Some Problems in Topology and Related Areas

FUZZY TOPOLOGICAL GAMES AND RELATED TOPICS

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CERTIFICATE

This is to certify that the thesis entitled "Fuzzy Topological Games and Related Topics" is an authentic record of research carried out by Sri. Sunil Jacob John under my supervision and guidance in the Department of Mathematics, Cochin University of Science and Technology for the PhD degree of the Cochin University of Science and Technology and no part of it has previously formed the basis for the award of any other degree or diploma in any other university.

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INTRODUCTION

Decision-making under uncertainty is as old as mankind. Just like most of the real world systems in which human perception and intuitive judgement play important roles, the conventional approaches to the analysis of large scale systems were ineffective in dealing with systems that are complex and mathematically ill defined. Thus an answer to capture the concept of imprecision in a way that would differentiate imprecision from uncertainty, the very simple idea put forward by the American cyberneticist L.A Zadeh [ZA] as the generalization of the concept of the characteristic function of a set to allow for immediate grades of membership was the genesis of the concept of a fuzzy set.

In mathematics a subset A of X can be equivalently represented by its characteristic function – a mapping χ_A from the universe X of discourse (region of consideration i.e., a larger set) containing A to the two element set $\{0,1\}$. That is to say x belongs to A if and only if $\chi_A(x) = 1$. But in the "fuzzy" case the "belonging to" relation $\chi_A(x)$ between x and A is no longer "either 0 or otherwise1", but it has a membership degree belonging to $[0,1]$ instead of $\{0,1\}$, or more generally, to a lattice L, because all membership degrees in mathematical view form an ordered structure, a lattice. A mapping from X to a lattice L is called a generalized characteristic function and it describes the fuzziness of the set in general. A fuzzy set on a universe X is simply just a mapping from X to a lattice L.

Even though Zadeh used $[0,1]$ as the value set of fuzzy sets, later many researchers working on different aspects of fuzzy sets especially in fuzzy topology modified the concept using different kinds of lattices for the membership value set. Important among them are L-Fuzzy sets of Goguen [GO], where L is an arbitrary lattice with minimum and maximum elements 0 and 1 respectively, complete distributive lattice with 0 and 1 by Gantner and others [G; S; W], complete and completely distributive lattice equipped with an order reversing involution by Bruce Hutton [HU], complete and completely distributive non-atomic Boolean algebra by Mira Sarkar [MI_2], complete

Brouwerian lattice with its dual also Brouwerian by Ulrich Hohle [HO] and complete distributive lattice by Rodabaugh [R].

Thus the fuzzy set theory extended the basic mathematical concept of a set. Owing to the fact that set theory is the corner stone of modern mathematics, a new and more general framework of mathematics was established. Fuzzy mathematics is just a kind of mathematics developed in this frame work . Hence in a certain sense, fuzzy mathematics is the kind of mathematical theory which contains wider content than the classical theory. Also it has found numerous applications in different fields such as Linguistics, Robotics, Pattern Recognition, Expert Systems, Military Control, Artificial Intelligence, Psychology, Taxonomy, and Economics.

Fuzzy topology is just a kind of topology developed on fuzzy sets and in his very first paper Chang [C] gives a strong basement for the development of fuzzy topology in the $[0,1]$ membership value framework. Compactness and its different versions are always important concepts in topology. In fuzzy topology, after the initial work of straight description of ordinary compactness in the pattern of covers of a whole space, many authors tried to establish various reasonable notions of compactness with consideration of various levels in terms of fuzzy open sets and obtained many important results. Since the level structures or in other words stratification of fuzzy open sets is involved, compactness in fuzzy topological spaces is one of the most complicated problems in this field. Many kinds of fuzzy compactness using different tools were raised, and each of them has its own advantages and shortcomings. In [LO] Lowen gives a comparative study of different compactness notions introduced by himself, Chang, T.E Gantner, R.C Steinlage, R.H Warren etc and all the value domains used in these notions are $[0,1]$.

Gantner and others [G;S;W] used the concept of shading families to study compactness and related topics in fuzzy topology. The shading families are a very natural generalization of coverings and in particular, a 1^*-shading family of fuzzy sets is a fuzzy covering in the sense of Chang [C]. Using these concepts Malghan and Benchalli [M; B_1]

defined point finite and locally finite families of fuzzy sets and introduced the concept of fuzzy paracompact spaces. Later Mao-Kang, Luo [MA] gives another version using quasi-coincidence relation and α-Q-covers. Arya and Purushottam [A;P] has some results regarding fuzzy metacompact spaces using fuzzy covers. As a continuation of these works, in the first chapter we introduce metacompactness in fuzzy topological spaces through α-shadings.

A combinatorial game in a mathematical way was first described in the beginning of the 17^{th} century. More particularly Bachet De Meziriac [BA] gave the following game called Nim. Two players alternatively choose numbers between *1* and *10*, the player on whose move the sum attains *100* is the winner. Bouten [BO] studied Nim and has many interesting results. A comprehensive study on the history of game theory is found in Worobjow [WO]. Game theoretic methods have found great many applications in topology. Topological studies in game theory arose from the famous Banach-Mazur game. This game related to the Baire category theorem was proposed by Mazur in 1935 and was solved by Banach in the same year. Hence the game came to be known as Banach-Mazur game.

The term 'topological game' was introduced by Berge [BE] . Following an analogy with topological groups, Berge originated the study of positional games of the form $G(X, \varphi)$ where X is a topological space and $\varphi : X \rightarrow P(X)$ is an upper and /or lower semi continuous multi-valued map assigning to a position x the set $\varphi(x)$ of the next legal position. ($P(X)$ has vietoris topology). If $\varphi(x) = \phi$ then x is a terminal position.

A somewhat different meaning for topological game was proposed by Telgarsky [T_2] (Who is the main initial contributor to this field). This term has an analogy with matrix games, differential games, statistical games etc, so that topological games are defined and studied within topology. In a topological game players choose some objects related to the topological structure of a space such as points, closed subsets, open covers etc. More over the condition on a play to be winning for a player may also include topological notions such as closure, convergence, etc. It turns out that topological games are related to (or can be used to define) the Baire property, Baire spaces, Completeness properties, Convergence properties, Separation properties, Covering and Base properties, Continuous images, Suslin sets, Singular spaces etc.

There are various frame works and notions for infinite positional game of perfect information. But the following are the most widely used ones. We shall always consider games of two players, called Player I and Player II where Player I starts the play (i.e., he makes the first move). Unless otherwise stated, a play of a game is a sequence of size ω, and the result of a play is either a win or loss for each player. A strategy of Player II is a function defined for each legal finite sequence of moves of Player I. A strategy for Player I is defined similarly. A stationary strategy is a strategy which depends on the opponent's last move only. A markov strategy is a strategywhich depends only on the ordinal number of the move and opponent's last move.

As we have stated earlier, a pursuit evasion game *G (K, X)* in which the pursuer and the evader choose certain subsets of a topological space *X* in a certain way is defined and studied by Telgarsky in [T_2]. Although the game resembles that of Banach-Mazur, it provides for completely different methods and problems to be introduced. Establishing the close relation between spaces of the class ***K*** and the space *X* in case of winning strategy for one of the player make it possible to prove many theorems for different types of topological spaces.

The main purpose of our study is to extend the concept of the topological game *G(**K**, X)* and some other kinds of games in to fuzzy topological games and to obtain some results regarding them. Owing to the fact that topological games have plenty of applications in covering properties, we have made an attempt to explore some inter relations of games and covering properties in fuzzy topological spaces. Even though our main focus is on fuzzy para-meta compact spaces and closure preserving shading families, some brief sketches regarding fuzzy *P*-spaces and Shading Dimension is also provided. As a pre-requisite to this study, we are compelled to do some work on fuzzy coverings also.

In the first chapter we collect the basic definitions and notions which are required in the succeeding sections. The main results obtained include a characterization of metacompactness in fuzzy topological spaces and a study of the behavior of α-metacompact spaces under perfect maps.

In Chapter II, fuzzy topological games, fuzzy winning strategies, stationary winning strategies, etc are defined and some results related to them are obtained. The main results are the equivalence of existence of winning strategies and existence of stationary winning strategies for player I in the game $G^*(\boldsymbol{K},X)$ and the equivalence of existence of fuzzy winning strategies of Player I in the game $G^*(\boldsymbol{K},X)$ and of that in $G^*(\boldsymbol{FK},X)$ .Again the behaviour of games under perfect maps is also investigated.

Chapter III deals with closure preserving shading families, countable α-compactness and some games associated with them. Also a complete characterisation of closure preserving shading families by fuzzy sets with finite support is provided. For this we introduce and make use of the concept of fuzzy $\boldsymbol{K}$-scattered spaces. Here we define the concepts of accumulation points and cluster points in a language which is closely related to that of shading families and in this frame work obtain a characterization for countable compactness in fuzzy topological spaces.

In Chapter IV, we have introduced and studied fuzzy P-spaces. The main result obtained is a characterization of fuzzy P-spaces in terms of a particular type of fuzzy topological game $G_\alpha(X)$.

Games in product fuzzy topological spaces are discussed in Chapter V. The main results are the existence of fuzzy winning strategies for Player I in $G^*(\boldsymbol{D}(\boldsymbol{K}_1 \times \boldsymbol{K}_2), X \times Y)$ if he has the same in both $G^*(\boldsymbol{K}_1, X)$ and $G^*(\boldsymbol{K}_2, Y)$.Here we make use of the concepts like fuzzy rectangles , D-products etc.

Chapter VI deals with some applications of games in product α-para, α-meta compact spaces and fuzzy covering dimension. Every product space discussed will have a winning strategy in some particular kind of fuzzy topological game.Further a fuzzy version of countable sum theorem for covering dimension in terms of fuzzy topological games is also obtained.

The idea of fuzzy sets introduced by Zadeh [ZA] using the unit interval $[0,1]$ to describe and deal with the non-crisp phenomena and procedures was generalized by Goguen [GO] using some lattice L instead of $[0,1]$. Through out the main body of the thesis we have been using the $[0,1]$ fuzzy set up. However all these discussions can be carried out in the L-fuzzy set up which in it self will yield interesting results. As a model , we give characterisations of metacompactness and covering dimension in the L-fuzzy context , where L is a complete and completely distributive lattice equipped with an order reversing involution. These constitute Appendixes I and II. Besides obtaining complete characterization of metacompactness and covering dimension in weakly induced L-fuzzy topological spaces, it is also shown that the extensions obtained are good in the sense of Ying-Ming and Mao-Kang [Y; M].

Chapter - I
PRELIMINARIES AND BASIC CONCEPTS

In this chapter we collect the basic definitions and obtain some pre-requisites, which will be used in the subsequent chapters.

1.1 Fuzzy Sets, Basic Operations and Fuzzy Topology

In his classical paper Zadeh [ZA] first introduced the concept of fuzzy sets as a class of objects with a continuum of grades of membership. Such a set is characterised by a membership function which assigns to each object a grade of membership ranging between *0* and *1*. An immediate application of this based on the operations of union, intersection, complementation of sets etc can be found in the theory of general topology. All these constitute a rich body of theory which is largely parallel to that of general topology and is called the theory of fuzzy topology. In fact general topology comes as a particular case of fuzzy topology and this theory was put forward by Chang [C].

We follow the original definitions of Zadeh [ZA] and Chang [C] for fuzzy sets and fuzzy topology respectively.

1.1.1 Definition [ZA] Let X be a set. A fuzzy set A in X is characterised by a membership function $x \to \mu_A(x)$ from X to the unit interval $I = [0,1]$.

Let A and B be fuzzy sets in X

Some Results mentioned in this Chapter are published in the paper titled *Fuzzy topological Games and α-metacompactness* in the Proceedings of the Annual Conference of the Kerala Mathematical Association and the National Conference on Analysis and Applications (1999) pp 89 - 91

Then

$A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ for all $x \in X$

$A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in X$

$C = A \vee B \Leftrightarrow \mu_C(x) = \text{Max } \{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

$C = A \wedge B \Leftrightarrow \mu_C(x) = \text{Min } \{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

Complement of A, $A' = E \Leftrightarrow \mu_E(x) = 1 \setminus \mu_A(x)$ for all $x \in X$

More generally, for a family of fuzzy sets $A = \{A_i : i \in I\}$,

The union $C = \cup_{i \in I} A_i$ and intersection $D = \cap_{i \in I} A_i$ are defined by

$$\mu_C(x) = \underset{i \in I}{Sup}\{\mu_{A_i}(x)\} \quad x \in X \text{ and}$$

$$\mu_D(x) = \underset{i \in I}{Inf}\{\mu_{A_i}(x)\} \quad x \in X$$

The symbol 0 and 1 will be used to to denote the empty fuzzy set ($\mu_\phi(x) = 0$ for all $x \in X$) and the full set X ($\mu_X(x) = 1$ for all $x \in X$) respectively.

1.1.2 Definition[C] A fuzzy topology on X is a family T of fuzzy sets in X which satisfies the following conditions.

(i) $0, 1 \in T$.

(ii) If $A, B \in T$, then $A \wedge B \in T$

(ii) If $A_i \in T$ for each $i \in I$, then $\cup_{i \in I} A_i \in T$.

T is called a fuzzy topology on X, and the pair (X,T) is a fuzzy topological space (fts).Every member of T is called a T-open fuzzy set (or simply open fuzzy set).A fuzzy set is called T-closed (or simply closed) if and only if its complement is T-open.

1.1.3 Definition [C] Let A be a fuzzy set in a fuzzy topological space (X,T).The largest open fuzzy set contained in A is called the interior of A and is denoted by $int\ A$. The smallest closed fuzzy set containing A is called the closure of A and is denoted by $cl\ A$.

1.1.4 Definition[C] Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function μ_B. Then the inverse of B, written as $f^{-1}[B]$, is a fuzzy set on X whose membership function is defined by $\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \forall x \in X$. On the other hand, let A be a fuzzy set in X with membership function μ_A. The image of A, written as $f[A]$, is a fuzzy set in Y whose membership function is given by

$$\mu_{f[A]}(y) = \underset{z \in f^{-1}(y)}{Sup}\{\mu_A(z)\} \quad \text{if } f^{-1}[y] \text{ is not empty}$$

$$= 0 \qquad \text{otherwise} \qquad \text{for all } y \in Y$$

where $f^{-1}(y) = \{x \mid f(x) = y\}$.

1.1.5 Definition[C] A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,U) is F-continuous iff the inverse of each U-open fuzzy set is T-open.

1.1.6 Result[C] A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,U) is F-continuous iff the inverse of each U-closed fuzzy set is T-closed.

1.1.7 Definition [C] A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,U) is F-open (resp.F-closed) iff it maps an open (resp. closed) fuzzy set in (X,T) on to an open (resp. closed) fuzzy set in (Y,U).

1.1.8 Definition [C] Let T be a fuzzy topology. A subfamily B of T is a base for T iff each member of T can be expressed as the join of some members of B.

1.1.9 Definition[C] Let T be a fuzzy topology. A subfamily S of T is a sub base for T iff the family of finite meets of S forms a base for T.

1.1.10 Definition[C] Let (X,T) be a fuzzy topological space. A family $\mathcal{A}$ of fuzzy sets is a cover of a fuzzy set B iff $B \le \cup\{A: A \in \mathcal{A}\}$. It is an open cover iff each member of $\mathcal{A}$ is an open fuzzy set. A sub cover of $\mathcal{A}$ is a sub family which is also a cover.

1.2 Shading Families

The notion of shading families was introduced in the literature by Gantner and others [G;S;W] as a very natural generalisation of coverings during the investigation of compactness in fuzzy topological spaces. In fact, In particular a 1^*-shading family of fuzzy sets is a covering in the sense of Chang [C].

1.2.1 Definition[G;S;W] Let (X,T) be a fuzzy topological space and $\alpha \in (0,1)$. A collection U of fuzzy sets is called an α-shading (resp. α^*- shading) of X if for each $x \in X$, there exists $g \in U$ with $g(x) > \alpha$ (resp. $g(x) \geq \alpha$). A sub-collection of an α-shading (resp. α^*- subshading) of X which is also an α-shading (resp. α^*- shading) is called an α-sub shading (resp. α^*- sub shading) of X. In a similar manner we can define 1^*-shading and 0-shading also.

1.2.2 Definition [G;S;W] A fuzzy topological space X is said to be α-compact (resp. α^*-compact) if each α- shading (resp. α^*- shading) of X by open fuzzy sets has a finite α-sub shading (resp. α^*- sub shading) , where $\alpha \in [0,1]$.

1.2.3 Definition [M;B_1] A fuzzy topological space X is said to be countably α-compact (resp.countably α^*- compact) if every countable α- shading (resp. α^*- shading) of X by open fuzzy sets has a finite α-sub shading (resp. α^*- sub shading) ,where $\alpha \in [0,1]$.

1.2.4 Definition [M;B_1] A fuzzy topological space X is said to be α -Lindelof (resp. α^*-Lindelof) if every α- shading (resp. α^*- shading) of X by open fuzzy sets has a countable α-subshading (resp. α^*- subshading) , where $\alpha \in [0,1]$.

1.2.5 Definition [M;B_1] Let X be a set. Let U and V be any two collections of fuzzy subsets of X. Then U is a refinement of V ($U < V$) if for each $g \in U$ there is an $h \in V$ such that $g \leq h$. If U , V, W are collections such that $U < V$ and $U < W$ then U is called a common refinement of V and W.

Note that any α-sub shading (resp. α^*-sub shading) of a given α-shading (resp. α^*-shading) is a refinement of that α-shading (resp. α^*-shading) .

1.2.6 Definition[M;B1] A refinement $\{b_t : t \in T\}$of $\{a_s : s \in S\}$is said to be precise if $T = S$ and $b_s \leq a_s$ for each $s \in S$.

1.2.7 Theorem[M;B1] Let $\{ a_s \}$ and $\{ b_t \}$ be two α-shadings (resp. α^*-shadings) of a fuzzy topological space (X,T) where $\alpha \in [0,1]$. Then

(i) $\{a_s \wedge b_t\}$is an α-shading of X which refines both $\{ a_s \}$ and $\{ b_t \}$. Further, if both $\{a_s\}$ and $\{ b_t \}$ are locally finite (point finite) so is $\{a_s \wedge b_t\}$.

(ii) Any common refinement of $\{ a_s \}$ and $\{ b_t \}$is also a refinement of $\{a_s \wedge b_t\}$.

1. 3 A Characterisation of α-Metacompactness

An approach to fuzzy paracompactness using the notion of shading families was introduced by Malghan and Benchalli [M;B1] . We extend in this section the concept of metacompactness to fuzzy topological spaces in terms of α-shadings and obtain a characterisation for the same.

1.3.1 Definition [M;B1] A family $\{a_s : s \in S \}$of fuzzy sets in a fuzzy topological space (X,T) is said to be locally finite if for each x in X there exists an open fuzzy set g with $g(x) = 1$ such that $a_s \leq 1 \setminus g$ holds for all but at most finitely many s in S.

1.3.2 Definition [M; B1] A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space (X,T) is said to be point finite if for each x *in* X, $a_s(x) = 0$ for all but at most finitely many s in S. Or equivalently as $a_s(x) > 0$ for at most finitely many s in S.

1.3.3 Proposition [M; B1] Let $\{a_s : s \in S\}$ be locally finite family of fuzzy sets in a fuzzy topological space (X,T). Then

(1) $\{cl\, a_s : s \in S\}$is also locally finite.

(2) For each $S' \subset S$, $\vee$ $\{cl\, a_s : s \in S'\}$ is a closed fuzzy set.

1.3.4 Definition [M; B_1] A collection $\{A_i : i \in I\}$ of fuzzy subsets of fuzzy topological space X is said to be closure preserving if for each $J \subseteq I$, cl $[\vee A_i: i \in J]$ $= \vee$ $[cl\, A_i: i \in J]$

1.3.5 Proposition [M; B_1] Every locally finite family of fuzzy sets in a fuzzy topological space is closure preserving.

1.3.6 Definition [M; B_1] A fuzzy topological space (X, T) is said to be α-paracompact (resp. α*-paracompact) if each α-shading (resp. α*- shading) of X by open fuzzy sets has a locally finite α-shading (resp. α*-shading) refinement by open fuzzy sets.

1.3.7 Definition.A fuzzy topological space (X, T) is said to be α-metacompact (resp.α*-metacompact) if each α-shading (resp. α*- shading) of X by open fuzzy sets has a point finite α-shading.(resp. α*- shading) refinement by open fuzzy sets.

1.3.8 Remark It is interesting to notice that α-metacompact will not imply β-metacompact and β-metacompact will not imply α-metacompact when $\alpha < \beta$ where α, $\beta \in [0,1]$. This stems from the fact that we are considering the relationship between two statements , each having two doubly quantified shadings.

1.3.9 Lemma Let $U = \{U_\lambda : \lambda \in \Delta\}$ be an α-shading of X by open fuzzy sets with Δ well ordered . Let $V_\lambda = \underset{\beta \le \lambda}{Sup} U_\beta$ for each $\lambda \in \Delta$. If $\{V_\lambda : \lambda \in \Delta\}$has a precise point finite refinement by open fuzzy sets $\{W_\lambda : \lambda \in \Delta\}$ and each of $1 \setminus \underset{\gamma > \lambda}{Sup} W_\gamma$ has a point finite α-shading by open fuzzy sets which is a partial refinement of $\{U_\beta : \beta \le \lambda\}$. Then U has a point finite open refinement.

Proof

Assume that $W_\lambda \neq 0$ implies $W_\lambda \neq W_\beta$ if $\lambda \neq \beta$. Let S_λ be a point finite α-shading of $1 \setminus \underset{\gamma>\lambda}{Sup} W_\gamma$ for each $\lambda \in \Delta$. Also S_λ is a partial refinement of $\{U_\beta : \beta \leq \lambda\}$. Therefore it follows that $S \in S_\lambda$ implies that $S < U_\beta$ for some $\beta \leq \lambda$.

Take $P_\lambda = \{W_\lambda \wedge S : S \in S_\lambda,\ S \leq U_\beta$ for some $\beta \leq \lambda\}$. Let $H = \cup\{P_\lambda : \lambda \in \Delta\}$. Any $h \in H$ is of the form $h = W_\lambda \wedge S$ for some $\lambda \in \Delta$ such that $S \in S_\lambda$ and $S \leq U_\beta$ for some $\beta \leq \lambda$. Therefore $h(x) = W_\lambda(x) \wedge S(x)$ for every x in X. Since $\{W_\lambda : \lambda \in \Delta\}$ and S_λ are point finite, so is their intersection. Therefore it follows that $h(x) > 0$ for at most finitely many $h \in H$. Thus H is a point finite open collection. Also $h \in H$ implies that $h < U_\beta$ for some β. For, $h = W_\lambda \wedge S$ for some $S \in S_\lambda$. Since S_λ is partial refinement of $\{U_\beta : \beta \leq \lambda\}$, $S \leq U_\beta$ for some $\beta \leq \lambda$. Therefore $h < U_\beta$ for some β.

Let $x \in X$. Now $\{\lambda \in \Delta : W_\lambda(x) > \alpha\}$ is finite since $\{W_\lambda : \lambda \in \Delta\}$ is point finite. Let δ be the greatest element. Therefore $[1 \setminus \underset{\gamma>\delta}{Sup} W_\gamma](x) > 0$. But S_δ is a point finite α–shading of $1 \setminus \underset{\gamma>\delta}{Sup} W_\gamma$. Therefore $t(x) > \alpha$ for some $t \in S_\delta$. Now take $h = W_\delta \wedge t$ where $t < U_\beta$ for some $\beta \leq \delta$. Then $h(x) = [W_\delta \wedge t](x)$

$$= W_\delta(x) \wedge t(x)$$

$> \alpha$ since $\{W_\lambda : \lambda \in \Delta\}$ is an α–shading of X and $\{S_\lambda : \lambda \in \Delta\}$ is an α–shading of $1 \setminus \underset{\gamma>\lambda}{Sup} W_\gamma$. Therefore it follows that H is an α–shading of X, which completes the proof.

1.3.10 Definition An α–shading V is said to be a point wise w-refinement of an α–shading U if for any $x \in X$, there is a finite $K \subset U$ such that if $V(x) > 0$ with $V \in V$, then $V < U$ for some $U \in K$.

1.3.11 Lemma If $\{U_n\}_1^{\infty}$ is a sequence of α–shadings of X by open fuzzy sets such that U_{n+1} is a point wise w-refinement of U_n for each $n \in N$, then U_1 has a σ–point finite refinement by open fuzzy sets.

Proof

Take $U_1 = \{U_\lambda : \lambda \in \Delta\}$ with Δ well ordered. If $U \in \cup\{U_n : n \in N\}$, we denote $\delta(U)$ as the smallest $\beta \in \Delta$ such that $U < U_\beta$. Then for each $n > 1$, take $W_n = \{ W \in U_n : \delta(W) = \delta(U)$ whenever $U \in U_{n+1}$ and $W < U \}$. We will prove that $\cup\{ W_n : n \in N\}$is an α–shading of X. Let $x \in X$ and for every $n > 1$,take $\lambda_n = Sup\{\delta(U) : U \in U_n$ and $U(x) > \alpha \}$. Clearly λ_n exists since U_n is a point wise w-refinement of U_{n-1}. Also $\lambda_1 > \lambda_2 > \lambda_3 > \ldots\ldots$, so there is some $\gamma \in \Delta$ and $m \in N$ such that $\lambda_k = \gamma$ or all $k \geq m$. Now U_{m+2} is a point wise w-refinement of U_{m+1}.Therefore for each $x \in X$, there is a finite $K \subset U_{m+1}$, such that if $U(x) > \alpha$ with $U \in U_{m+1}$ and $U < V$ for some $V \in K$. Therefore $\{ U \in U_{m+2} : U(x) > \alpha\}$is a partial refinement of K . Clearly there is some $K \in K$ with $\delta(K) = \gamma$, otherwise we are left with $\lambda_{m+2} > \lambda_{m+1}$ which is not possible.

If $U \in U_m$ with $K < U$, we have $\gamma = \delta(K) \leq \delta(U) \leq \lambda_m = \gamma$. Therefore $\delta(K) = \delta(U)$ and hence it follows that $K \in W_{m+1}$.Also $\{U \in W_{m+2} : U(x) > \alpha\}$ is a partial refinement of K. Therefore $U < K$ for some $U \in U_{m+2}$. Clearly $K(x) > \alpha$. Thus $\cup\{W_n : n \in N\}$ is an α-shading of X.

