Some Problems in Topology and Algebra

A STUDY OF FRAMES IN THE FUZZY AND INTUITIONISTIC FUZZY CONTEXTS

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CERTIFICATE

This is to certify that the work reported in the thesis entitled **"A Study of Frames in the Fuzzy and Intuitionistic Fuzzy Contexts"** that is being submitted by Sri. Rajesh K. Thumbakara for the award of Doctor of Philosophy to Cochin University of Science and Technology is based on bona fide research work carried out by him under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Frame theory

The first mathematician to take the notion of open set as basic to the study of continuity properties was Hausdorff in 1914. Using the lattice of open sets, Marshal stone $[ST]_1$ was able to give topological representation of Boolean algebras and distributive lattices and H. Wallman(1938) [WA] used lattice theoretic constructs to obtain the wallman compactification. In the 1940's McKinsey and Tarski [M; T] studied the "algebra of topology" that is topology studied from a lattice theoretical viewpoint. But a fundamental change in the outlook came in late fifties; Charles Ehresmann [EH] in 1959 first articulated the view that a complete lattice with an appropriate distributivity property deserved to be studied in their own right rather than simply as a means to study topological spaces. He called the lattice a local lattice. Dowker and Strauss($[D; P]_1$, $[D; P]_2$, $[D; P]_3$) introduced the term frame for a local lattice and extended many results of topology to frame theory. It was with the publication of John Isbell's "Atomless parts of spaces" $[IS]_1$ in 1972 that the real importance of the subject emerged. Since then Frame theory is studied extensively by many authors.

1.2 Fuzzy set theory

Among the various paradigmatic changes in science and mathematics in this century, one such change concerns the concept of uncertainty. According to the traditional view, science should strive for certainty in all its manifestations hence, uncertainty (vagueness) is regarded as unscientific. According to modern view, uncertainty is considered essential to science; it is not only an unavoidable plague, but it has, in fact, a great utility. L.A. Zadeh in 1965 introduced the notion of fuzzy sets $[ZA]_2$ to describe vagueness mathematically in its very abstractness and tried to solve such problems by giving a certain grade of membership to each member of a given set. This in fact laid the foundations of fuzzy set theory. Zadeh has defined a fuzzy set as a generalisation of the characteristic function of a subset. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse, a value representing its grade of membership in the fuzzy set. The membership grades are very often represented by real numbers in the closed interval between 0 and 1. The nearer the value of an element to unity, the higher the grade of its membership. The fuzzy set theory has a wider scope of applicability than classical set theory in solving various problems. Fuzzy set theory in the last three decades has developed along two lines:

1. as a formal theory which got formalised by generalizing the original ideas and concepts in classical mathematical areas.
2. as a very powerful modeling language, that can cope with a large fraction of uncertainties of real life situations.

1.3 Intuitionistic fuzzy set theory

In 1983, K. Atanassov proposed a generalization of the notion of fuzzy set, $[AT]_1$ known as Intuitionistic Fuzzy sets. He introduced a new component degree of non membership in addition to the degree of membership in the case of fuzzy sets with the requirement that their sum be less than or equal to one. The complement of the two degrees to one is regarded as a degree of uncertainty. Since then a great number of theoretical and practical results appeared in the area of Intuitionistic Fuzzy sets.

1.4 Summary of the Thesis

The main objective of this thesis is to study frames in Fuzzy and Intuitionistic Fuzzy contexts. The whole work is divided into six chapters. A brief chapter wise description of the thesis is given below.

Chapter 1

This is devoted to the basic definitions and results concerning Frames, Fuzzy sets and Intuitionistic Fuzzy sets which are required in the succeeding sections. All results here are quoted from existing literature.

Chapter 2

In this chapter we introduce the notion of fuzzy frames and we prove some results, which include

- If μ is a fuzzy subset of a frame F, then μ is a fuzzy frame of F iff each non-empty level subset μ_t of μ is a subframe of F.
- The category FuzzFrm of fuzzy frames has products.
- The category FuzzFrm of fuzzy frames is complete.

Chapter 3

In this chapter we introduce the notion of fuzzy quotient frames. The operation of binary meet and arbitrary join on a frame F induces, through Zadeh's extension principle new operations on the partially ordered set I^F. Here we define a fuzzy-quotient frame of F to be a fuzzy partition of F, that is, a subset of I^F and having a frame structure with respect to new operations. We also define and study fuzzy ideals over F. The results proved in this chapter include

- If μ and γ are fuzzy frames of a frame F having supremum property with respect to $\wedge$ and $\vee$ then $\mu \tilde{\wedge} \gamma$ and $\mu \tilde{\vee} \gamma$ are fuzzy frames of F.
- If R is an invariant fuzzy binary relation on a frame F then its fuzzy partition P_R is a fuzzy quotient frame of F.
- The set $I_F F$ of all fuzzy ideals of the frame F is a frame.

Chapter 4

In this chapter we define and study the notion of intuitionistic fuzzy frames and obtain some results, which include

- If A is an intuitionistic fuzzy set in F then A is an intuitionistic fuzzy frame of F iff $\Box A$ and $\Diamond A$ ('necessity' and 'possibility' operators) are intuitionistic fuzzy frames of F.
- If A is an intuitionistic fuzzy set on F then A is an intuitionistic fuzzy frame on F iff every non empty level set A_t, $t \in [0,1]$ of A is a subframe of the frame F.
- The category IFFrm of intuitionistic fuzzy frames has products.
- The category IFFrm of intuitionistic fuzzy frames is complete.

Chapter 5

In this chapter we introduce the concept of Intuitionistic fuzzy Quotient frames and has obtained the result:

- If R is an invariant intuitionistic fuzzy similarity relation on a frame F then its fuzzy partition P_R is an intuitionistic fuzzy quotient frame of F.

Chapter 6

Here we establish the categorical link between frames and intuitionistic fuzzy topologies. The main results include the following:

- Ʊ is a contravariant functor from the category IFTOP of intuitionistic fuzzy topological space to the category FRM of frames.
- Σ is a contravariant functor from the category FRM of frames to the category IFTOP of intuitionistic fuzzy topological spaces.
- Σ and Ω are adjoint on the right.

1.5 Basic Definitions and Results

1.5(a) Frames and Topological spaces

In the same way as the notion of Boolean algebra appears as an abstraction of the power set P(X) of a set X, the notion of frame arises as an abstraction from the topology τ of the topological space (X, τ).

The following definitions are adapted from $[BA]_1$, $[BA]_2$, $[BA]_3$, $[D; P]_2$, $[D; P]_4$, $[JO]_2$, [PI], [VIC]

Definition 1.5.1. A frame is a complete lattice L satisfying the distributive law $x \wedge (\bigvee S) = \vee \{x \wedge s \mid s \in S\}$ for all $x \in L$ and $S \subseteq L$, where $\wedge$ denotes binary meet and $\bigvee$ denotes arbitrary join.

Definition 1.5.2. A subset M of a frame L is a subframe of L if 0_L, $e_L \in M$ where 0_L and e_L are respectively bottom and top element of L, and M is closed under finite meets and arbitrary joins.

Note 1.5.3. Given a, b $\in$ L a frame, with a$\leq$ b then [a,b]= $\{x \in L \mid a \leq x \leq b\}$ is a frame but not a subframe of L.

Definition 1.5.4. For frames L, M a map h: L$\to$M is a frame homomorphism if h preserves finite meets (including top or unit element) and arbitrary joins (including bottom or zero element). That is $h(a \wedge b) = h(a) \wedge h(b)$ and $h(\vee X) = \vee h(x)$ for all a, b$\in$L and X$\subset$L.

Definition 1.5.5. For a family of frames $\{L_i \mid i \in I\}$, its product L is the Cartesian product of underlying sets with $\leq$ defined as $(a_i)_{i \in \Lambda} \leq (b_i)_{i \in \Lambda}$ iff $a_i \leq b_i$ for all $i \in I$.

Definition 1.5.6. For any frame F, a subset J $\subset$ F is an ideal if, J is a downset that is if $(a \in J, b \leq a) \Rightarrow b \in J$ and J is closed under finite joins.

Proposition 1.5.7. The set $\mathfrak{J}F$ of all ideals of a frame F is a frame, under inclusion order.

There is an important relation between frames and topological spaces which we describe below. The category of frames and frame homomorphisms will be denoted by Frm. The category of topological spaces and continuous maps will be denoted by Top.

Definition 1.5.8. The contravariant functor Ω : Top $\to$ Frm which assigns to each topological space (X, τ) its frame τ of open sets and to each continuous function

$f:(X,\tau)\to(X',\tau')$ the frame map $\Omega(f):\tau'\to\tau$ given by $\Omega(f)(u)=f^{-1}(u)$, where $u\in\tau'$ is called the open functor from Top to Frm.

Definition 1.5.9. Let L be a frame. The spectrum of L is the set ptL of all frame homomorphisms $p: L\to\{0, 1\}$ with the spectral topology $\mathcal{T}_{ptL}=\{\Sigma_x \mid x\in L\}$ where $\Sigma_x=\{p\in ptL \mid p(x)=1\}$. The contravariant functor Σ : Frm $\to$ Top which assigns to each frame its spectrum $\Sigma(L)=(ptL, \mathcal{T}_{ptL})$ and to each frame map $f: L\to L'$ the continuous map $\Sigma(f):\Sigma(L')\to\Sigma(L)$ given by $\Sigma(f)(p)=p\circ f$, where p is a point of L' is called the spectrum functor from Frm to Top.

Theorem 1.5.10. Σ and Ω are adjoint on the right with adjunctions $\eta_L: L\to\Omega\,\Sigma L$ given by $a\mapsto\Sigma_a$ and $\varepsilon_X: X\to\Sigma\,\Omega X$ given by $x\mapsto\bar{x}$ where $\bar{x}(U)=\text{card}(U\cap\{x\})$.

1.5(b) Fuzzy Sets

The following definitions are adapted from [DU; P], [K; Y] , [MO; M], [OV], $[ZA]_1$, [ZI].

Definition 1.5.11. A fuzzy set μ of a set X is a function from X to I where I = [0, 1].

Definition 1.5.12. The set all fuzzy sets of X, denoted by I^X is the set of all functions from X to [0, 1].

Definition 1.5.13. Let μ and γ be fuzzy sets of a non empty set X. Then

$$\mu=\gamma \Leftrightarrow \mu(x)=\gamma(x) \text{ for all } x\in X$$

$\mu \subseteq \gamma \Leftrightarrow \mu(x) \leq \gamma(x)$ for all $x \in X$

$\mu \vee \gamma = \delta \Leftrightarrow \delta(x) = \max\{\mu(x), \gamma(x)\}$ for all $x \in X$

$\mu \wedge \gamma = \delta \Leftrightarrow \delta(x) = \min\{\mu(x), \gamma(x)\}$ for all $x \in X$

Definition 1.5.14. Let $\{\mu_\alpha | \alpha \in \Lambda\} \subseteq I^X$. Then define $\bigcap_{i\in\Lambda} \mu_i(a) = \inf\{\mu_\alpha(a) \,|\, \alpha \in \Lambda\}$ and $\bigcup_{i\in\Lambda} \mu_i(a) = \sup\{\mu_\alpha(a) \,|\, \alpha \in \Lambda\}$.

Definition 1.5.15. If μ is fuzzy set of X, for any $t \in I$ the set $\mu_t = \{a \in X \,|\, \mu(a) \geq t\}$ and $\mu_t^{>} = \{a \in X \,|\, \mu(a) > t\}$ are respectively called level subset and strong level subset of μ.

Definition 1.5.16. If μ is fuzzy set of X then the height of μ is defined by $\text{hgt}(\mu) = \sup_{x\in X} \mu(x)$.

Proposition 1.5.17. Let μ and γ be fuzzy sets of a non empty set X. Then $(\mu \cup \gamma)_t = \mu_t \cup \gamma_t$.

Definition 1.5.18. Let X and Y be two non empty sets and μ any fuzzy set of X. Let f a function from X into Y. Then μ is said to be f- invariant if for all x, $y \in X$, $f(x) = f(y) \Rightarrow \mu(x) = \mu(y)$.

Proposition 1.5.19. Let f be a mapping from a set S to a set M and let $\{\mu_\alpha \,|\, \alpha \in \Lambda_1\}$ and $\{\lambda_\alpha \,|\, \alpha \in \Lambda_2\}$ be families of fuzzy sets in S and M respectively. Then we have,

i) $f(\bigcup_{\alpha\in\Lambda_1}\mu_\alpha) = \bigcup_{\alpha\in\Lambda_1} f(\mu_\alpha)$ ii) $f^{-1}(\bigcup_{\alpha\in\Lambda_2}\lambda_\alpha) = \bigcup_{\alpha\in\Lambda_2} f^{-1}(\lambda_\alpha)$ iii) $f f^{-1}(\lambda_\alpha) = \lambda_\alpha$ if f is surjective iv) $f^{-1}f(\mu_\alpha) = \mu_\alpha$ if μ_α is f-invariant.

Definition 1.5.20. Let $\otimes$ be any arithmetic operation and A, B any two fuzzy numbers then by Zadeh's extension principle $A\otimes B$ is a fuzzy set given by $A\otimes B(z) = \sup_{z=x\otimes y} \min[A(x), B(y)]$

Definition 1.5.21. A fuzzy binary relation R of a set X is a function from $X\times X$ to I where I = [0, 1].

Definition 1.5.22. A fuzzy binary relation R on a set X ($R\in I^{X\times X}$) is said to be a fuzzy similarity relation if it satisfies for all x, y, $z\in X$

1. R (x, x) = 1 (reflexive)
2. R (x, y) = R (y, x) (symmetric)
3. $R(x, y) \wedge R(y, z) \leq R(x, z)$ (transitive)

1.5(c) Intuitionistic Fuzzy Sets

The following definitions are adapted from $[AT]_1$,$[AT]_2$, $[B;B]_1$,$[CO]_1$, $[CO]_2$,[D; K]

Definition 1.5.23. An intuitionistic fuzzy set A in a nonempty set X is an object having the form A= $\{(x, \mu_A(x), \gamma_A(x)) \mid x\in X\}$ where the functions $\mu_A : X\rightarrow [0,1]$ and $\gamma_A : X\rightarrow [0,1]$ denote the degree of membership and degree of nonmembership respectively and $0\leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x\in X$.

Definition 1.5.24. Let X be a non empty set and let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in X\}$ be intuitionistic fuzzy sets in X. Then,

i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$

ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

iii) $\overline{A} = \{(x, \gamma_A(x), \mu_A(x)) \mid x \in X\}$

iv) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x)) \vee \gamma_B(x)) \mid x \in X\}$

v) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \gamma_A(x)) \wedge \gamma_B(x)) \mid x \in X\}$

vi) $\Box A = \{(x, \mu_A(x), 1 - \mu_A(x)) \mid x \in X\}$

vii) $\Diamond A = \{(x, 1 - \gamma_A(x), \gamma_A(x)) \mid x \in X\}$

Remark 1.5.25. Operators $\Box$ and $\Diamond$ are called $[AT]_1$ respectively 'necessity' and 'possibility' which will transform every intuitionistic fuzzy set in to a fuzzy set.

Definition 1.5.26. Let $\{A_i \mid i \in \Lambda\}$ be an arbitrary family of intuitionistic fuzzy sets in X then,

i) $\bigcap_{i\in\Lambda} A_i = \{(x, \wedge\mu_{Ai}(x), \vee\gamma_{Ai}(x)) \mid x \in X\}$

ii) $\bigcup_{i\in\Lambda} A_i = \{(x, \vee\mu_{Ai}(x), \wedge\gamma_{Ai}(x)) \mid x \in X\}$

Definition 1.5.27. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ be an intuitionistic fuzzy set in X. For any $t \in [0,1]$, $A_t = \{x \in X \mid \gamma_A(x) \leq t \leq \mu_A(x)\}$ is called a level subset of the intuitionistic fuzzy set A.

Result 1.5.28. Let A and B be intuitionistic fuzzy sets of a non empty set X. Then $(A \cup B)_t = A_t \cup B_t$.

Definition 1.5.29. Let X and Y be two non empty sets. An intuitionistic fuzzy relation R is an intuitionistic fuzzy set of $X \times Y$ given by,

$R = \{((x, y), \mu_R(x, y), \gamma_R(x, y)) \mid x \in X, y \in Y\}$ where $\mu_R : X \times Y \to [0, 1]$ and $\gamma_R : X \times Y \to [0, 1]$ satisfy the condition, $0 \le \mu_R(x, y) + \gamma_R(x, y) \le 1$ for every $(x, y) \in X \times Y$.

IFR($X \times Y$) denote the set of all intuitionistic fuzzy subsets of $X \times Y$.

1.5(d) Category Theory

The following definitions are adapted from [A; H; S], [BO], $[JO]_2$, [MA]

Definition 1.5.30. A category **C** consists of three things:

(b) A class of object, ob **C** denoted by capital letters

(c) For each ordered pair of objects (A, B), a set hom(A, B) whose elements are called morphisms with domain A and codomain B.

(d) For every ordered triple of objects (A, B, C) a map $(f, g) \mapsto g \circ f$ of the product set hom(A, B) $\times$ hom(B, C) into hom(A, C).

Also the objects and morphisms satisfy the following conditions

1. If (A, B) $\neq$ (C, D) then hom(A, B) and hom(C, D) are disjoint.
2. If $f \in$ hom(A, B), $g \in$ hom(B, C) and h $\in$ hom(C, D) then $(hg)f = h(gf)$.

3. For every object A we have an element $I_A \in$ hom(A, A) such that $f \circ I_A = f$ for every $f \in$ hom(A, B) and $I_A \circ g = g$ for every $g \in$ hom(B, A)

Definition 1.5.31. Let **C** be a category then the dual category of **C** is denoted by $\mathbf{C}^{OP}$ and is defined as,

(a) ob $\mathbf{C}^{OP}$ = ob **C**

(b) $\hom_{\mathbf{C}^{op}}(A, B) = \hom_{\mathbf{C}}(B, A)$

(c) If $f \in \hom_{\mathbf{C}^{op}}(A, B)$ and $g \in \hom_{\mathbf{C}^{op}}(B, D)$ then $g \circ f$ (in $\mathbf{C}^{OP}$) = $f \circ g$ (as given in **C**)

(d) I_A is as in **C**.

Definition 1.5.32. Let **C** and **D** be two categories, then a covariant functor F : **C** → **D** consists of,

(a) A map A ↦ F A of ob **C** into ob **D**

(b) For every pair of objects (A, B) of **C** a map $f \mapsto F(f)$ of $\hom_{\mathbf{C}}(A, B)$ into $\hom_{\mathbf{D}}(F\,A, F\,B)$.

Also these satisfy the following conditions:

(1) If $g \circ f$ is defined in **C** then $F(g \circ f) = F(g) \circ F(f)$

(2) $F(I_A) = I_{F\,A}$

Definition 1.5.33. A contravariant functor from **C** to **D** is defined to be a covariant functor from $\mathbf{C}^{OP}$ to **D**.

Definition 1.5.34. Let f and g be **C**- morphisms from A to B. A pair (E, e) is called an equalizer in **C** of f and g if (1) e: E $\to$A is a **C**- morphism (2) $f \circ e = g \circ e$ and (3) for any **C**- morphism e' : E$'$ $\to$A such that $f \circ e' = g \circ e'$, there exist a unique **C**- morphism $\bar{e}$: E$'$ $\to$E such that $e' = e \circ \bar{e}$

Definition 1.5.35. Let $\{A_\alpha | \alpha \in \Lambda\}$ be an indexed set of objects in a category **C** we define a product $\prod A_\alpha$ of the A_α to be a set $\{A, P_\alpha \mid \alpha \in \Lambda\}$ where A $\in$ ob **C**, $P_\alpha \in \hom_C$ (A, A_α) such that if B $\in$ ob **C** and $f_\alpha \in \hom_C$ (B, A_α), $\alpha \in \Lambda$ then there exist a unique f $\in \hom_C$ (B, A) such that $P_\alpha \circ f = f_\alpha$.

Result 1.5.36. A category **C** is complete if and only if it has equalizers and products over arbitrary sets of objects.

Definition 1.5.37. Let **C** and **D** be two categories and F and G be two functors from **C** to **D**. Then a natural transformation η from F to G is a map that assigns to each object A in **C** a morphism $\eta_A \in \hom_D$ (F A, G A) such that for any object A, B of **C** and any f $\in \hom_C$ (A, B) we have $G(f) \circ \eta_A = \eta_B \circ F(f)$.

Definition 1.5.38. Let **A** and **X** be categories. An adjunction from **X** to **A** is a triple $\langle F, G, \varphi \rangle : X \to A$, where F and G are functors $X \underset{G}{\overset{F}{\rightleftarrows}} A$ while φ is a function which assigns to each pair of objects x$\in$ **X**, $a \in$ **A** a bijection $\varphi = \varphi_{x,a}$: A(Fx, a) $\cong$ X(x, Ga).

Here A(Fx, a) is a bifunctor $X^{OP} \times A \xrightarrow{F \times Id} A^{OP} \times A \xrightarrow{hom}$ set which sends each pair of objects (x, a) to the hom-set A(Fx, a) and X(x, Ga) is a similar bifunctor $X^{OP} \times A \rightarrow$ set. The naturality of the bijection φ means that for all k: $a \rightarrow a'$ and h : $x' \rightarrow x$ both the diagrams:

$$\begin{array}{ccc} A(F\,x, a) & \xrightarrow{\varphi} & X(x, G\,a) \\ k_{*}\downarrow & & \downarrow(Gk)_{*} \\ A(F\,x, a') & \xrightarrow{\varphi} & X(x, G\,a') \end{array} \qquad \begin{array}{ccc} A(F\,x, a) & \xrightarrow{\varphi} & X(x, G\,a) \\ (Fh)^{*}\downarrow & & \downarrow h^{*} \\ A(F\,x', a) & \xrightarrow{\varphi} & X(x', G\,a) \end{array} \qquad (1)$$

commute. Here $k_{*} = A(F\,x, k)$ and $h^{*} = X(h, G\,a)$

Remark 1.5.39. Adjunction may also be described as bijections which assigns to each arrow f: F x $\rightarrow$ a an arrow φf : x $\rightarrow$ G a the right adjunct of f, such that the condition of (1) $\varphi(f \circ F\,h) = \varphi f \circ h$, $\varphi(k \circ f) = G\,k \circ \varphi f$ hold for all f and all arrows h : $x' \rightarrow x$ and k: $a \rightarrow a'$. Given such an adjunction, the functor F is said to be a left adjoint for G, while G is called a right adjoint for F.

CHAPTER 2

FUZZY FRAMES[⊗]

2.1 Introduction

In this chapter we generalise the concept of Frame in to a Fuzzy Frame and some results related to that are obtained.

2.2 Fuzzy Frame

We give the following definition for fuzzy frame.

Definition 2.2.1. Let F be a frame; then a fuzzy set $\mu : F \to [0,1]$ of F is said to be a fuzzy frame if ,

(F1) $\mu(\vee S) \geq \inf\{\mu(a) \mid a \in S\}$ for every arbitrary $S \subset F$

(F2) $\mu(a \wedge b) \geq \min\{\mu(a), \mu(b)\}$ for all $a, b \in F$

(F3) $\mu(e_F) = \mu(o_F) \geq \mu(a)$ for all $a \in F$, where e_F and o_F are respectively the unit and zero element of the frame F.

Example 2.2.2. Let μ^a be a fuzzy set of I=[0,1] defined by,

$$\mu^a(x) = \begin{cases} a, & x = 0,1 \\ x, & 0 < x \leq \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \end{cases}$$ where a is some chosen element in $(\frac{1}{2},1]$

Then μ^a is a fuzzy frame of I.

[⊗] Some of the results in this chapter were accepted for publication in the Journal Tripura Mathematical Society

Example 2.2.3. Consider the set R of real numbers with usual topology τ, which is a frame. Let μ be a fuzzy set in τ defined by,

$$\mu(u) = \begin{cases} 1, & u = R, \phi \\ \frac{1}{2}, & u \neq R, \phi \end{cases} \qquad \text{where } u \in \tau$$

Then μ is a fuzzy frame of τ.

Example 2.2.4. Let F be a frame with n elements. Let $(F_i)_{i = 1, 2, \ldots, 2m}$ be a strictly increasing chain of subframes of F where $F_1 = \{e_F, o_F\}$ and $F_{2m} = F$. Define fuzzy sets μ and λ on F as follows,

μ: F→ [0,1] such that

$$\mu(e_F) = \mu(o_F) = 1,\ \mu(x) = \begin{cases} \frac{1}{2k+1}, & \text{if } x \in F_{2k+1} - F_{2k-1} \text{ for } k = 1, 2, \ldots, m\text{-}1 \\ \frac{1}{2m+1}, & \text{if } x \in F_{2m} - F_{2m-1} \end{cases}$$

λ : F→ [0,1] such that

$$\lambda(e_F) = \lambda(o_F) = 1,\ \lambda(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in F_2 - F_1 \\ \frac{1}{2k}, & \text{if } x \in F_{2k} - F_{2k-2} \text{ for } k = 2, \ldots, m \end{cases}$$

Then μ and λ are fuzzy frames of F.

Proposition 2.2.5. If μ is a fuzzy frame of F then μ_t is a sub frame of F for any $t \in I$ with $t \leq \mu(e_F) = \mu(o_F)$.

Proof. For arbitrary $\{a_i\}_{i \in \wedge} \subseteq \mu_t$ we have $\mu(\vee a_i) \geq t$, since μ is a fuzzy frame and

$\mu(a_i) \geq t$ for all i . Hence $\vee a_i \in \mu_t$. Similarly for all a, b$\in \mu_t$ we have $a \wedge b \in \mu_t$. Also clearly e_F, $o_F \in \mu_t$. Therefore μ_t is a subframe of F.

Remark 2.2.6. If E is a subset of a frame F then E is a subframe of F if and only if χ_E is a fuzzy frame of F, where χ_E is the characteristic function of E.

Definition 2.2.7. Let μ be a fuzzy frame and μ_t be a level subset of the frame F for some $t \in I$ with $t \leq \mu(e_F)$. Then μ_t is called a level subframe of F.

Denote $\mu_t > \mu'_t$ if $\mu_t \supset \mu'_t$. Now since $t < t'$ if and only if $\mu_t > \mu'_t$ for any t, t' in μ(F) every fuzzy frame of a frame F gives a chain with level subframes of F,

$$\{o_F, e_F\} = \mu_{t_0} < \mu_{t_1} < \ldots < \mu_{t_r} = F \text{ where } t_i \in \operatorname{Im}\mu \text{ and } t_0 > t_1 > \ldots > t_r.$$

Since all subframes of a frame F usually do not form a chain we have not all subframes are level subframes of the same fuzzy frame.

We shall denote the chain of level subframes of a frame F by Γ_μ (F).

Definition 2.2.8. Let X be the set of all fuzzy frames of F, the relation "$\sim$" in X defined by $\mu \sim \mu'$ if and only if for all x, $y \in F$, $\mu(x) > \mu(y) \Leftrightarrow \mu'(x) > \mu'(y)$. Then "$\sim$" is an equivalence relation on X.

Proposition 2.2.9. Let μ and μ' be two fuzzy frames of a frame F then $\mu \sim \mu'$ if and only if $\Gamma_\mu(F) = \Gamma_{\mu'}(F)$.

Proof. Let $\mu_t \in \Gamma_\mu$ (F) and take $t' = \inf\ \{\mu'(a) \mid a \in \mu_t\}$ then $\mu_t = \mu'_{t}$. Similarly if $\mu'_t \in \Gamma_{\mu'}$ (F) and $t = \inf\ \{\mu\ (a) \mid a \in \mu'_t\}$ then $\mu'_t = \mu_t$. Hence Γ_μ (F) = $\Gamma_{\mu'}$ (F). Conversely for any x, y in F if $\mu(x) > \mu(y)$ then $y \notin \mu_{\mu(x)} = \mu'_t$ and $\mu'(y) < t \leq \mu(x)$ it follows that $\mu'(x) > \mu'(y)$. Similarly $\mu'(x) > \mu'(y)$ implies that $\mu(x) > \mu(y)$. Hence $\mu \sim \mu'$.

Note 2.2.10. Thus two fuzzy frames μ and η of a frame F are said to be equivalent if they have the same family of level subframes otherwise μ and η are non-equivalent.

We shall denote the equivalence class of μ by $[\mu]$.

Proposition 2.2.11. If two equivalent fuzzy frames μ and η of a frame have the same image sets then $\mu = \eta$.

Proof. Obvious.

Proposition 2.2.12. If each non-empty level subset μ_t, $t \in I$ of a fuzzy set μ is a subframe of F, then μ is a fuzzy frame of F.

Proof. Given $\mu_t = \{x \in F \mid \mu(x) \geq t\}$, $t \in I$ is a subframe of F. μ_t being a subframe o_F, $e_F \in \mu_t$, $t \in I$. In particular we have o_F, $e_F \in \mu_T$ where T the largest element of I such that $\mu_T \neq \phi$. Hence $\mu(e_F) = \mu\ (o_F) = T \geq \mu(a)$ for all $a \in F$. Now let S an arbitrary subset of F and let $t = \inf\{\mu\ (a) \mid a \in S\}$. Clearly we have $S \subset \mu_t$ hence

$\vee S \in \mu_t$ and therefore $\mu(\vee S) \geq \inf \{\mu(a)\} | a \in S\}$. Similarly for all a, $b \in F$ we have $\mu(a \wedge b) \geq \min \{\mu(a), \mu(b)\}$. Hence μ is a fuzzy frame of F.

Theorem 2.2.13. Let μ be a fuzzy subset of a frame F. Then μ is a fuzzy frame of F if and only if each non-empty level subset μ_t of μ is a subframe of F.

Proof. Follows from Proposition 2.2.5 and Proposition 2.2.12.

Theorem 2.2.14. Let F be a frame of finite order then there exists a fuzzy frame μ of F such that $\Gamma_\mu(F)$ is a maximal chain of all subframes of F.

Proof. Since F is frame of finite order, the number of subframes of F is finite. So there exists some maximal chain of subframes of F.

Take $F_0 = \{o_F, e_F\} < F_1 < F_2 < \ldots < F_n = F$. (1)

Now define $\mu(F_0)=\{1\}$ and $\mu(F_{i+1} \setminus F_i) = \{^1/_{i+1}\}$ for any i, $0 \leq i < n$. Clearly μ is a fuzzy frame of F and is given by the chain (1).

Remark 2.2.15. If F is a frame of finite order and μ a fuzzy frame of it then $\Gamma_\mu(F)$ is completely determined by μ and conversely for any finite frame F and the subframe chain $\{o_F, e_F\} < F_1 < F_2 < \ldots < F_n = F$ there exists an equivalence class of fuzzy frames of F such that $\Gamma_\mu(F)$ is the above chain.

Remark 2.2.16. If $[\mu] \neq [0]$ then there exists a fuzzy frame η of F in $[\mu]$ such that $\eta(e_F) = \eta(o_F) = 1$.

Theorem 2.2.17. If H is a subframe of F, μ a fuzzy frame of F and η is restriction of μ to H then η is a fuzzy frame of H.

Proof. Obvious

Theorem 2.2.18. Let $\{ I_\alpha \mid \alpha \in \Lambda \}$ be a collection of subframes of F such that

i) $F = \bigcup_{\alpha \in \Lambda} I_\alpha$

ii) s > t if and only if $I_s \subset I_t$ for all s, t$\in \Lambda$ where Λ a collection of elements in [0,1].

Then a fuzzy set μ defined on F by $\mu(x) = \sup \{ t \in \Lambda \mid x \in I_t \}$ is a fuzzy frame of F.

Proof. By Proposition 2.2.12 it is enough to show that non-empty level sets

$\mu_t = \{x \in F \mid \mu(x) \geq t\}$, $t \in I$ are subframes of F. We have the following two cases,

Case-I. $t = \sup \{ s \in \Lambda \mid s < t \}$

$a \in \mu_t \Leftrightarrow a \in \{x \in F \mid \mu(x) \geq t\} \Leftrightarrow a \in I_s$ for all $s < t \Leftrightarrow a \in \bigcap_{s<t} I_s$

Therefore $\mu_t = \bigcap_{s<t} I_s$ is a subframe of F.

Case-II. $t \neq \sup \{ s \in \Lambda \mid s < t \}$

In this case $\mu_t = \bigcup_{s \geq t} I_s$. For if $a \in \bigcup_{s \geq t} I_s$ then $a \in I_s$ for some $s \geq t$.

