_μ is defined from F into

$B(S)$, the set of all bounded complex-valued functions on S, by

$$T_\mu f(s) = \mu (L_s f) \text{ for all } f \in F \text{ and } s \in S.$$

The space F is called left introverted if $T_\mu f \in F$ for all $f \in F$ and $\mu \in F^*$. In the situation where F is a Banach sub-algebra of $C_b(S)$, F is said to be left m-introverted, if $T_\mu f \in F$ for all $f \in F$ and $\mu \in \Delta F$, the maximal ideal space of F.

(6) Suppose that F is left m-introverted and perfect. By an F-compactification of S, we mean a semigroup compactification (ψ, X) (i.e., a pair (ψ, X) such that X is a compact right topological semigroup and ψ is a continuous homomorphism from S into X) of S with the properties

$$(c_1): \psi(S) \subseteq \Lambda(X), \quad (c_2)\ F = \left\{ f \cdot \psi : f \in C(X) \right\}.$$

(Here $C(X)$ is the set of all continuous complex valued functions on X and $\Lambda(X)$ is the set of all $x \in X$ such that λ_x is continuous).

4.1.3. Theorem ([BE, JU, MI], p. 100 Corollary 2.6).

Let F be a left m-introverted perfect subalgebra of $C_b(S)$ and let (ψ_1, X_1) and (ψ_2, X_2) be F-compactifications of S. Then there exists an isomorphic homeomorphism

$\emptyset$ from X_1 onto X_2 such that $\emptyset.\psi_1 = \psi_2$.

4.1.4. Proposition ([HA],p. 6 , Proposition III 2.5).

Let (ψ, X) be a semigroup compactification with property (c_1). Then, $F = \{f.\psi : f \in C(X)\}$ is a left m-introverted perfect subalgebra of $C_b(S)$. Then (ψ, X) is an F-compactification of S.

4.1.5. Definition ([BE; JU, MI]).

We have the following subspace of $C_b(S)$.
$LMC(S) = \{f \in C_b(S) : s \mapsto \mu(L_s f)$ is continuous on S for all $\mu \in \Delta C_b(S)\}$. This is a left m-introverted perfect subalgebra of $C_b(S)$. Hence S has an LMC-compactification (p, pS). An important fact about (p, pS) is that it is maximal with property (c_1) in the sense that it has this property and given a semigroup compactification (ψ, X) of S with property (c_1), there exists a continuous homomorphism γ from pS onto X such that $\psi = \gamma . p$.

Theorem [HA]. Let F be a left m-introverted perfect subalgebra of $C_b(S)$ and let (ψ, X) be an F-compactification of S. Then ψ is a homeomorphism from S into X if and only if for every closed subset A of S and $s \in S \setminus A$, there exists $f \in F$ such that $f(s) = 1$ and $f(A) = \{0\}$.

LMC(R) satisfies the above properties and $p: R \longrightarrow pR$ is a homeomorphism where pR is the LMC-compactification of R.

§ 4.2. pR as a quotient of βR

By definition, [BE; JU; MI], we have,

$LMC(R) = \{ f \in C_b(R) : x \longmapsto \mu(L_x f)$ is continuous for every $\mu \in \beta R \}$, where $L_x f$ is defined by

$$L_x f(y) = (f.\lambda_x)(y) = f(xy).$$

So, $p: R \longrightarrow pR$ is defined as

$$p(x)\ (\mu(f)) = \mu(L_x f), \quad \text{for every } \mu \in \beta R,\ f \in C_b(R)$$

We can write

$$p(x)\ (\hat{f}(\mu)) = \mu(L_x f) = \mu(f.\lambda_x) = (f.\hat{\lambda}_x)(\mu),$$

for every $\mu \in \beta R$, $f \in C_b(R)$.

Given $t \in R$, define,

$$\mathfrak{Z}(t) = \{ \emptyset \neq Z(f \cdot \lambda_t) : f \in C_b(R),\ 1 \notin Z(f \cdot \lambda_t) \}.$$

Let $\mathfrak{Z}^*(t) = \{ \mathcal{B}_t : \mathcal{B}_t$ is a maximal Z-filter base in $\mathfrak{Z}(t) \}$.

Collect all the Z-ultrafilters generated by a $\mathcal{B}_t \in \mathfrak{Z}^*(t)$ and identify them. This identification is an equivalence relation in $\beta R \setminus R$.

If $\mathcal{B}_t \in \mathcal{F}^*(t)$ and $t \neq s$, then there exists $\mathcal{B}_s \in \mathcal{F}^*(s)$ that is "disjoint" from $\mathcal{B}_t$. (i.e., there exists $z_1 \in \mathcal{B}_t$ and $z_2 \in \mathcal{B}_s$ such that $z_1 \cap z_2 = \emptyset$ so that the Z-ultrafilters on R generated by $\mathcal{B}_t$ and $\mathcal{B}_s$ are different).

The principal Z-ultrafilter $e(t)$, $t \in R$, where $e: R \longrightarrow \beta R$ is the embedding, is generated by the Z-filter base

$$\begin{aligned} \mathcal{F}_1^*(t) &= \{Z(f.\lambda_t) : f \in C_b(R),\ 1 \in Z(f.\lambda_t)\} \\ &= e(t) = \{Z(f) : f \in C_b(R), t \in Z(f)\}. \end{aligned}$$

This extends the above equivalence to the whole of βR which is trivial on R.

4.2.2. Result. The equivalence defined above can be described alternatively as follows. Let, $\mu, \mu' \in \beta R$. Then $\mu \equiv \mu'$ in βR if and only if,

$$(f.\hat{\lambda}_t)(\mu) = (f.\hat{\lambda}_t)(\mu'),\ f \in C_b(R), \text{ for some } t \in R.$$

Proof: For $\mu \in \beta R$, $\overleftarrow{Z}(\mu) = \{f.\lambda_t : f \in C_b(R)$ and $\mu \in cl_{\beta R} Z(f.\lambda_t)$, $t \in R$ fixed$\}$ and $\mu \equiv \mu'$ if and only if $\overleftarrow{Z}(\mu) = \overleftarrow{Z}(\mu')$. This translated into Z-ultrafilters will give the result.

4.2.3. Result. Let Y be the set of all equivalence classes with the quotient topology. Then Y is the LMC-compactification pR of R.

Proof: With each $f.\lambda_t \in LMC(R)$, associate a function $g \in R^Y$ as follows. g(y) is the common value of $(f.\lambda_t)(\mu)$ at every point $\mu \in y$. Thus, $f = g.\tau$, where $\tau : \beta R \longrightarrow Y$ is the map which assigns to each $\mu \in \beta R$, its equivalence class τ_μ. Let C' denote the family of all such functions g. i.e., $g \in C'$ if and only if $g.\tau \in LMC(R)$. Now, the weak topology on Y induced by C' is the quotient topology, for, by definition, every function in C' is continuous on Y. Hence τ is continuous.

If y,y' are distinct points of Y, then there exists $g \in C'$ such that $g(y) \neq g(y')$. Thus Y is Hausdorff. Hence Y is completely regular.

Consider any function $h \in C_b(R)$. Since τ is continuous, $h.\tau$ is continuous on Y. This says that $h \in C'$. Therefore, $C' \supset C_b(R)$. Thus $C' = C_b(R)$ and this mapping $g \longmapsto g.\tau$ is an isomorphism.

Here τ is a quotient mapping, for, given any closed set $A \subseteq R$ and $x \in R \setminus A$, $p(x) \in pR$ and $A = F \cap pR$, for some closed subset F of pR. Since pR is compact Hausdorff, it is completely regular. So, there exists $g \in C_b(pR)$ such that $g(F) = \{0\}$ and $g(p(x))=1$. Let $f = g.p$. Then $f \in LMC(R)$ and $f(A) =\{0\}$ and $f(x) = 1$.

4.2.4. The members of pR will be denoted by ρ, γ, ζ etc where $\rho = [\mathcal{B}_t]$ for some $t \in R$ means that ρ is the equivalence class of all Z-ultrafilters generated by the Z-filter base $\mathcal{B}_t \in \mathcal{F}^*(t)$.

4.2.5. Definition. Let $\rho, \gamma \in pR$. Suppose that $\rho = [\mathcal{B}_t]$ and $\gamma = [\mathcal{B}_s]$ for some $t,s \in R$.

Define, $\mathcal{B}_t + \mathcal{B}_s = \left\{Z \subseteq R,\ Z \text{ closed} : \{x \in R : Z-x \in \mathcal{B}_t\} \in \mathcal{B}_s\right\}$.

Similarly,

$$\mathcal{B}_t \cdot \mathcal{B}_s = \left\{Z \subseteq R,\ Z \text{ closed} : \{x \in R : Z/_x \in \mathcal{B}_t\} \in \mathcal{B}_s\right\},$$

where,

$$\begin{aligned} Z-x &= \{z-x : z \in Z\} \text{ and} \\ Z/_x &= \{z/_x : z \in Z\}, \text{ when } x \neq 0 \\ &= R \quad \text{when } x = 0 \in Z \\ &= \emptyset \quad \text{when } x = 0 \notin Z \end{aligned}$$

4.2.6. Result. $\mathcal{B}_t + \mathcal{B}_s$ and $\mathcal{B}_t \cdot \mathcal{B}_s$ defined above are Z-filter bases.

Proof: We give the proof for '+' only, that for '.' being identical. Let $\mathcal{B}_t \in \mathfrak{Z}^*(t)$, $\mathcal{B}_s \in \mathfrak{Z}^*(s)$.

Let $Z(f.\lambda_t) \in \mathcal{B}_t$, $Z(g.\lambda_s) \in \mathcal{B}_s$. Then,

$$cl_R\,(Z(f.\lambda_t) + Z(g.\lambda_s)) \in \mathcal{B}_t + \mathcal{B}_s \text{ so that } \mathcal{B}_t + \mathcal{B}_s \neq \emptyset,$$

where,

$$Z(f.\lambda_t) + Z(g.\lambda_s) = \{x+y : x \in Z(f.\lambda_t), y \in Z(g.\lambda_s)\}.$$

Now, $\emptyset \notin \mathcal{B}_t + \mathcal{B}_s$, for

$$\begin{aligned} \emptyset \in \mathcal{B}_t + \mathcal{B}_s &\Rightarrow \{x \in R : \emptyset - x \in \mathcal{B}_t\} \in \mathcal{B}_s \\ &\Rightarrow \{x \in R : \emptyset \in \mathcal{B}_t\} \in \mathcal{B}_s \\ &\Rightarrow \emptyset \in \mathcal{B}_s \text{ which is not possible.} \end{aligned}$$

Let $Z, Z' \in \mathcal{B}_t + \mathcal{B}_s$. Then,

$$\{x \in R : Z - x \in \mathcal{B}_t\} \in \mathcal{B}_s \text{ and } \{x \in R : Z' - x \in \mathcal{B}_t\} \in \mathcal{B}_s.$$

Now, $\{x \in R : Z - x \in \mathcal{B}_t\} \cap \{x \in R : Z' - x \in \mathcal{B}_t\} = \{x \in R : (Z \cap Z') - x \in \mathcal{B}_t\}$.

$\mathcal{B}_s$, being a Z-filter base,

$\{x \in R : Z - x \in \mathcal{B}_t\} \cap \{x \in R : Z' - x \in \mathcal{B}_t\}$ contains a member of $\mathcal{B}_s$.

Thus, $\{x \in R : (Z \cap Z') - x \in \mathcal{B}_t\} \in \mathcal{B}_s$. So, $Z \cap Z' \in \mathcal{B}_t + \mathcal{B}_s$.

Thus, $\mathcal{B}_t + \mathcal{B}_s$ is a Z-filter base.

4.2.7. Definition. Let $\rho = [\mathcal{B}_t]$, $\eta = [\mathcal{B}_s]$, where $\mathcal{B}_t \in \mathfrak{Z}^*(t)$, $\mathcal{B}_s \in \mathfrak{Z}^*(s)$ for some $t, s \in R$. Define $\rho + \eta = [\mathcal{B}_{t+s}]$, where $\mathcal{B}_{t+s} \in \mathfrak{Z}^*(t+s)$ is such that every

$Z \in \mathcal{B}_t + \mathcal{B}_s$ is contained in some $Z' \in \mathcal{B}_{t+s}$. Define $\rho \cdot \gamma = [\mathcal{B}_{t.s}]$, where $\mathcal{B}_{t.s} \in \mathfrak{F}^*(t.s)$ is such that every $Z \in \mathcal{B}_t \cdot \mathcal{B}_s$ is contained in some $Z' \in \mathcal{B}_{t,s}$.

4.2.8. Result. Given $\mathcal{B}_t$, $\mathcal{B}_s$, there is some $\mathcal{B}_{t+s} \in \mathfrak{F}^*(t+s)$ such that for every $Z \in \mathcal{B}_t + \mathcal{B}_s$, there is some $Z' \in \mathcal{B}_{t+s}$ such that $Z \subset Z'$, and hence the addition in 4.2.7 is well defined.

Proof: Given $\mathcal{B}_t \in \mathfrak{F}^*(t)$, $\mathcal{B}_s \in \mathfrak{F}^*(s)$, let $Z_1 = Z(f.\lambda_t) \in \mathcal{B}_t$ $Z_2 = Z(g.\lambda_s) \in \mathcal{B}_s$. Then

$$cl_R\,(Z_1+Z_2) \in \mathcal{B}_t + \mathcal{B}_s \tag{1}$$

Let $z \in Z_1+Z_2$. Then $z = x+y$, where $x \in Z(f.\lambda_t)$, $y \in Z(g.\lambda_s)$. So $f(tx) = 0$, $g(sy) = 0$.

i.e., $f\left(\frac{t(t+s)zx}{(t+s)z}\right) = 0$ and $g\left(\frac{s(t+s)zy}{(t+s)z}\right) = 0$

if $s+t \neq 0$, $z \neq 0$.

i.e., $(f_1.\lambda_{t+s})(z) = 0$ and $(g_1.\lambda_{t+s})(z) = 0$, where

$$f_1 = f.\lambda_{tx/(t+s)z} \in C_b(R) \text{ and}$$

$$g_1 = g.\lambda_{sy/(t+s)z} \in C_b(R)\ .$$

Thus $z \in Z(f_1 \cdot \lambda_{t+s}) \cap Z(g_1 \cdot \lambda_{t+s}) = Z((f_1+g_1) \cdot \lambda_{t+s}) \in \mathcal{B}'_{t+s}$ for some $\mathcal{B}'_{t+s} \in \mathfrak{F}^*(t+s)$. Therefore,

$$cl_R\ (Z_1+Z_2) \subseteq Z((f_1+g_1) \cdot \lambda_{t+s}) \qquad (2)$$

Now, if Z_o is any member of $\mathcal{B}_t+\mathcal{B}_s$, then, $Z_1 = Z_o \cap cl_R(Z_1+Z_2) \in \mathcal{B}_t+\mathcal{B}_s$. As explained above, we get a member $Z' \in \mathcal{B}'_{t+s}$ such that $Z_1 \subset Z'$, since $\mathcal{B}'_{t+s}$ is maximal with respect to finite intersection property. Since $Z_1 \subset Z_o$, we obtain a $Z'' \in \mathcal{B}'_{t+s}$ such that $Z_o \subset Z''$, where, Z'' and $Z((f_1+g_1) \cdot \lambda_{t+s})$ meet in $Z' \in \mathcal{B}'_{t+s}$. Thus every $Z \in \mathcal{B}_t+\mathcal{B}_s$ is contained in some member of $\mathcal{B}'_{t+s} \in \mathfrak{F}^*(t+s)$.

If $t+s = 0$, then $s = -t$ so that $\mathcal{B}_s = \mathcal{B}_{-t}$. But $\mathcal{B}_{-t} = \mathcal{B}'_t$ for some $\mathcal{B}'_t \in \mathfrak{F}^*(t)$. So we need consider only $\mathcal{B}_t + \mathcal{B}'_t$ in which case also we get the above conclusion. Similarly when $z = 0$, we can obtain the same result.