Now we will construct a σ-point finite refinement. Let $V_{n\beta} = \cup\{W \in W_n : \delta(W) = \beta\}$ for any $n > 1$, $\beta \in \Delta$. If $V_n = \{ V_{n\beta} : \beta \in \Delta\}$. The collection $\cup\{V_n : n \in N\}$ is an α-shading of X and refines U_1. We will show that each V_n is point finite. Let $\Delta' \subset \Delta$ be such that $V_{n\beta}(x) > \alpha$ for every $\alpha \in \Delta'$. Pick corresponding $W_\beta \in W_n$ with $W_\beta(x) > \alpha$ and $\delta(W_\beta) = \beta$ for every $\beta \in \Delta'$. We know that each U_{n+1} is a point wise w-refinement of U_n and $W_n \subset U_n$. Therefore it follows that there is a finite $H \subset U_{n-1}$ such that $\{W_\alpha : \alpha \in \Delta'\}$is a partial refinement of H .By definition of W_n, we have $W_\alpha < H$ for $H \in H$ implies $\alpha = \delta(W_\alpha) = \delta(H)$. Now since H is finite, Δ' is finite and the lemma is complete.

1.3.12 Definition A collection U of fuzzy subsets of a fuzzy topological space X is said to be interior preserving if $Int\,(\wedge\{W{:}W\in W\}) = \wedge\,(Int\,\{W:W\in W\})$ for every $W\subset U$.

1.3.13 Definition A collection U of fuzzy subsets of a fuzzy topological space X is said to be well monotone if the subset relation ' < ' is a well order on U.

1.3.14 Definition A collection U of fuzzy subsets of a fuzzy topological space X is said to be directed if $U, V\in U$ implies there exists a $W\in U$ such that $U\vee V< W$.

1.3.15 Result A well monotone collection of open fuzzy sets is interior preserving and directed.

Proof

Proof follows from definitions 1.3.13, 1.3.14 and the fact that if U is a well monotone collection of open fuzzy sets, then so is $\{Int\,U : U\in U\}$.

1.3.16 Definition Let X be a fuzzy topological space and H be an α-shading of X. Then for any $x\in X$, we define $St(x, H) = \vee\,\{h\in H : h(x)>0\}$.

1.3.17 Lemma If an α-shading U of X by open fuzzy sets has a point finite α-shading refinement H such that $x\in Int\,(St(x, H))$ for every $x\in X$, then U has an open point wise w-refinement .

Proof

Since H is a refinement of U, for $h\in H$, take $U_h\in U$ such that $h < U_h$. For any $x\in X$, let $V_x = [Int\,(St(x,H)] \wedge Inf\,\{U_h{:}h\in H$ and $h(x)>0\}$. Now the collection $V=\{V_x{:}\ x\in X\}$ is the discrete point wise w-refinement of U by open fuzzy sets. For, each $x\in X$ we want to find out a finite $K\subset U$ such that if $V_x(x)>0$ with $V_x\in V$, then $V_x<K$ for some $K\in K$. Now take $K=\{U_h{:}\ U_h>h,\ h(x)>0\}$. Since H is point finite, clearly K is finite and $K\subset U$. This completes the proof.

1.3.18 Lemma If U is an interior preserving α–shading of X by open fuzzy sets, then U^F has a closure preserving closed refinement if and only if U has an interior preserving point wise w-refinement by open fuzzy sets, where U^F is the collection of all unions of finite sub-collections from U.

Proof

If F is a closure preserving closed refinement of U^F and $x \in X$, then let $V_x = [Inf\ \{U{:}U \in U \text{ and } U(x)>0\}] \setminus [Sup\ \{F{:}F \in F \text{ and } F(x)=0\}]$. Then the collection $\{V_x{:}x \in X\}$ is an interior preserving point wise w-refinement of U by open fuzzy sets .

Conversely suppose V is an interior preserving point wise w-refinement of U by open fuzzy sets. For $U \in U$, let $P_U = \{x \in X{:}\ St\ (x, V) < U\}$. Then $P = \{\ P_U{:}\ U \in U^F\ \}$ is a closure preserving closed refinement of U^F.

1.3.19 Lemma If U is a point finite α-shading of X, then U^F has a closure preserving closed refinement.

Proof

We know that a point finite α-shading of X by open fuzzy sets is an interior preserving open point wise w-refinement of itself. Therefore lemma follows from lemma 1.3.18 above.

1.4 Main Theorem

1.4.1 Theorem For any fuzzy topological space (X, T) the following are equivalent.

(i) X is α–metacompact

(ii) Every α–shading U of X by open fuzzy sets has a point finite refinement H such that $Int\ (St(x,H))(x) > 0$ for every $x \in X$.

(iii) Every α–shading U of X by open fuzzy sets has a point wise w-refinement by open fuzzy sets.

(iv) Every well-monotone α -shading of X by open fuzzy sets has a point finite open refinement.

(v) Every directed α-shading of X by open fuzzy sets has a closure preserving closed refinement.

(vi) For every α-shading $\boldsymbol{U}$ of X by open fuzzy sets, $\boldsymbol{U}^F$ has a closure preserving closed refinement.

Proof

Trivially (i) $\Rightarrow$(ii)

(ii) $\Rightarrow$(iii) follows from lemma 1.3.17

(iii) $\Rightarrow$ (i)

From repeated application of (iii) and lemma 1.3.11 it follows that if $\boldsymbol{U}$ is an α-shading of X by open fuzzy sets , then $\boldsymbol{U}$ has an α-shading refinement $\cup\{V_n : n \in N\}$ such that each V_n is a point finite collection of open fuzzy sets. For each $n \geq 1$ take $G_n = Sup\{V{:}V \in V_k , k \leq n\}$ and let $\boldsymbol{W}$ be a point wise w-refinement of $\boldsymbol{G} = \{G_n : n \in N\}$. Now $\boldsymbol{G}$ is directed and hence $\{St(x, \boldsymbol{W}) : x \in X\}$refines $\boldsymbol{G}$. Now if $\boldsymbol{P}_n = \{x : St(x, \boldsymbol{W}) < \boldsymbol{G}_n\}$ then $cl\, \boldsymbol{P}_n < G_n$ and $X = \vee \{\boldsymbol{P}_n : n \in N\}$.

Take $H_n = [\, V \setminus \vee P_k : N \in n, k < N\}$. Then $\boldsymbol{H} = \cup \{H_n : n \in N\}$ is a point finite open refinement of $\boldsymbol{U}$. This completes the proof of (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv)

Clearly follows from the definition of α-metacompactness.

(iv) $\Rightarrow$ (i).

Suppose that (iv) is true. Then if possible let X be not α-metacompact. Then there is a smallest cardinal number μ such that there exists an α-shading $\boldsymbol{U}$ of X by open fuzzy sets with no point finite open refinement and $|\, \boldsymbol{U} \,| = \mu$. Therefore every α-shading $\boldsymbol{W}$ of X by open fuzzy sets with $|\, \boldsymbol{W} \,| < |\, \boldsymbol{U} \,|$ has a point finite open refinement. Express $\boldsymbol{U}$ as $\boldsymbol{U} = \{U_\lambda : \lambda < \mu\}$ and take $V_\lambda = \text{Sup}\{\, U_\beta : \beta < \lambda\}$for each $\lambda < \mu$ .Clearly the collection $V = \{\, V_\lambda : \lambda < \mu \,\}$ is a well monotone α-shading of X. Then by (iv) we have point finite (precise) α-shading refinement $\{\, W_\lambda : \lambda < \mu \,\}$ by open fuzzy sets. Now let $F_\lambda = 1 \setminus \text{Sup}\{W_\beta : \beta > \lambda\}$ for every $\lambda < \mu$. Then $(1 \setminus F_\lambda) \cup \{\, U_\beta : \beta \geq \lambda\}$ is an open α-shading of X with cardinality less than μ..And by the minimality of μ it should have a point finite refinement by open fuzzy sets say $\boldsymbol{I}_\lambda$ .Take $\boldsymbol{S}_\lambda = \{\, I \in \boldsymbol{I}_\lambda : I \wedge F_\lambda \neq 0\}$ Then from lemma

1.3.9 it follows that U must have a point finite α-shading refinement by open fuzzy sets. This is a contradiction and hence the proof of (iv) $\Rightarrow$ (i) is complete.

(v)$\Leftrightarrow$ (vi)

U^F is the collection of all unions of finite sub collections from U. Clearly U^F is directed and hence has a closure preserving closed refinement.

Conversely let U be a directed α-shading of X by open fuzzy sets. Clearly U^F is a refinement of U and by (vi) U^F has a closure preserving closed refinement say V. Then $V< U^F< U$. Therefore it follows that V is a closure preserving closed refinement of U

(i) $\Rightarrow$ (vi)

Given that X is α-metacompact. Therefore every α-shading U of X by open fuzzy sets has a point finite α-shading refinement say V. Then by lemma 1.3.19, V^F has a closure preserving closed refinement. Since $V< U$ implies $V^F< U^F$ the proof of (i) $\Rightarrow$ (vi) is complete.

(vi) $\Rightarrow$ (iv)

Let U be a well monotone α-shading of X by open fuzzy sets. Then by Result 1.3.15 it follows that U is interior preserving. Now U^F is always directed and by (v) we get U^F has a closure preserving refinement by closed fuzzy sets. Then by lemma 1.3.18 , U^F has an interior preserving point wise w-refinement by open fuzzy sets, say U_2. Take $U_1 = U$. Then by repeated use of lemma 1.3.18, we get a sequence $(U_n)_1^\infty$ of α-shadings of X by open fuzzy sets such that U_{n+1} is an interior preserving point wise w- refinement of U_n. Then by lemma 1.3.11 U has an open refinement $\{V_n: n\in N\}$ where each V_n is point finite. For each $n\in N$, take $g_n = Sup\ \{ V : V\in V_k , k \leq n\}$. Now $\{g_n: n\in N\}$is a directed α-shading of X by open fuzzy sets and must have a closure preserving refinement by fuzzy closed sets say F. Then F may be expressed as $\{F_n: n\in N\}$. where $F_n < g_n$. Take $H_n = \{V \setminus \underset{k<n}{Sup} F_n : V\in V_n \}$. Consider $H = \cup \{ H_n : n\in N\}$. This is point finite for, let $x\in X$, we want to prove that $h(x)>0$ for at most finitely many $h\in H$. Now every $h\in H$ is of the form $V \setminus \underset{k<n}{Sup} F_n$ for some $V\in V_n$ .If possible let $h(x)>0$ for infinitely many $h\in H$. Now clearly $V(x)>0$ for infinitely many $V\in V_n$. This is a contradiction since each V_n is point finite. Therefore $\boldsymbol{H}$ is a point finite α-shading refinement of $\boldsymbol{U}$ by open fuzzy sets. This completes the proof.

Now we give an example of α-metacompact space which is not α- paracompact.

1.4.2 Example

Let X be the deleted Tychnoff plank $T_\infty = T \setminus \{(\omega_1, \omega)\}$ where T is the Tychnoff's plank given by $[0,\omega_1]\times[0,\omega]$ where ω_1 is the first uncountable ordinal and ω is the first infinite ordinal. Let $\alpha\in[0,1)$ be any number. Define for each $\varsigma\in[0,\omega)$ and $\beta\in[0,\omega_1)$, $U_\varsigma^\beta = \{(\beta,\gamma):\varsigma<\gamma\leq\omega\}$ and for each $\lambda\in[0,\omega_1)$ and $\delta\in[0,\omega)$ $V_\lambda^\delta = \{(\gamma,\delta):\lambda<\gamma\leq\omega_1\}$. Let T be the fuzzy topology generated by taking each point p of $[0,\omega_1)\times[0,\omega)$ as fuzzy points with value η where $\alpha<\eta\leq 1$ and characterestic functions of U_ς^β and V_λ^δ as the open sets. Now (X,T) is α-metacompact. For, any α-shading of X by open fuzzy sets has a refinement consisting of one basic neighbourhood for each $x\in X$. Any such α-shading refinement $\boldsymbol{U}$ is point finite, since an arbitrary point $x\in X$ can have at most three members of $\boldsymbol{U}$ such that $U(x)>\alpha$ where $U\in\boldsymbol{U}$.

Now the space (X, T) is not α -paracompact. For, consider the α -shading of X by sets $U_0 = X\setminus B$ and $U_n = V_0^{n-1}$ for $n = 1,2,3...$ where $B=\chi_A$ where $A=\{(\omega, n) : 0\leq n<\omega\}$ has no locally finite refinement . For, if possible let $\{W_\mu\}$ be a locally finite refinement. Now for each $n\in N$, we may define an ordinal α_n of to be the least ordinal such that characterestic function of $V_{\alpha_n}^n$ is contained in just one W_μ. If $\alpha = Sup\alpha_n < \omega_1$, every neighbourhood of (α , ω) will have non zero meet with infinitely many members of $\{W_\mu\}$.

1.5 Metacompactness and Mappings

1.5.1 Proposition Let $f: X \xrightarrow{onto} Y$ be an F-closed F-continuous mapping, where X and Y are fuzzy topological spaces. Then if $\{U_\alpha: \alpha\in\Lambda\}$ is a closure preserving family of fuzzy sets in X then so is $\{f(U_\alpha): \alpha\in\Lambda\}$

Proof

Since f is F-continuous, it follows clearly that $f(cl\ U_\alpha) \leq cl\, f(U_\alpha)$ for every $\alpha\in\Lambda$.

Now we have $U_\alpha \leq cl\ U_\alpha$ for every $\alpha\in\Lambda$.

Therefore $f(U_\alpha) \leq f(cl\ U_\alpha)$.

That is $cl\ [f(U_\alpha)] \leq$ cl $[f(cl\ U_\alpha)]$.

$= f(cl\ U_\alpha)$ since f is F-closed

Therefore we get $cl\ [f(U_\alpha)] = f(cl\ U_\alpha)$ for every $\alpha\in\Lambda$.

Now for any collection $\{f(U_\alpha): \alpha\in\Lambda\}$, clearly we have

$$\vee_{\alpha\in\Lambda} cl\ [f(U_\alpha)] \leq \text{cl}\ [\vee\{f(U_\alpha): \alpha\in\Lambda\}]$$

Again $f(U_\alpha) \leq cl\ [f(U_\alpha)]$

$= f(cl\ U_\alpha)$.

Therefore we have $\vee\{f(U_\alpha): \alpha\in\Lambda\} \leq \vee\{f(cl\ U_\alpha) : \alpha\in\Lambda\}$

That is $cl\ [\vee\{f(U_\alpha): \alpha\in\Lambda\}] \leq cl\ [\vee\{f(cl\ U_\alpha) : \alpha\in\Lambda\}]$

$= cl\ [f\ [\vee\ (cl\ U_\alpha)\ : \alpha\in\Lambda]\ ]$

$= cl\ [f(\,cl\ [\vee\ \{\ U_\alpha\ : \alpha\in\Lambda\})\ ]$ since $\{U_\alpha: \alpha\in\Lambda\}$ *is* closure preserving

$= f\ (cl\ [\vee\ \{U_\alpha\ : \alpha\in\Lambda\}]\ )$ since F is F-closed

$= f\ (\vee\ \{\ cl\ U_\alpha\ : \alpha\in\Lambda\})$

$= \vee\ \{f\ (cl\ U_\alpha) : \alpha\in\Lambda\}$

$= \vee\ \{\ cl\ [f\ (\ U_\alpha) : \alpha\in\Lambda]\ \}$

Thus we get $\vee_{\alpha\in\Lambda}\ cl\ [f(U_\alpha)] \geq$ cl $[\vee\{f(U_\alpha): \alpha\in\Lambda\}]$

And hence we have $\vee_{\alpha\in\Lambda}\ cl\ [f(U_\alpha)] =$ cl $[\vee\{f(U_\alpha): \alpha\in\Lambda\}]$

This completes the proof.

1.5.2 Proposition Let X and Y be two fuzzy topological spaces and let $f: X \xrightarrow{onto} Y$ be finite to one. If $\boldsymbol{U} = \{U_\alpha: \alpha \in \Lambda\}$ is a point finite collection of fuzzy sets in X, then $\{f(U_\alpha): \alpha \in \Lambda\}$ is also a point finite collection in Y.

Proof

Given that f is on to and finite to one. Therefore for every $y \in Y$, we have a finite (support) fuzzy subset $f^1(y)$ in X. Let $x \in f^{-1}(y)$. Then since $\{U_\alpha: \alpha \in \Lambda\}$ is a point finite collection in X, $U_\alpha(x) > 0$ for at most finitely many $\alpha \in \Lambda$. Now since $f^{-1}(y)$ is finite, we get a finite sub-collection $\boldsymbol{U_F}$ of $\boldsymbol{U}$. Now consider the collection $\{f(u_F): u_F \in \boldsymbol{U_F}\}$. This is finite and $f(u_F)(y) > 0$ for all $u_F \in \boldsymbol{U_F}$. Thus $\{f(U_\alpha): \alpha \in \Lambda\}$ is a point finite collection in Y.

1.5.3 Theorem Let X and Y be two fuzzy topological spaces and let $f: X \xrightarrow{onto} Y$ be a finite to one F-open F-continuous mapping. If X is α-metacompact then so is Y.

Proof:

Given that X is α-metacompact. Let $\boldsymbol{U}$ be an α-shading of Y by open fuzzy sets. Since f is F-continuous, it follows that $\boldsymbol{U'} = \{f^{-1}(U) : U \in \boldsymbol{U}\}$ is an α- shading of X by open fuzzy sets. Since X is α-metacompact , it follows that $\boldsymbol{U'}$ has a point finite α-shading refinement by open fuzzy sets say $\boldsymbol{V}$. Now clearly $\{f(V): V \in \boldsymbol{V}\}$ is a point finite α-shading of Y and it refines $\boldsymbol{U}$ also. Since f is F-open, $f(V)$ is also open. Hence Y is α-metacompact.

1.5.4 Theorem. Let $f: X \xrightarrow{into} Y$ be F-continuous, F-closed function. If X is α-metacompact , then Y is also α-metacompact.

Proof

Let $\boldsymbol{U}$ be an α-shading of Y by open fuzzy sets. Then by a characterization of α-metacompactness in 1.4.1, it is enough to prove $\boldsymbol{U^F}$ has a closure preserving α-shading refinement by closed fuzzy sets. Where $\boldsymbol{U^F}$ is the collection of all unions of finite sub collections from $\boldsymbol{U}$. Now since f is F-continuous $\boldsymbol{W} = \{f^{-1}(U) : U \in \boldsymbol{U}\}$ is an α -shading of X by open fuzzy sets. Since X is α-metacompact, it follows that $\boldsymbol{W^F}$ has a closure preserving α-shading refinement $\boldsymbol{F}$ by closed fuzzy sets. Since f is F-closed it follows that $f(F)$ is closed for each $F \in \boldsymbol{F}$ Thus $\{f(F) : F \in \boldsymbol{F}\}$ is the required closure preserving α-shading refinement of $\boldsymbol{U}^F$ by closed fuzzy sets.

1.5.6 Definition Let X and Y be two fuzzy topological spaces. Then $f: X \longrightarrow Y$ is F-open α-compact if f is F-open with α-compact fibers, where fibers of a mapping $f: X \longrightarrow Y$ are the sets $f^{-1}(y)$ for $y \in Y$.

1.5.7 Definition Let X and Y be two fuzzy topological spaces. $f: X \longrightarrow Y$ is pseudo F-open if whenever $f^{-1}(y) < U$, $y \in Y$ and U is an open fuzzy set in X, then $y \in Int\ (f(y))$.

1.5.8 Definition Let $\boldsymbol{U}$ be a collection of fuzzy subsets of a fuzzy topological space X. We say that $\boldsymbol{U}$ is α-compact finite if $\{U \in \boldsymbol{U} : U \wedge K \neq 0\}$ is finite for any α-compact subset K of X.

1.5.9 Lemma Locally finite families of fuzzy sets are α-compact finite.

Proof

Let $\boldsymbol{U}$ be a locally finite family of fuzzy subsets of a fuzzy topological space X. Let K be α-compact. Since $\boldsymbol{U}$ is locally finite, for any $x \in K$, we can find an open fuzzy set w_x such that $w_x(x)=1$ and $U_s \leq 1 \backslash w_x$ holds for all but at most finitely many s. Now clearly $\{ w_x : x \in X \}$ is a 1^*-shading of K and since K is α-compact we get a finite sub shading say $\{w_{x1}, w_{x2}, \ldots w_{xk}\}$ for some finite k where each of w_{xi} has non empty meet with at most finitely many $U \in \boldsymbol{U}$. Hence it follows that $\{U \in \boldsymbol{U} : U \wedge K \neq 0\}$ is finite.

1.5.10 Theorem If $f: X \longrightarrow Y$ be an F-continuous pseudo F-open α- compact with X α-paracompact, then Y is α- metacompact.

Proof

Consider an α-shading $\boldsymbol{U}$ of Y by open fuzzy sets. Now since f is F-continuous it follows that $\boldsymbol{U}'=\{f^{-1}(U): U \in \boldsymbol{U}\}$ is an α-shading of X by open fuzzy sets. Given that X is α-paracompact . So $\boldsymbol{U}'$ has a locally finite α-shading refinement by open fuzzy sets say

V. Now consider ***K***= $\{f(V) : V \in \mathcal{V}\}$. Since *f* is *F*-open α-compact and for every $y \in Y$, $f^{-1}(y)$ is α-compact, from Lemma 3.16 it follows that $f^{-1}(y)$ has non empty meet with at most finitely many members of ***V***. Also since every locally finite family is point finite, it follows that ***V*** is point finite and hence ***K*** is also point finite . Since *f* is pseudo *F*-open it follows clearly that $y \in Int(st(y,K))$ for every $y \in Y$. [where $st(x,U) = \vee\{U \in \mathcal{U} : U(x) > 0\}$]. Now from the characterization of α-metacompactness in theorem 4.1.1, the proof is complete.

CHAPTER – II

THE FUZZY TOPOLOGICAL GAME $G^*(K, X)$

2.1 Introduction

A pursuit evasion game G (K,X) in which the pursuer and the evader choose certain subsets of a topological space in a certain way is defined and studied by Telgarsky [T_2]. In this chapter we generalise the concept of topological games in to a fuzzy topological space and some results related to them are obtained. Just like in the case of $G(K,X)$, the fuzzy topological game $G^*(K,X)$ has plenty of applications in fuzzy topology especially in fuzzy metacompactness etc, which will be discussed in the succeeding chapters.

2.2 The Fuzzy Topological Game

2.2.1 Notation By K we denote a non empty family of fuzzy topological spaces, where all spaces are assumed to be T_1. That is all fuzzy singletons are fuzzy closed. $\underline{I}^x$ denote the family of all fuzzy closed subsets of X. Also $X \in K$ implies $\underline{I}^x \subseteq K$. DK (FK) denote the class of all fuzzy topological spaces which have a discrete (finite) fuzzy closed α-shading by members of K.

2.2.2 Definition Let K be a class of fuzzy topological spaces and let $X \in K$. Then the fuzzy topological game $G^*(K,X)$is defined as follows. There are two players Player I and Player II . They alternatively choose consecutive terms of the sequence$(E_1,F_1,E_2,F_2,...)$ of fuzzy subsets of X. When each player chooses his term he knows K, X and their previous choices. A sequence $(E_1,F_1,E_2,F_2,...)$ is a play for $G^*(K,X)$ if it satisfies the following conditions for each $n \geq 1$.

Some Results mentioned in this Chapter are published in the paper titled *Fuzzy Topological Games I* in the Far East J. Math. Sci.,Spl. Vol (1999) Part III (Geometry and Topology), 361-371.

(1) E_n is a choice of Player I

(2) F_n is a choice of Player II

(3) $E_n \in \underline{I}^X \cap K$

(4) $F_n \in \underline{I}^X$

(5) $E_n \vee F_n < F_{n-1}$ where $F_0 = X$

(6) $E_n \wedge F_n = 0$

Player I wins the play if $\underset{n\geq 1}{Inf}\, F_n = 0$. Otherwise Player II wins the Game.

2.2.3 Definition A finite sequence $(E_1,F_1,E_2,F_2,........,E_m,F_m)$ is admissible if it satisfies conditions (1) -- (6) for each $n \leq m$.

2.2.4 Definition Let S\` be a crisp function defined as follows

$$S`: \bigcup_{n\geq 1} (\underline{I}^X)^n \xrightarrow{\text{into}} \underline{I}^X \cap K$$

Let $S_1=\{X\}$

$S_2=\{F\in \underline{I}^X : (S`(X),F)$ is admissible for $G^*(K,X)\}$. Continuing like this inductively we get $S_n= \{(F_1,F_2,F_3,.......Fn) : (E_1,F_1,E_2,F_2,...E_m,F_n)$ is admissible for $G^*(K,X)$ where $F_0=X$ and $E_i = S`(E_1,F_1,E_2,F_2.....F_{i-1})$ for each $i \leq n\}$. Then the restriction S of $S`$ to $\cup_{n\geq 1} S_n$ is called a fuzzy strategy for Player I in $G^*(K,X)$.

2.2.5 Definition If Player I wins every play $(E_1,F_1,E_2,F_2,...E_n,F_n,............)$ such that $E_n = S(F_1,F_2,.........,F_{n-1})$, then we say that S is a fuzzy winning strategy.

2.2.6 Definition $S: \underline{I}^X \xrightarrow{\text{into}} \underline{I}^X \cap K$ is called a fuzzy stationery strategy for Player I in $G^*(K,X)$ if $S(F) < F$ for each $F \in \underline{I}^X$ .We say that S is a fuzzy stationary winning strategy if he wins every play ($S(X),F_1,S(F_1),F_2$.........)

From definitions above, we get

2.2.7 Result A function $S: \underline{I}^X \xrightarrow{\text{into}} \underline{I}^X \cap K$ is a fuzzy stationary winning strategy if and only if it satisfies

(i) For each $F \in \underline{I}^X$, $S(F) < F$

(ii) If $\{F_n: n \geq 1\}$ satisfies $S(X) \wedge F_1 = 0$ and $S(F_n) \wedge F_{n+1} = 0$ for each $n \geq 1$ then $\underset{n\geq 1}{Inf}\, F_n = 0$.

2.2.8 Theorem Player I has a fuzzy winning strategy in $G^*(K,X)$ if and only if he has a fuzzy stationary winning strategy in it.

Proof is similar to that of Yajima [Y_1] and for completeness we are including it.

Proof:

Sufficiency part follows clearly. Conversely let S be a fuzzy winning strategy of Player I for $G^*(K,X)$. Well order $\underline{I}^X \setminus \{0\}$ by $<$. Let H be any non empty closed fuzzy subset of X.