Hence we have $\mu(x) \geq s \geq t$. Therefore $x \in \mu_t$ and hence $\bigcup_{s \geq t} I_s \subseteq \mu_t$.

Now suppose $x \notin \bigcup_{s \geq t} I_s$. Then $x \notin I_s$ for all $s \geq t$.

Since $t \neq \sup \{ s \in \Lambda \mid s < t \}$ there exist $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Lambda = \phi$.

Hence $x \notin I_s$ for all $s \geq t-\varepsilon$. Thus $\mu(x) < t-\varepsilon < t$ and so $x \notin \mu_t$.

Therefore $\bigcup_{s \geq t} I_s \supseteq \mu_t$.

Thus $\mu_t = \bigcup_{s \geq t} I_s$ which is therefore a subframe of F.

Combining the two cases we have the required result.

Definition 2.2.19. Let μ be any fuzzy subset of the frame F then the fuzzy frame generated by μ in F is the least fuzzy frame of F containing μ and is denoted by $\langle \mu \rangle$.

Theorem 2.2.20. Let μ be a fuzzy set of the frame F then $\langle \mu \rangle (x) = \vee \{ t \mid x \in \langle \mu_t \rangle \}$ for all $x \in F$, where $\langle \mu_t \rangle$ is the subframe of F generated by μ_t .

Proof. Let η be any fuzzy frame of the frame F defined by $\eta(x) = \vee \{ t \mid x \in \langle \mu_t \rangle \}$ for all $x \in F$. Then for any arbitrary $S \subset F$ we have for all $x \in S$, $\eta(x) \geq \inf\{ \eta(y) \mid y \in S\}$.Now $S \subset \langle \mu_t \rangle \Rightarrow \vee S \in \langle \mu_t \rangle$, hence $\eta(\vee S) \geq \inf\{ \eta(y) \mid y \in E\}$. Also for $x, y \in F$ let $\eta(x) = t_1$ and $\eta(y) = t_2$. Suppose that $t_1 > t_2$. Then $y \in \langle \mu_t \rangle \Rightarrow x \in \langle \mu_t \rangle$ and so $x \wedge y \in \langle \mu_t \rangle$, hence $\eta(x \wedge y) \geq t_1 \wedge t_2$.

Again since e_F, $o_F \in \langle \mu_t \rangle$ for all t such that $\mu_t \neq \phi$ it follows that $\eta(e_F) = \eta(o_F) \geq \eta(x)$ for all $x \in F$. Thus η is a subframe of F.

Let $\mu(x) = t$, then $x \in \mu_t \subseteq \langle \mu_t \rangle$ and thus $\eta(x) \geq \mu(x)$. Hence $\eta \supseteq \langle \mu \rangle$ since

$\langle \mu \rangle$ is the smallest fuzzy frame of F which containing μ. Now let γ be any fuzzy frame of F such that $\gamma \supseteq \mu$ then $\gamma_t \supseteq \mu_t$ and so $\gamma_t \supseteq \langle \mu_t \rangle$ for all t. Hence $\gamma \supseteq \eta$.

Thus $\eta = \langle \mu \rangle$. Therefore the result follows.

2.3 Homomorphisms

Theorem 2.3.1. Let L and M be two frames, Φ a frame homomorphism from L onto M and μ a fuzzy frame of M, then $\lambda = \mu \circ \Phi$ is a fuzzy frame of L.

Proof. Let S be an arbitrary subset of L. Now $\Phi(\vee S) \in M$ and equal to $\vee \{\Phi(a) \mid a \in S\}$. Since μ is a fuzzy frame by Definition 2.2.1,

$$\mu \circ \Phi (\vee S) = \mu\{\vee \{\Phi(a) \mid a \in S\}\} \geq \inf \{\mu(\Phi(a)) \mid a \in S\}.$$

Also for all a, b$\in L$, $\mu \circ \Phi (a \wedge b) = \mu\{\Phi(a) \wedge \Phi(b)\} \geq \min \{\mu(\Phi(a)), \mu(\Phi(b))\}$.

Again $\mu(\Phi(o_L)) = \mu(\Phi(e_L))$. Therefore λ is a fuzzy frame of L.

Definition 2.3.2. Let λ, μ be fuzzy frames of frames L and M respectively. If there is a frame homomorphism f from L onto M such that $\lambda = \mu \circ f$ then we say λ is homomorphic to μ and is denoted by $f^{-1}(\mu)$.

If f is an isomorphism then we say that μ and λ are isomorphic.

Lemma 2.3.3. Let f be a homomorphism from a frame L on to a frame M and let μ be any fuzzy frame of M then $(f^{-1}(\mu))_t = f^{-1}(\mu_t)$ for every t$\in$I.

Proof. Let x$\in L$

Now $x \in (f^{-1}(\mu))_t \Leftrightarrow f^{-1}(\mu)(x) \geq t \Leftrightarrow \mu(f(x)) \geq t \Leftrightarrow f(x) \in \mu_t \Leftrightarrow x \in f^{-1}(\mu_t)$

Therefore $(f^{-1}(\mu))_t = f^{-1}(\mu_t)$ for every $t \in I$.

Remark 2.3.4. Theorem 2.3.1 follows also from above lemma since the homomorphic preimage of subframe is a subframe and again by Theorem 2.2.13 if μ is any fuzzy frame of the frame F then every non-empty level subset of μ is also a sub frame of F.

Theorem 2.3.5. Let $f : L \rightarrow M$ be a homomorphism between frames L and M. Then for every fuzzy frame μ of L, $f(\mu)$ is a fuzzy frame of M.

Proof. Define for all $y \in M$,

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

Now for any arbitrary $S \subset M$ we have,

$$f(\mu)(\vee S) = \sup\{\mu(x) \mid x \in f^{-1}(\vee S)\}$$

$$\geq \inf\{\sup\{\mu(x) \mid x \in f^{-1}(y)\}\} = \inf_{y \in S} \{f(\mu(y))\}.$$

Again for all $a, b \in M$ we have,

$$f(\mu)(a \wedge b) = \sup\{\mu(x) \mid x \in f^{-1}(a \wedge b)\}$$

$$\geq \min\{\sup(\mu(x) \mid x \in f^{-1}(a)), \sup(\mu(x) \mid x \in f^{-1}(b))\}$$

$$= \min\{f(\mu(a)), f(\mu(b))\}.$$

Also $f(\mu)$ preserves the unit and the zero elements of M.

Hence $f(\mu)$ is a fuzzy frame of M.

Theorem 2.3.6. Let F be a frame of finite order and f: F $\rightarrow$ F$'$ be an onto homomorphism. Let μ be a fuzzy frame of F with Im μ={t_0, t_1, . . . ,t_n} and $t_0 > t_1 > \ldots > t_n$. If the chain of level subframes of μ is $\{o_F, e_F\} = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \cdots \subseteq \mu_{t_n}$ = F. Then the chain of level subframe of f (μ) will be $\{o_F', e_F'\} = f(\mu_{t_0}) \subseteq f(\mu_{t_1}) \subseteq \cdots \subseteq f(\mu_{t_n}) = F'$.

Proof. Given F is a frame of finite order. We have f (μ) is a fuzzy frame of F$'$ by Theorem 2.3.5. Also clearly Imf (μ) $\subset$ Im μ. Now $f(\mu)_{t_i} = f(\mu_{t_i})$ for each $t_i \in$ Im f(μ).

For let $y \in f(\mu)_{t_i}$ then f (μ)(y) $\geq t_i$ by definition of level subset. Hence $\sup\{\mu(x) \mid x \in f^{-1}(y)\} \geq t_i$ follows from the proof of Theorem 2.3.5. Now choose $x_0 \in$ F such that $f(x_0) = y \in f(\mu_{t_i})$. It follows that $f(\mu)_{t_i} \subseteq f(\mu_{t_i})$ (1)

Let $f(x) \in f(\mu_{t_i})$. Then $x \in \mu_{t_i}$ hence $\mu(x) \geq t_i$ which implies $\sup\{\mu(z) \mid z \in f^{-1}(f(x))\} \geq t_i$ which implies $f(\mu)(f(x)) \geq t_i$ by Theorem 2.3.5. Hence $f(x) \in f(\mu)_{t_i}$ by definition of the level subset.

It follows that $f(\mu_{t_i}) \subseteq f(\mu)_{t_i}$ (2)

From (1) and (2) we have $f(\mu)_{t_i} = f(\mu_{t_i})$ (3)

Also if $\mu_{t_i} \subseteq \mu_{t_j}$ then $f(\mu_{t_i}) \subseteq f(\mu_{t_j})$ for t_i, $t_j \in$ Im μ. (4)

Combining (3) and (4) we have the required result.

2.4 Intersection and union of fuzzy frames

Let μ and λ be two fuzzy frames of F then $\mu \subseteq \lambda$ means $\mu(x) \leq \lambda(x)$ for all $x \in F$. Let $\mathbb{F}$ denote the set of all fuzzy frames of the frame F. We shall denote the supremum and infimum in $\mathbb{F}$ by $\bigcup$(union) and $\bigcap$(intersection) respectively. Thus $\bigcap_{i \in \Lambda} \mu_i(a) = \inf\{\mu_i(a) \mid i \in \Lambda\}$ and $\bigcup_{i \in \Lambda} \mu_i(a) = \sup\{\mu_i(a) \mid i \in \Lambda\}$ where $\mu_i \in \mathbb{F}$. The greatest element of $\mathbb{F}$ is F, which is the function χ_F and $\mathbb{F}$ has no least element.

Proposition 2.4.1. The intersection of any set of fuzzy frames on the frame F is a fuzzy frame.

Proof. We have $\bigcap_{i \in \Lambda} \mu_i(o_F) = \bigcap_{i \in \Lambda} \mu_i(e_F) \geq \bigcap_{i \in \Lambda} \mu_i(x)$ for all $x \in F$ clearly.

Also for all $x,y \in F$

$$\bigcap_{i \in \Lambda} \mu_i(x \wedge y) = \inf\{\mu_i(x \wedge y) \mid i \in \Lambda\} \geq \inf\{\min(\mu_i(x), \mu_i(y)) \mid i \in \Lambda\}$$

$$= \min(\inf\{\mu_i(x) \mid i \in \Lambda\}, \inf\{\mu_i(y) \mid i \in \Lambda\}) = \min(\bigcap_{i \in \Lambda} \mu_i(x), \bigcap_{i \in \Lambda} \mu_i(y))$$

Similarly for arbitrary $S \subseteq F$ we have,

$$\bigcap_{i \in \Lambda} \mu_i(\vee S) \geq \inf\{\inf_{x \in S}(\mu_i(x)) \mid i \in \Lambda\} = \inf_{x \in S}(\inf\{\mu_i(x) \mid i \in \Lambda\})$$

$$= \inf_{x \in S}(\bigcap_{i \in \Lambda} \mu_i(x))$$

Remark 2.4.2. The union of arbitrary family of fuzzy frames on a frame F need not be a fuzzy frame.

For consider the frame $F = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ where $X = \{a, b, c\}$ and the order is set inclusion.

Consider the fuzzy sets μ and λ defined on F by,

$$\mu(X) = \mu(\phi) = 1,\ \mu(\{a\}) = \frac{1}{5},\ \mu(\{b\}) = \frac{1}{2},\ \mu(\{a, b\}) = \frac{1}{3}$$

$$\lambda(X) = \lambda(\phi) = 1,\ \lambda(\{a\}) = \frac{2}{3},\ \lambda(\{b\}) = \frac{1}{5},\ \lambda(\{a, b\}) = \frac{1}{4}$$

Clearly μ and λ are fuzzy frames.

Here $(\mu \cup \lambda)(X) = (\mu \cup \lambda)(\phi) = 1$, $(\mu \cup \lambda)(\{a\}) = \frac{2}{3}$, $(\mu \cup \lambda)(\{b\}) = \frac{1}{2}$,

$(\mu \cup \lambda)(\{a,b\}) = \frac{1}{3}$. Now $\mu \cup \lambda$ is not a fuzzy frame as,

$$(\mu \cup \lambda)(\{a\} \vee \{b\}) = (\mu \cup \lambda)(\{a, b\}) = \frac{1}{3} < \inf\{(\mu \cup \lambda)(\{a\}), (\mu \cup \lambda)(\{b\})\}$$

Remark 2.4.3. The union of any chain of fuzzy frames is clearly a fuzzy frame. We can also have two non-comparable fuzzy frames such that their union is a fuzzy frame. For consider Example 2.2.4 where we have μ and λ are fuzzy frames of F such that neither $\mu \leq \lambda$ nor $\lambda \leq \mu$. Also $\mu \cup \lambda$ is given by $(\mu \cup \lambda)(e_F) = (\mu \cup \lambda)(o_F) = 1$, $(\mu \cup \lambda)(x) = \frac{1}{k}$ if $x \in F_k \setminus F_{k-1}$ for $k = 2,3,\ldots,2m$ Hence $\mu \cup \lambda$ is a fuzzy frame of F.

Theorem 2.4.4. Let $(\mu_i)_{i=1,2\ldots n}$ be a finite collection of fuzzy frames of a frame F.

Then $\bigcup_i \mu_i$ is a fuzzy frame if and only if for $t \in [0,1]$, $\mu_i(x) \geq t$ for all $x \in S$ an arbitrary subset of F and $\mu_i(x) \geq t$, $\mu_i(y) \geq t$ for all $x, y \in F$ implies $\mu_k(\vee S) \geq t$ and $\mu_k(x \wedge y) \geq t$ for some k, $1 \leq k \leq n$.

Proof. By Theorem 2.2.13 $\bigcup_i \mu_i$ is a fuzzy frame if and only if each nonempty level subset $(\bigcup_i \mu_i)_t$ is a subframe of F. Now $(\bigcup_i \mu_i)_t = \bigcup_i (\mu_i)_t$ for each $t \in [0,1]$.

But $\bigcup_i (\mu_i)_t$ is a subframe of F if and only if for any arbitrary $S \subset \bigcup_i (\mu_i)_t$ and $x, y \in \bigcup_i (\mu_i)_t$ we have $\vee S \in \bigcup_i (\mu_i)_t$ and $x \wedge y \in \bigcup_i (\mu_i)_t$.

That is $\mu_i(x) \geq t$ for all $x \in S$ an arbitrary subset of F and $\mu_i(x) \geq t$, $\mu_i(y) \geq t$ for all $x, y \in F$ implies $\mu_k(\vee S) \geq t$ and $\mu_k(x \wedge y) \geq t$ for some k, $1 \leq k \leq n$.

Proposition 2.4.5. $\mathbb{F}$ the set of all fuzzy frames of F under usual ordering of fuzzy set inclusion $\leq$ is not a complete lattice.

Proof. Since $\mathbb{F}$ has no infimum the result follows.

Theorem 2.4.6. Let S be the set of fuzzy frames of a frame F such that $\mu_i(e_F) = \mu_i(o_F) = 1$ for all $\mu_i \in S$. Then S forms a complete lattice under the usual ordering of fuzzy set inclusion $\leq$

Proof. Let $\{\mu_i \mid i \in \Lambda\}$ be a family of fuzzy frames of a frame F. Since $\bigcap_{i \in \Lambda} \mu_i$ is the largest fuzzy frame of F contained in each μ_i we set $\bigwedge_{i\in\Lambda} \mu_i = \bigcap_{i\in\Lambda} \mu_i$. Also since the fuzzy frame generated by the union $\bigcup_{i\in\Lambda} \mu_i$ is the largest fuzzy frame containing each μ_i we set $\bigvee_{i\in\Lambda} \mu_i = \left\langle \bigcup_{i\in\Lambda} \mu_i \right\rangle$, where $\left\langle \bigcup_{i\in\Lambda} \mu_i \right\rangle$ is the fuzzy frame generated by $\bigcup_{i\in\Lambda} \mu_i$. Also $\chi_{\{o_F,\, e_F\}}$ and χ_F are respectively the least and greatest element of S

Thus S is a complete lattice.

Remark 2.4.7. S is not atomic for if $\mu = \chi_{\{o_F,\, e_F\}} \vee a_t$ be an atom where a_t $(a \in F)$ is a fuzzy singleton, then we can find a $t' < t$ such that $\mu' = \chi_{\{o_F,\, e_F\}} \vee a_{t'} < \mu$.

Theorem 2.4.8. Let f be a homomorphism of a frame F into a frame F$'$. Let $\{\mu_i \mid i \in \Lambda\}$ be a family of fuzzy frames of F.

i) If $\bigcup_{i\in\Lambda} \mu_i$ is a fuzzy frame of F, then $\bigcup_{i\in\Lambda} f(\mu_i)$ is a fuzzy frame of F$'$.

ii) If $\bigcup_{i\in\Lambda} f(\mu_i)$ is a fuzzy frame of F$'$, then $\bigcup_{i\in\Lambda} \mu_i$ is a fuzzy frame of F, provided μ_i's are f-invariant.

Proof. i) Suppose $\bigcup_{i\in\Lambda} \mu_i$ is a fuzzy frame of F. Then the homomorphic image $f(\bigcup_{i\in\Lambda} \mu_i)$ is a fuzzy frame of F$'$ by Theorem 2.3.5.

Now since $f(\bigcup_{i\in\Lambda} \mu_i) = \bigcup_{i\in\Lambda} f(\mu_i)$ by Proposition 1.5.19 we have $\bigcup_{i\in\Lambda} f(\mu_i)$ is a fuzzy frame of F'.

ii) Suppose $\bigcup_{i\in\Lambda} f(\mu_i)$ is a fuzzy frame of F'. Then $f^{-1}(\bigcup_{i\in\Lambda} f(\mu_i))$ is a fuzzy frame of F by theorem 4.2. Also since $f^{-1}(\bigcup_{i\in\Lambda} f(\mu_i)) = \bigcup_{i\in\Lambda} f^{-1}f(\mu_i) = \bigcup_{i\in\Lambda} \mu_i$ by Proposition 1.5.19 we have $\bigcup_{i\in\Lambda} \mu_i$ is a fuzzy frame of F.

Theorem 2.4.9. Let f be a homomorphism of a frame F onto a frame F' and $\{\lambda_i \mid i\in \Lambda\}$ be a family of fuzzy frames of F' then the following are equivalent,

i) $\bigcup_{i\in\Lambda} \lambda_i$ is a fuzzy frame of F'.

ii) $\bigcup_{i\in\Lambda} f^{-1}(\lambda_i)$ is a fuzzy frame of F.

Proof. Suppose $\bigcup_{i\in\Lambda} \lambda_i$ is a fuzzy frame of F'. Now by Theorem 2.3.1 $f^{-1}(\bigcup_{i\in\Lambda} \lambda_i)$ is a fuzzy frame of F. Also by Proposition 1.5.19 we have $f^{-1}(\bigcup_{i\in\Lambda} \lambda_i) = \bigcup_{i\in\Lambda} f^{-1}(\lambda_i)$.

Therefore $\bigcup_{i\in\Lambda} f^{-1}(\lambda_i)$ is a fuzzy frame of F.

Conversely suppose $\bigcup_{i\in\Lambda} f^{-1}(\lambda_i)$ is a fuzzy frame of F. Now by Theorem 2.3.5 $f(\bigcup_{i\in\Lambda} f^{-1}(\lambda_i))$ is a fuzzy frame of F'. Also by Proposition 1.5.19 we have $f(\bigcup_{i\in\Lambda} f^{-1}(\lambda_i)) = \bigcup_{i\in\Lambda} \lambda_i$. Therefore $\bigcup_{i\in\Lambda} \lambda_i$ is a fuzzy frame of F'.

2.5 Product of Fuzzy frames

Definition 2.5.1. Let (μ, L) and (η, M) be fuzzy frames where L and M underlying sets which are frame. A morphism $\tilde{f}$: (μ, L) $\rightarrow$(η, M) is a frame homomorphism f: L $\rightarrow$M such that $\mu \le \eta \circ f$. That is the degree of membership of x in L does not exceed that of f(x) in M. The function f: L $\rightarrow$M is called the underlying function of $\tilde{f}$.

Definition 2.5.2. Let $\tilde{f}$: (μ, L) $\rightarrow$(η, M) and $\tilde{g}$: (η, M) $\rightarrow$ (γ, N) be morphisms then $\tilde{g} \circ \tilde{f}$: (μ, L) $\rightarrow$ (γ, N) is a frame homomorphism $g \circ f$: L $\rightarrow$ M such that $\mu \le \gamma \circ g \circ f$.

Let FFrm denote a category whose objects are fuzzy frames and morphisms as defined above. We have the following theorem

Theorem 2.5.3. The category FFrm of fuzzy frames has equalizers.

Proof. Let (μ,L) and (η,M) be fuzzy frames.

Let $\tilde{f}$: (μ,L) $\rightarrow$ (η,M) and $\tilde{g}$: (μ,L) $\rightarrow$(η,M) be two morphisms.

$$\text{Consider } L \underset{g}{\overset{f}{\rightrightarrows}} M$$

Let K = { x$\in$L | f(x) = g (x) } which is a subframe of L and let i:K$\rightarrow$L be the inclusion map. Then clearly $f \circ i = g \circ i$.

Define a fuzzy set λ on K as follows, for a $\in$K let $\lambda(a) = \mu(a)$.

Then $\tilde{i}$ is morphism from (λ, K) to (μ, L).

If for arbitrary fuzzy frame (ξ, N), $\tilde{h}$ is a morphism from (ξ, N) to (μ, L) such that $f \circ h = g \circ h$ then there exist $\theta: \text{N} \to \text{K}$ such that $i \circ \theta = h$.

Also $\xi \leq \lambda \circ \theta$ as for z $\in$N, ξ(z) $\leq \mu \circ$h (z) = μ(h (z)) = μ($i \circ \theta$(z)) = μ(i (θ(z))) = $\mu \circ$ i (θ(z)) = λ(θ(z)) = ($\lambda \circ \theta$)(z)

Thus $\tilde{\theta}$ is a morphism from (ξ, N) to (λ, K)

Now for z $\in$N

$$(\mu \circ i \circ \theta)(z) = (\mu \circ i)(\theta(z)) = \mu(i(\theta(z))) = \mu((i \circ \theta)(z)) = \mu(h(z)) = (\mu \circ h)(z) \geq \xi(z)$$

Hence $\mu \circ i \circ \theta \geq \xi$. Therefore the result follows.

Definition 2.5.4. Let μ_α be fuzzy frame of the frame F_α for $\alpha \in \Lambda$. The product of μ_α's is the function $\mu = \prod_{\alpha \in \Lambda} \mu_\alpha$ defined on the product $\text{F} = \prod_{\alpha \in \Lambda} \text{F}_\alpha$ with usual order by

$$\mu((x_\alpha)_{\alpha \in \Lambda}) = \inf_{\alpha \in \Lambda} \{ \mu_\alpha(x_\alpha) \}$$

Proposition 2.5.5. $\mu = \prod_{\alpha \in \Lambda} \mu_\alpha$ is a fuzzy frame of $\text{F} = \prod_{\alpha \in \Lambda} \text{F}_\alpha$

Proof. We have $\text{F} = \{ (a_\alpha)_{\alpha \in \Lambda} \mid a_\alpha \in \text{F}_\alpha \text{ for } \alpha \in \Lambda \}$

$e_\text{F} = (e_{F_\alpha})_{\alpha \in \Lambda}$ and $o_\text{F} = (o_{F_\alpha})_{\alpha \in \Lambda}$ are respectively the unit and zero element of F.

i) For arbitrary S$\subseteq$F we have,

$$\begin{aligned} \mu(\vee \text{S}) &= \mu(\bigvee_x \{ (x_\alpha) \mid \alpha \in \Lambda \}) \\ &= \mu((\bigvee_x x_\alpha)_{\alpha \in \Lambda}) \end{aligned}$$

$$= \inf_{\alpha\in\Lambda}\{\mu_\alpha(\bigvee_x x_\alpha)\}$$

$$\geq \inf_{\alpha\in\Lambda}\ \{\ \inf_x\ \{\mu_\alpha(x_\alpha)\}\}$$

$$= \inf_x\ \{\inf_{\alpha\in\Lambda}\{\mu_\alpha(x_\alpha)\}\}$$

$$= \inf_{x\in S}\ \mu(\mathrm{x})$$

ii) For all x = $(x_\alpha)_{\alpha\in\Lambda}$, y = $(y_\alpha)_{\alpha\in\Lambda} \in$ F

$$\mu((x_\alpha)_{\alpha\in\Lambda} \wedge (y_\alpha)_{\alpha\in\Lambda}) = \mu((x_\alpha \wedge y_\alpha)_{\alpha\in\Lambda})$$

$$= \inf_{\alpha\in\Lambda}\{\mu_\alpha(x_\alpha \wedge y_\alpha)\}$$

$$\geq \inf_{\alpha\in\Lambda}\ \{\min\{\mu_\alpha(x_\alpha),\ \mu_\alpha(y_\alpha)\}\}$$

$$= \min\{\inf_{\alpha\in\Lambda}\{\mu_\alpha(x_\alpha)\},\ \inf_{\alpha\in\Lambda}\{\mu_\alpha(y_\alpha)\}\}$$

$$= \min\{\mu(\mathrm{x}),\ \mu(\mathrm{y})\}$$

iii) $$\mu(e_{\mathrm{F}}) = \prod_{\alpha\in\Lambda}\mu_\alpha(e_{\mathrm{F}})$$

$$= \inf_{\alpha\in\Lambda}\{\mu_\alpha(e_{F_\alpha})\}$$

$$= \inf_{\alpha\in\Lambda}\{\mu_\alpha(o_{F_\alpha})\}$$

$$= \prod_{\alpha\in\Lambda}\mu_\alpha(o_{\mathrm{F}}) = \mu(o_{\mathrm{F}})$$

also $\mu(e_{\mathrm{F}}) = \prod_{\alpha\in\Lambda}\mu_\alpha(e_{\mathrm{F}}) = \inf\{\mu_\alpha(e_{F_\alpha})\}_{\alpha\in\Lambda}$

$$\geq \inf\{\mu_\alpha(a_\alpha)\}_{\alpha\in\Lambda}$$

$$= \prod_{\alpha\in\Lambda} \mu_\alpha (a) \text{ for all } a = (a_\alpha)_{\alpha\in\Lambda} \in F$$

$$= \mu(a)$$

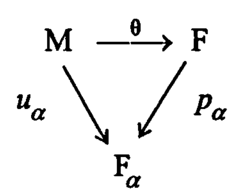
Hence we have the required result.

Theorem 2.5.6. The category FFrm of fuzzy frames has products.

Proof. Consider a family of fuzzy frames $\{(\mu_\alpha, F_\alpha) \mid \alpha \in \Lambda\}$. Corresponding to the product $F = \prod_{\alpha\in\Lambda} F_\alpha$ we have the fuzzy frame (μ, F) where $\mu = \prod_{\alpha\in\Lambda} \mu_\alpha$. Now consider the projection (homomorphism) $P_\alpha : F \to F_\alpha$. We have $\mu((x_\alpha)_{\alpha\in\Lambda}) = \inf_{\alpha\in\Lambda} \{\mu_\alpha(x_\alpha)\}$. Hence $\mu \le \mu_\alpha \circ P_\alpha$ for $\alpha \in \Lambda$.

Therefore $\tilde{P}_\alpha$ is morphism from (μ, F) to (μ_α, F_α) for $\alpha \in \Lambda$.

Now for arbitrary fuzzy frame (ξ, M) if $\tilde{u}_\alpha$ is a morphism from (ξ, M) to (μ_α, F_α). Then define θ : M→F as $(\theta(z))_\alpha = P_\alpha(\theta(z)) = (P_\alpha \circ \theta)(z) = u_\alpha(z)$ for all $\alpha \in \Lambda$ and $z \in M$. Now $\theta(z) = (u_\alpha(z))$ is a frame map as u_α for $\alpha \in \Lambda$ is a frame map.



Also for $z \in M$ we have $\xi(z) \le \mu_\alpha \circ u_\alpha(z)$ for all $\alpha \in \Lambda$ and hence,

$$\xi(z) \le \inf_{\alpha\in\Lambda} \mu_\alpha(u_\alpha(z)) = \inf_{\alpha\in\Lambda} \{\mu_\alpha(\theta(z))_\alpha\} = \mu(\theta(z)) = \mu \circ \theta(z).$$

Hence $\xi \le \mu \circ \theta$. Thus $\tilde{\theta}$ a morphism from (ξ, M) to (μ, F).

Clearly $P_\alpha \circ \theta = u_\alpha$ for all $\alpha \in \Lambda$

$$\text{Also } (\mu_\alpha \circ P_\alpha \circ \theta)(z) = (\mu_\alpha \circ P_\alpha)(\theta(z))$$

$$= \mu_\alpha(P_\alpha(\theta(z))$$

$$= \mu_\alpha(u_\alpha(z))$$

$$= (\mu_\alpha \circ u_\alpha)(z) \geq \xi(z)$$

Hence $\xi \leq \mu_\alpha \circ P_\alpha \circ \theta$.

Thus for each family $(\mu_\alpha, F_\alpha)_{\alpha\in\Lambda}$ of fuzzy frames there is a fuzzy frame (μ, F) and morphisms $\tilde{P}_\alpha : (\mu, F) \to (\mu_\alpha, F_\alpha)$ such that for any fuzzy frame (ξ, M) and family of morphisms $\tilde{u}_\alpha : (\xi, M) \to (\mu_\alpha, F_\alpha)$ there is a unique morphism $\tilde{\theta} : (\xi, M) \to (\mu, F)$ such that $P_\alpha \circ \theta = u_\alpha$ and $\xi \leq \mu_\alpha \circ P_\alpha \circ \theta$ for all $\alpha \in \Lambda$.

Therefore the result follows.

Theorem 2.5.7. The category FFrm of fuzzy frames is complete.

Proof. Follows from Theorem 2.5.3 and Theorem 2.5.6. □

Theorem 2.5.8. Let μ_1 and μ_2 be fuzzy sets of frames F_1 and F_2 respectively such that $\mu_1 \times \mu_2$ is a fuzzy frame of $F_1 \times F_2$. Then μ_1 or μ_2 is a fuzzy frame of F_1 or F_2 respectively.

Proof. We have $\mu_1 \times \mu_2 (e_{F_1}, e_{F_2}) \geq \mu_1 \times \mu_2(x, y)$ for all $(x, y) \in F_1 \times F_2$ also $\mu_1 \times \mu_2 (o_{F_1}, o_{F_2}) = \mu_1 \times \mu_2 (e_{F_1}, e_{F_2})$ where (e_{F_1}, e_{F_2}) and (o_{F_1}, o_{F_2}) are respectively the unit and zero elements of the frame $F_1 \times F_2$.

Now $\mu_1 \times \mu_2(x, y) = \inf \{\mu_1(x), \mu_2(y)\}$ for all $(x, y) \in F_1 \times F_2$ by Definition 2.5.3.

Then $\mu_1(x) \leq \mu_1(e_{F_1})$ or $\mu_2(y) \leq \mu_2(e_{F_2})$ also $\mu_1(e_{F_1})$ and $\mu_2(e_{F_2})$ are equal to either $\mu_1(o_{F_1})$ or $\mu_2(o_{F_2})$.

If $\mu_1(x) \leq \mu_1(e_{F_1})$ then $\mu_1(x) \leq \mu_2(e_{F_2})$ or $\mu_2(y) \leq \mu_2(e_{F_2})$.

