STOCHASTIC MODELLING ANALYSIS AND APPLICATIONS

RETRIAL QUEUES WITH ORBITAL SEARCH

THESIS SUBMITTED TO THE
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
UNDER THE FACULTY OF SCIENCE

BY
VARGHESE C. JOSHUA

DEPARTMENT OF MATHEMATICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
KOCHI – 682 022, INDIA

FEBRUARY 2003

CERTIFICATE.

This is to certify that thesis entitled ***Retrial queues with orbital search*** *is a bonafide record of the research work carried out by Mr. Varghese C. Joshua under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.*

Kochi–682 022,
25 February 2003

Dr. A. Krishnamoorthy
(Supervising Guide)
Professor, Dept. of Mathematics
Cochin Univ. of Sci. & Technology
Kochi.

Contents

1 Introduction **1**

1.1 Description of the queueing problem 1

1.2 Notation 4

1.3 Description of retrial queues 4

1.4 Areas of application 5

1.5 Measures of effectiveness 6

1.6 Literature survey 6

1.6.1 Standard queues 6

1.6.2 Retrial queues 7

1.6.3 Methods for solving queueing models 8

1.7 Author's contribution 9

2 $M|G|1$ **retrial queue with orbital search** **13**

2.1 The mathematical model 14

2.2 The $M|M|1$ case 18

2.3 The $M|G|1$ case 23

2.4 System performance Measures 32

2.5 Numerical examples 33

3 $M|G|1$ **retrial queue with nonpersistent customers and orbital search** **36**

3.1 The mathematical model 37

3.2 The case $H_2 = 1$ 38
3.2.1 Stability condition 38
3.2.2 Analysis of the steady state probabilities 40
3.2.3 Analysis of the busy period 43
3.2.4 Calculation of the first and second moment of the busy period 47
3.3 The care $H_2 < 1$ 52
3.3.1 Limiting distribution of the system state 52
3.4 Numerical examples 55

4 $MAP|M|c$ retrial queue with orbital search 61
4.1 The mathematical model 63
4.2 Steady state analysis of the model at an arbitrary epoch 66
4.2.1 Steady-state probability vector using direct truncation . . 68
4.2.2 Steady-state probability vector using Neuts-Rao method . 69
4.3 Algorithmic analysis 71
4.3.1 Computation of the matrix R 71
4.3.2 Computation of the vector **x** 73
4.4 System performance measures 73
4.5 Numerical examples 75

5 $M|PH|1$ retrial queue with service interruption and orbital search 94
5.1 The mathematical model 95
5.2 Algorithmic solution 97
5.2.1 Stability condition 99
5.2.2 Computation of the vector x 100
5.3 Other system characteristics 104

6 Excursion between classical and retrial queue 107
6.1 Mathematical modelling 108

6.2 Control problem 112
Bibliography 120

Chapter 1.

Introduction

1.1 Description of the queueing problem

Queueing theory is a branch of applied probability theory which serves to study service system prone to congestion. A queueing system can be described as customers arriving for service, waiting for service if service is not immediate, and if having waited for service, leaving the system after being served. An accurate representation of a queueing system require a detailed characterization of the underlying processes. The following are the six basis characteristics of a queueing process.

Arrival pattern of customers

The arrival pattern to a queueing system is often measured in terms of the average number of arrivals per unit time (mean arrival rate) or by the average time between successive arrivals (mean inter-arrival time). If the arrival pattern is deterministic, then it is fully determined by either the mean arrival rate or the mean inter-arrival time. On the other hand, if it is probabilistic, then their mean values provide only measures of central tendency for the arrival pattern, and further characterisation is required in the form of the probability distribution associated with the random process.

Arrivals may occur in batches instead of one at a time. In the bulk-arrival situations, not only may the time between successive arrivals of the batches be probabilistic, but also the number of customers in a batch.

