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CLIQUE IRREDUCIBILITY OF SOME ITERATIVE CLASSES OF GRAPHS

Aparna Lakshmanan S.

and

A. Vijayakumar

Department of Mathematics
Cochin University of Science and Technology
Cochin–682 022, India

e-mail: aparna@cusat.ac.in
e-mail: vijay@cusat.ac.in

Abstract

In this paper, two notions, the clique irreducibility and clique vertex irreducibility are discussed. A graph G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and it is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G. It is proved that $L(G)$ is clique irreducible if and only if every triangle in G has a vertex of degree two. The conditions for the iterations of line graph, the Gallai graphs, the anti-Gallai graphs and its iterations to be clique irreducible and clique vertex irreducible are also obtained.

Keywords: line graphs, Gallai graphs, anti-Gallai graphs, clique irreducible graphs, clique vertex irreducible graphs.

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1. Introduction

We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

A clique of a graph G is a maximal complete subgraph of G. A graph G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and it is clique reducible if it is not clique irreducible [7]. A graph G is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G and it is clique vertex reducible if it is not clique vertex irreducible.

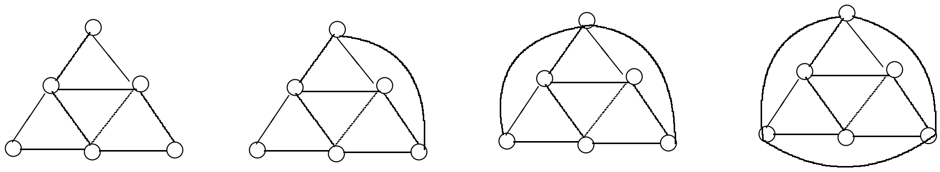
The line graph of a graph G, denoted by $L(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $L(G)$ are adjacent if the corresponding edges in G are incident. The iterations of $L(G)$ are recursively defined by $L^1(G) = L(G)$ and $L^{n+1}(G) = L(L^n(G))$, for $n \geqslant 1$ [5].

The Gallai graph of a graph G, denoted by $\Gamma(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $\Gamma(G)$ are adjacent if the corresponding edges in G are incident on a common vertex and they do not lie in a common triangle [4]. The anti-Gallai graph of a graph G, denoted by $\Delta(G)$, is a graph whose vertex set corresponds to the edge set of G and any two vertices in $\Delta(G)$ are adjacent if the corresponding edges lie in a triangle in G [4]. Both the Gallai graph and the anti-Gallai graph are spanning subgraphs of the line graph and their union is the line graph. Though $L(G)$ has a forbidden subgraph characterization, both these do not have the vertex hereditary property and hence cannot be characterized using forbidden subgraphs [4].

In [1], it is proved that there exist infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai and anti-Gallai graphs. The existence of a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be H-free for any finite graph H is proved. The relationship between the chromatic number, the radius and the diameter of a graph and its Gallai and anti-Gallai graphs are also obtained. In [4], it has been proved that $\Gamma(G)$ is isomorphic to G only for cycles of length greater than three. Also, computing the clique number and the chromatic number of $\Gamma(G)$ are NP-complete problems.

A graph G is clique-Helly if any family of mutually intersecting cliques has non-empty intersection [6]. It is hereditary clique-Helly if all the induced subgraphs of G are clique-Helly [6]. It is also proved in [6] that a graph G is hereditary clique-Helly, if it does not contain any Hajós' graph as an induced subgraph.

The complement of a graph G is denoted by G^c and the graph induced by a set of vertices $v_1, v_2, \ldots, v_n$ is denoted by $\langle v_1, v_2, \ldots, v_n \rangle$. A complete graph, a path and a cycle on n vertices are denoted by K_n, P_n and C_n respectively. The complete bipartite graph is denoted by $K_{m,n}$, where m and n are the number of vertices in each of the partition. A vertex of degree one is called a pendant vertex and an edge incident to a pendant vertex is called a pendant edge. A diamond is the graph $K_4 - \{e\}$, where e is any edge of K_4.