Let $\mu_1(x) \leq \mu_2(e_{F_2})$ for all $x \in F_1$

Then for all $x \in F_1$, $\mu_1 \times \mu_2 (x, e_{F_2}) = \inf\{\mu_1(x), \mu_2(e_{F_2})\} = \mu_1(x)$

Now for arbitrary $S \subseteq F_1$ we have

$$\begin{aligned}
\mu_1(\vee S) &= \mu_1 \times \mu_2(\vee S, e_{F_2}) \\
&= \mu_1 \times \mu_2(\bigvee_{x \in S} x, e_{F_2}) \\
&= \mu_1 \times \mu_2(\bigvee_{x \in S} (x, e_{F_2})) \\
&\geq \inf_{x \in S} \{\mu_1 \times \mu_2(x, e_{F_2})\} \\
&= \inf_{x \in S} \{\mu_1(x)\}
\end{aligned}$$

For all $x, y \in F_1$ we have,

$$\mu_1(x \wedge y) = \mu_1 \times \mu_2(x \wedge y, e_{F_2})$$

$$= \mu_1 \times \mu_2((x, e_{F_2}) \wedge (y, e_{F_2}))$$

$$\geq \min\{\mu_1 \times \mu_2(x, e_{F_2}), \mu_1 \times \mu_2(y, e_{F_2})\}$$

$$= \min\{\mu_1(x), \mu_1(y)\}$$

Now $\mu_1(e_{F_1}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2}) \geq \mu_1 \times \mu_2(x, e_{F_2}) = \mu_1(x)$ for all $x \in F_1$

Also $\mu_1(o_{F_1}) = \mu_1 \times \mu_2(o_{F_1}, e_{F_2}) = \inf\{\mu_1(o_{F_1}), \mu_2(e_{F_2})\}$

If $\mu_2(e_{F_2}) = \mu_1(o_{F_1})$ then

$\mu_1(o_{F_1}) = \inf\{\mu_2(e_{F_2}), \mu_2(e_{F_2})\} = \mu_2(e_{F_2}) \geq \mu_1(x)$ for all $x \in F_1$

If $\mu_2(e_{F_2}) = \mu_2(o_{F_2})$ then

$\mu_1(o_{F_1}) = \inf\{\mu_1(o_{F_1}), \mu_2(o_{F_2})\} = \mu_1 \times \mu_2(o_{F_1}, o_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2})$

$= \inf\{\mu_1(e_{F_1}), \mu_2(e_{F_2})\} = \mu_1(e_{F_1})$

Thus $\mu_1(e_{F_1}) = \mu_1(o_{F_1}) \geq \mu_1(x)$ for all $x \in F_1$

Therefore μ_1 is a fuzzy frame of F_1. (1)

Now let $\mu_1(x) \leq \mu_2(e_{F_2})$ is not true for all $x \in F_1$. That is if $\mu_1(x) > \mu_2(e_{F_2})$ for all $x \in F_1$ then $\mu_2(y) \leq \mu_2(e_{F_2})$ for all $y \in F_2$.

Therefore for all $y \in F_2$, $\mu_1 \times \mu_2(e_{F_1}, y) = \inf\{\mu_1(e_{F_1}), \mu_2(y)\} = \mu_2(y)$

Now for arbitrary $S \subseteq F_1$ we have

$$\mu_2(\vee S) = \mu_1 \times \mu_2(e_{F_1}, \vee S)$$

$$= \mu_1 \times \mu_2(e_{F_1}, \bigvee_{x \in S} x)$$

$$= \mu_1 \times \mu_2(\bigvee_{x \in S} (e_{F_1}, x))$$

$$\geq \inf_{x \in S} \{\mu_1 \times \mu_2(e_{F_1}, x)\} = \inf_{x \in S} \{\mu_2(x)\}$$

Similarly for all $x, y \in F_2$ we have $\mu_2(x \wedge y) \geq \min\{\mu_2(x), \mu_2(y)\}$

Now $\mu_2(e_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2}) \geq \mu_1 \times \mu_2(e_{F_1}, x) = \mu_2(x)$ for all $x \in F_2$.

Also $\mu_2(o_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, o_{F_2}) = \inf\{\mu_1(e_{F_1}), \mu_2(o_{F_2})\}$

If $\mu_1(e_{F_1}) = \mu_1(o_{F_1})$ then,

$\mu_2(o_{F_2}) = \inf\{\mu_1(o_{F_1}), \mu_2(o_{F_2})\} = \mu_1 \times \mu_2(o_{F_1}, o_{F_2}) = \mu_1 \times \mu_2(e_{F_1}, e_{F_2})$

$= \inf\{\mu_1(e_{F_1}), \mu_2(e_{F_2})\} = \mu_2(e_{F_2})$

If $\mu_1(e_{F_1}) = \mu_2(o_{F_2})$ then,

$\mu_2(o_{F_2}) = \inf\{\mu_1(e_{F_1}), \mu_1(e_{F_1})\} = \mu_1(e_{F_1}) \geq \mu_2(e_{F_2}) \geq \mu_2(x)$ for all $x \in F_2$

Thus $\mu_2(e_{F_2}) = \mu_2(o_{F_2}) \geq \mu_2(x)$ for all $x \in F_2$

Therefore μ_2 is a fuzzy frame of F_2. (2)

Hence from (1) and (2) either μ_1 or μ_2 is a fuzzy frame of F_1 or F_2 respectively.

Theorem 2.5.9. Let μ_α be fuzzy set of the frame F_α for $\alpha \in \Lambda$ such that $\prod_{\alpha\in\Lambda} \mu_\alpha$ is a fuzzy frame of $F = \prod_{\alpha\in\Lambda} F_\alpha$ .Now for $x_\alpha \in F_\alpha$ $(\alpha \in \Lambda)$ if $\mu_\alpha(e_{F_\alpha}) = \mu_\alpha(o_{F_\alpha}) \geq \mu_\alpha(x_\alpha)$ and $\mu_\alpha(e_{F_\alpha}) = \mu_\alpha(e_{F_\beta})$ for all $\alpha, \beta \in \Lambda$ where e_{F_α}, o_{F_α} are respectively the unit and zero element of the frame F_α then μ_α is a fuzzy frame of F_α for all $\alpha \in \Lambda$.

Proof. We have $\prod_{\alpha\in\Lambda} \mu_\alpha ((e_{F_\alpha})_{\alpha\in\Lambda}) = \prod_{\alpha\in\Lambda} \mu_\alpha ((o_{F_\alpha})_{\alpha\in\Lambda}) \geq \prod_{\alpha\in\Lambda} \mu_\alpha ((x_\alpha)_{\alpha\in\Lambda})$ for all $(x_\alpha)_{\alpha\in\Lambda} \in F$ where $(e_{F_\alpha})_{\alpha\in\Lambda}$ and $(o_{F_\alpha})_{\alpha\in\Lambda}$ are respectively the unit and zero elements of the frame F.

Now for $y \in F_\alpha$ consider $(y_\beta)_{\beta\in\Lambda} \in F$ where $y_\beta = \begin{cases} y & \text{if } \beta = \alpha \\ e_{F_\beta} & \textit{otherwise} \end{cases}$

Then for all $y \in F_\alpha$, $\prod_{\beta\in\Lambda} \mu_\beta((y_\beta)_{\beta\in\Lambda}) = \inf_{\beta\in\Lambda} \{\mu_\beta(y_\beta)\} = \mu_\alpha(y)$

Consider $\alpha \in \Lambda$

Now for arbitrary $S \subseteq F_\alpha$ we have,

$$\mu_\alpha(\vee S) = \prod_{\beta\in\Lambda} \mu_\beta((y_\beta)_{\beta\in\Lambda}) \text{ where } y_\beta = \begin{cases} \vee S & \text{if } \beta = \alpha \\ e_{F_\beta} & \textit{otherwise} \end{cases}$$

$$= \prod_{\beta\in\Lambda} \mu_\beta (\bigvee_{x\in S} (x_\beta)_{\beta\in\Lambda}) \text{ where } x_\beta = \begin{cases} x & \text{if } \beta = \alpha \\ e_{F_\beta} & \textit{otherwise} \end{cases}$$

$$\geq \inf_{x\in S} \{ \prod_{\beta\in\Lambda} \mu_\beta((x_\beta)_{\beta\in\Lambda})\}$$

$$= \inf_{x\in S} \{ \mu_\alpha(x)\}$$

Similarly it can be shown that for all x, y $\in F_\alpha$

$$\mu_\alpha(x \wedge y) = \prod_{\beta\in\Lambda} \mu_\beta((y_\beta)_{\beta\in\Lambda}) \text{ where } y_\beta = \begin{cases} x \wedge y & \text{if } \beta = \alpha \\ e_{F_\beta} & \textit{otherwise} \end{cases}$$

$$\geq \min\{ \mu_\alpha(x), \mu_\alpha(y)\}$$

Hence the result follows. □

Let f be a homomorphism on a frame F. If μ and σ are fuzzy frames of the frame f(F) then $\mu \times \sigma$ is a fuzzy frame of f(F) $\times$ f(F). The pre image $\mu \circ f$ and $\sigma \circ f$ are fuzzy frames of F and $(\mu \times \sigma)\circ(f,f)$ a fuzzy frame of F $\times$ F. We study this relation.

Theorem 2.5.10. Let F be a frame and f a homomorphism on F. Let μ and σ be fuzzy frames of the frame f(F) then $\mu \circ f \times \sigma \circ f = (\mu \times \sigma)\circ(f,f)$

Proof. For all $(x_1, x_2) \in$ F $\times$ F we have,

$$(\mu \times \sigma)\circ(f,f)\,(x_1, x_2) = (\mu \times \sigma)(f(x_1), f(x_2))$$

$$= \inf\{ \mu(f(x_1)), \sigma(f(x_2))\}$$

$$= \inf\{ \mu\circ f(x_1), \sigma\circ f(x_2)\}$$

$$= (\mu\circ f \times \sigma\circ f)\,(x_1, x_2)$$ □

The relation between images of product of fuzzy frames of a frame $\mathbb{F}$ is given as follows.

Theorem 2.5.11. Let μ and σ be fuzzy frames of the frame $\mathbb{F}$. If f is a homomorphism on $\mathbb{F}$, the product $f(\mu) \times f(\sigma)$ and $(f, f)(\mu \times \sigma)$ satisfies $(f, f)(\mu \times \sigma) \subseteq f(\mu) \times f(\sigma)$.

Proof. $f(\mu)$ and $f(\sigma)$ are fuzzy frames of $f(\mathbb{F})$ and $f(\mu) \times f(\sigma)$ is a fuzzy frame of $(f, f)(\mathbb{F} \times \mathbb{F}) = f(\mathbb{F}) \times f(\mathbb{F})$.

Now for each $y = (y_1, y_2) \in f(\mathbb{F}) \times f(\mathbb{F})$ we have,

$[(f, f)(\mu \times \sigma)](y) = \sup\{(\mu \times \sigma)(x) \mid x \in F^{-1}(y)\}$ where $F = (f, f)$ and $x = (x_1, x_2)$

$$= \sup\{\inf(\mu(x_1), \sigma(x_2)) \mid (x_1, x_2) \in F^{-1}(y)\}$$

$$\leq \inf(\sup\{\mu(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{\sigma(x_2) \mid x_2 \in f^{-1}(y_2)\})$$

$$= \inf\{f(\mu(y_1)), f(\sigma(y_2))\}$$

$$= (f(\mu) \times f(\sigma))(y)$$

That is $[(f, f)(\mathbb{F} \times \mathbb{F})](y) \leq (f(\mu) \times f(\sigma))(y)$ for all $y \in f(\mathbb{F}) \times f(\mathbb{F})$

Therefore the result follows. □

CHAPTER 3

FUZZY QUOTIENT FRAMES AND FUZZY IDEALS

3.1 Introduction

The operations of binary meet and arbitrary join on a frame F induce, through Zadeh's extension principle new operations on the partial ordered set I^F. We define a fuzzy quotient frame of F to be a fuzzy partition of F that is a subset of I^F and having a frame structure with respect to new operations. We also study regarding fuzzy ideals on a frame F.

3.2 Extended Operations

The operation of binary meet $\wedge$ and arbitrary join $\vee$ on a frame F can be extended by means of Zadeh's extension principle to operation $\tilde{\wedge}$ and $\tilde{\vee}$ on I^F as follows,

$$(\mu \,\tilde{\wedge}\, \gamma)(x) = \sup\{\mu(y) \wedge \gamma(z) \mid y, z \in F \text{ and } y \wedge z = x\}$$

$$(\tilde{\bigvee_{\alpha\in\Lambda}} \mu_\alpha)(x) = \sup\{\bigwedge_{\alpha\in\Lambda} \mu_\alpha(x_\alpha) \mid x_\alpha \in F \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x\} \qquad (1)$$

for all μ, γ, $\mu_\alpha \in I^F$ and $x \in F$.

The original operation $\wedge$ and $\vee$ on a frame F can be retrieved from $\tilde{\wedge}$ and $\tilde{\vee}$ by embedding F into I^F as the set of all fuzzy singletons each of which is a fuzzy set $1_x \in I^F$ which takes the value 1 at $x \in F$ and 0 elsewhere. Also μ_0, μ_e: $F \rightarrow I$ given by $\mu_0(x) = 0$

for all $x \in F$ and $\mu_e(x) = 1$ for all $x \in F$ are the largest and smallest elements of I^F. It can be observed that for every $\gamma \in I^F$, $\mu_0 = \mu_0 \tilde{\wedge} \gamma = \mu_0 \tilde{\vee} \gamma$ and $\mu_e \tilde{\wedge} \gamma \leq \mu_e$, $\mu_e \tilde{\vee} \gamma \leq \mu_e$ for all $\gamma \in I^F$

Note 3.2.1. I^F is a bounded partial ordered set .

Note 3.2.2. For any family $\{A_\alpha \mid \alpha \in \Lambda\}$ of subsets of a frame F, $\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha = \{\bigvee_{\alpha\in\Lambda} a_\alpha \mid a_\alpha \in A_\alpha\}$ and $A_\alpha \tilde{\wedge} A_\beta = \{a \wedge b \mid a \in A_\alpha, b \in A_\beta\}$ for all $\alpha, \beta \in \Lambda$.

We give the following definitions for supremum property with respect to binary meet $\wedge$ and arbitrary join $\bigvee$

Definition 3.2.3. A pair $\{\mu, \gamma\}$of fuzzy sets of a frame F has supremum property with respect to $\wedge$ if for any $x \in F$ there exist y_o, $z_o \in F$ with $y_o \wedge z_o = x$ such that $\sup\{\mu(y) \wedge \gamma(z) \mid y, z \in F \text{ and } y \wedge z = x\} = \mu(y_0) \wedge \gamma(z_0)$.

Definition 3.2.4. A family $\{\mu_\alpha \mid \alpha \in \Lambda\}$ of fuzzy sets of a frame F has supremum property with respect to $\bigvee$ if for any $x \in F$ there exist $\{a_\alpha \mid \alpha \in \Lambda\} \subseteq F$ with $x = \bigvee_{\alpha\in\Lambda} a_\alpha$ such that $\sup\{\bigwedge_{\alpha\in\Lambda} \mu_\alpha(x_\alpha) \mid x_\alpha \in F \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x\} = \bigwedge_{\alpha\in\Lambda} \mu_\alpha(a_\alpha)$.

Definition 3.2.5. A sub collection S of I^F is said to have supremum property with respect $\wedge$ and $\bigvee$ if every two elements of S has supremum property with respect to $\wedge$ and every arbitrary subset of S has supremum property with respect to $\bigvee$.

Proposition 3.2.6. Let $\{\mu, \gamma\}$ be a pair of fuzzy sets and $\{\mu_\alpha \mid \alpha \in \Lambda\}$ be a family of fuzzy sets of the frame F having supremum property with respect to $\wedge$ and $\vee$ respectively then for each $t \in [0, 1]$, $(\mu \tilde{\wedge} \gamma)_t = \mu_t \tilde{\wedge} \gamma_t$ and $(\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)_t = \tilde{\bigvee}_{\alpha\in\Lambda} (\mu_\alpha)_t$.

Proof. Let $x \in \mu_t \tilde{\wedge} \gamma_t$ then for some $y, z \in F$ with $x = y \wedge z$ we have $\mu(y) \geq t$ and $\gamma(z) \geq t$. Therefore $(\mu \tilde{\wedge} \gamma)(x) = \sup\{\mu(y) \wedge \gamma(z) \mid y, z \in F \text{ and } y \wedge z = x\} \geq t$.

Therefore $x \in (\mu \tilde{\wedge} \gamma)_t$. Hence $\mu_t \tilde{\wedge} \gamma_t \subseteq (\mu \tilde{\wedge} \gamma)_t$. (2)

Now let $x \in (\mu \tilde{\wedge} \gamma)_t$, then we have,

$(\mu \tilde{\wedge} \gamma)(x) = \sup\{\mu(y) \wedge \gamma(z) \mid y, z \in F \text{ and } y \wedge z = x\} \geq t$.

Since μ and γ has supremum property there exist $y_0, z_0 \in F$ with $y_0 \wedge z_0 = x$ such that $\sup\{\mu(y) \wedge \gamma(z) \mid y, z \in F \text{ and } y \wedge z = x\} = \mu(y_0) \wedge \gamma(z_0) \geq t$.

Therefore $y_0 \in \mu_t$, $z_0 \in \gamma_t$, thus $y_0 \wedge z_0 \in \mu_t \tilde{\wedge} \gamma_t$. Hence $(\mu \tilde{\wedge} \gamma)_t \subseteq \mu_t \tilde{\wedge} \gamma_t$. (3)

Therefore from (2) and (3) we have $(\mu \tilde{\wedge} \gamma)_t = \mu_t \tilde{\wedge} \gamma_t$.

Similarly if $x \in \tilde{\bigvee}_{\alpha\in\Lambda} (\mu_\alpha)_t$ then for $\{x_\alpha \mid \alpha \in \Lambda\} \subseteq F$ with $x = \bigvee_{\alpha\in\Lambda} x_\alpha$ we have $\mu_\alpha(x_\alpha) \geq t$ for all $\alpha \in \Lambda$.

Therefore $(\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)(x) = \sup\{\bigwedge_{\alpha\in\Lambda} \mu_\alpha(x_\alpha) \mid x_\alpha \in F \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x\} \geq t$.

Therefore $x \in (\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)_t$. Hence $\tilde{\bigvee}_{\alpha\in\Lambda} (\mu_\alpha)_t \subseteq (\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)_t$ (4)

Now let $x \in (\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)_t$. Since $\{\mu_\alpha \mid \alpha \in \Lambda\}$ has supremum property there exist

$\{a_\alpha \mid \alpha \in \Lambda\} \subseteq F$ with $x = \bigvee_{\alpha \in \Lambda} a_\alpha$ such that $\sup\{ \bigwedge_{\alpha \in \Lambda} \mu_\alpha(x_\alpha) \mid x_\alpha \in F$ and $\bigvee_{\alpha \in \Lambda} x_\alpha = x\}$

$= \bigwedge_{\alpha \in \Lambda} \mu_\alpha(a_\alpha) . \geq t$. Therefore $a_\alpha \in (\mu_\alpha)_t$ for all $\alpha \in \Lambda$. Thus $\bigvee_{\alpha \in \Lambda} a_\alpha \in \tilde{\bigvee}_{\alpha \in \Lambda} (\mu_\alpha)_t$.

$$\text{Hence } (\tilde{\bigvee}_{\alpha \in \Lambda} \mu_\alpha)_t \subseteq \tilde{\bigvee}_{\alpha \in \Lambda} (\mu_\alpha)_t \tag{5}$$

Therefore from (4) and (5) we have $(\tilde{\bigvee}_{\alpha \in \Lambda} \mu_\alpha)_t = \tilde{\bigvee}_{\alpha \in \Lambda} (\mu_\alpha)_t$

Theorem 3.2.7. Let $\{\mu, \gamma\}$ be a pair of fuzzy frames and $\{\mu_\alpha \mid \alpha \in \Lambda\}$ be a family of fuzzy frames of the frame F having supremum property with respect to $\wedge$ and $\vee$ respectively then $\mu \tilde{\wedge} \gamma$ and $\tilde{\bigvee}_{\alpha \in \Lambda} \mu_\alpha$ are fuzzy frames of F.

Proof. To show that $\mu \tilde{\wedge} \gamma$ is a fuzzy frame of F by Theorem 2.2.13 it is enough to show that each level subset $(\mu \tilde{\wedge} \gamma)_t$ of $\mu \tilde{\wedge} \gamma$ is a subframe for $t \in [0, 1]$.

By assumption μ and γ are fuzzy frames of F hence again by Theorem 2.2.13 the level subsets μ_t and γ_t are subframes of F. Since μ_t and γ_t are subframe $\mu_t \tilde{\wedge} \gamma_t$ is a subframe of F for $t \in [0, 1]$. Now by Proposition 3.2.6 we have $(\mu \tilde{\wedge} \gamma)_t = \mu_t \tilde{\wedge} \gamma_t$. Therefore $(\mu \tilde{\wedge} \gamma)_t$ is a subframe of F for all $t \in [0, 1]$. Hence $\mu \tilde{\wedge} \gamma$ is a fuzzy frame of F. Similarly we have $\tilde{\bigvee}_{\alpha \in \Lambda} \mu_\alpha$ is a fuzzy frame of F.

Theorem 3.2.8. Let μ, η and $\gamma \in I^F$. Then $\mu \tilde{\wedge} (\eta \tilde{\vee} \gamma) \leq (\mu \tilde{\wedge} \eta) \tilde{\vee} (\mu \tilde{\wedge} \gamma)$

Proof. Let $w \in F$

$$\text{Now } (\mu \tilde{\wedge} (\eta \tilde{\vee} \gamma))(w) = \sup\{\mu(u) \wedge (\eta \tilde{\vee} \gamma)(v) \mid u, v \in F, u \wedge v = w\} \tag{6}$$

For arbitrary u, v $\in$ F such that $u \wedge v = w$ consider,

$$\mu(u) \wedge (\eta \tilde{\vee} \gamma)(v) = \mu(u) \wedge \sup\{\eta(y) \vee \gamma(z) \mid y, z \in F, y \vee z = v\}$$

$$= \sup\{(\mu(u) \wedge \eta(y)) \wedge (\mu(u) \wedge \gamma(z)) \mid y, z \in F, y \vee z = v\}$$

$$\leq \sup\{(\mu(u) \wedge \eta(y)) \wedge (\mu(u) \wedge \gamma(z)) \mid y, z \in F, (u \wedge y) \vee (u \wedge z) = u \wedge v\}$$

$$\leq \sup\{(\mu \tilde{\wedge} \eta)(u \wedge y) \wedge (\mu \tilde{\wedge} \gamma)(u \wedge z) \mid y, z \in F, (u \wedge y) \vee (u \wedge z) = u \wedge v\}$$

$$= ((\mu \tilde{\wedge} \eta) \tilde{\vee} (\mu \tilde{\wedge} \gamma))(w) \qquad (7)$$

From (6) and (7) we have,

$$\{\mu \tilde{\wedge} (\eta \tilde{\vee} \gamma)\}(w) \leq \{(\mu \tilde{\wedge} \eta) \tilde{\vee} (\mu \tilde{\wedge} \gamma)\}(w)$$

Hence $\mu \tilde{\wedge} (\eta \tilde{\vee} \gamma) \leq (\mu \tilde{\wedge} \eta) \tilde{\vee} (\mu \tilde{\wedge} \gamma)$ □

Similarly we can have the following result

Theorem 3.2.9. For fuzzy set μ and the family of fuzzy sets $\{\mu_\alpha \mid \alpha \in \Lambda\}$ of the frame F, $\mu \tilde{\wedge} (\tilde{\bigvee}_{\alpha \in \Lambda} \mu_\alpha) \leq \tilde{\bigvee}_{\alpha \in \Lambda} (\mu \tilde{\wedge} \mu_\alpha)$.

Theorem 3.2.10. Let S be a sub collection of I^F which is closed with respect to $\tilde{\wedge}$ and $\tilde{\vee}$, and having supremum property with respect to $\wedge$ and $\vee$ then $\mu \tilde{\wedge} (\tilde{\bigvee}_{\alpha \in \Lambda} \mu_\alpha) = \tilde{\bigvee}_{\alpha \in \Lambda} (\mu \tilde{\wedge} \mu_\alpha)$.

Proof. By Proposition 3.2.6 we have,

$(\mu \tilde{\wedge} (\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha))_t = \mu_t \tilde{\wedge} (\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)_t = \mu_t \tilde{\wedge} (\tilde{\bigvee}_{\alpha\in\Lambda} (\mu_\alpha)_t) = \tilde{\bigvee}_{\alpha\in\Lambda} (\mu_t \tilde{\wedge} (\mu_\alpha)_t) =$

$\tilde{\bigvee}_{\alpha\in\Lambda} (\mu \tilde{\wedge} \mu_\alpha)_t = (\tilde{\bigvee}_{\alpha\in\Lambda} (\mu \tilde{\wedge} \mu_\alpha))_t$ for all $t \in [0, 1]$. Hence the result follows.

Remark 3.2.11. In terms of operations $\tilde{\wedge}$ and $\tilde{\vee}$ the conditions (F1) and (F2) for arbitrary fuzzy frame μ in Definition 2.2.1 can be rewritten as,

(F1)′ $\mu \geq \tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha$ where $\mu_\alpha = \mu$ (F2)′ $\mu \geq \mu \tilde{\wedge} \mu$

Proof. We have $(\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)(x) = \sup\{\bigwedge_{\alpha\in\Lambda} \mu_\alpha(a_\alpha) \mid a_\alpha \in F, \bigvee_{\alpha\in\Lambda} a_\alpha = x\}$

$= \sup\{\wedge \mu(a_\alpha) \mid a_\alpha \in F, \bigvee_{\alpha\in\Lambda} a_\alpha = x\}$

$\leq \sup\{\mu(\bigvee_{\alpha\in\Lambda} a_\alpha) \mid a_\alpha \in F, \bigvee_{\alpha\in\Lambda} a_\alpha = x\}$

$= \mu(x)$, for some $x \in F$

also $(\mu \tilde{\wedge} \mu)(x) = \sup\{\mu(y) \wedge \mu(z) \mid y, z \in F, y \wedge z = x\}$

$\leq \sup\{\mu(y \wedge z) \mid y, z \in F, y \wedge z = x\}$

$= \mu(x)$

Remark 3.2.12. In fact equality holds in the above result as,

$(\mu \tilde{\wedge} \mu)(x) = \sup\{\mu(y) \wedge \mu(z) \mid y, z \in F, y \wedge z = x\}$

$\geq \mu(x) \wedge \mu(e_F) = \mu(x)$

Also $(\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)(x) \geq \mu(x) \wedge \mu(o_F) \wedge \mu(o_F) \wedge \ldots = \mu(x)$

3.3 Fuzzy Quotient Frames

Definition 3.3.1. [OV], [MU] A fuzzy partition of F is a subcollection *P* of I^F whose members satisfy the following three conditions

i) Every $\gamma \in P$ is normalized i.e. $\gamma(x)=1$ for at least one $x \in F$.

ii) For each $x \in F$ there is exactly one $\gamma \in P$ with $\gamma(x) = 1$.

iii) If μ, $\gamma \in P$ and x, $y \in F$ are such that $\mu(x) = \gamma(y) = 1$ then $\mu(y) = \gamma(x) = \text{hgt}(\mu \wedge \gamma)$ where the height (hgt) of a fuzzy set $\lambda \in I^F$ is the real number,

$$\text{hgt}(\lambda) = \sup_{x \in F} \lambda(x)$$

Given a fuzzy partition P of F and an element $x \in F$, the unique member of P with value 1 at x is denoted by [x] and is called fuzzy similarity class of x.

A 1-1 correspondence between fuzzy partition and fuzzy similarity relation is defined by sending a fuzzy partition $P \subseteq I^F$ to its fuzzy similarity relation in $I^{F \times F}$, where for all $x, y \in F$ we have,

$$R_p(x, y) = [x](y) = [y](x) = \text{hgt}([x] \wedge [y])$$

The inverse correspondence is defined by sending a fuzzy similarity relation R on F to the fuzzy partition $P_R \subset I^F$ given by $P_R = \{R\langle x \rangle \mid x \in F\}$, where $R\langle x \rangle$ is the fuzzy set of F defined for all $y \in F$ by $R\langle x \rangle(y) = R(x, y)$.

Definition 3.3.2. We call a fuzzy partition P_R of a frame F a fuzzy quotient frame of F if P_R is a subset of I^F and (P_R, $\tilde{\wedge}, \tilde{\vee}$) is a frame.

Theorem 3.3.3. A fuzzy quotient frame P of a frame F satisfies the following properties for all x, y $\in$F and arbitrary $\{x_\alpha \mid \alpha \in \Lambda\} \subseteq F$

i) $[x] \tilde{\wedge} [y] = [x \wedge y]$

$\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha] = [\bigvee_{\alpha\in\Lambda} x_\alpha]$

ii) $1_x \tilde{\wedge} [y] = [x \wedge y] = [x] \tilde{\wedge} 1_y$

$1_x \tilde{\vee} [y] = [x \vee y] = [x] \tilde{\vee} 1_y$

iii) $1_x \tilde{\wedge} [e_F] = [x]$

$1_x \tilde{\vee} [o_F] = [x]$

iv) $[e_F]$ and $[o_F]$ are respectively the identity elements with respect to $\tilde{\wedge}$ and $\tilde{\vee}$.

v) $[x]^c = [x^c]$, where x^c the complement of x in F if it exits.

Proof. i) We have for all x, y$\in$F

$$([x] \tilde{\wedge} [y])(x \wedge y) = \sup\{[x](z) \wedge [y](w) \mid z \wedge w = x \wedge y\}$$

$$\geq [x](x) \wedge [y](y) = 1$$

P being a frame we have $[x] \tilde{\wedge} [y]$ is in P. Hence from the definition of fuzzy partition, we have $[x] \tilde{\wedge} [y] = [x \wedge y]$ as $[x \wedge y](x \wedge y)=1$.

Similarly $\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha](\bigvee_{\alpha\in\Lambda} x_\alpha) = \sup\{\bigwedge_{\alpha\in\Lambda} [x_\alpha](a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = \bigvee_{\alpha\in\Lambda} x_\alpha\}$

$$\geq \bigwedge_{\alpha\in\Lambda} [x_\alpha](x_\alpha) = 1$$

P being a frame we have $\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha]$ is in P. Hence from the definition of fuzzy partition,

we have $\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha] = [\bigvee_{\alpha\in\Lambda} x_\alpha]$ as $[\bigvee_{\alpha\in\Lambda} x_\alpha](\bigvee_{\alpha\in\Lambda} x_\alpha) = 1$.

ii) $(1_x \tilde{\wedge} [y])(x \wedge y) = \sup\{1_x(z) \wedge [y](w) \mid z \wedge w = x \wedge y\}$

$$\geq 1_x(x) \wedge [y](y) = 1$$

Therefore $1_x \tilde{\wedge} [y] = [x \wedge y]$.

Also $([y] \tilde{\wedge} 1_x)(x \wedge y) \geq 1$, hence $[x \wedge y] = [x] \tilde{\wedge} 1_y$.

Similarly we have $1_x \tilde{\vee} [y] = [x \vee y] = [x] \tilde{\vee} 1_y$.

iii) Clearly $(1_x \tilde{\wedge} [e_F])(x) \geq 1$ and $([e_F] \tilde{\wedge} 1_x)(x) \geq 1$

Hence $1_x \tilde{\wedge} [e_F] = [x] = [e_F] \tilde{\wedge} 1_x$

Similarly $1_x \tilde{\vee} [o_F] = [x] = [o_F] \tilde{\vee} 1_x$

iv) we have by (i) $[e_F] \tilde{\wedge} [x] = [e_F \wedge x] = [x] = [x] \tilde{\wedge} [e_F]$

also $[o_F] \tilde{\vee} [x] = [o_F \vee x] = [x] = [x] \tilde{\vee} [e_F]$

v) As $[e_F] = [x \vee x^c] = [x] \tilde{\vee} [x^c]$ and $[o_F] = [x \wedge x^c] = [x] \tilde{\wedge} [x^c]$

we have by (i) $[x]^c = [x^c]$

Remark 3.3.4. For all x, y, z $\in$ F we have,

$$\begin{aligned}[x](y \wedge z) &= [z \wedge y](x) \text{ by definition 3.11 of fuzzy partition} \\ &= ([z] \tilde{\wedge} [y])(x) \\ &\geq [z](x) \wedge [y](x) \\ &= [x](z) \wedge [x](y)\end{aligned}$$

Also for $x \in F$ and arbitrary $S \subset F$

$$[x](\vee S) = [\vee S](x) = [\bigvee_{\alpha \in \Lambda} x_\alpha](x)$$

$$\geq \bigwedge_{\alpha \in \Lambda} [x_\alpha](x)$$

$$= \bigwedge_{\alpha \in \Lambda} [x](x_\alpha) = \inf\{[x](x_\alpha) \mid x_\alpha \in S\}$$

3.4 Invariant fuzzy binary relation

We give the following definition for an invariant fuzzy binary relation.

Definition 3.4.1. A fuzzy binary relation R on a frame F is invariant if it satisfies for all $x, y, u, v \in F$

i) $R(x \wedge u, y \wedge v) \geq R(x, y)$ if $x \neq y$

$R(x \wedge u, y \wedge v) \leq R(x, y)$ if $x = y$

ii) $R(x \vee u, y \vee v) \geq R(x, y)$ if $x \neq y$

$R(x \vee u, y \vee v) \leq R(x, y)$ if $x = y$

Theorem 3.4.2. If R is an invariant fuzzy similarity relation on a frame F then its fuzzy partition P_R is a fuzzy quotient frame of F.