Claim-(1) Now we will prove that there is some $F(H)=(F_1,F_2,F_3,\ldots,F_m) \in (\underline{I}^X)^m$ satisfying

(i) $S(F_0,F_1,\ldots..F_i) \wedge H = 0$ for $0 \leq i \leq m-1$.

(ii) $S(F_0,F_1,\ldots..F_m) \wedge H \neq 0$

(iii) $F_{i+1}=\text{Min}\{F \in \underline{I}^X : H \leq F \leq F_i$ and $F \wedge S(F_1,F_2,\ldots F_i)=0\}$ for $0 \leq i \leq m-1$ where $F_0=X$ and $F(H)=0$ may occur.

To prove the above claim assume the contrary. Then we can inductively choose some $(F_1,F_2,\ldots.) \in (\underline{I}^X)^\omega$ such that $S(F_1,F_2,\ldots F_k) \wedge H = 0$ and $F_k=\text{Min}\{F \in \underline{I}^X : H \leq F \leq F_{k-1}$ and $S(F_1,F_2,\ldots F_{k-1}) \wedge H = 0\}$ for each $k \geq 1$.

Now $(E_1,F_1,E_2,F_2.....)$ where $E_k= S(F_1,F_2,...F_{k-1})$ is a play for each k≥1 for $G^*(\boldsymbol{K},X)$ and by definition of fuzzy strategy ,we have $\underset{k\geq 1}{Inf}\, F_k = 0$. Also $H \leq F_k$ for all $k\geq 1$. There fore

$H \leq \underset{k\geq 1}{Inf}\, F_k = 0$. This is a contradiction to $H \neq 0$. Thus claim- (1) holds.

Take $S^*(0) = 0$ and $S^*(H) = S(F_1,F_2,...F_m) \wedge H$ where $\boldsymbol{F}(H)= (F_1,F_2,...F_m)$ for each $H \in \underline{I}^X \setminus \{0\}$.Then S^* is a function from $\underline{I}^X$ into $\underline{I}^X \cap \boldsymbol{K}$ such that $S^*(H) \leq H$ for each $H \in \underline{I}^X$. We will prove that S^* is a fuzzy stationary winning strategy for Player I in $G^*(\boldsymbol{K},X)$.

Let $(E_1,H_1,E_2,H_2,.....)$ be a play such that $E_1=S^*(X)$ and $E_n=S^*(H_{n-1})$ for $n \geq 2$. We show that $\underset{n\geq 1}{Inf}\, H_n = 0$. For $n \leq m$, take $\boldsymbol{F}(H)\,/_n=(F_1,F_2,...F_n)$ and

$|\boldsymbol{F}(H)| = m$

Claim-(2)

We will show that there are some $(F_1,F_2,...) \in (\underline{I}^X)^\omega$ and a sequence $k(1)<k(2)<....$such that $k \geq k(n)$ implies $(F_1,F_2,...F_n) = \boldsymbol{F}(H_k)\,/_n$ for each $n \geq 1$.

Take $F_0=X$ and assume that $(F_1,F_2,...F_n) \in (\underline{I}^X)^n$ and $\{ k(i) : i \leq n\}$ has been already chosen . First we will prove that $|\boldsymbol{F}(H_k)| > n$ for each $k > k(n)$. Let $k > k(n)$, then by induction we have $\boldsymbol{F}(H_k)\,/_n = \boldsymbol{F}(H_{k(n)})\,/_n = (F_1,F_2,...F_n)$.

If $S(F_0,F_2,...F_n) \wedge H_{k(n)}=0$, then from $H_k < H_{k(n)}$ it follows that $S(F_0,F_2,...F_n) \wedge H_k = 0$. Otherwise if $S(F_0,F_2,...F_n) \wedge H_k \neq 0$ by (ii) of Claim-(1) above we have $\boldsymbol{F}(H_{k(n)}) =(F_0,F_2,...F_n)$ so that $S^*(H_{k(n)}) = S(F_0,F_2,...F_n) \wedge H_{k(n)}$.

$$\begin{aligned} \text{Hence } S(F_0,F_2,...F_n) \wedge H_k &= S^*(H_{k(n)}) \wedge H_k \\ &< E_{k(n)+1} \wedge H_{k(n)+1} \\ &= 0 \end{aligned}$$

Thus in both cases $S(F_0,F_2,...F_n)$ is disjoint from H_k. By the choice of $\boldsymbol{F}(H_k)$ this means $|\boldsymbol{F}(H_k)| > n$

Let $F_{n+1}(k)$ be the $(n+1)^{st}$ term of $F(H_k)$ for $k>k(n)$. This exists since we have already proved that $|F(H_k)| > n$. Now take $F_{n+1} =$ Min $\{ F_{n+1}(k) : k > k(n)\}$. Choose some $k(n+1) > k(n)$ such that $F_{n+1} = F_{n+1}(k(n+1))$. Let $k > k(n+1)$. Clearly $F_{n+1} \leq F_{n+1}(k)$. Also $F(H_k)/_n = F(H_{k(n+1)})/_n$

$= (F_1,F_2,...F_n)$ and $H_k < H_{k(n+1)}$

By (ii) of claim-(1) above we obtain $F_{n+1}(k) \leq F_{n+1}(k(n+1)) = F_{n+1}$. Hence $F_{n+1} = F_{n+1}(k)$ whenever $k \geq k(n+1)$. This means $(F_1,F_2,...F_{n+1}) = F(H_k)/_{n+1}$ for each $k > k(n+1)$. Thus claim - (2) holds.

Now consider $(E_1,F_1,E_2,F_2.......E_n,F_n)$ such that $E_i = S(F_0,F_1,F_2.......,F_{i-1})$ for $1 \leq i \leq n$ and $F_o = X$. This is an admissible sequence in $G^*(K,X)$. By the definition of fuzzy winning strategy we have $\underset{n\geq 1}{Inf}\, F_n = 0$. Also by claim-(2) , each F_n is in terms of some $F(H_k)$. Then from (ii) of claim- (1) , it follows that $H_k < F_n$ for each F_n . Therefore we have $\underset{n\geq 1}{Inf}\, H_n \leq \underset{n\geq 1}{Inf}\, F_n$. But $\underset{n\geq 1}{Inf}\, F_n = 0$. Therefore it follows that $\underset{n\geq 1}{Inf}\, H_n = 0$. Thus S^* is a fuzzy stationary winning strategy for Player I in $G^*(K,X)$.

2.2.9 Proposition Let K_1 and K_2 be two classes of fuzzy topological spaces with $K_1 \subset K_2$ and if Player I has a fuzzy winning strategy in $G^*(K_1, X)$, then he has a fuzzy winning strategy in $G^*(K_2, X)$.

Proof

From Theorem 2.2.8 it follows that Player I has a fuzzy stationary winning strategy in $G^*(K_1, X)$. say S. From theorem 2.2.8 it suffices to prove that PlayerI has a fuzzy stationary winning strategy in $G^*(K_2, X)$. Now $S: \underline{I}^X \xrightarrow{into} \underline{I}^X \cap K_1$. Then by Result 2.2.7 we have $S(F) < F$ where $F \in \underline{I}^X$ where and if $\{F_n : n \geq N\} \subseteq \underline{I}^X$ satisfies $S(X) \wedge F_1 = 0$ and $S(F_n) \wedge F_{n+1} = 0$ for all $n \geq 1$, then $\underset{n\geq 1}{Inf}\, F_n = 0$.

Now define $S^*: \underline{I}^X \xrightarrow{into} \underline{I}^X \cap K_2$ by $F \rightarrow S(F) \wedge K_2$. Now we will show that S^* is a fuzzy winning strategy for $G^*(K_2, X)$.

Now $S^*(F) = S(F) \wedge \boldsymbol{K_2}$,

$\leq S(F)$

$\leq F$

Therefore S^* is a stationary strategy for Player I in $G^*(\boldsymbol{K_2}, X)$.

Now to prove that S^* *is* winning, we want to prove that Player I wins every play of the form $(S^*(X), F_1, S^*(F_1), \ldots\ldots)$. For that we want to prove that $\operatorname{Inf}_{n\geq 1} F_n = 0$. Now we have $S^*(X) \wedge F_1 = [S(X) \wedge \boldsymbol{K_2}] \wedge F_1$

$= S(X) \wedge \boldsymbol{K_2} \wedge F_1$

$= 0$ Since S is a stationary winning strategy of Player I in $G^*(\boldsymbol{K_1}, X)$.

Also $S^*(F_n) \wedge F_{n+1} = S(F_n) \wedge \boldsymbol{K_2} \wedge F_{n+1}$

$= 0$

By Result 2.2.7 it follows that $\operatorname{Inf}_{n\geq 1} F_n = 0$. Therefore S^* is a fuzzy stationary winning strategy for Player I in $G^*(\boldsymbol{K_2}, X)$.

2.2.10 Proposition Let Y be a fuzzy closed subspace of a fuzzy topological space X. If Player I has a fuzzy winning strategy in $G^*(\boldsymbol{K}, X)$. Then he has a winning strategy in $G^*(\boldsymbol{K}, Y)$.

Proof

Let $S: \underline{I}^X \xrightarrow{\text{into}} \underline{I}^X \cap \boldsymbol{K}$ be a fuzzy stationary winning strategy of $G^*(\boldsymbol{K}, X)$.

Now define $S^*: \underline{I}^Y \xrightarrow{\text{into}} \underline{I}^Y \cap \boldsymbol{K}$ by $F' \rightarrow S(F) \wedge Y$ where $F' = F \wedge Y$ and $F \in \underline{I}^X$

Now $S^*(F') = S(F) \wedge Y$

$< F \wedge Y$

$= F'$

Thus S^* is a fuzzy stationary strategy of Player I in $G^*(\boldsymbol{K}, Y)$.

Let $\{ F_n' : n \geq 1\} \subset \underline{I}^Y$ where $F_n' = F_n \wedge Y$ for some $F_n \in \underline{I}^X$

Now $S^*(Y) \wedge F_1' = [S(X) \wedge Y] \wedge F_1'$

$= [S(X) \wedge Y] \wedge [F_1 \wedge Y]$

$= S(X) \wedge Y \wedge F_1$

$= 0$ since S is winning

Also $S^*(F_n') \wedge F_{n+1}' = 0$ follows clearly. Therefore from Result 2.2.7, it follows that $\underset{n\geq 1}{Inf}\, F_n = 0$. Therefore it follows that $\underset{n\geq 1}{Inf}\, F'_n = 0$. Thus proving S^* is a fuzzy stationary winning strategy of Player I in G^* $(\boldsymbol{K}, Y)$.

2.3 Finite and Countable Unions

Clearly we have $\boldsymbol{K} \subseteq \boldsymbol{FK}$ and $X \in \boldsymbol{FK}$ implies $\underline{I}^X \subseteq \boldsymbol{FK}$.

2.3.1 Proposition If Player I has a fuzzy winning strategy in G^* $(\boldsymbol{FK}, X)$, then he has a fuzzy winning strategy in G^* $(\boldsymbol{K}, X)$.

Proof

Let S be a fuzzy winning strategy for Player I in G^* $(\boldsymbol{FK}, X)$. We will try to define a fuzzy strategy t for G^* $(\boldsymbol{K}, X)$. Now take $E_0 = X$, $E_1 = S(E_0)$ and $F_0 = E_0$. Now $E_1 \in \underline{I}^X \cap \boldsymbol{FK}$. Therefore $E_1 = \vee\{H_{1,m} : m \leq k_1\}$ where $\{H_{1,m} : m \leq k_1\} \subseteq \underline{I}^X \cap \boldsymbol{K}$. We set $F_1 = H_{1,0}$ and $t(F_0) = F_1$. Also take $F_2 \in \underline{I}^X$ in such a way that $F_1 \wedge F_2 = 0$ and also set $F_3 = F_2 \wedge H_{1,1}$ and $t(F_0, F_1, F_2) = F_3$. Continuing like this we get an admissible sequence $(F_0, F_1, \ldots\ldots\ldots\ldots F_{2k1})$ for G^* $(\boldsymbol{K}, X)$. Take $F_{2k1+1} = t(F_0, F_1, \ldots\ldots\ldots\ldots F_{2k1}) = F_{2k1} \wedge H_{1,k1}$. Take $F_{2k1+2} \in \underline{I}^X$ with $F_{2k1+2} \leq F_{2k1}$ and $F_{2k1+2} \wedge F_{2k1+1} = 0$. Take $E_2 = F_{2k1+2}$. Now clearly $E_1 \wedge E_2 = 0$ and set $E_3 = S(E_0, E_1, E_2)$. Since $E_3 \in \underline{I}^X \cap \boldsymbol{FK}$, we have $E_3 = \vee\{H_{3,m} \quad m \leq k_3\}$ where each $H_{3,m} \in \underline{I}^X \cap \boldsymbol{K}$.

Continuing like this we get the Play $(E_0, E_1, E_2, \ldots\ldots)$ of $G^*(\boldsymbol{FK}, X)$ and $(F_0, F_1, F_2, \ldots\ldots)$ of G^* $(\boldsymbol{K}, X)$. Since S is a fuzzy winning strategy for $G^*(\boldsymbol{FK}, X)$, $\underset{n\geq 1}{Inf}\, E_{2n} = 0$. Now $\{E_{2n} : n \in N\} \subseteq \{F_{2n} : n \in N\}$. Therefore it follows that $\underset{n\geq 1}{Inf}\, F_{2n} = 0$.. Therefore t is a fuzzy winning strategy for Player I in G^* $(\boldsymbol{K}, X)$.

2.3.2 Remark From $K \subseteq FK$ and Proposition 2.2.9 it follows that if Player I has a fuzzy winning strategy in $G^*(K, X)$, then he has a fuzzy winning strategy in $G^*(FK, X)$.

From Remark 2.3.2 and Proposition 2.3.1 we get

2.3.3 Theorem Player I has a fuzzy winning strategy in $G^*(K, X)$ if and only if he has the same in $G^*(FK, X)$.

2.3.4 Proposition If a fuzzy topological space X has a fuzzy closed countable α-shading $\{X_n : n \in N\}$ such that Player I has a fuzzy winning strategy in $G^*(K, X_n)$ for each $n \in N$ then he has a fuzzy winning strategy in $G^*(K, X)$.

Proof

Let S_n be a fuzzy stationery winning strategy for Player I in $G^*(K, X_n)$ for each $n \in N$. Now it is enough if we prove that Player I has a fuzzy winning strategy in $G^*(FK, X)$. Now we take $S(X)=S_1(X)$ and assume that $(E_1,F_1,E_2,\ldots\ldots E_n,F_n)$ is an admissible sequence in $G^*(FK, X)$ such that $E_i = S(F_1,F_2,F_3,\ldots F_{i-1}.)$ for each $i \leq n$ where $F_0=X$. Take $E_{n+1} = S(F_1,F_2,F_3,\ldots F_n.) = \underset{k\leq n+1}{Sup}\, S_k(F_n \wedge X_k)$

Consider the Play $(E_1,F_1,E_2,\ldots\ldots)$ in $G^*(FK, X)$ such that $E_n=S(F_1,F_2,F_3,\ldots F_{n-1}.)$ for all $n \geq 1$. Now take an $m \geq 1$. By definition of Play we have $E_{n+1} \wedge F_{n+1} = 0$.-------- (1)

Here $E_{n+1} = \underset{k\leq n+1}{Sup}\, S_k(F_n \wedge X_k)$

$\geq S_m(F_n \wedge X_m)$

Also $F_{n+1} \wedge X_m \leq F_{n+1}$. Therefore from (1) it follows that

$[S_m(F_n \wedge X_m)] \wedge [F_{n+1} \wedge X_m] = 0$ for each $n \geq m$. Now since S_m is a stationary winning strategy for Player I in $G^*(K, X_m)$, we have

$S_m(F_n \wedge X_m) \leq F_n \wedge X_m$ for each $n \geq m$.

Therefore $[F_n \wedge X_m] \wedge [F_{n+1} \wedge X_m] = 0$ for each $n \geq m$. Thus $\wedge_{n \geq m} [F_n \wedge X_m] = 0$ We also have $F_{n+1} < F_n$ and hence it follows that $\underset{n\geq 1}{Inf}\, F_n = 0$. Thus Player I has a winning strategy in $G^*(FK, X)$, hence the proof is complete by Theorem 2.3.3.

2.3.5 Theorem Let X be a fuzzy topological space with a fuzzy subset E such that $E \in \underline{I}^x \cap$ E . If Player I has a fuzzy winning strategy in G^* (K, F) for each $F \in \underline{I}^x$ with $E \wedge F = 0$, then Player I has a fuzzy winning stratey in G^* $(K$, X)

Proof

For each $F \in \underline{I}^x$ with $E \wedge F = 0$, Let S_F be a fuzzy stationary winning strategy for Player I in G^* (K, F) . Now we will find out a fuzzy winning strategy S for Player I in G^* (K, X)

Define $S(X)=E$ and $(E_1,F_1,E_2,F_2,.....E_n,F_n)$ be an admissible sequence in G^* (K, X) such that $E_1 = S(F_0,F_1,,F_2,.....,F_{i-1})$ for each $i \leq n$ where $F_0 = X$. Take $E_{n+1} = S(F_0,F_1,,F_2,.....,F_n) = S_{F\,1}(F_n)$. Consider the play $(E_1,F_1,E_2,F_2,.....)$ Now clearly $E_{n+1} \wedge F_{n+1} = 0$. That is $S_{F\,1}(F_n) \wedge F_{n+1} = 0$. Also $S_{F\,1}(X) \wedge F_1 = E_1 \wedge F_1 = 0$ Since S_{F1} is a stationery winning strategy, it follows that $\underset{n\geq 1}{Inf}\, F_n = 0$. Thus Player I has a fuzzy winning strategy in G^* (K, X).

2.4 Games and Mappings

2.4.1 Theorem Let X and Y be two fuzzy topological spaces and K_1 and K_2 be two classes of fuzzy topological spaces such that $X \in K_1$ and $Y \in K_2$. If f is an F-continous function from X on to Y which maps all $E \in \underline{I}^x \cap K_1$ to $f(E) \in \underline{I}^x \cap K_2$ and if player I has a fuzzy winning strategy in G^* (K_1,X) , then Player I has a fuzzy winning strategy in G^* (K_2,Y).

Proof

Let S be a fuzzy stationary winning strategy for Player I in $G^*(K_1,X)$. Thus player I wins every play of the form $(S(X), F_1, S(F_1), \ldots..)$. Now we will define a stationary winning strategy t for Player$I$ in $G^*(K_2,Y)$. Now consider the play $(t(Y), P_1, t(P_1), P_2\ldots..)$ where $P_n = t(F_n)$ and $t : \underline{I}^Y \xrightarrow{into} \underline{I}^Y \cap K_2$ is defined by $t(P_n) = f[S(F_n)]$. Now t is a stationary winning strategy for $G^*(K_2,Y)$.

For, $t(F_n) = f[S(F_n)]$

$< f(F_n)$

$= P_n$. There fore t is a fuzzy stationery strategy.

Now $t(P_n) \wedge P_{n+1} = f[S(F_n)] \wedge f(F_{n+1})$

$= f[S(F_n) \wedge F_{n+1}]$

$= f(0)$

$= 0$

Also $t(Y) \wedge P_1 = f[S(X)] \wedge P_1$

$= f[S(X)] \wedge f(F_1)$

$= f[S(X) \wedge F_1]$

$= f(0)$

$= 0$

There fore it follows from Result 2.2.7 that $\operatorname{Inf}_{n\geq 1} F_n = 0$ and hence t is a stationary winning strategy for Player I in $G^*(K_2,Y)$.

2.4.2 Theorem Let $f: X \xrightarrow{into} Y$ be an F-continuous F-closed mapping such that $f^{-1}(E) \in \underline{I}^X \cap K_1$ whenever $E \in \underline{I}^X \cap K_2$. Then if Player I has a fuzzy winning strategy in $G^*(K_2,Y)$, then Player I has a fuzzy winning strategy in $G^*(K_1,X)$.

Proof

Let S be a fuzzy stationery winning strategy for Player I in $G^*(K_2,Y)$. Therefore Player I wins every play of the form $(S(Y), F_1, S(F_1), \ldots..)$. Now we will define a function

$t: \underline{I}^X \xrightarrow{\text{into}} \underline{I}^X \cap K_1$ as follows. Now $f: X \xrightarrow{\text{into}} Y$ is F-closed and hence we take $P_n = f^{-1}(F_n)$ where $P_n \in \underline{I}^X$ and $t(P_n) = f^{-1}[S(F_n)]$ for all $P_n \in \underline{I}^X$

Now $t(P_n) = f^{-1}[S(F_n)]$

$< f^{-1}(F_n)$

$= P_n$. Thus t is a fuzzy stationary strategy.

Now consider the play $(t(X), P_1, t(P_1), \ldots)$

$$\begin{aligned} t(P_n) \wedge P_{n+1} &= f^{-1}[S(F_n)] \wedge P_n \\ &= f^{-1}[S(F_n)] \wedge f^{-1}(F_{n+1}) \\ &= f^{-1}[S(F_n) \wedge F_{n+1}] \\ &= f^{-1}(0) \\ &= 0 . \end{aligned}$$

Also

$$\begin{aligned} t(X) \wedge P_1 &= f^{-1}[S(X)] \wedge P_1 \\ &= f^{-1}[S(X)] \wedge f^{-1}(F_1) \\ &= f^{-1}[S(X) \wedge F_1] \\ &= f^{-1}(0) \\ &= 0 . \end{aligned}$$

Therefore from Result 2.2.7 it follows that $Inf\, P_n = 0$ and hence t is a winning strategy also. Thus t is a fuzzy winning strategy for Player I in $G`(K_1,X)$. This completes the proof.

As an immediate consequence of Theorem 2.4.1 and Theorem 2.4.2 we get the following two Theorems.

2.4.3 Theorem Let X and Y are two fuzzy topological spaces and $f:X \xrightarrow{\text{into}} Y$ be an F-continuous function and $f^{-1}(E) \in \underline{I}^X \cap K_1$ whenever $E \in \underline{I}^X \cap K_2$. If Player II has a fuzzy winning strategy in $G^*(K_1,X)$. Then Player II has a fuzzy winning strategy in $G^*(K_2,Y)$.

2.4.4 Theorem Let $f: X \xrightarrow{\text{into}} Y$ be an F-continuous F-closed mapping such that $f^{-1}(E) \in \underline{I}^Y \cap K_2$ whenever $E \in \underline{I}^X \cap K_1$. If Player II has a fuzzy winning strategy in $G^*(K_2,Y)$, then Player II has a fuzzy winning strategy in $G^*(K_1,X)$.

2.4.5 Definition [M;B_2] Let $0 \leq \alpha < 1$ (resp. $0 < \alpha \leq 1$). An F-closed F-continuous function f from a fuzzy topological space X to a fuzzy topological space Y is said to be α-perfect (resp. α^*-perfect) if and only if $f^{-1}(y)$ is α-compact (resp. α^*-compact) for each $y \in Y$.

2.4.6 Definition A class K of fuzzy topological spaces is said to be α-perfect if $X \in K$ is equivalent to $Y \in K$, provided that there exists an α-perfect map from X onto Y.

From Theorems 2.4.1, 2.4.2, 2.4.3 and 2. 4.4 next theorem follows immediately.

2.4.7 Theorem Let K be an α-perfect class of fuzzy topological spaces and if there is an α-perfect map from X on to Y, Then

(i) If Player I has a fuzzy winning strategy in $G^*(K, X)$.then he has the same in $G^*(K_1, Y)$.

(ii) If Player II has a fuzzy winning strategy in $G^*(K, X)$.then he has the same in $G^*(K, Y)$.

CHAPTER – III

CLOSURE PRESERVING SHADING FAMILIES

In this chapter we study closure preserving shading families and weakly σ-discrete families and a complete characterisation of spaces with closure preserving shading families by fuzzy sets with finite support are obtained. This characterisation involves the concept of fuzzy K-scattered spaces and hereditarily metacompact spaces. Some close relationships of K-scattered α-metacompact spaces and countably α-compact α-metacompact spaces with the Game $G^*(DK,X)$ are investigated

3.1 Closure Preserving Shading Families

3.1.1 Definition A fuzzy topological space X said to be weakly σ--discrete if X is the supremum of a countable number of discrete subsets $\{X_n: n\geq 1\}$ such that $\vee\{X_i ; 1\leq i\leq n\}$ is a closed fuzzy set in X for each $n\geq 1$, where a subset D of a fuzzy topological space X is discrete if D is a discrete space given the subspace topology.

3.1.2 Lemma Let (X,T) be a fuzzy topological space and let $\mathbf{U}=\{U_x : x\in X\}$ be an α- shading of X by open fuzzy sets such that $U_x(x) > \alpha$ for each $x\in X$ and $U_x < U_y$ whenever $U_y(x) > \alpha$ .Then the collection $\mathcal{F}=\{F_x : x\in X\}$ is a closure preserving α- shading of X where F_x is defined as follows . $F_x = \{y\in X : U_y(x) > \alpha$ and $F_x(y) = U_y(y)\}$ for each x in X.

Some Results mentioned in this Chapter are included in a paper titled *FuzzyTopological Games, α-Matacompactneaa and α-Perfect maps* to appear in Glasnik Mathematicki Vol 35 (55) (2000), 63-72.

Proof

First we will show that $\boldsymbol{F}$ is an α-shading of X. Let $x \in X$, then clearly $U_x(x) > \alpha$. Now from the definition of F_x it follows that $F_x(x) = U_x(x) > \alpha$. There fore $\boldsymbol{F}$ is an α-shading of X.

Again to prove $\boldsymbol{F}$ is closure preserving , clearly we have $\vee_{y \in Y} cl\, F_y \leq cl\, [\vee_{y \in Y} F_y]$. Now let $cl\, [\vee_{y \in Y} F_y](x) > \alpha$ where $\alpha \in (0,1]$. clearly $U_x(x) > \alpha$ where $U_x \in \boldsymbol{U}$. Then there is a y_0 in Y such that $F_{y0} \wedge U_x \neq 0$. Let z be a point in $F_{y0} \wedge U_x$ with $[F_{y0} \wedge U_x](z) > \alpha$ .Then clearly $U_x(z) > \alpha$. Also $U_z(z) > \alpha$. Therefore $U_z < U_x$. Also $F_{y0}(z) > \alpha$ .Therefore $U_z(y_0) > \alpha$ by definition. But $U_z < U_x$. Therefore $U_x(y_0) > \alpha$. This implies $Fy_0(x) > \alpha$. Therefore $cl\, [F_{y0}(z)] > \alpha$. Thus $\vee_{y \in Y} [cl\, F_y] > \alpha$ .This completes the proof .