Another factor to be considered regarding the arrival pattern is the reaction of the customers in the queue. If the queue is too long, a customer may decide not to enter it upon arrival and in this situation he is said to have balked. On the other hand, a customer may enter the queue, but after some time lose patience and decide to leave. In this case, he is said to have reneged. In the event that there are more than one queue, customers may switch from one to another, that is jockey for position. These three situations are all examples of queues with impatient customers.

If the arrival pattern does not change with time, then it is called a stationary arrival pattern, otherwise, it is called non-stationary.

Service pattern of servers

Much of the discussions concerning the arrival pattern is appropriate in discussing service patterns. Service pattern can also be described by number of services per unit time (service rate) or by the time required to service customer (service time). Service may also be single or in batch, further it can be stationary or non-stationary with respect to time.

One important difference exists, however, between service and arrivals. The terms, service rate or service time are conditioned on the fact that the system is not empty; that is, if the system is empty, the server is idle. The servers that become idle may leave the system for a random period of time called vacation. These vacations may be utilised to perform additional work assigned to the servers.

The service rate may depend in the number of customers waiting for service.

In this situation, it is called state-dependent service. The problems of customer impatience can be looked upon as ones of state-dependent arrivals.

Queue discipline

Queue discipline refers to the rule by which customers are selected for service when a queue is formed. One of the most important and often practiced queue discipline is first in first out (FIFO). Some others in common usages are last in first out (LIFO). Another queue discipline is service in random order (RSS). In some cases, customers are given priorities up on entering the system, the ones with higher priorities to be selected for service ahead of those with lower priorities, regardless of their time of arrival to the system. There are two general situations in priority discipline. In the first, which is called preemptive, a customer with the highest priority is allowed to enter service immediately after suspending even the service in progress to a customer with lower priority. In the non-preemptive case, the highest priority customer goes to the head of the queue but cannot get into service until the customer presently in service is completed.

System capacity

In some queueing process, there is a finite upperbound to the queue size. In this situation, a customer is forced to balk if he arrives at a time when queue size is at its limit. This is a simple case of balking, since it is known exactly under what circumstances arriving customers must balk.

Service channels

The number of service channels refers to the number of parallel service stations which can provide identical service facilities to the customers simultaneously.

Stages of service

A service station may have several stages. That is, there may exist a series of service stages through which each customer must progress prior to leaving the system. They are called tandem queues.

1.2 Notation

A queueing process is described by the notation $A|B|X|Y|Z$ where A is the inter-arrival time distribution, B is the service time distribution, X the number of service channels, Y the system capacity and Z the queue discipline. It was introduced by Kendall (1953) and is widely used in books and papers.

1.3 Description of retrial queues

Queueing system in which arriving customers who find all servers and waiting positions (if any) occupied, may retry for service after a period of time, are called retrial queues or queues with repeated attempts. The most obvious example is provided by a person who desires to make a phone call. If the line is busy, then he can not queue up but tries again some time later. Thus, retrial queues are characterised by the following feature : a customer arriving when all servers accessible for him are busy, leaves the service area but after some random time repeats his demand. Retrial queues are a type of network with reservicing after blocking. Thus, this network contains two nodes: the main node where blocking is possible and a delay node for repeated attempts. As for other networks with blocking, the investigation of such systems presents great analytical difficulties. Nevertheless, the main feature of the theory of retrial queueing systems as an independent part of queueing theory are quite clearly drawn. In particular, the nature of results obtained, methods of analysis and areas of applications allow us to divide retrial queues into three large groups in a natural way: Single-channel systems, multi-channel fully available systems, and structurally complex systems.

The standard queueing models do not take into account the phenomenon of retrials and therefore cannot be applied in solving a number of practically important problems. Retrial queues have been introduced to solve this deficiency.