Proof. We have for all $x, y, z \in F$

$$([x] \tilde{\wedge} [y])(z) = \sup\{[x](u) \wedge [y](v) \mid u \wedge v = z\}$$

$$= \sup\{R(x, u) \wedge R(y, v) \mid u \wedge v = z\} \quad (1)$$

Case- I : if $x \neq u$, $y \neq v$

Then $R(x, u) \wedge R(y, v) \leq R(x \wedge (x \wedge y), u \wedge (u \wedge v)) \wedge R(y \wedge (x \wedge y), v \wedge (u \wedge v))$

$$= R(x \wedge y, u \wedge v) \wedge R(x \wedge y, u \wedge v)$$

$$= R(x \wedge y, u \wedge v)$$

Therefore, $\sup\{R(x, u) \wedge R(y, v) \mid u \wedge v = z\} \leq R(x \wedge y, z) = [x \wedge y](z)$ (2)

Case- II : if $x = u$, $y = v$

Then $R(x, u) \wedge R(y, v) = 1 \wedge 1 = 1 = R(x \wedge y, u \wedge v)$

Therefore, $\sup\{ R(x, u) \wedge R(y, v) \mid u \wedge v = z\} = R(x \wedge y, z) = [x \wedge y](z)$ (3)

Case- III : if $x = u,\ y \neq v$

Then $R(x, u) \wedge R(y, v) \leq 1 \wedge R(x \wedge y, u \wedge v) = R(x \wedge y, u \wedge v)$

Therefore, $\sup\{ R(x, u) \wedge R(y, v) \mid u \wedge v = z\} \leq R(x \wedge y, z) = [x \wedge y](z)$ (4)

A similar case when $x \neq u,\ y = v$

Combining (1) (2) (3) and (4) we have $[x] \tilde{\wedge} [y] \leq [x \wedge y]$ (5)

Now for all $x, y, z \in \mathcal{F}$ consider $[x \wedge y](z) = R(x \wedge y, z)$ (6)

Case- I : if $x \wedge y = z$

$$
\begin{aligned}
\text{Then } R(x \wedge y, z) = R(x \wedge y, x \wedge y) &= R(x, x) \wedge R(y, y) \\
&= [x](x) \wedge [y](y) \\
&\leq \sup\{ [x](u) \wedge [y](v) \mid u \wedge v = z\} \\
&= ([x] \tilde{\wedge} [y])(z) \qquad (7)
\end{aligned}
$$

Case- II : if $x \wedge y \neq z$

$$
\begin{aligned}
\text{Then } R(x \wedge y, z) &= R(x \wedge y, z) \wedge R(x \wedge y, z) \\
&\leq R((x \wedge y) \vee x, z \vee z) \wedge R((x \wedge y) \vee y, z \vee z) \\
&= R(x, z) \wedge R(y, z) \\
&= [x](z) \wedge [y](z) \\
&\leq \sup\{ [x](u) \wedge [y](v) \mid u \wedge v = z\} \\
&= ([x] \tilde{\wedge} [y])(z) \qquad (8)
\end{aligned}
$$

Combining (6) (7) and (8) we have $[x \wedge y] \leq [x] \tilde{\wedge} [y]$

Hence from (5) we have $[x] \tilde{\wedge} [y] = [x \wedge y]$.

Now for arbitrary $\{ x_\alpha \mid \alpha \in \Lambda \} \subseteq \mathcal{F}$ and $z \in \mathcal{F}$ we have

$$(\tilde{\bigvee}_{\alpha\in\Lambda}[x_\alpha])(z) = \sup\{\bigwedge_{\alpha\in\Lambda}[x_\alpha](a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = z\}$$

$$= \sup\{\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = z\} \quad (9)$$

Case- I : if $x_\alpha \neq a_\alpha$ for all $\alpha \in \Lambda$

Then $\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) \le \bigwedge_{\alpha\in\Lambda} R(x_\alpha \vee (\bigvee_{\alpha\in\Lambda} x_\alpha), a_\alpha \vee (\bigvee_{\alpha\in\Lambda} a_\alpha))$

$$= \bigwedge_{\alpha\in\Lambda} R(\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} a_\alpha) = R(\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} a_\alpha)$$

Therefore, $\sup\{\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = z\} = R(\bigvee_{\alpha\in\Lambda} x_\alpha, z) = [\bigvee_{\alpha\in\Lambda} x_\alpha](z)$ (10)

Case- II : if $x_\alpha = a_\alpha$ for all $\alpha \in \Lambda$

Then $\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) = 1 = R(\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} a_\alpha)$

Therefore, $\sup\{\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = z\} = R(\bigvee_{\alpha\in\Lambda} x_\alpha, z) = [\bigvee_{\alpha\in\Lambda} x_\alpha](z)$ (11)

Case- III : if $x_\alpha \neq a_\alpha$ for at least one $\alpha \in \Lambda$

If for $\beta \in \Lambda$, $x_\beta \neq a_\beta$ then $R(x_\beta, a_\beta) \le R(x_\beta \vee (\bigvee_{\alpha\in\Lambda} x_\alpha), a_\beta \vee (\bigvee_{\alpha\in\Lambda} a_\alpha))$

$$= R(\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} a_\alpha)$$

If for $\alpha \in \Lambda$, $x_\alpha = a_\alpha$ then $R(x_\alpha, a_\alpha) = 1$

Therefore $\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) \le R(\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} a_\alpha) \wedge 1 = R(\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} a_\alpha)$

Hence $\sup\{\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = z\} = R(\bigvee_{\alpha\in\Lambda} x_\alpha, z) = [\bigvee_{\alpha\in\Lambda} x_\alpha](z)$ (12)

Combining (9) (10) (11) and (12) we have $\tilde{\bigvee}_{\alpha\in\Lambda}[x_\alpha] \le [\bigvee_{\alpha\in\Lambda} x_\alpha]$ (13)

Now consider $[\bigvee_{\alpha\in\Lambda} x_\alpha](z) = R(\bigvee_{\alpha\in\Lambda} x_\alpha, z)$ (14)

Case- I : if $\bigvee_{\alpha\in\Lambda} x_\alpha = z$

$R(\bigvee_{\alpha\in\Lambda} x_\alpha, z) = R(\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} x_\alpha) = \bigwedge_{\alpha\in\Lambda} R(x_\alpha, x_\alpha) = \bigwedge_{\alpha\in\Lambda} ([x_\alpha](x_\alpha)) \leq$

$\sup\{\bigwedge_{\alpha\in\Lambda} [x_\alpha](a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = \bigvee_{\alpha\in\Lambda} x_\alpha = z\} = (\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha])(z)$ (15)

Case- II : if $\bigvee_{\alpha\in\Lambda} x_\alpha \neq z$

$R(\bigvee_{\alpha\in\Lambda} x_\alpha, z) \leq R((\bigvee_{\alpha\in\Lambda} x_\alpha) \wedge x_\alpha, z \wedge z) = R(x_\alpha, z)$ for all $\alpha \in \Lambda$

Therefore $R(\bigvee_{\alpha\in\Lambda} x_\alpha, z) \leq \bigwedge_{\alpha\in\Lambda} R(x_\alpha, z)$

$$\leq \sup\{\bigwedge_{\alpha\in\Lambda} R(x_\alpha, a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = z\}$$

$$= \sup\{\bigwedge_{\alpha\in\Lambda} [x_\alpha](a_\alpha) \mid \bigvee_{\alpha\in\Lambda} a_\alpha = z\}$$

$$= (\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha])(z) \quad (16)$$

Combining (14) (15) and (16) we have $[\bigvee_{\alpha\in\Lambda} x_\alpha] \leq \tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha]$

Hence from (13) we have $\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha] = [\bigvee_{\alpha\in\Lambda} x_\alpha]$

Clearly $[e_F]$ and $[o_F]$ are respectively the unit and zero element of element of P_R.

Now for any $[x] \in P_R$ and arbitrary $S \subset P_R$ we have,

$$[x] \tilde{\wedge} (\tilde{\vee} S) = [x] \tilde{\wedge} [\bigvee_{\alpha\in\Lambda} x_\alpha]$$

$$= [x \wedge (\bigvee_{\alpha\in\Lambda} x_\alpha)]$$

$$= [\bigvee_{\alpha\in\Lambda} (x \wedge x_\alpha)]$$

$$= \tilde{\bigvee}_{\alpha\in\Lambda} [x \wedge x_\alpha]$$

$$= \tilde{\bigvee}_{\alpha\in\Lambda} ([x] \mathbin{\tilde{\wedge}} [x_\alpha])$$

Hence P_R satisfies infinite distributive law. Thus P_R is a frame.

Therefore P_R is a fuzzy quotient frame of F.

Remark 3.4.3. Let F be a frame, then the transformation Ω from the set of invariant fuzzy similarity relations on F to the set of fuzzy quotient frames P of F sends an invariant fuzzy similarity relation R on F to its fuzzy partition $P_R \subset I^F$ given by $P_R = \{R\langle x\rangle \mid x \in F\}$

Remark 3.4.4. Let F be a frame such that every element of it has complement. Then for some $x \in F$, we have $[x] \mathbin{\tilde{\wedge}} [x^c] = [x \wedge x^c] = [o_F]$ and

$$[x] \mathbin{\tilde{\vee}} [x^c] = [x \vee x^c] = [e_F]$$

Also
$$[e_F](x) = [x \vee x^c](x)$$
$$= ([x] \mathbin{\tilde{\vee}} [x^c])(x)$$
$$\geq [x](x) \wedge [x^c](x)$$
$$= [x](x^c)$$

That is $[e_F](x) \geq [x](x^c)$. Similarly we have $[o_F](x) \geq [x](x^c)$ for all $x \in F$.

Example 3.4.5. Consider the frame F = {{a, b, c}, {a, b}, {b, c}, {a}, {b}, {c}, ∅} under set inclusion. Define a fuzzy similarity relation R_F on F by,

$$R_F(x, y) = \begin{cases} 1 & \text{if } x = y \\ \frac{1}{2} & \text{otherwise} \end{cases} \qquad \text{which is invariant.}$$

Now P_R = {[x] | x ∈ F} where [x](y) = R_F(x, y) is a fuzzy partition of F, hence a fuzzy quotient frame.

3.5 Fuzzy Ideal of a Frame

We give the following definition for fuzzy ideal.

Definition 3.5.1. Let F be a frame, then a fuzzy set μ on F is said to be a fuzzy ideal of F if

(F1) $\mu(a \vee b) \geq \min\{\mu(a), \mu(b)\}$, for all a, b ∈ F

(F2) $\mu(a \wedge b) \geq \max\{\mu(a), \mu(b)\}$, for all a, b ∈ F

(F3) $\mu(O_F) = 1$ where O_F the zero element of F

Example 3.5.2. Consider the frame F = {{a,b}, {a}, {b}, ϕ} where the order is set theoretic inclusion. Let μ and γ be two fuzzy sets defined on F by,

$\mu(\{a,b\}) = 0.2$, $\mu(\{a\}) = 0.5$, $\mu(\{b\}) = 0.2$, $\mu(\phi) = 1$ and

$\gamma(\{a,b\}) = 0.3$, $\gamma(\{a\}) = 0.3$, $\gamma(\{b\}) = 0.4$, $\gamma(\phi) = 1$.

Then μ and γ are fuzzy ideals on F.

Theorem 3.5.3. If μ and γ are two fuzzy ideals of F then $\mu \cap \gamma$ is a fuzzy ideal of F.

Proof. (F1) $(\mu \cap \gamma)(a \vee b) = \min\{\mu(a \vee b), \gamma(a \vee b)\}$

$$\geq \min\{\min(\mu(a), \mu(b)), \min(\gamma(a), \gamma(b))\}$$

$$\geq \min\{\min(\mu(a), \gamma(a)), \min(\mu(b), \gamma(b))\}$$

$$= \min\{(\mu \cap \gamma)(a), (\mu \cap \gamma)(b)\}, \text{ for all } a, b \in F$$

(F2) $(\mu \cap \gamma)(a \wedge b) = \min\{\mu(a \wedge b), \gamma(a \wedge b)\}$

$$\geq \min\{\mu(a), \gamma(a)\} \text{ by Definition 3.5.1 (F2)}$$

$$\geq (\mu \cap \gamma)(a), \text{ for all } a, b \in F$$

Similarly $(\mu \cap \gamma)(a \wedge b) \geq (\mu \cap \gamma)(b)$

Therefore $(\mu \cap \gamma)(a \wedge b) \geq (\mu \cap \gamma)(a) \vee (\mu \cap \gamma)(b)$, for all $a, b \in F$

(F3) $(\mu \cap \gamma)(0_F) = \min\{\mu(0_F), \gamma(0_F)\} = 1$

Therefore $\mu \cap \gamma$ is a fuzzy ideal.

Result 3.5.4. If $\{\mu_i \mid i \in \Lambda\}$ a family of fuzzy ideals of F then $\bigcap_{i \in \Lambda} \mu_i$ is a fuzzy ideal of F.

Proof. As above.

Remark 3.5.5. Union of fuzzy ideals on a frame F need not be a fuzzy ideal on F.

For consider fuzzy ideals of Example 3.5.2

Now $(\mu \cup \gamma)(\{a,b\})=0.3$, $(\mu \cup \gamma)(\{a\})=0.5$, $(\mu \cup \gamma)(\{b\})=0.4$, $(\mu \cup \gamma)(\phi)=1$.

$\mu \cup \gamma$ is not a fuzzy ideal of F since,

$(\mu \cup \gamma)(\{a\} \vee \{b\}) = (\mu \cup \gamma)(\{a, b\}) = 0.3 < \min\{(\mu \cup \gamma)(\{a\}), (\mu \cup \gamma)(\{b\})\}$

Definition 3.5.6. Let $\mu \in I^{F}$. Let $\langle \mu \rangle = \bigcap\{\gamma \mid \mu \subseteq \gamma,\ \gamma$ a fuzzy ideal of F }, where $\mu \subseteq \gamma$ means $\mu(x) \leq \gamma(x)$ for all $x \in F$. Then $\langle \mu \rangle$ is called the fuzzy ideal of F generated by μ.

Note. $\langle \mu \rangle$ is the smallest fuzzy ideal of F containing μ.

We state without proof the following result.

Theorem 3.5.7. Let μ and γ be any two fuzzy subset of F, then

i) $\langle \mu \rangle = \mu$, if μ a fuzzy ideal

ii) $\mu \subseteq \gamma \Rightarrow \langle \mu \rangle \subseteq \langle \gamma \rangle$

iii) $\langle \mu / _S \rangle \subseteq \langle \mu \rangle / _S$ where S a subframe of F and $\mu / _S$ and $\langle \mu \rangle / _S$ are the restriction of μ and $\langle \mu \rangle$ to S. □

Theorem 3.5.8. Let μ, γ be two fuzzy ideals of F, then $\mu \tilde{\vee} \gamma$ is a fuzzy ideal of F and $\mu \tilde{\vee} \gamma = \langle \mu \cup \gamma \rangle$.

Proof. Let x, y $\in$ F, then $(\mu \tilde{\vee} \gamma)(x) = \sup\{\mu(u) \wedge \gamma(v) \mid u, v \in F, u \vee v = x\}$

$$\geq \mu(x) \wedge \gamma(0_F) = \mu(x)$$

Hence $\mu \tilde{\vee} \gamma \supseteq \mu$. Similarly we have $\mu \tilde{\vee} \gamma \supseteq \gamma$.

Thus $\mu \tilde{\vee} \gamma \supseteq \mu \cup \gamma$

(F1) $(\mu \tilde{\vee} \gamma)(a \vee b) = \sup\{\mu(u) \wedge \gamma(v) \mid u, v \in F, u \vee v = a \vee b\}$

$\geq \sup\{\mu(u_1 \vee v_1) \wedge \gamma(u_2 \vee v_2) \mid u_1, u_2, v_1, v_2 \in F, u_1 \vee u_2 = a,$
$v_1 \vee v_2 = b\}$

$\geq \sup\{(\mu(u_1) \wedge \mu(v_1)) \wedge (\gamma(u_2) \wedge \gamma(v_2)) \mid u_1, u_2, v_1, v_2 \in F,$
$u_1 \vee u_2 = a,\ v_1 \vee v_2 = b\}$

$= \sup\{(\mu(u_1) \wedge \gamma(u_2)) \wedge (\mu(v_1) \wedge \gamma(v_2)) \mid u_1, u_2, v_1, v_2 \in F,$
$u_1 \vee u_2 = a, v_1 \vee v_2 = b\}$

$= \sup\{\mu(u_1) \wedge \gamma(u_2) \mid u_1, u_2 \in F,\ u_1 \vee u_2 = a\} \wedge$
$\sup\{\mu(v_1) \wedge \gamma(v_2) \mid v_1, v_2 \in F, v_1 \vee v_2 = b\}$

$= (\mu \tilde{\vee} \gamma)(a) \wedge (\mu \tilde{\vee} \gamma)(b)$ for all $a, b \in F$

(F2) $(\mu \tilde{\vee} \gamma)(a \wedge b) \geq \sup\{\mu(a \wedge u) \wedge \gamma(a \wedge v) \mid u, v \in F, u \vee v = b\}$

$\geq \sup\{\mu(u) \wedge \gamma(v) \mid u, v \in F, u \vee v = b\}$

$= (\mu \tilde{\vee} \gamma)(b)$ for all $a, b \in F$

Similarly we have $(\mu \tilde{\vee} \gamma)(a \wedge b) \geq (\mu \tilde{\vee} \gamma)(a)$, for all $a, b \in F$.

Therefore $(\mu \tilde{\vee} \gamma)(a \wedge b) \geq (\mu \tilde{\vee} \gamma)(a) \vee (\mu \tilde{\vee} \gamma)(b)$, for all $a, b \in F$.

(F3) If $\mu(O_F) = \gamma(O_F) = 1$ then clearly $(\mu \tilde{\vee} \gamma)(O_F) \geq \mu(O_F) = 1$ and $(\mu \tilde{\vee} \gamma)(O_F) \geq \gamma(O_F)$. Therefore $(\mu \tilde{\vee} \gamma)(O_F) = 1$.

Thus if μ, γ are any two fuzzy ideals of F then $\mu \tilde{\vee} \gamma$ is a fuzzy ideal of F.

Now let ξ be any fuzzy ideal of F such that $\mu \cup \gamma \subseteq \xi$ then,

$$(\mu \tilde{\vee} \gamma)(x) = \sup\{\mu(u) \wedge \gamma(v) \mid u, v \in F, u \vee v = x\}$$

$$\leq \sup\{\xi(u) \wedge \xi(v) \mid u, v \in F, u \vee v = x\}$$

$$\leq \xi(x)$$

Therefore $\mu \tilde{\vee} \gamma \subseteq \xi$.

Thus $\mu \tilde{\vee} \gamma$ is the smallest ideal of F such that $\mu \cup \gamma \subseteq \mu \tilde{\vee} \gamma$.

Therefore $\mu \tilde{\vee} \gamma = \langle \mu \cup \gamma \rangle$.

Proposition 3.5.9. Let μ, $\gamma \in I^F$. If μ, γ be any two fuzzy ideals of F, then $\mu \tilde{\wedge} \gamma \subseteq \mu \cap \gamma$

Proof. $(\mu \tilde{\wedge} \gamma)(x) = \sup\{\mu(u) \wedge \gamma(v) \mid u, v \in F, u \wedge v = x\}$

$\leq \{\mu(x) \wedge \gamma(x)\}$ [$\because \mu(u \wedge v) = \mu(x) \geq \mu(u) \vee \mu(v)$ and

$\gamma(u \wedge v) = \gamma(x) \geq \gamma(u) \vee \gamma(v)$]

$= (\mu \cap \gamma)(x)$

Therefore $\mu \tilde{\wedge} \gamma \subseteq \mu \cap \gamma$.

Proposition 3.5.10. Let μ be a fuzzy set of a frame F such that $\mu(O_F) = 1$. Then μ is a fuzzy ideal of F if and only if each level subset μ_t of μ is an ideal of F for $t \in I$.

Proof. Let μ be a fuzzy ideal of a frame to show that μ_t is a fuzzy ideal for $t \in I$.

For arbitrary $a, b \in \mu_t$ we have $\mu(a) \geq t$, $\mu(b) \geq t$ hence $\mu(a \vee b) \geq t$, as $\mu(a \vee b) \geq \min\{\mu(a), \mu(b)\}$.Hence $a \vee b \in \mu_t$.

Also $O_F \in \mu_t$ as $\mu(O_F) = 1$.

Again $a \wedge b \in \mu_t$ for all $a, b \in \mu_t$ since $\mu(a \wedge b) \geq \mu(a) \vee \mu(b)$.

Conversely suppose every strong level subset μ_t of the fuzzy set μ an ideal of F to show that μ is a fuzzy ideal of F.

Given $\mu(O_F) = 1$, therefore $\mu(O_F) \geq \mu(x)$ for all $x \in F$.

Now consider arbitrary a, b $\in F$. Let $t = \inf\{\mu(a), \mu(b)\}$. Clearly we have $a, b \in \mu_t$ hence $a \vee b \in \mu_t$. Therefore $\mu(a \vee b) \geq t = \inf\{\mu(a), \mu(b)\}$.

Now let $\mu(a) = t_1$ and $\mu(b) = t_2$ then we have $a \wedge b \in \mu_{t_1}$ and $a \wedge b \in \mu_{t_2}$. Hence $\mu(a \wedge b) \geq t_1$ and $\mu(a \wedge b) \geq t_2$. Therefore $\mu(a \wedge b) \geq t_1 \vee t_2 = \mu(a) \vee \mu(b)$.

Proposition 3.5.11. Let μ and η be fuzzy ideal of a frame F then $(\mu \cap \eta)_t = \mu_t \cap \eta_t$ for all $t \in I$.

Proof. We have $x \in \mu_t \cap \eta_t \Rightarrow x \in \mu_t$ and $x \in \eta_t$

$\Rightarrow \mu(x) \geq t$ and $\eta(x) \geq t$

$\Rightarrow \inf\{\mu(x), \eta(x)\} \geq t$

$\Rightarrow (\mu \cap \eta)(x) \geq t$

$\Rightarrow x \in (\mu \cap \eta)_t$

Therefore $\mu_t \cap \eta_t \subseteq (\mu \cap \eta)_t$

Also $\mu \cap \eta \leq \mu$ and $\mu \cap \eta \leq \eta$. Hence $(\mu \cap \eta)_t \subseteq \mu_t$ and $(\mu \cap \eta)_t \subseteq \eta_t$.

Therefore $(\mu \cap \eta)_t \subseteq \mu_t \cap \eta_t$. Hence the result follows.

Proposition 3.5.12. Let μ and η be fuzzy ideal of a frame F with supremum property with respect to $\vee$, then $(\mu \tilde{\vee} \eta)_t = \mu_t \tilde{\vee} \eta_t$ for all $t \in I$.

Proof. We have $x \in \mu_t \tilde{\vee} \eta_t \Rightarrow$ there exist $y \in \mu_t$ and $z \in \eta_t$ such that $x = y \vee z$

$\Rightarrow \sup\{\mu(y) \wedge \gamma(z) \mid x = y \vee z\} \geq t$

$\Rightarrow (\mu \tilde{\vee} \eta)(x) \geq t$

$\Rightarrow x \in (\mu \tilde{\vee} \eta)_t$ for all $x \in F$.

Therefore $\mu_t \tilde{\vee} \eta_t \subseteq (\mu \vee \eta)_t$

Since μ and η has supremum property we have, $(\mu \tilde{\vee} \eta)_t \subseteq \mu_t \tilde{\vee} \eta_t$

Therefore the result follows.

Proposition 3.5.13. Let μ, η and γ be fuzzy ideals of a frame F then $\mu \cap (\eta \tilde{\vee} \gamma) = (\mu \cap \eta) \tilde{\vee} (\mu \cap \gamma)$.

Proof. $(\mu \cap (\eta \tilde{\vee} \gamma))(w) = \mu(w) \wedge (\eta \tilde{\vee} \gamma)(w)$

$= \mu(w) \wedge \sup\{\eta(y) \wedge \gamma(z) \mid y, z \in F, y \vee z = w\}$

$= \sup\{(\mu(w) \wedge \eta(y)) \wedge (\mu(w) \wedge \gamma(z)) \mid y, z \in F, y \vee z = w\}$

$\leq \sup\{(\mu(y) \wedge \eta(y)) \wedge (\mu(z) \wedge \gamma(z)) \mid y, z \in F, y \vee z = w\}$

$[\because y = y \wedge w, z = w \wedge z]$

$= \sup\{(\mu \cap \eta)(y) \wedge (\mu \cap \gamma)(z) \mid y, z \in F, y \vee z = w\}$

$= ((\mu \cap \eta) \tilde{\vee} (\mu \cap \gamma))(w)$ for all $w \in F$

Therefore $\mu \cap (\eta \tilde{\vee} \gamma) \leq (\mu \cap \eta) \tilde{\vee} (\mu \cap \gamma)$ (1)

Now $\mu \cap \eta \leq \mu$ and $\mu \cap \gamma \leq \mu$, also $\mu \cap \eta \leq \eta \leq \eta \tilde{\vee} \gamma$ and $\mu \cap \gamma \leq \gamma \leq \eta \tilde{\vee} \gamma$.

Therefore $\mu \cap \eta \leq \mu \cap (\eta \tilde{\vee} \gamma)$ and $\mu \cap \gamma \leq \mu \cap (\eta \tilde{\vee} \gamma)$ and hence,

$(\mu \cap \eta) \tilde{\vee} (\mu \cap \gamma) \leq \mu \cap (\eta \tilde{\vee} \gamma)$ (2)

From (1) and (2) we have, $\mu \cap (\eta \tilde{\vee} \gamma) = (\mu \cap \eta) \tilde{\vee} (\mu \cap \gamma)$.

Theorem 3.5.14. Given an arbitrary collection $(\mu_i)_{i\in\Lambda}$ and η of fuzzy ideals of a frame $\mathbb{F}$ then $\eta \cap (\tilde{\bigvee}_{i\in\Lambda} \mu_i) = \tilde{\bigvee}_{i\in\Lambda} (\eta \cap \mu_i)$.

Proof. For $w \in \mathbb{F}$,

$$
\begin{aligned}
(\eta \cap (\tilde{\bigvee}_{i\in\Lambda} \mu_i))(w) &= \eta(w) \wedge (\tilde{\bigvee}_{i\in\Lambda} \mu_i)(w) \\
&= \eta(w) \wedge \sup\{\bigwedge_{i\in\Lambda} \mu_i(a_i) \mid a_i \in \mathbb{F}, \bigvee_{i\in\Lambda} a_i = w\} \\
&= \sup\{\bigwedge_{i\in\Lambda} (\eta(w) \wedge \mu_i(a_i)) \mid a_i \in \mathbb{F}, \bigvee_{i\in\Lambda} a_i = w\} \\
&\leq \sup\{\bigwedge_{i\in\Lambda} (\eta(a_i) \wedge \mu_i(a_i)) \mid a_i \in \mathbb{F}, \bigvee_{i\in\Lambda} a_i = w\} \\
&\qquad [\because a_i = a_i \wedge w] \\
&= \sup\{\bigwedge_{i\in\Lambda} (\eta \cap \mu_i)(a_i)) \mid a_i \in \mathbb{F}, \bigvee_{i\in\Lambda} a_i = w\} \\
&= (\tilde{\bigvee}_{i\in\Lambda} (\eta \cap \mu_i))(w)
\end{aligned}
$$

Therefore $\eta \cap (\tilde{\bigvee}_{i\in\Lambda} \mu_i) \leq \tilde{\bigvee}_{i\in\Lambda} (\eta \cap \mu_i)$ (1)

Now $\eta \cap \mu_i \leq \mu_i$ for all $i \in \Lambda$ hence $\tilde{\bigvee}_{i\in\Lambda} (\eta \cap \mu_i) \leq \tilde{\bigvee}_{i\in\Lambda} \mu_i$ also $\eta \cap \mu_i \leq \eta$ hence $\tilde{\bigvee}_{i\in\Lambda} (\eta \cap \mu_i) \leq \eta$.

Therefore $\tilde{\bigvee}_{i\in\Lambda} (\eta \cap \mu_i) \leq \eta \cap (\tilde{\bigvee}_{i\in\Lambda} \mu_i)$ (2)

From (1) and (2) we have $\tilde{\bigvee}_{i\in\Lambda} (\eta \cap \mu_i) = \eta \cap (\tilde{\bigvee}_{i\in\Lambda} \mu_i)$

Theorem 3.5.15. The set $I_F F$ of all fuzzy ideals of the frame F is a frame.

Proof. I_F F is a complete lattice, which is bounded above by χ_F and below by $\chi_{\{O_F\}}$ where the intersection of fuzzy ideals gives the meet and the join is given by the operation $\tilde{\vee}$. Also here finite meet is distributed over arbitrary join. Hence $I_F F$ is a frame.

CHAPTER 4

INTUITIONISTIC FUZZY FRAMES[⊗]

4.1 Introduction

In this chapter we generalise the concept of Frame into an Intuitionistic fuzzy frame and some results related to that are obtained.

4.2 Intuitionistic Fuzzy Frame

We give the following definition for an intuitionistic fuzzy frame.

Definition 4.2.1. Let F be a frame, then an intuitionistic fuzzy set A = $\{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ in F is called an intuitionistic fuzzy frame of F if it satisfies the following conditions,

i) $\mu_A(\vee S) \geq \inf\{\mu_A(a) \mid a \in S\}$

$\gamma_A(\vee S) \leq \sup\{\gamma_A(a) \mid a \in S\}$ for every arbitrary $S \subset F$

ii) $\mu_A(a \wedge b) \geq \min\{\mu_A(a), \mu_A(b)\}$

$\gamma_A(a \wedge b) \leq \max\{\gamma_A(a), \gamma_A(b)\}$ for all a, b $\in F$

iii) $\mu_A(e_F) = \mu_A(o_F) \geq \mu_A(a)$

$\gamma_A(e_F) = \gamma_A(o_F) \leq \gamma_A(a)$ for all $a \in F$, where e_F and o_F are respectively the unit and zero element of the frame F.

[⊗] Some of the results in this chapter were accepted for publication in The Journal of Fuzzy Mathematics

Example 4.2.2. Consider the set R of real numbers with usual topology τ which is a frame. Let A be an intuitionistic fuzzy set in τ defined by, A={(x, μ_A(x), γ_A(x))| x∈F}

$$\text{Where } \mu_A(x)=\begin{cases}1, & x=R,\ \phi\\ \dfrac{1}{2}, & x\neq R,\ \phi\end{cases} \quad , \gamma_A(x)=\begin{cases}0, & x=R,\ \phi\\ \dfrac{1}{3}, & x\neq R,\ \phi\end{cases}$$

Then A is an intuitionistic fuzzy frame of τ .

Example 4.2.3. Consider an intuitionistic fuzzy set A of I = [0,1] defined by,

A= {(x, $\mu_A^a(x)$, $\gamma_A^a(x)$) | x∈[0,1]} where a is some chosen element in ($\frac{1}{2}$,1] and

$$\mu_A^a(x)=\begin{cases}a, & x=0,1\\ \dfrac{x}{3}, & 0<x\le\dfrac{1}{2}\\ 1-x, & \dfrac{1}{2}<x<1\end{cases} \quad , \quad \gamma_A^a(x)=\begin{cases}1-a, & x=0,1\\ 1-x, & 0<x\le\dfrac{1}{2}\\ \dfrac{x}{3}, & \dfrac{1}{2}<x<1\end{cases}$$

Then A is an intuitionistic fuzzy frame of I.

Theorem 4.2.4. Let F be a frame and A_1, A_2 two intuitionistic fuzzy frame in F then $A_1\cap A_2$ is an intuitionistic fuzzy frame of F.