Hajós' graphs

In this paper, the graphs G for which $L(G)$ and $L^2(G)$ are clique vertex irreducible are characterized and it is deduced that $L^n(G)$ for $n \geqslant 3$ is clique vertex irreducible if and only if G is $K_3, K_{1,3}$ or P_k where $k \leqslant n+3$. After characterizing the graphs G such that $L(G)$, $L^2(G)$, $L^3(G)$ and $L^4(G)$ are clique irreducible, we prove that $L^n(G)$, $n \geqslant 5$, is clique irreducible if and only if it is non-empty and $L^4(G)$ is clique irreducible. The Gallai graphs which are clique irreducible and clique vertex irreducible are characterized. A forbidden subgraph characterization for clique vertex irreducibility of $\Gamma(G)$ is obtained. Also, the forbidden subgraphs for the anti-Gallai graphs and all its iterations to be clique irreducible and clique vertex irreducible are obtained.

All graph theoretic terminology and notations not mentioned here are from [2].

2. The Iterations of the Line Graph

Theorem 1. *Let G be a graph. The line graph $L(G)$ is clique vertex irreducible if and only if G satisfies the following conditions*

(1) *Every triangle in G has at least two vertices of degree two,*

(2) *Every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two lying in a triangle.*

Proof. Let G be a graph which satisfies the conditions (1) and (2). The cliques of $L(G)$ are induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three, the edges in G which are incident on a vertex of degree two and which do not lie in a triangle and by the edges in G which lie in a triangle. By (2), the cliques in $L(G)$

induced by the vertices corresponding to the edges in G which are incident on a vertex, have a vertex which does not lie in any other clique of $L(G)$. By (1), the cliques in $L(G)$ induced by the vertices which correspond to the edges in G which lie in a triangle, have a vertex which does not lie in any other clique of $L(G)$. Therefore, G is clique vertex irreducible.

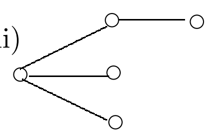
Conversely, assume that $L(G)$ is a clique vertex irreducible graph. Let $\langle u_1, u_2, u_3 \rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 in G. $T = \langle e_1, e_2, e_3 \rangle$ is a clique in $L(G)$. If $d(u_i) > 2$ for two u_is, u_1 and u_2, then there exist v_1 and v_2 (not necessarily different, but different from u_3) such that u_i is adjacent to v_i for $i = 1, 2$. But then, the vertices e_1 and e_3 will be present in the clique induced by the edges incident on the vertex u_1 and the vertices e_2 and e_3 will be present in the clique induced by the edges incident on the vertex u_2. Therefore, every vertex in T belongs to another clique in $L(G)$ which is a contradiction to the assumption that $L(G)$ is clique vertex irreducible. Hence every triangle in G has at least two vertices of degree two.

Now, let $u \in V(G)$ and $N(u) = \{u_1, u_2, \ldots, u_p\}$, where $p \geqslant 2$ and if $p = 2$ then u_1 is not adjacent to u_2. Let e_i be the vertex in $L(G)$ corresponding to the edge uu_i in G for $i = 1, 2, \ldots, p$. Let C be the clique $\langle e_1, e_2, \ldots, e_p \rangle$ in $L(G)$. If u has no pendant vertex attached to it then every u_i has a neighbor $v_i \neq u$ for $i = 1, 2, \ldots, p$. The v_is are not necessarily pairwise different. Moreover, some v_i can be equal to some u_j with $j \neq i$, except in the case $p = 2$. Therefore, for each i, every e_i in $L(G)$ will be present in another clique, either induced by the edges incident on the vertex u_i in G or by the edges in a triangle containing u and u_i in G. But this is a contradiction to the assumption that $L(G)$ is clique vertex irreducible. Hence, every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

■

Theorem 2. *Let G be a connected graph. The second iterated line graph $L^2(G)$ is clique vertex irreducible if and only if G is one of the following graphs.*

(i) K_2 (ii) K_3 (iii) P_3 (iv) P_4 (v) P_5 (vi) $K_{1,3}$ (vii)



Proof. By Theorem 1, $L^2(G)$ is clique vertex irreducible if and only if

(1) Every triangle in $L(G)$ has at least two vertices of degree two,

(2) Every vertex of degree greater than one in $L(G)$ has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

By (2), every non-pendant edge in G must have a pendant edge attached to it on one end vertex and the degree of that end vertex must be two.

Case 1. $L(G)$ has a triangle.
A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. Let it correspond to a triangle in G. If any of the vertices of this triangle has a neighbor outside the triangle, then two vertices in the corresponding triangle in $L(G)$ have neighbors outside the triangle, which is a contradiction. Therefore, since G is connected, in this case G must be K_3.

