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ON SOME INFINITE CONVEX INVARIANTS

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VIJAYAKRISHNAN.S

DEPARTMENT OF MATHEMATICS
COCHIN UNIVERSITY OF SCIENCE TECHNOLOGY
COCHIN- 682022, INDIA

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CERTIFICATE

This is to certify that the thesis entitled **"On Some Infinite Convex Invariants"** is an authentic record of research carried out by **Sri. Vijayakrishnan. S,** under our supervision and guidance in the Department of Mathematics, Cochin University of Science and Technology, Cochin - 22 for the Ph.D degree of the Cochin University of Science and Technology and no part of it has previously formed the basis for the award of any other degree or diploma in any other University.

Dr. R.S. Chakravarthi
(Supervisor)
Reader, Department of Mathematics
CUSAT, Cochin-682022

Dr. T. Thrivikraman
(Co-Supervisor)
Professor
Department of Mathematics
CUSAT, Cochin-682022

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INTRODUCTION
ON SOME INFINITE CONVEX INVARIANTS

The origin of convexity can be traced back to the period of Archimedes and Euclid. The major developments in the eighteenth century are Kepler's works on Archimedean solids, determination of the densest lattice packing of circular discs in E^2 by Lagrange, and Legendre's proof of the Euler's relations between number of vertices, edges and faces of a convex polytope in E^3.Cauchy's proof of Euclid's statement that two convex polytopal surfaces in E^3 coincide up to proper or improper rigid motions if there is a homeomorphism between these surfaces the restriction of which to any face is a rigid motion is a major contribution to convexity in the nineteenth centaury. Other contribution came from Steiner who gave a series of proofs of the isoperimetric property of circles and balls using Steiner symmetrisation and the four-hinge method. The solution of the isoperimetric problem was achieved by Edler and by Schwarz and Weierstrass. A second contribution of Steiner to convexity is his formula for the volume of parallel bodies of a convex body.

At the turn of the nineteenth centaury, convexity became an independent branch of Mathematics with its own problems, methods and theories. Minkowski (1864- 1909) systematically developed convexity theory. His theorem on mixed volumes and lattice point theorem are of great importance. The contributions of Blaschke (1885-1902) include characterization of balls and ellipsoids, Blaschke's selection theorem and the affine isoperimetric inequality.

The early papers of abstract convexity can be sorted out into two kinds. The first type deals with generalization of particular problems such as separation of convex sets [EL], extremality [FA], [DAV] or continuous selection Michael [MI]. Papers of the second type are involved with a multi-purpose system of axioms. Schmidt [SC] and Hammer [HA_1, HA_3, HA_4] discuss the viewpoint of generalized topology, which enters into convexity via the closure operator. The arising of convexity from algebraic operations, and the related property of domain finiteness receive attention in Birkhoff and Frink [BI, F], Schmidt [SC] and Hammer [HA_3].

The classical theorems of Helly, Radon and Caratheodory stand at the origin of what is known today as the combinatorial geometry of convex sets. "Helly's theorem on the intersection of convex sets" was discovered by Helly in 1913 and communicated to Radon who published a first proof in 1921.Helly's own proof came in 1923. Helly's theorem may be formulated as follows.

Let K be a family of convex sets in R^d, and suppose that K is finite or each member of K is compact. If every d+1 or fewer members of K have a common point, then there is a point common to all members of K.

Radon's theorem turned out to be extremely useful in combinatorial convexity theory. Radon's theorem is as follows.

Let X be a set of d+2 or more points in R^d. Then X contains two disjoint subsets of X whose convex hulls have a common point.

Radon's theorem has seen numerous applications, frequently in proofs, led to a rich body of variants, refinements, and deep generalizations such as Tverberg's theorem [TV_1, TV_2]. Caratheodory's theorem is the fundamental dimensionality result in convexity and one of the corner stones in combinatorial geometry. The theorem is formulated as follows.

Let X be a set in R^d and p be a point in the convex hull of X. Then there is a subset Y of X consisting of d+1 or fewer points such that p lies in the convex hull of Y.

These three classical theorems are not only closely related, but in fact, each of them can be derived from each of the others. In abstract convexity theory, this has been the main incentive to study the inter relationships between the three classical results in an axiomatic setting.

The viewpoint of combinatorial geometry originates in Levi [LE], where the relationship between Helly's and Radon's theorem is discussed. The survey papers of Danzer, Grunbaum and Klee [DA, GR, KL] stimulated the investigations on abstract convexity. The other major contributors to the theory of abstract convexity are Tverberg who extended Radon's theorem in R^d, Eckhoff, Jamison, Sierksma and Soltan. An elegant survey has been done by Van de vel [VAD_9] whose work has been acclaimed as remarkable.

The theory convex invariants have grown out of the classical results of Helly, Radon and Caratheodory in Euclidean spaces. Levi gave the first general definition of the invariants Helly number and Radon number. A general theory convex invariants was first developed by Kay and Womble [KA, WO].

Most of the results mentioned above are relevant in finite dimensional Euclidean spaces. To study the geometrical and topological implications in the infinite dimensional set up, we introduce the concept of infinite convex invariants in an abstract convexity setting and study the different relations among them. We also introduce the notion of transfinite convex dimension of a topological convex structure.

The thesis is divided into five chapters.

In chapter 0 we give the basic definitions and results, which we are using in the succeeding chapters.

Based on the works of Kay and Womble [KA, WO] and Soltan [SOL_1], Van de vel [VAD_9] considered Helly dependence of subsets (not necessarily finite) and the convex invariant called Helly number (which is finite) in a general convex structure. We felt that the restriction on Helly number to be finite is rather too much of a handicap and started investigating in this direction. In the first chapter we introduce the concepts of infinite Helly number, infinite star Helly number and infinite compact Helly number and then obtain extensions of compact intersection theorem [JA_3] and countable intersection theorem [JA_3] to the infinite situation. A nonempty subset F of a convex structure X is Helly dependent if $\cap_{a\in F}$ co (F\ {a}) $\neq \phi$. If α is an infinite cardinal, we say that h (X) $\leq \alpha$ if and only if each F $\subseteq$ X with $|F| > \alpha$ is Helly dependent. The infinite star Helly number h*(X) is defined as the least cardinal α such that each collection of convex sets in X with α intersection property has nonempty intersection. The infinite Helly number of an H – convex structure in terms of the degree of minimal dependence of functionals is obtained.

In chapter 2 we introduce the infinite Caratheodory number, infinite Radon number and the infinite exchange number of a convex structure. We obtain relations between Radon, Caratheodory, Helly and exchange dependence for arbitrary subsets of a convex structure. The inequalities of Levi [LE] and Sierksma [SI_1] are discussed in the infinite context. We investigate the behaviour of convex invariants under convexity preserving images. We also extend the Eckhoff-Jamison [SI_2] inequality.

The notion of rank of a convex structure was introduced by Jamison [JA_4] and that of a generating degree was introduced by Van de Vel [VAD_8]. In chapter 3 we obtain a relationship between rank and generating degree in the infinite situation. The generating degree is defined using the following generalization of Dilworth's theorem [DIL]. If P is a poset such that every set of elements of order greater than α be dependent while there is at least one set of α independent elements, then P is a set sum of α disjoint chains. We also prove that for a non-coarse convex structure, rank is less than or equal to the generating degree. We also generalize Tverberg's theorem using infinite partition numbers.

Van de Vel introduces the notion of convex dimension cind for a topological convex structure [VAD_1]. In chapter 4, we introduce the notion of transfinite convex dimension trcind. We compare the transfinite topological and transfinite convex dimensions (Prop.4.2.3). We obtain the following characterization of trcind in terms of hyperplanes. For an FS_3 convex structure X with connected convex sets the following statements are equivalent.

1. trcind (X) $\leq \alpha$,where α is an ordinal
2. Corresponding to each hyper plane H $\subseteq$ X, there exists a $\beta < \alpha$ such that trcind(H) $\leq \beta$.

We also obtain a characterization of tricind in terms of mappings to cubes [Prop.4.3.1].

CHAPTER 0

PRELIMINARIES

In this chapter we give the basic definitions and results, which we use in the succeeding chapters. These are adapted from [VAD_9] and [CH_2].

0.1 CONVEXITY THEORY: BASIC CONCEPTS

0.1.1 Definition

A family $\mathcal{C}$ of subsets of a set X is called a convexity on X if

(1) ϕ and X are in $\mathcal{C}$

(2) $\mathcal{C}$ is closed under intersections, that is, if $\mathcal{D} \subset \mathcal{C}$ is non -empty, then $\cap \mathcal{D}$ is in $\mathcal{C}$.

(3) C is closed under nested unions, that is, if $\mathcal{D} \subset \mathcal{C}$ is non -empty and totally ordered by inclusion, then $\cup \mathcal{D}$ is in $\mathcal{C}$.

The pair (X, $\mathcal{C}$) is called a convex structure (convexity space, aligned space). The members of $\mathcal{C}$ are called convex sets and their complements are called concave sets. It is customary to denote the convex structure (X, $\mathcal{C}$) by the symbol X.

0.1.2 Definition

For a subset A of X, the convex hull of A, denoted by co (A) is the smallest convex set containing A, that is, co (A) = $\cap$ {C | A $\subset$ C $\in$ $\mathcal{C}$}. The convex hull of a finite set is called a polytope.

The axioms (1) and (2) in definition (0.1.1) are first used by Levi [LE] in 1951,and later on by many authors, Eckhoff [EC_1], Jamison [JA_1], Kay and

Womble [KA, WO] and Sierksma [SI_1]. The concept of alignment is introduced by Jamison [JA_1]. Hammer [HA_3] has shown that axiom (3) is equivalent to "domain finiteness condition" which says that for each A in X and for each point $p \in co(A)$ there is a finite set $F \subset A$ with $p \in co(F)$. Instead of the term alignment we find in the literature the terms " algebraic closure system" and "domain finite convexity space".

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two convexities on X and if $\mathcal{C}_1 \subset \mathcal{C}_2$, then we say that $\mathcal{C}_1$ is coarser than $\mathcal{C}_2$ and $\mathcal{C}_2$ is finer than $\mathcal{C}_1$.The power set 2^X is the finest convexity and $\{\Phi, X\}$ is the coarsest convexity on X.

0.1.3 Definition

A collection **S** of sets in X is a subbase of a convex structure (X, $\mathcal{C}$) provided $\mathbf{S} \subset \mathcal{C}$ and $\mathcal{C}$ is the coarsest among all convexities that include **S**. In this case we say that **S** generates the convexity $\mathcal{C}$. A collection **B** of sets in X is a base of a convex structure (X, $\mathcal{C}$) provided $\mathbf{B} \subset \mathcal{C}$ and each member of $\mathcal{C}$ is the union of an up directed sub collection of **B.** In this case **B** is said to generate the convexity $\mathcal{C}$.

0.1.4 Proposition

Let $\mathcal{C}$ be a convexity on X. Then $\mathbf{B} \subset \mathcal{C}$ is a base for (X, $\mathcal{C}$) if and only if it contains all polytopes.

0.1.5 Proposition

A collection $\mathbf{S} \subset \mathcal{C}$ is a sub base for $(X, \mathcal{C})$ if and only if each nonempty polytope is the intersection of a sub family of **S**.