Proof. Let A_1= {(x, μ_{A1} (x), γ_{A1} (x)) | x∈F} and A_2= {(x, μ_{A2} (x), γ_{A2} (x)) | x∈F}

Then $A_1\cap A_2$ = {(x, μ_{A1} (x) ∧ μ_{A2} (x), γ_{A1} (x) ∨ γ_{A2} (x)) | x∈F}

Let $\mu_{A1\cap A2}(x) = \mu_{A1}(x)\wedge\mu_{A2}(x)$ and $\gamma_{A1\cap A2}(x) = \gamma_{A1}(x)\vee\gamma_{A2}(x)$ for x∈F

i) $\mu_{A1\cap A2}(x\wedge y) = \mu_{A1}(x\wedge y)\wedge\mu_{A2}(x\wedge y)$

$$\geq \min\{\mu_{A1}(x), \mu_{A1}(y)\} \wedge \min\{\mu_{A2}(x), \mu_{A2}(y)\}$$

$= \min \{\mu_{A1}(x) \wedge \mu_{A2}(x), \mu_{A1}(y) \wedge \mu_{A2}(y)\}$

$= \min \{\mu_{A1\cap A2}(x), \mu_{A1\cap A2}(y)\}$ for all x, y$\in$F

$\gamma_{A1\cap A2}(x\wedge y) = \gamma_{A1}(x\wedge y) \vee \gamma_{A2}(x\wedge y)$

$\leq \max \{\gamma_{A1}(x), \gamma_{A1}(y)\} \vee \max \{\gamma_{A2}(x), \gamma_{A2}(y)\}$

$= \max \{\gamma_{A1}(x) \vee \gamma_{A2}(x), \gamma_{A1}(y) \vee \gamma_{A2}(y)\}$

$= \max \{\gamma_{A1\cap A2}(x), \gamma_{A1\cap A2}(y)\}$ for all x, y$\in$F

ii) $\mu_{A1\cap A2}(\vee S) = \mu_{A1}(\vee S) \wedge \mu_{A2}(\vee S)$

$\geq \underset{i=1,2}{\wedge} \inf \{\mu_{Ai}(x) \mid x\in S\}$

$= \inf \{\underset{i=1,2}{\wedge} \mu_{Ai}(x) \mid x\in S\}$

$= \inf \{\mu_{A1\cap A2}(x) \mid x\in S\}$ for every arbitrary S$\subset$F

$\gamma_{A1\cap A2}(\vee S) = \gamma_{A1}(\vee S) \vee \gamma_{A2}(\vee S)$

$\leq \underset{i=1,2}{\vee} \sup \{\gamma_{Ai}(x) \mid x\in S\}$

$= \sup \{\underset{i=1,2}{\vee} \gamma_{Ai}(x) \mid x\in S\}$

$= \sup \{\gamma_{A1\cap A2}(x) \mid x\in S\}$ for every arbitrary S$\subset$F

iii) For unit element e_F and zero element o_F of the frame F we have,

$\mu_{A1\cap A2}(o_F) = \mu_{A1\cap A2}(e_F) \geq \mu_{A1\cap A2}(x)$ and $\gamma_{A1\cap A2}(o_F) = \gamma_{A1\cap A2}(e_F)$

$\leq \gamma_{A1\cap A2}(x)$ for all x$\in$F

For $\mu_{A1\cap A2}(o_F) = \mu_{A1}(o_F) \wedge \mu_{A2}(o_F)$

$\geq \mu_{A1}(x) \wedge \mu_{A2}(x) = \mu_{A1\cap A2}(x)$ for all x$\in$F

Similarly $\mu_{A1 \cap A2}(e_F) \geq \mu_{A1 \cap A2}(x)$ for all $x \in F$.

Also $\gamma_{A1 \cap A2}(o_F) = \gamma_{A1}(o_F) \vee \gamma_{A2}(o_F)$

$$\leq \gamma_{A1}(x) \vee \gamma_{A2}(x) = \gamma_{A1 \cap A2}(x) \text{ for all } x \in F.$$

Similarly $\gamma_{A1 \cap A2}(e_F) \leq \gamma_{A1 \cap A2}(x)$ for all $x \in F$

Thus by definition $A_1 \cap A_2$ is an intuitionistic fuzzy frame of F.

In a similar way we can prove the following result.

Result 4.2.5. If $\{A_i \mid i \in \Lambda\}$ a family of intuitionistic fuzzy frames of F then $\bigcap_{i \in \Lambda} A_i$ is an intuitionistic fuzzy frame of F.

Theorem 4.2.6. Let F be a frame. If A is an intuitionistic fuzzy frame of F then $\Box A$ is also an intuitionistic fuzzy frame of F.

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$. Then $\Box A = \{(x, \mu_A(x), 1-\mu_A(x)) \mid x \in F\}$

Let $\delta_A(x) = 1-\mu_A(x)$ where $x \in F$. Since A is an intuitionistic fuzzy frame for all x, y $\in F$ we have $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$ also for every arbitrary $S \subseteq F$ we have $\mu_A(\vee S) \geq \inf\{\mu_A(x) \mid x \in S\}$.

Now $\delta_A(x \wedge y) = 1-\mu_A(x \wedge y) \leq 1-\min\{\mu_A(x), \mu_A(y)\}$

$$= \max\{1-\mu_A(x), 1-\mu_A(y)\}$$

$$= \max\{\delta_A(x), \delta_A(y)\} \text{ for all } x, y \in F.$$

Also $\delta_A(\vee S) = 1-\mu_A(\vee S) \leq 1-\inf\{\mu_A(x) \mid x \in F\} = \sup\{1-\mu_A(x) \mid x \in F\}$

$= \sup\{\delta_A(x) \mid x \in F\}$ for arbitrary $S \subseteq F$.

Also clearly $\mu_A(o_F) = \mu_A(e_F) \geq \mu_A(x)$for all $x \in F$.

Also $\delta_A(o_F) = 1 - \mu_A(o_F) \leq 1 - \mu_A(x) = \delta_A(x)$ for all $x \in F$. Similarly $\delta_A(e_F) \leq \delta_A(x)$ for all $x \in F$. Therefore the result follows.

Theorem 4.2.7. Let F be a frame if A is an intuitionistic fuzzy frame of F then $\Diamond A$ is also an intuitionistic fuzzy frame of F.

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$. Then $\Diamond A = \{(x, 1-\gamma_A(x), \gamma_A(x)) \mid x \in F\}$

Let $\delta_A(x) = 1 - \gamma_A(x)$ for $x \in F$. Since A is an intuitionistic fuzzy frame for all $x, y \in F$ we have $\gamma_A(x \wedge y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$ and for every arbitrary $S \subseteq F$ we have $\gamma_A(\vee S) \leq \sup\{\gamma_A(x) \mid x \in S\}$.

Now $\delta_A(x \wedge y) = 1 - \gamma_A(x \wedge y) \geq 1 - \max\{\gamma_A(x), \gamma_A(y)\}$

$= \min\{1-\gamma_A(x), 1-\gamma_A(y)\}$

$= \min\{\delta_A(x), \delta_A(y)\}$ for all $x, y \in F$.

Also $\delta_A(\vee S) = 1 - \gamma_A(\vee S) \geq 1 - \sup\{\mu_A(x) \mid x \in F\}$

$= \inf\{1 - \mu_A(x) \mid x \in F\}$

$= \inf\{\delta_A(x) \mid x \in F\}$ for arbitrary $S \subseteq F$.

Again $\gamma_A(o_F) = \gamma_A(e_F) \leq \gamma_A(x)$ for all $x \in F$ as A is an intuitionistic fuzzy frame, hence clearly $\delta_A(o_F) = \delta_A(e_F) \leq \delta_A(x)$ for all $x \in F$. Therefore the result follows.

Theorem 4.2.8. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ be an intuitionistic fuzzy set in F. Then A is an intuitionistic fuzzy frame of F if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy frames of F.

Proof. If A= $\{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ is an intuitionistic fuzzy frame of F, then $\Box$A and $\Diamond$A are intuitionistic fuzzy frames of F by Theorem 4.2.6 and Theorem 4.2.7.

Conversely if $\Box$A and $\Diamond$A are intuitionistic fuzzy frames of F then the fuzzy sets μ_A and $\bar{\gamma}_A = 1-\gamma_A$ are fuzzy frames of F. Now for arbitrary $S \subseteq F$ we have,

$\mu_A(\vee S) \geq \inf\{\mu_A(x) \mid x \in S\}$ and

$\bar{\gamma}_A(\vee S) \geq \inf\{\bar{\gamma}_A(x) \mid x \in S\}$

$= \inf\{1-\gamma_A(x) \mid x \in S\}$

$= 1-\sup\{\gamma_A(x) \mid x \in S\}$.

Also $\bar{\gamma}_A(\vee S) = 1-\gamma_A(\vee S)$.

Hence $\gamma_A(\vee S) \leq \sup\{\gamma_A(x) \mid x \in S\}$

Similarly we have for arbitrary x, $y \in F$, $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(x \wedge y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$.

Again $\bar{\gamma}_A(o_F) = \bar{\gamma}_A(e_F) \geq \bar{\gamma}_A(x)$ for all $x \in F$.

That is $1-\gamma_A(o_F) = 1-\gamma_A(e_F) \geq 1-\gamma_A(x)$ for all $x \in F$. Hence $\gamma_A(o_F) = \gamma_A(e_F) \leq \gamma_A(x)$ for all $x \in F$. Also $\mu_A(e_F) = \mu_A(o_F) \geq \mu_A(x)$ for all $x \in F$.

Hence A is an intuitionistic fuzzy frame of F.

Remark 4.2.9. If A is an intuitionistic fuzzy frame of F then $\bar{A}$ cannot be an intuitionistic fuzzy frame of F, which follows from Definition 1.5.24 and Definition 4.2.1.

Remark 4.2.10. If A and B are two intuitionistic fuzzy frames of F then $A \cup B$ need not be an intuitionistic fuzzy frame of F.

For example consider the frame $F = \{X, \phi, \{a, b\}, \{a, c\}, \{a\}\}$ where $X = \{a, b, c\}$ and the intuitionistic fuzzy frames A and B defined as below $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in F\}$ where,

$$\mu_A(X) = \mu_A(\phi) = 1,\ \mu_A(\{a,b\}) = 0.5,\ \mu_A(\{a,c\}) = 0.4,\ \mu_A(\{a\}) = 0.4$$

$$\gamma_A(X) = \gamma_A(\phi) = 0,\ \gamma_A(\{a,b\}) = 0.3,\ \gamma_A(\{a,c\}) = 0.2,\ \gamma_A(\{a\}) = 0.3$$

and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in F\}$ where,

$$\mu_B(X) = \mu_B(\phi) = 1,\ \mu_B(\{a,b\}) = 0.4,\ \mu_B(\{a,c\}) = 0.5,\ \mu_B(\{a\}) = 0.4$$

$$\gamma_B(X) = \gamma_B(\phi) = 0,\ \gamma_B(\{a,b\}) = 0.4,\ \gamma_B(\{a,c\}) = 0.3,\ \gamma_B(\{a\}) = 0.3$$

Consider $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)) \mid x \in F\}$

Let $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ and $\gamma_{A\cup B}(x) = \gamma_A(x) \vee \gamma_B(x)$

We shall show that $\mu_{A\cup B}(x \wedge y) < \min\{\mu_{A\cup B}(x), \mu_{A\cup B}(y)\}$

For $x = \{a, b\}$, $y = \{a, c\}$ we have $\mu_{A\cup B}(x \wedge y) = \mu_A(x \wedge y) \vee \mu_B(x \wedge y) = \mu_A(\{a\}) \vee \mu_B(\{a\}) = 0.4$. Now $\mu_{A\cup B}(x) = 0.5$, $\mu_{A\cup B}(y) = 0.5$, $\min\{\mu_{A\cup B}(x), \mu_{A\cup B}(y)\} = 0.5$. Therefore $\mu_{A\cup B}(x \wedge y) < \min\{\mu_{A\cup B}(x), \mu_{A\cup B}(y)\}$.

Hence here $A \cup B$ is not an intuitionistic fuzzy frame of F.

Theorem 4.2.11. If H is a sub frame of F, $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ an intuitionistic fuzzy frame of F and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in F\}$ the restriction of A to H then B is an intuitionistic fuzzy frame of H.

Proof. Let S be an arbitrary subset of H.

Now $\mu_B(\vee S) = \mu_A(\vee S) \geq \inf \{ \mu_A(x) \mid x \in S \} = \inf \{ \mu_B(x) \mid x \in S \}$ and

$\gamma_B(\vee S) = \gamma_A(\vee S) \leq \sup \{ \gamma_A(x) \mid x \in S \} = \sup \{ \gamma_B(x) \mid x \in S \}$

Similarly for arbitrary $x, y \in H$ we have,

$$\mu_B(x \wedge y) \geq \min\{ \mu_B(x), \mu_B(y) \}, \quad \gamma_B(x \wedge y) \leq \max\{ \gamma_B(x), \gamma_B(y) \}$$

Again since $o_H = o_F$ and $e_H = e_F$ we have,

$\mu_B(e_H) = \mu_A(e_H)$, $\mu_B(o_H) = \mu_A(o_H)$ and $\gamma_B(e_H) = \gamma_A(e_H)$, $\gamma_B(o_H) = \gamma_A(o_H)$

Hence B is an intuitionistic fuzzy frame of H.

Proposition 4.2.12. If A an intuitionistic fuzzy frame of a frame F then every non empty level set A_t of A for $t \in [0,1]$ is a subframe of F.

Proof. If A is an intuitionistic fuzzy frame on F then for arbitrary $x, y \in A_t$, we have

$\gamma_A(x) \leq t \leq \mu_A(x)$, $\gamma_A(y) \leq t \leq \mu_A(y)$.

Now by Definition 4.2.1 we have for all $x, y \in A_t$

$\mu_A(x \wedge y) \geq \min\{ \mu_A(x), \mu_A(y) \} \geq t$, $\gamma_A(x \wedge y) \leq \max\{ \gamma_A(x), \gamma_A(y) \} \leq t$

Again for arbitrary $F_t \subseteq A_t$ we have $\gamma_A(x) \leq t \leq \mu_A(x)$ for all $x \in F_t$.

Hence $\mu_A(\vee F_t) \geq \inf \{ \mu_A(x) \mid x \in F_t \} \geq t$, $\gamma_A(\vee F_t) \leq \sup\{ \gamma_A(x) \mid x \in F_t \} \leq t$

Also clearly o_F and $e_F \in A_t$. Therefore A_t is a subframe of F.

Lemma 4.2.13. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ be an intuitionistic fuzzy frame of the frame F then $(\mu_A)_t = \{ x \in F \mid \mu_A(x) \geq t \}$ and $(\gamma_A)_t = \{ x \in F \mid \gamma_A(x) \leq t \}$ where $t \in [0,1]$ are either empty or subframes of F.

Proposition 4.2.14. If every non empty level set A_t, $t \in [0,1]$ of an intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ is a subframe of the frame F then A is an intuitionistic fuzzy frame on F.

Proof. Suppose $A_t = \{x \in X \mid \gamma_A(x) \le t \le \mu_A(x)\}$ for $t \in [0,1]$ is a subframe on F.

A_t being a subframe we have e_F, $o_F \in A_t$ for all $t \in [0,1]$.

Hence $\gamma_A(e_F) = \gamma_A(o_F) \le t \le \mu_A(e_F) = \mu_A(o_F)$ for all $t \in [0,1]$. In particular we have e_F and o_F belong to both $(\mu_A)_{T_1}$ and $(\gamma_A)_{T_1}$ also e_F and o_F belongs to both $(\mu_A)_{T_2}$ and $(\gamma_A)_{T_2}$, where T_1 and T_2 are respectively the largest and smallest element of $[0,1]$ such that $(\mu_A)_{T_1}, (\gamma_A)_{T_1}, (\mu_A)_{T_2}, (\gamma_A)_{T_2}$ are non empty.

Hence $\mu_A(e_F) = \mu_A(o_F) \ge \mu_A(x)$ and $\gamma_A(e_F) = \gamma_A(o_F) \le \gamma_A(x)$ for all $x \in F$.

Now let S be an arbitrary subset of F then

$$\mu_A(\vee S) \ge \inf\{\mu_A(x) \mid x \in S\} \text{ and } \gamma_A(\vee S) \le \sup\{\gamma_A(x) \mid x \in S\} \qquad (1)$$

Otherwise there exists some $S_0 \subset F$ such that

$$\mu_A(\vee S_0) < \inf\{\mu_A(x) \mid x \in S_0\} \text{ or } \gamma_A(\vee S) > \sup\{\gamma_A(x) \mid x \in S_0\}$$

Taking $t_0 = \frac{1}{2}\,[\mu_A(\vee S_0) + \inf\{\mu_A(x) \mid x \in S_0\}]$ we have $\mu_A(\vee S_0) < t_0 < \inf\{\mu_A(x) \mid x \in S_0\}$. Hence $t_0 < \mu_A(x)$ for all $x \in S_0$. Therefore $x \in (\mu_A)_{t_0}$ for all $x \in S_0$ and hence $S_0 \subseteq (\mu_A)_{t_0}$. Since $(\mu_A)_{t_0}$ is a subframe by Lemma 4.2.13 we have $\vee S_0 \in (\mu_A)_{t_0}$. Therefore $\mu_A(\vee S_0) > t_0$ a contradiction.

Similarly taking $t_0 = \frac{1}{2}\,[\gamma_A(\vee S_0) + \sup\{\gamma_A(x) \mid x \in S_0\}]$ we have $\gamma_A(\vee S_0) > t_0 > \sup\{\gamma_A(x) \mid x \in S_0\}$. Hence $t_0 > \gamma_A(x)$ for all $x \in S_0$. Therefore

$x \in (\gamma_A)_{t_0}$ for all $x \in S_0$ and hence $S_0 \subseteq (\gamma_A)_{t_0}$. Since $(\gamma_A)_{t_0}$ is a subframe by Lemma 4.2.13 we have $\vee S_0 \in (\gamma_A)_{t_0}$. Therefore $\gamma_A(\vee S_0) < t_0$ a contradiction. Hence (1) holds.

Also for arbitrary $x, y \in F$ we can show similarly that $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(x \wedge y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$.

Therefore $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ is an intuitionistic fuzzy frame on F.

Theorem 4.2.15. Let F be a frame then an intuitionistic fuzzy set A on F is an intuitionistic fuzzy frame on F if and only if every non empty level set A_t, $t \in [0,1]$ of A is a subframe of the frame F.

Proof. Follows from Proposition 4.2.12 and Proposition 4.2.14.

Definition 4.2.16. Let A be an intuitionistic fuzzy set of the frame F then the intuitionistic fuzzy frame generated by A in F is the least intuitionistic fuzzy frame B of F with $A \subseteq B$ and is denoted by $\langle A \rangle$.

Theorem 4.2.17. Let F be a frame and $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ an intuitionistic fuzzy set of F then $\langle A \rangle = \{(x, \langle \mu_A \rangle(x), \langle \gamma_A \rangle(x)) \mid x \in F\}$ where,

$\langle \mu_A \rangle(x) = \vee \{t \mid x \in \langle (\mu_A)_t \rangle\}$, $\langle \gamma_A \rangle(x) = \wedge \{t \mid x \in \langle (\gamma_A)_t \rangle\}$ for all $x \in F$ is an intutionistic fuzzy frame generated by A.

Proof. Consider $\langle A \rangle = \{(x, \langle \mu_A \rangle(x), \langle \gamma_A \rangle(x)) \mid x \in F\}$ where,

$\langle \mu_A \rangle(x) = \vee \{t \mid x \in \langle (\mu_A)_t \rangle\}$, $\langle \gamma_A \rangle(x) = \wedge \{t \mid x \in \langle (\gamma_A)_t \rangle\}$ for all $x \in F$.

Now for any arbitrary $S \subseteq F$ we have for all $x \in S$, $\langle \mu_A \rangle (x) \geq \inf\{ \langle \mu_A \rangle (y) \mid y \in S\}$ and $\langle \gamma_A \rangle (x) \leq \sup \{ \langle \gamma_A \rangle (y) \mid y \in S\}$.

Now $S \subseteq \langle (\mu_A)_t \rangle \Rightarrow \vee S \in \langle (\mu_A)_t \rangle$ and $S \subseteq \langle (\gamma_A)_t \rangle \Rightarrow \vee S \in \langle (\gamma_A)_t \rangle$.

Hence $\langle \mu_A \rangle (\vee S) \geq \inf\{ \langle \mu_A \rangle (y) \mid y \in S\}$ and

$$\langle \gamma_A \rangle (\vee S) \leq \sup \{ \langle \gamma_A \rangle (y) \mid y \in S\} \qquad (1)$$

Also for $x, y \in F$ let $\langle \mu_A \rangle (x) = t_1$, $\langle \gamma_A \rangle (x) = t_1'$ and

$$\langle \mu_A \rangle (y) = t_2, \ \langle \gamma_A \rangle (y) = t_2'$$

Suppose that $t_1 > t_2$ and $t_1' < t_2'$.

Then $y \in \langle (\mu_A)_t \rangle \Rightarrow x \in \langle (\mu_A)_t \rangle$ and so $x \wedge y \in \langle (\mu_A)_t \rangle$

Similarly $y \in \langle (\gamma_A)_t \rangle \Rightarrow x \in \langle (\gamma_A)_t \rangle$ and so $x \wedge y \in \langle (\gamma_A)_t \rangle$

Hence $\langle \mu_A \rangle (x \wedge y) \geq t_1 \wedge t_2$ and $\langle \gamma_A \rangle (x \wedge y) \leq t_1' \vee t_2'$. (2)

Again since e_F, o_F belongs to both $\langle (\mu_A)_t \rangle$ and $\langle (\gamma_A)_t \rangle$ for all t such that $(\mu_A)_t \neq \phi$, $(\gamma_A)_t \neq \phi$ it follows that,

$\langle \mu_A \rangle (e_F) = \langle \mu_A \rangle (o_F) \geq \langle \mu_A \rangle (x)$ and $\langle \gamma_A \rangle (e_F) = \langle \gamma_A \rangle (o_F) \geq \langle \gamma_A \rangle (x)$ for all $x \in F$. (3)

From (1), (2) and (3) we have thus $\langle A \rangle$ is an intutionistic fuzzy frame of F.

Now let B be any intutionistic fuzzy frame of F such that $B \supseteq A$ then $(\mu_B)_t \supseteq (\mu_A)_t$ and $(\gamma_A)_t \supseteq (\gamma_B)_t$ and so $(\mu_B)_t \supseteq \langle (\mu_A)_t \rangle$ and $\langle (\gamma_A)_t \rangle \supseteq (\gamma_B)_t$ for all t.

Hence $B \supseteq \langle A \rangle$. Therefore the result follows.

Theorem 4.2.18. Let $(A_i)_{i=1,2...n}$ where $A_i = \{(x, \mu_{A_i}(x), \gamma_{A_i}(x)) \mid x \in F\}$ be a finite collection of intuitionistic fuzzy frames of a frame F. Then $\bigcup_{i\in\Lambda} A_i$ is an intuitionistic fuzzy frame if and only if for $t\in[0,1]$ $\gamma_{A_i}(x) \le t \le \mu_{A_i}(x)$ for all $x\in S$ an arbitrary subset of F and $\gamma_{A_i}(x) \le t \le \mu_{A_i}(x)$, $\gamma_{A_i}(y) \le t \le \mu_{A_i}(y)$ for all x, $y\in F$ implies $\gamma_k(\vee S) \le t \le \mu_k(\vee S)$ and $\gamma_k(x\wedge y) \le t \le \mu_k(x\wedge y)$ for some k, $1\le k \le n$.

Proof. By Theorem 4.2.15 $\bigcup_{i\in\Lambda} A_i$ is an intuitionistic fuzzy frame if and only if each nonempty level subset $(\bigcup_i A_i)_t$ is a subframe of F. Now $(\bigcup_i A_i)_t = \bigcup_i (A_i)_t$ for each $t\in[0,1]$. But $\bigcup_i (A_i)_t$ is a subframe of F if and only if for any arbitrary $S \subset \bigcup_i (A_i)_t$ we have $\vee S \in \bigcup_i (A_i)_t$ and for all a, $b \in \bigcup_i (A_i)_t$, $a\wedge b \in \bigcup_i (A_i)_t$. That is $\gamma_{A_i}(x) \le t \le \mu_{A_i}(x)$ for all $x\in S$ an arbitrary subset of F and $\gamma_{A_i}(x) \le t \le \mu_{A_i}(x)$, $\gamma_{A_i}(y) \le t \le \mu_{A_i}(y)$ for all x, $y\in F$ implies $\gamma_k(\vee S) \le t \le \mu_k(\vee S)$ and $\gamma_k(x\wedge y) \le t \le \mu_k(x\wedge y)$ for some k, $1\le k \le n$.

Theorem 4.2.19. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x\in F\}$ be a intuitionistic fuzzy set of a frame F with Card Im $\mu_A < \infty$ and Card Im $\gamma_A < \infty$. Define subframes F_i of F inductively as follows $F_0 = (K_0)$ frame generated by K_0, where $K_0 = \{x\in F \mid \mu_A(x) = \sup\{\mu_A(y) \mid y\in F\}, \gamma_A(x) = \inf\{\gamma_A(y) \mid y\in F\}\}$, $F_i = (K_i)$ frame generated by K_i, where $K_i = F_{i-1} \cup \{x\in F \mid \mu(x) = \sup\{\mu(y) \mid y\in F - F_{i-1}\}, \gamma(x) = \inf\{\gamma(y) \mid y\in F - F_{i-1}\}\}$ for all i = 1,2, . . . ,n such that $F_n = F$. Then the intuitionistic fuzzy set A^* of F defined by $\mu_{A^*}(x) = \sup\{\mu_A(y) \mid y\in F\}$, $\gamma_{A^*}(x) = \inf\{\gamma_A(y) \mid y\in F\}$ for all $x\in F_0$ and $\mu_{A^*}(x) = \sup\{\mu_A(y) \mid y\in F-F_{i-1}\}$, $\gamma_{A^*}(x) = \inf\{\gamma_A(y) \mid y\in F-F_{i-1}\}$ for all $x\in F_i-F_{i-1}$ where i = 1,2, . . . ,n is the intuitionistic fuzzy frame generated by A in F.

Proof. Clearly $A \le A^*$ as $\mu_A \le \mu_{A^*}$ and $\gamma_A \ge \gamma_{A^*}$. Also F_i's form the chain, $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq F_n = F$ of all level subframes of the intuitionistic fuzzy set A^* of F. Hence by Theorem 4.2.15, A^* is an intuitionistic fuzzy frame of F.

Also we have A^* is generated by A. For let $B= \{(x, \mu_B(x), \gamma_B(x)) \mid x\in F\}$ be any intuitionistic fuzzy frame of F such that $A\le B$.

If $x\in K_0$ then $\sup\{\mu_A(y) \mid y\in F\} = \mu_A(x) \le \mu_B(x)$, $\inf\{\gamma_A(y) \mid y\in F\} = \gamma_A(x) \ge \gamma_B(x)$ hence $\sup\{\mu_A(y) \mid y\in F\} \le \inf\{\mu_B(x) \mid x\in K_0\}$, $\inf\{\gamma_A(y) \mid y\in F\} \ge \sup\{\gamma_B(x) \mid x\in K_0\}$. Choosing $\inf\{\mu_B(x) \mid x\in K_0\}= t_0$ and $\sup\{\gamma_B(x) \mid x\in K_0\}= t_0'$ we have $\mu_B(x)\ge t_0$ and $\gamma_B(x) \le t_0'$ for all $x\in K_0$. Therefore $K_0\subseteq(\mu_B)_{t_0}$ the subframe of μ_B and $K_0\subseteq(\gamma_B)_{t_0'}$ the subframe of γ_B by Lemma 4.2.13. Since $(\mu_B)_{t_0}$ and $(\gamma_B)_{t_0'}$ are subframes we have $F_0\subset(\mu_B)_{t_0}$ and $F_0\subset(\gamma_B)_{t_0'}$. Therefore $\mu_B(x)\ge t_0$ and $\gamma_B(x) \le t_0'$ for all $x\in F_0$. Hence for all $x\in F_0$,

$$\mu_{A^*}(x) = \sup\{\mu_A(y) \mid y\in F\} \le \inf\{\mu_B(x) \mid x\in K_0\}= t_0 \le \mu_B(x),$$

$$\gamma_{A^*}(x) = \inf\{\gamma_A(y) \mid y\in F\} \ge \sup\{\gamma_B(x) \mid x\in K_0\}= t_0' \ge \gamma_B(x)$$

Also if $x\in K_1- F_0$ then $\sup\{\mu_A(y) \mid y\in F-F_0\} = \mu_A(x) \le \mu_B(x)$ and $\inf\{\gamma_A(y) \mid y\in F-F_0\} = \gamma_A(x) \ge \gamma_B(x)$. Hence $\sup\{\mu_A(y) \mid y\in F-F_0\} \le \inf\{\mu_B(x) \mid x\in K_1-F_0\}$,

$\inf\{\gamma_A(y) \mid y \in F - F_0\} \geq \sup\{\gamma_B(x) \mid x \in K_1 - F_0\}$. Choosing $\inf\{\mu_B(x) \mid x \in K_1 - F_0\} = t_1$ and $\sup\{\gamma_B(x) \mid x \in K_1 - F_0\} = t_1'$ we have $\mu_B(x) \geq t_1$ and $\gamma_B(x) \leq t_1'$ for all $x \in K_1 - F_0$. Therefore $K_1 - F_0 \subseteq (\mu_B)_{t_1}$ the subframe of μ_B and $K_1 - F_0 \subseteq (\gamma_B)_{t_1'}$ the subframe of γ_B. Also $F_0 \subseteq (\mu_B)_{t_0} \subseteq (\mu_B)_{t_1}$ and $F_0 \subseteq (\gamma_B)_{t_0'} \subseteq (\gamma_B)_{t_1'}$. Hence $F_1 \subseteq (\mu_B)_{t_1}$, $F_1 \subseteq (\gamma_B)_{t_1'}$ and so $\mu_B(x) \geq t_1$ and $\gamma_B(x) \leq t_1'$ for all $x \in F_1$. Therefore for all $x \in F_1 - F_0$,

$$\mu_{A^*}(x) = \sup\{\mu_A(y) \mid y \in F - F_0\} \leq \inf\{\mu_B(x) \mid x \in K_1 - F_0\} = t_1 \leq \mu_B(x),$$

$$\gamma_{A^*}(x) = \inf\{\gamma_A(y) \mid y \in F - F_0\} \geq \sup\{\gamma_B(x) \mid x \in K_1 - F_0\} = t_1' \geq \gamma_B(x).$$

Proceeding as above we have $\mu_{A^*}(x) \leq \mu_B(x)$ and $\gamma_{A^*}(x) \geq \gamma_B(x)$ for all $x \in F_i - F_{i-1}$, $i = 2,3\ldots n$. Hence $A^* \leq B$ for all $x \in F$. Therefore the result follows.

4.3 Homomorphisms

Theorem 4.3.1. Let L and M be two frames $f: L \rightarrow M$ be a frame homomorphism and $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ an intuitionistic fuzzy frame of L. Then the image of A under f denoted by $f(A) = \{(y, \mu_{f(A)}(y), \gamma_{f(A)}(y)) \mid y \in M\}$

where $\mu_{f(A)}(y) = \begin{cases} \sup\{\mu_A(x) \mid x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$

$$\gamma_{f(A)}(y) = \begin{cases} \inf\{\gamma_A(x) \mid x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{otherwise} \end{cases}$$

is an intuitionistic fuzzy frame of M.

Proof. For arbitrary $S \subset M$ we have,

$$\mu_{f(A)}(\vee S) = \sup\{\mu_A(x) \mid x \in f^{-1}(\vee S)\}$$

$$\geq \inf\{\sup\{\mu_A(x) \mid x \in f^{-1}(y)\}\}$$

$$= \inf\{\mu_{f(A)}(y)\} \text{ for } y \in S.$$

$$\gamma_{f(A)}(\vee S) = \inf\{\gamma_A(x) \mid x \in f^{-1}(\vee S)\}$$

$$\leq \sup\{\inf\{\gamma_A(x) \mid x \in f^{-1}(y)\}\}$$

$$= \sup\{\gamma_{f(A)}(y)\} \text{ for } y \in S.$$

Again for x, y $\in M$ we have, $\mu_{f(A)}(x \wedge y) \geq \min\{\mu_{f(A)}(x), \mu_{f(A)}(y)\}$ and $\gamma_{f(A)}(x \wedge y) \leq \max\{\gamma_{f(A)}(x), \gamma_{f(A)}(y)\}$.

Also $\mu_{f(A)}(o_M) = \mu_{f(A)}(e_M) \geq \mu_{f(A)}(x)$ and $\gamma_{f(A)}(o_M) = \gamma_{f(A)}(e_M) \geq \gamma_{f(A)}(x)$ for all x $\in M$.

Hence f(A) is an intuitionistic fuzzy frame of M.

Theorem 4.3.2. Let L and M be two frames, f a frame homomorphism from L onto M and B = $\{(x, \mu_B(x), \gamma_B(x)) \mid x \in F\}$ an intuitionistic fuzzy frame of M. Then the preimage of B under f denoted by $f^{-1}(B) = \{(x, \mu_{f^{-1}(B)}(x), \gamma_{f^{-1}(B)}(x)) \mid x \in F\}$ where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x))$ is an intuitionistic fuzzy frame of L.