Now, if $\mathcal{B}_t = \mathcal{B}_{t'}$, and $\mathcal{B}_s = \mathcal{B}_{s'}$, then by the above argument, every $Z \in \mathcal{B}_t + \mathcal{B}_s$ is contained in some $Z' \in \mathcal{B}_{t+s}$. Similarly every $Z \in \mathcal{B}_{t'} + \mathcal{B}_{s'}$ is contained in some $Z' \in \mathcal{B}_{t'+s'}$. But $\mathcal{B}_t+\mathcal{B}_s = \mathcal{B}_{t'} + \mathcal{B}_{s'}$. So, every member of $\mathcal{B}_{t+s}$ should meet some member of $\mathcal{B}_{t'+s'}$ and the maximality of

$\mathcal{B}_{t+s}$, $\mathcal{B}_{t'+s'}$ imply that $\mathcal{B}_{t+s} = \mathcal{B}_{t'+s'}$. Thus the addition in 4.2.7 is well-defined.

4.2.9. Note 1. We have a similar result for the product defined in 4.2.7.

Note 2. Given $\mathcal{B}_{t+s}$, there is some $\mathcal{B}_t \in \mathcal{J}^*(t)$, say $\mathcal{B}'_t$, and some $\mathcal{B}_s \in \mathcal{J}^*(s)$, say, $\mathcal{B}'_s$ such that every $Z \in \mathcal{B}_{t+s}$ is contained in some $Z' \in \mathcal{B}'_t + \mathcal{B}'_s$.

Proof is similar to that of result 4.2.8.

4.2.10. Result. Addition and multiplication defined in pR are left-continuous.

Proof: We give the proof for '+' only.

Define f: pR $\longrightarrow$ pR by $f(\eta) = \rho + \eta$, where $\rho = [\mathcal{B}_t]$ and $\eta = [\mathcal{B}_s]$ for some $t,s \in R$ so that $\rho + \eta = [\mathcal{B}_{t+s}]$, where, $\mathcal{B}_{t+s} \in \mathcal{J}^*(t+s)$ is such that every $Z \in \mathcal{B}_{t+s}$ contains some Z' in $\mathcal{B}_t + \mathcal{B}_s$. If q: $\beta R \longrightarrow$ pR is the quotient map and U is an open neighbourhood of $\rho + \eta$ then $q^{-1}(U)$ is open in βR and every member in $[\mathcal{B}_{t+s}]$ lies in $q^{-1}(U)$. Then $\beta R - q^{-1}(U)$ is closed in βR and we have $\beta R - q^{-1}(U) = \bigcap \overline{Z}$, for some zero-sets Z in R, where, $\overline{Z} = \{$All Z-ultrafilters on R containing Z as a member$\}$. So, no member of $[\mathcal{B}_{t+s}]$

belongs to $\cap \bar{Z}$. So some zero set in this closed set $\cap \bar{Z}$, say Z_o, does not belong to $\mathcal{B}_{t+s}$. So, $Z_o \notin \mathcal{B}_t + \mathcal{B}_s$ by definition of $\mathcal{B}_{t+s}$. i.e., $B = \{x \in R: Z_o - x \in \mathcal{B}_t\} \notin \mathcal{B}_s$. i.e., $B \notin \mathcal{B}_s$. So no member of $\beta R \setminus R$ generated by $\mathcal{B}_s$ belongs to $\bar{B} = cl_{\beta R} B$. So, every Z-ultrafilter generated by $\mathcal{B}_s$ belongs to $\beta R - \bar{B}$, which is an open set in βR. So, $\zeta = [\mathcal{B}_s] \in q(\beta R - \bar{B}) = W$ (say), which is open in pR. If $\xi \in pR$ is such that $\xi \in W$, say, $\xi = [\mathcal{B}_r]$, for some $r \in R$, then $B \notin \mathcal{B}_r$. i.e., $\{x \in R: Z_o - x \in \mathcal{B}_t\} \notin \mathcal{B}_r$. i.e., $Z_o \notin \mathcal{B}_t + \mathcal{B}_r$, and so $Z_o \notin \mathcal{B}_{t+r}$. So, no member belonging to $[\mathcal{B}_{t+r}]$ belongs to $\cap \bar{Z}$. i.e., Every member generated by $\mathcal{B}_{t+r}$ belongs to $\beta R - \cap \bar{Z} = q^{-1}(U)$. i.e., $[\mathcal{B}_{t+r}] \in U$. i.e., $\rho + \xi \in U$. Thus, f is continuous.

4.2.11. Result. The operations + and . in pR are associative.

Proof: We give the proof for '+' only.

Let $\rho = [\mathcal{B}_t]$, $\zeta = [\mathcal{B}_s]$, $\xi = [\mathcal{B}_r]$ be members of pR, for some $t, s, r \in R$.

$$Z \in \mathcal{B}_t+(\mathcal{B}_s+\mathcal{B}_r) \Leftrightarrow \{g \in R: Z-g \in \mathcal{B}_t\} \in \mathcal{B}_s+\mathcal{B}_r$$

$$\Leftrightarrow \{h \in R: \{g \in R: Z-g \in \mathcal{B}_t\}- h \in \mathcal{B}_s\} \in \mathcal{B}_r$$

$$\Leftrightarrow \{h \in R: \{g \in R: (Z-h)-g \in \mathcal{B}_t\} \in \mathcal{B}_s\} \in \mathcal{B}_r$$

$$\Leftrightarrow \{h \in R: Z-h \in \mathcal{B}_t + \mathcal{B}_s\} \in \mathcal{B}_r$$

$$\Leftrightarrow Z \in (\mathcal{B}_t+\mathcal{B}_s) + \mathcal{B}_r.$$

Thus the two Z-filter bases $\mathcal{B}_t + (\mathcal{B}_s+\mathcal{B}_r)$ and $(\mathcal{B}_t+\mathcal{B}_s)+\mathcal{B}_r$ are the same. But $\rho + (\zeta+\xi) = [\mathcal{B}_{t+(s+r)}]$, where, $\mathcal{B}_{t+(s+r)} \in \mathcal{F}^*(t+(s+r))$ is such that every $Z \in \mathcal{B}_{t+(s+r)}$ contains some $Z' \in \mathcal{B}_t + (\mathcal{B}_s+\mathcal{B}_r)$ and $(\rho+\zeta)+\xi = [\mathcal{B}'_{(t+s)+r}]$, where $\mathcal{B}'_{(t+s)+r} \in \mathcal{F}^*((t+s)+r)$ is such that every $Z \in \mathcal{B}'_{(t+s)+r}$ contains some member of $(\mathcal{B}_t+\mathcal{B}_s)+\mathcal{B}_r$. So, we must have $\mathcal{B}_{t+(s+r)} = \mathcal{B}'_{(t+s)+r}$, because the two Z-filter bases belong to $\mathcal{F}^*(t+s+r)$. Thus, $\rho+(\zeta+\xi) = (\rho+\zeta)+\xi$.

4.2.12. The operations + and . are associative left-continuous operations on pR which extend ordinary addition and multiplication on R.

Proof: For any $x, y \in R$, we have by definition, $p(x)$ = the equivalence class consisting of the Z-ultrafilter e(x) [$e:R \longrightarrow \beta R$ is the embedding of R into βR] generated by the Z-filter base,

$$\mathfrak{F}_1^*(x) = \{\emptyset \neq Z(f.\lambda_x) : 1 \in Z(f.\lambda_x),\ f \in C_b(R)\}$$
$$= e(x) = \{\emptyset \neq Z(f) : x \in Z(f),\ f \in C_b(R)\}.$$

Similarly we have $p(y) = e(y)$.

4.2.13. **Result.** The centers of the semigroups $(pR,+)$ and $(pR,.)$ contain R.

Proof: Let $x \in R$ and $\rho = [\mathcal{B}_t] \in pR \setminus R$ for some $t \in R$. We shall show that $\rho + p(x) = p(x) + \rho$, the proof for . being essentially identical.

Consider $\rho + p(x)$, where $\rho = [\mathcal{B}_t]$ and $p(x) = [\mathfrak{F}_1^*(x)]$

Now, $Z \in \mathcal{B}_t + \mathfrak{F}_1^*(x) \Rightarrow \{y \in R : Z - y \in \mathcal{B}_t\} \in \mathfrak{F}_1^*(x)$

$\Rightarrow Z-1 \in \mathcal{B}_t$, because 1 belongs to every member of $\mathfrak{F}_1^*(x)$.

But $Z-1 = \{z \in R:\ 1+z \in Z\}$

$= \{z \in R:\ 1 \in Z-z\}$. i.e., $Z-z$ intersects every member of $\mathfrak{F}_1^*(x)$.

$= \{z \in R : Z-z \in \mathfrak{F}_1^*(x)\}$.

Therefore $\{z \in R:\ Z-z \in \mathfrak{F}_1^*(x)\} \in \mathcal{B}_t$.

Therefore $Z \in \mathfrak{F}_1^*(x) + \mathcal{B}_t$.

Thus $\mathcal{B}_t + \mathfrak{F}_1^*(x) \subseteq \mathfrak{F}_1^*(x) + \mathcal{B}_t$. A similar argument yields $\mathfrak{F}_1^*(x) + \mathcal{B}_t \subseteq \mathcal{B}_t + \mathfrak{F}_1^*(x)$. Thus every member of $\mathcal{B}_{t+x}$ which generates members of $\rho + p(x)$ (and $\mathcal{B}_{x+t}$) contains members of $\mathcal{B}_t + \mathfrak{F}_1^*(x)$ and $\mathfrak{F}_1^*(x) + \mathcal{B}_t$. So $[\mathcal{B}_{t+x}] = [\mathcal{B}_{x+t}]$. i.e., $\rho + p(x) = p(x) + \rho$.

4.2.14. **Result.** Let $\rho = [\mathcal{B}_t] \in pR$, where $t \in R$ and $m \in R$, $m \neq 0$. Then, $\rho + m = \rho + p(m)$, where, $p(m) = [\mathfrak{F}_1^*(m)]$ is such that $\rho + p(m)$ is the class of all Z-ultrafilters generated by Z-filter base $\mathcal{B}_t + m = \{Z+m : Z \in \mathcal{B}_t\}$.

Proof: By m we mean p(m), where p(m) is the class consisting of the Z-ultrafilter e(m) [where $e: R \longrightarrow \beta R$ is the embedding] generated by the Z-filter base.

$$\mathfrak{F}_1^*(m) = \{\emptyset \neq Z(f.\lambda_m) : 1 \in Z(f.\lambda_m),\ f \in C_b(R)\}$$

Let $\rho + p(m) = [\mathcal{B}_{t+m}]$. Then $Z \in \mathcal{B}_{t+m} \Rightarrow Z \supset Z'$, where,

$$Z' \in \mathcal{B}_t + \mathfrak{F}_1^*(m) \Rightarrow Z' \in \mathcal{B}_t + e(m)$$

$$\Rightarrow \{x \in R : Z' - x \in \mathcal{B}_t\} \in e(m)$$

$$\Rightarrow Z' - m \in \mathcal{B}_t \Rightarrow Z' \in \mathcal{B}_t + m \text{ (defined as above)}$$

Conversely,

$$Z' \in \mathcal{B}_t + m \Rightarrow Z' - m \in \mathcal{B}_t \text{ and } m \in e(m)$$

$$\Rightarrow \{m\} \subseteq \{x \in R : Z' - x \in \mathcal{B}_t\}$$
$$\Rightarrow Z' \in \mathcal{B}_t + e(m) \Rightarrow Z' \in \mathcal{B}_t + \mathcal{F}_1^*(m)$$

So, every member of $\mathcal{B}_{t+m}$ which generates $\rho + p(m)$ contains some member of $\mathcal{B}_t + m = \{Z+m : Z \in \mathcal{B}_t\}$ and $\mathcal{B}_t + m$ is a Z-filter base. Hence, $\rho + m = [\mathcal{B}_{t+m}] = [\mathcal{B}_t + m]$.

4.2.3. Result. Let $\rho = [\mathcal{B}_t] \in pR$ for some $t \in R$ and $n \in R$, $n \neq 1$. Then $\rho.n = \rho.p(n)$, where, $p(n) = [\mathcal{F}_1^*(n)]$, is such that $\rho.p(n)$ is the class of all Z-ultrafilters generated by the Z-filter base $\mathcal{B}_t.n = \{Z.n : Z \in \mathcal{B}_t\}$.

Proof is similar to that for addition.

We now have the following characterizations of sums and products in pR.

§ 4.3 Sums and Products in pR

4.3.1. Result. Let $\rho, \zeta \in pR$, where $\rho = [\mathcal{B}_t]$, $\zeta = [\mathcal{B}_s]$ for some $t, s \in R$. Let $Z \subseteq R$ be closed. Then,

(a) $Z \in \mathcal{B}_t + \mathcal{B}_s$ if and only if there exists $Z' \in \mathcal{B}_s$ and a family $\{B_x : x \in Z'\} \subset \mathcal{B}_t$ such that $\left(\bigcup_{x \in Z'} V_x\right) \subset Z$, where, for each $x \in Z'$, V_x is closed locally finite subset of R such that $V_x \subset \mathrm{Int}(B_x + x)$ and $\{V_x - x : x \in Z'\} \subset \mathcal{B}_t$.

(b) $Z \in \mathcal{B}_t \cdot \mathcal{B}_s$ if and only if there exists $Z' \in \mathcal{B}_s$ and a family $\{B_x : x \in Z'\} \subset \mathcal{B}_t$ such that $\left(\bigcup_{x \in Z'} V_x\right) \subset Z$, where, for each $x \in Z'$, V_x is a closed locally finite subset of R such that $V_x \subset \text{Int}\,(B_x . x)$ and $\{V_{x/x} : x \in Z'\} \subset \mathcal{B}_t$.

Proof: We establish (a) only.

Necessity: Suppose that $Z \in \mathcal{B}_t + \mathcal{B}_s$. Then by definition, $Z' = \{x \in R : Z-x \in \mathcal{B}_t\} \in \mathcal{B}_s$. i.e., $Z' \in \mathcal{B}_s$. Put $B_x = Z-x, x \in Z'$. Then, $\{B_x : x \in Z'\} \subset \mathcal{B}_t$. Now,

$\mathcal{U} = \{\text{Int}\,(R - (\text{Int}\,(B_x+x)) : x \in Z'\} \cup \{\text{Int}(B_x+x) : x \in Z'\}$

is an open cover of R. Since R is paracompact, $\mathcal{U}$ has a closed locally finite refinement, say $\mathcal{V}$. Let

$\mathcal{V}' = \{V_x \in \mathcal{V} : V_x \subset \text{Int}\,(B_x+x),\ x \in Z'\}$. Then $\mathcal{V}'$ is a family of locally finite closed sets. So $\bigcup_{x \in Z'} V_x$ is closed and $\left(\bigcup_{x \in Z'} V_x\right) \subset Z$. Here, for each $x \in Z'$, we have $V_x \subset \text{Int}\,(B_x+x)$. So, $V_x - x \subset \text{Int}\, B_x \subset Z-x$, where $Z-x \in \mathcal{B}_t$, which means that $Z-x = Z(f.\lambda_t)$ for some $f \in C_b(R)$. Also, for each $x \in Z'$, $V_x - x$ is a closed set in R and hence a zero-set, say $V_x - x = Z(h)$ for some $h \in C_b(R)$. So, $\emptyset \neq Z(h) \subset Z(f.\lambda_t)$. So, $Z(h) = Z(h) \cap Z(f.\lambda_t)$, which is a zero-set in R belonging to $\mathcal{B}_t$, since, $Z(h) \cap Z(f.\lambda_t) \supset Z((g^2+f^2).\lambda_t)$, where, $g = h.\lambda_{1/t} \in C_b(R)$. Thus, $\{V_x - x : x \in Z'\} \subset \mathcal{B}_t$.