1.4 Areas of application

Queueing theory have been the subject of considerable research since the appearance of the first telephone systems. Telephone systems remained the principal application of the theory through about 1950. But the trend began to change immediately after the second world war and numerous other applications have been found and much work in the area. During this time investigations in another branch of applied probability, namely 'reliability' also began. The 'machine interference model which is a special case of queueing and reliability also developed around the same time. It was also discovered that models of the reliability of complex systems could be formulated in terms of queues (arrivals of breakdowns and repair services). These three areas of applied probability have much in common and can be handled by the same mathematical techniques and procedures. In the 60's the modelling of computer systems and data transmissions systems opened the way to studies of queues characterized by complex service disciplines and have created the need to analyze inter connected systems. Progress in this area has been rapid and so many industrial applications have been widely accepted since the 70's. The methods of queueing networks have always been a basic component of the study of communication systems. The widespread introduction of computers into these systems has introduced the use of new results on queueing networks in studies of the performance of large communication networks. Some of the other prominent applications of the queueing theory are landing of aircrafts, loading and unloading of ships, machine repair, taxi services and toll booths.

1.5 Measures of effectiveness

Queueing models can be classified into two general types — descriptive or prescriptive. Descriptive models, describe some current "real-world" situation. In this model for given types of arrival and service patterns and specified queue discipline and configuration, the state probabilities and expected-value measures of effectiveness which desirable the system are obtained. That is, to describe a queueing model we must relate quantities such as queue length, waiting time of a customer etc. :- to the known quantities such as arrival rate, retrial rate service rate etc:- On the other hand, the prescriptive model prescribe what the real-world situation should be, that is, the 'optimal' behaviour at which to aim. This effort is generally referred to under the title of design and control of queues. Generally, the controllable parameters are the service pattern, number of channels, and queue discipline, or some combination of these.

1.6 Literature survey

1.6.1 Standard queues

The first work on queueing theory was *The Theory of Probabilities and Telephone Conversations* by A. K. Erlang who published this paper in 1909. In his later works, he observed that a telephone system was generally characterized by either (1) Poisson input, exponential service time, and multiple channel, or (2) Poisson input, constant service time and a single channel. He was also responsible for the notion of stationary equilibrium, for the introduction of the so-called balance-of-state equations, and for the first consideration of the optimization of a queueing system. In 1927, Molina published his *Application of the Theory of Probability to Telephone Trunking Problems*. Thornton Fry's *Probability and Its Engineering Uses* (1928) expanded much of Erlang's earlier works. In the early 1930's some pioneering works were done by Felix Pollaczek, Kolmogorov, Kleinrock, Crommelin and Palm. For a comprehensive review of the main results and literature, refer Gross and Harris [38], Kleinrock [45], Saaty [66] and Hideki Takaji [39, 40, 41].

1.6.2 Retrial queues

One of the earliest papers in Retrial queues was *On the Influence of Repeated Calls in the Theory of Probabilities of Blocking* by Kosten [46]. In 1957, J. W. Cohen [26] published *Basic Problems of Telephone Traffic Theory and the Influence of Repeated Calls* in which he considered the more general $M|M|c$ retrial queue with impatient customers. He also obtained the necessary and sufficient condition for the ergodicity of retrial queues. But the approach was based on explicit solution of the Kolmogorov equations for the stationary distribution which leads to very cumbersome arguments. Shortly after this paper, the first criteria based on mean drifts were published. In 1968, Keilson, Cozzolino and Young published the first result on $M|G|1$ retrial queue using the method of supplementary variable. In the early 1970's Jonin and Sedol independently obtained explicit formulas for P_{0n} and P_{1n}, as well as the blocking probability and expected number in the orbit. In the late 70's, Falin [32, 33] considered the Markov chain embedded at service completion, the busy period and the functioning of the system in a non-stationary regime and the method for obtaining the distribution of the virtual waiting time. Methods of numerical calculations of the steady state distribution were developed by de Kok [47, 48]. Wilkinson [71] suggested the use of a truncated model for numerical solution of the Kolmogorov equations for the original model with unlimited orbit capacity. An approximation with the help of the model where the retrial rate equals infinity when the number of customers in orbit exceeds some level was suggested by Falin (1983). In 1990, Neuts and Rao [62] suggested an approximation with the help of the model where the retrial rate stays constant when the number of customers in orbit exceeds some level. Two extensive survey articles in retrial queues are due to Yang and Templeton [74] and Falin [36], covering, respectively, the developments upto mid 80's and late 80's. The only monograph on this topic is by Falin and Templeton [37] which provides an excellent scenario of retrial queues. For a systematic account of the results published in retrial queues, we refers to the bibliographical information in [5, 6].