Proof. Let S be an arbitrary subset of L, then $f(\vee S) \in M$ and is equal to $\vee\{f(x) \mid x \in S\}$.

Now $\mu_{f^{-1}(B)}(\vee S) = \mu_B(f(\vee S)) = \mu_B(\vee \{f(x) \mid x \in S\}) \geq \inf\{\mu_B(f(x)) \mid x \in S\}$

$= \inf\{\mu_{f^{-1}(B)}(x) \mid x \in S\}$

$\gamma_{f^{-1}(B)}(\vee S) = \gamma_B(f(\vee S)) = \gamma_B(\vee \{f(x) \mid x \in S\}) \leq \sup\{\gamma_B(f(x)) \mid x \in S\}$

$= \sup\{\gamma_{f^{-1}(B)}(x) \mid x \in S\}$

Also for arbitrary x, y $\in L$

$\mu_{f^{-1}(B)}(x \wedge y) = \mu_B(f(x \wedge y)) = \mu_B(f(x) \wedge f(y)) \geq \min\{\mu_B(f(x)), \mu_B(f(y))\}$

$= \min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\}$

$\gamma_{f^{-1}(B)}(x \wedge y) = \gamma_B(f(x \wedge y)) = \gamma_B(f(x) \wedge f(y)) \leq \max\{\gamma_B(f(x)), \gamma_B(f(y))\}$

$= \max\{\gamma_{f^{-1}(B)}(x), \gamma_{f^{-1}(B)}(y)\}$

Now since $f(e_L) = f(o_L) \geq f(x)$ for all $x \in L$ we have,

$\mu_{f^{-1}(B)}(e_L) = \mu_B(f(e_L)) = \mu_B(f(o_L)) = \mu_{f^{-1}(B)}(o_L)$ also

$\mu_{f^{-1}(B)}(e_L) = \mu_B(f(e_L)) \geq \mu_B(f(x)) = \mu_{f^{-1}(B)}(x)$ for all $x \in L$

Therefore $\mu_{f^{-1}(B)}(e_L) = \mu_{f^{-1}(B)}(o_L) \geq \mu_{f^{-1}(B)}(x)$ for all $x \in L$.

Similarly we have $\gamma_{f^{-1}(B)}(e_L) = \gamma_{f^{-1}(B)}(o_L) \leq \gamma_{f^{-1}(B)}(x)$ for all $x \in L$.

Therefore $f^{-1}(B)$ is an intuitionistic fuzzy frame of L.

Theorem 4.3.3. Let L and M be two sets $f: L \rightarrow M$ a bijection. If A= $\{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ is an intuitionistic fuzzy set of L then $f^{-1}(f(A)) = A$.

Proof. For arbitrary $x \in L$ let $f(x) = y$. Since f is a bijection we have $f^{-1}(y) = x$. Now $\mu_{f^{-1}(f(A))}(x) = \mu_{f(A)}(f(x)) = \mu_{f(A)}(y) = \sup\{\mu_A(x) \mid x \in f^{-1}(y)\} = \mu_A(x)$ and $\gamma_{f^{-1}(f(A))}(x) = \gamma_{f(A)}(f(x)) = \gamma_{f(A)}(y) = \inf\{\gamma_A(x) \mid x \in f^{-1}(y)\} = \gamma_A(x)$ since f is bijective. Therefore $f^{-1}(f(A)) = A$.

Corollary 4.3.4. Let L and M be two sets $f: L \rightarrow M$ be an isomorphism. If B an intuitionistic fuzzy set of M then $f(f^{-1}(B)) = B$.

Lemma 4.3.5. Let f be a homomorphism from a frame L to a frame M and let A = $\{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ be an intuitionistic fuzzy frame of M then $(f^{-1}(A))_t = f^{-1}(A_t)$ for every $t \in [0, 1]$.

Proof. Let $x \in L$.

Now $x \in (f^{-1}(A))_t \Leftrightarrow \gamma_{f^{-1}(A)}(x) \leq t \leq \mu_{f^{-1}(A)}(x) \Leftrightarrow \gamma_A(f(x)) \leq t \leq \mu_A(f(x))$

$\Leftrightarrow f(x) \in f^{-1}(A_t)$

Therefore $(f^{-1}(A))_t = f^{-1}(A_t)$ for every $t \in I$.

Remark 4.3.6. Theorem 4.3.1 follows by above lemma also since by a theorem the homomorphic preimage of subframe is a subframe, again by Theorem 4.2.15 if A an intuitionistic fuzzy frame of any frame F then every non-empty level subset A_t, $t \in [0,1]$ of A is also a sub frame of F.

Proposition 4.3.7. Let L and M be two frames, $f: L \to M$ a homomorphism. Let A be an intuitionistic fuzzy frame of L then $f(A)_t = f(A_t)$ for every $t \in [0, 1]$.

Proof. We have by Theorem 4.3.1 f(A) an intuitionistic fuzzy frame of M. Also clearly $\operatorname{Im} f(A) \subset \operatorname{Im} A$. Now $f(A)_t = f(A_t)$ for each $t \in \operatorname{Im} f(\mu)$.

For, let $y \in f(A)_t$ then $\gamma_{f(A)}(y) \leq t \leq \mu_{f(A)}(y)$ hence $\inf\{\gamma_A(x) \mid x \in f^{-1}(y)\} \leq t$ and $\sup\{\mu_A(x) \mid x \in f^{-1}(y)\} \geq t$. Choose $x_0 \in L$ such that $f(x_0) = y \in f(A_t)$.

Therefore $f(A)_t \subseteq f(A_t)$. (1)

Let $f(x) \in f(A_t)$ then $x \in A_t$ and hence $\gamma_A(f(x)) \leq t \leq \mu_A(f(x))$. Which implies $\inf\{\gamma_A(z) \mid z \in f^{-1}(f(x))\} \leq t \leq \sup\{\mu_A(z) \mid z \in f^{-1}(f(x))\}$.

Hence $\gamma_{f(A)}(f(x)) \leq t \leq \mu_{f(A)}(f(x))$.

Therefore $f(x) \in f(A)_t$ and hence $f(A_t) \subseteq f(A)_t$ (2)

From (1) and (2) we have $f(A)_t = f(A_t)$.

4.4 Product of Intuitionistic Fuzzy frames

We use notation (A, L) to denote the intuitionistic fuzzy frame A of the frame L.

Definition 4.4.1. Let (A, L) and (B, M) be intuitionistic fuzzy frames where $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in L\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in M\}$. A morphism

$\tilde{f}$: (A, L) →(B, M) is a homomorphism f : L →M such that $\mu_A \leq \mu_B \circ f$ and $\gamma_A \geq \gamma_B \circ f$, that is the degree of membership of x in L does not exceed that of f(x) in M and the degree of non membership of x in L exceed that of f(x) in M. We call the function f: L →M the underlying function of $\tilde{f}$.

Definition 4.4.2. Let $\tilde{f}$: (A, L) →(B, M) and $\tilde{g}$: (B, M) →(C, N) be morphisms then $\tilde{g} \circ \tilde{f}$: (A, L) →(C, N) is a frame homomorphism $g \circ f$: L →M such that $\mu_A \leq \mu_C \circ g \circ f$ and $\gamma_A \geq \gamma_C \circ g \circ f$

Let IFFrm denote a category whose objects are intuitionistic fuzzy frames and morphism as defined above. We have the following theorem

Theorem 4.4.3. The category IFFrm of intuitionistic fuzzy frame has equalizers.

Proof. Let (A, L) and (B, M) be intuitionistic fuzzy frames where A= {(x, μ_A(x), γ_A(x)) | x∈ L } and B = {(x, μ_B(x), γ_B(x)) | x∈ M }.

Let $\tilde{f}$: (A, L) → (B, M) and $\tilde{g}$: (A, L) → (B, M) be two morphisms.

$$\text{Consider } L \underset{g}{\overset{f}{\rightrightarrows}} M$$

Let K = { x∈L | f(x) = g(x) } which is a subframe of L and let $i: K \to L$ be the inclusion map. Then clearly $f \circ i = g \circ i$.

Define an intuitionistic fuzzy set C = {(x, μ_C(x), γ_C(x)) | x∈ K } on K as follows, for a ∈K. let μ_C(a) = μ_A(a) and γ_C(x) = γ_A(x). Then $\tilde{i}$ is morphism from (C, K) to (A, L).

If for arbitrary intuitionistic fuzzy frame (D, N), D $=\{(x, \mu_D(x), \gamma_D(x)) \mid x \in N\}$ $\tilde{h}$ is a morphism from (D, N) to (A, L) such that $f \circ h = g \circ h$ then there exist $\theta : N \rightarrow K$ such that $i \circ \theta = h$.

Also $\mu_D \leq \mu_C \circ \theta$ and $\gamma_D \geq \gamma_C \circ \theta$, as for $z \in N$

$\mu_D(z) \leq (\mu_A \circ h)(z) = (\mu_A \circ i \circ \theta)(z) = (\mu_A \circ i)(\theta(z)) = \mu_C(\theta(z)) = (\mu_C \circ \theta)(z)$

and $\gamma_D(z) \geq (\gamma_A \circ h)(z) = (\gamma_A \circ i \circ \theta)(z) = (\gamma_A \circ i)(\theta(z)) = \gamma_C(\theta(z)) = (\gamma_C \circ \theta)(z)$

Thus $\tilde{\theta}$ is a morphism from (D, N) to (C, K)

Now for $z \in N$,

$(\mu_A \circ i \circ \theta)(z) = (\mu_A \circ i)(\theta(z)) = \mu_A(i(\theta(z))) = \mu_A((i \circ \theta)(z)) = \mu_A(h(z)) = (\mu_A \circ h)(z) \geq \mu_D(z)$ again we have $(\gamma_A \circ i \circ \theta)(z) = (\gamma_A \circ i)(\theta(z)) = \gamma_A(i(\theta(z))) = \gamma_A((i \circ \theta)(z)) = \gamma_A(h(z)) = (\gamma_A \circ h)(z) \leq \gamma_D(z)$

Hence $\mu_A \circ i \circ \theta \geq \mu_D$ and $\gamma_A \circ i \circ \theta \leq \gamma_D$. Therefore the result follows.

Definition 4.4.4. Let $A_\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) \mid x \in F_\alpha\}$ be an intuitionistic fuzzy frame of the frame F_α for $\alpha \in \Lambda$. The product of A_α's is $A = \prod_{\alpha \in \Lambda} A_\alpha$ defined on the product $\mathrm{F} = \prod_{\alpha \in \Lambda} F_\alpha$ with usual order by $A = \{(x, \mu(x), \gamma(x)) \mid x \in \mathrm{F}\}$ where,

$\mu((x_\alpha)_{\alpha \in \Lambda}) = \inf_{\alpha \in \Lambda} \{\mu_\alpha(x_\alpha)\}$ and $\gamma((x_\alpha)_{\alpha \in \Lambda}) = \sup_{\alpha \in \Lambda} \{\gamma_\alpha(x_\alpha)\}$ for $x = (x_\alpha)_{\alpha \in \Lambda} \in \mathrm{F}$.

Proposition 4.4.5. $A = \prod_{\alpha \in \Lambda} A_\alpha$ is an intuitionistic fuzzy frame of $\mathrm{F} = \prod_{\alpha \in \Lambda} F_\alpha$

Proof. We have $\mathrm{F} = \{(a_\alpha)_{\alpha \in \Lambda} \mid a_\alpha \in \mathrm{F}_\alpha \text{ for } \alpha \in \Lambda\}$

$e_{\mathrm{F}} = (e_{F_\alpha})_{\alpha \in \Lambda}$ and $o_{\mathrm{F}} = (o_{F_\alpha})_{\alpha \in \Lambda}$ are respectively the unit and zero element of F.

i) For arbitrary $S \subseteq \mathrm{F}$ we have,

$$\begin{aligned}
\mu(\vee S) &= \mu(\bigvee_x \{(x_\alpha) \mid \alpha \in \Lambda\}) \\
&= \mu((\bigvee_x x_\alpha)_{\alpha \in \Lambda}) \\
&= \inf_{\alpha \in \Lambda}\{\mu_\alpha(\bigvee_x x_\alpha)\} \\
&\geq \inf_{\alpha \in \Lambda}\{\inf_x \{\mu_\alpha(x_\alpha)\}\} \\
&= \inf_x \{\inf_{\alpha \in \Lambda} \{\mu_\alpha(x_\alpha)\}\} = \inf_{x \in S} \mu(x)
\end{aligned}$$

$$\begin{aligned}
\gamma(\vee S) &= \gamma(\bigvee_x \{(x_\alpha) \mid \alpha \in \Lambda\}) \\
&= \gamma((\bigvee_x x_\alpha)_{\alpha \in \Lambda}) \\
&= \sup_{\alpha \in \Lambda}\{\gamma_\alpha(\bigvee_x x_\alpha)\} \\
&\leq \sup_{\alpha \in \Lambda}\{\sup_x \{\gamma_\alpha(x_\alpha)\}\} \\
&= \sup_x \{\sup_{\alpha \in \Lambda} \{\gamma_\alpha(x_\alpha)\}\} = \sup_{x \in S} \{\gamma(x)\}
\end{aligned}$$

ii) For all $x = (x_\alpha)_{\alpha \in \Lambda}$, $y = (y_\alpha)_{\alpha \in \Lambda} \in \mathrm{F}$ we have

$$\mu((x_\alpha)_{\alpha \in \Lambda} \wedge (y_\alpha)_{\alpha \in \Lambda}) = \mu((x_\alpha \wedge y_\alpha)_{\alpha \in \Lambda})$$

$$= \inf_{\alpha\in\Lambda} \{ \mu_\alpha (x_\alpha \wedge y_\alpha) \}$$

$$\geq \inf_{\alpha\in\Lambda} \{ \min\{ \mu_\alpha(x_\alpha), \mu_\alpha(y_\alpha)\}\}$$

$$= \min\{ \inf_{\alpha\in\Lambda} \{\mu_\alpha(x_\alpha)\}, \inf_{\alpha\in\Lambda} \{\mu_\alpha(y_\alpha)\}\}$$

$$= \min\{ \mu(\mathrm{x}), \mu(\mathrm{y})\}$$

$$\gamma((x_\alpha)_{\alpha\in\Lambda} \wedge (y_\alpha)_{\alpha\in\Lambda}) = \gamma((x_\alpha \wedge y_\alpha)_{\alpha\in\Lambda})$$

$$= \sup_{\alpha\in\Lambda} \{ \gamma_\alpha (x_\alpha \wedge y_\alpha) \}$$

$$\leq \sup_{\alpha\in\Lambda} \{ \max\{ \gamma_\alpha(x_\alpha), \gamma_\alpha(y_\alpha)\}\}$$

$$= \max\{ \sup_{\alpha\in\Lambda} \{\gamma_\alpha(x_\alpha)\}, \sup_{\alpha\in\Lambda} \{\gamma_\alpha(y_\alpha)\}\}$$

$$= \max\{ \mu(\mathrm{x}), \mu(\mathrm{y})\}$$

iii) $$\mu(e_{\mathrm{F}}) = \prod_{\alpha\in\Lambda} \mu_\alpha (e_{\mathrm{F}})$$

$$= \inf_{\alpha\in\Lambda} \{\mu_\alpha(e_{F_\alpha})\}$$

$$= \inf_{\alpha\in\Lambda} \{\mu_\alpha(o_{F_\alpha})\}$$

$$= \prod_{\alpha\in\Lambda} \mu_\alpha (o_{\mathrm{F}})$$

$$= \mu(o_{\mathrm{F}})$$

also $$\mu(e_{\mathrm{F}}) = \prod_{\alpha\in\Lambda} \mu_\alpha (e_{\mathrm{F}}) = \inf\{\mu_\alpha(e_{F_\alpha})\}_{\alpha\in\Lambda}$$

$$\geq \inf \{ \mu_\alpha (a_\alpha) \}_{\alpha\in\Lambda}$$

$$= \prod_{\alpha\in\Lambda} \mu_\alpha(a) \text{ for all } a = (a_\alpha)_{\alpha\in\Lambda} \in F$$

$$= \mu(a)$$

Now $\gamma(e_F) = \prod_{\alpha\in\Lambda} \gamma_\alpha(e_F) = \sup_{\alpha\in\Lambda} \{\gamma_\alpha(e_{F_\alpha})\}$

$$= \sup_{\alpha\in\Lambda} \{\gamma_\alpha(o_{F_\alpha})\}$$

$$= \gamma(o_F)$$

also $\gamma(e_F) = \prod_{\alpha\in\Lambda} \gamma_\alpha(e_F) = \sup\{\gamma_\alpha(e_{F_\alpha})\}_{\alpha\in\Lambda}$

$$\leq \sup\{\gamma_\alpha(a_\alpha)\}_{\alpha\in\Lambda}$$

$$= \prod_{\alpha\in\Lambda} \gamma_\alpha(a) \text{ for all } a = (a_\alpha)_{\alpha\in\Lambda} \in F$$

$$= \gamma(a)$$

Hence we have the required result.

Theorem 4.4.6. The category IFFrm of intuitionistic fuzzy frames has products.

Proof. Consider a family of intuitionistic fuzzy frame $\{(A_\alpha, F_\alpha) \mid \alpha \in \Lambda\}$ where $A_\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) \mid x \in F_\alpha\}$. Corresponding to the product $F = \prod_{\alpha\in\Lambda} F_\alpha$ we have an intuitionistic fuzzy frame (A, F) where $A = \prod_{\alpha\in\Lambda} A_\alpha = \{(x, \mu(x), \gamma(x)) \mid x \in F\}$.

Now consider the projection (homomorphism) $P_\alpha : F \to F_\alpha$.

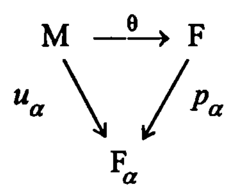
We have $\mu((x_\alpha)_{\alpha\in\Lambda}) = \inf_{\alpha\in\Lambda} \{\mu_\alpha(x_\alpha)\}$ and $\gamma((x_\alpha)_{\alpha\in\Lambda}) = \sup_{\alpha\in\Lambda} \{\gamma_\alpha(x_\alpha)\}$ hence,

$\mu \le \mu_\alpha \circ P_\alpha$ and $\gamma_A \ge \gamma_B \circ f$, for $\alpha \in \Lambda$.

Therefore $\tilde{P}_\alpha$ is morphism from (A, F) to (A_α, F_α) for $\alpha \in \Lambda$.

Now for arbitrary intuitionistic fuzzy frame (B, M) where B = $\{(x, \mu_M(x), \gamma_M(x)) \mid x \in M\}$ if $\tilde{u}_\alpha$ is a morphism from (B, M) to (A_α, F_α). Then define $\theta : M \to F$ as $(\theta(z))_\alpha = P_\alpha(\theta(z)) = (P_\alpha \circ \theta)(z) = u_\alpha(z)$ for all $\alpha \in \Lambda$ and $z \in M$.

Now $\theta(z) = (u_\alpha(z))$ is a frame map as u_α for $\alpha \in \Lambda$ is a frame map.



Also for $z \in M$ we have $\mu_M(z) \le \mu_\alpha \circ u_\alpha(z)$ and $\gamma_M(z) \ge \gamma_\alpha \circ u_\alpha(z)$ for all $\alpha \in \Lambda$ and hence,

$$\mu_M(z) \le \inf_{\alpha\in\Lambda} \mu_\alpha(u_\alpha(z)) = \inf_{\alpha\in\Lambda} \{\mu_\alpha((\theta(z))_\alpha\} = \mu(\theta(z)) = \mu \circ \theta(z) \text{ and}$$

$$\gamma_M(z) \ge \sup_{\alpha\in\Lambda} \gamma_\alpha(u_\alpha(z)) = \sup_{\alpha\in\Lambda} \{\gamma_\alpha((\theta(z))_\alpha\} = \gamma(\theta(z)) = \gamma \circ \theta(z)$$

Hence $\mu_M \le \mu \circ \theta$ and $\gamma_M \ge \gamma \circ \theta$

Thus $\tilde{\theta}$ is a morphism from (B, M) to (A, F)

Clearly $P_\alpha \circ \theta = u_\alpha$ for all $\alpha \in \Lambda$

Also $(\mu_\alpha \circ P_\alpha \circ \theta)(z) = (\mu_\alpha \circ P_\alpha)(\theta(z))$

$$= \mu_\alpha(P_\alpha(\theta(z))$$

$$= \mu_\alpha(u_\alpha(z))$$

$$= (\mu_\alpha \circ u_\alpha)(z)$$

$$\geq \mu_M(z)$$

again $(\gamma_\alpha \circ P_\alpha \circ \theta)(z) = (\gamma_\alpha \circ P_\alpha)(\theta(z))$

$$= \gamma_\alpha(P_\alpha(\theta(z))$$

$$= \gamma_\alpha(u_\alpha(z))$$

$$= (\gamma_\alpha \circ u_\alpha)(z)$$

$$\leq \gamma_M(z)$$

Hence $\mu_M \leq \mu_\alpha \circ P_\alpha \circ \theta$ and $\gamma_M \geq \gamma_\alpha \circ P_\alpha \circ \theta$.

Thus for each family $\{(A_\alpha, F_\alpha) | \alpha \in \Lambda\}$ of intuitionistic fuzzy frames there is an intuitionistic fuzzy frame (A, F) and morphisms $\tilde{P}_\alpha$: (A, F) $\to (A_\alpha, F_\alpha)$ such that for any intuitionistic fuzzy frame (B, M) and family of morphisms $\tilde{u}_\alpha$: (B, M) $\to$ (A_α, F_α) there is a unique morphism $\tilde{\theta}$: (B, M) $\to$ (A, F) such that $P_\alpha \circ \theta = u_\alpha$ and $\mu_M \leq \mu_\alpha \circ P_\alpha \circ \theta$, $\gamma_M \geq \gamma_\alpha \circ P_\alpha \circ \theta$ for all $\alpha \in \Lambda$.

Therefore the result follows.

Theorem 4.4.7. The category IFFrm of intuitionistic fuzzy frames is complete.

Proof. Follows from Theorem 4.4.3 and Theorem 4.4.6. □

Theorem 4.4.8. Let $A_\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) \mid x \in F_\alpha\}$ be an intuitionistic fuzzy set of the frame F_α for $\alpha \in \Lambda$ such that $\prod_{\alpha\in\Lambda} A_\alpha = \{(x, \mu(x), \gamma(x)) \mid x \in F\}$ is an intuitionistic fuzzy frame of $F = \prod_{\alpha\in\Lambda} F_\alpha$ .Now for $x_\alpha \in F_\alpha$ $(\alpha \in \Lambda)$ if $\mu_\alpha(e_{F_\alpha}) = \mu_\alpha(o_{F_\alpha}) \geq \mu_\alpha(x_\alpha)$, $\gamma_\alpha(e_{F_\alpha}) = \gamma_\alpha(o_{F_\alpha}) \leq \gamma_\alpha(x_\alpha)$ for all $\alpha \in \Lambda$ and $\mu_\alpha(e_{F_\alpha}) = \mu_\alpha(e_{F_\beta})$,

$\gamma_\alpha(e_{F_\alpha}) = \gamma_\alpha(e_{F_\beta})$ for all $\alpha, \beta \in \Lambda$ where e_{F_α}, o_{F_α} are respectively the unit and zero element of the frame F_α then A_α is an intuitionistic fuzzy frame of F_α for all $\alpha \in \Lambda$.

Proof. We have $\mu((e_{F_\alpha})_{\alpha\in\Lambda}) = \mu((o_{F_\alpha})_{\alpha\in\Lambda}) \geq \mu((x_\alpha)_{\alpha\in\Lambda})$ and $\gamma((e_{F_\alpha})_{\alpha\in\Lambda}) = \gamma((o_{F_\alpha})_{\alpha\in\Lambda}) \leq \gamma((x_\alpha)_{\alpha\in\Lambda})$ for all $(x_\alpha)_{\alpha\in\Lambda} \in F$, where $(e_{F_\alpha})_{\alpha\in\Lambda}$ and $(o_{F_\alpha})_{\alpha\in\Lambda}$ are respectively the unit and zero elements of the frame F.

Now for $y \in F_\alpha$ consider $(y_\beta)_{\beta\in\Lambda} \in F$ where $y_\beta = \begin{cases} y & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$

Then for all $y \in F_\alpha$, $\mu((y_\beta)_{\beta\in\Lambda}) = \inf_{\beta\in\Lambda}\{\mu_\beta(y_\beta)\} = \mu_\alpha(y)$ and

$\gamma((y_\beta)_{\beta\in\Lambda}) = \sup_{\beta\in\Lambda}\{\gamma_\beta(y_\beta)\} = \gamma_\alpha(y)$

Consider $\alpha \in \Lambda$.

Now for arbitrary $S \subseteq F_\alpha$ we have,

$$\mu_\alpha(\vee S) = \mu((y_\beta)_{\beta\in\Lambda}) \text{ where } y_\beta = \begin{cases} \vee S & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$$

$$= \mu(\bigvee_{x\in S} (x_\beta)_{\beta\in\Lambda}) \text{ where } x_\beta = \begin{cases} x & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$$

$$\geq \inf_{x\in S} \{ \mu((x_\beta)_{\beta\in\Lambda})\}$$

$$= \inf_{x\in S} \{\mu_\alpha(x)\}$$

$$\text{Also } \gamma_\alpha(\vee S) = \gamma((y_\beta)_{\beta\in\Lambda}) \text{ where } y_\beta = \begin{cases} \vee S & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$$

$$= \gamma(\bigvee_{x\in S} (x_\beta)_{\beta\in\Lambda}) \text{ where } x_\beta = \begin{cases} x & \text{if } \beta = \alpha \\ e_{F_\beta} & \text{otherwise} \end{cases}$$

$$\leq \sup_{x\in S} \{ \gamma((x_\beta)_{\beta\in\Lambda})\}$$

$$= \sup_{x\in S} \{\gamma_\alpha(x)\}$$

Similarly it can be shown that for all x, y $\in F_\alpha$,

$\mu_\alpha(x\wedge y) \geq \min\{\mu_\alpha(x), \mu_\alpha(y)\}$ and $\gamma_\alpha(x\wedge y) \leq \max\{\gamma_\alpha(x), \gamma_\alpha(y)\}$

Hence the result follows. □

Let f be a homomorphism on a frame F. If A and B are intuitionistic fuzzy frames of the frame f(F) then A × B is an intuitionistic fuzzy frame of f(F) × f(F). The preimage f^{-1}(A) and f^{-1}(B) are intuitionistic fuzzy frames of F and $(f, f)^{-1}$(A×B) an intuitionistic fuzzy frame of F × F. We study this relation.

Theorem 4.4.9. Let F be a frame and f a homomorphism on F. Let A and B be intuitionistic fuzzy frames of the frame f(F) then $f^{-1}(A) \times f^{-1}(B) = (f, f)^{-1}(A\times B)$.

Proof. Let A = {(y, $\mu_A(y)$, $\gamma_A(y)$) | y∈ f(F)} and B = {(y, $\mu_B(y)$, $\gamma_B(y)$) | y ∈ f(F)}

For all $(x_1, x_2) \in F \times F$ we have,

$$\begin{aligned}
(\mu_A \times \mu_B)\circ(f, f)(x_1, x_2) &= (\mu_A \times \mu_B)(f(x_1), f(x_2)) \\
&= \inf\{\mu_A(f(x_1)), \mu_B(f(x_2))\} \\
&= \inf\{\mu_A\circ f(x_1), \mu_B\circ f(x_2)\} \\
&= (\mu_A\circ f\times \mu_B\circ f)(x_1, x_2)
\end{aligned}$$

also
$$\begin{aligned}
(\gamma_A \times \gamma_B)\circ(f, f)(x_1, x_2) &= (\gamma_A \times \gamma_B)(f(x_1), f(x_2)) \\
&= \sup\{\gamma_A(f(x_1)), \gamma_B(f(x_2))\} \\
&= \sup\{\gamma_A\circ f(x_1), \gamma_B\circ f(x_2)\} \\
&= (\gamma_A\circ f\times \gamma_B\circ f)(x_1, x_2)
\end{aligned}$$

Hence $(\mu_A \times \mu_B)\circ(f, f) = \mu_A\circ f\times \mu_B\circ f$ and $(\gamma_A \times \gamma_B)\circ(f, f) = \gamma_A\circ f\times \gamma_B\circ f$

Therefore the result follows. □

The relation between images of product of intuitionistic fuzzy frames of a frame F is given as follows.

Theorem 4.4.10. Let A and B be intuitionistic fuzzy frames of the frame F. If f is a homomorphism on F, the product f (A) × f (B) and (f, f) (A × B) satisfies $(f, f)(A \times B) \subseteq f(A)\times f(B)$.

Proof. Let $A = \{(y, \mu_A(y), \gamma_A(y)) \mid y \in f(\mathbb{F})\}$ and $B = \{(y, \mu_B(y), \gamma_B(y)) \mid y \in f(\mathbb{F})\}$

$f(A)$ and $f(B)$ are intuitionistic fuzzy frames of $f(\mathbb{F})$ and $f(A) \times f(B)$ is an intuitionistic fuzzy frame of $(f, f)(\mathbb{F} \times \mathbb{F}) = f(\mathbb{F}) \times f(\mathbb{F})$.

Now for each $y = (y_1, y_2) \in f(\mathbb{F}) \times f(\mathbb{F})$ we have,

$[(f, f)(\mu_A \times \mu_B)](y) = \sup\{(\mu_A \times \mu_B)(x) \mid x \in F^{-1}(y)\}$ where $F = (f, f)$ and $x = (x_1, x_2)$

$$= \sup\{\inf(\mu_A(x_1), \mu_B(x_2)) \mid (x_1, x_2) \in F^{-1}(y)\}$$

$$\leq \inf(\sup\{\mu_A(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{\mu_B(x_2) \mid x_2 \in f^{-1}(y_2)\})$$

$$= \inf\{f(\mu_A(y_1)), f(\mu_B(y_2))\}$$

$$= (f(\mu_A) \times f(\mu_B))(y)$$

Also

$[(f, f)(\gamma_A \times \gamma_B)](y) = \sup\{(\gamma_A \times \gamma_B)(x) \mid x \in F^{-1}(y)\}$ where $F = (f, f)$ and $x = (x_1, x_2)$

$$= \sup\{\sup(\gamma_A(x_1), \gamma_B(x_2)) \mid (x_1, x_2) \in F^{-1}(y)\}$$

$$= \sup(\sup\{\gamma_A(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{\gamma_B(x_2) \mid x_2 \in f^{-1}(y_2)\})$$

$$= \sup\{f(\gamma_A(y_1)), f(\gamma_B(y_2))\}$$

$$= (f(\gamma_A) \times f(\gamma_B))(y)$$

That is $(f, f)(\mu_A \times \mu_B) \leq f(\mu_A) \times f(\mu_B)$ and $(f, f)(\gamma_A \times \gamma_B) = f(\gamma_A) \times f(\gamma_B)$

Therefore the result follows. □

CHAPTER 5

INTUITIONISTIC FUZZY QUOTIENT FRAMES

5.1 Introduction

The operations of binary meet and arbitrary join on a frame F can be extended to obtain new operations on the collection of all intuitionistic fuzzy set IFS of F. We define an intuitionistic fuzzy quotient frame of F to be an intuitionistic fuzzy partition of F that is a subset of IFS and having a frame structure with respect to new operations.

5.2 Extended Operations

We extend the operation of binary meet $\wedge$ and arbitrary join $\vee$ on a frame F to operations $\tilde{\wedge}$ and $\tilde{\vee}$ on the set of all intuitionistic fuzzy set IFS of F as follows.