Sufficiency: Suppose that there exists $Z' \in \mathcal{B}_t$ and a family $\{B_x : x \in Z'\} \subset \mathcal{B}_t$ such that $\left(\bigcup_{x \in Z'} V_x\right) \subset Z$, where, for each $x \in Z'$, V_x is a closed locally finite subset of R such that $V_x \subset \mathrm{Int}(B_x + x)$ and $\{V_x - x : x \in Z'\} \subset \mathcal{B}_t$. It suffices to show that $\left(\bigcup_{x \in Z'} V_x\right) \in \mathcal{B}_t + \mathcal{B}_s$. Suppose not. Then, if $\mu \in [\mathcal{B}_{t+s}]$, then there exists $A \subseteq R \setminus \left(\bigcup_{x \in Z'} V_x\right)$ such that $A \in \mu$. But μ is generated by $\mathcal{B}_{t+s}$. So, there exists $B \in \mathcal{B}_t + \mathcal{B}_s$ such that $B \subseteq A$. Now, $B \in \mathcal{B}_t + \mathcal{B}_s \Rightarrow C = \{x \in R : B - x \in \mathcal{B}_t\} \in \mathcal{B}_s$. Also, $Z' \in \mathcal{B}_s$. So, there exists $D \in \mathcal{B}_s$ such that $D \subset C \cap Z'$. Pick $n \in D$. Then, $B - n \in \mathcal{B}_t$. Also $V_n - n \in \mathcal{B}_t$. Pick $y \in (B-n) \cap (V_n - n)$. Then $y + n \in B \cap V_n$, a contradiction.

4.3.2. Result. Let $\rho = [\mathcal{B}_t] \in pR \setminus R$ and $m \in R$. Then,

(a) There exists $\zeta \in pR \setminus R$ such that $\zeta + m = \rho$.

(b) There exists $\zeta \in pR \setminus R$ such that $\zeta \cdot m = \rho$, $m \neq 0$.

Proof:

(a) Define $\mathcal{B}_s = \{Z - m : Z \in \mathcal{B}_t\}$. Note that $\mathcal{B}_s$ is contained in $\mathcal{B}_{t-m} = \mathcal{B}_t + \mathcal{B}_{-m}$.

(b) Define $\mathcal{B}_s = \{Z/_m : Z \in \mathcal{B}_t\}$. Then $\mathcal{B}_s$ is contained in $\mathcal{B}_{t/m} = \mathcal{B}_t \cdot \mathcal{B}_{1/m}$.

4.3.3. Result. Let $p \in pR\setminus R$, where $p = [\mathcal{B}_t]$ for some $t \in R$ and let $m \in R$. The following statements are equivalent.

(a) There is some $q = [\mathcal{B}_s]$ such that $\mathcal{B}_t + m = \mathcal{B}_{t.s}$.

(b) For each $Z \in \mathcal{B}_t$, there exists $n \in R$ such that $(Z+m)/_n \in \mathcal{B}_t$.

(c) For each function $f: R \longrightarrow \mathcal{B}_t$, there exists $n \in R$ such that $(f(m)+m) \cap (f(n).n) \neq \emptyset$.

Proof: (a) $\Longrightarrow$ (b). Note first that $q \in pR\setminus R$. We have, $\mathcal{B}_t + m = \mathcal{B}_{t.s}$. Let $Z \in \mathcal{B}_t$. Then $Z+m \in \mathcal{B}_t + m$. So, $Z+m \in \mathcal{B}_{t.s}$. So, $Z+m \supset Z'$, where $Z' \in \mathcal{B}_t . \mathcal{B}_s$ so that $Z' = Z_o + m \in \mathcal{B}_t . \mathcal{B}_s$, where $Z_o \in \mathcal{B}_t$. So $\{x \in R : (Z_o+m)/_x \in \mathcal{B}_t\} \in \mathcal{B}_s$, and hence is infinite. So, there exists $n \in R$ such that $(Z_o+m)/_n \in \mathcal{B}_t$, which means that $(Z+m)/_n \in \mathcal{B}_t$, by maximality of $\mathcal{B}_t$.

(b) $\Longrightarrow$ (c). Let $f: R \to \mathcal{B}_t$. Then $f(m) \in \mathcal{B}_t$. So pick $n \in R$ such that $(f(m)+m)/_n \in \mathcal{B}_t$. Let $y \in f(n) \cap (f(m)+m)/_n$. Then $(f(m)+m) \cap (f(n).n) \neq \emptyset$.

(c) $\Longrightarrow$ (b). Let $Z \in \mathcal{B}_t$. Suppose that for each $n \in R$, one has $(Z+m)/_n \notin \mathcal{B}_t$. Consider,

$\mathcal{U} = \left\{ \mathrm{Int}(Z+m)/_n \; : \; n \in R \right\} \cup \left\{ R-(Z+m)/_n \; : \; n \in R \right\}$. Then $\mathcal{U}$ is an open cover of R. Since R is paracompact, $\mathcal{U}$ has a closed locally finite refinement, say $\mathcal{V}$. Let

$\mathcal{V}' = \left\{ V_n \in \mathcal{V} : V_n \subset R - (Z+m)/_n \;, \; n \in R \right\}$. Then $\mathcal{V}'$ is a family of locally finite closed sets. So $\left(\bigcup_{V_n \in \mathcal{V}'} V_n \right)$ is closed. Define a function $f: R \longrightarrow \mathcal{B}_t$, by

$$f(n) = Z \text{ if } n = m$$

$$= \left(\bigcup_{V_n \in \mathcal{V}'} V_n \right) \text{if } n \neq m.$$

Then a contradiction is obtained.

(b) $\Longrightarrow$ (a). Let $\mathcal{Q} = \left\{ \left\{ x \in R : (Z+m)/_x \in \mathcal{B}_t \right\} : Z \in \mathcal{B}_t \right\}$.
Then $\mathcal{Q}$ is a Z-filter base in which each member contains a member of the Z-filter base $\mathcal{B}_{(\frac{t+m}{t})} = \mathcal{B}_s$.

Let $\mathcal{Z}$ be the class of all Z-ultrafilters generated by $\mathcal{B}_s$.
[If $B = \left\{ x \in R : (Z+m)/_x \in \mathcal{B}_t \right\}$ is a member of $\mathcal{Q}$ for some $Z \in \mathcal{B}_t$, then $B \supset B'$ where $B' \in \mathcal{B}_s$.

So, $\left\{x \in R : (Z+m)/_x \in \mathcal{B}_t\right\} \in \mathcal{B}_s$. i.e., $Z+m \in \mathcal{B}_t \cdot \mathcal{B}_s$ and hence $Z+m \in \mathcal{B}_{t.s}$. But $Z \in \mathcal{B}_t$ and so $Z+m \in \mathcal{B}_t + \mathcal{F}_1^*(m)$. i.e., $Z+m \in \mathcal{B}_{t+m} = \mathcal{B}_t + m$. So $\mathcal{B}_t + m = \mathcal{B}_{t.s}$].

4.3.4. **Result.** Let $\rho \in pR \setminus R$, where $\rho = [\mathcal{B}_t]$ for some $t \in R$. Let $m \in R$, where $m \neq 0$. Then there exists $\gamma = [\mathcal{B}_s] \in pR$ such that $\rho + m = \rho \cdot \gamma$ if and only if for each $Z \in \mathcal{B}_t$ and each function $f: R \longrightarrow \mathcal{B}_t$, there exists $n \in R$ such that $(Z+m) \cap (f(n).n) \neq \emptyset$.

Proof: Necessity. Let $Z \in \mathcal{B}_t$ and $f: R \longrightarrow \mathcal{B}_t$ given. Then $Z+m \in \mathcal{B}_t + m$. We have $\rho + m = \rho \cdot \gamma$. Let $Z+m \in \mathcal{B}_t + m$. Then $Z+m \in \mathcal{B}_{t.s}$, so, $Z+m$ contains Z', where $Z' \in \mathcal{B}_t \cdot \mathcal{B}_s$, and $Z' = Z_o + m$ for $Z_o \in \mathcal{B}_t$. i.e., $\left\{x \in R : (Z_o+m)/_x \in \mathcal{B}_t\right\} \in \mathcal{B}_s$. So, pick $n \in R$ such that $(Z_o+m)/_n \in \mathcal{B}_t$. Also $f(n) \in \mathcal{B}_t$. So, $(Z_o+m)/_n \cap f(n) \neq \emptyset$. If $x_o \in (Z_o+m)/_n \cap f(n)$, then $x_o.n \in (Z_o+m) \cap (f(n).n)$. Thus $(Z_o+m) \cap (f(n).n) \neq \emptyset$. Therefore, $(Z+m) \cap (f(n).n) \neq \emptyset$.

Sufficiency: Suppose that for each $Z \in \mathcal{B}_t$ and each $f: R \to \mathcal{B}_t$, $\mathcal{C}(Z,f) = \left\{n \in R : (Z+m) \cap (f(n).n) \neq \emptyset\right\}$. We claim $\mathcal{D} = \left\{\mathcal{C}(Z,f) : Z \in \mathcal{B}_t,\ f: R \longrightarrow \mathcal{B}_t\right\}$ is a Z-filter base. In fact, given $Z_1, Z_2 \in \mathcal{B}_t$ and $f_1, f_2: R \longrightarrow \mathcal{B}_t$, we have $Z \in \mathcal{B}_t$ such that $Z \subset Z_1 \cap Z_2$ and $f: R \longrightarrow \mathcal{B}_t$ defined by $f(n) \subset f_1(n) \cap f_2(n)$. Then $\mathcal{C}(Z,f) \subset \mathcal{C}(Z_1,f_1) \cap \mathcal{C}(Z_2,f_2)$

and by assumption, $\mathcal{C}(Z,f)' \neq \emptyset$. In fact each member of $\mathcal{D}$ contains a member of the Z-filter base $\mathcal{B}_{\frac{t+m}{t}} = \mathcal{B}_s$. Let $\mathcal{Y}$ be the class of all Z-ultrafilters generated by $\mathcal{B}_s$. We claim that for this $\mathcal{Y}$, $\rho+m = \rho \cdot \mathcal{Y}$. For this, we prove that $\mathcal{B}_t+m = \mathcal{B}_{t.s}$. Suppose instead that there is some $Z_o \in \mathcal{B}_t+m \setminus \mathcal{B}_{t.s}$. Then $Z_o-m \in \mathcal{B}_t$ and there exists $Z_1 \subsetneq R \setminus Z_o$ such that $Z_1 \in \mathcal{B}_{t.s}$ and so $Z_1 \supset Z_2$ where, $Z_2 \in \mathcal{B}_t \cdot \mathcal{B}_s$. Let $B = \left\{ x \in R : Z_{2/x} \in \mathcal{B}_t \right\}$. Then $B \in \mathcal{B}_s$

Define $f: R \longrightarrow \mathcal{B}t$ by

$$f(n) = Z_{2/n} \text{ if } n \in B$$
$$= \bigcup_{n \in B} V_n \text{ if } n \notin B,$$

where $\{V_n : n \in B\}$ is a family of locally finite closed sets such that $V_n \subsetneq R - Z_{2/n}$, for $n \in B$.

[$\mathcal{U} = \left\{ \text{Int } Z_{2/n} : n \in B \right\} \cup \left\{ R - Z_{2/n} : n \in B \right\}$ is an open cover of R. Since R is paracompact, $\mathcal{U}$ has a closed locally finite refinement say $\mathcal{V}$. Let $\mathcal{V}' = \left\{ V_n \in \mathcal{V} : V_n \subset R - Z_{2/n} : n \in B \right\}$. Then $\mathcal{V}'$ is a family of locally finite closed sets and $\bigcup_{n \in B} V_n$ is a closed set and hence a zero-set in R]. Then $\mathcal{C}(Z_o-m, f)$ belongs to the family $\mathcal{D}$. So, pick

$n \in B \cap \mathcal{C}(Z_o - m, f)$. Then we have,

$(Z_o - m+m) \cap (f(n).n) = Z_o \cap Z_1 \neq \emptyset$, a contradiction.

4.3.5. Result. Let $\rho \in pR \setminus R$ and $n \in R$, $n \neq 1$, where $\rho = [\mathcal{B}_t]$ for some $t \in R$. Then there exists $\eta \in pR$ such that $\rho + \eta = \rho.n$ if and only if for each $Z \in \mathcal{B}_t$ and each function $f:R \longrightarrow \mathcal{B}_t$, there exists $m \in R$ such that $(f(m)+m) \cap (Z.n) \neq \emptyset$.

Proof: The proof is essentially identical to that of the previous result.

4.3.6. Result. Let $\rho, \eta \in pR \setminus R$, where $\rho = [\mathcal{B}_t]$, $\eta = [\mathcal{B}_s]$ for some $t,s \in R$. Then $\rho + \eta \neq \rho \cdot \eta$ if and only if there exists $Z \in \mathcal{B}_s$ and a family $\{A_x : x \in Z\} \subset \mathcal{B}_t$ such that $(n.A_n) \cap (m+A_m) = \emptyset$, whenever $n,m \in Z$.

Proof: Necessity. Since $\rho + \eta \neq \rho \cdot \eta$ we have $\mathcal{B}_{t+s} \neq \mathcal{B}_{t.s}$. In fact no member of $\mathcal{B}_{t+s}$ contains a member of $\mathcal{B}_{t.s}$ and no element of $\mathcal{B}_{t.s}$ contains an element of $\mathcal{B}_{t+s}$.
Pick $D \in \mathcal{B}_{t+s} \setminus \mathcal{B}_{t.s}$. Then there exists $Z_o \subseteq R \setminus D$ such that $Z_o \in \mathcal{B}_{t.s}$. So, $D \supset D_o$, $Z_o \supset Z_1$ such that, $D_o \in \mathcal{B}_t + \mathcal{B}_s$ and $Z_1 \in \mathcal{B}_t \cdot \mathcal{B}_s$. i.e., $\{x \in R : D_o - x \in \mathcal{B}_t\} \in \mathcal{B}_s$ and

$\{x \in R : Z_{1/x} \in \mathcal{B}_t\} \in \mathcal{B}_s$. Let $B = \{x \in R : D_o - x \in \mathcal{B}_t$ and $Z_{1/x} \in \mathcal{B}_t\}$.
Since $\mathcal{B}_s$ is a Z-filter base, B contains a member $B_o \in \mathcal{B}_s$.
For each $n \in B_o$, let $G_n = (D_o - n) \cap (Z_{1/n})$. Since $\mathcal{B}_t$ is a Z-filter base, for each $n \in B_o$, there exists $A_n \in \mathcal{B}_t$ such that $A_n \subset G_n$. Then $\{A_n : n \in B_o\} \subset \mathcal{B}_t$ and for $n \in B_o$, $n + A_n \subseteq D_o$ and $n . A_n \subseteq Z_1$, and so $(n + A_n) \cap (n . A_n) = \emptyset$.

Sufficiency: Suppose that there exists $Z \in \mathcal{B}_s$ and a family $\{A_n : n \in Z\} \subseteq \mathcal{B}_t$.
$\mathcal{U} = \{R - (A_n + n) : n \in Z\} \cup \{Int(A_n + n) : n \in Z\}$ is an open cover of R. Since R is paracompact, $\mathcal{U}$ has a closed locally finite refinement, say $\mathcal{V}$. Let $\mathcal{V}' = \{V_n \in \mathcal{V} : V_n \subset Int(A_n + n),\ n \in Z\}$.
Then $\mathcal{V}'$ is a family of locally finite closed sets. So, $D = \bigcup_{n \in Z} V_n$ is closed. Also, $\{V_n - n : n \in Z\} \subset \mathcal{B}_t$, since $V_n - n \subset A_n$ for each $n \in Z$ and $V_n - n$ is a zero-set contained in A_n which belongs to $\mathcal{B}_t$. We claim that $D \in \mathcal{B}_{t+s}$.
Suppose instead that $D \notin \mathcal{B}_{t+s}$. Then every member of $[\mathcal{B}_{t+s}]$ contains some zero-set Z_1 in the complement of D.
If $\mu \in [\mathcal{B}_{t+s}]$, then there exists $Z_o \in \mathcal{B}_{t+s}$ such that $Z_o \in \mu$, $Z_o \subseteq Z_1 \subseteq R \setminus D$. Now, Z_o contains $Z_2 \in \mathcal{B}_t + \mathcal{B}_s$. So, $\{x \in R : Z_2 - x \in \mathcal{B}_t\} \in \mathcal{B}_s$. Also $Z \in \mathcal{B}_s$. So pick $m \in Z$ such

that $Z_2 - m \in \mathcal{B}_t$. Pick $x \in Z_2 - m$ such that $x \in V_m - m \subset \text{Int } A_m$. Then $x+m \in V_m$ while $x+m \in Z_2$, a contradiction.