0.1.6 Definition

A subbase of X is called an intersectional subbase if each convex set is the intersection of subbasic sets.

0.1.7 Definition

An H-convexity on a vector space V over a totally ordered field K is the convexity generated by the family $\mathbf{S} = \{f^{-1}(\leftarrow, t] \,/\, t \in K,\ f \in \mathbf{F}\}$, where **F** is a collection of linear functionals from V to K.

If **F** is symmetric, that is, **F** contains –f whenever it contains f, then **S** also contains all sets of the type $f^{-1}[t, \rightarrow)$ with $t \in K$ and $f \in \mathbf{F}$ and the convexity generated by S is called the symmetric H – convexity.

0.1.8 Definition

Let $(X, \mathcal{C})$ be a convex structure and let Y be a subset of X. The family of sets $\mathcal{C}|_Y = \{C \cap Y \mid C \in \mathcal{C}\}$ is a convexity on Y called the relative convexity of Y and the resulting convex structure $(Y, \mathcal{C}|_Y)$ is a subspace of X.

0.1.9 Definition

Let $(X_i, \mathcal{C}_i)$ for $i \in I$ be a family of convex structures, let X be the product of the sets X_i and let $\pi_i : X \longrightarrow X_i$ denote the i^{th} projection. The product convexity $\mathcal{C}$ of X is generated by the subbase $\{\pi_i^{-1}(C_i) \mid C_i \in \mathcal{C}_i, i \in I\}$. The resulting convex structure $(X, \mathcal{C})$ is called the product of the spaces $(X_i, \mathcal{C}_i)$ for $i \in I$ and is denoted by $\Pi_{i \in I} (X_i, \mathcal{C}_i)$.

0.1.10 Definition

Let $f: X_1 \longrightarrow X_2$ be a function between two convex structures X_1 and X_2. Then f is said to be

(1) a convexity preserving function (cp function) provided for each convex set C in X_2, $f^{-1}(C)$ is convex in X_1.

(2) a convex to convex function (cc function) provided for each convex set C in X_1, $f(C)$ is convex in X_2.

The function f is an isomorphism if it is a bijection and is both cp and cc.

0.1.11 Definition

Let $(X, \mathcal{C})$ be a convex structure. A subset H of X is called a half space provided H is both convex and concave.

Note that ϕ and X are half spaces in any convexity of X. Also if $f : X \longrightarrow Y$ is a convexity preserving function and if H is a half space of Y, then $f^{-1}(H)$ is a half space of X.

0.1.12 Definition

Let $(X, \mathcal{C})$ be a convex structure. It is said to be

(1) S_1 if all singletons are convex.

(2) S_2 if $x_1 \neq x_2$ are points in X, then there is a half space $H \subset X$ with $x_1 \in H$ and $x_2 \notin H$.

(3) S_3 if $C \subset X$ is convex and if $x \in X \setminus C$, then there is a half space H of X with $C \subset H$, $x \notin H$.

(4) S_4 if C, $D \subset X$ are disjoint convex sets, then there is a half space H of X with $C \subset H$ and $D \subset X \setminus H$.

If X satisfies axiom S_i then X is called an S_i convex structure and $\mathcal{C}$ is called an S_i convexity. We have S_2 implies S_1 and under the assumption of S_1, $S_4 \Rightarrow S_3 \Rightarrow S_2$.

0.1.13 Proposition

(1) A convex structure is S_3 if and only if it is generated by half spaces.

(2) A point convex space is S_3 if and only if it embeds in a Cantor cube.

0.1.14 Definition

(a) A convex structure X is said to be a join hull commutative space (JHC Space) if $C \subseteq X$ is a non empty convex set and if $a \in X$, $co\{\{a\} \cup C\} = \cup \{co\{a,x\} \mid x \in C\}$.

(b) X satisfies ramification property if for all $b,c,d \in X$, $c \notin co\{b, d\}$ and $d \notin co\{b,c\}$ imply $co\{b,c\} \cap co\{b,d\} = \{b\}$.

(c) An interval ab of X is decomposible provided for each $x \in ab$, $ax \cup xb = ab$ and $ax \cap xb = \{x\}$.

(d) X satisfies cone-union property if C, C_1, C_2 ...C_n are convex sets with $C \subseteq \cup_i C_i$ and if $a \in X$, then co $\{\{a\} \cup C\} \subseteq \cup_i \{$ co $\{\{a\} \cup C_i\}\}$.

0.1.15 Definition

Let X be a set and let $I : X \times X \longrightarrow 2^X$ be a function with the following properties.

(1) Extensive law: $a, b \in I(a, b)$

(2) Symmetry law: $I(a, b) = I(b, a)$

Then I is called an interval operator on X, and I (a, b) is the interval between a and b. The resulting pair (X, I) is called an interval space. A subset C of X is (interval) convex provided $I(x, y) \subset C$ for every $x, y \in C$.

0.1.16 Definition

An interval operator I on X is geometric provided the following hold.

(1) Idempotent law: $I(b, b) = \{b\}$ for all $b \in X$.

(2) Monotone law: If a,.b, $c \in X$ and $c \in I(a, b)$, then $I(a, c) \subset I(a, b)$.

(3) Inversion law: If $a, b \in X$ and $c, d \in I(a, b)$, then $c \in I(a, d)$ implies $d \in I(c, b)$.

A set with a geometric interval operator is called a geometric interval space.

0.1.17 Definition

Let X be a geometric interval space, $C \subset X$ and $b \in X$. A point $c \in C$ is called the gate of b in C provided $c \in bx$ for each $x \in C$. If each point of X has a gate in C, then C is a gated set within X, and the resulting function $X \longrightarrow C$, which assigns to a point of X its gate in C, is the gate map of C.

0.1.18 Proposition

In a geometric interval space X

(1) gated sets are convex.

(2) if X is S_3, then all gate maps of X are cp and cc.

0.1.19 Theorem

Let (X_i, I_i) for i =1,2 be geometric interval spaces such that $X_1 \cap X_2$ is a gated subset of X_1 and of X_2, on which the respective interval operators coincide. Then there is a unique geometric interval operator I on $X_1 \cup X_2$ subject to the following two conditions.

(1) I extend I_1 and I_2.

(2) If $a \in X_1$ and $b \in X_2$, then I (a, b) meets $X_1 \cap X_2$.

If p_i: $X_i \longrightarrow X_1 \cap X_2$ for i = 1,2 are the gate maps, and if $a \in X_1$ and $b \in X_2$, then (3) $I(a, b) = I_1(a, p_2(b)) \cup I_2(p_1(a), b)$.

Moreover X_1 and X_2 are gated, and the gate map $X_1 \cup X_2 \longrightarrow X_i$ extends p_i for i =1,2. The resulting interval space is called the gated amalgam of X_1, X2.

0.1.20 Theorem

Let X_1 and X_2 be S_3 convex structures of arity two meeting in a nonempty gated subspace. Then there is one and only one S_3 convexity on $X = X_1 \cup X_2$ which is of arity two and takes X_1 and X_2 as convex subspaces. This convexity is derived from the gated amalgamation of the summands. Moreover if $F_i \subset X_i$ for i= 1,2 are sets with F_1 nonempty and if $p_i: X \longrightarrow X_i$ for i= 1,2 denotes the gate map, then co $(F_1 \cup F_2) \cap X_1 = co(F_1 \cup p_1 (F_2))$.

0.1.21 Definition

A median operator on a set X is a function $m: X^3 \rightarrow X$ satisfying the following properties.

(1) Absorption law, that is m (a, a, b) = a

(2) Symmetry law, if σ is any permutation of a, b, c then m $(\sigma(a),\sigma(b),\sigma(c))$ = m (a, b, c).

(3) Transitive law, m (m (a,b,c),d,c) = m(a,m(b,c,d),c).

The point m (a, b, c) is called the median of a,b,c and the resulting pair (X, m) is called a median algebra. A subset C of a median algebra is convex if $m (C \times C \times X) \subseteq C$.

Let (X_i, m_i) for $i \in I$ be median algebras and let $X = \Pi_{i\in I} X_i$, then $m: X^3 \rightarrow X$ defined by $m (a, b, c) = (m_i (a_i, b_i, c_i))_{i\in I}$, where $a = (a_i)_{i\in I}$, $b = (b_i)_{i\in I}$, $c = (c_i)_{i\in I}$ is a median operator on X and the resulting convexity on X is precisely the product of the median convexities produced by m_i on X_i for $i \in I$.

0.2 RELATIONSHIPS BETWEEN HELLY, RADON, CARATHEODORY AND EXCHANGE NUMBERS

The theory of convex invariants developed out of the classical results of Helly, Radon and Caratheodory. Here we give the definitions of the invariants. (See [LE], [KA,WO], [SI_1] and [SI_2])

0.2.1 Definition

Let X be a convex structure and F be any non empty finite subset of X. Then,

(a) F is said to be Helly dependent if $\cap_{a\in F}$ co $(F\backslash\{a\}) \neq \varnothing$, where co (A) denotes the convex hull of A. F is said to be Helly independent if it not Helly dependent.

(b) F is said to be Caratheodory dependent if co (F) $\subseteq \cup_{a\in F}$ co $(F\backslash\{a\})$. F is said to be Caratheodory independent if it is not Caratheodory dependent.

(c) F is Radon dependent if there is a partition $\{F_1, F_2\}$ of F such that co $(F_1) \cap$ co $(F_2) \neq \varnothing$. F is said to be Radon independent if it is not Radon dependent.

(d) F is called exchange dependent if for each $p\in F$, co $(F\backslash\{p\}) \subseteq \cup\{$co $(F\backslash\{a\}) \mid a \in F, a \neq p\}$. F is said to be exchange independent otherwise.

0.2.2 Proposition

Let X be a JHC space and $F \subseteq X$ be any set.

1. If X has ramification property and if F is Radon independent, then for each pair of subsets F_1, F_2 of F, $co(F_1) \cap co(F_2) = co(F_1 \cap F_2)$
2. If X has decomposible segments and F has at least two points, then for all $x \in co(F)$, $co(F) = \cup_{a \in F} co\{\{x\} \cup F \setminus \{a\}\}$.

0.2.3 Proposition

For a non-empty finite subset of a convex structure the following are true.

(1) Radon dependence implies Helly dependence.

(2) If X is join hull commutative and has the ramification property, then Radon dependence is equivalent to Helly dependence.

(3) If X has cone union property, then exchange dependence implies Caratheodory dependence.

(4) If X is join hull commutative and has decomposable segments, then Helly dependence implies exchange dependence.

0.2.4 Definition

Let X be a convex structure and $0 \leq n < \infty$, then

(1) $h(X) \leq n$ if and only if each finite set $F \subset X$ with cardinality greater than n is Helly dependent.

(2) the Caratheodory number $c(X) \leq n$ if and only if each $F \subseteq X$ with $|F| > n$ is Caratheodory dependent.

(3) The Radon number $r(X) \leq n$ if and only if each $F \subseteq X$ with $|F| > n$ is Radon dependent.

(4) The exchange number $e(X) \leq n$ if and only if each $F \subseteq X$ with $|F| > n$ is exchange dependent.

0.2.5 Theorem

Let X be a convex structure and let $n < \infty$.

(1) $h(X) \leq n$ if and only if each finite collection of convex sets in X meeting n by n has a non empty intersection.