If the triangle in $L(G)$ corresponds to a $K_{1,3}$ in G, then two of the edges of this $K_{1,3}$ cannot have any other edge incident on any of its end vertices. Therefore, G cannot have a vertex of degree greater than three. Moreover, two vertices of $K_{1,3}$ in G must be pendant vertices. Again, by (2) and since G is connected, we conclude that G is either $K_{1,3}$ or the graph (vii).

Case 2. $L(G)$ has no triangle.
Since $L(G)$ has no triangle, G cannot have a K_3 or a vertex of degree greater than or equal to 3. Therefore, since G is connected, G must be a path or a cycle of length greater than three. Again, by (2), G cannot be a path of length greater than five or a cycle. Therefore G is K_2, P_3, P_4 or P_5. ■

Corollary 3. *Let G be a connected graph. The n^{th} iterated line graph $L^n(G)$ is clique vertex irreducible if and only if G is $K_3, K_{1,3}$ or P_k where $n+1 \leqslant k \leqslant n+3$, for $n \geqslant 3$.*

Theorem 4. *The line graph $L(G)$ is clique irreducible if and only if every triangle in G has a vertex of degree two.*

Proof. Let G be a graph such that every triangle in G has a vertex of degree two. Let C be a clique in $L(G)$.

Case 1. The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three.

An edge of C can be present in another clique of $L(G)$ if and only if the corresponding pair of edges in G lies in a triangle. Thus, if every edge of C lies in another clique of $L(G)$, then G has an induced K_p, where p is at least four. But, this contradicts the assumption that every triangle in G has a vertex of degree two.

Case 2. The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree two and which do not lie in a triangle.

In this case, C is K_2 which always has an edge of its own.

Case 3. The clique C is induced by the vertices corresponding to the edges which lie in a triangle T in G.

Since T has a vertex v of degree two, the vertices corresponding to the edges which are incident on v induce an edge in C which does not lie in any other clique of $L(G)$. Therefore, G is clique irreducible.

Conversely, assume that G is a clique irreducible graph. Let $\langle u_1, u_2, u_3\rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 of G. $T = \langle e_1, e_2, e_3\rangle$ is a clique in $L(G)$. If $d(u_i) > 2$ for each i, there exist v_1, v_2, v_3 such that u_i is adjacent to v_i for $i = 1, 2, 3$ (v_1, v_2 and v_3 are not necessarily different, but they are different from u_1, u_2 and u_3). Then the edges e_1e_2, e_2e_3 and e_3e_1 of $L(G)$ will be present in the cliques induced by edges which are incident on the vertices u_1, u_2 and u_3 respectively. Therefore, every edge in T is in another clique of $L(G)$, which is a contradiction. ■

Theorem 5. *The second iterated line graph $L^2(G)$ is clique irreducible if and only if G satisfies the following conditions*

(1) *Every triangle in G has at least two vertices of degree two,*

(2) *Every vertex of degree three has at least one pendant vertex attached to it,*

(3) *G has no vertex of degree greater than or equal to four.*

Proof. Let G be a graph such that $L^2(G)$ is clique irreducible. By Theorem 4, every triangle in $L(G)$ has a vertex of degree two. Then, we have the following cases.

Case 1. The triangle in $L(G)$ corresponds to a triangle in G.

Let $\langle u_1, u_2, u_3 \rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 of G. At least one of the vertices of the triangle $\langle e_1, e_2, e_3 \rangle$ in $L(G)$ must be of degree two. Let e_1 be a vertex of degree two in $L(G)$. Since e_2 and e_3 belong to $N(e_1)$ in $L(G)$, e_1 has no other neighbors in $L(G)$. Therefore, the corresponding end vertices, u_1 and u_2 in G have no other neighbors. Hence (1) holds.

Case 2. The triangle in $L(G)$ corresponds to a $K_{1,3}$ (need not be induced) in G.

Let e_1, e_2, e_3 be the vertices in $L(G)$ corresponding to the edges uu_1, uu_2, uu_3 in G. At least one of the vertices of the triangle $\langle e_1, e_2, e_3 \rangle$ in $L(G)$ must be of degree two. Let e_1 be a vertex of degree two in $L(G)$. Vertices e_2 and e_3 belong to $N(e_1)$ in $L(G)$ and hence e_1 has no other neighbors in $L(G)$. Therefore, the corresponding end vertices, u and u_1 in G have no other neighbors. Since u has no other neighbors (3) holds and since u_1 has no other neighbors (2) holds.