We now show that $D \notin \mathcal{B}_{t.s}$. Suppose instead that $D \in \mathcal{B}_{t.s}$. Then D contains $D' \in \mathcal{B}_t \cdot \mathcal{B}_s$. So $\{x \in R : D'/_x \in \mathcal{B}_t\} \in \mathcal{B}_s$. Also $Z \in \mathcal{B}_s$. So, pick $n \in Z$ such that $D'/_n \in \mathcal{B}_t$. Pick $y \in A_n \cap D'/_n$. Then $y.n \in D'$. So pick $m \in Z$ such that $y.n \in V_m \subset m + A_m$. Then $y.n \in (n.A_n) \cap (m+A_m)$, a contradiction.

4.3.7. Corollary. Let $p, q \in pR \setminus R$, where $p = [\mathcal{B}_t]$, $q = [\mathcal{B}_s]$ for some $t, s \in R$. Then $p + q = p \cdot q$ if and only if whenever $Z \in \mathcal{B}_s$ and a family $\{A_n : n \in Z\} \subset \mathcal{B}_t$ there exist $m, n \in Z$ such that $(n.A_n) \cap (m+A_m) \neq \emptyset$.

§ 4.4 Solutions of Equations

4.4.1. Result. Given $p = [\mathcal{B}_t] \in pR \setminus R$ for some $t \in R$ and $n \in R$, $n \neq 1$, there exists $q \in pR \setminus R$ such that $p + q = p \cdot n$.

Proof: Given $p = [\mathcal{B}_t] \in pR \setminus R$, where $t \in R$, consider $Z \in \mathcal{B}_t$. We construct a set $B_Z \subseteq R$ as follows.

Let $\mathcal{C} = \{Z.n - x : x \in Z\}$ be a family of closed sets. Then,

$\mathcal{U} = \{R \setminus C : C \in \mathcal{C}\} \cup \{\text{Int } C : C \in \mathcal{C}\}$ is an open cover of R.

Since R is paracompact, there exists a closed locally finite refinement $\mathcal{V}$ of $\mathcal{U}$.

Let $\mathcal{V}' = \{V_x \in \mathcal{V} : V_x \subset \text{Int } C,\ x \in Z,\ C \in \mathcal{C}\}$. Then $\mathcal{V}'$ is a family of locally finite closed sets. Let

$B_Z = \bigcup_{x \in Z} V_x$. Then B_Z is a closed set and hence a zero-set.

Let $\mathcal{Q} = \{B_Z : Z \in \mathcal{B}_t\}$. Then $\mathcal{Q}$ is a Z-filter base, each member of which contains a member of the Z-filter base $\mathcal{B}_{t(n-1)} = \mathcal{B}_s$. Let $\mathcal{Y}$ be the class of all Z-ultrafilters generated by $\mathcal{B}_s$. We claim that for this $\mathcal{Y}$ we get $\rho + \mathcal{Y} = \rho \cdot n$. We use the characterization result 4.3.5 to prove this equality. For this, we first prove that for at least one $m \in B_Z$, Z.n-m belongs to $\mathcal{B}_t$. Suppose not. Then for no $m \in B_Z$, Z.n-m belongs to $\mathcal{B}_t$. So no member of βR generated by $\mathcal{B}_t$ contains Z.n-m for $m \in B_Z$. So every member of βR generated by $\mathcal{B}_t$ must contain zero-sets contained in $R \setminus$ Z.n-m, for every $m \in B_Z$. If $\mu \in [\mathcal{B}_t]$, then there exists $Z_0 \subseteq R \setminus (Z.n-m)$ for some $m \in B_Z$ such that $Z_0 \in \mu$. But μ is generated by $\mathcal{B}_t$, so, there exists $Z_1 \in \mathcal{B}_t$ such that $Z_1 \subseteq Z_0 \subseteq R \setminus Z.n-m$. Also $Z \in \mathcal{B}_t$. So $Z \cap Z_0 \supset Z_2$, where $Z_2 \in \mathcal{B}_t$ and $Z_0 \subseteq R \setminus Z.n-m$. If $y \in Z_2$, then $y \in Z$ and $y \in R \setminus Z.n-m$. So y = d.n-m, where $y \in Z$ and

$y \in R\setminus Z.n-m$, $d \in R\setminus Z$. i.e., $m=d.n-y$, where $y \in Z$, $d \in R\setminus Z$. (Here $R\setminus Z.n-m = (R\setminus Z).n-m)$. Thus every $m_i \in B_Z$ has an expression $m_i = d_i.n-y_i$, where $d_i \in R\setminus Z$, $y_i \in Z$. So, we would have,

$B_Z \subseteq \bigcup_{x \in Z} ((R\setminus Z).n-x)$. But, $B_Z = \left(\bigcup_{x \in Z} V_x\right)$, where

$V_x \subset \mathrm{Int}(Z.n-x)$, $x \in Z$. Thus we have,

$B_Z \subset \bigcup_{x \in Z} \mathrm{Int}(Z.n-x) \subset \bigcup_{x \in Z} ((R\setminus Z).n-x)$, which is not possible. Thus for at least one $m \in B_Z$, say m_o, where $m_o \in \left(\bigcup_{x \in Z} V_x\right) \setminus \bigcup_{x \in Z} ((R\setminus Z).n-x)$, we have $Z.n-m_o \in \mathcal{B}_t$. Also, for any function $f:R \longrightarrow \mathcal{B}_t$, $f(m_o) \in \mathcal{B}_t$. So, $f(m_o) \cap (Z.n-m_o) \neq \emptyset$. Therefore, $(f(m_o)+m_o)) \cap (Z.n) \neq \emptyset$.

4.4.2. Result. Given $p = [\mathcal{B}_t] \in pR\setminus R$, where $t \in R$, $m \in R$, $m \neq 0$, there exists $\zeta \in pR\setminus R$ such that $p+m = p.\zeta$.

Proof: Here, we can obtain ζ as the class of all Z-ultrafilters generated by the Z-filter base $\mathcal{B}_s = \mathcal{B}_{(\frac{t+m}{t})}$ each member of which is contained in some member of the Z-filter base $\mathcal{Q} = \left\{B_Z : Z \in \mathcal{B}_t\right\}$, where B_Z for each $Z \in \mathcal{B}_t$ can be constructed as follows:

We consider $\mathcal{C} = \left\{C = (Z+m)/_x,\ x \neq 0,\ x \in Z\right\}$, a family of closed sets. Then $\mathcal{U} = \left\{R \setminus C : C \in \mathcal{C}\right\} \cup \left\{\text{Int } C : C \in \mathcal{C}\right\}$ is an open cover of R. R being paracompact, $\mathcal{U}$ has a closed locally finite refinement, say $\mathcal{V}$. Take $\mathcal{V}' = \left\{V_x \subset \text{Int } (Z+m)/_x : x \in Z\right\}$. Then $\mathcal{V}'$ is a family of locally finite closed sets. Let $B_Z = \left(\bigcup_{x \in Z} V_x\right)$. Then B_Z is a closed set and hence a zero-set. Let $\mathcal{Q} = \left\{B_Z : Z \in \mathcal{B}_t\right\}$. Let η be the class of all Z-ultrafilters generated by $\mathcal{B}_s = \mathcal{B}_{\left(\frac{t+m}{t}\right)}$ each member of which is contained in a member of $\mathcal{Q}$. For this η we can prove that $p+m = p \cdot \eta$. We proceed as in Result 4.4.1 and use the characterization Result 4.3.4 to obtain this result.

4.4.3. Result. Given $p = [\mathcal{B}_t] \in pR \setminus R$ for some $t \in R$, $t \neq 1$, there exists $\eta = [\mathcal{B}_s] \in pR \setminus R$ such that $p + \eta = p \cdot \eta$.

Proof: Given $p = [\mathcal{B}_t] \in pR \setminus R$, for some $t \in R$. Consider $\mathcal{B}_t$. Let $Z \in \mathcal{B}_t$. Let $\mathcal{C} = \left\{C = \frac{Z}{x-1},\ x \neq 1 : x \in Z\right\}$. Then $\mathcal{C}$ is a family of closed sets in R. $\mathcal{U} = \left\{R \setminus C : C \in \mathcal{C}\right\} \cup \left\{\text{Int } C : C \in \mathcal{C}\right\}$ is an open cover of R. Since R is paracompact, $\mathcal{U}$ has a closed locally finite refinement, say $\mathcal{V}$. Let

$\mathcal{V}' = \left\{V \in \mathcal{V} : V \subset \text{Int } C,\ C \in \mathcal{C}\right\}$. Then $\mathcal{V}'$ is a family of locally finite closed sets. Let $B_Z = \left(\bigcup_{V \in \mathcal{V}'} V\right)$.

Then B_Z is a closed set and hence a Zero-set. Let $\mathcal{Q} = \left\{B_Z : Z \in \mathcal{B}_t\right\}$. Then $\mathcal{Q}$ is a Z-filter base, each member of which contains a member of the Z-filter base $\mathcal{B}_s = \mathcal{B}_{(\frac{t}{t-1})}$. Let $\mathcal{Y}$ be the class of all Z-ultrafilters generated by $\mathcal{B}_s$. We claim that for this $\mathcal{Y}$, $p + \mathcal{Y} = p \cdot \mathcal{Y}$. To prove this we use the characterization in Corollary 4.3.7.

Let $Z' \in \mathcal{B}_s$ and $\left\{A_m : m \in Z'\right\} \subset \mathcal{B}_t$ be given. Then $Z' = B_Z$ for some $Z \in \mathcal{B}_t$ and there exists $C_m \in \mathcal{B}_t$ such that $C_m \subseteq A_m \cap Z$ for each $m \in B_Z$. So, $\left\{C_m : m \in B_Z\right\} \subset \mathcal{B}_t$. Now, $C_m \in \mathcal{B}_t \Rightarrow B_{C_m} \in \mathcal{Q}$ and hence $B_{C_m} \in \mathcal{B}_s$ and $B_{C_m} \subset B_{A_m} \cap B_Z$ for each $m \in B_Z$. So, $\left\{C_m : m \in B_{C_m}\right\} \subset \mathcal{B}_t$. Let the members $m \in B_Z$ be such that $m_j \in B_Z \Rightarrow m_j \in B_{C_{m_j}}$. Then $m_j \in B_{C_{m_j}} \Rightarrow m_j = a_j/b_{j-1}$, $b_j \neq 1$, $a_j, b_j \in C_{m_j}$, $C_{m_j} \in \mathcal{B}_t$.

i.e., $\quad m_j b_j = m_j + a_j$

i.e., $\quad (m_j + C_{m_j}) \cap (m_j \cdot C_{m_j}) \neq \emptyset$.

i.e., $\quad (m_j + A_{m_j}) \cap (m_j \cdot A_{m_j}) \neq \emptyset$.

Thus, there exists $m \in B_Z$ such that $(m + A_m) \cap (m \cdot A_m) \neq \emptyset$. Hence, $p + \mathcal{Y} = p \cdot \mathcal{Y}$.

Chapter-V

REMOTE POINTS IN pR

§ 5.0. Introduction

In this chapter, we prove the existence of remote points in the LMC-compactification pR of R, where R, the set of real numbers with usual topology is considered as a semitopological semigroup. The existence of remote points in $\beta R \setminus R$ was proved, assuming CH, by Fine and Gillman [FI, GI]. More information on remote points in $\beta R \setminus R$ can be obtained from [PL], [WO_1], [WO_2].

In section 5.1 we prove the existence of remote and non-remote points in pR.

In section 5.2 the arithmetic in pR, as described in chapter four is applied to the class of remote points and incidentally we prove that the extended addition and multiplication in pR are non-commutative.

§ 5.1. Existence of remote points in pR

5.1.1. Definition. A remote point of pR is a point which does not belong to the closure of any discrete subspace of R. It is clear that any remote point of pR must lie in $pR \setminus R$. A point of $pR \setminus R$ which is not a remote point is called a non-remote point of pR.

We have the following results concerning the remote points in βX, for a topological space X.

5.1.2. Theorem [WA]. Let p be in X^* where X is a metric space of non-measurable cardinal and consider the following conditions:-

(a) p is a C-point of X^*

(b) p has no member which is nowhere dense

(c) $M^p = O^p$

(d) p is a remote point in βX.

Conditions (a), (b) and (c) are equivalent and are implied by (d). All the four conditions are equivalent if the set of isolated points of X has compact closure in X.

5.1.3. Theorem [PL]. If X is a non-compact separable metric space in which the set of isolated points has compact closure, then βX contains 2^c remote points which form a dense subspace of X^* (under CH).

5.1.4. Theorem [PL]. Consider the space R. Let R_o and P denote the set of remote points of βR and P-points of R^* respectively. Then R_o' and P' will denote the non-remote points and the non-P-points. The set $P \cap R_o$, $P \cap R_o'$, $P' \cap R_o$ and $P' \cap R_o'$ are each dense subsets of R^* and each has cardinal 2^c (under CH).

5.1.5. Result. There exist remote and non-remote points in pR.

Proof: The remote points in βR are generated by the maximal Z-filter bases in the collection,

$$\mathcal{F}_R = \left\{\emptyset \neq Z(f) \subseteq R : Z(f) \text{ is not nowhere dense, } f \in C_b(R)\right\}.$$

For each $t \in R$, the collection

$$\mathcal{B}_t = \left\{\emptyset \neq Z(f.\ \lambda_t) \in \mathcal{F}_R,\ f \in C_b(R)\right\}$$

is a Z-filter base in R. The class of all z-ultrafilters generated by this Z-filter base is a remote point in pR.

βR also contains 2^c non-remote points which form a dense subspace of $\beta R \setminus R$ [PL]. This subspace in βR is generated by the maximal Z-filter bases containing members of the collection

$$\mathcal{F}_N = \left\{\emptyset \neq Z(f) \subseteq R:\ Z(f) \text{ is nowhere dense}\right\}.$$

For each $t \in R$, the maximal Z-filter bases $\mathcal{B}_t$ in $\mathcal{F}^*(t)$, where,

$$\mathcal{B}_t = \left\{\emptyset \neq Z(f.\lambda_t) : f \in C_b(R) \text{ and at least one } Z(f.\lambda_t) \text{ is nowhere dense}\right\}$$

generate Z-ultrafilters that are non-remote points in βR.

5.1.6. Result. Let $\rho = [\mathcal{B}_t] \in pR \setminus R$ for some $t \in R$. Let $q: \beta R \longrightarrow pR$ be the quotient map. Then every member belonging to $q^{-1}(\rho)$ is remote in βR, if ρ is a remote point in pR .

Proof: Suppose that $q^{-1}(\rho)$ contains at least one non-remote point of βR which belongs to the closure in βR of some discrete subspace D of R. Since q is continuous, it follows that ρ would belong to the closure in pR of the discrete subspace D of R so that, by definition, ρ is non-remote.

5.1.7. Result. If $\rho = [\mathcal{B}_t] \in pR\setminus R$ for some $t \in R$ is remote in pR, then no member of $\mathcal{B}_t$ is nowhere dense.

Proof: If some $Z \in \mathcal{B}_t$ were nowhere dense, then Z would be the boundary of its complement. So, there would exist a discrete space $D \subset R\setminus Z$ such that $D \cup Z = cl_R D$. Thus $\rho \in cl_{pR} Z \subset cl_{pR} D$ so that ρ is non-remote.

5.1.8. Corollary. The class consisting of all the Z-ultrafilters on R which represents a non-remote point in $pR\setminus R$ contains non-remote points in $\beta R \setminus R$.

5.2. Applications of Arithmetic on pR on Remote Points.

5.2.1. Result. Let $\rho = [\mathcal{B}_t]$ for some $t \in R$ be a remote point in pR. If $m \in R$, then $\rho + m$ and $\rho . m$ are remote points in pR.