3.1.3 Lemma The following are equivalent for a fuzzy topological space X.

(a) X has a closure preserving α.-shading by fuzzy sets having finite support.

(b) X has an α-shading $\boldsymbol{U} = \{U_x : x \in X\}$ by open fuzzy sets such that

(i) $U_x(x) > \alpha$ for each $x \in X$.

(ii) $U_x < U_y$ whenever $U_y(x) > \alpha$ and

(iii) $\boldsymbol{U}$ is point finite in X.

Proof

Let $\boldsymbol{F} = \{ F_\lambda : \lambda \in \Lambda\}$ be a closure preserving α-shading by fuzzy sets having finite support. Now we define $U_x = Inf\{F_\lambda : F_\lambda(x) > \alpha\}$. Now clearly $U_x(x) > \alpha$ and hence $\boldsymbol{U} = \{U_x : x \in X\}$ is an α-shading.

Now $U_y = Inf\{F_\lambda : F_\lambda(y) > \alpha\}$. Now if possible let $U_y(x) < \alpha$. Clearly $U_x(x) > \alpha$ and hence $U_y(x) < U_x(x)$ and hence (ii).

Again we will prove that $\boldsymbol{U}$ is a point finite collection in X. If possible let for any $x \in X$, $U_x(x) > \alpha$ for infinitely many $U \in \boldsymbol{U}$. Thus we can choose an infinite number of points $\{x_\lambda : \lambda \in \Lambda\}$ of X such that $Ux_\lambda(x) > \alpha$ for each $\lambda \in \Lambda$. Now since $\boldsymbol{F}$ is an α-shading we have an $F_\lambda \in \boldsymbol{F}$ such that $F_\lambda(x_\lambda) > \alpha$ for each $\lambda \in \Lambda$. This is a contradiction since F_λ is a fuzzy set with finite support.

Converse part follows from Lemma 3.1.2.

3.1.4 Lemma Let X be a fuzzy topological space with a point finite α-shading $\boldsymbol{U} = \{U_x : x \in X\}$ by open fuzzy sets with $U_x(x) > \alpha$ for every $x \in X$ and $U_x < U_y$ whenever $U_y(x) > \alpha$, then X has a countable pair wise disjoint α-shading $\{X_n : n \in N\}$ such that each X_n is discrete and $\overset{n}{\underset{i=1}{\vee}} X_i$ is fuzzy closed in X for each $n \in N$.

Proof

Let $\boldsymbol{U}$ be a point finite α-shading of X by open fuzzy sets, we define α-Ord $(x, \boldsymbol{U})$ = Card$\{U \in \boldsymbol{U} : U(x) > \alpha\}$ and let X_n be the collection of all fuzzy points with α-Ord $(x, \boldsymbol{U}) = n$ and values defined by $X_n(x) = Sup\{U(x) : U \in \boldsymbol{U}$ and $U(x) > \alpha\}$. This is possible since $\boldsymbol{U}$ is point finite. Clearly X_n' s are pair wise disjoint. Now we will prove that $K_n = \overset{n}{\underset{i=1}{\vee}} X_i$ is fuzzy closed for each $n \geq 1$. For, if possible let K_n have a cluster point t_η which does not belong to K_n. Then every nbd of t_η contains some point of K_n. Now $U_i(t) > \alpha$ for at most finitely many $U_i \in \boldsymbol{U}$. Now U_t is the smallest of all such U's. Consider the neighbourhood U_t of t_η. Then U_t contains some point of K_n say s_λ. Now clearly U_t is a nbd of s_λ and hence $U_s < U_t$. Now $t_\eta \in X_k$ for some $k > n$ and $s_\lambda \in X_p$ for some $p \leq n$. Since $t_\eta \in X_k$, it follows that $U_i(t) > \alpha$ for $i = 1, 2, \ldots k$. and $U_t < U_i$ for $i = 1, 2, \ldots k$ where $U_s(s) > \alpha$ and $U_s < U_t < U_i$. Thus $U_i(s) > \alpha$ for $i = 1, 2, \ldots k$. Thus $s_\lambda \in$

X_k for some $k>n$. This is a contradiction since X_i are disjoint. Thus $\overset{n}{\underset{i=1}{\vee}} X_i$ is closed for each $n\geq 1$.

Now let $n\geq 1$ and $x\in X_n$. Now α-$Ord\,(x, \boldsymbol{U})=n$. Therefore we can find $x_1,x_2,\ldots,x_n \in X$ such that $U_{xi}(x) > \alpha$ for $i =1, 2,\ldots n$. Let y_η be a fuzzy point in $U_x \wedge X_n$ where $\eta = U_y(y)$. Clearly $\eta > \alpha$. Now $y_\eta \in U_x$ implies $\alpha < \eta < \in U_x(y)$. Thus $U_x(y) > \alpha$. Also $U_y(y) > \alpha$. There fore $U_y < U_x < U_{xi}$ for $i =1, 2,\ldots n$. Now $\alpha < U_y(y) < U_{xi}(y)$. Thus $U_y(y) > \alpha$ and $y\in X_n$ implies that $y = x_i$ for some i .

Now consider the set of fuzzy singletons with support $\{x_i: i =1,2\ldots n\}$. Now clearly $U_x \wedge X_n < \{p_1,p_2,\ldots\ldots p_n\}$, where p_i are fuzzy singletons with support x_i . Now since X is assumed to be fuzzy T_1, singletons are fuzzy closed and hence it follows that X_n's are discrete for each $n\geq 1$.

3.1.5 Proposition If a fuzzy topological space X has a closure preserving α-shading by fuzzy closed and α-compact sets, then X is α-metacompact.

Proof

From the characterization of α-metacompactness, it is enough if we prove that every directed α-shading by open fuzzy sets of X has a closure preserving closed refinement.

Let $\boldsymbol{U}$ be a directed α-shading of X by open fuzzy sets. Let $\boldsymbol{C}$ be any closure preserving α-shading by closed α-compact fuzzy sets. Now $\boldsymbol{U}$ is an α-shading of C for any $C\in \boldsymbol{C}$. Now since C is α-compact, it has a finite α-sub shading say $\{U_1, U_2,\ldots\ldots U_k\}$.

Now $C < U_1 \vee U_2 \vee \ldots\ldots \vee U_k$

$< U$ for some $U\in \boldsymbol{U}$ since $\boldsymbol{U}$ is directed.

Thus $C < U$ for some $U \in \boldsymbol{U}$. Therefore $\boldsymbol{C}$ is an α-shading refinement of $\boldsymbol{U}$. This completes the proof.

3.2 Fuzzy K- Scattered Spaces

Let X be a fuzzy topological space and $F \in \underline{I}^X$. Then we define the ***K***- derivative of F of order 1, $F^{(1)}$ as the collection of all fuzzy points in X whose support set is given by Supp $F^{(1)} = \{x \in X : F(x) > 0$ and $\exists$ no fuzzy nbd $g \in \boldsymbol{K}$ with $g(x) > 0$ and $g < F\}$. Where ***K*** is a collection of spaces such that we always assume that ***K*** is non empty and $X \in \boldsymbol{K}$ implies $\underline{I}^X \subseteq \boldsymbol{K}$. The value of x at $F^{(1)}$ is the same as that at F. That is $F^{(1)}(x) = F(x)$ for all $x \in Supp F^{(1)}$. Inductively we define $F^{(\lambda+1)} = (F^{(\lambda)})^{(1)}$ for each ordinal λ. If λ. is a limit ordinal $F^{(\lambda)} = \bigwedge_{\mu < \lambda} F^{(\mu)}$.

Now take $\varsigma(X) = Inf\{\lambda : X^{(\lambda)} = 0\}$ if it exists

$= \infty$ other wise.

3.2.1 Definition A fuzzy topological space X is said to be ***K***-scattered if $\varsigma(X) = \eta$ for some ordinal η. Or equivalently for every $0 \neq F \in \underline{I}^X$ there exists a point $x \in F$ and a fuzzy nbd N of X with $N(x) > 0$ where $N < F$ and $N \in \boldsymbol{K}$.

3.2.2 Definition Let $\boldsymbol{U}$ be an α-shading of a fuzzy topological space X. We say that $\boldsymbol{U}$ is α-disjoint if $U \wedge V < \alpha$ *for all* $U, V \in \boldsymbol{U}$ and $U \neq V$.

3.2.3 Definition An α-disjoint α-shading $\{L_\lambda : \lambda < \eta\}$ of a fuzzy topological space X is called a fuzzy ***K***-scattered partition if $L_\lambda(x) \leq N(x)$ *for all* $x \in X$ and for some $N \in \boldsymbol{K}$ and $\bigwedge_{\mu < \delta} L_\mu$ is open in X for each $\lambda < \eta$.

3.2.4 Lemma A ***K***-scattered fuzzy topological space has a ***K***-scattered partition

Proof

Let X be a $\boldsymbol{K}$-scattered fuzzy topological space. There fore by definition, we have $\varsigma(X) = \eta$ for some ordinal η. Let $\lambda<\eta$ and take $Y_\lambda = X^{(\lambda)} \backslash X^{(\lambda+1)}$. Therefore clearly it follows that each $x \in Y_\lambda$ has a fuzzy neighbourhood N_x in Y_λ with $N_x(x)>0$ and $N_x \in \boldsymbol{K}$. Well order Y_λ by $<_\lambda$. For each $x \in Y_\lambda$, take $L_x = [\underset{Y\lambda}{Int} N_x] \setminus [\vee [\underset{Y\lambda}{Int} N_y : y <_\lambda x]$. Clearly each of L_x is open and $L_x < N_x$ $\forall x$ and $N_x \in \boldsymbol{K}$. Also since each L_x is fuzzy open, so is their arbitrary union. Thus $\underset{\mu(\delta}{\vee} \{ L_x : x \in Y_\lambda$ and $\lambda < \varsigma(X) \}$ is open . Therefore it follows that $\{ L_x : x \in Y_\lambda$ and $\lambda < \varsigma(X) \}$ is a $\boldsymbol{K}$-scattered partition of X.

3.2.5 Definition Let A be a fuzzy set. Then a fuzzy point $p \in A$ is called an isolated point if it has a fuzzy neighbourhood U such that $U(x)=0$ *for all* $x \in A$ with $x \neq p$.

3.2.6 Definition A fuzzy topological space X is said to be scattered if each fuzzy subset A of X has an isolated point in X.

3.2.7 Remark Clearly $\boldsymbol{1}$-scattered and scattered spaces coincide. Where $\boldsymbol{1}$ is the class consisting of all one point spaces and the empty space.

3.2.8 Remark The converse of the Lemma 3.2.4 is in general not true. This follows from the next example.

3.2.9 Example Take $X=\boldsymbol{R}^2$. Define a fuzzy topology T on $\boldsymbol{R}^2$ by declaring each point in $\boldsymbol{R}^2$ with rational co-ordinates as fuzzy open singletons (we denote it by Q^*) together with the sets of the form $\{z\} \cup \{Q^* \wedge \chi_U\}$, where $z \in \chi_U$ and U is the usual crisp open set in $\boldsymbol{R}^2$.

Let A be any fuzzy subset of X. If $A \wedge Q^* \neq 0$, from the definition of Q^* it follows that A has an isolated point. Now if $A \wedge Q^* = 0$, then $A < (Q^*)'$. Then every fuzzy point $p \in A$ is contained in $\{p\} \cup \{Q^*\}$. This cannot contain any other point of A, $A < (Q^*)'$. Hence (X, T) is $\boldsymbol{1}$-scattered.

Now for every α-disjoint α-shading of X, any member of this shading cannot be contained in some $N \in \mathbf{1}$. Thus X has no $\mathbf{1}$-scattered partition even though it is $\mathbf{1}$-scattered.

3.2.10 Definition [G;K;M] A fuzzy topological space X is said to be fuzzy regular if and only if for every fuzzy point p in X and for every open fuzzy set U containing p , there exists an open fuzzy set W such that $p \leq W \leq clW \leq U$.

The converse of the Lemma 3.2.4 is true only when X is fuzzy regular.

3.2.11 Lemma A regular fuzzy topological space with a $\boldsymbol{K}$-scattered partition is $\boldsymbol{K}$-scattered.

Proof

Let X be a fuzzy regular space with a $\boldsymbol{K}$-scattered partition $\{L_\lambda : \lambda < \eta\}$. Let F be any fuzzy closed set in X. Let $\delta = \text{Min}\ \{ \lambda : L_\lambda \wedge F \neq 0\}$. And take some $x \in L_\lambda \wedge F$. Now clearly from the definition of the $\boldsymbol{K}$-scattered partition, it follows that $\bigvee_{\lambda \leq \delta} L_\lambda$ is an open fuzzy set containing x. Since X is fuzzy regular , it is possible to find an open fuzzy set U such that $x \in U <$ cl $U < \bigvee_{\lambda \leq \delta} L_\lambda$. Now $L_\delta < N$ for some $N \in \boldsymbol{K}$. Thus for each closed fuzzy set F there exists $x \in F$ such that $x \in F < cl\ U \wedge F < L_\delta \wedge F < N \wedge F \in \boldsymbol{K}$. Thus X is $\boldsymbol{K}$-scattered.

3.2.12 Proposition A fuzzy topological space X is hereditarily α-metacompact if and only if every α-disjoint α-shading $\{L_\lambda : \lambda < \eta\}$ of X such that $\bigvee_{\mu \langle \lambda} L_\mu$ is open in X for each $\lambda < \eta$ has a point finite expansion $\{U_\lambda : \lambda < \eta\}$ of open fuzzy sets.(i.e., $L_\lambda < U_\lambda$ *for each* $\lambda < \eta$.)

Proof

Let X be hereditarily α-metacompact and let$\{ L_\lambda : \lambda<\eta\}$be an α-disjoint α-shading of X such that $\vee_{\mu<\lambda} L_\mu$ is open for each $\lambda<\eta$. We give the proof by the method of induction. Now clearly the statement holds for $\eta=1$. Let η be a fixed ordinal. Assume that the statement is true for all $\varsigma<\eta$. If η is not a limit ordinal, take $\eta = \varsigma+1$ Now $\{L_\mu : \mu<\varsigma\}$is an α-disjoint α-shading of $\vee_{\mu<\varsigma} L_\mu$. Since $\vee_{\mu<\varsigma} L_\mu$ is open , from hereditarily α-metacompactness and induction hypothesis it follows that$\{ L_\lambda : \lambda< \varsigma\}$ has a point finite expansion$\{ U_\lambda : \lambda< \varsigma\}$ by open fuzzy sets. Now put $U_\varsigma=X$. Then clearly $\{U_\lambda : \lambda<\eta\}$ is a point finite expansion of $\{ L_\lambda : \lambda<\eta\}$ by open fuzzy sets.

Now if η is a limit ordinal, we define $S_\delta = \vee_{\lambda<\delta} L_\lambda$ for every $\lambda<\eta$. Now clearly $\{ S_\delta : \delta<\eta\}$ is an α-shading of X by open fuzzy sets. Now since X is α-metacompact , there is an α-shading refinement $\{ U_\delta : \delta<\eta\}$ by open fuzzy sets such that $U_\delta < S_\delta$ for each$\delta<\eta$. Now consider the collection $\{ L_\lambda \wedge U_\delta : \lambda<\delta\}$. This is an α-disjoint α-shading of U_δ of length $\delta<\eta$. Hence from hereditarily α-metacompactness and induction hypothesis it follows that $\{ L_\lambda \wedge U_\delta : \lambda<\delta\}$ has a point finite expansion $\{ W_{\lambda,\delta} : \lambda<\delta\}$ such that $W_{\lambda,\delta} < U_\delta$ for each $\lambda<\delta$. Take $W_\lambda = \vee_{\lambda<\delta<\eta} W_{\lambda,\delta}$ for each $\lambda<\eta$. Now clearly $\{ W_\lambda : \lambda<\eta\}$is point finite expansion of $\{ L_\lambda : \lambda<\eta\}$ by open fuzzy sets.

Conversely to prove every open subspace of X is α-metacompact , let O be an open subspace of X. Let $\mathbf{U} = \{U_\lambda : \lambda<\eta\}$be an α-shading of O by open fuzzy sets. We define $\{ L_\lambda : \lambda<\eta\}$ as follows. $Supp(L_\lambda)= \{x\in X : U_\lambda(x)> \alpha\} \setminus \{ x\in X : \sup_{\mu<\lambda} U_\mu(x)> \alpha \}$ and $L_\lambda(x)= U_\lambda(x)$ *for all* $x\in X$. Also take $L_\eta = [Sup\{ U : U \in \mathbf{U} \}]'$. Now consider $\{L_\lambda : \lambda\leq\eta\}$. Also this is an α-disjoint α-shading of X such that $\vee_{\mu<\lambda} L_\mu$ is open for each $\lambda<\eta$ and hence has a point finite expansion $\{ W_\lambda : \lambda\leq\eta\}$ by open fuzzy sets such that $W_\lambda < U_\lambda$ for each $\lambda<\eta$. Now clearly $\mathbf{W}= \{ W_\lambda : \lambda<\eta\}$ is a point finite refinement of $\mathbf{U}$ by open fuzzy sets. Hence O is α-metacompact.

3.2.13 Definition A class of fuzzy topological spaces K is said to be finitely additive if every space with a finite α-shading by closed fuzzy sets of K belongs to K.

3.2.14 Lemma Let K be a finitely additive family of fuzzy topological spaces and suppose that each X belongs to K has a countable α-shading $\{X_n: n\geq 1\}$ and for each $n\geq 1$ there exists a point finite α-shading $\mathbf{U}_n$ of X such that $X_n \setminus \text{Sup}\ \{\ \mathbf{U}_n \setminus \mathbf{V}\ \} \in K$ for each finite $\mathbf{V}\subset\mathbf{U}_n$. Then X has a closure preserving α-shading by closed fuzzy sets which belongs to K.

Proof

Consider the collection $\mathbf{W}= \cup_{n\geq 1}\mathbf{W}_n$ where $\mathbf{W}_n$ are defined as follows. Take

$\mathbf{W}_1= \mathbf{U}_1$

$\mathbf{W}_2=\{\ W_{2,U}: U\in \mathbf{U}_2\ \}$ where

$Supp(W_{2,U})= \{x\in X: U(x)>\alpha\}\setminus \{x\in X: X_1(x)>\alpha\}$ and

$W_{2,U}(x)= U(x)$ *for all* $x\in X$.

Proceeding like this we get

$\mathbf{W}_{n+1} = \{W_{n+1,U} : U \in \mathbf{U}_{n+1}\}$ where

$Supp(W_{n+1,U}) = \{x\in X: U(x)>\alpha\}\setminus \{x\in X: [X_1\vee X_2 \vee X_3\vee \ldots\ldots\ldots \vee X_n](x)>\alpha\}$

Now clearly $\mathbf{W}$ is an α-shading of X by open fuzzy sets and is also pont finite. For, let $x\in X$. Let k be the smallest integer such that $X_k(x) > \alpha$. Now clearly $W(x)=0$ for all $W\in \mathbf{W}_m$ for $m= k+1\ ,\ k+2\ ,\ldots\ldots$ Also since $\mathbf{U}_i$ are point finite , if follows that each of $\mathbf{W}_m\ \mathbf{W}_1\ ,\mathbf{W}_2\ \ldots\ldots\ \mathbf{W}_k$. has at most finitely many members with membership values of x greater than zero. Thus $\mathbf{W}$ is also point finite.

Now we know that if an α-shading, $\mathbf{U}$ of a fuzzy topological space X is interior preserving then the collection $\mathbf{F}= \{X \setminus \underset{U\in \mathbf{U}}{Sup}\{U : U(x)=0\}: x\in X\}$is a closure preserving α-shading of X by closed fuzzy sets. Now consider the collection $\mathbf{K} = \{K_x: x\in X\ \}$ where $K_X = X \setminus \underset{w\in W}{Sup}\{W: W(x)=0\}$ Now since $\mathbf{W}$ is point finite , it is interior preserving and hence **K** is a closure preserving α-shading of X by closed fuzzy sets. Take an $x\in X$. Let $n_x = \text{Min}\{n : X_n(x) > \alpha\}$. Now if $y\in K_x$, then clearly $n_y \leq n_x$. Also $K_x < \vee_{n\leq nx} [X_n \setminus Sup_{U\in Un} \{U : U(x)=0\}] \in K$. Since K_x is closed it follows that $K_x \in K$

3.2.15 Theorem Let K be a finitely additive class of fuzzy topological space. If X is a hereditarily α-metacompact space with countable α-shading by closed and K-scattered fuzzy sets , then X has a closure preserving α-shading by closed fuzzy sets which belong to K.

Proof

Take a countable α-shading $\{X_n : n\geq 1\}$ of X by closed fuzzy sets where each X_n is K-scattered. Let $n\geq 1$. Then X_n has a K-scattered partition say $\{L_{n,\lambda}: 0<\lambda<\eta_n\}$.Also take $L_{0,\lambda} = X_n'$. Now take$\mathbf{L}_n=\{L_{n,\lambda}: 0\leq\lambda<\eta_n\}$. Clearly $\mathbf{L}_n$ is an α-shading of X. Since $\mathbf{L}_n$ is a K-scattered partition of X_n together with X_n', it is disjoint and hence has an open expansion say $\mathbf{U}_n=\{U_{n,\lambda}: 0\leq\lambda\leq\eta_n\}$. Let $\mathbf{V}$ be any finite sub collection of $\mathbf{U}_n$.Then from the fact $\mathbf{L}_n$is an α-shading of X and $L_{n,\lambda} < U_{\eta,\lambda}$ *for all* $0\leq\lambda\leq\eta_n$, it follows that

$$F_n(\mathbf{V}) = X_n \setminus \vee\{\mathbf{U}_n \setminus \mathbf{V}\}$$
$$< \vee\{L_{n,\lambda} \in \mathbf{L}_n : U_{n,\lambda} \in \mathbf{V}, \lambda>0\}$$
$$\in K$$

Thus $F_n(\mathbf{V})$ is contained in some member of K and $F_n(\mathbf{V}) \in \underline{I}^X$.It follows that $F_n(\mathbf{V}) \in K$. Thus by Lemma 3.2.14, X has a closure preserving α-shading by members of K which are closed fuzzy sets.

Now we give a complete characterisation of spaces with closure preserving α-shading by fuzzy sets with finite support.

3.2.16 Theorem The following are equivalent for a fuzzy topological Space X.

(a) X has a closure preserving α-shading by fuzzy sets with finite support.

(b) X is hereditarily α-metacompact and has a σ- closure preserving α-shading by fuzzy sets with finite support.

(c) X is hereditarily α-metacompact and weakly σ-discrete.

(d) X has a countable α-shading by fuzzy *1*-scattered subsets and is hereditarily α-metacompact.

Proof

(a) $\Rightarrow$ (b)

If X has a closure preserving α-shading by fuzzy sets with finite support , then every subspace of X also should have such a shading. Then from Proposition 3.1.5 it follows that X is hereditarily α-metacompact.

(a) $\Rightarrow$ (c)

This follows from Lemma 3.1.3 and Lemma 3.1.4.

(b) $\Rightarrow$ (c)

Given that X has a σ-closure preserving α-shading by fuzzy sets with finite support . Thus there is an α-shading $\{X_n: n\in N\}$ of X such that each of X_n has a c-p α-shading *by* fuzzy sets with finite support. Thus each of X_n is weakly σ-discrete. (by (a) $\Rightarrow$ (c)). Thus each of X_n is the supremum of a countable number of discrete subsets $\{X_{n,k} : n,k\in N\}$ such that $\vee_{k\leq m} X_{n,k}$ is closed in X_n for each $m\in N$. Since a countable collection of countable sets is countable, it follows that $\{X_{n,k} : n,k\in N\}$ is a countable α-shading which satisfies (c).

(c) $\Rightarrow$ (d)

Consider the set $\{Y_n : n\geq 1\}$ where $Y_n = \vee_{k\leq n} X_k$, for each $n\geq 1$. Then clearly $\{Y_n : n\geq 1\}$ is a countable closed α-shading of X. Now each X_n is discrete and hence is *1*-scattered. Also the union of two *1*-scattered spaces is also *1*-scattered. Therefore it follows that each Y_n is scattered.

(d) $\Rightarrow$ (a)

Follows from Theorem 3.2.15.

3.2.17 Lemma Let X be a fuzzy topological spaces and $U=\{U_\lambda : \lambda\in\Lambda\}$ be a point finite α-shading by open fuzzy sets. Let $B_n = \{x\in X : \alpha\text{-}Ord(x, U) \leq n\}$. Then $\{B_n : n\geq 0\}$ is an α-shading of X by closed fuzzy sets. If $n>0$ and F is a closed fuzzy set with $F<B_n$ and $F\wedge B_{n-1} = 0$, then F has a discrete α-shading by closed fuzzy sets where each member is contained in some $U \in U$.

Proof

For any $x\in X$ with $B_n(x)=0$ for some n, by the definition of B_n it follows that there is some $\Lambda'\subset\Lambda$ with $n+1$ numbers such that $U_\lambda(x) > \alpha$ *for all* $\lambda\in\Lambda'$. Now since each U_λ is fuzzy open , so is $\wedge\{U_\lambda : \lambda\in\Lambda\}$. This is an open fuzzy nbd of x disjoint from B_n. Therefore it follows that $1\setminus Bn$ is fuzzy open and so B_n are closed fuzzy sets.

Also given that U is a point finite α-shading of X. Therefore there exists at most finitely many $U\in U$ with $U(x)>\alpha$ for any $x\in X$. Then clearly $B_n(x)>\alpha$ for some n. Thus $\{B_n : n\geq 0\}$ is an α-shading of X

Take F as in the statement of the Lemma. Let Ω be the set of all subsets of Λ which have n elements and for each $\gamma\in\Omega$ define $V_\gamma = \wedge\{U_\lambda : \lambda\in\gamma\}$. Now clearly $V_\gamma \wedge F < U_\lambda$ for each λ in γ and the collection $\{V_\gamma \wedge F : \gamma\in\Omega\}$ is disjoint and hence a discrete α-shading of X by closed fuzzy sets.