0.2.6 Theorem

The following hold for all convex structures.

(1) $h(X) \leq r(X)$. (Levi inequality) [LE].

(2) $e(X) - 1 \leq c(X) \leq \max\{ h(X), e(X) - 1 \}$. (Sierksma inequality) [SI1]

(3) $r(X) \leq c(X)(h(X) - 1) + 1$ if $h(X) \neq 1$ or $c(X) < \infty$ (Eckhoff –Jamison inequality) [SI1].

0.2.7 Theorem

Let X be the gated amalgam of S_3 spaces X_1 and X_2 of arity two. Then $c(X) = \max\{c(X_1), c(X_2)\}$ unless X_1 and X_2 are free convex structures with more than one point. In this situation the Caratheodory number one larger.

0.2.8 Theorem

Let X be the gated amalgam of S_3 spaces X_1 and X_2 of arity two. Then $e(X) = \max\{ e(X_1), e(X_2) \}$.

0.2.9 Theorem

Let X be the gated amalgam of S_3 spaces X_1 and X_2. Then $h(X) = \max \{h(X_1), h(X_2)\}$.

0.2.10 Theorem

Let X be the gated amalgam of two geometric interval spaces X_1, X_2. Then $\max \{r(X_1), r(X_2)\} \leq r(X) \leq \max \{r(X_1), r(X_2)\}+1$.

0.2.11 Definition

Let X be a set and $F, G \subseteq X$. Let $E(F,G) = \{Y \in 2^X \mid F \subseteq Y, G \cap Y = \phi\}$. The family of all sets of the type $E(F,G)$ where F,G are finite, is an open base for the topology of 2^X. The resulting topology on 2^X is known as the inclusion - exclusion topology.

0.2.12 Theorem

A convexity on X is a compact subset of 2^X, relative to the inclusion-exclusion topology.

0.2.13 Theorem

Let X be a convex structure.

(1) The collection of all half spaces in X is a compact subset of 2^X, relative to the inclusion- exclusion topology.

(2) The closure in 2^X of a subbase includes all co-points of X.

0.2.14 Theorem

Let X be a convex structure of Helly number $h(X) < \infty$, and let **D** be a family of convex sets compact in 2^X. If $\cap$ **D** = ϕ, then some subfamily containing at most h(X) sets from D has an empty intersection.

0.2.15 Definition

A convex structure X is said to have a σ - finite Helly number provided there is sequence $(X_n)_{n \in N}$ of subspaces of X such that $\cup X_n = X$ and each X_n has a finite Helly number.

0.2.16 Theorem

Let X be an S_3 convex structure of sigma finite Helly number, such that for each half space H in X there is a countable subset A of H with H\ext (X) $\subseteq c_o(A)$. Then each collection of convex sets in X with an empty intersection has a countable sub collection with an empty intersection.

0.2.17 Definition

A subset F of a convex structure X is convexly independent if $a \notin$ co (F\{a}), the convex hull of F\{a}, for each $a \in F$. It is said to be convexly dependent otherwise.

0.2.18 Definition

The rank of a convex structure X is defined to be the number d(X) as $d(X) \leq n$ if and only if each subset of X with more than n points is convexly dependent, where $0 \leq n < \infty$.

0.2.19 Definition

The generating degree of a convex structure X is defined as the number gen(X), satisfying $gen(X) \leq n$ if and only if there is a subbase of X of width less than or equal to n, where $n < \infty$.

0.2.20 Theorem

Let X be a poset and $0 < n < \infty$. Then the width of X is atmost n if and only if there exists n totally ordered families $X_1, X_2 \ldots\ldots\ldots X_n \subset X$ with $X_1 \cup X_2 \cup \ldots \cup X_n = X$.

0.2.21 Proposition

For a non coarse convex structure X, $d(X) \leq gen(X)$.

0.2.22 Definition

Let X be a convex structure and let $F \subseteq X$ be a non empty indexed set. A partition $\{F_1, F_2 \ldots\ldots F_k\}$ of F is called a Tverberg k-partition provided $\bigcap_{i=1}^{k} co(Fi) \neq \phi$. The k^{th} Tverberg number ρ_k of X is defined as follows.

If $n < \infty$, then $\rho_k(X) \leq n$ if and only if each finite indexed set with more than n points has a Tverberg partition in k+1 parts.

0.2.23 Theorem

For each $k \geq 1$,the k^{th} partition number $\rho_k(R^d)$ satisfies $\rho_k = k\,(d+1)$.

0.3 TOPOLOGICAL CONVEX STRUCTURES AND CONVEX DIMENSION

The notion of topological convex structures was introduced by Jamison [JA_1]. Restricted or deviating notions were formulated by Deak [DE], Bryant [BR_2], Kay [KA], Guay [GU], Van Mill and Van de vel [ML_1, VAD]. In this section we give some basic definitions and results, which will be using in Chapter four.

0.3.1 Definition

Let X be a set equipped with a topology τ and a convexity **C**. We say that τ is compatible with the convex structure (X, *C*) provided all polytopes are closed in τ. Then X is called a topological convex structure, and is denoted as (X,τ, C).

Note that a topology is compatible with the convexity on the same underlying set if and only if the convexity is generated by closed sets.

0.3.2 Definition

Let X be a topological convex structure and let the subset Y of X be equipped with the relative topology $\tau \mid_Y$ and the relative convexity $\mathbf{C} \mid_Y$. The resulting topological convex structure is called a subspace of X.

0.3.3 Definition

A topological convex structure X is closure stable provided the closure of each convex subset is convex. X is called interior stable provided the interior of each convex subset is convex.

0.3.4 Definition

Let (X, τ, C) be a topological convex structure. The weak topology of X is the topology generated by the collection of convex closed sets as subbase of closed sets. It is denoted as (X, τ_W, C).

0.3.5 Definition

A function f: X $\to$ Y, where X and Y are topological convex structures, is a convexity preserving function (c.p function) if $f^{-1}(C)$ is convex in X for each convex set C in Y.

The following functional separation axioms were introduced by Van de vel [VAD_4].

0.3.6 Definition

A topological convex structure X is said to be

(1) FS_2, if for each pair of distinct points p, q $\in$ X there exists a continuous c p functional of X separating p and q.

(2) FS_3, if for each convex closed set C and for each point q $\notin$ C, there exists a continuous cp functional of X separating C and q.

(3) FS_{3+}, if for each pair, consisting of a convex closed set C and a polytope P disjoint from C, there exists a continuous cp functional of X separating P and C.

(4) FS_4, if for each pair of disjoint and non-empty convex closed sets C, D, there exists a continuous cp functional of X separating C and D.

0.3.7 Definition

A convex closed screening of two sets A and B is a pair (C, D) of convex closed sets such that $A \subset C \setminus D$, $B \subset D \setminus C$ and $C \cup D = X$.

A set $C \subset X$ is a separator of two non-empty sets A, $B \subset X$ provided there exists disjoint open sets $O \supset A$ and $P \supset B$ such that $X \setminus C = O \cup P$.

0.3.8 Definition

The convex small inductive dimension of a topological convex structure X is the number cind(X), satisfying the following rules.

(1) cind(X) = -1 if and only if $X = \phi$.

(2) cind(X) $\leq n + 1$ (where $n < \infty$) if and only if each pair consisting of a convex closed set C and a point $p \in X \setminus C$, has a convex closed screening (A, B) such that cind $(A \cap B) \leq n$.

Note that cind(C) $\leq$ cind(X) for each convex subset C of a topological convex structure X. Also cind $(X \times Y)$ = cind(X) + cind(Y).

The transfinite small inductive dimension of a toplogical space was studied by Toulmin [TO]. The transfinite small inductive dimension trind is defined as follows.

0.3.9 Definition

Let X be a topological space (separable, metrizable). Then,

(1) trind(X) = - 1 if and only if $X = \phi$.

(2) trind(X) $\leq \alpha$, where α is an ordinal number, if for every point $p \in X$ and each open set $V \subset X$ which contains p, there exists an pen set $U \subset X$ such that $p \in U \subset V$ and trindBd(U) $< \alpha$.

(3) trind(X) $= \alpha$ if and only if trind(X) $\leq \alpha$ and the inequality trind(X) $\leq \beta$ holds for no $\beta < \alpha$.

Chatyrko [CH_2] obtained the following revision of Toulmin's finite sum theorem for trind [TO].

0.3.10 Theorem

Let $X = X_1 \cup X_2$, where X_i is closed in X and trind $(X_i) = \alpha_i$ for i = 1, 2 (αi's are ordinals). Then

(a) for any two closed subsets A and B of X, there exists a partition C between A and B such that trind(C) $\leq \max \{ \alpha_1, \alpha_2 \}$.

(b) $\max \{ \alpha_1, \alpha_2 \} \leq$ trind(X) $\leq \max\{ \alpha_1, \alpha_2 \} + 1$.

0.3.11 Proposition

Let X be a topological convex structure of which the weak topology is separable and metrizable. Then ind $(X_W) \leq$ cind(X).

0.3.12 Lemma

Let X be a topological convex structure.

(1) If (C_1,C_2) is a screening pair of convex closed sets, then there is a minimal convex closed screening pair (D_1, D_2) with $D_i \subset C_i$ for i = 1,2.

(2) Let X be closure stable, FS_3, and let all convex sets be connected. If (C_1, C_2) is a minimal convex closed screening pair and if $C = C_1 \cap C_2$, then for each dense convex subset $B \subset X$, the set $B \cap C$ is dense in C.

0.3.13 Proposition

Let X be a non empty, closure stable and FS_3 space with connected convex sets. If $H \subset X$ is a half space, then cind (cl (H) \ H) $\leq$ cind (X) - 1.

The following is a characterisation of cind in terms of hyperplanes.

0.3.14 Corollary

In a closure stable FS_3 space with connected convex sets the following statements are equivalent for each number n.

(1) cind(X) $\leq n + 1$.

(2) cind(H) $\leq n$ for each hyperplane H of X.

0.3.15 Corollary

In a closure stable FS_3 space with connected convex sets, a convex set and its closure have the same convex dimension.

0.3.16 Corollary

In a closure stable FS_3 space with connected convex sets and of finite dimension, each dense half space has a non – empty interior. In fact, its interior meets every non- empty convex open set of the space.

0.3.17 Proposition

Let X and Y be closure stable FS_3 spaces with connected convex sets, and let f: $X \to Y$ be a closed continuous and cp function of X onto Y. Then $\text{cind}(X) \geq \text{cind}(Y)$.

0.3.18 Theorem

Let X be a closure stable FS_3 space with connected convex sets, and let $0 \leq n < \infty$. If $C \subset X$ is a convex set with $\text{cind}(C) \geq n$, then there is a continuous cp function f: $X \to [0,1]^n$ with $f(C) = [0,1]^n$. If all polytopes of X are compact, then the converse is also true.

CHAPTER 1

* HELLY DEPENDENCE AND INFINITE HELLY NUMBERS

1.1 INTRODUCTION

Based on the works of Levi [LE], Kay and Womble [KA, WO] and Soltan [SOL_1], Van de vel [VAD_3] considered Helly dependence of subsets (not necessarily finite) and the convex invariant called Helly number (which is finite) in a general convex structure. We felt that the restriction on Helly number to be finite is rather too much of a handicap and started investigating in this direction.