Proof: We give the proof for '+' only. $\rho = [\mathcal{B}_t]$ is a remote point in $pR\setminus R$ means that every Z-ultrafilter belonging to $[\mathcal{B}_t]$ is a remote point in βR. Suppose that $\rho + m \in cl_{pR}D$, where D is a discrete subspace of R. Then some member belonging to $[\mathcal{B}_t+m]$ belongs to $cl_{\beta R}D$, where $[\mathcal{B}_t+m] = \rho + m$. Suppose that $\mu \in [\mathcal{B}_t+m]$ is such that $\mu \in cl_{\beta R}D$. Then $cl_R D \in \mu$. Then $\rho \longmapsto \rho+m$ being a homeomorphism, there exists $\mu' \in [\mathcal{B}_t]$ such that, $cl_R D-m \in \mu' \Longrightarrow \mu' \in cl_{\beta R}(cl_R D-m)$. i.e., μ' which belongs to $[\mathcal{B}_t]$ belongs to the closure of the discrete subspace $cl_R D-m$. This means that ρ is a non-remote point, which is a contradiction. So, no member of $[\mathcal{B}_t+m]$ can belong to the closure of a discrete subspace of R. So, $\rho+m$ is a remote point in pR.

5.2.2. Result. Let $\rho = [\mathcal{B}_t]$, $\zeta = [\mathcal{B}_s]$, $s,t \in R$ be remote points in pR. Then $\rho+\zeta$ and $\rho.\zeta$ are remote points in pR.

Proof: We give the proof for '+' only, that for '.' is identical. Given that $\rho = [\mathcal{B}_t]$, $\zeta = [\mathcal{B}_s]$, $s,t \in R$ are remote in pR, by definition, no member of $\mathcal{B}_t$ and no member of $\mathcal{B}_s$ are nowhere dense. We have, $\rho+\zeta = [\mathcal{B}_{t+s}]$. Let $Z \in \mathcal{B}_{t+s}$. Then $Z \in \mathcal{B}_t + \mathcal{B}_s$. So, $Z' = \{x \in R: Z-x \in \mathcal{B}_t\} \in \mathcal{B}_s$. Let $m \in Z'$. Then $Z-m \in \mathcal{B}_t$. i.e., $Z \in \mathcal{B}_t+m$. By result 5.2.1, $\mathcal{B}_t+m$ has no member that is nowhere dense because $[\mathcal{B}_t]$

remote in pR $\Longrightarrow$ $[\mathcal{B}_t+m]$ is remote in pR. So, Z is not nowhere dense. Since Z taken from $\mathcal{B}_{t+s}$ is arbitrary, this means that no member of $\mathcal{B}_{t+s}$ is nowhere dense. So, $[\mathcal{B}_{t+s}] = \rho + \zeta$ is a remote point in pR.

5.2.3. Result. The set of remote points in pR form a subsemigroup under the extended operations + and . .

Proof: The result follows from 5.2.2 and the fact that the extended operations are associative in pR.

5.2.4. Result. Let $\rho = [\mathcal{B}_t]$ for some $t \in R$ be a non-remote point in pR and let $m \in R$. Then $\rho+m = [\mathcal{B}_t+m]$ and $\rho \cdot m = [\mathcal{B}_t \cdot m]$ are non-remote points in pR.

Proof: We give the proof for '+' only, that for '.' is identical. $\rho \in pR$ is a non-remote point in pR means that ρ belongs to the closure in pR of some discrete subspace D of R. i.e., $\rho \in cl_{pR}D$. If $q: \beta R \longrightarrow pR$ is the quotient map, then $q^{-1}(cl_{pR}D)$ is a closed set in βR and every member of $[\mathcal{B}_t]$ belongs to $q^{-1}(cl_{pR}D)$. If $\mu \in [\mathcal{B}_t]$, then, $\mu \in q^{-1}(cl_{pR} D) \Longrightarrow cl_R D \in \mu$. i.e., $(cl_R D)+m \in \mathcal{B}_t+m$. i.e., $\rho+m = [\mathcal{B}_t+m]$ belongs to the closure in pR of the discrete space D+m of R. So, $\rho+m$ is a non-remote point.

5.2.5. Result. Let $p = [\mathcal{B}_t]$, $q = [\mathcal{B}_s]$ for some $t, s \in R$ be non-remote points of pR. Then $p + q$ and $p \cdot q$ are both non-remote points of pR.

Proof: We establish the result for '+' only. Given that $p = [\mathcal{B}_t]$, $q = [\mathcal{B}_s]$ are non-remote points of pR, p, q belong to closure in pR of discrete subspaces, say, D_1, D_2 respectively of R. i.e., $[\mathcal{B}_t] \in cl_{pR} D_1$ and $[\mathcal{B}_s] \in cl_{pR} D_2$. If $\mu_1 \in [\mathcal{B}_t]$ and $\mu_2 \in [\mathcal{B}_s]$, then we have $\mu_1 \in cl_{\beta R} D_1$ and $\mu_2 \in cl_{\beta R} D_2$.

Therefore, $cl_R D_1 \in \mu_1$ and $cl_R D_2 \in \mu_2$. Since $\mu_1 \in [\mathcal{B}_t]$, $\mu_2 \in [\mathcal{B}_s]$, there exist $z_1 \in \mathcal{B}_t$, $z_2 \in \mathcal{B}_s$ such that $z_1 \subset cl_R D_1$ and $z_2 \subset cl_R D_2$. Consider $z_1 + z_2 = \{x+y : x \in Z_1, y \in Z_2\}$. Then, $cl_R (z_1+z_2) \in \mathcal{B}_{t+s}$. So $cl_R(z_1+z_2) \in \mathcal{B}_t + \mathcal{B}_s$. Also, $z_1 + z_2 \subset cl_R (z_1+z_2)$ so that $p + q = [\mathcal{B}_{t+s}] \in cl_{pR}(z_1+z_2)$, where z_1+z_2 is a discrete subspace of R. So, $p + q$ is a non-remote point in pR .

5.2.6. Corollary. The set of non-remote points form a subsemigroup of pR under the extended operations in pR .

Proof: The result follows from 5.2.5 and the fact that the extended operations in pR are associative.

5.2.7. Result. Let $p = [\mathcal{B}_t]$, $q = [\mathcal{B}_s]$ for some $s, t \in R$ be members of pR, where p is a remote point and q, a non-remote point. Then $p + q$ is a remote point and $q + p$ is a non-remote point. Thus addition in pR is non-commutative.

Proof: $p = [\mathcal{B}_t]$ is a remote point and $q = [\mathcal{B}_s]$, a non-remote point in pR, for some $t, s \in R$. Let $Z \in \mathcal{B}_{t+s}$. Then $Z \in \mathcal{B}_t + \mathcal{B}_s$. Then $Z' = \{x \in R : Z - x \in \mathcal{B}_t\} \in \mathcal{B}_s$. Let $m \in Z'$. Then, $Z - m \in \mathcal{B}_t \Rightarrow Z \in \mathcal{B}_t + m$. $[\mathcal{B}_t]$ is a remote point $\Rightarrow [\mathcal{B}_t + m]$ is a remote point by result 5.2.1 $\Rightarrow Z$ is not nowhere dense. Thus $Z \in \mathcal{B}_{t+s}$ is not nowhere dense. Since $Z \in \mathcal{B}_{t+s}$, was taken arbitrary, it follows that no member of $\mathcal{B}_{t+s}$ is nowhere dense. So, $[\mathcal{B}_{t+s}] = p + q$, is a remote point of pR.

We have $q = [\mathcal{B}_s]$ non-remote and $p = [\mathcal{B}_t]$, remote in pR.

Let $Z \in \mathcal{B}_{s+t}$. Then $Z \in \mathcal{B}_s + \mathcal{B}_t$. So, $B = \{x \in R : Z - x \in \mathcal{B}_s\} \in \mathcal{B}_t$. For $m \in B$, $Z - m \in \mathcal{B}_s$. So, $Z \in \mathcal{B}_s + m$. Since $[\mathcal{B}_s + m]$ is non-remote by Result 5.2.4, $q + m$ is non-remote. So, $q + m$ belongs to the closure of some discrete space D of R. So $cl_R D \in \mathcal{B}_s + m$. So, $Z \cap cl_R D \supset Z_1$, where $Z_1 \in \mathcal{B}_s+m$ and Z_1 is nowhere dense. So, $Z_1-m\in\mathcal{B}_s$. Thus for each $m_j \in B$, we get $Z_{m_j} \in \mathcal{B}_s + m_j$, where Z_{m_j} is nowhere dense. So, $Z_{m_j}- m_j \in \mathcal{B}_s$ and $Z_{m_j}-m_j$ is nowhere dense. If $\mathcal{C} = \{Z_{m_j}: m_j \in B\}$, then $\mathcal{C}$ is a family of closed nowhere dense sets in R.

$\mathcal{U} = \{R \setminus C : C \in \mathcal{C}\} \cup \{\text{Int } C : C \in \mathcal{C}\}$, is an open cover of R. Since R is paracompact, $\mathcal{U}$ has a closed locally finite subcover $\mathcal{V}$. Let $\mathcal{V}' = \{V \in \mathcal{V} : V \subset \text{Int } C,\ C \in \mathcal{C}\}$. Then $\mathcal{V}'$ is a family of locally finite, nowhere dense, closed sets. Then $\left(\bigcup_{V \in \mathcal{V}'} V\right)$ is closed and nowhere dense and $\left(\bigcup_{V\in\mathcal{V}'} V\right) \subset Z$ where $Z \in \mathcal{B}_{s+t}$. So, $\left(\bigcup_{V\in\mathcal{V}'} V\right) \in \mathcal{B}_{s+t}$.

(Since $\left(\bigcup_{V\in\mathcal{V}'} V\right)$ is a closed set in R, it is a zero-set in R, say Z(h), and $Z(h) \subset Z \in \mathcal{B}_{s+t}$. So $Z(h) = Z(h) \cap Z$ and if $Z = Z(f.\lambda_{s+t})$, then, $Z(h) \supset Z((h^2+f^2).\lambda_{(s+t).1}) \in \mathcal{B}(s+t)$ and so $Z(h) \in \mathcal{B}_{s+t}$. This is possible for every $Z \in \mathcal{B}_{s+t}$. So, $\mathcal{B}_{s+t}$ contains nowhere dense zero-sets. So $\eta + \rho$ is non-remote.

5.2.8. Result. Let $\rho = [\mathcal{B}_t]$, $\eta = [\mathcal{B}_s]$ belong to pR for some $t, s \in R$. If ρ is remote and η is non-remote, then, $\rho \cdot \eta$ is remote and $\eta \cdot \rho$ is non-remote and thus multiplication in pR is non-commutative.

Proof: The proof is identical to that for addition.

5.2.9. **Result.** Given $\rho = [\mathcal{B}_t] \in pR\setminus R$, a remote point in $pR\setminus R$ and $n \in R$, there exists $\gamma \in pR\setminus R$, where γ is remote in pR such that $\rho + \gamma = \rho . n$. If ρ given is non-remote in $pR\setminus R$, then γ is also non-remote.

Proof: $\rho = [\mathcal{B}_t] \in pR\setminus R$ for some $t \in R$ and $n \in R$ are given. We have shown that there exists $\gamma \in pR\setminus R$ such that $\rho + \gamma = \rho . n$ (Chapter IV, Result 5.4.1). We have obtained γ as the class of all Z-ultrafilters on R generated by the Z-filter base $\mathcal{B}_s = \mathcal{B}(t(n-1))$ each member of which is contained in members of the Z-filter base $\mathcal{B} = \{B_Z : Z \in \mathcal{B}_t\}$, where for each $Z \in \mathcal{B}_t$, $B_Z = \left(\bigcup_{V \in \mathcal{V}'} V\right)$, where, $\mathcal{V}' = \{V \subsetneq R$, V locally finite and closed: $V \subset Int(Z.n-x)$, $x \in Z\}$.

Suppose that ρ is a remote point in $pR\setminus R$. Then, by definition, no member of $\mathcal{B}_t$ is nowhere dense. So, for every $Z \in \mathcal{B}_t$, B_Z constructed as above is also not nowhere dense. Thus $\mathcal{B}$ is a Z-filter base consisting of not nowhere dense members. So, $\mathcal{B}_s$ also contains no nowhere dense members. Hence γ, the class of all Z-ultrafilters generated by $\mathcal{B}_s$ is a remote point in pR.

Suppose that $\rho = [\mathcal{B}_t]$ is a non-remote point in pR. Then, by definition, $\mathcal{B}_t$ contains at least one nowhere

dense number. Let $Z \in \mathcal{B}_t$ be such that Z is nowhere dense. Then, $\text{Int}_{\beta R}(\text{cl}_{\beta R} Z) = \emptyset$. So, $\text{Int}_{\beta R}(\text{cl}_{\beta R}Z) \cap R = \emptyset$. Then, $\mathcal{C} = \{Z.n-x : x \in Z\}$ is a family of closed nowhere dense sets in R. So, for each $x \in Z$, $Z.n-x$ is the frontier of an open set, say, C_x. So, $\mathcal{U} = \{R \setminus Z.n-x : x \in Z\} \cup \{C_x : x \in Z\}$ is an open cover of R. Since R is paracompact, $\mathcal{U}$ has a closed locally finite refinement, say, $\mathcal{V}$.
Let $\mathcal{V}' = \{V \in \mathcal{V} : V \subset C_x,\ x \in Z\}$. Then, $\mathcal{V}'$ is a family of locally finite closed sets that are nowhere dense. If $B_Z = \left(\bigcup_{V \in \mathcal{V}'} V\right)$, then B_Z is a closed nowhere dense set. Then $\mathcal{B} = \{B_Z : Z \in \mathcal{B}_t\}$ contains nowhere dense sets. Also, $\mathcal{B}$ is a Z-filter base, each member of $\mathcal{B}$ containing a member of the Z-filter base, $\mathcal{B}_s = \mathcal{B}_{t(n-1)}$ which also contains nowhere dense sets. So, η, which is the class of all Z-ultrafilters generated by $\mathcal{B}_s$, is therefore, a non-remote point in pR.

5.2.10. Result. Let $\rho = [\mathcal{B}_t] \in pR \setminus R$ for some $t \in R$ be a remote point in $pR \setminus R$ and let $m \in R$, $m \neq 0$. Then there exists $\eta \in pR \setminus R$ such that η is remote and $\rho + m = \rho \cdot \eta$. If ρ is a non-remote point in $pR \setminus R$, then η is also non-remote.

Proof: We proceed as in Result 5.2.9 to prove this result, once we know that η is generated by the Z-filter base

$\mathcal{B}_s = \mathcal{B}_{(\frac{t+m}{t})}$, each member of which is contained in a member of the Z-filter base $\mathcal{B} = \left\{B_Z : Z \in \mathcal{B}_t\right\}$ where, for each $Z \in \mathcal{B}_t$, B_Z is constructed as follows:

$B_Z = \left(\bigcup_{V \in \mathcal{V}'} V\right)$, where $\mathcal{V}' = \left\{V \subset \text{Int}(Z+m)/_x : x \in Z,\ x \neq 0\right\}$ is a family of locally finite closed sets in R. (Result 4.4.2).

5.2.11. Result. Let $\rho = [\mathcal{B}_t] \in pR \setminus R$ be a remote point in $pR \setminus R$ for some $t \in R$. Then there exists $\zeta \in pR \setminus R$, remote in $pR \setminus R$ such that $\rho + \zeta = \rho \cdot \zeta$. If ρ is non-remote then so is ζ.

Proof: The proof is similar to that of Result 5.2.9 and 5.2.10, once we know that given $\rho = [\mathcal{B}_t] \in pR \setminus R$, for some $t \in R$, there exists ζ, the class of all Z-ultrafilters generated by $\mathcal{B}_s = \mathcal{B}_{(\frac{t}{t-1})}$, each member of which is contained in a member of the Z-filter base $\mathcal{B} = \left\{B_Z : Z \in \mathcal{B}_t\right\}$, where for each $Z \in \mathcal{B}_t$, B_Z is obtained as follows:

$$B_Z = \left(\bigcup_{V \in \mathcal{V}'} V\right), \text{ where,}$$

$\mathcal{V}' = \left\{V \subset R,\ V \text{ locally finite and closed: } V \subset \text{Int}\ (\frac{Z}{x-1}),\ x \neq 1, x \in Z\right\}$.

(Result 4.4.3).