3.2.18 Corollary Let $U = \{U_\lambda : \lambda < \eta\}$ be a point finite α-shading of a fuzzy topological spaces X by open fuzzy sets and $X_n = \{x\in X : \alpha\text{-}Ord(x, U) \leq n\}$ for each $n\geq 1$. Then $\{X_n : n\geq 1\}$ is a countable α-shading of X by closed fuzzy sets and $B_n=\{B(\lambda_1,\lambda_2,\lambda_3\ldots\lambda_n) : \lambda_1 < \lambda_2 < \lambda_3\ldots\eta\}$ is a discrete clopen α-shading of $X_n \setminus X_{n-1}$ for each $n\geq 1$ where $B(\lambda_1,\lambda_2,\lambda_3\ldots\ldots\lambda_n) = \wedge_{i\leq n} U_{\lambda i} \wedge (X_n \setminus X_{n-1})$.

Proof

Take $F = X_n \setminus X_{n-1}$ in Lemma 3.2.17 the corollary follows.

3.2.19 Definition An α-disjoint α-shading $\{L_\lambda: \lambda<\eta\}$ of a fuzzy topological space is a ***K***-scattered partition if $L_\lambda(x) \leq N(x)$ *for all* $x\in X$. and for some $N\in \boldsymbol{K}$ and $\vee\{L_\mu : \mu < \eta\}$ is fuzzy open in X for each $\lambda<\mu$.

3.2.20 Theorem Let ***K*** be a finitely additive class of fuzzy topological spaces. If a hereditarily α-metacompact space X is K-scattered then Player I has a winning strategy in $G^*(\boldsymbol{DK},X)$.

Proof

Since X is fuzzy ***K***-scattered, X has a fuzzy ***K***-scattered partition. Say $\boldsymbol{V}= \{V_\lambda: \lambda<\eta\}$ .Now from proposition 3.2.11, it follows that there exists a point finite fuzzy open expansion $\boldsymbol{U}=\{U_\lambda: \lambda<\eta\}$ of $\boldsymbol{V}$. Now since $\boldsymbol{V}$ is an α-shading of X, it follows that $\boldsymbol{U}$ is also an α-shading of X . Let X_n and B_n , $n \geq 1$ be taken as in Corollary 3.2.18 . For each $F\in \underline{I}^X$, take $k(F) = Min\{ k \geq 1 : F\wedge X_k \neq 0\}$ and $\boldsymbol{B(F)} = \{ B(\lambda_1,\lambda_2,\lambda_3...,\lambda_n)\wedge F : B(\lambda_1,\lambda_2,\lambda_3...,\lambda_n) \in \boldsymbol{B_k}$ *and* $k = k(F)\}$ and $\boldsymbol{B(0)}=\{0\}$. Now by Corollary 3.2.18 it follows that each member of $\boldsymbol{B(F)}$ is fuzzy closed in X and $\boldsymbol{B(F)}$ is discrete in X.

We have $B(\lambda_1,\lambda_2,\lambda_3...,\lambda_k) = \bigwedge_{i\leq k} U_{\lambda i} \wedge (X_k \setminus X_{k-1})$. Thus $B(\lambda_1,\lambda_2,\lambda_3...,\lambda_k) < \bigvee_{i\leq k} U_{\lambda i} < \bigvee_{i\leq k} V_{\lambda i}$. Also since each $\boldsymbol{B(F)}$ is fuzzy closed and ***K*** is finitely additive

$\cup \boldsymbol{B(F)} = \cup \{ B(\lambda_1,\lambda_2,\lambda_3...,\lambda_k) \wedge F : B(\lambda_1,\lambda_2,\lambda_3...,\lambda_k) \in \boldsymbol{B_k} , k=k(F) \}$. Also by Corollary 3.2.18, $\boldsymbol{B_k}$ is a discrete α-shading of $X_k \setminus X_{k-1}$ by closed fuzzy sets. Hence $(Xk \setminus X_{k-1}) \wedge F\in DK \cap \underline{I}^X$ where $k=k(F)$.

Now we define a fuzzy stationery winning strategy S of Player I for $G^*(\boldsymbol{DK},X)$ as follows

$S : \underline{I}^X \rightarrow DK \cap \underline{I}^X$, where $S(F) = (X_{k(F)} \setminus X_{k(F)-1}) \wedge F$

Consider the play $(S(X), F_1, S(F_1), F_2, ...)$ of $G^*(DK,X)$. We have clearly $S(Fn)<Fn$ and hence S is stationery. Now we want to prove S is winning, that is $Inf\ F_n= 0$. Now since $\{X_n : n\geq 1\}$ is an α-shading of X and $F_n \wedge X_n = 0$ for all $k=1,2,3...$ It follows that it is enough to prove $F_n \wedge X_n = 0$ for all $n \geq 0$. We will prove this by

induction. Let $F_n \wedge X_n = 0$ and assume that $F_n \wedge X_{n+1} \neq 0$. Therefore by definition of $k(F_n)$ we get $k(F_n) = n+1$.

Now $S(F_n) \wedge F_{n+1} = ((X_{n+1} \setminus X_n) \wedge F_n) \wedge F_{n+1}$

$$= (X_{n+1} \setminus X_n) \wedge F_{n+1}$$

$$= 0$$

Now clearly $X_n \wedge F_n = 0$ and $F_{n+1} < F_n$. Hence $F_n \wedge X_{n+1} = 0$. Therefore it follows that $F_{n+1} \wedge X_{n+1} = 0$. Thus the proof is complete by induction.

3.3 Countably α–compact Spaces and the Game $G^*(DK, X)$

We know that most of the properties of countably compact spaces in general topology are discussed in terms of cluster points and accumulation points. So we define α-cluster points and α–accumulation points in fuzzy topological spaces in a language which is closely related to shading families and in this framework we obtain a characterisation for countable compactness in fuzzy topological spaces and later use this to obtain some relations of countably α–compact spaces and the fuzzy topological Game $G^*(DK, X.)$.

3.3.1 Definition. Let $\alpha \in [0,1]$. An α-cluster point (resp. α^*- cluster point) of a set A in a fuzzy topological space X is a fuzzy point x_λ such that each fuzzy neighbourhood U of x_λ with $U(x) > \alpha$ (resp. $U(x) \geq \alpha$) contains some fuzzy point of A with distinct support.

3.3.2 Definition. A sequence $(x^n_{\lambda n})$ of fuzzy points with distinct support in a fuzzy topological space X α-accumulates at x_λ (resp. α^*-accumulates) at x_λ if and only if for every fuzzy neighbourhood U of x_λ with $U(x) > \alpha$ (resp. $U(x) \geq \alpha$) and for every $n \in N$, there is an $m \geq n$ such that $x^m_{\lambda m} < U$ and (λ_n)accumulates at λ in the crisp sense in $[0,1]$

3.3.3 Theorem . The following are equivalent in a fuzzy topological space.

(i) X is countably α - compact.

(ii) Every fuzzy subset of X with countably infinite support has at least one α-cluster point

(iii) Every sequence of fuzzy points in X with distinct support has an α -accumulation point.

Proof:

(i) $\Rightarrow$ (ii)

If possible let A be a fuzzy subset of X with countably infinite support and has no α-cluster point. Then it follows that every fuzzy point $x^{i}_{\lambda i}$ in A has a fuzzy neighbourhood U_i with $U_i\ (x^{i}) > \alpha$ which contains no other fuzzy point of A with distinct support. Now $Supp(A)$ clearly closed and $X \setminus Supp(A)$ is open . Now consider the collection $\chi_{X \setminus Supp(A)} \cup \{U_i : i \in N\}$.This is clearly a countable α-shading of X by open fuzzy sets which has no finite α-subshading.

(ii) $\Rightarrow$ (iii)

Let $(x^{n}_{\lambda n})$ be a sequence of fuzzy points in X with distinct support. Then there are two possibilities.

(a) Cardinality of the support of the range set is countably infinite. Then by (ii) this has atleast one α-cluster point say x_λ .Now every fuzzy nbd U of x_λ with $U(x)>\alpha$ contains infinitely many points of the sequence other than x_λ. Clearly this x_λ is an α - accumulation point of the sequence. For, For any $n \in N$ the set $\{x^{n}_{\lambda n} : 1 \leq n \leq N\}$ is finite .Therefore it follows that for any neighbourhood U of x_λ with $U(x)>\alpha$ and for any $n \in N$, there is an $m \geq n$ such that $x^{m}_{\lambda m} < U$ and (λ_n) accumulates at λ .

(b) If cardinality of range set is finite , then there should be some fuzzy point x_λ with $x^n_{\lambda n} = x_\lambda$ for infinitely many $n \in N$. Then clearly this x_λ is an α-accumulation point.

(iii) $\Rightarrow$ (i)

Let X be not countably α-compact . Let $U=\{U_1, U_2, U_3 \ldots\ldots\}$ be a countable α-shading of X by open fuzzy sets which has no finite α- subshading. Therefore $\{U_1, U_2, U_3 \ldots\ U_k\}$ cannot α-shade X for any finite k. Therefore corresponding to each finite k we can find an $x^k \in X$ such that $U_j(x^k) > \alpha$ for some $j>k$ and $U_i(x^k) \leq \alpha$ for $1 \leq i \leq k$. Let $U_j(x^k) = \eta_k$ where $\eta_k \in (\alpha\ ,1]$. Now the sequence $(x^k{}_{\eta k})$has no α-accumulation point. For, if possible let $x_\eta$be an α-accumulation point of $(x^k{}_{\eta k})$.Now since U is an α-shading of X, we can find a minimum $l \in N$ such that $U_l(x) > \alpha$ and $U_i(x) \leq \alpha$ for all $1 \leq i \leq l$. Now take $n=l+1$ and consider the neighbourhood U_l of x. Then for any $m \geq n$ we have $x^m{}_{\eta m} > U_l$ .For corresponding to any m, we can find some U_j such that $U_j\ (x^m) > \alpha$ for some $j>m$.and $U_i(x^m) \leq \alpha$ for $1 \leq\ i \leq m$. Here $m \geq n = l+1$. Therefore $l<m$ and it follows that $U_l(x^m) \leq \alpha$.. But $\eta_m \in (\alpha\ ,1]$. Thus $x^m{}_{\eta m} < U_l$ which is a contradiction. This completes the proof.

3.3.4 Definition [M;B1].A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space (X,T) is said to be discrete if for each x in X , there exists an open fuzzy set g of X with $g(x) = 1$ such that $a_s \leq 1-g$ holds for all but at most one s in S.

3.3.5 Theorem. If C is a closure preserving α-shading of a fuzzy topological space X by fuzzy closed and countably α-compact sets and if K is a class of fts with $C \subset K$, then Player I has a fuzzy stationary winning strategy in $G^*\ (DK,X)$

Proof

Corresponding to each fuzzy closed set F in X, consider the collection$\{C \wedge F: C \in C\}$ and let $D\ (F)$ be the maximal disjoint sub-collection of this. This is possible since C is an α-shading of X. Clearly $D(F)$ is closure preserving and disjoint and hence it is discrete. Now define $S: \underline{I}^X \xrightarrow{\text{into}} \underline{I}^X \cap DK$ by $F \to \vee D(F)$. We will show that S is a fuzzy stationary winning strategy for Player I in $G^*(DK,X)$.

Let $\{F_n : n \in N\}$ be a decreasing ($F_1 > F_2 > F_3 \ldots$) sequence with $S(X) \wedge F_1 = 0$ and $S(F_n) \wedge F_{n+1} = 0$. If possible let if $\underset{n\geq 1}{Inf} F_n \neq 0$. Then there exists $C_0 \in C$ such that C_0 has non empty meet with each of F_n. Now $C_0 \wedge F_n \notin D(F_n)$ for each $n \geq 1$. For, If $C_0 \wedge F_n \in D(F_n)$ for some n, then

$C_0 \wedge F_n = (C_0 \wedge F_n) \wedge F_{n+1}$

$< [\vee D(F_n)] \wedge F_{n+1}$

$= S(F_n) \wedge F_{n+1}$

$= 0$. This is a contradiction. Therefore $C_0 \wedge F_n \notin D(F_n)$ for each for every $n \geq 1$.

Fix some $n \geq 1$. $D(F_n)$ is maximal and disjoint. Also $C_0 \wedge F_n \notin D(F_n)$. Therefore we can take some $C_n \in C$ such that $C_n \wedge F_n \in D(F_n)$ and $(C_n \wedge C_0) \wedge F_n \neq 0$. For each $n \geq 1$, take some $x^n \in X$ such that $[(C_0 \wedge F_n) \wedge C_n](x^n) > \alpha$ where $\alpha \in (0,1]$. Let Min $\{C_0(x), F_n(x), C_n(x) = \lambda_n$. Now clearly we have $[S(F_n)](x^n) > \alpha$. Also $S(F_n) \wedge F_{n+1} = 0$. Therefore $F_{n+1}(x^n) = 0$. Now consider the sequence $(x^n{}_{\lambda n})$ in C_0. Now C_0 is countably α-compact. There fore it has an α-cluster point say x_λ in C_0. This follows from Theorem 3.3.3.

Now we have $\underset{n\geq 1}{Inf} F_n(x) > \alpha$. For, if $F_n(x) \leq \alpha$ for some n, then we can choose some $m \geq n$ with $\lambda_m > F_n(x^m)$. But $F_m < F_n$. Therefore $F_m(x^m) < F_n(x^m)$. Now $\lambda_m \leq F_m(x^m) < F_n(x^m)$. There fore $\lambda_m < F_n(x^m)$. This is a contradiction.

Now claim $Sup_{n\geq 1} C_n(x) = 0$. For, let $C_n(x) > 0$ for some n. Now $C_0 \wedge F_n \in D(F_n)$ and $F_{n+1}(x) > \alpha$. Then $(C_n \wedge F_n \wedge F_{n+1})(x) < (S(Fn) \wedge F_{n+1})(x)$

$= 0$. Therefore $C_n(x) = 0$. This is a contradiction.

Since C is closure preserving , we have $cl\{x^n{}_{\lambda n} : n\geq 1\}(x) > \alpha$. Also $cl\ \{x^n{}_{\lambda n} : n\geq 1\} < cl\ Sup_{n\geq 1} C_n = Sup_{n\geq 1} C_n$. There fore $Sup_{n\geq 1} C_n(x) > \alpha$, where $\alpha \in (0,1]$. This is a contradiction to $Sup_{n\geq 1} C_n(x) = 0$. This completes the proof.

From Theorem 3.3.5 and Theorem 2.3.4 , next corollary follows clearly.

3.3.6 Corollary If a fuzzy topological space X has a σ–closure preserving α–shading by α–compact closed fuzzy sets , then player I has a winning strategy in $G^*(DC,X)$

Chapter - IV

FUZZY P-SPACES AND THE GAME $G_\alpha(X)$

The concept of *P*-spaces was introduced by K.Morita and as a generalization of this in this chapter we define fuzzy *P*-spaces (P_α-spaces) and some results regarding them are obtained. Again a characterization of P_α-spaces in terms of a particular type of fuzzy topological game $G_\alpha(X)$ is also obtained.

4.1 Fuzzy P-Spaces

4.1.1 Definition A collection $\{U_i: i = 1,2,3, \ldots\ldots .\}$ of fuzzy subsets of a set X is called an increasing family if $U_i < U_{i+1}$ for every $i = 1,2,3, \ldots..$

4.1.2Definition A fuzzy topological space X is said to be a P_α-space if for every increasing family $\mathcal{U} = \{ U(a_1,a_2, \ldots..a_i): a_1,a_2, \ldots..a_i \in A,\ i=1,2,3..\}$ of open fuzzy sets in X, there exists a precise refinement $F = \{ F(a_1,a_2, \ldots..a_i): a_1,a_2, \ldots..a_i \in A,\ i=1,2,3..\}$ by closed fuzzy sets satisfying the condition that if $\mathcal{U}$ is an α-.shading of X, then F is also an α-shading of X where $\alpha \in [0,1)$.

4.1.3Theorem A fuzzy topological space X is a P_α-space if and only if there exists a crisp function $p: \cup G^n \to \underline{I}^X$ such that

(i) If $(G_1,G_2,G_3 \ldots\ldots\ldots) \in G^n$, $n \in N$ then $p(G_1,G_2,G_3 \ldots\ldots\ldots) < Sup\{G_k: 1 \le k \le n\}$

(ii) If $\{G_1,G_2,G_3 \ldots\ldots\ldots\ldots\ldots\}$ is an α-shading of X, then so is $\{p(G_1), p(G_1,G2), p(G_1,G_2,G_3), \ldots\ldots\ldots\}$.Where G represent the family of all open fuzzy subsets of X.

Some Results mentioned in this Chapter are accepted for publication in the paper titled *Fuzzy P-Spaces* in the Journal *Discourse(s)* No.3 Vol. I (2000)

Proof

Let X be a P_α-space. Let $(G_1,G_2,G_3........) \in G^n$ and take $a_i = G_i$ *in* the definition of P_α-spaces and define $U(a_1,a_2,.....a_n) = U(G_1,G_2,G_3...G_n) = Sup\ \{G_i : 1 \leq k \leq n\}$.Then clearly $U(G_1,G_2,G_3...G_n) < U(G_1,G_2,G_3...G_{n+1})$. Then from the definition of P_α-spaces the remaining follows.

Conversely let $U = \{U(a_1,a_2,.....a_i): a_i \in A,\ i = 1,2,3.....\}$ be an increasing family of open fuzzy sets in X. Now corresponding to each $U(a_1,a_2,a_3,......a_i)$ *in* U , we define

$$F(a_1,a_2,.....a_i) = p(U(a_1), U(a_1,a_2)\ U(a_1,a_2,a_3),....\ U(a_1,a_2,a_3,......a_n))$$
$$< Sup\ \{U(a_1,a_2,.....a_i) : 1 \leq i \leq n\}$$
$$= U(a_1,a_2,.....a_i) \text{ since } U \text{ is increasing.}$$

Now if U is an α-shading of X, for every $x \in X$, there exists a $U(a_1,a_2,a_3,......a_k)$ such that $U(a_1,a_2,a_3,......a_k)(x) > \alpha$. Now clearly by definition, we have $F(a_1,a_2,a_3,......a_k)(x) > \alpha$ and hence $\{F(a_1,a_2,a_3,......a_i): a_i \in A,\ i=1,2,3,....\}$is an α-shading of X. Hence X is a P_α-space.

From the definition of P_α-Spaces and Theorem 4.1.3 next theorem follows clearly.

4.1.4 Theorem A fuzzy topological space X is a P_α-space if and only if there is a crisp function defined from the family of all increasing finite sequences of open fuzzy sets G to the collection of all closed fuzzy sets $\underline{I}^X$ with $p(G_1,G_2,G_3...G_n) < G_n$ where $(G_1,G_2,G_3...G_n) \in G^n$ and if $G_n < G_{n+1}$for each $n \in N$ and if $\{ G_1,G_2,G_3...G_n \}$is an α-shading then so is $\{p(G_1), p(G_1,G2), p(G_1,G_2,G_3), ...\}$.

4.1.5 Theorem A fuzzy topological space X is a P_α-space if and only if there exists a crisp function $p: \cup (\underline{I}^X)^n \to \underline{I}^X$ such that

(i) For each $(F_0, F_1, \ldots\ldots\ldots F_n) \in (\underline{I}^X)^n$, $n \geq 0$

$p(F_0, F_1, \ldots\ldots\ldots F_n) \wedge \mathrm{Inf}_{i \leq n} F_i = 0$

(ii) For each $(F_0, F_1, \ldots\ldots\ldots) \in (\underline{I}^X)^\omega$ with $\mathrm{Inf}_{n \geq 1} F_n = 0$;

the collection $\{p(F_0, F_1, \ldots\ldots\ldots F_n) : n \geq 0\}$ is an α-shading of X.

Proof

Let $(F_1, \ldots\ldots\ldots F_n) \in (\underline{I}^X)^n$ Then F_1', $F_1' \vee F_2'$, $F_1'^c \vee F_2' \vee F_3'$, ... is an increasing family of open sets . Take $U(a_1) = F_1'$, $U(a_1, a_2) = F_1' \vee F_2'$ ... $U(a_1, a_2 \ldots\ldots, a_n) = F_1' \vee F_2' \vee \ldots\ldots\ldots \vee F_n'$. Now since X is a P_α -space, there exists a collection $\{F(a_1), F(a_1, a_2) \ldots\ldots\}$ such that $F(a_1, a_2 \ldots\ldots, a_i) < U(a_1, a_2 \ldots\ldots, a_i)$ for each $i = 1, 2, 3, \ldots$

Now define $p(F_1, \ldots\ldots\ldots F_n) = 0$ if $\mathit{Inf}_{i \leq n} F_i \neq 0$

$= F(a_1, a_2, \ldots\ldots\ldots a_n)$ otherwise

Clearly p has properties (i) and (ii)

Conversely let $(G_1, G_2, \ldots\ldots\ldots\ldots G_n) \in G^n$. Then $F_1 = G_1'$, $F_2 = G_2'$, ... $F_n = G_n'$ are all closed and hence there exists a function $p' : (\underline{I}^X)^n \rightarrow \underline{I}^X$ such that

$$p'(F_1, \ldots\ldots\ldots F_n) \wedge \mathit{Inf}_{i \leq n} F_i = 0.$$

Take $p(G_1, G_2, \ldots\ldots\ldots\ldots G_n) = p'(F_1, \ldots\ldots\ldots F_n)$ in Theorem 4.1.3, then

$p(G_1, G_2, \ldots\ldots\ldots\ldots G_n) \wedge \mathit{Inf}_{i \leq n} F_i = 0$

Therefore $p(G_1, G_2, \ldots\ldots\ldots\ldots G_n) < (\mathit{Inf}_{i \leq n} F_i)'$

$= \mathit{Sup}_{i \leq n} F_i'$

$= \mathit{Sup}_{i \leq n} G_i$ and hence p satisfies (i) and (ii) of

Theorem 4.3.1 and hence X is a P_α-space.

4.1.6 Theorem If a fuzzy topological space X has a σ-closure preserving fuzzy closed α-shading by countably α-compact sets, then X is a P_α - Space.

Proof

Let $F = \cup\{F_n : n \in N\}$ be an α-shading of X such that each F_n is closure preserving and countably α-compact. Let $\{U(a_1,a_2,\ldots\ldots a_n): a_i \in A,\ i = 1,2,3,\ldots\}$ be an increasing sequence of open fuzzy sets.Now corresponding to each $U(a_1,a_2,\ldots\ldots a_n)$we define $F(a_1,a_2,\ldots\ldots a_n) = Sup\ \{F:F< U(a_1,a_2,\ldots\ldots a_n),\ F\in \overset{n}{\underset{i=1}{\cup}} F_i\}$. Since $\overset{n}{\underset{i=1}{\cup}} F_i$ is closure preserving it follows that $F(a_1,a_2,\ldots\ldots a_n)$ is fuzzy closed and $F(a_1,a_2,\ldots\ldots a_n) < U(a_1,a_2,\ldots\ldots a_n)$ for each $n\geq 1$.

Again let $\{U(a_1,a_2,\ldots\ldots a_i) : i = 1,2,3\ldots\}$be an α-shading of X. Let $x\in X$, Now since F is an α-shading of X, there exists an $F_0 \in F$ such that $F_0(x)>\alpha$.Let $F_0\in F_k$ for some k. Since F_0 is countably α-compact, and $U(a_1,a_2,\ldots)$'s are increasing we can find out some $j \in N$ *such* that $j \geq k$ and $F_0 < U(a_1,a_2,\ldots,a_j)$.

Now $F(a_1,a_2,\ldots,a_j).(x) = \underset{F<U(a1,a2,\ldots,aj)}{Sup}\ \{F(x): F\in \overset{n}{\underset{i=1}{\cup}} F_i\} \geq F_0(x) > \alpha$

Thus $\{F(a_1,a_2,\ldots,a_j) : a_i \in A,\ i = 1,2,3\ldots\}$ is also an α -shading of X. This completes the proof.

4. 2 A Characterisation of P_α-spaces using the Game $G_\alpha(X)$

In this section we describe a game associated with P_α-spaces. Here $G_\alpha(X)$ denote the following infinite positional fuzzy topological game. Let G and F denote the collection of all open (resp. closed) fuzzy subsets of a fuzzy topological space X. There are two players Player I and Player II. Players alternatively choose fuzzy subsets of X so that each player knows X and first k elements when he is choosing the $(k+1)^{th}$ element.

We say that a sequence $(G_1,F_1,\ldots\ldots G_n,F_n)$ is a play for $G_\alpha(X)$ if for each $n \geq 1$, we have

(i) $G_n \in \boldsymbol{G}$ is a choice of Player I.

(ii) $F_n \in \boldsymbol{F}$ and $F_n < Sup\ \{G_k : 1 \le k \le n\}$ is a choice of Player II.

Player I wins the play $(G_1, F_1, \ldots\ldots G_n, F_n)$ if $\{G_n : n \in N\}$ is an α-shading of X and $\{F_n : n \in N\}$ is not. And Player II wins if $\{F_n : n \in N\}$ or both $\{G_n : n \in N\}$ and $\{F_n : n \in N\}$ are α-shadings of X.

A strategy for Player I is a crisp function S: $\{0\} \bigcup_{n=1}^{\infty} \boldsymbol{F}^{\mathbf{n}} \to \boldsymbol{G}$ and that of Player II is t: $\bigcup_{n=1}^{\infty} \boldsymbol{G}^{\mathbf{n}} \to \boldsymbol{F}$ such that $t\ (G_1, G_2, \ldots\ldots G_n) <$ Sup $\{ G_i: 1 \le i \le n\}$ for each $(G_1, G_2, \ldots\ldots G_n) \in \boldsymbol{G}^{n}$ and $n \ge 1$.

Now clearly for each pair of strategies (s, t) there exists a unique Play$(G_1, F_1, \ldots\ldots G_n, F_n)$ of $G_\alpha (X)$ defined as follows.
Take $G_1 = s(0)$, $F_1 = t(G_1)$, $G_2 = s(F_1)$, $F_2 = t(G_1, G_2)$ and so on.

A strategy s (resp.t) is winning for Player I (resp. Player II) if he wins every play of $G_\alpha (X)$ using it.

From Theorem 4.1.6 and definition of $G_\alpha (X)$, we get the following game theoretic characterization of P_α-spaces.

4.2.1 Theorem A fuzzy topological space,is a P_α-space if and only if Player II has a winning strategy in $G_\alpha (X)$.

4.3 Remarks

Just like the applications of P-spaces in general topology, P_α-spaces help the study of covering properties in fuzzy topological spaces. In fact the results regarding the product of α-metacompact spaces is discussed in Chapter VI and there it is shown that the product of two α–metacompact spaces need not be α–metacompact and if we impose some conditions on one of these spaces, we can make the product α–metacompact and this is done in terms of P_α-spaces.