For $\{A_\alpha \mid \alpha \in \Lambda\} \subseteq$ IFS where $A_\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) \mid x \in F\}$ we have,

$$A_\alpha \tilde{\wedge} A_\beta = \{(x, (\mu_\alpha \tilde{\wedge}_1 \mu_\beta)(x), (\gamma_\alpha \tilde{\wedge}_2 \gamma_\beta)(x)) \mid x \in F\}$$

where $(\mu_\alpha \tilde{\wedge}_1 \mu_\beta)(x) = \sup\{\mu_\alpha(y) \wedge \mu_\beta(z) \mid y, z \in F \text{ and } y \wedge z = x\}$ and

$(\gamma_\alpha \tilde{\wedge}_2 \gamma_\beta)(x) = \inf\{\gamma_\alpha(y) \vee \gamma_\beta(z) \mid y, z \in F \text{ and } y \wedge z = x\}$.

$$\tilde{\bigvee_{\alpha\in\Lambda}} A_\alpha = \{(x, (\tilde{\bigvee_{\alpha\in\Lambda}}{}_1\, \mu_\alpha)(x), (\tilde{\bigvee_{\alpha\in\Lambda}}{}_2\, \gamma_\alpha)(x)) \mid x \in F\}$$

where $(\tilde{\bigvee_{\alpha\in\Lambda}}{}_1\, \mu_\alpha)(x) = \sup\{\bigwedge_{\alpha\in\Lambda} \mu_\alpha(x_\alpha) \mid x_\alpha \in F \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x\}$ and

$(\tilde{\bigvee_{\alpha\in\Lambda}}{}_2\, \gamma_\alpha)(x) = \inf\{\bigvee_{\alpha\in\Lambda} \gamma_\alpha(x_\alpha) \mid x_\alpha \in F \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x\}$

Proposition 5.2.1. $A_\alpha \tilde{\wedge} A_\beta$ and $\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha$ are intuitionistic fuzzy set of $\mathbb{F}$.

Proof. Consider $A_\alpha \tilde{\wedge} A_\beta$

Clearly for all $x\in\mathbb{F}$, $0\le (\mu_\alpha \tilde{\wedge}_1 \mu_\beta)(x) \le 1$ and $0\le (\gamma_\alpha \tilde{\wedge}_2 \gamma_\beta)(x) \le 1$. Now for $x\in\mathbb{F}$,

$$
\begin{aligned}
(\mu_\alpha \tilde{\wedge}_1 \mu_\beta)(x) + (\gamma_\alpha \tilde{\wedge}_2 \gamma_\beta)(x) &= \sup(\inf\{\mu_\alpha(y), \mu_\beta(z)\} \mid y, z \in \mathbb{F} \text{ and } y\wedge z = x) \\
&\quad + \inf(\sup\{\gamma_\alpha(y), \gamma_\beta(z)\} \mid y, z \in \mathbb{F} \text{ and } y\wedge z = x) \\
&\le \sup(\inf\{\mu_\alpha(y), \mu_\beta(z)\} \mid y, z \in \mathbb{F} \text{ and } y\wedge z = x) \\
&\quad + \sup(\sup\{\gamma_\alpha(y), \gamma_\beta(z)\} \mid y, z \in \mathbb{F} \text{ and } y\wedge z = x) \\
&= \sup(\inf\{\mu_\alpha(y), \mu_\beta(z)\} + \sup\{\gamma_\alpha(y), \gamma_\beta(z)\} \mid y, z \in \mathbb{F} \text{ and } y\wedge z = x) \\
&\le 1, \text{ for all } x\in\mathbb{F}
\end{aligned}
$$

Again for all $x\in\mathbb{F}$, $0\le (\tilde{\bigvee}_{1\,\alpha\in\Lambda} \mu_\alpha)(x) \le 1$ and $0\le (\tilde{\bigvee}_{2\,\alpha\in\Lambda} \gamma_\alpha)(x) \le 1$. Now for $x\in\mathbb{F}$,

$$
\begin{aligned}
(\tilde{\bigvee}_{1\,\alpha\in\Lambda} \mu_\alpha)(x) + (\tilde{\bigvee}_{2\,\alpha\in\Lambda} \gamma_\alpha)(x) &= \sup(\inf_\alpha \{\mu_\alpha(x_\alpha)\} \mid x_\alpha \in \mathbb{F} \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x) \\
&\quad + \inf(\sup_\alpha \{\gamma_\alpha(x_\alpha)\} \mid x_\alpha \in \mathbb{F} \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x) \\
&\le \sup(\inf_\alpha \{\mu_\alpha(x_\alpha)\} \mid x_\alpha \in \mathbb{F} \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x) \\
&\quad + \sup(\sup_\alpha \{\gamma_\alpha(x_\alpha)\} \mid x_\alpha \in \mathbb{F} \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x) \\
&= \sup(\inf_\alpha \{\mu_\alpha(x_\alpha)\} + \sup_\alpha \{\gamma_\alpha(x_\alpha)\} \mid x_\alpha \in \mathbb{F} \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x) \\
&\le 1, \text{ for all } x\in\mathbb{F}
\end{aligned}
$$

(since for $0\le a \le 1$ if $\inf_\alpha \{\mu_\alpha(x_\alpha)\} \to a$ then we have $\sup_\alpha \{\gamma_\alpha(x_\alpha)\} \to b \le 1-a$ as $\gamma_\alpha(x_\alpha) \le 1-\mu_\alpha(x_\alpha)$)

Hence the result follows. □

The original operation $\wedge$ and $\vee$ on a frame F can be retrieved from $\tilde{\wedge}$ and $\tilde{\vee}$ by embedding F into IFS as the set of all intuitionistic fuzzy singletons each of which is an intuitionistic fuzzy set $I_x = \{(y, \mu_x(y), \gamma_x(y)) \mid x \in F\}$ where $\mu_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$ and $\gamma_x(y) = \begin{cases} 0 & \text{if } y = x \\ 1 & \text{otherwise} \end{cases}$. The largest and smallest elements of IFS are respectively $A_e = \{ (x, \mu_e(x), \gamma_e(x)) \mid x \in F \}$ where $\mu_e(x) = 1$, $\gamma_e(x) = 0$ for all $x \in F$ and $A_o = \{(x, \mu_o(x), \gamma_o(x)) \mid x \in F \}$ where $\mu_o(x) = 0$, $\gamma_o(x) = 1$ for all $x \in F$. It can be observed that for every $A \in$ IFS, $A_o = A_o \,\tilde{\wedge}\, A = A_o \,\tilde{\vee}\, A$ and $A_e \,\tilde{\wedge}\, A \le A_e$, $A_e \,\tilde{\vee}\, A \le A_e$.

Note 5.2.2. IFS is a partial ordered set with largest and smallest element.

Note 5.2.3. For any family $\{A_\alpha \mid \alpha \in \Lambda\}$ of subsets of a frame F, $A_\alpha \,\tilde{\wedge}\, A_\beta = \{a \wedge b \mid a \in A_\alpha, b \in A_\beta\}$ for all $\alpha, \beta \in \Lambda$ and $\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha = \{ \bigvee_{\alpha\in\Lambda} a_\alpha \mid a_\alpha \in A_\alpha \}$.

Definition 5.2.4. **[AT]₂** Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F \}$ be an intuitionistic fuzzy set in F. For any $t \in [0, 1]$ $A_t = \{ x \in F \mid \gamma_A(x) \le t \le \mu_A(x)\}$ is called the level subset of the intuitionistic fuzzy set A.

Proposition 5.2.5. Let $\{A, B\}$ be a pair of intuitionistic fuzzy sets and $\{A_\alpha \mid \alpha \in \Lambda\}$ be a family of intuitionistic fuzzy set of the frame F then for each $t \in [0, 1]$, $A_t \,\tilde{\wedge}\, B_t \subseteq (A \,\tilde{\wedge}\, B)_t$ and $\tilde{\bigvee}_{\alpha\in\Lambda} (A_\alpha)_t \subseteq (\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha)_t$.

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in F\}$

Let $x \in A_t \tilde{\wedge} B_t$ then for some $y, z \in F$ with $x = y \wedge z$ we have $\gamma_A(y) \leq t \leq \mu_A(y)$ and $\gamma_B(z) \leq t \leq \mu_B(z)$.

Then $(\mu_A \tilde{\wedge}_1 \mu_B)(x) = \sup\{\mu_A(y) \wedge \mu_B(z) \mid y, z \in F \text{ and } y \wedge z = x\} \geq t$ and

$$(\gamma_A \tilde{\wedge}_2 \gamma_B)(x) = \inf\{\gamma_A(y) \vee \gamma_B(z) \mid y, z \in F \text{ and } y \wedge z = x\} \leq t$$

Therefore $x \in (A \tilde{\wedge} B)_t$. Hence $A_t \tilde{\wedge} B_t \subseteq (A \tilde{\wedge} B)_t$.

Now for $A_\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) \mid x \in F\}$, if $x \in \tilde{\bigvee}_{\alpha\in\Lambda} (\mu_\alpha)_t$ then for $\{x_\alpha \mid \alpha \in \Lambda\} \subseteq F$ with $x = \bigvee_{\alpha\in\Lambda} x_\alpha$ we have $\gamma_\alpha(x_\alpha) \leq t \leq \mu_\alpha(x_\alpha)$ for all $\alpha \in \Lambda$.

Hence $(\tilde{\bigvee}_{1\,\alpha\in\Lambda} \mu_\alpha)(x) = \sup\{\bigwedge_{\alpha\in\Lambda} \mu_\alpha(x_\alpha) \mid x_\alpha \in F \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x\} \geq t$ and

$$(\tilde{\bigvee}_{2\,\alpha\in\Lambda} \gamma_\alpha)(x) = \inf\{\bigvee_{\alpha\in\Lambda} \gamma_\alpha(x_\alpha) \mid x_\alpha \in F \text{ and } \bigvee_{\alpha\in\Lambda} x_\alpha = x\} \leq t.$$

Therefore $x \in (\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha)_t$. Hence $\tilde{\bigvee}_{\alpha\in\Lambda} (A_\alpha)_t \subseteq (\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha)_t$. □

We give the following definitions for supremum property with respect to binary meet $\wedge$ and arbitrary join $\bigvee$.

Definition 5.2.6. A pair $\{A, B\}$ of intuitionistic fuzzy sets of a frame F where $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in F\}$ has supremum property with respect to $\wedge$ if for any $x \in F$ there exist $y_o, z_o \in F$ with $y_o \wedge z_o = x$ such that $(\mu_A \tilde{\wedge}_1 \mu_B)(x) = \mu_A(y_0) \wedge \mu_B(z_0)$ and $(\gamma_A \tilde{\wedge}_2 \gamma_B)(x) = \gamma_A(y_0) \vee \gamma_B(z_0)$.

Definition 5.2.7. A family $\{A_\alpha \mid \alpha \in \Lambda\}$ of intuitionistic fuzzy sets of a frame F where $A_\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) \mid x \in F\}$ has supremum property with respect to $\vee$ if for any $x \in F$ there exist $\{a_\alpha \mid \alpha \in \Lambda\} \subseteq F$ with $x = \bigvee_{\alpha\in\Lambda} a_\alpha$ such that $(\tilde{\bigvee}_{1\,\alpha\in\Lambda} \mu_\alpha)(x) = \bigwedge_{\alpha\in\Lambda} \mu_\alpha(a_\alpha)$ and $(\tilde{\bigvee}_{2\,\alpha\in\Lambda} \gamma_\alpha)(x) = \bigvee_{\alpha\in\Lambda} \gamma_\alpha(a_\alpha)$.

Definition 5.2.8. A sub collection S of IFS is said to have supremum property with respect $\wedge$ and $\vee$ if every two elements of S has supremum property with respect to $\wedge$ and every arbitrary subset of S has supremum property with respect to $\vee$.

Proposition 5.2.9. Let $\{A, B\}$ be a pair of intuitionistic fuzzy sets and $\{A_\alpha \mid \alpha \in \Lambda\}$ be a family of intuitionistic fuzzy set of the frame F having supremum property with respect to $\wedge$ and $\vee$ respectively then for each $t \in [0, 1]$, $(A \tilde{\wedge} B)_t = A_t \tilde{\wedge} B_t$ and $(\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha)_t = \tilde{\bigvee}_{\alpha\in\Lambda} (A_\alpha)_t$.

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in F\}$. Let $x \in (A \tilde{\wedge} B)_t$, then $(\gamma_A \tilde{\wedge}_2 \gamma_B)(x) \le t \le (\mu_A \tilde{\wedge}_1 \mu_B)(x)$. Since the pair $\{A, B\}$ has supremum property with respect to $\wedge$ there exist $y_o, z_o \in F$ with $y_o \wedge z_o = x$ such that $\gamma_A(y_o) \vee \gamma_B(z_o) \le t \le \mu_A(y_o) \wedge \mu_B(z_o)$.

Hence $\gamma_A(y_o) \le t \le \mu_A(y_o)$, $\gamma_B(z_o) \le t \le \mu_B(z_o)$.

Therefore $y_0 \in A_t$ and $z_0 \in B_t$. Thus $x = y_0 \wedge z_0 \in A_t \tilde{\wedge} B_t$.

Hence $(A \tilde{\wedge} B)_t \subseteq A_t \tilde{\wedge} B_t$.

Therefore by Proposition 5.2.5 we have $(\mu \tilde{\wedge} \gamma)_t = \mu_t \tilde{\wedge} \gamma_t$.

Similarly for $A_\alpha = \{(x, \mu_\alpha(x), \gamma_\alpha(x)) \mid x \in F\}$ if $x \in (\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha)_t$ since $\{A_\alpha \mid \alpha \in \Lambda\}$ has supremum property there exist $\{a_\alpha \mid \alpha \in \Lambda\} \subseteq F$ with $x = \bigvee_{\alpha\in\Lambda} a_\alpha$ such that,

$$\bigvee_{\alpha\in\Lambda} \gamma_\alpha(a_\alpha) \le t \le \bigwedge_{\alpha\in\Lambda} \mu_\alpha(a_\alpha).$$

It follows $\gamma_\alpha(a_\alpha) \le t \le \mu_\alpha(a_\alpha)$ for all $\alpha \in \Lambda$.

Therefore $a_\alpha \in (A_\alpha)_t$ for all $\alpha \in \Lambda$. Thus $x = \bigvee_{\alpha\in\Lambda} a_\alpha \in \tilde{\bigvee}_{\alpha\in\Lambda} (A_\alpha)_t$.

Hence $(\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha)_t \subseteq \tilde{\bigvee}_{\alpha\in\Lambda} (A_\alpha)_t$.

Then by Proposition 5.2.5 we have $(\tilde{\bigvee}_{\alpha\in\Lambda} \mu_\alpha)_t = \tilde{\bigvee}_{\alpha\in\Lambda} (\mu_\alpha)_t$.

Theorem 5.2.10. Let $\{A, B\}$ be a pair of intuitionistic fuzzy sets and $\{A_\alpha \mid \alpha \in \Lambda\}$ be a family of intuitionistic fuzzy set of the frame F having supremum property with respect to $\wedge$ and $\vee$ respectively then $A \tilde{\wedge} B$ and $\tilde{\bigvee}_{\alpha\in\Lambda} A_\alpha$ are intuitionistic fuzzy frame of F.

Proof. To show that $A \tilde{\wedge} B$ is an intuitionistic fuzzy frame of F by Proposition 4.2.12 it is enough to show that each level subset $(A \tilde{\wedge} B)_t$ of $A \tilde{\wedge} B$ is a subframe for $t \in [0,1]$. Since A and B are intuitionistic fuzzy frames of F we have by Proposition 4.2.12 the level subsets A_t and B_t are subframes of F. Since A_t and B_t are subframe $A_t \tilde{\wedge} B_t$ is a subframe of F for $t \in [0, 1]$. Now by Proposition 5.2.9 we have $(A \tilde{\wedge} B)_t = A_t \tilde{\wedge} B_t$.

Therefore $(A \tilde{\wedge} B)_t$ is a subframe of F for all $t \in [0, 1]$. Hence $A \tilde{\wedge} B$ is an intuitionistic fuzzy frame of F. Similarly we have $\tilde{\bigvee}_{\alpha \in \Lambda} A_\alpha$ is an intuitionistic fuzzy frame of F.

Theorem 5.2.11. Let A, B and C be intuitionistic fuzzy sets of a frame F. Then $A \tilde{\wedge} (B \tilde{\vee} C) \leq (A \tilde{\wedge} B) \tilde{\vee} (A \tilde{\wedge} C)$

Proof. Let $A = \{ (x, \mu_A(x), \gamma_A(x)) \mid x \in F \}$, $B = \{ (x, \mu_B(x), \gamma_B(x)) \mid x \in F \}$ and $C = \{(x, \mu_C(x), \gamma_C(x)) \mid x \in F \}$.

$A \tilde{\wedge} (B \tilde{\vee} C) = \{(x, (\mu_A \tilde{\wedge}_1 (\mu_B \tilde{\vee}_1 \mu_C))(x), (\gamma_A \tilde{\wedge}_2 (\gamma_B \tilde{\vee}_2 \gamma_C))(x)) \mid x \in F \}$

Now $(\mu_A \tilde{\wedge}_1 (\mu_B \tilde{\vee}_1 \mu_C))(x) = \sup\{ \mu_A(y) \wedge (\mu_B \tilde{\vee}_1 \mu_C)(z) \mid y, z \in F, y \wedge z = x\}$ (1)

$(\gamma_A \tilde{\wedge}_2 (\gamma_B \tilde{\vee}_2 \gamma_C))(x) = \inf\{ \gamma_A(y) \vee (\gamma_B \tilde{\vee}_2 \gamma_C)(z) \mid y, z \in F, y \wedge z = x\}$ (2)

For arbitrary $y, z \in F$ such that $y \wedge z = x$

$$\begin{aligned}
\mu_A(y) \wedge (\mu_B \tilde{\vee}_1 \mu_C)(z) &= \mu_A(y) \wedge \sup\{ \mu_B(u) \wedge \mu_C(v) \mid u, v \in F, u \vee v = z\} \\
&= \sup\{(\mu_A(y) \wedge \mu_B(u)) \wedge (\mu_A(y) \wedge \mu_C(v)) \mid u, v \in F, u \vee v = z\} \\
&\leq \sup\{(\mu_A(y) \wedge \mu_B(u)) \wedge (\mu_A(y) \wedge \mu_C(v)) \mid u, v \in F, (y \wedge u) \vee (y \wedge v) = y \wedge z\} \\
&\leq \sup\{(\mu_A \tilde{\wedge}_1 \mu_B)(y \wedge u) \wedge (\mu_A \tilde{\wedge}_1 \mu_C)(y \wedge v) \mid u, v \in F, (y \wedge u) \vee (y \wedge v) = y \wedge z\} \\
&= ((\mu_A \tilde{\wedge}_1 \mu_B) \tilde{\vee}_1 (\mu_A \tilde{\wedge}_1 \mu_C))(x) \qquad (3)
\end{aligned}$$

From (1) and (3) we have,

$$(\mu_A \tilde{\wedge}_1 (\mu_B \tilde{\vee}_1 \mu_C))(x) \leq ((\mu_A \tilde{\wedge}_1 \mu_B) \tilde{\vee}_1 (\mu_A \tilde{\wedge}_1 \mu_C))(x) \text{ for all } x \in F \qquad (4)$$

Also $\gamma_A(y) \vee (\gamma_B \tilde{\vee}_2 \gamma_C)(z) = \gamma_A(y) \vee \inf\{\gamma_B(u) \vee \gamma_C(v) \mid u, v \in F, u \vee v = z\}$

$$= \inf\{(\gamma_A(y) \vee \gamma_B(u)) \vee (\gamma_A(y) \vee \gamma_C(v)) \mid u, v \in F, u \vee v = z\}$$

$$\geq \inf\{(\gamma_A(y) \vee \gamma_B(u)) \vee (\gamma_A(y) \vee \gamma_C(v)) \mid u, v \in F, (y \wedge u) \vee (y \wedge v) = y \wedge z\}$$

$$\geq \inf\{(\gamma_A \tilde{\wedge}_2 \gamma_B)(y \wedge u) \wedge (\gamma_A \tilde{\wedge}_2 \gamma_C)(y \wedge v) \mid u, v \in F, (y \wedge u) \vee (y \wedge v) = y \wedge z\}$$

$$= ((\gamma_A \tilde{\wedge}_2 \gamma_B) \tilde{\vee}_2 (\gamma_A \tilde{\wedge}_2 \gamma_C))(x) \qquad (5)$$

From (2) and (5) we have,

$$(\gamma_A \tilde{\wedge}_2 (\gamma_B \tilde{\vee}_2 \gamma_C))(x) \geq ((\gamma_A \tilde{\wedge}_2 \gamma_B) \tilde{\vee}_2 (\gamma_A \tilde{\wedge}_2 \gamma_C))(x) \text{ for all } x \in F \qquad (6)$$

From (4) and (6) we have $A \tilde{\wedge} (B \tilde{\vee} C) \leq (A \tilde{\wedge} B) \tilde{\vee} (A \tilde{\wedge} C)$ □

Similarly we can have the following result

Theorem 5.2.12. For an intuitionistic fuzzy set A and the family of intuitionistic fuzzy set $\{A_\alpha \mid \alpha \in \Lambda\}$ of a frame F $A \tilde{\wedge} (\tilde{\bigvee}_{\alpha \in \Lambda} A_\alpha) \leq \tilde{\bigvee}_{\alpha \in \Lambda} (A \tilde{\wedge} A_\alpha)$.

Theorem 5.2.13. Let S be a sub collection of IFS which is closed with respect to $\tilde{\wedge}$ and $\tilde{\vee}$, and having supremum property with respect to $\wedge$ and $\bigvee$ then $A \tilde{\wedge} (\tilde{\bigvee}_{\alpha \in \Lambda} A_\alpha) = \tilde{\bigvee}_{\alpha \in \Lambda} (A \tilde{\wedge} A_\alpha)$.

Proof. By Proposition 5.2.9 we have $(A \tilde{\wedge} B)_t = A_t \tilde{\wedge} B_t$ and $(\tilde{\bigvee}_{\alpha \in \Lambda} A_\alpha)_t = \tilde{\bigvee}_{\alpha \in \Lambda} (A_\alpha)_t$

Now $(A \tilde{\wedge} (\underset{\alpha\in\Lambda}{\tilde{\vee}} A_\alpha))_t = A_t \tilde{\wedge} (\underset{\alpha\in\Lambda}{\tilde{\vee}} A_\alpha)_t = A_t \tilde{\wedge} (\underset{\alpha\in\Lambda}{\tilde{\vee}} (A_\alpha)_t) = \underset{\alpha\in\Lambda}{\tilde{\vee}} (A_t \tilde{\wedge} (A_\alpha)_t) =$

$\underset{\alpha\in\Lambda}{\tilde{\vee}} (A \tilde{\wedge} A_\alpha)_t = (\underset{\alpha\in\Lambda}{\tilde{\vee}} (A \tilde{\wedge} A_\alpha))_t$ for all $t \in [0, 1]$. Hence the result follows.

Remark 5.2.14. In terms of operations $\tilde{\wedge}$ and $\tilde{\vee}$ the conditions (i) and (ii) for arbitrary intuitionistic fuzzy frame $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$ in Definition 4.2.1 can be rewritten as,

(i)′ $A \supseteq \underset{\alpha\in\Lambda}{\tilde{\vee}} A_\alpha$ where $A_\alpha = A$

(ii)′ $A \supseteq A \tilde{\wedge} A$

Proof. $\underset{\alpha\in\Lambda}{\tilde{\vee}} A_\alpha = \{(x, (\underset{\alpha\in\Lambda}{\tilde{\vee}_1} \mu_\alpha)(x), (\underset{\alpha\in\Lambda}{\tilde{\vee}_2} \gamma_\alpha)(x)) \mid x \in F\}$ where $\mu_\alpha = \mu_A$ and $\gamma_\alpha = \gamma_A$ for $\alpha \in \Lambda$.

For arbitrary $x \in F$ we have,

$$(\underset{\alpha\in\Lambda}{\tilde{\vee}_2} \gamma_\alpha)(x) = \inf\{\underset{\alpha\in\Lambda}{\vee} \gamma_\alpha(a_\alpha) \mid a_\alpha \in F, \underset{\alpha\in\Lambda}{\vee} a_\alpha = x\}$$

$$= \inf\{\underset{\alpha\in\Lambda}{\vee} \gamma_A(a_\alpha) \mid a_\alpha \in F, \underset{\alpha\in\Lambda}{\vee} a_\alpha = x\}$$

$$\geq \inf\{\gamma_A(\underset{\alpha\in\Lambda}{\vee} a_\alpha) \mid a_\alpha \in F, \underset{\alpha\in\Lambda}{\vee} a_\alpha = x\}$$

$$= \gamma_A(x)$$

Hence $\underset{\alpha\in\Lambda}{\tilde{\vee}_2} \gamma_\alpha \geq \gamma_A$. Similarly it can be shown that $\mu_A \geq \underset{\alpha\in\Lambda}{\tilde{\vee}_1} \mu_\alpha$

Also $A \tilde{\wedge} A = \{(x, (\mu_A \tilde{\wedge}_1 \mu_A)(x), (\gamma_A \tilde{\wedge}_2 \gamma_A)(x)) \mid x \in F\}$

For arbitrary $x \in F$ we have,

$(\gamma_A \tilde{\wedge}_2 \gamma_A)(x) = \inf\{\gamma_A(y) \tilde{\vee} \gamma_A(z) \mid y, z \in F \text{ and } y \wedge z = x\}$

$\geq \inf\{\gamma_A(y \wedge z) \mid y, z \in F \text{ and } y \wedge z = x\}$

$= \gamma_A(x)$

Hence $\gamma_A \tilde{\wedge}_2 \gamma_A \geq \gamma_A$. Similarly it can be shown that $\mu_A \tilde{\wedge}_1 \mu_A \leq \mu_A$.

Therefore the result follows.

Remark 5.2.15. Infact equality holds in the above result as,

$(\underset{\alpha\in\Lambda}{\tilde{\vee}_2} \gamma_\alpha)(x) = \inf\{\underset{\alpha\in\Lambda}{\vee} \gamma_\alpha(a_\alpha) \mid a_\alpha \in F, \underset{\alpha\in\Lambda}{\vee} a_\alpha = x\}$

$\leq \gamma_A(x) \vee \gamma_A(o_F) \vee \gamma_A(o_F) \vee \ldots$

$= \gamma_A(x)$

Also $(\gamma_A \tilde{\wedge}_2 \gamma_A)(x) = \inf\{\gamma_A(y) \tilde{\vee} \gamma_A(z) \mid y, z \in F \text{ and } y \wedge z = x\}$

$\leq \gamma_A(x) \vee \gamma_A(e_F) = \gamma_A(x)$

Hence $\underset{\alpha\in\Lambda}{\tilde{\vee}_2} \gamma_\alpha \leq \gamma_A$ and $\gamma_A \tilde{\wedge}_2 \gamma_A \leq \gamma_A$.

Similarly we have $\underset{\alpha\in\Lambda}{\tilde{\vee}_1} \mu_\alpha \geq \mu_A$ and $\mu_A \tilde{\wedge}_1 \mu_A \geq \mu_A$.

5.3 Intuitionistic Fuzzy Quotient Frame

Definition 5.3.1. An intuitionistic fuzzy binary relation R on F Definition 1.5.29 ($R \in IFR(F \times F)$) where $R = \{((x, y), \mu_R(x, y), \gamma_R(x, y)) \mid x, y \in F\}$ is said to be an intuitionistic fuzzy similarity relation if it satisfies for all $x, y, z \in F$

i) Reflexive : $\mu_R(x, x) = 1$, $\gamma_R(x, x) = 0$

ii) Symmetric : $\mu_R(x, y) = \mu_R(y, x)$, $\gamma_R(x, y) = \gamma_R(y, x)$

iii) Transitive : $R \geq R \circ R$ where,

$$R \circ R = \{((x, z), \mu_{R \circ R}(x, z), \gamma_{R \circ R}(x, z)) \mid x \in F, z \in F\}$$

$$\mu_{R \circ R}(x, z) = \sup_y \inf\{\mu_R(x, y), \mu_R(y, z)\}$$

$$\gamma_{R \circ R}(x, z) = \inf_y \sup\{\gamma_R(x, y), \gamma_R(y, z)\}$$

Analogous to that of Definition 3.3.1 we define an intuitionistic fuzzy partition of F as follows

Definition 5.3.2. An intuitionistic fuzzy partition of F is a subcollection P of the collection of intuitionistic fuzzy sets (IFS) whose members satisfy the following three conditions

i) Every $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\} \in P$ is normalized, that is $\mu_A(x) = 1$ for at least one $x \in F$.

ii) For each $x \in F$ there is exactly one $A \in P$ with $\mu_A(x) = 1$ and $\gamma_A(x) = 0$.

iii) If $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in F\}$, $B = \{(x, \mu_B(x), \gamma_B(x)) \mid x \in F\} \in P$ and $x, y \in F$ are such that $\mu_A(x) = \mu_B(y) = 1$, $\gamma_A(y) = \gamma_B(x) = 0$ then

$\mu_A(y) = \mu_B(x) = hgt_1(\mu_A \wedge \mu_B)$ and $\gamma_A(y) = \gamma_B(x) = hgt_2(\gamma_A \vee \gamma_B)$

where $hgt_1(\mu_A \wedge \mu_B) = \sup_{x \in F}(\mu_A(x) \wedge \mu_B(x))$ and

$hgt_2(\gamma_A \vee \gamma_B) = \inf_{x \in F}(\gamma_A(x) \vee \gamma_B(x))$

Given an intuitionistic fuzzy partition P of F and an element $x \in F$, we denote by $[x] = \{(y, \mu_{[x]}(y), \gamma_{[x]}(y)) \mid y \in F\}$ the unique member of P with $\mu_{[x]}(x) = 1, \gamma_{[x]}(x) = 0$ and is called intuitionistic fuzzy similarity class of x.

A 1-1 correspondence between intuitionistic fuzzy partition and intuitionistic fuzzy similarity relation is defined by sending a fuzzy partition $P \subseteq$ IFS to its intuitionistic fuzzy similarity relation R in IFR($F \times F$), where for all $x, y \in F$ we have,

$$\mu_R(x, y) = \mu_{[x]}(y) = \mu_{[y]}(x) = \text{hgt}_1(\mu_{[x]} \wedge \mu_{[y]})$$

$$\gamma_R(x, y) = \gamma_{[x]}(y) = \gamma_{[y]}(x) = \text{hgt}_2(\gamma_{[x]} \vee \gamma_{[y]})$$

The inverse correspondence is defined by sending an intuitionistic fuzzy similarity relation R in IFR($F \times F$) to the intuitionistic fuzzy partition $P_R = \{ R\langle x\rangle \mid x \in F \} \subseteq$ IFS where $R\langle x\rangle$ is the intuitionistic fuzzy set of F defined by ,

$$R\langle x\rangle = \{(y, \mu_R\langle x\rangle(y), \gamma_R\langle x\rangle(y)) \mid y \in F\} \text{ with}$$

$$\mu_R\langle x\rangle(y) = \mu_R(x, y) \text{ and } \gamma_R\langle x\rangle(y) = \gamma_R(x, y).$$

Definition 5.3.3. We call an intuitionistic fuzzy partition P_R of a frame F an intuitionistic fuzzy quotient frame of F if P_R is a subset of IFS and $(P_R, \tilde{\wedge}, \tilde{\vee})$ is a frame .

Theorem 5.3.4. An intuitionistic fuzzy quotient frame P of a frame F satisfies the following properties for all $x, y \in F$ and arbitrary $\{x_\alpha \mid \alpha \in \Lambda\} \subseteq F$

i) $[x] \tilde{\wedge} [y] = [x \wedge y]$

$$\tilde{\bigvee}_{\alpha \in \Lambda} [x_\alpha] = [\bigvee_{\alpha \in \Lambda} x_\alpha]$$

ii) $I_x \tilde{\wedge} [y] = [x \wedge y] = [x] \tilde{\wedge} I_y$

$I_x \tilde{\vee} [y] = [x \vee y] = [x] \tilde{\vee} I_y$

iii) $I_x \tilde{\wedge} [e_F] = [x]$

$I_x \tilde{\vee} [o_F] = [x]$

iv) $[e_F]$ and $[o_F]$ are respectively the identity elements with respect to $\tilde{\wedge}$ and $\tilde{\vee}$.

v) $[x]^c = [x^c]$, where x^c the complement of x in F if it exits.

Proof. i) We have for all $x, y \in F$, $[x \wedge y] = \{(z, \mu_{[x \wedge y]}(z), \gamma_{[x \wedge y]}(z)) \mid z \in F\}$

where $\mu_{[x \wedge y]}(x \wedge y) = 1$, $\gamma_{[x \wedge y]}(x \wedge y) = 0$.

Since P is a frame we have $[x] \tilde{\wedge} [y]$ is in P.

Now $[x] \tilde{\wedge} [y] = \{(z, (\mu_{[x]} \tilde{\wedge}_1 \mu_{[y]})(z), (\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]})(z)) \mid z \in F\}$

where $(\mu_{[x]} \tilde{\wedge}_1 \mu_{[y]})(x \wedge y) = \sup\{\mu_{[x]}(z) \wedge \mu_{[y]}(w) \mid z \wedge w = x \wedge y\}$

$$\geq \mu_{[x]}(x) \wedge \mu_{[y]}(y) = 1$$

$(\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]})(x \wedge y) = \inf\{\gamma_{[x]}(z) \vee \gamma_{[y]}(w) \mid z \wedge w = x \wedge y\}$

$$\leq \gamma_{[x]}(x) \vee \gamma_{[y]}(y) = 0$$

Hence from the definition of intuitionistic fuzzy partition, we have $[x] \tilde{\wedge} [y] = [x \wedge y]$.