Chapter VI

k-UNIFORM Z-ULTRAFILTERS ON A SEMITOPOLOGICAL SEMIGROUP S

§ 6.0. Introduction

We take S of infinite cardinality to be a completely regular and Hausdorff semitopological semigroup. Suppose that S is locally compact. Then pS, the LMC-compactification of S is the family of all equivalence classes of Z-ultrafilters on S, where, a member of pS is of the form $p = [\mathcal{B}_t]$ for some $t \in S$, where $\mathcal{B}_t$ is a maximal Z-filter base in the family $\mathcal{F}^*(t) = \left\{\emptyset \neq Z(f.\lambda_t): 1 \notin Z(f.\lambda_t), f \in C_b(S)\right\}$ and $[\mathcal{B}_t]$ is the set of Z-ultrafilters on S generated by $\mathcal{B}_t$. The Z-filterbase, $\mathcal{F}_1^*(t) = \left\{\emptyset \neq Z(f.\lambda_t): 1 \in Z(f.\lambda_t), f \in C_b(S)\right\}$ generates the principal Z-ultrafilter e(t), where, $e:S \longrightarrow \beta S$ is the embedding. The construction of pS is analogous to pR, where R is the set of real numbers with usual topology, considered as a semitopological semigroup (see Chapter IV). In this chapter we define and discuss various properties of k-uniform Z-ultrafilters analogous to k-uniform ultrafilters [CO; NE]. Given a set A and a cardinal k, $[A]^k = \left\{B \subseteq A: |B|=k\right\}$ and $[A]^{<k} = \left\{B \subseteq A : |B| < k\right\}$.

In section 6.1 we give the necessary definitions and results concerning k-uniform Z-ultrafilters in pS.

In section 6.2 we study the ideal structure of the space of k-uniform Z-ultrafilters in pS with respect to the arithmetic defined in pS (analogous to that for pR).

§ 6.1. k-Uniform Z-ultrafilters on S.

6.1.1. Definition. Let $\rho = [\mathcal{B}_t]$ for some $t \in S$ be a member of pS. The norm of $\mathcal{B}_t$ denoted by $\|\mathcal{B}_t\|$ is defined by $\|\mathcal{B}_t\| = \min\{|Z| : Z \in \mathcal{B}_t\}$. The norm of ρ is defined as $\|\rho\| = \|\mathcal{B}_t\|$. ρ is said to be k-uniform if $\|\rho\| \geq k$. We denote by $\mathcal{U}_k(S)$, the set of k-uniform Z-ultrafilters on S.

6.1.2. Note. When $\rho \in pS$ is such that ρ is ω-uniform, then $\rho \in pS \setminus S$, otherwise $\rho \in S$. A $|S|$- uniform member of pS is called uniform. The set of all $|S|$-uniform Z-ultrafilters on S is denoted by $\mathcal{U}(S)$. Thus, we have $\beta S \setminus S = \mathcal{U}_\omega(S)$.

6.1.3. Definition. Let $\mathcal{A}$ be a non-empty family of zero sets. Then $\mathcal{A}$ has k-uniform finite intersection property if $|\bigcap_{i \leq n} A_i| \geq k$, whenever $n < \omega$ and $A_i \in \mathcal{A}$ for $i \leq n$.

6.1.4. Note. If $\mathcal{A}$ has $|S|$-uniform finite intersection property, then $\mathcal{A}$ is said to have uniform finite intersection property. It is clear that $\mathcal{A}$ has the finite intersection property if and only if $\mathcal{A}$ has the 1-uniform finite intersection property.

6.1.5. Result. Define $\mathcal{Z}^k(S) = \{A \subseteq S \mid A \text{ is a zero set and } |S \setminus A| < k\}$. If $\omega \leqslant k \leqslant |S|$, then $\mathcal{Z}^k(S)$ is a Z-filter on S.

Proof: Now $\mathcal{Z}^k(S) \neq \emptyset$, since $S \in \mathcal{Z}^k(S)$, and $\emptyset \notin \mathcal{Z}^k(S)$ is clear. If $Z_1, Z_2 \in \mathcal{Z}^k(S)$, then $|S \setminus Z_1| < k$, $|S \setminus Z_2| < k$. Since, $|S \setminus (Z_1 \cap Z_2)| = |(S \setminus Z_1) \cup (S \setminus Z_2)| < k$, we get $Z_1 \cap Z_2 \in \mathcal{Z}^k(S)$. Further if $Z \in \mathcal{Z}^k(S)$ and $Z' \supset Z$ then $|S \setminus Z'| \leqslant |S \setminus Z| < k$. Therefore, $Z' \in \mathcal{Z}^k(S)$. Thus $\mathcal{Z}^k(S)$ is a Z-filter on S.

6.1.6. Result. Let $\omega \leqslant k \leqslant |S|$. Then,

(a) A Z-ultrafilter p on S is k-uniform if and only if $\mathcal{Z}^k(S) \subset p$.

(b) There is a k-uniform Z-ultrafilter on S. i.e., $\mathcal{U}_k(S) \neq \emptyset$.

(c) Each family of zero-sets on S with k-uniform finite intersection property is contained in a k-uniform Z-ultrafilter on S.

Proof:

(a) Let p be k-uniform. Let $A \in \mathcal{Z}^k(S)$. Then $|S \setminus A| < k$ and so any zero-set $Z \subset S \setminus A$ also has cardinality less than k. So, $Z \notin p$. Since p is a Z-ultrafilter, we have, $A \in p$. Conversely, let $\mathcal{Z}^k(S) \subset p$ and let $A \in p$. If $|A| < k$, then $S \setminus A$ contains a zero-set B, where $B \in \mathcal{Z}^k(S) \subset p$ and so $\emptyset \neq A \cap B \in p$; thus $|A| \geqslant k$.

(b) Since $\omega \leqslant k \leqslant |S|$, the family $\mathcal{Z}^k(S)$ is a Z-filter on S. So, there is a Z-ultrafilter p on S such that $\mathcal{Z}^k(S) \subset p$ and p is k-uniform by (a).

(c) Let $\mathcal{Z}$ be a family of zero-sets of S with k-uniform finite intersection property. We claim that $\mathcal{Z} \cup \mathcal{Z}^k(S)$ has finite intersection property. Let $m, n \in \omega$ and let $\{A_k : k \leqslant n\} \subseteq \mathcal{Z}$ and $\{B_i : i \leqslant m\} \subset \mathcal{Z}^k(S)$. Then $|\bigcap_{k \leqslant n} A_k| \geqslant k$ and $|S \setminus \bigcap_{i \leqslant m} B_i| < k$.

So, $(\bigcap_{k \leqslant n} A_k) \cap (\bigcap_{i \leqslant m} B_i) \neq \emptyset$. So, there is a Z-ultrafilter p on S such that $\mathcal{Z} \cup \mathcal{Z}^k(S) \subset p$ and p is k-uniorm by (a) above.

6.1.7. Note: We have the following special cases of this lemma.

(1) There is a uniform Z-ultrafilter on S.

(2) A Z-ultrafilter p on S is uniform if and only if $\mathcal{C}(S)=\{A \in \mathcal{P}(S),\ A,$ a zero set$: |S\setminus A|<|S|\} \subset p$.

(3) If $\mathcal{F}$ is a non-empty family of closed subsets of S with the uniform finite intersection property, then $\mathcal{F}$ is contained in a uniform Z-ultrafilter on S.

§ 6.2. The Semigroup $V_k(S)$

6.2.1. Definition. Let $p \in pS$, where $p = [\mathcal{B}_t]$ for some $t \in S$. Let k=1 or $k \geq \omega$. Define,

$C_k(\mathcal{B}_t) = \{A \subseteq S :$ A a zero set and $|\{x \in S: A-x \notin \mathcal{B}_t\}| < k\}$

i.e., $C_k(\mathcal{B}_t)$ is the set of zero-sets of S which k-almost always translate to a member of $\mathcal{B}_t$. $(C_1(\mathcal{B}_t))$ is the set of zero-sets, which always translate to a member of $\mathcal{B}_t$.

6.2.2. Result. Let $p = [\mathcal{B}_t]$ for some $t \in S$ and $k \leq |S|$ (with k=1 or $k \geq \omega$). Then

(1) $C_k(\mathcal{B}_t)$ has finite intersection property.

(2) $\mathcal{B}_t + U_k(S) = \{\text{All Z-filter bases } \mathcal{B} : C_k(\mathcal{B}_t) \subset \mathcal{B}\}$ where $U_k(S)$ is the family of all Z-filter bases in S, where the members of each Z-filter base have cardinality $\geqslant k$.

Proof: Given $Z_1, Z_2 \in C_k(\mathcal{B}_t)$, $|\{x \in S: Z_1 - x \notin \mathcal{B}_t\}| < k$ and $|\{x \in S: Z_2 - x \notin \mathcal{B}_t\}| < k$. Now,

$$\{x \in S: (Z_1 \cap Z_2) - x \notin \mathcal{B}_t\} = \{x \in S: Z_1 - x \notin \mathcal{B}_t\} \cup \{x \in S: Z_2 - x \notin \mathcal{B}_t\}$$

So $Z_1 \cap Z_2 \in C_k(\mathcal{B}_t)$. Thus $C_k(\mathcal{B}_t)$ has finite intersection property.

(2) Let $\mathcal{B} \in \mathcal{B}_t + U_k(S)$. Pick $\mathcal{B}_s \in U_k(S)$ such that $\mathcal{B} = \mathcal{B}_t + \mathcal{B}_s$. Let $Z \in C_k(\mathcal{B}_t)$. Then $|\{x \in S: Z - x \notin \mathcal{B}_t\}| < k$. So, $\{x \in S: Z - x \notin \mathcal{B}_t\} \notin \mathcal{B}_s$. Thus there exists $Z' \subset S \setminus \{x \in S: Z - x \notin \mathcal{B}_t\}$ such that $Z' \in \mathcal{B}_s$. i.e., $\{x \in S: Z - x \in \mathcal{B}_t\} \in \mathcal{B}_s \Rightarrow Z \in \mathcal{B}_t + \mathcal{B}_s = \mathcal{B}$

Thus $C_k(\mathcal{B}_t) \subset \mathcal{B}_t + \mathcal{B}_s \Rightarrow \mathcal{B}_t + \mathcal{B}_s \in$ RHS.

Thus, $\mathcal{B}_t + U_k(S) \subseteq \{\text{All Z-filter bases } \mathcal{B} : C_k(\mathcal{B}_t) \subset \mathcal{B}\}$.
Conversely, let $\mathcal{C}$ be a Z-filter base in S such that $C_k(\mathcal{B}_t) \subset \mathcal{C}$. For each $Z \in \mathcal{C}$, let $D(Z) = \{x \in S : Z - x \in \mathcal{B}_t\}$.
Observe that if $Z_1, Z_2 \in \mathcal{C}$, then $D(Z_1 \cap Z_2) = D(Z_1) \cap D(Z_2)$.
Further, if $Z \in \mathcal{C}$, then $S \setminus Z$ has no zero set belonging to $C_k(\mathcal{B}_t)$ (since $C_k(\mathcal{B}_t) \subset \mathcal{C}$). So, $|D(Z)| \geqslant k$. Thus $\{D(Z) : Z \in \mathcal{C}\}$ has k-uniform finite intersection property.
Pick $\mathcal{B}_s \in U_k(S)$ such that $\{D(Z) : Z \in \mathcal{C}\} \subset \mathcal{B}_s$. Then, $\mathcal{C} \subseteq \mathcal{B}_t + \mathcal{B}_s$, and conversely, if $Z \in \mathcal{B}_t + \mathcal{B}_s$, where, $\mathcal{B}_s \in U_k(S)$. Then, $\{x \in S : Z - x \in \mathcal{B}_t\} \in \mathcal{B}_s$.
Since $\mathcal{B}_s \in U_k(S)$, $|\{x \in S : Z - x \in \mathcal{B}_t\}| \geqslant k$.
i.e., $|\{x \in S : Z - x \notin \mathcal{B}_t\}| < k$. So, $Z \in C_k(\mathcal{B}_t) \subset \mathcal{C}$.
So, $\mathcal{B}_t + \mathcal{B}_s \subseteq \mathcal{C}$. Thus $\mathcal{C} = \mathcal{B}_t + \mathcal{B}_s$.
Hence, $\mathcal{B}_t + U_k(S) = \{\text{All Z-filter bases } \mathcal{B} : C_k(\mathcal{B}_t) \subset \mathcal{B}\}$.

6.2.3. Definition. Let $V_k(S)$ be the equivalence classes consisting of all Z-ultrafilters generated by members of $U_k(S)$. Then $V_k(S)$ is a semigroup. Evidently, $V_1(S) = pS$.

6.2.4. Result. Let $\omega \leqslant k \leqslant |S|$. The following statements are equivalent.

(a) $V_k(S)$ is a subsemigroup of pS.

(b) For all $p \in V_k(S)$, where $p = [\mathcal{B}_t]$, $t \in S$ and all zero-sets $A \in [S]^{<k}$, $S \setminus A$ contains members belonging to $C_k(\mathcal{B}_t)$.

(c) For all zero-sets $A \in [S]^{<k}$ and all zero-sets $B \in [S]^{k}$, there exists $F \in [B]^{<\omega}$ such that $|\bigcap_{x \in F} A-x| < k$.

Proof: To see that (a) $\Rightarrow$ (b).

Let $p = [\mathcal{B}_t] \in V_k(S)$ and let $A \in [S]^{<k}$, where A is a zero-set in S. Suppose that $S \setminus A$ contains no zero-set belonging to $C_k(\mathcal{B}_t)$. Then $C_k(\mathcal{B}_t) \cup \{A\}$ has finite intersection property. (If $B \in C_k(\mathcal{B}_t)$ and $B \cap A = \emptyset$, then $B \subset S \setminus A$. So $S \setminus A \in C_k(\mathcal{B}_t)$). Pick $\mathcal{B}_s$, a Z-filter base in S such that $C_k(\mathcal{B}_t) \cup \{A\} \subseteq \mathcal{B}_s$. Pick by the previous result, $\mathcal{B}_r \in U_k(S)$ such that $\mathcal{B}_{t+r} = \mathcal{B}_s$. Since $A \in \mathcal{B}_s$, $\mathcal{B}_s \notin U_k(S)$. So, $\mathcal{B}_{t+r} \notin U_k(S)$, a contradiction.

To see that (b) $\Longrightarrow$ (c).

Let $A \in [S]^{<k}$ and let $B \in [S]^{k}$. Suppose that for each $F \in [B]^{<\omega}$, $|\bigcap_{x \in F} A-x| \geq k$. Then $\{A-x : x \in B\}$ has the k-uniform finite intersection property. So, pick

$\mathcal{B}_s \in U_k(S)$ such that $\{A-x : x \in B\} \subseteq \mathcal{B}_s$. Then $B \subseteq \{x \in S : A-x \in \mathcal{B}_s\}$. So, $S \setminus A$ does not contain members belonging to $C_k(\mathcal{B}_s)$, a contradiction.

To see that (c) $\Longrightarrow$ (a).

Let $\mathcal{B}_t, \mathcal{B}_r \in U_k(S)$. Let $\mathcal{B}_s = \mathcal{B}_{t+r}$. Then by the previous result, $C_k(\mathcal{B}_t) \subseteq \mathcal{B}_s$. Suppose that $\mathcal{B}_s \notin U_k(S)$ and pick $A \in \mathcal{B}_s$ such that $|A| < k$. Let $D = \{x \in S : A-x \in \mathcal{B}_t\}$. Then $D \in \mathcal{B}_r$. So, $|D| \geq k$. Pick $B \in [D]^k$, where B is a zero set. Pick $F \in [B]^{<\omega}$ such that $|\bigcap_{x \in F} A-x| < k$. Then $\bigcap_{x \in F} A-x \in \mathcal{B}_t$. So, $\mathcal{B}_t \notin U_k(S)$, a contradiction.

6.2.5. **Definition.** Let $p = [\mathcal{B}_t] \in pS$ for some $t \in S$. Then $\mathcal{B}_t$ is (k,γ)-regular, where γ is an infinite cardinal if there is a family $\{A_\xi : \xi < \gamma\}$ of zero-sets in S, contained in $\mathcal{B}_t$ such that if $\mathcal{J} \subset \gamma$ and $|\mathcal{J}| = k$, then $\bigcap_{\xi \in \mathcal{J}} A_\xi = \emptyset$. The family $\{A_\xi : \xi \in \gamma\}$ is called (k,γ)-regular family for $\mathcal{B}_t$.