CHAPTER - V

GAMES IN PRODUCT SPACES

In this chapter we study topological games in product fuzzy topological spaces. For two classes of fuzzy topological spaces K_1 and K_2 we define $K_1 \times K_2$ as the set of all product spaces $X \times Y$ such that $X \in K_1$, $Y \in K_2$ and all closed subsets of them. Here we explore the possibility of having a winning strategy for Player I in $G^*(D(K_1, \times K_2), X \times Y)$ if he has the same in $G^*(K_1, X)$ and $G^*(K_2, Y)$. Here we make use of concepts like fuzzy rectangles, fuzzy D-products etc.

5.1 Preliminaries

5.1.1 Definition [WON2] Let $\{X_i\}_{i\in I}$ be a family of fuzzy topological spaces. Let $X = \prod_{i\in I} X_i$ be the usual Cartesian product and let P_i be the projection from X on to X_i for each $i\in I$. The set X with fuzzy topology having the family $F = \{ P_i^{-1}(B) : B \in T_i, i \in I \}$ as a sub base is called the product fuzzy topological space.

5.1.2 Definition Let $X\times Y$ be a fuzzy product space. A fuzzy subset of the form $R = R` \times R``$ where $R`$ and $R``$ are projection of R in to X and Y respectively is called a fuzzy rectangle in $X\times Y$.

5.1.3 Definition Let X be a fuzzy topological space. A fuzzy subset U of X is called a co-zero fuzzy set if there is an F-continuous function $C:X:\to[0, 1]$ such that $C^{-1}(0) = 1\text{-}U$.

5.2 Fuzzy Games in Product Spaces

5.2.1 Definition A product fuzzy topological space $X \times Y$ is called a fuzzy D-product if for any disjoint pair $\{E,F\}$ of a closed fuzzy rectangle E and a closed fuzzy set F in $X \times Y$, there exists a σ-discrete collection $\boldsymbol{R}$ of closed fuzzy rectangles such that $F <$ Sup $\{R : R \in \boldsymbol{R}\} < (X \times Y) \setminus E$

5.2.2 Theorem Let X and Y be two fuzzy topological spaces such that Y is α-compact and player I has a fuzzy winning strategy in G^{*} $(\boldsymbol{C}, X)$ then player I has a fuzzy winning strategy in G^{*} $(\boldsymbol{C}, X \times Y)$ where $\boldsymbol{C}$ is the class of all α-compact spaces.

Proof

Let P be the projection map from $X \times Y$ on to X. Now since Y is α-compact, it follows that $P^{-1}(x)$ is α-compact for each $x \in X$. Since P is F-continuous, it follows that P is an α-perfect map. Also since $\boldsymbol{C}$ is an α-perfect class, from Theorem 2.4.7 it follows that player I has a fuzzy wining strategy in G^{*} $(\boldsymbol{C}, X \times Y)$.

Now from the fact that the class of all fuzzy $\boldsymbol{C}$ –scattered spaces ($\widetilde{S}$ $\boldsymbol{C}$) forms an α- perfect class and an argument similar to that in proof of Theorem 5.2.2 it follows that

5.2.3Theorem Let X and Y be two fuzzy topological spaces such that Y is α-compact and player I has a fuzzy winning strategy in G^{*} ($\widetilde{S}$ $\boldsymbol{C}$, X), then player I has a fuzzy winning strategy in G^{*} ($\widetilde{S}$ $\boldsymbol{C}$, $X \times Y$).

5.2.4 Theorem: Let $X \times Y$ be a fuzzy D-product. If player I has fuzzy winning strategies in G^{*} $(\boldsymbol{K}_1, X)$ and G^{*} $(\boldsymbol{K}_2, X)$ then he has a fuzzy winning strategy in $G^{*}(D(\boldsymbol{K}_1, \times \boldsymbol{K}_2), X \times Y)$.

Proof

For convenience we use the following notations.

$N_0 = \{0, 3, 6, 9, \ldots\}$, $N_1 = \{1, 4, 7, \ldots\}$ $N_2 = \{2, 5, 8, \ldots\}$ so that $N_0 \cup N_1 \cup N_2 = \omega \cup \{0\}$. $T = (k_1, k_2, \ldots, k_n) \in \omega^n$ $n \geq 1$ and $\Sigma T = k_1 + k_2 + \ldots + k_n$. For each $T = (k_1, k_2, \ldots, k_n)$ with $k_n \notin N_0$, take $T^* = (k_1, k_2, \ldots, k_i)$ if $k_n, k_{n-1}, \ldots, k_{i+1} \in N_1$ ($\in N_2$) and $k_i \notin N_1$($\notin N_2$) for some $i<n$. if $k_n, k_{n-1}, \ldots, k_1 \in N_1$ ($\in N_2$) put $T^* = (0)$.

Now by Theorem 2.3.3, it is enough.if we construct a fuzzy winning strategy t for $G^*(L, X\times Y)$ where $L = F[D(K_1 \times K_2)]$. Take $E_1 = t(X\times Y) = S_1(X) \times S_2(Y)$ where S_1 and S_2 are fuzzy stationery winning strategies for player I in G^* $(K_1,\ X)$ and G^* $(K_2,\ X)$ receptively. Take $A_0 = \{0\}$. $R(0) = \{X\times Y\}$ and $F_0 = R(0) = X\times Y$. Player II chooses some $F_1 \in \underline{I}^{X\times Y}$ with $F_1 \wedge E_1 = 0$.

Assume that we have already constructed an admissible sequence $(E_1, F_1, \ldots, E_m, F_m)$ in $G^*(L, X\times Y)$ such that $E_i = t(F_1, F_2, \ldots, F_{i-1})$ for each $i \leq m$ and that there exists a discrete collection $\{R(a)\colon\ a = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in A_{T\oplus k}\}$ of closed fuzzy rectangles for each $T \in \omega^n$, $n \geq 0$ with $\Sigma T \leq m\text{-}1$ and $k \geq 1$ satisfying

(1) $a \oplus \alpha \in A_{T\oplus k}$ implies $a \in A_T$

(2) For each $a \in A_T$, $R(a) \wedge F_{\Sigma T+1} < Sup\{R(a \oplus \alpha) : a \oplus \alpha \in A_{\Sigma T+1}, k \geq 1\} < R(a)$

(3) For each $a \oplus \alpha \in A_{T\oplus k}$ where $T = (k_1, k_2, \ldots, k_n) \in \omega^n$, $n \geq 0$, $k \geq 1$.

(i) $S_1(R(a)') \wedge R(a\oplus\alpha)' = 0$ if $k_n, k \in N_0 \cup N_1$

(ii) $S_2(R(a)'') \wedge R(a\oplus\alpha)'' = 0$ if $k_n, k \in N_0 \cup N_2$

(iii) $S_1(R(a^*)') \wedge R(a\oplus\alpha)' = 0$ if $k_n \in N_2$ and $k \in N_0 \cup N_1$

(iv) $S_2(R(a^*)'') \wedge R(a\oplus\alpha)'' = 0$ i f $k_n \in N_1$ and $k \in N_0 \cup N_2$

where $a^* = a/i \in A_{T^*}$ if $a \in A_{T^*}$, $T^* \in \omega^i$.

If T=0 consider $k_n = 0 \in N_0$.

Take $T = (k_1, k_2, \ldots, k_n)) \in w^n$, $n \geq 0$, with $\Sigma T = m$. Now $T = T\_ \oplus k_n$ and hence $\Sigma T = m - k_n \leq m\text{-}1$. Therefore $R(T)$ is constructed.

For each $a \in A_T$ take $E(a) = S_1(R(a)') \times S_2(R(a)'')$ if $k_n \in N_0$

$E(a) = S_1(R(a)') \times S_2(R(a^*)'') \wedge R(a)''$ if $k_n \in N_1$

$E(a) = S_1(R(a^*)') \wedge R(a`) \times S_2(R(a)')$ if $k_n \in N_2$

Then $E(a) \in K_1 \times K_2$ and take

$$E_{m+1} = t(F_0, F_1, \ldots, F_m) = \sup_{\substack{a \in A_T \\ T \in \cup_{n \geq o} \omega^n \\ \sum T = m}} E(a) \wedge F_m$$

and clearly $E_{m+1} \in L$ since $\{R(a): a \in A_T\}$ is discrete in $X \times Y$. Now player II choose an $F_{m+1} \in \underline{I}^{X \times Y}$ with $F_{m+1} < F_m$ and $F_{m+1} \wedge E_{m+1} = 0$.

Again take some $T \in \cup_{n \geq 0} \omega^n$ with $\Sigma\ T = m$ and $a \in A_T$. Since $R(a)$ is a fuzzy D-product and $E(a)$ is a closed fuzzy rectangle in $R(a)$ with $E(a) \wedge F_{m+1} = 0$ there exists a σ-discrete collection $\{R(a \oplus \alpha): \alpha \in B(a, k), k \geq 1\}$ of closed fuzzy rectangles in $R(a)$.

Now, $R(a) \wedge F_{m+1} < \sup_{\substack{\alpha \in B(a,k) \\ k \geq 1}} R(a \oplus \alpha) < R(a) \setminus E(a)$.

Now from the fact that $E(a)$ and $R(a \oplus \alpha)$ are disjoint fuzzy rectangles in $X \times Y$, we may assume that for each $\alpha \in B(a, k)$.

$E(a)' \wedge R(a \oplus \alpha)' = 0$ and $E(a)'' \wedge R(a \oplus \alpha)'' = 0$ implies $k \in N_0$

$E(a)' \wedge R(a \oplus \alpha)' = 0$ and $E(a)'' \wedge R(a \oplus \alpha)'' \neq 0$ implies $k \in N_1$

$E(a)' \wedge R(a \oplus \alpha)' \neq 0$ and $E(a)` \wedge R(a \oplus \alpha)'' = 0$ implies $k \in N_2$.

Define $A_{T \oplus k} = \{ a \oplus \alpha \mid \alpha \in B(a, k), a \in A_T\}$ for each $k \geq 1$. Now clearly $\{R(b): b \in A_{T \oplus k}\}$ is discrete in $X \times Y$ and (1)-(3) are satisfied.

Now we will prove that t is the required winning strategy by showing that $\inf_{n \geq 1} F_m = 0$. For, if possible let $\inf_{n \geq 1} F_m(x,y) = \eta$ for some $(x, y) \in X \times Y$ and $\eta \in (0, 1]$. Then by (2) it follows that we can choose some $(k_1, k_2, \ldots) \in \omega^\omega$ and $(\alpha_1, \alpha_2, \ldots)$ such that $a_n = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in A_{T_n}$ where $\mathrm{T}_n = (k_1, k_2, \ldots, k_n)$ and $R(a_n)(x, y) = \eta$ for each $n \geq 1$ for each $n \geq 1$. Then $R(a_n)'(x) \geq \eta$ and $R(a_n)''(y) \geq \eta$ for each $n \geq 1$.

$\therefore \inf_{n\geq 1} R(a_n)'(x) \geq \eta$ and $\inf_{n\geq 1} R(a_n)''(y) \geq \eta$. (a)

Now assume that $(k_1, k_2, \ldots)$ contains an infinite sequence $(k_{i(1)}, k_{i(2)}, \ldots)$ consisting of all nos in $N_0 \cup N_1$. Let $\mathrm{T}(0)=a(0)=0$ and $\mathrm{T}(j) = T_{i(j)} = (k_{i(1)}, \ldots, k_{i(j)})$ and $a(j) = a_{i(j)} = (\alpha_{i(1)}, \ldots, \alpha_{i(j)})$ for each $j \geq 1$. Now claim $\mathrm{S}_1(R(a(j))') \wedge R(a(j+1))' = 0$ for each $j \geq 0$. For if $k_{i(j)+1} \in N_0 \cup N_1$, $\mathrm{T}(j+1) = \mathrm{T}(j) \oplus k_{i(j)+1}$ follows from $k_{i(j)+1} = k_{i(j+1)}$ and hence $a(j+1) = a(j) \oplus \alpha_{i(j)+1} \in A_{\mathrm{T}(j+1)}$. Now claim follows from (3)(i). Similar argument holds for $k_{i(j+1)} \in N_2$ also.

Now from the claim above and (2) it follows that $(S_1(X), R(a(1))', S(R(a(1)')), R(a(2))', \mathrm{S}(R(a(1))') \ldots)$ is a play of $G^*(K_1, X)$ and hence $\inf_{n\geq 1} R(a(j))'=0$. This is a contradiction to (a). Now a similar argument holds good for the case of $(k_1, k_2, \ldots)$ contains an infinite sequence $(k_{i(1)}, k_{i(2)}, \ldots)$ consisting of all natural numbers k_n belonging to $N_0 \cup N_2$ also. This completes the proof.

Chapter VI

APPPLICATIONS OF FUZZY TOPOLOGICAL GAMES

In this chapter we discuss some applications of fuzzy topological games in covering properties and dimension theory. We mainly focus our attention on α-para (meta) compact spaces and shading dimension. Every product space discussed will have a winning strategy in some particular kind of fuzzy topological game. Also a fuzzy version of countable sum theorem for covering dimension in terms of fuzzy topological games is obtained.

6.1.Games and Product in α-para (meta) Compact Spaces

First we obtain a characterisation of fuzzy regular α-paracompact spaces , which will be useful in proving the main theorems in this section.

6.1.1 Theorem For a fuzzy regular space X, the following are equivalent

(i) X is α-paracompact.

(ii) Every α-shading of X by open fuzzy sets has a σ-locally finite α-shading refinement by open fuzzy sets.

(iii) Every α-shading refinement of X by open fuzzy sets has a locally finite α-shading refinement.

(iv) Every α-shading of X by open fuzzy sets has a locally finite α-shading refinement by closed fuzzy sets.

Proof

(i) $\Rightarrow$ (ii)

Follows from the fact that every locally finite α-shading is σ-locally finite.

(ii) $\Rightarrow$ (iii)

Let U be an α-shading of X by open fuzzy sets. Let V be the σ- locally finite α-shading refinement of U by open fuzzy sets. Therefore $V = \bigcup_{i=1}^{\infty} V_i$ where each $V_i = \{V_{i\beta}: \beta \in \Lambda\}$ is locally finite. Now take $W_i = \underset{v \in V_i}{Sup}\, V$. Now $W = \{ Wi : i = 1,2,3......\}$is clearly an

α-shading of X. Take $A_1 = W_1$ and $A_i = W_i \setminus \bigcup_{i<n} W_i$ for $i = 1, 2, 3, \ldots$. Now $\{A_i : i = 1, 2, \ldots\}$ is a locally finite α-shading refinement of W. Then by Theorem 1.2.7 it follows that $\{ A_i \Lambda V_{i\beta}:\ i =1, 2, \ldots, \beta \in \Lambda\}$ is a locally finite α-shading refinement of V and hence of U.

(iii) $\Rightarrow$(iv)

Let U be an α-shading of X by open fuzzy sets. For any $x \in X$, take some $U_x \in U$ with $U_x(x) > \alpha$. Now since X is fuzzy regular, it is possible to find a fuzzy open nbd V_x of X such that $V_x(x) > \alpha$ and $x \in V_x < \overline{V}_x < U_x$. Now by (iii) we have $\{V_x:\ x \in X\}$ has a locally finite α-shading refinement $\{A_r:\ r \in \Gamma\}$ (say). Then by Theorem 1.3.3 $\{\overline{A}_r:\ r \in \Gamma\}$ is also locally finite. Now for each $r \in \Gamma$, if $A_r < V_x$ then $\overline{A}_r < V_x < U$ for some $U \in U$. Hence $\{\overline{A}_r:\ r \in \Gamma\}$ is the required α-shading refinement by closed fuzzy sets.

(iv) $\Rightarrow$ (i).

Let U be an α-shading of X by open fuzzy sets and V be a locally finite α-shading refinement of U by closed fuzzy sets. For each $x \in X$, let W_x be an open fuzzy set such that $W_x(x) = 1$ and $V_i = 1 \setminus W_x$ holds for all but almost finitely many i. This is possible since V is locally finite where $V_i \in V$. Now clearly $W = \{W_x:\ x \in X\}$ is an α-shading of X and let A be a locally finite α-shading refinement of W by closed fuzzy sets. Now we take $V^* = X \setminus\ Sup\ \{A:\ A \in A\ ,\ A \wedge V = 0\}$. Clearly each V^* is fuzzy open and contains V. Consider $V^* = \{\ V^*:\ V \in V\}$. Now V^* is a locally finite α-shading of X. For let $x \in X$, now we can find an open fuzzy set U such that $U(x) = 1$ and $A_i \leq 1\text{-}U$ holds for all but at most finitely many i. (since A is locally finite). i.e., $A_i \wedge U \neq 0$ for almost finitely many i. Now if $U \wedge V^* \neq 0$ for some $V^* \in V^*$, then $V^* \wedge A_i \neq 0$ for some $i = 1, 2, 3, \ldots, n$. By the definition of V^* it follows that $A_i \wedge V = 0$ for some $i = 1, 2, \ldots, n$. Now if A is a refinement of $\{W_x:\ x \in X\}$ and each W_x meet only finitely many $V \in V$ implies that A_i meets only finitely many members of V and hence we have $U \wedge V^* = 0$ for all but finitely many $V^* \in V^*$. And hence V^* is locally finite.

Now for every $V \in \mathbf{V}$ take $U \in \mathbf{U}$ such that $V \leq U$ and consider $\{ U \wedge V^* : V \in \mathbf{V} \}$. This is clearly an α-shading refinement of $\mathbf{U}$ by open fuzzy sets which is locally finite. Hence X is α-paracompact.

6.1.2Theorem If Player I has a fuzzy winning strategy in $G^*(\mathbf{DC}, X)$ and X is α-paracompact, then $X \times Y$ is α-paracompact for every α-paracompact space Y. Where $\mathbf{DC}$ denote the class of all fuzzy topological spaces which have a finite fuzzy closed α-shading by members of $\mathbf{C}$, where $\mathbf{C}$ is the collection of all α-compact spaces.

Proof

Let S be a fuzzy stationery winning strategy for player I in $G^*(\mathbf{DC}, X)$. Let $\mathbf{G}$ be any α-shading of $X \times Y$ by open fuzzy sets. Then from the characterization of α-para compactness in theorem 6.1.1 it suffices to prove that $\mathbf{G}$ has σ-locally finite refinement by open fuzzy sets. Takes $\mathbf{U}_0 = \{0\}$, $A_0 = \{0\}$ and $R(0) = H(0) = X \times Y$, we shall construct a collection $\mathbf{U}_n$ of fuzzy co-zero rectangles and a collection $\{(R(a), H(a)) | a = (a_1, a_2, \ldots, a_n) \in A_n\}$ of pairs of closed rectangles $R(a)$ and open rectangle $H(a)$ for each $n \geq 1$, satisfying the following conditions

(i) $\mathbf{U}_n$ is locally finite in $X \times Y$

(ii) Each $U \times V$ in $\mathbf{U}_n$ is contained in some $G \in \mathbf{G}$.

(iii) $\{H(a): a \in A_n\}$ is locally finite in $X \times Y$ for each $a \in A_n$ and $n \geq 1$.

(iv) $a_- \in A_{n-1}$, where $a_- = (a_1, a_2, \ldots a_{n-1})$

(v) $R(a) < R(a_-) \wedge H(a)$

(vi) $S(R(a_-))' \wedge R(a)' = 0$

(vii) $R(a) \backslash \mathrm{Sup}\, \mathbf{U}_{n+1} < \mathrm{Sup}\{R(a \oplus \alpha) | a \oplus \alpha \in A_{n+1}\}$.

Where ` and `` represents projections on X and Y axes respectively.

The construction of $\mathbf{U}_i$ are similar to that of Yajima [Y_1] in crisp case and hence we omit it.

Now take $\mathbf{U} = \bigcup_{n \geq 1} \mathbf{U}_n$. From (i) and (ii) we get that $\mathbf{U}$ is a σ-locally finite collection of co-zero rectangles and $U \times V \in \mathbf{U}$ is contained in some $G \in \mathbf{G}$.

Now from (v) and (vi) we get if (a_n) is a sequence such that $a_n \in A_n$ and $(a_n)_. \in A_{n-1}$ for each $n \geq 1$ where $a_0 = \phi$, then $\sup_{n\geq 1} R(a_n)' = 0$. For ($S(X)$, $R(a_1)'$, $S(R(a_1))'$,..., $S(R(a_n)')$, $R(a_n)'$, ...) is a play of G^* (DC, X) and player I wins this play and hence $\sup_{n\geq 1} R(a_n)' = 0$.

Now it is enough if we prove that ***U*** is an α-shading of $X \times Y$. If possible let ***U*** be not an α-shading of $X \times Y$. Therefore $U(x, y) \leq \alpha$ for every $U \in$ ***U***. Then by (vii) we can find an infinite sequence $(\alpha_1, \alpha_2, \ldots)$ such that $a_n = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(x, y) \in R(a_n)$ $\forall$ $n \geq 1$. Now clearly $R(a_n)'(x) > 0$ and hence $\sup_{n\geq 1} R(a_n)' \neq 0$. This is a contradiction and hence the proof is complete.

Using the notion of the P_α-spaces discussed in Chapter IV, we obtain an analogue of theorem 6.1.2 for α-metacompact spaces as follows .

6.1.3Theorem Let X be a fuzzy regular α-metacompact P_α-space and Player I has a winning strategy in $G^*(DC,X)$, then $X \times Y$ is α-metacompact for every α-metacompact space Y. Where ***DC*** denote the class of all fts which have a discrete fuzzy closed α-shading by members of C,where C is the collection of all α-compact spaces.

Proof

We use the following notations. If a = $(a_1, a_2 \ldots\ldots, a_n)$ then $a \oplus \zeta = (a_1, a_2 \ldots\ldots, a_n, \zeta)$, $a/k = (a_1, a_2 \ldots\ldots, a_k)$ and $a_. = a/n\text{-}1$ .Also ` and `` represents the projections on X and Y respectively.

Given that Player I has a fuzzy winning strategy in $G^*(DC,X)$. Therefore by Theorem 2.2.8 it follows that Player I has a stationary winning strategy and let this be s. Let p be a function defined as in Theorem 4.1.5.We will prove that every α-shading G of $X \times Y$ by open fuzzy sets has a point finite α-shading refinement by open fuzzy rectangles.

Let $U_0=\{0\}$, $A_0=\{0\}$ and $R(0) = H(0) = X\times Y$. For each $n\geq 1$, we shall construct a collection U_n of open fuzzy rectangles and a collection $\{\{R(a),H(a)\}: a = (a_1,a_2,\ldots\ldots a_n) \in A_n\}$ of pairs consisting of fuzzy closed × open rectangle $R(a)$ and open rectangle $H(a)$ satisfying the following conditions

For each $n \geq 1$

(i) U_n is a point finite collection in $X\times Y$

(ii) For every $U\times V\in U_n$, there is a $G\in G$ such that $U\times V < G$

(iii) $\{ H(a): a\in A_n\}$ is point finite in $X\times Y$.

(iv) Sup $\{U: U\in U_n \} <$ Sup $\{H(a): a\in A_{n-1}\}$

(v) $a\_ \in A_{n-1}$

(vi) $R(a) < R(a\_)$ and $R(a) < H(a) < H(a\_)$

(vii) $S(R(a\_)') \wedge R(a)' = 0$

(viii) $R(a) \setminus Sup\{U: U\in U_{n+1}\} < Sup\{ R(a+\xi); \{a+\xi)\in A_{n+1} \}$

(ix) $p(R(a/1)', \ldots\ldots\ldots R(a/n-1), R(a)') \wedge H(a)''=0$

Assume that for each $i \leq n$, the collections U_i and $\{R(a), H(a) ; a\in A_i\}$ have been constructed.

Now for any $a\in A_n$, let $\{C_\gamma : \gamma\in\Gamma(a) \}$ be a discrete collection of α-compact sets whose supremum is $S(R(a)')$. From the fact that X is fuzzy regular α-metacompact it follows that there exists point finite collections $\{W_\gamma : \gamma\in\Gamma(a) \}$ and $\{O_\gamma : \gamma\in\Gamma(a) \}$ of open fuzzy sets such that $C_\gamma < W_\gamma < cl\ W_\gamma < O_\gamma < H(a)' \setminus Sup\{ C_\beta: \beta\in\Gamma(a), \beta\neq\gamma\}$ for each $\gamma\in\Gamma(a)$. Now Y is α-metacompact and $R(a)''$ is open in Y. Now $R(a)''$ is α-metacompact (Since α-metacompact is hereditary with respect to open subsets) and hence for each $\gamma\in\Gamma(a)$, there exists a collection $U_\gamma = \{U_{\delta,j}\times V_\delta : j = 1,2,3\ldots m_\delta \text{ and } \delta\in\Delta(\gamma)\}$ such that

(i) $C_\gamma < U_\delta = \underset{i\leq m\delta}{Sup}\ U_{\delta,j} < W_\delta$ for each $\delta\in\Delta(\gamma)$.

(ii) Each $U_{\delta,j}\times V_\delta$ is contained in some $G \in G$.

(i) $\{ V_\delta: \delta\in\Delta(\gamma)\}$ is point finite α-shading of $R(a)''$.

Set $U_{n+1} = \cup\{ U_\gamma : \gamma\in\Gamma(a) \text{ and } a\in A_n\}$ and

$A_{n+1} = \{a\oplus\delta: \delta\in\Delta(\gamma), \gamma\in\Gamma(a), a\in A_n\} \cup \{a+\theta : a\in A_n\}$

Take any $a+\xi \in A_{n+1}$. Then observe that $a/i \in A_i$ for all $i \leq n$.

If $\xi = \delta$ for some $\delta \in \Delta(\gamma)$ and, $\gamma \in \Gamma(a)$, put $R(a \oplus \delta) = [\,(cl\ W_\gamma \setminus V_\delta) \wedge R(a)'\,] \times V_\delta$ and $H(a \oplus \delta) = O_\gamma \setminus p\,(R(a/1)', \ldots\ldots R(a/n)', R(a+\theta)') \times V_\delta$

If $\xi = \theta$, *put* $R(a+\theta) = R(a) \setminus \underset{\gamma \in \Gamma(a)}{Sup}\ W_\gamma \times R(a)''$ and

$H(a+\theta) = H(a)' \setminus p\,(R(a/1)', \ldots\ldots R(a/n)', R(a+\theta)') \times H(a)''$

Then clearly U_{n+1} *and* $\{R(a), H(a) : a \in A_{n+1}\}$ *satisfies conditions (i) --- (ix)*

Now take $U = \bigcup_{n \geq 1} U_n$. Now it can be shown that U is an α-shading of X and we will prove that U is also point finite. Also by (ii) U is a collection of open fuzzy rectangles in $X \times Y$ and any $U \times V \in U$ is contained in some $G \in G$.