In this chapter we introduce the concepts of (infinite) Helly number, (infinite) star Helly number and (infinite) compact Helly number and then obtain extensions of intersection theorem (Prop.1.2.7) and countable intersection theorem (Prop.1.2.9) to the infinite situation. The infinite Helly number of an H-convex structure in terms of the degree of minimal dependence of functionals is obtained (Prop.1.2.11).

* *Some of the results in this chapter are included in the paper " On Helly dependence and infinite Helly numbers" published in the journal of the Tripura Mathematical society, 4, (2002), 7-12.*

1.2 HELLY DEPENDENCE AND EXTENSION OF COMPACT INTERSECTION THEOREM

1.2.1 Definition

Let X be a convex structure and F be any non empty subset of X. Then F is said to be Helly dependent if $\cap_{a \in F} co(F \setminus \{a\}) \neq \phi$, where co(A) denotes the convex hull of A. Also F is said to be Helly independent if it is not Helly dependent.

1.2.2 Definition

Let X be a convex structure. Then we say that $h(X) \leq \aleph_0$ if and only if each $F \subseteq X$ with $|F| > \aleph_0$ is Helly dependent. In addition to this, if for any finite cardinal α, there exists a Helly independent subset F with $|F| \geq \alpha$, we say $h(X) = \aleph_0$.

More generally if α is an infinite cardinal, then we say that $h(X) \leq \alpha$ if and only if each $F \subseteq X$ with $|F| > \alpha$ is Helly dependent. In addition if for any cardinal β less than α, there is a Helly independent subset F with $|F| \geq \beta$, then we say h(X) is precisely equal to α.

1.2.3 Definition

Let V be a vector space over a totally ordered field K and F be a collection of linear functionals from V to K and α be any infinite cardinal. Then the degree of minimal dependence md(F) is defined as $md(F) \leq \alpha$ if and only if for each $\beta > \alpha$ and for each collection $\{f_i\}$ of linearly dependent elements of F with cardinality β, there exists a subfamily of linearly dependent functionals with cardinality less than or equal to β.

Note

A collection ***C*** of subsets of X is said to satisfy α- intersection property if the intersections of each sub collection of ***C*** containing α or less members is non-empty.

1.2.4 Definition

Let X be a convex structure. Then the infinite star Helly number h*(X) is defined as the least cardinal α such that each collection of convex sets in X with α - intersection property has nonempty intersection.

1.2.5 Proposition

For a convex structure X with the star Helly number $h^*(X) = \alpha$, an infinite cardinal, $h(X) \leq \alpha$.

Proof

Since $h^*(X) = \alpha$, each collection of convex sets in X with α - intersection property has non-empty intersection. Let $F \subseteq X$ with $|F| = \beta > \alpha$. Then the family $\{co(F\backslash\{x_i\}) \mid x_i \in F\}$ satisfies α-intersection property and hence $\cap\, co(F\backslash\{x_i\}) \neq \phi$. Therefore $F \subseteq X$, with $|F| = \beta > \alpha$, is Helly dependent.

Note

By a compact family ***D***, we mean a family ***D*** of subsets of X compact in the inclusion -exclusion topology of 2^X.

A family of subsets of X is said to satisfy compact α-intersection property if intersection of each compact sub-collection of sets with α or less members is non-empty.

1.2.6 Definition

Let X be a convex structure. Then the infinite compact Helly number $h_C(X)$ is defined as the least cardinal α such that each collection of convex sets in X with compact α intersection property has nonempty intersection.

1.2.7 Proposition

Let X be a convex structure and α be an infinite cardinal. Let $h_C(X) = \alpha$. If **D** is a family of convex sets compact in 2^X having empty intersection, then **D** possesses a subfamily containing at most α sets having empty intersection.

Proof

Consider a decreasing chain $(\boldsymbol{D}_i)_{i\in I}$ of compact families $\boldsymbol{D}_i \subseteq \boldsymbol{D}$, each with an empty intersection. Let $\boldsymbol{D}_\infty = \cap_i \boldsymbol{D}_i$. Then $\boldsymbol{D}_\infty$ is the lower bound of this chain. By Zorn's lemma, the family of all compact sub collections of $\boldsymbol{D}$ having empty intersection has a minimal member $\boldsymbol{D}_o$. By the definition of $h_C(X)$ this family $\boldsymbol{D}_o$ cannot have more than α members.

Note

X and ϕ are half spaces in any convexity on X. All the other half spaces are called nontrivial.

1.2.8 Proposition

Let X be a non-empty S_3 convex structure with $h_C(X) = \alpha$, an infinite cardinal. Then each non-trivial half space $H_o \subseteq X$ is the intersection of at most α sets in $\mathcal{H}$, where $\mathcal{H}$ is a subbase consisting of half spaces.

Proof

Consider the family $K = \{H \in \mathcal{H} \mid H_o \subseteq H\}$. Since the collection of all half spaces is a compact subset of 2^X, the family K is compact. The assumption that X is S_3 gives $\cap K = H_o$. By proposition (1.2.7) $K \cup \{X \setminus H_o\}$ admits a subfamily of at most α sets having empty intersection.

1.2.9 Proposition

Let X be an S_3 convex structure of infinite compact Helly number α such that for each half space H in X there is a subset A of H with $|A| \leq \alpha$ and $H \setminus ext(X) \subseteq co(A)$. Then each collection of convex sets in X with empty intersection admits a subfamily of at most α sets having empty intersection.

Proof

First we can see that if **C** is convex set included in ext (X), then $|C| \leq \alpha$. This is because $h_C(C) \leq \alpha$, and C is free. Since X is S_3 it is enough to prove the result for families $\mathcal{H}$ consisting of half spaces such that $\cap \mathcal{H} = \phi$.

Suppose there is a net $\langle H_n \rangle$ in $\mathcal{H}$ with $\cap H_n \subseteq ext(X)$ (here the directed set is $W(\omega_\alpha)$). Since $\cap H_n$ is a convex set $|\cap H_n| \leq \alpha$. For each element in $\cap H_n$, there is an $H \in \mathcal{H}$, which does not contain that element. The sets in the net together with these $H^{\prime s}$ form a subfamily of $\mathcal{H}$ having empty intersection.

Now assume that there is no net in $\mathcal{H}$ whose intersection is in ext (X). Consider $\overline{H}$ (the closure of H). By proposition (1.2.7), $\overline{H}$ has a subfamily H' with

$|\mathcal{H}'| \leq \alpha$ and $\cap \mathcal{H}' = \phi$. Choose a net $<H_k>$ in $\overline{H}$ with empty intersection. Fix k. Choose sets A and B with $A \subseteq H_k$ and $B \subseteq X \backslash H_k$ with $H_k \backslash ext(X) \subseteq co(A)$, $X \backslash (H_k \cup ext(X)) \subseteq co(B)$ and $|A| = |B| = \beta \leq \alpha$. Let $<A_l>$and $<B_l>$ be nets in 2^A and 2^B such that $<co(A_l)> \rightarrow co(A)$ and $<co(B_l)> \rightarrow co(B)$. Since H_k is adherent to $\mathcal{H}$, there is a set H_{kl} in $\mathcal{H}$ with $A_l \subseteq H_{kl}$ and $B_l \subseteq X \backslash H_{kl}$. By compactness of $\overline{H}$, the net $<H_{kl}>$clusters at some H_k' in $\overline{H}$. Similarly the net $<X \backslash H_{kl}>$ clusters at $X \backslash H_k'$. From these we have,

$$H_k \backslash ext(X) = co(A) \backslash ext(X) \subseteq H_k' \backslash ext(X).$$

$$X \backslash (H_k \cup ext(X)) = C_o(B) \backslash ext(X) \subseteq X \backslash (H_k' \cup ext(X)).$$

This shows that $H_k \backslash ext(X) = H_k' \backslash ext(X)$. Then $H_k \backslash ext(X)$ is a cluster point of the net $<H_{kl} \backslash ext(X)>$. By assumption there is a point $p \in \cap_{k,l} <H_{kl} \backslash ext(X)>$. Then $\{H_{kl} \backslash ext(X)\} \subseteq \boldsymbol{E}\{\{p\},\phi\}$. Then all cluster points $H_k \backslash ext(X)$ contain p contradicts that $\cap H_k = \phi$.

Note

Recall that an H-convexity on a vector space V over a totally ordered field K is the convexity generated by the family $S = \{f^{-1}(\leftarrow, t] \,/\, t \in K, f \in F\}$, where F is a collection of linear functionals from V to K.

A subbase of X is called an intersectional subbase if each convex set is the intersection of subbasic sets.

1.2.10 Proposition

Let S be any intersectional subbase of an S_3 space and if any sub collection *of sets in S meeting $\aleph_0$ by $\aleph_0$ has a non empty intersection, then h(X) $\leq \aleph_0$.*

Proof

Suppose $h(X) = \beta > \aleph_0$. Then there exists a Helly independent set $F \subseteq X$ with $|F| = \beta > \aleph_0$. That is $\cap_{a\in F} co(F\setminus\{a\}) = \phi$. For each $a\in F$, let S_a be the set of all $S\in$ ***S*** containing $F\setminus\{a\}$. Since X is S_3 $co(F\setminus\{a\}) = \cap_{a\in F} S_a$. Consider $\cup_{a\in F} S_a$. Then $\cup_{a\in F} S_a$ meet $\aleph_0$ by $\aleph_0$ and hence $\cap_{a\in F} S_a \neq \phi$. This implies that is $\cap_{a\in F} co(F\setminus\{a\}) \neq \phi$, a contradiction.

1.2.11 Proposition

Let V be a vector space over R and let **C** be the H- convexity on V generated symmetrically by a set F of linear functionals with md(F) = $\aleph_0$. Then $h(V,C) = \aleph_0$.

Note: We say md(*F*)= $\aleph_0$ if the supremum of the lengths of all minimally dependent sub collections of *F* is $\aleph_0$.

Proof

First we show that md(*F*) $\leq$ h(V,C). We have md(*F*) > n for each n. Then for any n, there exists a set of n+1 minimally dependent functionals generating a convexity coarser than C having Helly number greater than n. Then the original Helly number h(V,C) > n. Therefore md(*F*) $\leq$ h(V,C). Now to show that $h(V,C) \leq \aleph_0$. Suppose $h(V,C) > \aleph_0$. Then by (prop 1.2.10) there is collection $\{f_i^{-1}(H_i)\}$ of half spaces meeting N_0 by N_0 whose intersection is empty. Let $\boldsymbol{D} = \{f_i\}$. Consider the class $\boldsymbol{E}$ of all subfamilies of $\boldsymbol{D}$ such that $\cap_i f_i^{-1}(H_i) = \phi$. $\boldsymbol{E}$ is a partially ordered set under inclusion. Every chain in $\boldsymbol{E}$ has a lower bound. By Zorn's lemma $\boldsymbol{E}$ possesses a minimal element say $\boldsymbol{E}_0$. Then $\boldsymbol{E}_0$ is a sub family of $\boldsymbol{D}$ such that $\cap_i f_i^{-1}(H_i) = \phi$. Also $|\boldsymbol{E}_0| > \aleph_0$ and $\boldsymbol{E}_0$ is a minimally dependent subfamily of $\boldsymbol{F}$. This contradicts that $md(\boldsymbol{F}) = \aleph_0$.