If $\mathcal{B}_t$ is an $(\omega,|S|)$-regular Z-filter base, we simply say that $\mathcal{B}_t$ is regular. We get a family $\mathcal{A} = \{A_\xi : \xi < |S|\}$ of members of $\mathcal{B}_t$ such that if

$|\mathcal{G}| < |S|$ and $|\mathcal{G}| = \omega$, then $\bigcap_{\zeta \in \omega} A_\zeta = \emptyset$.
i.e., countable intersection of members of the family is empty. Then $\mathcal{B}_t$ is said to be simply regular.

6.2.6. Result. Let $\omega \leqslant k \leqslant |S|$. Statements (a) and (b) are equivalent and imply statement (c). If k is regular, all three statements are equivalent.

(a) $V_k(S)$ is a right ideal of pS

(b) For all $A \in [S]^{<k}$, A zero set, and for all $x \in S$,
$|A-x| < k$.

(c) For all $x,y \in S$, $|\rho_x^{-1}[\{y\}]| < k$.

Proof: To see that (a) $\Rightarrow$ (b).

Let $A \in [S]^{<k}$ where A is a zero-set and let $x \in S$. Suppose that $|A-x| \geqslant k$ and pick $\rho \in V_k(S)$, where $\rho = [\mathcal{B}_t]$ for some $t \in S$ with $\mathcal{B}_t \in U_k(S)$ such that $A-x \in \mathcal{B}_t$. Then $A \in \mathcal{B}_{t+x} \Rightarrow A \in \mathcal{B}_t + \mathcal{B}_x$. So, $\mathcal{B}_t+x \subseteq \mathcal{B}_{t+x} \notin U_k(S)$, a contradiction.

To see that (b) $\Rightarrow$ (a).

Let $\mathcal{B}_t \in U_k(S)$ and $\eta \in pS$, where $\eta = [\mathcal{B}_s]$, for some

$s \in S$. Suppose that $\mathcal{B}_{t+s} \notin U_k(S)$. Pick $A \in \mathcal{B}_{t+s}$ such that $|A| < k$. Since $A \in \mathcal{B}_{t+s} \Rightarrow A \in \mathcal{B}_t + \mathcal{B}_s$, $\{x \in S : A-x \in \mathcal{B}_t\} \neq \emptyset$. Pick $x \in S$ such that $A-x \in \mathcal{B}_t$. Then $|A-x| \geqslant k$.

To see that (b) $\Longrightarrow$ (c).

We have $\rho_x^{-1}[\{y\}] = \{y\}-x$. Assume that k is regular.

To see that (c) $\Longrightarrow$ (b).

Let $A \in [S]^{<k}$, where A is a zero-set and let $x \in S$.

Then $A-x = \rho_x^{-1}[A] = \bigcup_{y \in A} \rho_x^{-1}[\{y\}]$. Since k is regular, $|A| < k$ and for each $y \in A$, $|\rho_x^{-1}[\{y\}]| < k$, we have $|A-x| < k$.

6.2.7. Corollary. Let $\omega \leqslant k \leqslant |S|$. If right cancellation holds in S, $V_k(S)$ is a right ideal of βS.

Proof: Since ρ_x is one-to-one, for each $A \subseteq S$, A, a zero set, $|A-x| \leqslant |A|$.

6.2.8. Theorem. Let $\omega \leqslant k \leqslant |S|$. The following statements are equivalent.

(a) $V_k(S)$ is a left ideal of pS.

(b) For all $\rho = [\mathcal{B}_t] \in V_k(S)$, $t \in S$, and all zero-sets $A \in [S]^{<k}$, $S \setminus A$ contains members belonging to $C_k(\mathcal{B}_t)$.

(c) For all zero-sets $A \in [S]^{<k}$ and all $B \in [S]^k$, there exists $F \in [B]^{<\omega}$ such that $\bigcap_{x \in F} A-x = \emptyset$.

Proof:

The proof is similar to that of Result 6.2.4.

6.2.9. Corollary. $V_k(S)$ is an ideal of pS.

Appendix

ON E-COMPACT SPACES

We consider here the more general situation of E-compact spaces, for a topological space E. By taking E as a topological field, we can construct $\beta_E X$, the maximal E-compactification of X as the collection of all E-Z-ultrafilters on X. Having obtained $\beta_E X$ in this manner, and assuming further that E is a topological field, we can study the problem of extending the semi-group operation on X to $\beta_E X$ and also various situations analogous to what have been studied in the various chapters of this thesis. Still more general situation arises if we consider E-compact spaces of Herrlich (E being an epireflective subcategory of the category of all Hausdorff spaces). We do not propose to embark on this, in this thesis.

A.O. Introduction

In [EN; MR] the idea that any compact Hausdorff space can be characterized as a space that is homeomorphic to some closed subspace of a topological product of the closed unit interval $\{x:0 \leq x \leq 1\}$ in the real line is generalized and the class of topological spaces, the members of which are homeomorphic to any closed subspace of topological powers of some given space E, is considered. Further investigations have appeared in the papers [BL], [HE], [MR] and so on. One special instance of that generalization is the case in which the space E is the real line. This class of spaces is necessarily the class of real compact spaces.

For our purpose, we have considered E to be a (Hausdorff) topological field.

§ A.1. Preliminary Concepts.

A.1.1. Definition [EN; MR]. A space X is E-completely regular if X is homeomorphic to a subspace of a product of copies of E and X is called E-compact if X is homeomorphic to a closed subspace of a product of copies of E.

A.1.2. Definition [EN; MR]. A subset U of X is called E-open if it is of the form $f^{-1}(V)$, where V is an open subset of some finite power E^n and $f \in C(X,E^n)$. A subset F of X is E-closed if and only if its complement is E-open.

A.1.3. Theorem [PO; WO]. Let X and E be spaces. The following are equivalent.

(1) X is E-completely regular

(2) For each closed subset A of X and each $p \in X \setminus A$, there is a positive integer n and $f \in C(X,E^n)$ such that $f(p) \notin cl_{E^n} f(A)$.

i.e., $\bigcup \{C(X,E^n) : n \in N\}$ separates points and closed sets of X.

(3) E-open subsets of X form a base for the open subsets of X.

In general, we cannot replace $\bigcup \{C(X,E^n) : n \in N\}$ by C(X,E).

§ A.2. Some Definitions and Results.

A.2.1. Convention.

(1) We take ω copies of E and name them $\{E_i : i \in \omega\}$. Then by E^n, we mean $E_1 \times E_2 \times \ldots \times E_n$. If $n < m$, there is an obvious embedding of E^n in E^m namely, $(x_1, x_2, \ldots, x_n) \longmapsto (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0)$. This convention is needed for defining algebraic operations in our further developments. However, this does not conflict with notation used in [PO; WO] in situations like the theorem A.1.3, since any rearrangement of coordinates is a homeomorphism.

(2) We consider the class of all spaces X such that for each closed set $A \subset X$ and a point $x \in X \setminus A$, there is a positive integer n and $f \in C(X,E^n)$ such that $f(A) = 0$ and $f(x) \neq 0$.

A.2.2. Definition. $C_E(X) = \bigcup\{C(X,E^n) : n \in N\}$. $Z_{E^n}(f) = \{x \in X: f(x) = 0\}$, where $f \in C(X,E^n)$, is called an E-zero set of f. For $f,g \in C_E(X)$, define $(f+g)(x)=f(x)+g(x)$, for every $x \in X$. If $f \in C(X,E^n)$, $g \in C(X,E^m)$ and if $n > m$, then, since E^m is embedded in E^n as described above, g(x) can be taken as a member of E^n. i.e., $g \in C(X,E^n)$. So, $f(x)+g(x)$ makes sense. Likewise, $(f.g)(x)=f(x).g(x)$ for every $x \in X$.

A.2.3. **Result.** By convention (2), X is E-completely regular.

Proof. By convention (2), given X, for each closed set $A \subset X$ and a point $x \in X \setminus A$, there is a positive integer n and $f \in C(X,E^n)$ such that $f(A) = 0$ and $f(x) \neq 0$. Then, $f(x) \notin cl_{E^n} f(A)$, so that X is E-completely regular.

A.2.4. **Result.** X is E-completely regular if and only if its topology is the weakest for which each $f \in C_E(X)$ is continuous.

Proof. Suppose that (X,τ) is E-completely regular and $\tau' \leqslant \tau$ and each $f \in C_E(X,\tau)$ is continuous with respect to τ'. If F is a closed set with respect to τ, then for each $x \in X\setminus F$, there is some $f_x \in C_E(X)$ such that $f_x(F) = \{0\}$ and $f_x(x) \neq 0$. Since f_x is continuous, with respect to τ', $Z_{E^n}(f_x)$ is closed for any $n \in N$ with respect to τ'. Thus $F = \bigcap_{x \in X\setminus F} Z_{E^n}(f_x)$ is closed with respect to τ' and so $\tau' = \tau$.

Conversely suppose that τ is the weakest topology on X for which each $f \in C_E(X)$ is continuous. Then, a subbase for the closed sets (with respect to τ) is the family $\left\{\{x \in X: f(x) = r\} : f \in C_E(X),\ r \in E^n,\ n \in N\right\}$. We show that the base this family generates is the family of all E-zero sets of members of $C_E(X)$. The result then follows from A.2.3.

First, every E-zero set $Z_{E^n}(f)$, $n \in N$, $f \in C(X,E^n)$ is in this family. A typical member in this family is

$$\left\{x \in X: f(x)=r,\ r \in E^m,\ f \in C(X,E^m)\right\} = Z_{E^m}(g).$$

Now a finite union of E-zero-sets is an E-zero-set

$$Z_{E^{k_1}}(f_1) \cup Z_{E^{k_2}}(f_2) \cup \dots \cup Z_{E^{k_m}}(f_m) = Z_{E^k}(f_1 \cdot f_2 \cdot \dots \cdot f_m),$$

where, $k = \max(k_1, k_2, \dots, k_m)$.

It follows that the base generated by the family above is simply, the family of all $Z_{E^n}(f)$, $n \in N$, $f \in C(X,E^n)$.

A.2.5. Definition. Two subsets A and B of X are said to be E-completely separated (from one another) in X, if there exists a positive integer n such that for $f \in C(X,E^n)$, $f(x) = 0$ for every $x \in A$ and $f(x) \neq 0$, for every $x \in B$.

Evidently, two sets contained in E-completely separated sets are E-completely separated.

When an E-zero set Z is a neighbourhood of a set A, we refer to Z as an E-zero-set neighbourhood of A.

A.2.6. Result. If two sets are contained in disjoint E-zero-sets, then, they are E-completely separated.

Proof: If $Z_{E^n}(f) \cap Z_{E^m}(g) = \emptyset$, then, we may define $h(x) = f(x)$, $x \in X$. Then, $h \in C(X,E^n)$ or $h \in C(X,E^m)$ depending on whether $n > m$ or $m > n$. Also, h is equal to 0 on $Z_{E^n}(f)$ $(Z_{E^m}(f))$ and non-zero on $Z_{E^n}(g)$ $((Z_{E^m}(g))$.

A.2.7. Result. If A, A' are E-completely separated, then there exists E-zero-sets F,Z such that $A \subset X-Z \subset F \subset X-A'$.

Proof: If A, A' are E-completely separated, then, there exists a positive integer n such that $f \in C(X,E^n)$ and $f(x) = 0$ for every $x \in A$ and $f(x) \neq 0$ for every $x \in A'$. The set $F = \{x \in X : f(x) = 0\}$ is a zero-set neighbourhood of A. Let $Z = \text{cl}\{x \in X : f(x) \neq 0\}$. Then $A \subset X-Z \subset F \subset X-A'$.

A.2.8. Definition. A subspace S of X is C_E-embedded in X if every function in $C_E(S) = \bigcup\{C(S,E^n) : n \in N\}$ can be extended to a function in $C_E(X) = \bigcup\{C(X,E^n) : n \in N\}$.

A.2.9. Result. If a subspace S of X is C_E-embedded in X, then, any two E-completely separated sets in S are E-completely separated in X.

Proof: If A and B are E-completely separated in S, then, there exists a positive integer n such that $f \in C(S,E^n)$, where f is 0 on A and non-zero on B. By hypothesis, f has an extension to a function g in $C_E(X)$, particularly, $g \in C(X,E^n)$. Since g is 0 on A and non-zero on B, they are E-completely separated in X.

§ A.3. E-Z-Filters

A.3.1. Definition. An E-Z-filter on X is a collection $\mathcal{F}$ of E-zero-sets of X with the properties:

(1) $\emptyset \notin \mathcal{F}$.

(2) $Z_1, Z_2 \in \mathcal{F} \Rightarrow Z_1 \cap Z_2 \in \mathcal{F}$.

(3) If Z is an E-zero-set in X and $Z \supset Z_1$, where $Z_1 \in \mathcal{F}$, then, $Z \in \mathcal{F}$.

If in addition, the following condition is satisfied, we say $\mathcal{F}$ is an E-Z-ultrafilter.

(4) $\mathcal{F}$ is not properly contained in an E-Z-filter.

Every family $\mathcal{B}$ of E-zero-sets that has finite intersection property is contained in an E-Z-filter: the smallest such is a family $\mathcal{F}$ of all E-zero-sets containing finite intersections of members of $\mathcal{B}$. We say that $\mathcal{B}$ generates the E-Zfilter $\mathcal{F}$. When $\mathcal{B}$ itself is closed under finite intersection, it called the E-base for $\mathcal{F}$.

Clearly, every family $\mathcal{B}$ of E-zero-sets that has finite intersection property is contained in an E-Z-ultra-filter. Thus an E-Z-ultrafilter is a maximal subfamily of $Z_E(X)$ with finite intersection property.

A.3.2. Definition. By a prime E-Z-filter, we shall mèan an E-Z-filter with the following property:
Whenever the union of two E-zero-sets belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

A.3.3. Result. Let $\mathcal{A}$ be an E-Z-ultrafilter on X. If an E-Zero-set Z meets every member of $\mathcal{A}$, then $Z \in \mathcal{A}$.

Proof: $\mathcal{A} \cup \{Z\}$ generates an E-Z-filter. As this contains the maximal E-Z-filter $\mathcal{A}$, it must be $\mathcal{A}$.

A.3.4. Result. Every E-Z-ultrafilter is a prime E-Z-filter.

Proof: If E-zero sets Z and Z' do not belong to an E-Z-ultrafilter $\mathcal{A}$, then by the previous result, there exist $A, A' \in \mathcal{A}$ such that $Z \cap A = Z' \cap A' = \emptyset$. Then $Z \cup Z'$ does not meet the member $A \cap A'$ of $\mathcal{A}$, and hence does not belong to $\mathcal{A}$.

A.3.5. Result. Let $\mathfrak{F}$ be a non-empty collection of E-zero-sets in X such that $\emptyset \notin \mathfrak{F}$ and $\mathfrak{F}$ has finite intersection property. Then $\mathfrak{F}$ is an E-Z-ultrafilter if and only if whenever Z is an E-zero-set such that $Z \notin \mathfrak{F}$, then $(X \setminus Z) \supset Z'$, an E-Zero-set such that $Z' \in \mathfrak{F}$.

Proof: Suppose that $\mathfrak{F}$ is an E-Z-ultrafilter. We have $X \in \mathfrak{F}$ and since $X = Z \cup (X \setminus Z)$, and $\mathfrak{F}$ is a prime E-Z-filter, the result follows.

Conversely, assume that either $Z \in \mathfrak{F}$ or $(X \setminus Z) \supset Z'$, where $Z' \in \mathfrak{F}$, for every $Z \in Z_E(X)$. Since $\mathfrak{F}$ is closed under finite intersection and $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ has finite intersection property. Suppose that $\mathcal{F}$ is not maximal and pick $\mathcal{F}' \subset Z_E(X)$ such that $\mathcal{F}'$ has finite intersection property and $\mathcal{F} \subsetneq \mathcal{F}'$. Pick $Z_1 \in \mathcal{F}' \setminus \mathcal{F}$. Then $Z_1 \notin \mathcal{F}$. So $X \setminus Z_1$ contains Z_2 such that $Z_2 \in \mathcal{F}$ which implies $Z_2 \in \mathcal{F}'$. But then $Z_1 \cap Z_2 = \emptyset$, which is false.