Similar to the proof of claim in Theorem 6.1.2 , we get if $\{a_n\}$ is a sequence such that $a_n \in A_n$ and $(a_n)_- = a_{n-1}$ for each $n \geq 1$, where $a_0 = 0$, then $Inf_{n \geq 1} H(a_n)' = 0$ -----(1)

Again we claim that $Inf_{n \geq 1} (Sup H_n) = 0$. Where $H_n = \{H(a) : a \in A_n\}$. For if possible let there be an z_0 such that $Inf_{n \geq 1} (Sup H_n)(z_0) > \eta$ = for some $\eta > 0$. Take $A_n(z_0) = \{ a \in A_n : H(a)(z_0) \geq \eta\}$. By (iii) we get $A_n(z_0)$ is finite and by (v) and (vi), $a \in A_n(z_0) \Rightarrow a_- \in A_{n-1}(z_0)$. Then by Konings' Lemma [Lemma 2.8 of [Y_1]] there exists (β_1, β_2, β_3,......) such that $a_n \in (\beta_1, \beta_2 \ldots\ldots \beta_n) \in A_n(z_0)$ for each $n \geq 1$. Then $H(a_n)(z_0) \geq \eta$ for each $n \geq 1$. Hence $Inf_{n \geq 1} H(a_n)(z_0) \geq \eta$. This is a contradiction to our claim.

Let $z \in X \times Y$ then by claim above we can find an $m \geq 1$ such that $Sup\ H_m(z) = 0$. Now from (v) and (vi) it follows that $Sup\ H_{n+1} < Sup\ H_n$ for each $n \geq 1$. Since $Sup\ H_n(z) = 0$ for each $n \geq m$, from (iv) we get that $Sup\ U_n(z) = 0$ whenever $n > m$. Hence it follows from (i) that U is point finite in $X \times Y$. This completes the proof.

From Theorems 4.1.6 , 3.3.6 , and 6.1.3 next corollary follows easily.

6.1.4 Corollory If X is a fuzzy regular α-metacompact space with a σ-closure preserving α-shading by α-compact sets, then $X \times Y$ is α-metacompact for every α-metacompact space Y.

6.2 Games and Shading Dimension

6.2.1 Notation Through out this section by shading we mean *0*-shading and assume that every shading is essential.

6.2.2 Definition Let X be a non empty set and $\boldsymbol{U}=\{U_\lambda: \lambda\in\Lambda\}$ be a non empty family of fuzzy subsets of X. Then the order of $\boldsymbol{U}$ is the largest integer n such that there exists a subset M of Λ having $n+1$ elements such that $\underset{\lambda\in M}{Inf}\ U_\lambda > 0$. And if there is no such integer, order is ∞. If the collection is void, its order is defined to be -1.

6.2.3 Definition The shading dimension of a fuzzy topological space X ($Shad\ X$) is the least integer n such that every finite open shading of X has an open shading refinement of order not exceeding n. If there is no such integer, the shading dimension is said to be ∞.

6.2.4 Definition A shading $\boldsymbol{U}=\{U_\alpha: \alpha\in\Lambda\}$ is essential if for every $\beta\in\Lambda$, $U_\beta > 1\backslash \underset{\alpha\neq\beta}{Sup}\{U_\alpha: \alpha\in\Lambda\}$.

6.2.5 Theorem If X is a fuzzy topological space, then the following are equivalent

(i) $Shad\ X\leq n$

(ii) For every finite open shading $\{U_1, U_2, U_3..U_k\}$ of X, there is an open shading $\{V_1, V_2, V_3 \ldots\ldots V_k\}$ of order not exceeding n such that $Vi< Ui$ for each i $=1,2,3\ldots k$.

(iii) If $\{U_1, U_2, U_3..U_{n+2}\}$ is an open shading of X, there is an open shading $\{V_1, V_2, V_3 \ldots\ldots V_{n+2}\}$ such that $Vi< Ui$ and $\underset{1\leq i\leq n}{Inf}\ V_i=0$.

Proof

(i) $\Rightarrow$ (ii)

Let $Shad\ X\leq n$. Therefore the shading $\{U_1, U_2, U_3..U_k\}$ has an open shading refinement say $\boldsymbol{W}$ with order not exceeding n. Now if $W\in\boldsymbol{W}$, there is some i such that $W<U_i$ and suppose that each W is associated with one of the U_i containing it and take $V_i=Sup\{W: W<U_i\}$. Clearly each V_i is open and $V_i<U_i$ for every i. Since order of $\boldsymbol{W}$ is not

exceeding n, it follows that for any $x \in X$, $W(x)>0$ for at most $n+1$ members of $\boldsymbol{W}$ and each $W \in \boldsymbol{W}$ is associated with a unique U_i. Therefore it follows that $V(x)>0$ for at most $n+1$ members of $\{V_i\}$ and hence $\{V_i\}$ is a shading of X with order not exceeding n.

(ii) $\Rightarrow$ (iii) is clear

(iii)$\Rightarrow$ (ii)

Let $\boldsymbol{U} = \{U_1, U_2, U_3..U_k\}$ be a finite open shading of X. Assume $k > n+1$,
We define a collection $\{G_i: 1 \leq i \leq n+2\}$ as follows
$G_i = U_i$ for $1 \leq n+1$ and
$G_{n+2} = \operatorname{Sup}_{n+2 \leq i \leq k} U_i$
Clearly each G_i is open and $\{ G_i \}$ is a shading of X. Then by (iii) there is an open shading $\{H_1, H_2, H_3....H_{n+2}\}$ such that $H_i < U_i$ and $\operatorname{Inf}_{1 \leq i \leq n+2} H_i = 0$. Now take $W_i = U_i$ if $1 \leq n+1$ and $W_i = U_i \wedge H_{n+2}$ if $i > n+1$. Then clearly $\boldsymbol{W} = \{W_1, W_2, W_3..W_k\}$ is an open shading of X with the property that $W_i \leq U_i$ and $\operatorname{Inf}_{1 \leq i \leq n+2} W_i = 0$. Now if there exists a subset B of $\{1,2,....k\}$ with $n+2$ elements such that $\operatorname{Inf}_{i \in B} W_i \neq 0$, we will renumber the family $\boldsymbol{W}$ to give a family $\boldsymbol{P} = \{P_1, P_2, P_3..P_k\}$ such that $\operatorname{Inf}_{1 \leq i \leq n+2} P_i \neq 0$. By proceeding in a manner similar to the construction above, we can obtain an open shading $\boldsymbol{W}' = \{W'_1, W'_2, W'_3..W'_k\}$ such that $W'_i < P_i$ and $\operatorname{Inf}_{1 \leq i \leq n+2} W'_i = 0$. By repeating this process for a finite number of times, we will end up with an open shading $\{V_1, V_2, V_3..V_k\}$ of X with order not exceeding n and $V_i < U_i$.

(ii)$\Rightarrow$ (i) is obvious.

6.2.6 Theorem If A is a closed fuzzy subset of a fuzzy topological space X, then *Shad* $A \leq$ *Shad* X.

Proof

Let $Shad\, X \leq n$. Now it is enough to prove that $Shad\, A \leq n$. For let $\boldsymbol{U} = \{U_1, U_2, U_3..U_k\}$ be a finite open shading of A. Then clearly for each i, $U_i = A \wedge V_i$ for some V_i open in X. Then it follows clearly that $\{V_1, V_2, V_3..V_k, 1\backslash A\}$ is an open shading of X . Now since $Shad\ X \leq n$, this collection has an open shading refinement $\boldsymbol{W}$ with order not exceeding n. Now consider $\{W \wedge A: W \in \boldsymbol{W}\}$. This is an open refinement of $\boldsymbol{U}$ with order not exceeding n and hence $Shad\, X \leq n$.

6.2.7 Definition [HU] A fuzzy topological space (X,T) is normal if for every closed fuzzy set k and open fuzzy set b such that $k\leq b$, there exists a fuzzy set a such that $k\leq int\ a \leq cl\ a \leq b$.

6.2.8 Definition An open shading $\boldsymbol{U} = \{U_\alpha: \alpha \in \Lambda\}$ is said to be shrinkable if there exists an open shading $\boldsymbol{V} = \{V_\alpha: \alpha \in \Lambda\}$ such that $cl\ V_\alpha < U$ for every $\alpha \in \Lambda$. Then $\boldsymbol{V}$ is called a shrinking of $\boldsymbol{U}$.

6.2.9 Definition Two families $\{A_\lambda: \lambda \in \Lambda\}$ and $\{B_\lambda: \lambda \in \Lambda\}$ of fuzzy subsets of a set X is similar if for each $\gamma \subset \Lambda$, $\underset{\lambda\in\gamma}{Inf}\ A_\lambda$ and $\underset{\lambda\in\gamma}{Inf}\ B_\lambda$ are both zero or both non zero.

6.2.10 Proposition A fuzzy topological space X is normal if and only if every point finite shading of X by open fuzzy sets is shrinkable.

Proof

Suppose that X is fuzzy normal. Let $\boldsymbol{U} = \{U_\alpha: \alpha \in \Lambda\}$ be a point finite shading of X by open fuzzy sets. For convenience take $\Lambda = \{1,2,3,\ldots\}$. Construct $\boldsymbol{V} = \{V_\alpha: \alpha \in \Lambda\}$ by transfinite induction as follows.

Put $F_1 = 1 \backslash \underset{\alpha>1}{Sup} U_\alpha$. Now clearly $F_1 < U_1$. Then by normality [Theorem 1.12 of [M;B1]], there exists an open set V_1 such that $F_1 < V_1$ and $cl\ V_1 < U_1$. Let U_β be constructed for each $\beta < \alpha$ and let $F_\alpha = 1 \backslash [\underset{\beta<\alpha}{Sup} V_\beta \vee \underset{\gamma>\alpha}{Sup} U_\gamma]$. Now clearly F_α is closed and $F_\alpha < U_\alpha$. Also by normality there exists an open fuzzy subset V_α with $F_\alpha < V_\alpha$

$< cl\ V_\alpha < U_\alpha$ .Now $V = \{ V_\alpha: \alpha \in \Lambda\}$ is the required shrinking provided that it is a shading of X. For, let $x \in X$, since U is a shading and point finite, it follows that $U(x)>0$ for at most finitely many $U \in U$. Say $U_{k1}, U_{k2}, \ldots\ldots\ldots U_{kn}$. Take $k = Max\{k_1, k_2 \ldots\ldots k_n\}$. Now clearly $x \notin U_\gamma$ for any $\gamma > k$ .And hence if $x \notin U_\beta$ for any $\beta < k$, then clearly $x \notin V_\gamma$ for any $\gamma > k$. Therefore $x \in F_\gamma < V_\gamma$. Hence $x \in V_\gamma$ Thus in any case $V_\beta(x)>0$ for some $\beta \in \Lambda$. Hence$\{ V_\alpha : \alpha \in \Lambda\}$ is the required shrinking.

Conversely let A and B be two closed subsets of X such that $A \leq 1 \setminus B$. Then clearly $\{1 \setminus A , 1 \setminus B\}$ is a point finite open shading of X. For any shrinking $\{U, V\}$ of $\{1 \setminus A , 1 \setminus B\}$ we have the open fuzzy sets $1 \setminus clA$ and $1 \setminus clB$ containing B and A respectively. Hence X is fuzzy normal. [By theorem 1.12 of [M;B1]]

6.2.11 Proposition Let X be a fuzzy normal space and $\{G_\alpha: \alpha \in \Lambda\}$ a family of locally finite collection of open fuzzy sets in X and $\{F_\alpha: \alpha \in \Lambda\}$ be a collection of closed fuzzy sets of X such that $F_\alpha < G_\alpha$ for each α. Then there is an open collection $\{H_\alpha: \alpha \in \Lambda\}$ similar to $\{F_\alpha: \alpha \in \Lambda\}$ and $F_\alpha < H_\alpha < G_\alpha$ for each α .

Proof

Well order Λ and construct the set F as follows . Take all finite intersections from $\{F_\alpha: \alpha \in \Lambda\}$ which does not meet F_0. Take F as the supremum of all these and from normality it follows that there exists an open fuzzy set H_0 such that $F_o < H_o < cl\ H_o$ $< (1 \setminus F) \wedge G_0$. Now $\{ cl\ H_0, F_1, F_2, \ldots \}$ is similar to $\{F_\alpha: \alpha \in \Lambda\}$. Continuing transfinitely we get a collection $\{H_\alpha: \alpha \in \Lambda\}$ which has the required property.

6.2.12 Proposition (A characterisation for shading dimension for fuzzy normal spaces) The following are equivalent for a fuzzy normal space.

(i) *Shad* $X \leq n$

(ii) Every finite open shading of X can be refined by a finite closed fuzzy shading of order not exceeding $n+1$.

(iii) Every finite open shading $\{G_1, G_2, \ldots\ldots\ldots G_k\}$ of X can be refined by a finite closed fuzzy shading $\{H_1, H_2, \ldots\ldots\ldots H_k\}$of order not exceeding $n+1$ such that $H_i < G_i$ for each i.

Proof follows from theorem 6.2.6 and Proposition 6.2.10 above.

6.2.13 Lemma If E is a closed fuzzy subset of a fuzzy normal space X with $Shad\ E \leq n$, then for each finite open shading $\{U_i: i \leq k\}$ of X, there exists a finite shading $V = \{V_i: i \leq k\}$ of open fuzzy sets and an open set G containing E such that

(i) $V_i < U_i$ for each $i \leq k$.

(ii) $Ord(x, V) \leq n+1$ for each $x \in cl\ G$

Proof

From the characterisation in Proposition 6.2.12, clearly there exists a a closed collection $F = \{F_i: i \leq k\}$ which is a shading of E and $F_i < U_i \wedge E$ for each i with order $Ord(x, F) \leq n+1$ for each $x \in X$. Now proceeding in a similar manner as in theorem 7.14 of Engeling [E] in the crisp case, there exists a finite open collection of fuzzy sets $W = \{W_i: i \leq k\}$ such that $F_i < U_i < W_i$ for each $i \leq k$ and $Ord(x, W) \leq n+1$ for each $x \in X$. Take an open set G in X such that $E < G < cl\ G < \underset{i \leq k}{Sup} W_i$(This is possible by normality). Put $V_i = W_i \vee (U_i \setminus cl\ G\}$. Then $V = \{V_i: i \leq k\}$ and G has the required properties.

6.2.14 Notation $Shad_n$ denote the family of all fuzzy topological spaces with shading dimension $\leq n$.

6.2.15 Remark By Theorem 6.2.6, it follows that $X \in Shad_n \Rightarrow \underline{I}^X \subset Shad_n$.

6.2.16 Main Theorem Let X be a fuzzy normal space and Player I has a winning strategy in $G^*(Shad_n, X)$, then $Shad\ X \leq n$.

Proof

Let $\{U_j: j \leq k\}$ be a shading of X by open fuzzy sets and S be a fuzzy stationary winning strategy for Player I in $G^*(Shad_n, X)$.

For each i construct an open shading $\mathbf{U}_i = \{U_{i,j}: j \leq k\}$ of X and open fuzzy sets G_i with the following properties for each $i \geq 1$.

(i) $cl\ U_{i,j} < U_{i-1,j} < U_j \ : j = 1,2,3,\ldots..k$

(ii) $Ord(x, \mathbf{U}_i) \leq n+1$ for each $x \in cl\ G_i$

(iii) $S(1 \setminus G_{i-1}) \vee G_{i-1} < G_i$

Take $U_{0,j} = U_j$ for each $j \leq k$ and $G_0 = 0$. Let G_l and U_i are constructed for each $i \leq m$. Let $E = S(1 \setminus G_m)$. Them clearly $1 \setminus G_m$ is closed and hence $Shad\ E \leq n$. Then by lemma 6.2.13 we have an open shading $\mathbf{U}_{m+1} = \{ U_{m+1,j}: j \leq k \}$of X and an open set $G > E$ with $U_{m+1,j} < U_{m,j}$ for each $j \leq k$ and $Ord(x, \mathbf{U}_{m+1}) \leq n+1$ for each $x \in cl\ G$. Also $U_{m+1,j} < cl\ U_{m+1,j} < U_{m,j}$, since X is fuzzy normal. Let $G_{m+1} = G \vee G_m$. Then $\mathbf{U}_{m+1}$ and G_{m+1} satisfy (i) --- (iii).

Now take $F_j = \underset{i \geq 1}{Inf}\ clU_{i,j}$ for each $j \leq k$. Take an $x \in X$, now since each $\mathbf{U}_i$ is a finite shading of X, we can choose $j(x) \leq k$ such that $x \in U_{i,j(x)}$ for infinitely many i. But by (i) we have $x \in \underset{i \geq 1}{Inf}\ U_{i,j(x)} = F_j(x)$. Thus the collection $\mathbf{F} = \{F_j: j \leq k\}$ is a shading of X by closed fuzzy sets such that $F_j < U_j$ for each $j \leq k$. Now $\{G_i: i \geq 1\}$ is a shading of X. Take some $i \geq 1$ such that $x \in G_{i,0}$. Now by (ii) we have $Ord(x, \mathbf{F}) \leq Ord(x, \mathbf{U}_{i,0}) \leq n+1$. Therefore it follows from Proposition 6.2.12 that $Shad\ X \leq n$.

APPENDIX- I

L - FUZZY METACOMPACTNESS

Mao-Kang [MA] introduced a reasonable notion for paracompactnes in fuzzy topology and systematically studied paracompactness for the case $[0,1]$. Later Jiu-Lan [JI] continued the investigation in the L-fuzzy topological spaces(L-*fts*) using α-Q-covers and quasi-coincidence relation. In chapter I, we have already discussed metacompactness in $[0,1]$ fuzzy topological spaces and obtained a characterization for the same.In this section we define point finite collections and metacompactness in L-*fts*. Besides getting a characterization for metacompactness in weakly induced L-*fts* that involves the concepts of well-monotone and directed α-Q-covers, it is also seen that metacompactness obtained is a good L-extension of ordinary metacompactness and it is hereditary with respect to closed subsets.

For basic notions and definitions we follow Ying-Ming and Mao-Kang [Y;M] .We let q denote the quasi coincidence relation. Also χ denote the characteristic function and $Pt(L^X)$ is the collection of all L-fuzzy points in the L-*fts* (L^X,δ). A molecule in a lattice L is a join irreducible element in L and the set of all molecules of L is denoted by $M(L)$.Also we denote $A_{(\alpha)} = \{x \in X : A(x) \nleq \alpha\}$ and $A_{[\alpha]} = \{x \in X : A(x) \leq \alpha\}$.

A.1.1 Definition [Y;M] Let (L^X,δ) be an L-*fts* . A fuzzy point x_α is quasi coincident with $A \in L^X$ (and write $x_\alpha \prec A$) if $x_\alpha \nleq A'$. Also A quasi coincides with B at x (AqB at x) if $A(x) \nleq B'(x)$. We say A quasi coincident with B and write AqB if AqB at x for some $x \in X$. Further $A \neg qB$ means A not quasi coincides with B. We say $U \in \delta$ is a quasi coincident *nbd* of x_α (Q-*nbd*) if $x_\alpha \prec U$. The family of all Q-*nbd*s of x_α is denoted by $Q_\delta(x_\alpha)$ or $Q(x_\alpha)$.

Some Results mentioned in Appendix I are included in a paper titled *Metacompactness in L-Fuzzy Context* to appear in the Journal of Fuzzy Mathematics, Vol. 8 No.3 (2000).

A.1.2 Definition [Y;M] Let (L^X,δ) be an *L-fts* , $A\in L^X$. $\Phi \subset L^X$ is called a *Q*-cover of *A* if for every $x\in Supp(A)$, there exists $U \in \Phi$ such that $x_{A(x)} \prec U$. Φ is a *Q*-cover of (L^X,δ) if Φ is a *Q*-cover of $\underline{1}$. If $\alpha\in M(L)$, then $C\in\delta$ is an *α-Q-nbd* of *A* , if $C\in Q(x_\alpha)$ for every $x_\alpha \leq A$. Φ is called an *α-Q* cover of *A* if for every $x_\alpha \leq A$, there exists $U \in \Phi$ such that $x_\alpha \prec U$.

A.1.3 Definition [Y;M] Let (L^X,δ) be an *L-fts*. $A = \{A_t : t\in T\} \subset L^X$, $x_\lambda \in M(L^X)$. *A* is called locally finite at x_λ if there exists $U\in Q(x_\lambda)$ and a finite subset T_0 of T such that $t\in T\setminus T_0 \Rightarrow A_t \neg qU$. And *A* is *-locally finite at x_λ if $t\in T\setminus T_0 \Rightarrow \chi_{At_{(0)}} \neg qU$, where $At_{(0)}$ $=\{ x\in X : A_t(x) \nleq 0 \}$. *A* is called locally finite (resp.*-locally finite) for short , if *A* is locally finite (resp. *-locally finite) at every molecule x_λ of L^X.

A.1.4 Definition [Y;M] Let (L^X,δ) be an *L-fts*. $A\in L^X$, $\alpha\in M(L)$. *A* is called *α*-paracompact (resp.*α**-paracompact) if for every open *α-Q*-cover Φ of *A*, there exists an open refinement Ψ of Φ such that Ψ is locally finite (resp. *-locally finite)in *A* and Ψ is also an *α-Q*-cover of *A*. *A* is called paracompact (resp.*-paracompact) if *A* is a *α*-paracompact (resp.*α**-paracompact) for every $\alpha\in M(L)$. (L^X,δ) is paracompact (resp.*-paracompact) if $\underline{1}$ is paracompact (resp.*-paracompact). Where a collection *A* refines **B** ($A < B$) if for every $A\in A$, $\exists\, B\in B$ such that $A \leq B$.

Now we define point finite families and metacompactness.

A.1.5 Definition Let (L^X,δ) be an *L-fts*. $A = \{A_t : t\in T\} \subset L^X$, $x_\lambda \in M(L^X)$. *A* is called point finite at x_λ if $x_\lambda \prec A_t$ for at most finitely many $t\in T$. And *A* is *-point finite at x_λ if there exists at most finitely many $t\in T$ such that $x_\lambda \prec \chi_{At_{(0)}}$, where $At_{(0)} = \{ x\in X : A_t(x) \nleq 0 \}$. *A* is called point finite (resp. *-point finite) for short , if *A* is point finite (resp. *-point finite) at every molecule x_λ of L^X.

A.1.6 Definition Let (L^X,δ) be an *L-fts*. $A \in L^X$, $\alpha \in M(L)$. A is called α-metacompact (resp. α*-metacompact) if every open α-Q-cover of A has a point finite (resp. *-point finite) open refinement which is also an α-Q-cover of A . A is called metacompact (resp.*-metacompact) if A is α-metacompact (resp. α*-metacompact)for every $\alpha \in M(L)$. And (L^X,δ) is metacompact (resp.*-metacompact) if $\underline{1}$ is metacompact (resp.*-metacompact).

A.1.7 Remark Clearly we have *-point finite $\Rightarrow$ point finite and hence

α*-metacompact $\Rightarrow$ α -metacompact and

*-metacompact $\Rightarrow$ metacompact

A.1.8 Proposition Every locally finite (resp. *-point finite) family is point finite (resp. *-point finite)

Proof of Proposition A.1.8 follows immediately from the definitions.

Remark From the Proposition A.1.8 it follows that paracompact (resp.*-para compact) $\Rightarrow$ metacompact (resp.*-metacompact)

A.1.9 Definition [Y;M] Let (L^X,δ) be an *L-fts*. $\boldsymbol{A} = \{A_t : t \in T\} \subset L^X$ is a closure preserving collection if for every subfamily $\boldsymbol{A_0}$ of $\boldsymbol{A}$, $cl\,[\vee \boldsymbol{A_0}] = \vee\ cl\ \boldsymbol{A_0}$.

A.1.10 Proposition A point finite closure preserving closed collection is always locally finite.

Proof

Let $\{A_t : t \in T\}$ be a point finite closure preserving closed collection and let $x_\lambda \in M(L^X)$. There fore $x_\lambda \prec A_t$ for $t \in T_0$ say an at most finite subset of T.

Now take $V = \text{cl}\,\{\vee A_t : t \notin T_0\}$

$= \vee\,\{\text{cl}\,A_t : t \notin T_0\}$ since the collection is closure preserving.

$= \vee\,\{A_t : t \notin T_0\}$ since each A_t is closed.

Take $U = V' = (\vee \{ A_t : t \notin T_0 \})'$

$= \wedge \{ A_t' : t \notin T_0 \}$

Now if $t \in T \setminus T_0$, $x_\lambda \neg q A_t$ and hence $x_\lambda\, q\, A_t'$ for every $t \in T \setminus T_0$. Therefore it follows that $x_\lambda\, q\, (\vee \{ A_t : t \notin T_0 \})'$. That is $x_\lambda\, qU$.-----------*(1)*

Now since $x_\lambda \neg q A_t$ it follows that $x_\lambda \leq A_t'$.Again by (1) we have x_λ U'. Combining these two we get $A_t' \geq x_\lambda \not\leq U'$. That is $A_t' \not\leq U'$ and hence $A_t \neg q\, U$. This completes the proof.

A.1.11 Remark

(i) A collection $\boldsymbol{U}=\{U : U \in \boldsymbol{U}\}$ is locally finite implies that so is $\{cl\ U : U \in \boldsymbol{U}\}$. But this does not hold for point finite families.

(ii) Similar to the Proposition A.1.10 it can be shown that a *-point finite closure preserving collection is always *-locally finite.

A.1.12 Proposition Let (L^X, δ) be an *L-fts.* $\alpha \in M(L)$, $A \in L^X$, $B \in \delta'$. Then

(i) If A is α-metacompact then so is $A \wedge B$

(ii) If A is metacompact then so is $A \wedge B$

Proof

(i) Let $\boldsymbol{U}$ be an α-Q-cover of $A \wedge B$. Let $\boldsymbol{V}$ be the cover $\boldsymbol{U} \cup \{B'\}$. Now $\boldsymbol{V}$ is an α-Q-cover of A. For, Let $x_\alpha \leq A$. Now if $x_\alpha \leq B$, $x_\alpha \in M(\downarrow(A \wedge B))$ then there exists $U \in \boldsymbol{U}$ such that $x_\alpha \prec U$ .Where if $A \subset L$, then $\downarrow A = \{p \in L : \exists\, a \in A \text{ s.t } p \leq a\}$. If $x_\alpha \not\leq B$, then $x_\alpha \prec B' \in \boldsymbol{V}$. Thus $\boldsymbol{V}$ is an α-Q-cover of A.