An extension of compact intersection theorem is obtained for a family of sets in X, which is K- compact in 2^X using K- compact Helly number $h_K(X)$.

1.2.12 Definition

The infinite K- compact Helly number $h_K(X)$ is defined as the least cardinal α such that each collection convex sets in X with "K- compact α intersection property" has non- empty intersection.

1.2.13 Proposition

Let X be a convex structure and α be an infinite cardinal. Let $h_K(X) = \alpha$. If **D** is a family of convex sets K- compact in 2^X having empty intersection, then **D** possesses a subfamily containing at most α sets having empty intersection.

Proof

Let P be the collection of all K- compact sub families of **D** having empty intersection. Consider a decreasing chain $(\boldsymbol{D}_i)_{i\in I}$ of K- compact families $\boldsymbol{D}_i \subseteq \boldsymbol{D}$, each with an empty intersection. Let $\boldsymbol{D}_\infty = \cap_i \boldsymbol{D}_i$. Then $\boldsymbol{D}_\infty$ is K- compact in 2^X and is the lower bound of this chain. By Zorn's lemma, the family of all K- compact sub collections of ***D*** having empty intersection has a minimal member $\boldsymbol{D}_o$. Since $h_K(X) = \alpha$, the family $\boldsymbol{D}_o$ cannot have more than α members.

CHAPTER 2
RELATIONS BETWEEN INFINITE CONVEX INVARIANTS

2.1 INTRODUCTION

This chapter deals with relations among various invariants of a convex structure. Levi [LE] proved that for a finite subset of a convex structure, Radon dependence implies Helly dependence. In [HAR] Hammer proved that if X is a join hull commutative space and has ramification property, then Radon dependence is equivalent to Helly dependence. In this chapter we first introduce the infinite Caratheodory number, infinite Radon number and infinite exchange number. We obtain relations between Radon, Caratheodory, Helly and exchange dependence for arbitrary subsets of a convex structure (prop 2.3.2). The inequalities of Levi [LE] and Sierksma [SI_1] are discussed in the infinite context in (Prop 2.3.4). In (Prop 2.3.5), we investigate the behavior of convex invariants under convexity preserving images. We extend the Eckhoff-Jamison inequality [SI_1] in (Prop 2. 3.7).

2.2 INFINITE CONVEX INVARIANTS

In this section we introduce various infinite invariants.

2.2.1 Definition

Let X be a convex structure and F be any non empty subset of X. Then,

(1) F is said to be Caratheodory dependent if $co(F) \subseteq \cup_{a \in F} co(F \setminus \{a\})$. F is said to be Caratheodory independent if it is not Caratheodory dependent.

(2) F is Radon dependent if there is a partition $\{F_1, F_2\}$ of F such that $co(F_1) \cap co(F_2) \neq \varnothing$. F is said to be Radon independent if it is not Radon dependent.

(3) F is called exchange dependent if for each $p \in F$, $co(F \setminus \{p\}) \subseteq \cup \{co(F \setminus \{a\}) \mid a \in F, a \neq p\}$. F is said to be exchange independent otherwise.

2.2.2 Definition

Let X be a convex structure. Then we say that the Caratheodory number $c(X) \leq \aleph_0$ if and only if each $F \subseteq X$ with $|F| > \aleph_0$ is Caratheodory dependent. In addition to this, if for any finite cardinal α, there is a Caratheodory independent subset F with $|F| \geq \alpha$, we say $c(X) = \aleph_0$. Generally if α is an infinite cardinal, then we say that $c(X) \leq \alpha$ if and only if each $F \subseteq X$ with $|F| > \alpha$ is Caratheodory dependent. In addition if for any cardinal β less than α, there is a Caratheodory independent subset F with $|F| \geq \beta$, then we say c(X) is equal to α.

2.2.3 Definition

The Radon number $r(X) \leq \alpha$ if and only if each $F \subseteq X$ with $|F| > \alpha$ is Radon dependent, where α is any infinite cardinal. In addition if for any cardinal β less than α, there is a Radon independent subset F with $|F| \geq \beta$, then we say r(X) is equal to α.

2.2.4 Definition

The exchange number $e(X) \leq \alpha$ if and only if each $F \subseteq X$ with $| F | > \alpha$ is exchange dependent, where α is any infinite cardinal. In addition if for any cardinal β less than α, there is an exchange independent subset F with $| F | \geq \beta$, then we say e(X) is equal to α.

2.3 RELATIONS BETWEEN INFINITE CONVEX INVARIANTS

The properties given below are available for finite subsets of a join hull commutative space X. Here we prove them for arbitrary subsets of X.

2.3.1 Proposition

Let X be a JHC space and $F \subseteq X$ be any set.

1. If X has ramification property and if F is Radon independent, then for each pair of subsets F_1, F_2 of F, $co(F_1) \cap co(F_2) = co(F_1 \cap F_2)$
2. If X has decomposable segments and F has at least two points, then for all $x \in co(F)$, $co(F) = \cup_{a \in F} co\{\{x\} \cup F \setminus \{a\}\}$.

Proof

Case 1.

$F_1 \cap F_2 = \emptyset$, then the result follows from Radon independence. We obviously have $co(F_1 \cap F_2) \subseteq co(F_1) \cap co(F_2)$. Now to show that

$$co(F_1) \cap co(F_2) \subseteq co(F_1 \cap F_2).$$

Assume that the inclusion is not true. That is there exists

$x \in co(F_1) \cap co(F_2)$ but $x \notin co(F_1 \cap F_2)$.

Since $x \in co(F_1)$, by domain finiteness of the hull operator, there exists a finite subset $F_1^1 \subseteq F_1$ such that $x \in co(F_1^1)$. Similarly there is a finite subset $F_2^1 \subseteq F_2$ such that $x \in co(F_2^1)$. Thus $x \in co(F_1^1) \cap co(F_2^1)$ and since the inclusion is true in the finite case $x \in co(F_1^1 \cap F_2^1)$. But $F_1^1 \cap F_2^1 \subseteq F_1 \cap F_2$ and this contradict that $x \notin co(F_1 \cap F_2)$.

Case 2.

Let $F \subseteq X$ be any subset. Fix $x \in co(F)$. By domain finiteness we can find a finite set $F_1 \subseteq F$ with $x \in co(F_1)$. But we have $co(F_1) = \cup_{a \in F_1} co(\{z\} \cup F_1 \setminus \{a\})$ for every $z \in co(F_1)$. Let $y \in co(F)$. We will show that $y \in co(\{x\} \cup F \setminus \{a\})$ for some $a \in F$. If $y \in co(F_1)$, then

$y \in co(F_1) = \cup_{a \in F_1} co(\{x\} \cup F_1 \setminus \{a\}) \subseteq \cup_{a \in F} co(\{x\} \cup F \setminus \{a\})$.

If $y \notin co(F_1)$, then we can find a finite subset $F_2 \subset F$ such that $y \in co(F_2)$. Take $F_3 = F_1 \cup F_2$. Then $x, y \in co(F_3)$ and the result follows as in the above case.

2.3.2 Proposition

Let F be any subset of a convex structure X. Then

1. Radon dependence implies Helly dependence.
2. If X is JHC and has ramification property, then Radon dependence is equivalent to Helly dependence.
3. If X is JHC and has decomposable segments, then Helly dependence implies exchange dependence.

4. If X has the cone-union property, then exchange dependence implies Caratheodory dependence.

Proof

1. Let F be Radon dependent.

Then there is a partition $\{F_1, F_2\}$ of F such that $co(F_1) \cap co(F_2) \neq \emptyset$. Let $p \in co(F_1) \cap co(F_2)$. For each $a \in F$, either $F_1 \subseteq F\setminus\{a\}$ or $F_2 \subseteq F\setminus\{a\}$. Then $p \in co(F\setminus\{a\})$. That is $\cap_{a\in F}\, co(F\setminus\{a\}) \neq \emptyset$. Therefore F is Helly dependent.

2. Suppose X is JHC and has ramification property.

Let F be Radon independent. Then $\cap_{a\in F}\, co(F\setminus\{a\}) = co(\cap_{a\in F} F\setminus\{a\}) = co(\emptyset) = \emptyset$ (by prop 2.3.1 (1)). Therefore F is Helly independent.

3. Let $F \subseteq X$ be Helly dependent. Then $\cap_{a\in F}\, co(F\setminus\{a\}) \neq \emptyset$.

Take $x \in \cap_{a\in F} co(F\setminus\{a\})$. Then for each $p \in F$,

$co(F\setminus\{p\}) = \cup\, co(\{x\} \cup F\setminus\{a, p\} / a \in F\setminus\{p\}) \subseteq \cup\, co(F\setminus\{a\} / a \in F\setminus\{p\})$.

Therefore F is exchange dependent.

4. Let X satisfy the cone union property and let $F \subseteq X$ be exchange dependent.

Fix a point $p \in F$. By exchange dependence we have

$co(F\setminus\{p\}) \subseteq \cup\, co(F\setminus\{a\} / a \in F\setminus\{p\})$. Then by using cone union property,

$co(F) = co(\{p\} \cup F\setminus\{p\}) \subseteq \cup\, \{co(\{p\} \cup co(F\setminus\{a\}) / a \in F\setminus\{p\}\}$

$= \cup\, co(F\setminus\{a\} / a \in F\setminus\{p\})$.

Therefore F is Caratheodory dependent.

The following proposition can be used as an alternative definition for the Caratheodory number of a convex structure.

2.3.3 Proposition

Let X be a convex structure, and α any infinite cardinal. Then $c(X) \leq \alpha$ if and only if for each $A \subseteq X$ and $p \in co(A)$, there is a subset F of A with $|F| \leq \alpha$ and $p \in co(F)$.

Proof

Suppose that $c(X) \leq \alpha$ and $p \in co(A)$. By domain finiteness of the hull operator there is a finite set $F \subseteq A$ satisfying the condition. Now assume that the condition is true. Suppose $c(X) > \alpha$. Then there is a set $A \subseteq X$ with $|A| > \alpha$ which is Caratheodory independent. That is $co(A) \not\subset \cup_{a \in A} co(A \setminus \{a\})$. That is there is a point $x \in co(A)$ which is not in any of the sets $co(A \setminus \{a\})$ and in particular there is no subset F of cardinality less than or equal to α containing x.

2.3.4 Proposition

Let X be a convex structure with Helly number h(X), Caratheodory number c(X), Radon number r(X) and exchange number e(X) all infinite cardinals, then

1. $h(X) \leq r(X)$
2. $e(X) \leq c(X) \leq \max\{h(X), e(X)\}$

Proof

1. Let $r(X) = \alpha$, an infinite cardinal. We will show that $h(X) \leq \alpha$.

Let $F \subseteq X$ with $|F| = \beta > \alpha$. Since F is Radon dependent there is a partition $\{F_1, F_2\}$ of F such that $co(F_1) \cap co(F_2) \neq \phi$. Let $p \in co(F_1) \cap co(F_2)$. For each $a \in F$, either $F_1 \subseteq F\setminus\{a\}$ or $F_2 \subseteq F\setminus\{a\}$. Then $p \in co(F\setminus\{a\})$. That is $\cap_{a\in F} co(F\setminus\{a\}) \neq \phi$. Hence F is Helly dependent. Therefore $h(X) \leq \alpha$.