Conversely, assume that G is a graph which satisfies all the three conditions. A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. A triangle in $L(G)$ which corresponds to a triangle in G has at least one vertex of degree two by (1). Again, a triangle in $L(G)$ which corresponds to a $K_{1,3}$ in G has at least one vertex of degree two by (2) and (3). Therefore, every triangle in $L(G)$ has at least one vertex of degree two and by Theorem 4, $L^2(G)$ is clique irreducible. ∎

Theorem 6. *Let G be a connected graph. If $G \neq K_3$ then, $L^3(G)$ is clique irreducible if and only if G satisfies the following conditions*

(1) *G is triangle free,*

(2) *G has no vertex of degree greater than or equal to four,*

(3) *At least two of the vertices of every $K_{1,3}$ in G are pendant vertices,*

(4) *If uv is an edge in G, then either u or v has degree less than or equal to two.*

Proof. Let $L^3(G)$ be clique irreducible. By Theorem 5, $L(G)$ satisfies:

(1′) Every triangle in $L(G)$ has at least two vertices of degree 2,

(2′) Every vertex of degree three in $L(G)$ has at least one pendant vertex attached to it,

(3′) $L(G)$ has no vertex of degree greater than or equal to 4.

A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. Every triangle in $L(G)$ has at least two vertices of degree two implies that every triangle in G has its three vertices of degree two. i.e., G is a triangle, because G is connected. Since $G \neq K_3$, G must be triangle free. Also, every $K_{1,3}$ in G has at least two pendant vertices and the degree of a vertex cannot exceed three. Therefore (1), (2) and (3) hold. Again (3′) implies that no edge in G can have more than three edges incident on its end vertices. Therefore, (4) holds.

Conversely, assume that the given conditions hold. Since G is triangle free, a triangle in $L(G)$ corresponds to a $K_{1,3}$ (need not be induced) in G. Therefore, by (2) and (3) every triangle in $L(G)$ has at least two vertices of degree two.

Let e be a vertex of degree three in $L(G)$ and let uv be the corresponding edge in G. Since e is of degree three in $L(G)$, the number of edges incident on u in G together with the number of edges incident on v in G is three. If u (or v) has three more edges incident on it then u (or v) will be of degree at least four which is a contradiction to the condition (2). Therefore, u has two neighbors and v has one neighbor (or vice versa) in G. Let u_1 and u_2 be the neighbors of u, and let v_1 be the neighbor of v in G. Then $\langle u, v, u_1, u_2 \rangle = K_{1,3}$ in G and hence at least two of v, u_1 and u_2 must be pendant vertices. Since v is not a pendant vertex, u_1 and u_2 must be pendant vertices. Therefore, e has two pendant vertices attached to it in $L(G)$ corresponding to the edges uu_1 and uu_2 in G. Hence (2′) is satisfied.

Again, (2), (3) and (4) together imply (3′). Since the conditions (1′), (2′) and (3′) are satisfied, by Theorem 5, $L^3(G)$ is clique irreducible. ■

Theorem 7. *Let G be a connected graph. The fourth iterated line graph $L^4(G)$ is clique irreducible if and only if G is $K_3, K_{1,3}, P_n$ with $n \geqslant 5$ or C_n with $n \geqslant 4$.*

Proof. Let $L^4(G)$ be clique irreducible. Then by Theorem 6, if $L(G) \neq K_3$ then $L(G)$ must be triangle free. If $L(G) = K_3$ then G is either K_3 or $K_{1,3}$. If $L(G)$ is triangle free then G is triangle free and cannot have vertices of degree greater than or equal to three. Therefore, G is either a path or a cycle of length greater than three.

Conversely, if G is $K_3, K_{1,3}, P_n$ or C_n then $L^4(G)$ is either a triangle, a path or a cycle and all of them are clique irreducible.