§A.4. Convergence of E-Z-Filters.

We now discuss the convergence of E-Z-filters on an E-completely regular space. It is analogous to the standard theory of convergence of Z-filters or Z-filter bases on an arbitrary Hausdorff space.

A.4.1. **Definition.** Let X be an E-completely regular space. A point $p \in X$ is said to be a cluster point of an E-Z-filter $\mathcal{F}$ if every E-neighbourhood of p meets every member of $\mathcal{F}$. Thus, since the members of $\mathcal{F}$ are E-closed sets, p is a cluster point of $\mathcal{F}$ if and only if $p \in \bigcap \mathcal{F}$.

If S is a non-empty subset of X, then E-cl S(the E-closure of S in X) is the set of cluster points of the E-Z filter $\mathcal{F}$ of all E-zero sets containing S, because, the E-zero sets in the E-completely regular space X form a base for the E-closed sets.

A.4.2. **Definition.** The E-Z-filter $\mathcal{F}$ is said to converge to the limit p if every E-neighbourhood of p contains a member of $\mathcal{F}$. If $\mathcal{F}$ converges to p, then p is a cluster point of $\mathcal{F}$.

A.4.3. Result. $\mathcal{F}$ converges to p if and only if $\mathcal{F}$ contains the E-Z-filter of all E-zero-set neighbourhoods of p.

Proof: In the E-completely regular space X, every E-neighbourhood of p contains an E-zero-set neighbourhood of p.

A.4.4. Result. If p is a cluster point of $\mathcal{F}$, then at least one E-Z-ultrafilter containing $\mathcal{F}$ converges to p.

Proof: Let $\mathcal{E}$ be the E-Z-filter of all E-zero-set neighbourhoods of p. Then $\mathcal{F} \cup \mathcal{E}$ has the finite intersection property and so it is embeddable in an E-Z-ultrafilter $\mathcal{A}$. Since $\mathcal{A}$ contains $\mathcal{E}$, it converges to p. In particular, an E-Z-ultrafilter converges to any cluster point.

A.4.5. Result. Let $p \in X$, where X is E-completely regular and $\mathcal{F}$ be a prime E-Z-filter on X. The following are equivalent:

(1) p is a cluster point of $\mathcal{F}$.

(2) $\mathcal{F}$ converges to p.

(3) $\bigcap \mathcal{F} = \{p\}$.

Proof: It suffices to show that (1) $\Longrightarrow$ (2). Let V be any E- zero-set neighbourhood of p. Since X is E-completely regular, V contains an E-neighbourhood of p of the form X-Z, where Z is an E-zero-set. Since $V \cup Z = X$, either $V \in \mathcal{F}$ or $Z \in \mathcal{F}$, since $\mathcal{F}$ is prime. But Z cannot belong to $\mathcal{F}$ because $p \notin Z$. So $V \in \mathcal{F}$. Thus $\mathcal{F}$ converges to p.

A.4.6. Notation. The family of all E-zero-sets containing a given point p is denoted by $\mathcal{A}_p$. $\mathcal{A}_p$ is an E-Z-filter. Since any E-zero-set not containing p is completely separated from $\{p\}$, $\mathcal{A}_p$ is an E-Z-ultrafilter.

A.4.7. Result. p is a cluster point of E-Z-filter $\mathcal{F}$ if and only if $\mathcal{F} \subset \mathcal{A}_p$.

Proof: p is a cluster point of $\mathcal{F}$ if and only if p belongs to every member of $\mathcal{F}$.

A.4.8. Corollary:

(1) $\mathcal{A}_p$ is the unique E-Z-ultrafilter converging to p.

(2) Distinct E-Z-ultrafilters cannot have a common cluster point.

(3) If $\mathcal{F}$ is an E-Z-filter converging to p, then $\mathcal{A}_p$ is the unique E-Z-ultrafilter containing $\mathcal{F}$.

A.4.9. Definition. The mapping τ_E^* : Let τ be a continuous mapping from X to E^k for some $k \in N$. Let $\mathfrak{F}$ be an E-Z-filter on X. The image of $\mathfrak{F}$ under τ is not an E-Z-filter. The total pre-image of an E-Zero-set, however is an E-zero-set, since,

$$\tau^{-1} [Z_{E^k}(g)] = Z_X(g.\tau).$$

The collection of all $Z_{E^k}(g)$, $k \in N$, whose pre-images belong to $\mathfrak{F}$, is an E-Z-filter on E^k, denoted by $\tau^*\mathfrak{F}$. i.e., $\tau^*\mathfrak{F} = \left\{ Z_E \in Z_E(E^k) : \tau^{-1}(Z_E) \in \mathfrak{F} , k \in N \right\}$. Clearly, $\tau^*\mathfrak{F}$ is an E-Z-filter on $\cup\left\{E^n : n \in N\right\}$. It need not be an E-Z-ultrafilter, even when $\mathfrak{F}$ itself is. But when $\mathfrak{F}$ is an E-Z-ultrafilter, then $\tau^*\mathfrak{F}$ will be prime.

A.4.10. Result. Let Z be an E-zero-set in X. If $p \in cl_T Z$, where T is an E-compact space, then at least one E-Z-ultrafilter on X contains Z and converges to p.

Proof: Let $\mathfrak{T}$ be the E-Z-filter on T of all E-zero-set neighbourhoods (in T) of p and $\mathcal{B}$ be the trace of $\mathfrak{T}$ on X. Since $p \in cl_T Z$, $\mathcal{B} \cup \{Z\}$ has finite intersection property and so is contained in E-Z-ultrafilter $\mathcal{A}$. Then $\mathcal{A}$ converges to p.

A.4.11. Result. Let X be dense in an E-compact space T. The following statements are equivalent.

(1) Every continuous mapping τ from X into any E-compact space Y has an extension to a continuous mapping from T into Y.

(2) X is C_E-embedded in T.

(3) Any two disjoint E-zero-sets have disjoint E-closures in T.

(4) For any two E-zero-sets Z_1, Z_2 in X, $cl_T(Z_1 \cap Z_2) = cl_T Z_1 \cap cl_T Z_2$.

(5) Every point of T is the limit of a unique E-Z-ultrafilter on X.

Proof: (1) $\Longrightarrow$ (2). A function $f \in C_E(X)$, say $f \in C(X,E^k)$, for some $k \in N$, is a continuous mapping into the E-compact subset $cl_{E^k}[f(X)]$. Hence (2) is a special case of (1).

(2) $\Longrightarrow$ (3). This follows from A.2.9.

(3) $\Longrightarrow$ (4). If $p \in cl\ Z_1 \cap cl\ Z_2$, then for every E-zero-set neighbourhood V (in T) of p, we have $p \in cl(V \cap Z_1)$ and $p \in cl(V \cap Z_2)$. i.e., V meets $Z_1 \cap Z_2$. Therefore, $p \in cl(Z_1 \cap Z_2)$. Thus, $cl\ Z_1 \cap cl\ Z_2 \subset cl(Z_1 \cap Z_2)$. The reverse inclusion is always true.

(4) $\Longrightarrow$ (5) Since X is dense in T, each point of T is the limit of at least one E-Z-ultrafilter. On the other hand, distinct E-Z-ultrafilters have disjoint E-zero-sets and (3) implies that a point p cannot belong to the closures of both these E-zero-sets. Hence, the two E-Z-ultrafilters cannot both converge to p.

(5) $\Longrightarrow$ (1) Given $p \in T$, let $\mathcal{A}$ denote the unique E-Z-ultrafilter on X with limit p. We write,

$$\tau_E^* \mathcal{A} = \left\{ F_E \in Z_E(Y) : \tau^{-1}(F_E) \in \mathcal{A} \right\}.$$

This is an E-Z-filter on the E-compact space Y and so has a cluster point. Moreover, since $\mathcal{A}$ is a prime E-Z-filter, so is $\tau_E^* \mathcal{A}$. So, $\tau_E^* \mathcal{A}$ has a limit in Y. Denote this family by $\overline{\tau}_E p$. Then,

$$\bigcap \tau_E^* \mathcal{A} = \{ \overline{\tau}_E \; p \} \qquad \text{(A)}$$

This defines a mapping $\overline{\tau}_E : T \longrightarrow Y$. In case $p \in X$, we have $p \in \bigcap \mathcal{A}$ so that $\tau_E p \in \tau_E^* \mathcal{A}$. Therefore $\overline{\tau}_E$ agrees with τ_E on X. For F_E, F_E' on $Z_E(Y)$, let us write $Z_E = \tau_E^{-1}(F_E)$, $Z_E' = \tau_E^{-1}(F_E')$. If $p \in cl_T Z_E$, Z_E belongs to $\mathcal{A}$ and so $F_E \in \tau_E^* \mathcal{A}$. Thus, $p \in cl\, Z_E \Rightarrow \tau_E p \in F_E$. To establish continuity of τ_E at the point p, we consider an arbitrary E-zero-set neighbourhood F_E of $\overline{\tau}_E p$ and exhibit an

E-neighbourhood of p that is carried by $\overline{\tau}_E$ into F_E. Let F'_E be an E-zero-set whose complement is an E-neighbourhood of $\overline{\tau}_E p$ contained in F. Then $F_E \cup F'_E = Y$ so that $Z_E \cup Z'_E = X$ and therefore $cl\ Z_E \cup cl\ Z'_E = T$. Since $\overline{\tau}_E p \notin F'_E$, we have $p \notin cl\ Z_E'$. Therefore, $T - cl\ Z'_E$ is an E-neighbourhood of p. Also, every point q in this neighbourhood belongs to $cl\ Z_E$, whence $\overline{\tau}_E q \in F_E$.

A.4.12. Result. The E-completely regular space X has an E-compactification $\beta_E X$ with the following properties.

(1) Every continuous mapping τ_E from X into any E-compact space Y has a continuous extension from $\beta_E X$ into Y.

(2) Every function f in $C_E(X)$ has an extension to a function f in $C_E(\beta_E X)$.

(3) Any two disjoint E-zero-sets in X have disjoint E-closures in βX.

(4) For any two E-zero sets Z_E, Z'_E in X,

$$cl_{\beta_E X}(Z_E \cap Z'_E) = cl_{\beta_E X}(Z_E) \cap cl_{\beta_E X}(Z'_E)$$

(5) Distinct E-Z-ultrafilters have distinct limits in $\beta_E X$.

Furthermore, $\beta_E X$ is unique in the following sense. If an E-compactification T of X satisfies any one of the listed conditions, then there exists a homeomorphism of $\beta_E X$ onto T that leaves X pointwise fixed.

Proof: We first prove the uniqueness. By theorem A.4.11, if T satisfies (1) - (4) it satisfies all of them.

By (1), the identity mapping on X, which is continuous mapping into the E-compact space T has an extension from all of $\beta_E X$ into T. Similarly, it has an extension from T into $\beta_E X$. Hence these extensions are homeomorphisms.

We now consider the construction of $\beta_E X$. There is a one-one correspondence between the E-Z ultrafilters on X and the points of $\beta_E X$, each E-Z ultrafilter converging to its corresponding point. We have a correspondence between the fixed E-Z-ultrafilters and the points of X. Hence X constitutes an index set for the fixed E-Z-ultrafilters. The points of $\beta_E X$ are defined to be the elements of the enlarged index set in order to include all the E-Z-ultrafilters on X.

The family of all E-Z-ultrafilters on X is written $(\mathcal{A}^p)_{p \in \beta_E X}$, where for $p \in X$, $\mathcal{A}^p$ is the family of all E-zero-sets containing p. The topology on $\beta_E X$ is defined in such a way that p is the limit of the E-Z-ultrafilter $\mathcal{A}^p$, for every $p \in \beta_E X$, not only for $p \in X$.

For an E-zero set $Z_E \subset X$, let $\overline{Z}_E$ denote all elements of $\beta_E X$ of which Z_E is a member. We claim that the set $\mathcal{B} = \{\overline{Z}_E : Z_E$ is an E-zero set in $X\}$ is a base for the E-closed sets for a topology on $\beta_E X$.

(1) $\emptyset$ is an E-zero set in X and so $\overline{\emptyset} \in \mathcal{B}$. However, $\overline{\emptyset} = \{p \in \beta_E X : \emptyset \in p\} = \emptyset$. $\therefore \emptyset \in \mathcal{B}$. Also, $\overline{X} = \{p \in \beta_E X : X \in p\} = \beta_E X$. $\therefore \beta_E X \in \mathcal{B}$.

(2) Suppose that $\overline{Z_E}, \overline{Z_E'} \in \mathcal{B}$. $Z_E \cup Z_E' \in \mathcal{A}^p$ if and only if $Z_E \in \mathcal{A}^p$ or $Z_E' \in \mathcal{A}^p$. Thus the elements of $\beta_E X$ which contain $Z_E \cup Z_E'$ are precisely those which contain Z_E or Z_E'. So $\overline{Z}_E \cup \overline{Z}_E' = \overline{Z_E \cup Z_E'}$ and so is closed under finite unions.

Give $\beta_E X$ the topology having $\mathcal{B}$ as a base for the closed sets. Define $\beta_E : X \longrightarrow \beta_E X$ by $\beta_E(x) = \mathcal{F}_x$, where $\mathcal{F}_x = \{Z_E : x \in Z_E\}$. Then $\mathcal{F}_x$ is an E-Z-ultrafilter and hence belongs to $\beta_E X$. Now, $\mathcal{F}_x \in \overline{Z}_E \cap \beta_E(X)$ if and only if $Z_E \in \mathcal{F}_x$ if and only if $x \in Z_E$.

Thus, $\overline{Z}_E \cap \beta_E(X) = \beta_E(Z_E)$. This says that β_E is a continuous and closed mapping. For $x,y \in X$, if $\beta_E(x) = \beta_E(y)$, then $\mathfrak{Z}_x = \mathfrak{Z}_y$ so that every E-zero-set containing x also contains y. i.e., x=y. Thus $\beta_E : X \longrightarrow \beta_E X$ is a topological embedding. We have, $\beta_E(Z_E) = \overline{Z}_E \cap \beta_E(X)$. Therefore,

$$cl_{\beta_E X}\ (\beta_E(Z_E)) \subset \overline{Z}_E \qquad (1)$$

For any basic E-closed set Z_E' containing $\beta_E(Z_E)$, it follows that $\beta_E(Z_E') = Z_E' \cap \beta_E(X) \supset \beta_E(Z_E)$. Thus,

$$\overline{Z_E'} \supset \overline{Z_E} \text{ and so } cl_{\beta_E X}\ (\beta_E(Z_E)) \supset \overline{Z}_E \qquad (2)$$

Thus from (1) and (2),

$$cl_{\beta_E X}(\beta_E(Z_E)) = \overline{Z}_E .$$

This gives us that $cl_{\beta_E X}\ (\beta_E(X)) = \overline{X} = \beta_E X$, so that $\beta_E(X)$ is dense in $\beta_E X$.

To show that $\beta_E X$ is an E-compactification of X, we prove that it is an E-compact Hausdorff space.

To see that $\beta_E X$ is Hausdorff space, consider any two distinct point p and p'. Choose disjoint E-zero-sets $A \in \mathcal{A}^p$

and $A' \in \mathcal{A}^{p'}$. Now, there exist an E-zero set Z_E disjoint from A and an E-zero-set Z_E' disjoint from A' such that $Z_E \cup Z_E' = X$, (Result A.2.7). So, $p \notin \text{cl } Z_E$, $p' \notin \text{cl } Z_E'$. Since $\text{cl } Z_E \cup \text{cl } Z_E' = \beta_E X$, the neighbourhoods $\beta_E X - \text{cl } Z_E$ of p and $\beta_E X - \text{cl } Z_E'$ of p' are disjoint.

Finally, consider any collection of basic E-closed sets $\overline{Z}_E$ with finite intersection property, Z_E ranging over some family $\mathcal{B}$. Now, $\mathcal{B}$ itself has finite intersection property so that $\mathcal{B}$ is embeddable in a E-Z-ultrafilter $\mathcal{F}$. Then,

$$p \in \bigcap_{\overline{Z}_E \in \mathcal{F}} \overline{Z}_E \subset \bigcap_{Z_E \in \mathcal{B}} \overline{Z}_E$$

so that the latter intersection is non-empty. Therefore, $\beta_E X$ is compact.

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