1.6.3 Methods for solving queueing models

One of the methods used for solving Markovian queueing models is to build up a difference-differential equation. There are several methods for the solution of equation of this type. The simplest method, which is applicable only in very special cases is to solve recursively. Usually the best method to deal with such equations is to convert them, if possible into a single equation for a generating function. This method is discussed in detail by D. R. Cox and H. D. Miller [27]. Embedded Markov chain technique is commonly used when one among the service time and inter-arrival time is exponentially distributed while the other is not. This was introduced by Kendall [44]. Cox has analysed non-Markovian models by converting them into Markovian models through the introduction of one or more supplementary variables. A stable recursive scheme for the computation of the limiting probabilities can be developed based on a versatile regenerative approach, Tijms [69].

The investigation of many of the retrial queues is essentially more difficult than that of queueing models without retrials. Since the equilibrium distribution of the system state is expressed in terms of contour integrals or as limit of extended continued fractions, they are not convenient for practical applications. More useful is the implementation of 'computational probability'. By computational probability we mean the study of stochastic models with a genuine added concern for algorithmic feasibility over a wide, realistic range of parameter values. The matrix geometric methods comes under broader heading of computational probability. For a wide variety of stochastic models, the steady-state and occasionally the transient measures of the underlying process can be expressed in terms of a matrix R or G. That matrix is the minimal non-negative solution to a non-linear equation. Such matrix solutions to stochastic models were first proposed in the early 1970's by Marcel F. Neuts [59]. Marcel, with his students, and several other researchers have since then provided much impetus to the mathematical development of this method. By the introduction of the matrix geometric methods, the "Laplacian curtain" which covers the solution and hides the structural properties of many interesting queueing models is effectively lifted. This technique finds wide spread application, particularly, in telecommunication performance analysis. This is developed in the context of a two dimensional Markov process (X_t, Y_t) on the state space $\{0, 1, \ldots\} \times \{0, \ldots m\}$ with the property that the first co-ordinate of the process is skip-free upward. Calling by level i the set of states $\{i\} \times \{0, \ldots m\}$, and denoting the $(m+1) \times (m+1)$ submatrix of transient probabilities (infinitesimal rates) from i to j by $P(i, j)$, the process were also assumed to posses the spatial homogeneity property $P(i, i+1-j) = A_j$; $0 \leq j \leq i$, $i \geq 0$, in addition to the skip-free upward property for levels given by $P(i, j) = 0$; for $j > i + 1$.

1.7 Author's contribution

Retrial queues considered by researchers so far have the characteristic that each service is preceded and followed by an idle period which is terminated either by the arrival of a customer from the orbit (secondary customer) or by a primary (first attempt) customer. However, we consider retrial queueing models in which, even without a waiting room, each service completion epoch need not necessarily be followed by an idle time. This is achieved as follows: immediately on a service completion, the server picks up a customer from the orbit with probability p_j, when there are j customers in the orbit (it is assumed that server is aware of the orbital status, for example there is a register with him of customers in orbit, where as the orbital customers are ignorant of the server status.) With probability $1 - p_j$, no search is made on a service completion epoch and in this case, as in the classical retrial queue, a competition takes place between primary and secondary customers for service. Thus, if search is made, a service is followed by another service and if not, a service is followed by an idle time.

Our study has two main objectives. The first one is to introduce orbital search in retrial queueing models which allows to minimize the idle time of the server. If the holding costs and cost of implementing the search of customers are introduced, the results are obtained can be used for the optimal tuning of the parameters of the search mechanism. The second objective is to provide insight of the link between the corresponding retrial queue and the standard queue (without retrials). To this end, we observe that when $p_j = 1$, our model reduces to the corresponding classical queueing models (without retrials) and when $p_j = 0$, it becomes the corresponding retrial queueing model.