Corollary 8. *For $n \geqslant 5$, $L^n(G)$ is clique irreducible if and only if it is non-empty and $L^4(G)$ is clique irreducible.*

3. The Gallai Graphs

Theorem 9. *The Gallai graph $\Gamma(G)$ is clique vertex irreducible if and only if for every $v \in V(G)$, every maximal independent set I in $N(v)$ with $|I| \geqslant 2$ contains a vertex u such that $N(u) - \{v\} = N(v) - I$.*

Proof. Let G be a graph such that its Gallai graph $\Gamma(G)$ is clique vertex irreducible. A clique C in $\Gamma(G)$ of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex $v \in V(G)$ whose other end vertices form a maximal independent set I of size at least two in $N(v)$. Let $I = \{v_1, v_2, \ldots, v_p\}$, where $p \geqslant 2$, be a maximal independent set in $N(v)$. Let e_i be the vertex in $\Gamma(G)$ corresponding to the edge vv_i in G for $i = 1, 2, \ldots, p$. Let C be the clique $\langle e_1, e_2, \ldots, e_p \rangle$ in $\Gamma(G)$. Let e_i be the vertex in C which does not belong to any other clique in G. Therefore, e_i has no neighbors in $\Gamma(G)$ other than those in C. Hence, $N(v_i) - \{v\} = N(v) - I$.

Conversely, assume that for every $v \in V(G)$, every maximal independent set $I = \{v_1, v_2, \ldots, v_p\}$ in $N(v)$ contains a vertex u such that $N(u) - \{v\} = N(v) - I$. If C is a clique of size one, it contains a vertex of its own. Otherwise, let C be defined as above. By our assumption, there exists a vertex $u = v_i$ such that $N(u) - \{v\} = N(v) - I$. Therefore, e_i has no neighbors outside C. Hence C has a vertex e_i of its own. ■

Theorem 10. *If $\Gamma(G)$ is clique vertex reducible, then G contains one of the graphs in Figure 1 as an induced subgraph.*

Proof. Let G be a graph such that $\Gamma(G)$ is clique vertex reducible and let C be a clique in $\Gamma(G)$ such that each vertex of C belongs to some other clique in $\Gamma(G)$. Consider the order relation $\preceq$ among the vertices of C where $e \preceq e'$ if $N[e] \preceq N[e']$. If $\preceq$ is a total ordering, then every vertex adjacent to the minimum vertex e is also adjacent to all the vertices in C. Therefore, by maximality of C, e cannot have neighbors outside C. This is a contradiction to the assumption that e belongs to some other clique of $\Gamma(G)$. So, there exist two vertices e_1 and e_2 in C which are not comparable. That is, there exist vertices f_1 and f_2 of $\Gamma(G)$ such that e_i is adjacent to f_j

if and only if $i = j$. Let vv_1 and vv_2 be the edges corresponding to e_1 and e_2, respectively. Then v_1 and v_2 are non-adjacent. Let u_1 and u_2 be the end points of f_1 and f_2, respectively, which are both different from v, v_1 and v_2.

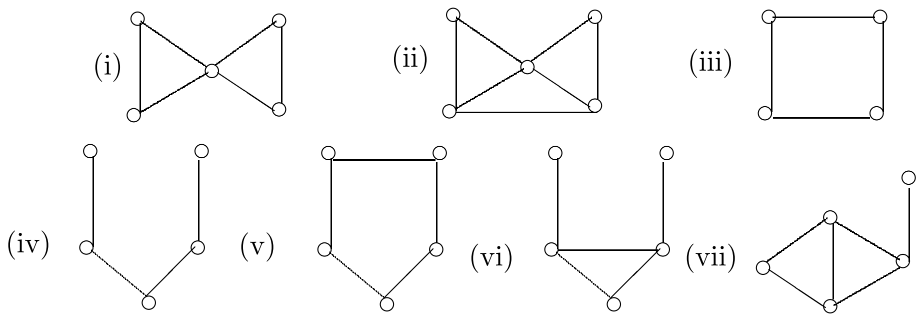


Figure 1

Case 1. Both f_1 and f_2 correspond to the edges incident to v.
In this case, u_1 and u_2 are adjacent to v, u_i is adjacent to v_j if and only if $i \neq j$ and u_1 and u_2 can be either adjacent or not. Therefore $\langle v, v_1, v_2, u_1, u_2 \rangle$ is the graph (i) or (ii) in Figure 1.

Case 2. None of f_1 and f_2 correspond to the edges incident to v.
In this case, u_1 and u_2 are adjacent to v_1 and v_2, respectively, and not to v. If $u_1 = u_2$ then G contains an induced C_4. If $u_1 \neq u_2$ and G does not contain an induced C_4, then $\langle v, v_1, v_2, u_1, u_2 \rangle$ is either P_5 or C_5.