2. Let $c(X) = \alpha$, an infinite cardinal and let F be a subset of X with $|F| > \alpha$. Take $p \in F$. Then $|F \setminus \{p\}| > \alpha$ and $co(F\setminus\{p\}) \subseteq \cup_{a\neq p} co(F\setminus\{a,p\}) \subseteq \cup_{a\neq p} co(F\setminus\{a\})$. This shows that F is exchange dependent. Therefore $e(X) \leq c(X)$.

To prove the other inequality, let $\max\{h(X),e(X)\} = \alpha$ and $F \subseteq X$ with $|F| > \alpha$. Then F is Helly dependent. Then there is a point $p \in \cap_{a\in F} co(F\setminus\{a\})$. Consider $F \cup \{p\}$. Now $|F \cup \{p\}| > \alpha$ and is exchange dependent. Therefore $co(F) \subseteq \cup_{a\in F} co(F\cup\{p\}\setminus\{a\}) = \cup_{a\in F} co(F\setminus\{a\})$.

2.3.5 Proposition

Let X and Y be convex structures and $f: X \to Y$ be a convexity preserving surjection, then $h(X) \geq h(Y)$ and $r(X) \geq r(Y)$. If f is also convex to convex then $c(X) \geq c(Y)$ and $e(X) \geq e(Y)$.

Proof

Let $h(X) = \alpha$ an infinite cardinal and let $G \subseteq Y$ with $|G| = \beta > \alpha$. We will show that G is Helly dependent in Y. For each $b \in G$ there exists a $\in X$

such that $f(a) = b$. Denote $F = f^{-1}(G)$. Since f is convexity preserving, $f(co(F\setminus\{a\})) \subseteq co(G\setminus\{b\}) \neq \emptyset$. But $\cap_{a\in F} co\, F\setminus\{a\} \neq \emptyset$. Therefore $\cap_{a\in G} co(G\setminus\{b\}) \neq \emptyset$.

Suppose $r(x) = \alpha$. Let $G \subseteq Y$ with $|G| = \beta > \alpha$. For each $b \in G$ there exists $a \in X$ with $f(a) = b$. Take $F = f^{-1}(G)$. Since $|F| > \alpha$, there is a partition $\{F_1, F_2\}$ of F such that $co(F_1) \cap co(F_2) \neq \phi$. Since f is convexity preserving,

$$f(co\, F_1) \subseteq co(f(F_1)) = co(G_1) \text{ and}$$

$$f(co\,(F_2) \subseteq co(f(F_2)) = co(G_2).$$

$$\text{Since } co(F_1) \cap co(F_2) \neq \phi,$$

$$co(G_1) \cap co(G_2) \neq \phi.$$

Now suppose that f is both convexity preserving, convex to convex and $c(X) = \alpha$, an infinite cardinal. Let $G \subseteq Y$ with $|G| = \beta > \alpha$. For each $b\in G$ there is an element $a \in X$ such that $f(a) = b$. Take $F = f^{-1}(G)$. Since f is both convexity preserving and convex to convex,

$$f(co(F)) = co\, f(F) = co(G) \text{ and}$$

$$f(co\, F\setminus\{a\}) = co(f(F\setminus\{a\})) = co(G\setminus\{b\}).$$

This shows that

$$co\,(G) \subseteq \cup\, co\,(G\setminus \{b\}).$$

Therefore G is Caratheodory dependent and $c(Y) \leq \alpha$. If we take $e(X) = \alpha$ and $G \subseteq Y$ with $|G| = \beta > \alpha$, we can see that $co(G\setminus\{p\}) \subseteq \cup_{b\in G} co(G\setminus \{b\}, b \neq p)$ for each $p \in G$ and hence G is exchange dependent. Therefore $e(Y) \leq \alpha$.

2.3.6 Proposition

Let X and Y be convex structures and $f: X \to Y$ be an isomorphism. Let $h(X) = \alpha$, then $h(Y) = \alpha$.

Proof

Let $G \subseteq Y$ with $|G| = \beta > \alpha$. Since f is a bijection, for each $b \in G$, there exists $a \in F \subseteq X$ such that $f(a) = b$. Since f is both convexity preserving and convex to convex, $f(\text{co } F\setminus\{a\}) = \text{co } (G \setminus \{b\})$. Since $\cap_{a\in F} \text{co}(F\setminus\{a\}) \neq \phi$, we have $f(\cap_{a\in F} \text{co}(F\setminus\{a\}) = \cap \text{co}(G\setminus\{b\}) \neq \phi$. Therefore G is Helly dependent.

The following proposition is an extension of Eckhoff-Jamison inequality [SI_1]. See proposition (0.2.6).

2.3.7 Proposition

Let X be a convex structure with the infinite star Helly number $h^*(X) = \alpha$ and Caratheodory number $c(X) = \beta$, both infinite cardinals, then the Radon number r(X) satisfies $r(X) \leq \max\{\alpha,\beta\}$

Proof

Let $F \subseteq X$ with $|F| > \max\{\alpha,\beta\}$. We will show that F is Radon dependent. Take $p \in F$. Then the sets $\text{co}(F\setminus\{p\})$ and $\text{co}(F\setminus A)$ for $p \notin A \subseteq F$ and $|A| \leq \beta$ meet α by α. Suppose $\text{co}(F\setminus\{p\})$ belongs to the collection of α sets chosen. Among the remaining sets of the type $\text{co}(F\setminus(A_i))$, note that $|\cup A_i| \leq \max\{\alpha,\beta\}$. Then there exists a point $q\in F$, such that $q \neq p$ and $q \notin \cup A_i$,

$q \in co(F\backslash\{p\}) \cap \cap[co(F\backslash\{A_i\})/A_i \subseteq F, |A_i| \leq \beta]$. If $co(F\backslash\{p\})$ is not in the collection, then $p \in \cap\{ co(F\backslash Ai)) / | Ai | \leq \beta\}$. Since $h^*(X) = \alpha$, each collection of convex sets meeting α by α has non empty intersection. Therefore there is a point $x \in co(F\backslash\{p\}) \cap \cap[co(F\backslash (A) / p \notin A \subseteq F, | A | \leq \beta]$. Also, since the Caratheodory number of X is β, there is a set $A \subseteq F\backslash\{p\}$ with $| A | \leq \beta$ and $x \in co (A)$ (By prop.2.3.3). Then $\{A, F\backslash A\}$ is a partition of F.

CHAPTER 3

*RANK, GENERATING DEGREE AND GENERALISED PARTITION NUMBERS

3.1 INTRODUCTION

The notion of rank of a convex structure was introduced by Jamison [JA_4] and that of a generating degree was introduced by Van de vel [VAD_8]. In this chapter we consider the invariants rank and generating degree (both infinite) of a convex structure X. The generating degree was defined using a generalisation of Dilworth's theorem for posets (Prop.3.2.2). For a non-coarse convex structure, rank is less than or equal to the generating degree (Prop.3.2.4). In section 3.3 we generalize Tverberg's theorem using (infinite) partition numbers.

Following closely the results on gated amalgams by Bandelt, Chepoi and Van de vel [VAD_9], we consider some infinite convex invariants for gated amalgams in section 3.4.

3.2 RANK AND GENERATING DEGREE

3.2.1 Definition

The rank of a convex structure X is defined to be the number d(X) as $d(X) \leq \alpha$ (any cardinal finite or infinite) if and only if each subset of X with cardinality greater than α is convexly dependent.

** Some of the results in this chapter are included in the paper "Relationship between rank and generating degree" presented in the national conference on Mathematical modeling conducted by Kerala Mathemaical Association at Baselius College, Kottayam, 2002 Jan.*

The following result is an extension of Dilworth theorem [DIL]. See proposition (0.2.20)

3.2.2 Proposition

Let every set of elements of a poset P of order greater than α be dependent while at least one set of α elements is independent, then P is a set sum of α disjoint chains. (Here α is any infinite cardinal)

Proof

First we show that the result is true when $\alpha = \aleph_0$.

Let **P** be a poset with every set of elements of cardinality greater than $\aleph_0$ be dependent while there exists at least one set of independent elements with cardinality $\aleph_0$. Suppose that **P** is not the set sum of $\aleph_0$ disjoint chains. With each set of $\aleph_0$ independent elements in **P**, we have a set sum of the form $C_1 + C_2 + \ldots$ of $\aleph_0$ disjoint chains properly contained in **P**. Let **K** be the class of all such set sums. Define a partial order $\leq$ on **K** as follows. $C_1 + C_2 + \ldots \leq D_1 + D_2 + \ldots$ if and only if $C_i \subseteq D_i$ for all i. Then **K** is a poset under the partial order of inclusion. In **K** every chain has an upper bound. By Zorn's lemma **K** has a maximal element say $C_1' + C_2' + \ldots$, consisting of $\aleph_0$ sets. Consider **P** $\setminus C_1' + C_2' \ldots$ This set contains no set of $\aleph_0$ independent elements. For if $z_1, z_2 \ldots$ be a set of $\aleph_0$ independent elements in **P** $\setminus C_1' + C_2' + \ldots$, then the corresponding set sum $Z_1 + Z_2 + \ldots$ of $\aleph_0$ disjoint chains is contained in $C_1' + C_2' + \ldots$. Let the maximum number of independent elements in **P** $\setminus C_1' + C_2' + \ldots$ be m, a finite number. Then by Dilworth theorem, $\mathbf{P} \setminus C_1' + C_2' + \ldots\ldots = C_1 + C_2 + \ldots\ldots C_m$. But then $\mathbf{P} = C_1' + C_2' + \ldots\ldots + C_1 + C_2 + \ldots\ldots C_m$, a contradiction.

We prove the general case by transfinite induction. Assume that the result is true for all infinite cardinals less than α, and that every set of elements of **P** of order greater than α is dependent, while at least one set of α elements is independent. Suppose that **P** is not the set sum of α disjoint chains. As in the previous case we can find a maximal element $C_1' + C_2' + \ldots\ldots$ of α sets such that $\mathbf{P} \setminus C_1' + C_2' + \ldots\ldots$ contains no set of α independent elements. Let $\beta < \alpha$ be the maximum number of independent elements in $\mathbf{P} \setminus C_1' + C_2' + \ldots\ldots$ Then $\mathbf{P} = C_1' + C_2' + \ldots\ldots + C_1 + C_2 + \ldots\ldots$, a contradiction

This theorem can be reformulated as follows.

Let X be a poset and α any infinite cardinal. Width of X is less than or equal to α if and only if there exists a family of α totally ordered sets X_i such that $\cup X_i = X$. We use this to define the generating degree, gen(X) of a convex structure X.

3.2.3 Definition

Let α be any infinite cardinal. Then gen (X) $\leq \alpha$ if and only if there is a subbase of X of width less than or equal to α.

3.2.4 Proposition

If a convex structure X is not the coarse one, then d(X) $\leq$ gen (X)

Proof

Let α be any infinite cardinal and let gen (X) $\leq \alpha$. To show that d(X) $\leq \alpha$. Let **S** be a subbase of X of width less than or equal to α and F $\subseteq$ X with $|F| > \alpha$. Without loss of generality we can assume that **S** is an intersectional subbase (ie, each convex set in **X** is the intersection of a subfamily of **S**). We have to show that F is convexly dependent. If not, then to each a$\in$F we can find a subbasic element S(a) such that if b $\neq$ a $\in$ F, b$\in$ S(a) and a $\notin$ S(a). These subbasic sets are incomparable and the cardinality of the set of incomparable subbasic sets is greater than α. This contradicts the fact that width of **S** is less than or equal to α.