Since A is α-metacompact, it has a point finite refinement $\boldsymbol{W}$ such that $\boldsymbol{W}$ is an α-Q-cover of A. Now clearly $\boldsymbol{W}' = \{W \in \boldsymbol{W} : W \leq U$ for some $U \in \boldsymbol{U}\}$ is point finite in $A \wedge B$. Now we will show that $\boldsymbol{W}'$ is an α-Q -cover of $A \wedge B$.

Let $x_\alpha \leq A \wedge B \leq A$. Then there exists $W \in \boldsymbol{W}$ such that $x_\alpha \prec W$. Now since $x_\alpha \leq B$, we have $B \geq x_\alpha \not\leq W'$. And hence $W \not\leq B'$. Since $\boldsymbol{W}$ refines $\boldsymbol{V} = \boldsymbol{U} \cup \{B'\}$, there exists a $U \in \boldsymbol{U}$ such that $W \leq U$. There fore $x_\alpha \prec W \in \boldsymbol{W}'$. Thus $\boldsymbol{W}'$ is an open α-Q-cover of $A \wedge B$. And hence $A \wedge B$ is α-metacompact.

(ii) can be deduced easily from (i)

A.1.13 Remark A result similar to that of Proposition A.1.12 can be obtained for α*-metacompact and *-metacompact spaces also.

From the proposition A.1.12 and remark A.1.13 it follows clearly that

A.1.14 Theorem α-metacompactness , α*-metacompactness, α-metacompactness and *-metacompactness are all closed hereditary.

A.1.15 Definition [Y;M] Let (L^X,δ) be an *L-fts*. Then by [δ] we denote the family of support sets of all crisp subsets in δ. (X, [δ]) is a topology and it is the background space. (L^X,δ) is weakly induced if each $U\in\delta$ is a lower semi continuous function from the background space (X, [δ]) to L.

A.1.16 Definition [Y;M] For a property P of ordinary topological space, a property P* of *L-fts* is called a good *L*-extension of *P*, if for every ordinary topological space (X,T) , (X,T) has the property P if and only if $(X,\omega_L(T))$ has property P*. In particular when L = $[0,1]$ we say P* is a good extension of P. Where $\omega_L(T)$ is the family of all lower semi continuous functions from (X,T) to L.

A.1.17 Theorem Let (L^X,δ) be a weakly induced *L-fts*. Then the following conditions are equivalent.

(i) (L^X,δ) is metacompact.

(ii) There exists $\alpha\in M(L)$ such that (L^X,δ) is α-metacompact.

(iii) (X, [δ]) is metacompact

Proof

(i)$\Rightarrow$ (ii) clear

(ii)$\Rightarrow$(iii) Let $\mathcal{U}\subset[\delta]$ be an open cover of X. Then clearly $\{\chi_U : U\in \mathcal{U}\}$ is an open α-Q-cover of $\underline{1}$. Then by (ii) it has a point finite refinement which is also an α-Q-cover of $\underline{1}$ say

V. Take $V_{(\alpha')} = \{x \in X : V(x) \nleq \alpha'\}$. Consider the collection $W = \{ V_{(\alpha')} : V \in \mathbf{V}\}$. Then by the weakly induced property of (L^X, δ), W is an open cover of $(X, [\delta])$.

Now we will prove that W is a point finite open refinement of U. Now for any $V \in V$, there is a $U \in U$ such that $V < \chi_U$. Let $x \in W$ for some $W \in W$. That is $x \in V_{(\alpha')}$ for some $V \in V$. That is $V(x) \nleq \alpha$. So $\chi_U(x) \neq 0$ and hence $x \in U$ and $W \subset U$. So W is a refinement of U.

Again if possible let $x \in W$ for infinitely many $W \in W$. That is $x \in \{x \in X : V(x) \nleq \alpha'\}$ for infinitely many $V \in V$. There fore it follows that $V(x) \nleq \alpha$ and hence $x_\alpha \nleq V'$ for infinitely many $V \in V$. That is $x_\alpha \prec V$ for infinitely many $V \in V$. This is a contradiction since V is point finite. Hence (ii)$\Rightarrow$(iii).

(iii) $\Rightarrow$ (i) Let $U \subset \delta$ be an α-Q cover of $\underline{1}$. where $\alpha \in M(L)$. Since (L^X, δ) is weakly induced $\{U_{(\alpha')} : U \in U\}$ is an open cover of X and there exists a refinement V of this which is also point finite cover of X. For every $V \in \mathbf{V}$, let U_V be such that $V \subset U_{V(\alpha')}$. Let $W = \{\chi_V \wedge U_V : V \in V \text{ and } V \subset U_{V(\alpha')}\}$. Now clearly W is an open refinement of U.

Now we will prove that W is point finite. Let $x_\lambda \in M(L^X)$. Then since V is point finite it follows clearly that $x \in V_1, V_2, \ldots\ldots V_n$ for some $n \in N$ and $V_i \in V$ for $i = 1, 2, \ldots n$. Now we will show that $x_\lambda \prec \chi_{V_i} \wedge U_{V_i}$ for at most finitely many i. For, if possible let $x_\lambda \prec \chi_V \wedge U_V$ for infinitely many $V \in V$. Then $x_\lambda \prec \chi_V$ or $x_\lambda \prec U_V$ for infinitely many $V \in V$. In both cases $x \in V$ for infinitely many $V \in V$. This is a contradiction and hence W is point finite. Thus (iii) $\Rightarrow$ (i). This completes the proof.

A.1 .18 Theorem Let (L^X, δ) be a weakly induced L-*fts*. Then the following conditions are equivalent.

(i) (L^X, δ) is $*$-metacompact.

(ii) There exists $\alpha \in M(L)$ such that (L^X, δ) is α^*-metacompact.

(iii) $(X, [\delta])$ is metacompact

Proof

(i)$\Rightarrow$ (ii) clear

(ii)$\Rightarrow$(iii) Let $\boldsymbol{U}\subset[\delta]$ be an open cover of X. Then clearly $\{\chi_U : U\in \boldsymbol{U}\}$ is an open α-Q-cover of $\underline{1}$and it has a *-point finite refinement V which is also an α-Q-cover of $\underline{1}$. We take $\boldsymbol{W} = \{ V_{(\alpha')}: V\in \mathbf{V}\}$. Now clearly $\boldsymbol{W}$ is a refinement of $\boldsymbol{U}$ and a cover of X. Since (L^X,δ) is weakly induced, $\boldsymbol{W}\subset[\delta]$. Now it is enough if we prove that $\boldsymbol{W}$ is a point finite collection.

We want to prove that for any $x\in X$, $x\in V_{i(\alpha')}$ for at most finitely many i. By (ii) we have $x_\alpha \prec \chi_{V_{i(0)}}$ for at most finitely many i and hence we have $\alpha\nleq\chi_{V_{i(0)}}'(x)$ for at most finitely many i. Now we know that $V_{i(0)}\nleq V_{i(\alpha')}$ and hence $\chi_{V_{i(0)}}\nleq\chi_{V_{n(\alpha')}}$. Therefore $\alpha'\nleq\chi_{V_{i(0)}}(x)\nleq\chi_{V_{i(\alpha')}}(x)$ for at most finitely many i. That is $\chi_{V_{i(\alpha')}}(x)\nleq\alpha'$ and thus $\chi_{V_{i(\alpha')}}(x)\neq 0$ and hence it follows that $x\in V_{i(\alpha')}$ for at most finitely many i.

(iii) $\Rightarrow$(i) Let $\alpha\in M(L)$ and $\boldsymbol{U}\subset\delta$ be an open α-Q-cover of $\underline{1}$. Now $\{ U_{(\alpha')} : U\in \boldsymbol{U}\}$ is an open cover of X, since (L^X,δ) is weakly induced. Given that (X, $[\delta]$) is metacompact and hence there exists an open point finite refinement V of $\{ U_{(\alpha')} : U\in \boldsymbol{U}\}$ which is a cover of X. For every $V\in V$ let $U_V \in U$ be such that $V\subset U_{V(\alpha')}$ and take W=$\{\chi_V\wedge U_V : V\in V\}$. Clearly $\boldsymbol{W}$ is an open α-Q -cover of $\underline{1}$ which is *-point finite.

For let $x_\lambda\in M(L^X)$. If possible let $x_\lambda\prec\chi\ (\chi_V\wedge U_V)_{(0)}$ for infinitely many $V\in V$.

That is $x_\lambda\prec\chi_V\wedge\chi_{U_V(0)}$ for infinitely many $V\in V$.

And hence $x_\lambda\prec\chi_V$ or $x_\lambda\prec\chi\ U_{V(0)}$ for infinitely many $V\in V$. In both cases $x\in V$ for infinitely many $V\in V$. This is a contradiction that V is point finite and hence $\boldsymbol{W}$ is *-point finite.

A.1.19 Theorem If (L^X,δ) is a weakly induced *L-fts*, then the following are equivalent

(i) (L^X,δ) is metacompact.

(ii) For every $\alpha\in M(L)$, every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

(iii) There exists an $\alpha\in M(L)$ such that every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

Proof

(i)$\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious

(iii)$\Rightarrow$(i) It is enough if we prove that (X, [δ]) is metacompact. By a characterization of [DE], it is enough to prove that every well monotone open cover of (X, [δ]) has a point finite open refinement.

Let $\{U_t : t\in T\}$ be a well monotone open cover of $(X, [\delta])$. Then clearly $\{\chi_{U_t} : t\in T\}$ is an open well monotone α-Q-cover of $\underline{1}$. So it has a point finite refinement say $\mathcal{A} = \{A_t : t\in T\}$. Let $\mathcal{B} = \{A_{t(\alpha')} : t\in T\}$. Now clearly $\mathcal{B}\subset [\delta]$ since(L^X,δ) is weakly induced. Now we show that $\mathcal{B}$ is the required point finite refinement .

If possible let for any $x\in X$, $x\in \mathcal{B}$ for infinitely many $B\in \mathcal{B}$. That is $A_t(x) \not\leq \alpha'$ for infinitely many $t\in T$. Thus $x_\alpha \prec A_t$ for infinitely many $t\in T$. This is a contradiction to that $\mathcal{A}$ is point finite. Also $U_t \supset A_{t(\alpha')}$. For, let , $x\in A_{t(\alpha')}$ for some $t\in T$. Now since$\{A_t : t\in T\}$refines $\{\chi_{U_t} : t\in T\}$ it follows that $\alpha' \not\geq A_t(x) \leq \chi_{U_t}$ and this implies $\chi_{U_t}(x) \neq 0$.Thus $x\in U_t$ and hence $\mathcal{B}$ is a refinement of $\{U_t : t\in T\}$also .This completes the proof.

A.1.20 Lemma Let (L^X,δ) be a weakly induced L-*fts* and $\alpha\in M(L)$. Then if every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$ then(L^X,δ) is metacompact.

Proof

By the characterization of metacompactness [DE] , it is enough to prove that every directed open cover of (X, [δ]) has a closure preserving closed refinement.

Let $\mathcal{U}= \{U_t : t\in T\}$ be a directed open cover of X. Then $\{\chi_{U_t} : t\in T\}$ is clearly a directed open α-Q-cover of $\underline{1}$ and hence it has a closure preserving closed refinement say

$A = \{A_t : t \in T\}$ which is also an α-Q-cover of $\underline{1}$. Now consider $B = \{A_{t\,[\alpha']} : t \in T\}$. This is the required collection where for any L-Fuzzy set A, $A_{[\alpha]} = \{x \in X : A(x) \leq \alpha\}$. Now since (L^X, δ) is a weakly induced, we have $B \subset \delta'$. Now since $\{A_t : t \in T\}$ refines $\{\chi_{U_t} : t \in T\}$, it follows that if $x \in A_{t\,[\alpha']}$ we have $\chi_{U_t}(x) \geq A_t(x) \nleq \alpha$. Thus $\chi_{U_t}(x) \neq 0$. Therefore $x \in U_t$ and $A_{t\,[\alpha']} \subset U_t$.

Now we will prove that B is closure preserving. For always we have $\vee\ cl\ A_0 \leq cl\ \vee A_0$ for any sub collection A_0 of B. For convenience take $A_0 = \{A_{t\,[\alpha']} : t \in T_0\}$ where $T_0 \subset T$. We will show that $\{cl\,[\vee A_{t[\alpha']}] : t \in T_0\} \leq \vee\{cl\,[A_{t[\alpha']}] : t \in T_0\}$. Let $x \in cl\,[\vee A_{t[\alpha']}]$. Then $cl\,[\vee A_t](x) \leq \alpha'$. That is $x_{\alpha'} \geq cl\,[\vee A_t] = \vee\ cl\,[A_t]$ since $\{A_t : t \in T\}$ is closure preserving. Therefore $\alpha' \geq \vee\ cl\,[A_t](x)$. This implies $x \in \vee\{cl\,[A_{t[\alpha']}] : t \in T_0\}$. This completes the proof.

A.1.21 Lemma Let (L^X, δ) be a weakly induced metacompact L-*fts* and $\alpha \in M(L)$. Then every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an open α-Q-cover of $\underline{1}$.

Proof

Let Φ be a directed open α-Q-cover of $\underline{1}$. Now $U = \{U_{(\alpha')} : \alpha \in \Phi\}$ is a directed open cover of $(X, [\delta])$. Since $(X, [\delta])$ is metacompact, it follows that U has a closure preserving closed refinement say V. Now consider $\{\chi_V : V \in V\}$. This is the required closure preserving closed refinement of Φ which is also an open α-Q-cover of $\underline{1}$.

A.1.22 Lemma Let (L^X, δ) be an L-*fts* and $\alpha \in M(L)$. Then the following are equivalent.

(i) Every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an open α-Q-cover of $\underline{1}$.

(ii) For every α-Q-cover U of $\underline{1}$, U^F has a closure preserving closed refinement which is also an open α-Q-cover of $\underline{1}$. Where U^F is the collection of all unions of finite sub collections from U.

Proof

(i) $\Rightarrow$ (ii) Clearly U^F is directed and hence has a closure preserving closed refinement.

(ii) $\Rightarrow$ (i) Let U be directed open α-Q-cover of $\underline{1}$. Now since U is directed, U^F is a refinement of U. Then by (ii) U^F has a closure preserving closed refinement say V which is an α-Q-cover of $\underline{1}$. Now $V < U^F < U$. Hence it follows that V is the required closure preserving closed refinement of U.

Combining Theorem A.1.19, Lemmas A.1.20 , A.1.21, and A.1.22 we get the following characterization of metacompactness in a weakly induced *L-fts*.

A.1.23 Theorem Let (L^X, δ) be a weakly induced *L-fts* . Then the following are equivalent

(i) (L^X, δ) is metacompact.

(ii) There exists $\alpha \in M(L)$ such that (L^X, δ) α-metacompact.

(iii) $(X, [\delta])$ is metacompact

(iv) For every $\alpha \in M(L)$, every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

(v) There exists an $\alpha \in M(L)$ such that every well monotone open α-Q-cover of $\underline{1}$ has a point finite open refinement which is also an α-Q-cover of $\underline{1}$.

(vi) For every $\alpha \in M(L)$, every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

(vii) There exists an $\alpha \in M(L)$ such that every directed open α-Q-cover of $\underline{1}$ has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

(viii) For every $\alpha \in M(L)$, every open α-Q-cover U of $\underline{1}$, U^F has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

(ix) There exists an $\alpha \in M(L)$ such that for every open α-Q-cover U of $\underline{1}$, U^F has a closure preserving closed refinement which is also an α-Q-cover of $\underline{1}$.

APPENDIX – II

L-FUZZY COVERING DIMENSION

In this section a characterisation of covering dimension in L-fts which is good in the sense of Ying-Ming and Mao-Kang [Y;M] is obtained.

A.2.1Definition The order of a fuzzy point x_α in a family $\boldsymbol{U} = \{U_\alpha : \alpha \in A\}$ is the number of elements of $\boldsymbol{U}$ which are quasi coincident with x_α. We denote it by Ord $(x_\alpha, \boldsymbol{U})$. Order of the collection $\boldsymbol{U}$ is the supremum of all Ord $(x_\alpha, \boldsymbol{U})$ with x_α running over $M(L^X)$.

A.2.2 Definition Let (L^X,δ) be an *L-fts*, $A \in L^X$. Then α-dim A is the least integer n such that every finite open α-Q-cover of A has an open α-Q-cover refinement of order not exceeding n. And dim $A = n$ if α-dim $A = n$ for every $\alpha \in M(L)$. $\dim(L^X,\delta) = n$ if $\dim \underline{1} = n$.

A.2.3 Remark $\dim(L^X,\delta) = -1$ if and only if X is void and $\dim(L^X,\delta) = n$ if it is true that $\dim(L^X,\delta) \le n$ and $\dim(L^X,\delta) \le n-1$ is not true. Also $\dim(L^X,\delta) = \infty$ if it is not true for any integer n that $\dim(L^X,\delta) \le n$.

A.2.4 Theorem Let (L^X,δ) be an *L-fts*. Then the following are equivalent

(i) $\dim(L^X,\delta) \le n$

(ii) For every $\alpha \in M(L)$, every finite α-Q-cover $\{U_1,U_2,\ldots U_k\}$ of $\underline{1}$ by open fuzzy subsets , there is an open α-Q-cover$\{V_1,V_2,\ldots V_k\}$ of order not exceeding n such that $V_i<U_i$ for $i = 1,2,3,\ldots k$.

(iii) If$\{U_1,U_2,\ldots U_{n+2}\}$ is an open α-Q-cover of $\underline{1}$, then there exists an open α-Q-cover $\{V_1,V_2,\ldots V_{n+2}\}$ $\underline{1}$ *such* that $V_i<U_i$ and $\operatorname{Inf}_{1\le i\le n+2} V_i < \alpha$. Where $\alpha \in M(L)$.

Proof

(i) $\Rightarrow$ (ii)

Let $\dim(L^X,\delta) \le n$, $\alpha \in M(L)$ and $\boldsymbol{U} = \{U_1, U_2, \ldots U_k\}$ be a finite open α-Q-cover of $\underline{1}$. Now $\boldsymbol{U}$ has a refinement say $\boldsymbol{W}$ with order not exceeding n. Now if $W \in \boldsymbol{W}$, there exists some i such that $W < U_i$ and suppose that each W is associated with a unique U_i containing it and take $V_i = Sup\{W : W < U_i\}$. Clearly each U_i is open and $V_i < U_i$ for some i. Now since order of $\boldsymbol{W}$ is not exceeding n, it follows that for each $x_\alpha \in M(L^X)$ quasi coincides with atmost $n+1$ members of $\boldsymbol{W}$ and each $W \in \boldsymbol{W}$ is associated with a unique U_i. And hence x_α quasi coincides with atmost $n+1$ members of $\{V_i\}$. Hence $\{V_i\}$.is an α-Q-cover of $\underline{1}$ with order not exceeding n.

(i)$\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are obvious

(iii) $\Rightarrow$ (ii)

Let $\boldsymbol{U} = \{U_1, U_2, \ldots U_k\}$ be a finite open an α-Q-cover of $\underline{1}$. Assume that $k > n+1$. Define the collection $\{G_i : 1 \le i \le n+2\}$ as follows.

$G_i = U_i$ if $i \le n+1$ and $G_{n+2} = \underset{n+2 \le i \le k}{Sup} U_i$. Now clearly $\{G_i : 1 \le i \le n+2\}$ is an open an α-Q-cover of $\underline{1}$ and by hypothesis of (iii) there is an open an α-Q-cover $\{H_1, H_2, \ldots H_{n+2}\}$. such that $H_i < G_i$ and $\underset{1 \le i \le n+2}{Inf} H_i < \alpha$.

Now take $W_i = U_i$ if $i \le n+1$ and $W_i = U_i \wedge H_{n+2}$ if $i > n+1$. Then clearly the collection $\boldsymbol{W} = \{W_1, W_2, \ldots W_k\}$ is an open an α-Q-cover of $\underline{1}$ with the property that $W_i < U_i$ and $\underset{1 \le i \le n+2}{Inf} W_i < \alpha$. Now if there exists a subset B of $\{1, 2, 3, \ldots k\}$ with $n+2$ elements such that, $\underset{i \in B}{Inf} W_i > \alpha$ we will renumber the family $\boldsymbol{W}$ to give a family $\boldsymbol{P} = \{P_1, P_2, \ldots P_k\}$ such that $\underset{1 \le i \le n+2}{Inf} P_i > \alpha$ .Now proceeding in a manner similar to the construction above we obtain an an α-Q-cover $\boldsymbol{W}' = \{W'_1, W'_2, \ldots W'_k\}$ by open fuzzy sets with $W_i < P_i$ and

$\underset{1\le i\le n+2}{Inf}W'_i < \alpha$ .Now again if $\boldsymbol{C}$ is a subset of $\{1,2,....k\}$ with $n+2$ elements such that $\underset{i\in P}{Inf}P_i > \alpha$ then $\underset{i\in P}{Inf}W'_i > \alpha$. By repeating this process for a finite number of times we will end up with an open α-Q-cover $\{V_1,V_2,.....V_k\}$ of $\underline{1}$ with order not exceeding n and $V_i < U_i$. This completes the proof.

A.2.5 Theorem In a weakly induced *L-fts* the following are equivalent

(i) $\dim(L^X,\delta) \le n$

(ii) There exists an $\alpha \in M(L)$ such that α–$\dim(L^X,\delta) \le n$

(iii) $\dim(X,[\delta]) \le n$

Proof

(i)$\Rightarrow$ (ii) clear

(ii)$\Rightarrow$ (iii)

Let $\boldsymbol{U} = \{U_1,U_2,.....U_k\} \subset [\delta]$ be a finite open cover of X. Then $\{\chi_U : U \in \boldsymbol{U}\}$ is an open α-Q-cover of $\underline{1}$. Then by α–$\dim(L^X,\delta) \le n$ it follows that $\{\chi_U : U \in \boldsymbol{U}\}$ has an open refinement $\boldsymbol{V}$ of order not exceeding n. Now consider $\boldsymbol{W} = \{V_{(\alpha')} : V \in \boldsymbol{V}\}$ where $V_{(\alpha')} = \{x \in X: V(x) \not\le \alpha'\}$. Now by the weakly induced property , $\boldsymbol{W}$ is an open cover of X .Now we will prove that $\boldsymbol{W}$ has order not exceeding n.

For, if possible let order of $\boldsymbol{W}$ be greater than n. Therefore there exists an $x \in X$ which belongs to at least $n+2$ members of $\boldsymbol{W}$.

ie, $x \in \{x \in X: V(x) \not\le \alpha'\}$ for at least $n+2$ members of $\boldsymbol{V}$

ie , $V(x) \not\le \alpha'$ for at least $n+2$ members of $\boldsymbol{V}$

or $x_\alpha \prec \boldsymbol{V}$ for at least $n+2$ members of $\boldsymbol{V}$. This is a contradiction to order of $\boldsymbol{V}$ is not exceeding n .

(iii)$\Rightarrow$ (i)

Let $\boldsymbol{U} \subset [\delta]$ be an open α-Q-cover of $\underline{1}$ where $\alpha \in M(L)$. Since (L^X,δ) is weakly induced, it follows that $\{U_{(\alpha')} : U \in \boldsymbol{U}$ is an open cover of X and it has an open refinement of order not exceeding n say $\boldsymbol{V}$. For every $V \in \boldsymbol{V}$, let U_V be such that $V < U_{V_{(\alpha')}}$. Consider $\boldsymbol{W} = \{\chi_V \wedge U_V : V \in \boldsymbol{V},\ V < U_{V_{(\alpha')}}\}$. This is an open refinement of $\boldsymbol{U}$ with order not exceeding n. If possible let order of $\boldsymbol{W}$ be greater than n. Therefore there exists an $x_\alpha \in M(L^X)$ which quasi coincides with at least $n+2$ members of $\boldsymbol{W}$.

ie, $x_\alpha \not\leq (\chi_V \wedge U_V)'$ for at least $n+2$ members of $\boldsymbol{W}$

ie, $x_\alpha \not\leq \chi_V' \vee U_V'$ for at least $n+2$ members of $\boldsymbol{W}$

ie, $x_\alpha \prec \chi_V$ or $x_\alpha \prec U_V$ for at least $n+2$ members of $\boldsymbol{V}$

In both cases $x \in V$ for at most $n+2$ members of $\boldsymbol{V}$ and this is a contradiction. This completes the proof.

A.2.6 Remark The L-Fuzzy covering dimension defined is a good L-extension of ordinary covering dimension.

A.2.7 Main Theorem In a weakly induced *L-fts* the following are equivalent

(i) $\dim(L^X,\delta) \leq n$

(ii) For every $\alpha \in M(L)$, every finite α-Q-cover of $\underline{1}$ by open fuzzy sets has a precise open refinement of order not exceeding n

(iii) There exists an $\alpha \in M(L)$ such that every finite α-Q-cover of $\underline{1}$ by open fuzzy sets has a precise open refinement of order not exceeding

(iv) If $\{U_1, U_2, \ldots U_{n+2}\}$ is an open α-Q-cover of $\underline{1}$, then there exists an open α-Q-cover $\{V_1, V_2, \ldots V_{n+2}\}$ $\underline{1}$ such that $V_i < U_i$ and $\operatorname{Inf}_{1\leq i\leq n+2} V_i < \alpha$. Where $\alpha \in M(L)$.

(v) There exists an $\alpha \in M(L)$ such that α–$\dim(L^X,\delta) \leq n$

(vi) $\dim(X,[\delta]) \leq n$

Proof

Equivalence of (i) ,(v) and (vi) follows from theorem A.2.5. All other implications except (iii) $\Rightarrow$ (i) follows from theorem A.2.4.

(iii) $\Rightarrow$ (i)

By theorem A.2.5 it is enough to prove that $\dim(X,[\delta]) \leq n$ .Let $\boldsymbol{U} \subset [\delta]$ be a finite open cover of X. Then $\{\chi_U : U \in \boldsymbol{U}\}$ is a finite open α-Q-cover of $\underline{1}$ and it has a precise open refinement of order not exceeding n. Let it be $\boldsymbol{V} = \{V_1, V_2 \ldots V_k\}$. Let $\boldsymbol{W} = \{V_{i(\alpha')}: i = 1,2,3,\ldots\ldots\}$. By weakly induced property, $\boldsymbol{W}$ is an open cover of X. Also it is easy to show that order of $\boldsymbol{W}$ is not exceeding n and hence $\dim(X,[\delta]) \leq n$.

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