Similarly we have $\tilde{\bigvee}_{\alpha \in \Lambda} [x_\alpha] = [\bigvee_{\alpha \in \Lambda} x_\alpha]$ as $\tilde{\bigvee}_{\alpha \in \Lambda} [x_\alpha]$ is in P and

$$(\tilde{\bigvee_1}_{\alpha \in \Lambda} \mu_{[x_\alpha]})(\bigvee_{\alpha \in \Lambda} x_\alpha) = 1, \ (\tilde{\bigvee_2}_{\alpha \in \Lambda} \gamma_{[x_\alpha]})(\bigvee_{\alpha \in \Lambda} x_\alpha) = 0$$

ii) $I_x \tilde{\wedge} [y] = \{(z, (\mu_x \tilde{\wedge}_1 \mu_{[y]})(z), (\gamma_x \tilde{\wedge}_2 \gamma_{[y]})(z)) \mid z \in F\}$ where

$$(\mu_x \tilde{\wedge}_1 \mu_{[y]})(x \wedge y) = \sup\{\mu_x(z) \wedge \mu_{[y]}(w) \mid z \wedge w = x \wedge y\}$$

$$\geq \mu_x(x) \wedge \mu_{[y]}(y) = 1$$

$$(\gamma_x \tilde{\wedge}_2 \gamma_{[y]})(x \wedge y) = \inf\{\gamma_x(z) \vee \gamma_{[y]}(w) \mid z \wedge w = x \wedge y\}$$

$$\leq \gamma_x(x) \wedge \gamma_{[y]}(y) = 0$$

Hence from the definition of intuitionistic fuzzy partition, we have $I_x \tilde{\wedge} [y] = [x \wedge y]$.

Also we have $[x] \tilde{\wedge} I_y = [x \wedge y]$ as $(\mu_{[x]} \tilde{\wedge}_1 \mu_y)(x \wedge y) = 1$ and $(\gamma_{[x]} \tilde{\wedge}_2 \gamma_y)(x \wedge y) = 0$

Similarly we have $I_x \tilde{\vee} [y] = [x \vee y] = [x] \tilde{\vee} I_y$.

iii) Clearly $(\mu_x \tilde{\wedge}_1 \mu_{[e_F]})(x) \geq 1$ and $(\gamma_x \tilde{\wedge}_2 \gamma_{[e_F]})(x) \leq 0$

Hence $I_x \tilde{\wedge} [e_F] = [x]$.

Similarly we have $I_x \tilde{\vee} [o_F] = [x]$.

iv) we have by (i) $[e_F] \tilde{\wedge} [x] = [e_F \wedge x] = [x] = [x] \tilde{\wedge} [e_F]$

also $[o_F] \tilde{\vee} [x] = [o_F \vee x] = [x] = [x] \tilde{\vee} [e_F]$

v) As $[e_F] = [x \vee x^c] = [x] \tilde{\vee} [x^c]$ and $[o_F] = [x \wedge x^c] = [x] \tilde{\wedge} [x^c]$

we have by (i), $[x]^c = [x^c]$

Remark 5.3.5. For all x, y, z $\in F$ we have,

$\mu_{[x]}(y \wedge z) = \mu_{[y \wedge z]}(x)$ by definition of intuitionistic fuzzy partition

$$= (\mu_{[y]} \tilde{\wedge}_1 \mu_{[z]})(x) \geq \mu_{[y]}(x) \wedge \mu_{[z]}(x) = \mu_{[x]}(y) \wedge \mu_{[x]}(z)$$

also $\gamma_{[x]}(y \wedge z) = \gamma_{[y \wedge z]}(x)$ by definition of intuitionistic fuzzy partition

$$= (\gamma_{[y]} \tilde{\wedge}_2 \gamma_{[z]})(x) \leq \gamma_{[y]}(x) \wedge \gamma_{[z]}(x) = \gamma_{[x]}(y) \wedge \gamma_{[x]}(z)$$

Now for $x \in F$ and arbitrary $S = \{x_\alpha \mid \alpha \in \Lambda\} \subseteq F$

$$\mu_{[x]}(\vee S) = \mu_{[\vee S]}(x) = \mu_{[\underset{\alpha\in\Lambda}{\vee} x_\alpha]}(x) = (\underset{\alpha\in\Lambda}{\tilde{\vee}_1}\, \mu_{[x_\alpha]})(x) \geq \underset{\alpha\in\Lambda}{\wedge} \mu_{[x_\alpha]}(x)$$

$$= \underset{\alpha\in\Lambda}{\wedge} \mu_{[x]}(x_\alpha)$$

also $\gamma_{[x]}(\vee S) = \gamma_{[\vee S]}(x) = \gamma_{[\underset{\alpha\in\Lambda}{\vee} x_\alpha]}(x) = (\underset{\alpha\in\Lambda}{\tilde{\vee}_2}\, \gamma_{[x_\alpha]})(x) \leq \underset{\alpha\in\Lambda}{\vee} \gamma_{[x_\alpha]}(x)$

$$= \underset{\alpha\in\Lambda}{\vee} \gamma_{[x]}(x_\alpha)$$

5.4 Invariant fuzzy binary relation

We give the following definition for an invariant intuitionistic fuzzy binary relation.

Definition 5.4.1. An intuitionistic fuzzy binary relation,

$R = \{((x, y), \mu_R(x, y), \gamma_R(x, y)) \mid x, y \in F\}$ on a frame F is invariant if it satisfies for all $x, y, u, v \in F$

i) $\mu_R(x\wedge u, y\wedge v) \geq \mu_R(x, y)$ and $\gamma_R(x\wedge u, y\wedge v) \leq \gamma_R(x, y)$ if $x \neq y$

$\mu_R(x\wedge u, y\wedge v) \leq \mu_R(x, y)$ and $\gamma_R(x\wedge u, y\wedge v) \geq \gamma_R(x, y)$ if $x = y$

ii) $\mu_R(x\vee u, y\vee v) \geq \mu_R(x, y)$ and $\gamma_R(x\vee u, y\vee v) \leq \gamma_R(x, y)$ if $x \neq y$

$\mu_R(x\vee u, y\vee v) \leq \mu_R(x, y)$ and $\gamma_R(x\vee u, y\vee v) \geq \gamma_R(x, y)$ if $x = y$

Theorem 5.4.2. If R is an invariant intuitionistic fuzzy similarity relation on a frame F then its intuitionistic fuzzy partition P_R is a fuzzy quotient frame of F.

Proof. We have for all x, y, z $\in$ F

$$[x] \tilde{\wedge} [y] = \{(z, (\mu_{[x]} \tilde{\wedge}_1 \mu_{[y]})(z), (\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]})(z)) \mid z \in F\}$$

where $(\mu_{[x]} \tilde{\wedge}_1 \mu_{[y]})(z) = \sup\{\mu_{[x]}(u) \wedge \mu_{[y]}(v) \mid u \wedge v = z\}$

$$= \sup\{\mu_R(x, u) \wedge \mu_R(y, v) \mid u \wedge v = z\} \quad \text{(a)}$$

and $(\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]})(z) = \inf\{\gamma_{[x]}(u) \vee \gamma_{[y]}(v) \mid u \wedge v = z\}$

$$= \inf\{\gamma_R(x, u) \vee \gamma_R(y, v) \mid u \wedge v = z\} \quad \text{(b)}$$

Proceeding as in the proof of Theorem 3.4.2 it can be shown that,

$$\mu_{[x]} \tilde{\wedge}_1 \mu_{[y]} = \mu_{[x \wedge y]}$$

Now we shall show that $\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]} = \gamma_{[x \wedge y]}$ for x, y $\in$ F. Consider (b)

Case- I: if $x \neq u$, $y \neq v$

Then $\gamma_R(x, u) \vee \gamma_R(y, v) \geq \gamma_R(x \wedge (x \wedge y), u \wedge (u \wedge v)) \vee \gamma_R(y \wedge (x \wedge y), v \wedge (u \wedge v))$

$$= \gamma_R(x \wedge y, u \wedge v) \vee \gamma_R(x \wedge y, u \wedge v)$$

$$= \gamma_R(x \wedge y, u \wedge v)$$

Therefore, $\inf\{\gamma_R(x, u) \vee \gamma_R(y, v) \mid u \wedge v = z\} \geq \gamma_R(x \wedge y, z) = \gamma_{[x \wedge y]}(z)$ (1)

Case- II : if $x = u$, $y = v$

Then $\gamma_R(x, u) \vee \gamma_R(y, v) = 0 \vee 0 = 0 = \gamma_R(x \wedge y, u \wedge v)$

Therefore, $\inf\{\gamma_R(x, u) \vee \gamma_R(y, v) \mid u \wedge v = z\} = \gamma_R(x \wedge y, z) = \gamma_{[x \wedge y]}(z)$ (2)

Case- III : if $x = u$, $y \neq v$

Then $\gamma_R(x, u) \vee \gamma_R(y, v) \geq 0 \vee \gamma_R(x \wedge y, u \wedge v) = \gamma_R(x \wedge y, u \wedge v)$

Therefore, $\inf\{\gamma_R(x, u) \vee \gamma_R(y, v) \mid u \wedge v = z\} \geq \gamma_R(x \wedge y, z) = \gamma_{[x \wedge y]}(z)$ (3)

A similar case when $x \neq u,\ y = v$

Combining (1) (2) (3) and (b) we have $\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]} \geq \gamma_{[x \wedge y]}$ (4)

Now for all $x, y, z \in F$ consider $\gamma_{[x \wedge y]}(z) = \gamma_R(x \wedge y, z)$ (5)

Case- I : if $x \wedge y = z$

Then $\gamma_R(x \wedge y, z) = \gamma_R(x \wedge y, x \wedge y) = \gamma_R(x, x) \vee \gamma_R(y, y)$

$$= \gamma_{[x]}(x) \vee \gamma_{[y]}(y)$$

$$\geq \inf\{\gamma_{[x]}(u) \vee \gamma_{[y]}(v) \mid u \wedge v = z\}$$

$$= (\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]})(z) \quad (6)$$

Case- II : if $x \wedge y \neq z$

Then $\gamma_R(x \wedge y, z) = \gamma_R(x \wedge y, z) \vee \gamma_R(x \wedge y, z)$

$$\geq \gamma_R((x \wedge y) \vee x, z \vee z) \vee \gamma_R((x \wedge y) \vee y, z \vee z)$$

$$= \gamma_R(x, z) \vee \gamma_R(y, z)$$

$$= \gamma_{[x]}(z) \vee \gamma_{[y]}(z)$$

$$\geq \inf\{\gamma_{[x]}(u) \vee \gamma_{[y]}(v) \mid u \wedge v = z\}$$

$$= (\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]})(z) \quad (7)$$

Combining (5) (6) and (7) we have $\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]} \leq \gamma_{[x \wedge y]}$

Hence from (4) we have $\gamma_{[x]} \tilde{\wedge}_2 \gamma_{[y]} = \gamma_{[x \wedge y]}$.

Therefore $[x] \tilde{\wedge} [y] = [x \wedge y]$

Similarly it can be shown that for arbitrary $\{x_\alpha \mid \alpha \in \Lambda\} \subseteq F$,

$\tilde{\bigvee}_{1\,\alpha\in\Lambda} \mu_{[x_\alpha]} = \mu_{[\bigvee_{\alpha\in\Lambda} x_\alpha]}$ and $\tilde{\bigvee}_{2\,\alpha\in\Lambda} \gamma_{[x_\alpha]} = \gamma_{[\bigvee_{\alpha\in\Lambda} x_\alpha]}$

Therefore $\tilde{\bigvee}_{\alpha\in\Lambda} [x_\alpha] = [\bigvee_{\alpha\in\Lambda} x_\alpha]$.

Now clearly $[e_F]$ and $[0_F]$ are respectively the unit and zero element of element of P_R.

For any $[x] \in P_R$ and arbitrary $S \subset P_R$ we have,

$$[x] \tilde{\wedge} (\tilde{\vee} S) = [x] \tilde{\wedge} [\bigvee_{\alpha\in\Lambda} x_\alpha]$$

$$= [x \wedge (\bigvee_{\alpha\in\Lambda} x_\alpha)]$$

$$= [\bigvee_{\alpha\in\Lambda} (x \wedge x_\alpha)]$$

$$= \tilde{\bigvee}_{\alpha\in\Lambda} [x \wedge x_\alpha]$$

$$= \tilde{\bigvee}_{\alpha\in\Lambda} ([x] \tilde{\wedge} [x_\alpha])$$

Hence P_R satisfies infinite distributive law. Thus P_R is a frame.

Therefore P_R is an intuitionistic fuzzy quotient frame of F.

Remark 5.4.3. Let F be a frame, then the transformation Ω from the set of invariant intuitionistic fuzzy similarity relation on F to the set of intuitionistic fuzzy quotient frames P of F sends an invariant intuitionistic fuzzy similarity relation R on F to its intuitionistic fuzzy partition $P_R \subset$ IFS given by $P_R = \{R\langle x\rangle \mid x \in F\}$

Example 5.4.4. Consider the frame F = {{a, b, c}, {a, b}, {b, c}, {a}, {b}, {c}, ∅} under set inclusion. Define an intuitionistic fuzzy similarity relation R_F on F by,

$$R_F = \{((x, y), \mu_R(x, y), \gamma_R(x, y)) \mid x, y \in F\}$$

where $\mu_R(x, y) = \begin{cases} 1 & \text{if } x = y \\ \frac{1}{2} & \text{otherwise} \end{cases}$ and $\gamma_R(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{4} & \text{otherwise} \end{cases}$

Which is invariant.

Now $P_R = \{[x] \mid x \in F\}$ where $[x](y) = R_F(x, y)$ is an intuitionistic fuzzy partition of F, hence an intuitionistic fuzzy quotient frame.

CHAPTER 6

INTUITIONISTIC FUZZY TOPOLOGICAL SPACES AND FRAMES

6.1 Introduction

The notion of Intutionistic fuzzy topology (Dogen Coker[CO]$_2$) generalises the notion of topology but is still an example of a frame. In this chapter Intutionistic fuzzy "open" and "spectrum" functors, adjoint on the right are constructed. This categorical link between frames and Intutionistic fuzzy topologies are established.

6.2 Preliminaries

Definition 6.2.1.[CO]$_2$ An intutionistic fuzzy topology on a nonempty set X is a family $\mathcal{T}$ of intutionistic fuzzy sets in X satisfying the following axioms,

i) $0_\sim, 1_\sim \in \mathcal{T}$ where $0_\sim = \{(x, 0, 1) \mid x \in X\}$ and $1_\sim = \{(x, 1, 0) \mid x \in X\}$

ii) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$

iii) $\bigcup G_i \in \mathcal{T}$ for any arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \mathcal{T}$

Any intutionistic fuzzy set in $\mathcal{T}$ is known as an intutionistic fuzzy open set in X.

Remark 6.2.2.[CO]$_2$ If $\mathcal{T}$ is an intutionistic fuzzy topology on X, then $(X, \mathcal{T})$ is called an intutionistic fuzzy topological space.

Definition 6.2.3.[CO]$_2$ Let X and Y be two nonempty sets and $f: X \rightarrow Y$ be a function. If $B = \{(y, \mu_B(y), \gamma_B(y)) \mid y \in Y\}$ is an intutionistic fuzzy set in Y, then the preimage of B under f, denoted by $f^{-1}(B)$ is the intutionistic fuzzy set in X defined by

$f^{-1}(B) = \{ (x, \mu_{f^{-1}(B)}(x), \gamma_{f^{-1}(B)}(x)) \mid x \in X \}$ where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x))$

Remark 6.2.4. We also have $f^{-1}(B) = \{(x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)) \mid x \in X\}$ where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$.

Definition 6.2.5. **[CO]₂** Let $f : X \to Y$ be a function and $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two intutionistic fuzzy topological spaces then f is said to be fuzzy continuous if and only if the preimage of each intutionistic fuzzy set in $\mathcal{T}_2$ is an intutionistic fuzzy set in $\mathcal{T}_1$.

Remark 6.2.6. An intutionistic fuzzy topology is infact a frame. The distributivity property is easily verified.

6.3 Intutionistic fuzzy topological spaces and frames

Let Ω be a functor from the category IFTOP of intutionistic fuzzy topological *spaces and fuzzy continuous maps to the category FRM of frames.*

Definition 6.3.1. For each intutionistic fuzzy topological space $(X, \mathcal{T})$ define $\Omega(X, \mathcal{T}) = \mathcal{T}$ if $f: (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is fuzzy continuous define $\Omega f = f^{\leftarrow}(A) = f^{-1}(A)$

Theorem 6.3.2. If $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is a fuzzy continuous map then $f^{\leftarrow} : \mathcal{T}_2 \to \mathcal{T}_1$ is a frame map.

Proof. Let $A = \{(y, \mu_A(y), \gamma_A(y)) \mid y \in Y\}$, $B = \{(y, \mu_B(y), \gamma_B(y)) \mid y \in Y\}$,

$B_i = \{(y, \mu_{B_i}(y), \gamma_{B_i}(y)) \mid y \in Y\}$ $(i \in \Lambda)$ be members of τ_2.

Then
$$
\begin{aligned}
f^{\leftarrow}(A \cap B) &= f^{-1}(A \cap B) \\
&= \{(x, \mu_{f^{-1}(A\cap B)}(x), \gamma_{f^{-1}(A\cap B)}(x)) \mid x \in X\} \\
&= \{(x, \mu_{A\cap B}(f(x)), \gamma_{A\cap B}(f(x))) \mid x \in X\} \\
&= \{(x, \mu_A(f(x)) \wedge \mu_B(f(x)), \gamma_A(f(x)) \vee \gamma_B(f(x))) \mid x \in X\} \\
&= \{(x, \mu_{f^{-1}(A)}(x) \wedge \mu_{f^{-1}(B)}(x), \gamma_{f^{-1}(A)}(x) \vee \gamma_{f^{-1}(B)}(x)) \mid x \in X\} \\
&= \{(x, \mu_{f^{-1}(A)}(x), \gamma_{f^{-1}(A)}(x)) \mid x \in X\} \cap \\
&\qquad \{(x, \mu_{f^{-1}(B)}(x), \gamma_{f^{-1}(B)}(x)) \mid x \in X\} \\
&= f^{\leftarrow}(A) \cap f^{\leftarrow}(B)
\end{aligned}
$$

Also
$$
\begin{aligned}
f^{\leftarrow}\Big(\bigcup_{i\in\Lambda} B_i\Big) &= f^{-1}\Big(\bigcup_{i\in\Lambda} B_i\Big) \\
&= \{(x, \mu_{f^{-1}(\bigcup_{i\in\Lambda} B_i)}(x), \gamma_{f^{-1}(\bigcup_{i\in\Lambda} B_i)}(x)) \mid x \in X\} \\
&= \{(x, \mu_{\bigcup_{i\in\Lambda} B_i}(f(x)), \gamma_{\bigcup_{i\in\Lambda} B_i}(f(x))) \mid x \in X\} \\
&= \{(x, \bigvee_{i\in\Lambda} \mu_{B_i}(f(x)), \bigwedge_{i\in\Lambda} \gamma_{B_i}(f(x))) \mid x \in X\} \\
&= \bigcup_{i\in\Lambda} \{(x, \mu_{B_i}(f(x)), \gamma_{B_i}(f(x))) \mid x \in X\} \\
&= \bigcup_{i\in\Lambda} \{(x, \mu_{f^{-1}(B_i)}(x), \gamma_{f^{-1}(B_i)}(x)) \mid x \in X\} \\
&= \bigcup_{i\in\Lambda} f^{\leftarrow}(B_i)
\end{aligned}
$$

Hence $f^{\leftarrow}: \tau_2 \rightarrow \tau_1$ is a frame map.

Corollary 6.3.3. Ω is a contravariant functor from the category IFTOP of intutionistic fuzzy topological space to the category FRM of frames.

Definition 6.3.4. i) Let $\mathbb{F}$ be a frame define $\Sigma\mathbb{F} = \hom(\mathbb{F}, [0,1])$ where [0,1] is a frame in its usual ordering ($\leq$)

ii) For each $a \in \mathbb{F}$, define $\Sigma_a, \Sigma_a' : \Sigma\mathbb{F} \to [0,1]$ by $\Sigma_a(p) = p(a)$ and $\Sigma_a'(p) = 1 - p(a)$

iii) Let $E_{\Sigma F} = \{ E_a \mid a \in \mathbb{F} \}$ where $E_a = \{(p, \Sigma_a(p), \Sigma_a'(p)) \mid p \in \Sigma\mathbb{F} \}$

Theorem 6.3.5. $(\Sigma\mathbb{F}, E_{\Sigma F})$ is an intutionistic fuzzy topological space.

Proof. We have $\Sigma_1(p) = p(1) = 1$ and $\Sigma_1'(p) = 0$ for all $p \in \Sigma\mathbb{F}$. Also $\Sigma_0(p) = p(0) = 0$ and $\Sigma_0'(p) = 1$ for all $p \in \Sigma\mathbb{F}$. So the top and bottom elements $\{(p, \Sigma_1(p), \Sigma_1'(p)) \mid p \in \Sigma\mathbb{F}\}$ and $\{(p, \Sigma_0(p), \Sigma_0'(p)) \mid p \in \Sigma\mathbb{F} \}$ of the set of all intutionistic fuzzy subsets of $\Sigma\mathbb{F}$ are members of $E_{\Sigma F}$.

Now for $a, b \in \mathbb{F}$ we have,

$$\begin{aligned}
E_a \cap E_b &= \{(p, \Sigma_a(p), \Sigma_a'(p)) \mid p \in \Sigma\mathbb{F} \} \cap \{(p, \Sigma_b(p), \Sigma_b'(p)) \mid p \in \Sigma\mathbb{F} \} \\
&= \{(p, \Sigma_a(p) \wedge \Sigma_b(p), \Sigma_a'(p) \vee \Sigma_b'(p)) \mid p \in \Sigma\mathbb{F} \} \\
&= \{(p, p(a) \wedge p(b), (1 - p(a)) \vee (1 - p(b)) \mid p \in \Sigma\mathbb{F} \} \\
&= \{(p, p(a \wedge b), 1 - p(a \wedge b)) \mid p \in \Sigma\mathbb{F} \} \\
&= \{(p, \Sigma_{a\wedge b}(p), \Sigma'_{a\wedge b}(p)) \mid p \in \Sigma\mathbb{F} \} \\
&= E_{a\wedge b}
\end{aligned}$$

Therefore $E_a \cap E_b \in E_{\Sigma F}$

Also for $a_i \in F$ ($i \in \Lambda$) we have,

$$\bigcup_{i\in\Lambda} E_{a_i} = \bigcup_{i\in\Lambda} \{(p, \Sigma_{a_i}(p), \Sigma'_{a_i}(p)) \mid p \in \Sigma F\}$$

$$= \{(p, \bigvee_{i\in\Lambda} \Sigma_{a_i}(p), \bigwedge_{i\in\Lambda} \Sigma'_{a_i}(p)) \mid p \in \Sigma F\}$$

$$= \{(p, \bigvee_{i\in\Lambda} p(a_i), \bigwedge_{i\in\Lambda} (1 - p(a_i)) \mid p \in \Sigma F\}$$

$$= \{(p, \bigvee_{i\in\Lambda} p(a_i), 1 - \bigvee_{i\in\Lambda} p(a_i)) \mid p \in \Sigma F\}$$

$$= \{(p, p(\bigvee_{i\in\Lambda} a_i), 1 - p(\bigvee_{i\in\Lambda} a_i)) \mid p \in \Sigma F\}$$

$$= \bigcup_{i\in\Lambda} \{(p, \Sigma_{\bigvee_{i\in\Lambda} a_i}(p), \Sigma'_{\bigvee_{i\in\Lambda} a_i}(p)) \mid p \in \Sigma F\}$$

$$= E_{\bigvee_{i\in\Lambda} a_i}$$

Therefore $\bigcup_{i\in\Lambda} E_{a_i} \in E_{\Sigma F}$

Hence $E_{\Sigma F}$ is an intutionistic fuzzy topology.

Theorem 1.14. Let $f: L \to M$ be a frame homomorphism. Define $\Sigma f: \Sigma M \to \Sigma L$ by $\Sigma f(p) = p \circ f$, then $\Sigma f: (\Sigma M, E_{\Sigma M}) \to (\Sigma L, E_{\Sigma L})$ is fuzzy continuous.

Proof. Let $a \in L$ then $E_a = \{(p, \Sigma_a(p), \Sigma_a'(p)) \mid p \in \Sigma L\} \in E_{\Sigma L}$

$$\text{Now } (\Sigma f)^{-1}(E_a) = \{(q, (\Sigma f)^{-1}(\Sigma_a)(q), (\Sigma f)^{-1}(\Sigma_a')(q)) \mid q \in \Sigma M\}$$

$$= \{(q, \Sigma_a(\Sigma f(q)), \Sigma_a'(\Sigma f(q)) \mid q \in \Sigma M\}$$

$= \{(q, \Sigma_a q \circ f, \Sigma_a' q \circ f) \mid q \in \Sigma M\}$

$= \{(q, q \circ f(a), 1 - q \circ f(a)) \mid q \in \Sigma M\}$

$= \{(q, q(f(a)), 1 - q(f(a))) \mid q \in \Sigma M\}$

$= \{(q, \Sigma_{f(a)}(q), \Sigma'_{f(a)}(q) \mid q \in \Sigma M\}$

Therefore $(\Sigma f)^{-1}(E_a) \in E_{\Sigma M}$

Hence Σf is fuzzy continuous.

Corollary 1.15. Σ is a contravariant functor from the category FRM of frames to the category IFTOP of intutionistic fuzzy topological spaces.

Theorem 1.16. Σ and Ω are adjoint on the right

Proof. Let $f \in \hom((X, E), (\Sigma F, E_{\Sigma F}))$ and $g \in \hom(F, \Omega(X, E))$ where ,

$g(a) = \{(x, \mu_{g(a)}(x), \gamma_{g(a)}(x)) \mid x \in X\}$

Define $\bar{f} : F \rightarrow \Omega(X, E)$ by $a \mapsto \bar{f}(a)$

where $\bar{f}(a) = \{(x, \mu_{\bar{f}(a)}(x), \gamma_{\bar{f}(a)}(x)) \mid x \in X\}$

$= \{(x, f(x)(a), 1 - f(x)(a)) \mid x \in X\}$

Claim: $\bar{f}$ is a frame map

For $a, b \in F$,

$\bar{f}(a \wedge b) = \{(x, f(x)(a \wedge b), 1 - f(x)(a \wedge b)) \mid x \in X\}$

$= \{(x, f(x)(a) \wedge f(x)(b), (1 - f(x)(a)) \wedge (1 - f(x)(b))) \mid x \in X\}$

$= \{(x, f(x)(a), 1 - f(x)(a)) \mid x \in X\} \cap$
$\{(x, f(x)(b), 1 - f(x)(b)) \mid x \in X\}$

$= \bar{f}(a) \cap \bar{f}(b)$

Similarly it can be shown that for arbitrary $\{ a_i \mid i \in \Lambda \} \subseteq F$,

$$\bar{f}\,(\bigvee_{i\in\Lambda} a_i) = \bigcup_{i\in\Lambda} \bar{f}\,(a_i)$$

Therefore $\bar{f}$ is a frame map.

Define $\tilde{g} : (X, E) \to (\Sigma F, E_{\Sigma F})$ by $x \mapsto \tilde{g}(x)$, where $\tilde{g}(x)(a) = \mu_{g(a)}(x)$

Claim: $\tilde{g}(x) \in \Sigma F$

Since g is a homomorphism we have, $g(a \wedge b) = g(a) \cap g(b)$

Now it follows from the equality of intuitionistic fuzzy set,

$$\tilde{g}(x)(a \wedge b) = \mu_{g(a \wedge b)}(x) = \mu_{g(a)}(x) \wedge \mu_{g(b)}(x) = \tilde{g}(x)(a) \wedge \tilde{g}(x)(b)$$

Similarly $\tilde{g}(x)(\bigvee_{i\in\Lambda} a_i) = \bigvee_{i\in\Lambda}(\tilde{g}(x)(a_i))$

Claim: $\tilde{g}$ is a fuzzy continuous.

For $a \in F$ we have $E_a \in E_{\Sigma F}$ and

$$\tilde{g}^{-1}(E_a) = \{(x, \tilde{g}^{-1}(\Sigma_a)(x), \tilde{g}^{-1}(\Sigma_a')(x)) \mid x \in X\}$$

$$= \{(x, \Sigma_a \circ \tilde{g}(x), \Sigma_a' \circ \tilde{g}(x)) \mid x \in X\}$$

$$= \{(x, \tilde{g}(x)(a), 1 - \tilde{g}(x)(a)) \mid x \in X\}$$

$$= \{(x, \mu_{g(a)}(x), 1 - \mu_{g(a)}(x)) \mid x \in X\}$$

Now $g(a) \in E$. Hence $\tilde{g}^{-1}(E_a) \in E$

Also $\tilde{\bar{f}}(x)(a) = \mu_{\bar{f}(a)}(x) = f(x)(a)$

$$\tilde{\tilde{g}}(a) = \{ (x,\ \tilde{g}(x)(a),\ 1 - \tilde{g}(x)(a)) \mid x \in X \}$$

$$= \{ (x,\ \mu_{g(a)}(x),\ 1 - \mu_{g(a)}(x)) \mid x \in X \}$$

$$= g(a)$$

So $\tilde{\tilde{f}} = f$ and $\tilde{\tilde{g}} = g$

Thus $\hom(F, \Omega(X, E)) \cong \hom((X, E), (\Sigma F, E_{\Sigma F}))$

Now to check naturality condition

Let $h \in \hom(L, M)$ and $g \in \hom((X, E), (Y, E'))$

Consider

$$\begin{array}{cccccc}
L & & \hom(L, \Omega(X,E)) & \xrightarrow{\widetilde{(-)}} & \hom((X,E), (\Sigma L, E_{\Sigma L})) & \\
h \big\downarrow & (-)\circ h & \big\uparrow & & \big\uparrow & \Sigma h \circ (-) \\
M & & \hom(M, \Omega(X,E)) & \xrightarrow[\widetilde{(-)}]{} & \hom((X,E), (\Sigma M, E_{\Sigma M})) &
\end{array}$$

Then for $k \in \hom(M, \Omega(X, E))$ we have,

$\widetilde{k \circ h}(x)(a) = \mu_{k \circ h(a)}(x)$ and $\Sigma h \circ \tilde{k}(x)(a) = \Sigma h \circ \mu_{k(a)}(x) = \mu_{k \circ h(a)}(x)$

Hence $\widetilde{k \circ h} = \Sigma h \circ \tilde{k}$.

Now consider

$$
\begin{array}{ccccc}
(X, E) & & \hom((X, E), (\Sigma L, E_{\Sigma L})) & \xrightarrow{\overline{(-)}} & \hom(L, \Omega(X, E)) \\
g \downarrow & & \uparrow (-)\circ g & & \uparrow \Omega g \circ (-) \\
(Y, E') & & \hom((Y, E'), (\Sigma L, E_{\Sigma L})) & \xrightarrow[\overline{(-)}]{} & \hom(L, \Omega(Y, E'))
\end{array}
$$

Then for $f \in \hom((Y, E'), (\Sigma L, E_{\Sigma L}))$ we have,

$$\overline{f \circ g}(a) = \{ (x, \mu_{\overline{f \circ g}(a)}(x), 1 - \mu_{\overline{f \circ g}(a)}(x)) \mid x \in X \}$$

$$= \{ (x, f \circ g(x)(a), 1 - f \circ g(x)(a)) \mid x \in X \}$$

and $\Omega g \circ \bar{f}(a) = g^{-1} \{ (x, \mu_{\bar{f}(a)}(x), 1 - \mu_{\bar{f}(a)}(x)) \mid x \in X \}$

$$= \{ (x, \mu_{\bar{f}(a)}(g(x)), 1 - \mu_{\bar{f}(a)}(g(x))) \mid x \in X \}$$

$$= \{ (x, f \circ g(x)(a), 1 - f \circ g(x)(a)) \mid x \in X \}$$

Hence $\overline{f \circ g} = \Omega g \circ \bar{f}$

Therefore naturality condition holds.

Hence the result follows.

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