3.3. AN EXTENSION OF TVERBERG'S THEOREM

In this section we extend the Tverberg's theorem [TV_1] regarding partition numbers of a convex structure. See proposition (0.2.23)

3.3.1 Definition

Let X be a convex structure. We say that the k^{th} partition number $\rho_k(X) \leq \aleph_0$ if and only if each indexed set of X with cardinality greater than $\aleph_0$ has a Tverberg partition in k + 1 parts.

More generally if α is any infinite cardinal, then $\rho_k(X) \leq \alpha$ if and only if each indexed set of X with cardinality greater than α has a Tverberg partition in k+1 parts.

3.3.2 proposition

For each $k \geq 1$, $\rho_k(R^\alpha) = \alpha$ for any limit cardinal α.

Proof

First we prove the theorem when $\alpha = \aleph_0$. Suppose that $\rho_k(R^{\aleph_0}) = m$, a finite number. Then each indexed subset of $R^{\aleph_0}$ with more than m points has a partition in k+1 parts. Consider a subspace A of $R^{\aleph_0}$ of dimension m. Since $\rho_k(A) = k(m+1)$ (by Tverberg theorem), there exists a subset of A with more than m points which does not have a Tverberg partition. So no finite number is the partition number of $R^{\aleph_0}$.

For the general case assume that the result is true for all spaces with dimension less than α. Suppose $\rho_k(R^\alpha) = \beta < \alpha$. Then each indexed set in R^α with more than β points has a partition in k+1 parts. Consider a subspace A of R^α of dimension Υ where $\beta < \Upsilon < \alpha$. Then $\rho_k(A) = \Upsilon$. In A we can find a set of Υ points having no Tverberg partition. That is there is a set in R^α with Υ points having no Tverberg partition.

3.4 CONVEX INVARIANTS IN GATED AMALGAMS

3.4.1 Proposition

Let X be the gated amalgam of S_3 spaces X_1 and X_2 having Helly numbers α and β (both infinite cardinals). Then $h(X) = \max\{\alpha,\beta\}$.

Proof

Since X_1 and X_2 are convex subspaces of X, we have $\alpha, \beta \leq h(X)$. Let F be a subset of X with $|F| = \gamma > \max\{\alpha, \beta\}$. If F is included in X_1 or X_2, then F is a Helly dependent set. Now define $F_1 = F \cap X_1$ and $F_2 = F \setminus F_1$. Then, $\cap_{a\in F} co(F\setminus\{a\}) \supseteq \cap_{a\in F1}\{co(F_1\setminus\{a\} \cup p_1(F_2)\} \cap \cap_{a\in F2}\{co(F_1 \cup p_1(F_2 \setminus \{a\})\}$ where $p_i : X \rightarrow X_i$ be the gate map.

We regard $F_1 \cup p_1(F_2)$ as a set indexed by $F_1 \cup F_2$. Then the right side is nonempty since $\alpha \leq \Upsilon$ and $F_1 \cup p_1(F_2)$ is a subset of X_1 with cardinality greater than Υ.

3.4.2 Proposition

Let X be the gated amalgam of S_3 spaces X_1 and X_2 of arity two, having Caratheodory numbers α and β (both infinite cardinals). Then $c(X) = \max\{\alpha, \beta\}$.

Proof

We have $\alpha, \beta \leq c(X)$, since X_1 and X_2 are convex subspaces of X. To prove the other inequality, let $F \subset X$ with $|F| = \Upsilon > \max\{\alpha,\beta\}$. To show that F is a Caratheodory dependent set. If $F \subset X_i$ for some i, it is caratheodory dependent. On the other hand decompose F as $F_1 \cup F_2$ with $F_i \subset X_i$.

We have $|F_1 \cup p_1(F_2)| > \gamma$.

Also,

$$co(F_1 \cup F_2) \cap X_1 = co(F_1 \cup p_1(F_2))$$

$= \cup_{a \in F_1} co\{ F_1 \setminus \{a\} \cup p_1(F_2) \} \cup \cup_{a \in F_2} co\, (F_1 \cup p_1(F_2 \setminus \{a\}))$

by Caratheodory dependence.

Now by hull formula, (Theorem 0.1.20)

$co\, (F_1 \cup p_1(F_2 \setminus \{a\}) = co(F_1 \cup F_2 \setminus \{a\}) \cap X_1$ and

$co\{ F_1 \setminus \{a\} \cup p_1(F_2)\} = co\{ F_1 \setminus \{a\} \cup F_2\} \cap X_1$. Then,

$co(F_1 \cup F_2) \cap X_1 =$

$$\cup_{a \in F_1}(co\, (F_1 \setminus \{a\} \cup F_2) \cap X_1) \cup \cup_{a \in F_2} (co\, (F_1 \cup F_2 \setminus \{a\}) \cap X_1) \quad .... (1)$$

Similarly, $co\, (F_1 \cup F_2) \cap X_2 = co\, (p_2(F_1) \cup F_2)$. We consider $p_2(F_1) \cup F_2$ as a set indexed by $F_1 \cup F_2$. Therefore,

$co\, (p_2(F_1) \cup F_2) = \cup_{a \in F_1} co(p_2\, (F_1 \setminus \{a\}) \cup F_2) \cup \cup_{a \in F_2} co(p_2\, (F_1) \cup F_2 \setminus \{a\})$.

But $co(p_2\, (F_1 \setminus \{a\}) \cup F_2) = co\, (F_1 \setminus \{a\} \cup F_2) \cap X_2$ and

$co(p_2\, (F_1) \cup F_2 \setminus \{a\}) = co(F_1 \cup F_2 \setminus \{a\}) \cap X_2$. Therefore,

$co(F_1 \cup F_2) \cap X_2 =$

$$\cup_{a \in F_1}(co\, (F_1 \setminus \{a\} \cup F_2) \cap X_2) \cup \cup_{a \in F_2} (co\, (F_1 \cup F_2 \setminus \{a\}) \cap X_2) \,......(2).$$

From (1) and (2) $co(F_1 \cup F_2) = \cup_{a \in F_1}(co\, (F_1 \setminus \{a\} \cup F_2) \cup \cup_{a \in F_2} co\, (F_1 \cup F_2 \setminus \{a\})$, which implies that $F_1 \cup F_2$ is Caratheodory dependent.

3.4.3 Proposition

Let X be the gated amalgam of S_3 spaces X_1 and X_2 of arity two, having exchange numbers α and β (both infinite cardinals). Then $e(X) = \max\ \{\alpha, \beta\}$.

Proof

As X_1 and X_2 are convex sub spaces $\alpha, \beta \leq e(X)$. To show that $e(X) \leq \max \{\alpha, \beta\}$. Let $F \subset X$ with $|F| = \Upsilon > \max \{\alpha,\beta\}$. Take $p \in F$. Assume that $p \in X_2$. Decompose $F = F_1 \cup F_2$ with $F_1 \subset X_1$, $p \in F_2 \subset X_2$.

As $F_1 \cup p_1 (F_2)$ is exchange dependent. We have $co (F \setminus \{p\}) \cap X_1$

$= co(F_1 \cup F_2 \setminus \{p\}) \cap X_1 = co (F_1 \cup p_1(F_2 \setminus \{p\}))$

$= \cup_{a \in F1} co ((F_1 \setminus \{a\}) \cup p_1 (F_2 \setminus \{p\})) \cup \cup_{a \in F2\setminus\{p\}} co (F_1 \cup p_1 (F_2 \setminus \{a\}))$

$= \cup_{a \in F1} co ((F_1 \setminus \{a\}) \cup F_2 \setminus \{p\}) \cap X_1 \cup \cup_{a \in F2\setminus\{p\}} co (F_1 \cup F_2 \setminus \{a\}) \cap X_1$

$= \cup_{a \in F\setminus\{p\}} co (F \setminus \{a\}) \cap X_1$. Similarly $co(F \setminus \{p\}) \cap X_2 = \cup_{a \in F\setminus\{p\}} co (F \setminus \{a\}) \cap X_2$.

Therefore $co (F \setminus \{p\}) = \cup_{a \in F\setminus\{p\}} co (F \setminus \{a\})$.

CHAPTER 4

*ON TRANSFINITE CONVEX DIMENSION

4.1 INTRODUCTION

In [VAD_l] Van de vel introduced the notion of convex dimension cind for a topological convex structure. In this chapter we introduce the notion of transfinite convex dimension trcind. In section 4.2 we compare the transfinite topological dimension and transfinite convex dimension (Prop 4.2.3). A characterization of trcind in terms of hyperplanes (Cor 4.2.5) is obtained. In section 4.3 we obtain a characterization of trcind in terms of mappings to cubes (Prop 4.3.1). Throughout this chapter we assume that the convex structure is S_1 and closure stable.

4.2 TRANSFINITE CONVEX DIMENSION

4.2.1. Definition

Let X be a topological convex structure. Then:

1. trcind (X) = -1 if and only if $X = \Phi$
2. trcind $(X) \leq \alpha$, where α is an ordinal if and only if to each pair consisting of a convex closed set C and a point $p \in X \setminus C$, there exists a convex closed screening (A, B) and an ordinal $\beta < \alpha$ such that trcind $(A \cap B) \leq \beta$.

We say that trcind (X) = α if and only if trcind (X) $\leq \alpha$ but trcind (X) $\not\leq \beta$ for any $\beta < \alpha$.

* *Some of the results in this chapter are presented in the international conference on Transform Techniques and their applications at St. Joseph's College, Irinjalkuda, 2000 Dec.*

4.2.2 Proposition

trcind (C) $\leq$ trcind (X) for each convex subset C of a topological convex structure X.

Proof

Let trcind (X) = α, an ordinal. Assume that the result is true for spaces with dimension less than α. Let p $\in$C and D $\subseteq$ C, where D is a convex closed subset of C such that p $\notin$ D. Then cl (D), the closure of D is a convex closed set in X and D = cl(D) $\cap$ C with p $\notin$ cl(D). Since trcind (X) = α, there exists a $\beta < \alpha$ and a screening (A, B) of convex closed sets in X such that trcind (A $\cap$ B) $\leq \beta$. Take A' = A $\cap$ C and B' = B $\cap$ C. Then (A', B') is a pair of convex closed sets in C which screen p and D. Also A' $\cap$ B' $\subseteq$ A $\cap$ B, then by assumption trcind (A' $\cap$ B') $\leq \beta$.

4.2.3 Proposition

Let X be a topological convex structure of which the weak topology is separable and metrizable. Then trind (X_w) $\leq$ trcind (X) + k, for some integer k. (Here trind denotes the transfinite small inductive dimension).