Case 3. Exactly one of f_1 and f_2 correspond to the edges incident to v, say f_1.

In this case, u_1 is adjacent to both v and v_2 and is not adjacent to v_1. The vertex u_2 is adjacent to v_2 and is not adjacent to v. If u_2 is adjacent to v_1 then G contains an induced C_4. Otherwise, $\langle v, v_1, v_2, u_1, u_2 \rangle$ is the graph (vi) or (vii) in Figure 1. ■

Theorem 11. *The Gallai graph* $\Gamma(G)$ *is clique irreducible if and only if for every* $v \in V(G)$, $\langle N(v) \rangle^c$ *is clique irreducible.*

Proof. A clique C in $\Gamma(G)$ of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex $v \in V(G)$ whose other end vertices form a maximal independent set I of size at least two in $N(v)$. Therefore, C has an edge which does not belong to any other clique of $\Gamma(G)$ if and only if I has a pair of vertices both of which together does not belong to any other maximal independent set in $N(v)$. But, this happens if and only if every clique of size at least two in $\langle N(v)\rangle^c$ has an edge which does not belong to any other clique in $\langle N(v)\rangle^c$, since a maximal independent set in a graph corresponds to a clique in its complement. ■

Theorem 12. *The second iterated Gallai graph* $\Gamma^2(G)$ *is clique irreducible if and only if for every* $uv \in E(G)$, *either* $\langle N(u)-N(v)\rangle$ *and* $\langle N(v)-N(u)\rangle$ *are clique vertex irreducible or one among them is a clique and the other is clique irreducible.*

Proof. By Theorem 11, $\Gamma^2(G)$ is clique irreducible if and only if for every $e \in V(\Gamma(G))$, $\langle N(e)\rangle^c$ is clique irreducible.

Let $e = uv \in E(G)$, $N(u)-N(v) = \{u_1, u_2, \ldots, u_p\}$ and $N(v)-N(u) = \{v_1, v_2, \ldots, v_l\}$. Also let $e_i = uu_i$ for $i = 1, 2, \ldots, p$ and $f_j = vv_j$ for $j = 1, 2, \ldots, l$. $N_{\Gamma(G)}(e) = \{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_l\}$. $\langle N(e)\rangle^c$ is clique irreducible if and only if every maximal independent set I in $\langle N(e)\rangle$ has a pair of vertices of its own. e_i is not adjacent to e_j if and only if u_i is adjacent to u_j. Similarly, f_i is not adjacent to f_j if and only if v_i is adjacent to v_j. So, $I = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}, f_{j_1}, f_{j_2}, \ldots, f_{j_l}\}$ if and only if $\{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ is a clique in $\langle N(u) - N(v)\rangle$ and $\{v_{j_1}, v_{j_2}, \ldots, v_{j_l}\}$ is a clique in $N(v) - N(u)$. Therefore, every maximal independent set I in $N_{\Gamma(G)}(e)$ has a pair of vertices of its own if and only if either both $\langle N(u) - N(v)\rangle$ and $\langle N(v) - N(u)\rangle$ are clique vertex irreducible or one among them is a clique and the other is clique irreducible. ■

Theorem [6]**.** *If* G *is hereditary clique-Helly, then it is clique irreducible.*

Theorem 13. *If* $\Gamma(G)$ *is clique reducible then* G *contains one of the graphs in Figure* 2 *as an induced subgraph.*

Proof. Let $\Gamma(G)$ be a clique reducible graph. By Theorem [6], $\Gamma(G)$ contains at least one of the Hajós' graph as an induced subgraph. The Hajós' graphs is an induced subgraph of $\Gamma(G)$ if and only if G contains one of the graphs in Figure 2 as an induced subgraph. Hence the theorem. ■

Note. The converse is not necessarily true.

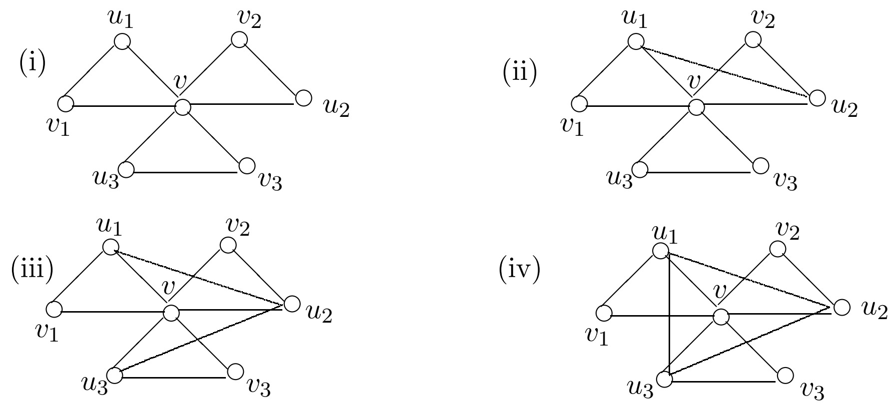


Figure 2

Let G be the graph in Figure 3. $V(G) = \{v, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5, w_7, w_7, w_8\}$. Let $\langle v, v_1, v_2, v_3, u_1, u_2, u_3 \rangle$ be the graph (i) in Figure 2 and let w_is for $i = 1, 2, \ldots, 8$ induce a complete graph. Also, let w_1 be adjacent to $\{v_1, v_2, v_3\}$, w_2 be adjacent to $\{v_1, v_2, u_3\}$, w_3 be adjacent to $\{v_1, u_2, v_3\}$, w_4 be adjacent to $\{v_1, u_2, u_3\}$, w_5 be adjacent to $\{u_1, v_2, v_3\}$, w_6 be adjacent to $\{u_1, v_2, u_3\}$, w_7 be adjacent to $\{u_1, u_2, v_3\}$, w_8 be adjacent to $\{u_1, u_2, u_3\}$ and v adjacent to w_i for $i = 1, 2, \ldots, 8$.

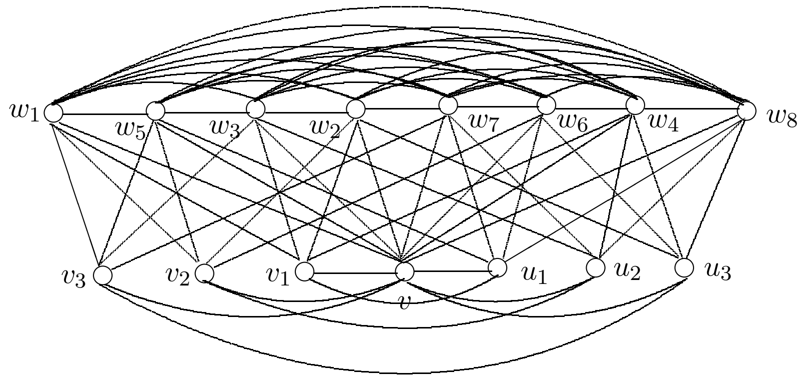


Figure 3

In $\Gamma(G)$ the vertices corresponding to the edges with one end vertex v induces K_6 minus a perfect matching in which the vertices of each of the eight triangles are adjacent to another vertex each. The remaining vertices induce the graph $H = 4K_{1,8}$. Therefore, $\Gamma(G)$ is clique irreducible.

4. The Iterations of the Anti-Gallai Graphs

Theorem 14. *The anti-Gallai graph $\Delta(G)$ is clique vertex irreducible if and only if G neither contains K_4 nor one of the Hajós' graphs as an induced subgraph.*

Proof. Let G be a graph which does neither contain K_4 nor one of the Hajós' graphs as an induced subgraph. The cliques of $\Delta(G)$ are induced by the vertices corresponding to the edges of G incident on a vertex of degree at least 3 whose other end vertices induce a complete graph and by the vertices corresponding to the edges which lie in a triangle. In the first case G contains an induced K_4, which is a contradiction. Therefore, the cliques of $\Delta(G)$ are induced by the edges which lie in a triangle. Let $\langle u_1, u_2, u_3 \rangle$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $\Delta(G)$ corresponding to the edges u_1u_2, u_2u_3, u_3u_1 in G. Then $\langle e_1, e_2, e_3 \rangle$ is a clique in $\Delta(G)$. If a vertex e_i for $i = 1, 2, 3$ lies in another clique of $\Delta(G)$, then the edge corresponding to e_i lies in another triangle. Therefore, the end vertices of the edge corresponding to e_i in G has a neighbor v_i for $i = 1, 2, 3$. $v_i \neq v_j$ if $i \neq j$ and v_1, v_2, v_3 are not adjacent to u_3, u_1, u_2, respectively, since otherwise G contains a K_4, which is a contradiction. Then, $\langle u_1, u_2, u_3, v_1, v_2, v_3 \rangle$ is one of the Hajós' graphs, a contradiction. Hence, G is clique vertex irreducible.