Proof

Let α be an infinite ordinal and let trcind (X) $\leq \alpha$. Assume that the result is true if trcind $< \alpha$. Let A $\subseteq$ X be a closed set and p $\notin$ A. Since we are considering the weak topology, there exists convex closed sets C_1, $C_2 \ldots C_m$ such that $A \subseteq \bigcup_{i=1}^{m} C_i$ and $p \notin \bigcup_{i=1}^{m} C_i$. By the definition of trcind, for each i = 1, 2......... m, there is a convex closed screening (D_i, E_i) of p and C_i such that trcind $(D_i \cap E_i) \leq \beta_i$ where $\beta_i < \alpha$. Take $D = \bigcup_{i=1}^{m} D_i$. Then D is a closed neighbourhood of p disjoint from A and Bd (D) $\subseteq$ $(D_i \cap E_i)$. This is because

$$\begin{aligned}
\text{Bd}(D) \subseteq \bigcup_{i=1}^{m} \text{Bd}(D_i) &= \bigcup_{i=1}^{m} \text{cl}(D_i) \cap \text{cl}(X \backslash D_i) \\
&= \bigcup_{i=1}^{m} (\text{cl}(D_i) \cap \text{cl}(E_i \backslash D_i)) \subseteq \bigcup_{i=1}^{m} (D_i \cap E_i).
\end{aligned}$$

By induction hypothesis, trind $(D_i \cap E_i) \leq \beta_i + m_i$ for every i, where each m_i is an integer. Now by the sum theorem for trind [CH_2], (See prop 0.3.10),

$$\begin{aligned}
\text{trind Bd}(D) &\leq \text{trind}\Big(\bigcup_{i=1}^{m} (D_i \cap E_i)\Big). \\
&\leq \max(\beta_i + m_i) + m \\
&< \max\{\alpha + m_i\} + m \\
&= \alpha + k \text{ for some integer } k.
\end{aligned}$$

4.2.4 Proposition

Let X be a non empty FS_3 space with connected convex sets. If $H \subseteq X$ is a half space, then trcind (cl (H) \ H) < trcind (X).

Proof

Assume that trcind (X) $\leq \alpha$, an ordinal and that the result is true if trcind $< \alpha$. Let $C \subseteq$ cl (H) \ H be a convex closed set and $p \notin C$. Consider cl (H).

Since X is closure stable, cl (H) is convex in X. Then trcind (cl (H)) $\leq$ trcind (X) $\leq \alpha$. Also C = cl (C) $\cap$cl (H)\ H, where cl (C) is convex closed in cl (H) and p$\notin$cl(C). Then there exists a convex closed screening (A, B) of cl (C) and p in cl(H) such that trcind (A $\cap$ B) $\leq \beta < \alpha$. Without loss of generality we can assume that (A, B) is a minimal screening pair. Also since H is dense in cl (H), we can conclude that H $\cap$ A $\cap$ B is dense in A $\cap$ B. Therefore,

$$(\text{cl}(H) \setminus H) \cap A \cap B$$

$$= \text{cl}(H) \cap (A \cap B) \setminus H$$

$$= \text{cl}(H) \cap (\text{cl}(A \cap B)) \setminus H$$

$$= A \cap B \setminus H = (A \cap B) \setminus H \cap A \cap B$$

$$= \text{cl}(H \cap A \cap B) \setminus (H \cap A \cap B).$$

Since (H $\cap$ A $\cap$ B) is a relative half space of A $\cap$ B, by inductive hypothesis, trcind ((cl (H) \ H) $\cap$A $\cap$ B) $\leq \gamma$, where $\gamma < \beta$. This shows that each relatively convex closed set C of cl (H) \ H and each point p$\notin$C of cl (H) \H, can be screened by convex closed sets of the form

(cl (H)\H) $\cap$ A and (cl (H) \H) $\cap$ B and trcind ((cl (H) \H) $\cap$ A $\cap$ B) $\leq \gamma < \beta$. Thus trcind (cl (H) \H) $\leq \beta < \alpha$.

A set of the type cl (H) \H where H is an open half space of X is called a hyperplane.

4.2.5 Corollary

Let X be an FS_3 space with connected convex sets. The following statements are equivalent.

1. trcind (X) $\leq \alpha$, where α is an ordinal.
2. Corresponding to each hyper pane H $\subseteq$ X, there exists a $\beta < \alpha$ such that trcind (H) $\leq \beta$

Proof

(1) $\Rightarrow$ (2) by using prop (4.2.4) above. Now assume (2). Let C be a convex closed set in X and p $\notin$C. By FS_3, there exists a continuous cp functional f : X $\rightarrow$ R separating p and C. Let f (C) $\subseteq (-\infty,0]$ and f (p) > 0.

Take H = f^{-1} $(-\infty, f(p)/2)$. Then cl (H) and cl (X\H) is a convex closed screening of p and C and cl (H) $\cap$ cl (X\H) = Bd(H) and trcind(Bd (H)) $\leq \beta < \alpha$.

4.2.6 Proposition

Let X be an FS_3 space with connected convex sets. If C is a non-empty convex subset of X of dimension $\alpha > 0$, an ordinal, then the intersection of all relatively dense convex subsets of C is relatively dense in C.

Proof

Let trcind (C) = α. Assume that the result is true for all convex sets with dimension less than α. Let E = $\cap$ A_i, where each A_i is a relatively dense convex subset of C. To show that cl (E) = C. Let p $\in$ C \ cl (E). Then cl (E) $\cap$ C is a convex closed set in C and p $\notin$ cl (E) $\cap$ C. Then there exists a minimal convex closed screening of cl (E) $\cap$ C and p whose intersection D satisfies trcind (D) $\leq \beta$, where $\beta < \alpha$. Also each dense convex subset of C induces a dense convex subset of D. Therefore the sets $A_i \cap D$ are all dense in D. Therefore $\cap_i \{A_i \cap D\} = E \cap D$ is relatively dense D. Thus $E \cap D \neq \Phi$, which is a contradiction.

4.2.7 Proposition

In an FS_3 space with connected convex sets, a convex set and its closure have the same convex dimension.

Proof

Let X be the space and $C \subseteq X$ be convex in X. Without loss of generality assume that C is dense in X. We will show that trcind (C) = trcind (X). We have trcind (C) $\leq$ trcind (X). To show that trcind (X) $\leq$ trcind (C). Let trcind (C) $\leq \alpha$, an ordinal. We prove the result by transfinite induction.

Assume that the result is true for all convex sets with dimension less than α. Let D be a convex closed set in X and $p \in X \setminus D$. By FS_3, there exists an open half space $O \subseteq X$ such that $D \subseteq$ cl (O) and $p \notin$ cl (O). Now consider a minimal convex closed screening D_1, D_2 of D and p with $D_1 \subseteq$ cl (O) and $D_2 \subseteq X \setminus O$. Now $D_1 \cap D_2 \subseteq$ cl (O) $\cap X \setminus O \subseteq$ Bd (O). Since C is dense in X, $cl_C (O \cap C) = cl_Y(O) \cap C$. Similarly $cl_C(X \setminus O \cap C) = X \setminus O \cap C$. Therefore Bd(O) $\cap$ C is the relative boundary of $O \cap C$ in C. Then, trcind ($D_1 \cap D_2 \cap C$) $\leq$ trcind (Bd(O) $\cap$ C) $\leq \beta < \alpha$. (By corollary (4.2.5)). Since C is dense and convex,

$D_1 \cap D_2 \cap C$ is a dense subset of $D_1 \cap D_2$. By induction hypothesis trcind $(D_1 \cap D_2 \cap C)$ = trcind $(D_1 \cap D_2)$.

Thus trcind $(X) \leq \alpha$.

4.2.8 Proposition

In an FS_3 space with connected convex sets and of dimension α, an ordinal, each dense half space has a non-empty interior. In fact, its interior meets every non-empty convex open set of the space.

Proof

Let X be the space and let $H \subseteq X$ be a dense half space. Let $O \neq \Phi$ be a convex open set in X. Then $H \cap O$ is a relatively dense half space of O. By corollary (4.2.4), trcind $(O \setminus H)$ < trcind (O). Now by prop (4.2.7), $O \setminus H$ is not dense in O. Then $\Phi \neq \text{int}_O (O \cap H) \subseteq \text{int}(H)$.

4.3 TRANSFINITE CONVEX DIMENSION AND CONVEXITY PRESERVING MAPS

4.3.1 Proposition

Let X be an FS_3 space with connected convex sets and let $[0,1]^{\aleph_0}$ be equipped with the standard median convexity. If $C \subseteq X$ is a convex set with trcind(C) $\geq \aleph_0$, then there exists a continuous convexity preserving function $f = (f_n) : X \rightarrow [0,1]^{\aleph_0}$, where for each n, f_n is a continuous convexity preserving function from $X \rightarrow [0,1]^n$ such that $f_n(C) = [0,1]^n$.

Proof

Since trcind (C) $\geq \aleph_o$, trcind (C) > n for all n. Then for each n, there exists a continuous convexity preserving function $f_n : X \rightarrow [0,1]^n$ satisfying $f_n(C) = [0,1]^n$ (See theorem[0.3.16]).Then the function $f = (f_n)$ is a continuous convexity preserving function from X to $[0,1]^{\aleph_o}$. For, let C be any subbasic convex set in $[0,1]^{\aleph_o}$. Then $C = \pi_i^{-1}(C_i)$, where C_i is convex in $[0,1]^i$.

$$\begin{aligned} \text{Therefore } f^{-1}(\pi_i^{-1}(C_i)) &= \{x \in X : f_n(x) \in \pi_n(\pi_i^{-1}(C_i)) \text{ for all } n\} \\ &= \cap_n f_n^{-1}(\pi_n(\pi_i^{-1}(C_i)), \text{ which is convex in } X. \end{aligned}$$

4.3.2 Proposition

Let X and Y be FS_3 spaces with connected convex sets and let $f : X \rightarrow Y$ be a closed, continuous and convexity preserving function of X onto Y. Then trcind (X) $\geq$ trcind(Y).

Proof

We will show that trcind (Y) $\geq \alpha$ implies that trcind (X) $\geq \alpha$, where α is any ordinal. Assume that the statement is valid for all $\beta < \alpha$. Now if trcind (Y) $\geq \alpha$, then by corollary (4.2..5), there is an open half space O of Y such that for any ordinal $\beta < \alpha$, trcind (Bd(O)) $\geq \beta$. Then the set $P = f^{-1}(O)$ is an open half space of X and since f is closed and surjective f(Bd(P)) = Bd(O). Hence f induces a closed convexity preserving map from Bd(P) to Bd(O) which is onto and by inductive assumption, trcind(Bd(P)) $\geq \beta$. This implies that trcind (X) $\geq \alpha$.

4.3.3 Corollary

Let X be an FS_3 space with connected convex sets and with compact polytopes. Then trcind (X) $\geq \aleph_0$ if and only if for each n, there exists a polytope P_n such that trcind $(P_n) \geq n$.

Proof

If for each n, there exists a polytope P_n such that trcind $(P_n) \geq n$, then trcind(X) $\geq$ n for all n, and hence trcind(X) $\geq \aleph_0$. On the other hand if trcind (X) $\geq \aleph_0$, then there exists a continuous convexity preserving function $f = (f_n): X \to [0,1]^{\aleph_0}$, where each $f_n : X \to [0,1]^n$ is continuous, convexity preserving and onto. For each f_n, take one pre- image of each corner point. Let F_n be the resulting set. Then f_n maps $co(F_n)$ onto $[0,1]^n$. Thus trcind$(co(F_n)) \geq n$.

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