Conversely, assume that G is clique vertex irreducible. If G contains K_4 or one of the Hajós' graphs as an induced subgraph, then there exists a clique in $\Delta(G)$, corresponding to a triangle in G, which shares each of its vertices with some other clique of $\Delta(G)$. ■

Lemma 1. *If G is K_4-free then $\Gamma(G)$ is diamond free.*

Proof. Let G be a graph which does not contain K_4 as an induced subgraph. Therefore, a triangle in $\Delta(G)$ can only be induced by a triangle in G. If two vertices of the triangle in $\Delta(G)$ have a common neighbor, then it forces G to have a K_4, a contradiction. Therefore, $\Delta(G)$ is diamond free. ■

Theorem 15. *The second iterated anti-Gallai graph $\Delta^2(G)$ is clique vertex irreducible if and only if G does not contain K_4 as an induced subgraph.*

Proof. By Theorem 14, $\Delta^2(G)$ is clique vertex irreducible if and only if $\Delta(G)$ does neither contain K_4 nor one of the Hajós' graphs as an induced subgraph.

Let G be a graph which does not contain K_4 as an induced subgraph. Therefore, G does not contain K_5 as an induced subgraph and hence $\Delta(G)$ does not contain K_4 as an induced subgraph. Again, by Lemma 1, $\Delta(G)$ cannot have diamond as an induced subgraph and hence it does not contain any of the Hajós' graph as an induced subgraph. Hence, $\Delta^2(G)$ is clique vertex irreducible.

Conversely, assume that $\Delta^2(G)$ is clique vertex irreducible. If G contains K_4 as an induced subgraph then in $\Delta(G)$ the vertices corresponding to the edges of this K_4 induce K_6 minus a perfect matching which is the fourth Hajós' graph, a contradiction. Therefore, G does not contain K_4 as an induced subgraph. ■

Theorem 16. *The n^{th} iterated anti-Gallai graph $\Delta^n(G)$ is clique vertex irreducible if and only if G does not contain K_{n+2} as an induced subgraph.*

Proof. By Theorem 15, $\Delta^n(G)$ is clique vertex irreducible if and only if $\Delta^{n-2}(G)$ does not contain K_4 as an induced subgraph. $\Delta^{n-2}(G)$ does not contain K_4 as an induced subgraph if and only if $\Delta^{n-3}(G)$ does not contain K_5 as an induced subgraph. Proceeding like this, we get that $\Delta(G)$ does not contain K_{n+1} as an induced subgraph if and only if G does not contain K_{n+2} as an induced subgraph. Therefore, $\Delta^n(G)$ is clique vertex irreducible if and only if G does not contain K_{n+2} as an induced subgraph. ■

Theorem [3]**.** *If a graph G has no induced diamond, then every edge of G belongs to exactly one clique.*

Theorem 17. *The anti-Gallai graph $\Delta(G)$ is clique irreducible if and only if G does not contain K_4 as an induced subgraph.*

Proof. Let G be a graph which does not contain K_4 as an induced subgraph. By Lemma 1 and Theorem [3], $\Delta(G)$ is clique irreducible.

Conversely, if G contains a $K_4 = \langle u_1, u_2, u_3, u_4\rangle$, then it follows that the clique in $\Delta(G)$, corresponding to the triangle $\langle u_1, u_2, u_3\rangle$ in G, shares each of its edges with some other clique. Therefore, if $\Delta(G)$ is clique irreducible, then G cannot have K_4 as an induced subgraph. ■

Theorem 18. *The n^{th} iterated anti-Galli graph $\Delta^n(G)$ is clique irreducible if and only if G does not contain an induced K_{n+3}.*

Proof. By Theorem 17, $\Delta^n(G)$ is clique irreducible if and only if $\Delta^{n-1}(G)$ does not contain an induced K_4. $\Delta^{n-1}(G)$ does not contain an induced K_4 if and only if $\Delta^{n-2}(G)$ does not contain an induced K_5. Proceeding like this, we get, $\Delta(G)$ does not contain an induced K_{n+2} if and only if G does not contain an induced K_{n+3}. Therefore, $\Delta^n(G)$ is clique irreducible if and only if G does not contain an induced K_{n+3}. ■

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