In chapter 2, we concentrate on the performance evaluation of a single server retrial queue with orbital search as follows: we consider a single server queueing system to which primary customers arrive according to a Poisson stream of rate λ. If the server is free at the arrival time of a primary customer, the arriving customer begins to be served immediately and leaves the system after service completion. Otherwise, if the server is busy, the arriving customer becomes a source of repeated calls. Every such source produces a Poisson process of repeated calls with intensity μ. The service times are independent with common probability function $B(x)$ $(B(0) = 0)$. Immediately after completing each service, the server goes for search of customers in the orbit with probability p_j and remains idle with probability $q_j = 1 - p_j$. The stability condition of the system is obtained. Limiting distribution of the system state is investigated. Explicit expressions of the limiting probabilities and their moments are obtained.

In chapter 3, a single server retrial queueing model with nonpersistent customers and orbital search is considered. If the server is busy at the time of arrival of a primary call then with probability $1 - H_1$ the call leaves the system without service and with probability $H_1 > 0$, forms a source of repeated calls. Similarly, if the server is occupied at time of arrival of a repeated call, with probability $1 - H_2$ the customer leaves the system without service and with probability H_2 goes back to the orbit. All other assumptions and notations introduced in chapter 2 hold in this chapter as well. An important feature of the model under consideration is that for many problems the cases $H_2 < 1$ and $H_2 = 1$ yield essentially different solutions. In the case $H_2 = 1$, the model is analysed in full detail using supplementary variable method. Stability condition is obtained. The joint distribution of the server state and the orbit length in steady state is studied. The structure of the busy period and its analysis in term of Laplace transformation have been discussed. This chapter also provides a direct method of calculation for the first and second moment of the busy period. The case $H_2 < 1$ is far more complicated and so closed form solution is obtained only in the case of exponential service time distribution.

In chapter 4, we consider a multiserver retrial queueing model (MAP$|M|c$) with search of customers from the orbit. The Markovian arrival process (MAP), a special class of tractable Markov renewal process, is a rich class of point processes that includes many well-known process such as Poisson, PH-renewal process, and Markov-modulated Poisson processes. One of the most significant features of the MAP is the underlying Markovian structure and fits ideally in the context of matrix -analytic solutions to stochastic models. The idea of the MAP is to significantly generalize the Poisson process and still keep the tractability for modeling purposes. In many practical applications, notably in communications engineering, production and manufacturing engineering, the arrivals do not usually form a renewal process. MAP is a convenient tool to model both renewal and non-renewal arrivals. The steady-state analysis of the model using direct truncation and Neuts-Rao truncation are performed. Efficient algorithms for computing various steady state performance measures and illustrative numerical examples are presented.

In chapter 5, we consider an $M|PH|1$ retrial queue with service interruptions and orbital search. In addition to the assumptions of the model described in chapter 2, the unit undergoing service is subject to interruptions. Service interruption occur according to a Poisson process with rate σ. Here the service times are assumed to be of phase type. PH-distributions and PH-renewal process were introduced by M. F. Neuts. The class of PH-distributions includes many wellknown distributions such as generalized Erlang, hyper exponential etc., as special cases and has a number of interesting closure properties. A detailed discussions of the properties of PH-distributions and their uses in stochastic modelling may be found in Neuts [59]. Efficient algorithm procedures for the steady-state analysis of the model are presented.

In chapter 6, we consider an excursion between classical and the retrial queue in the following way: For the present model as long as the number of customers in the orbit remains less than or equal to N, the server immediately on a service completion, picks up, the next customer from the orbit with probability 1 and starts service. When the orbit size reaches $N + 1$, no more search is made for customers until it comes on to N at a service completion epoch. That is, during the period of no search, customers from orbit have to make trials on their own. Hence the present model deals with a back and forth movement between classical queue and retrial queue. The motivation behind this model is that when orbit size increases, retrial rate also correspondingly increases thereby reducing the idle time of the server between services. By assigning costs to customer search and cost for switching to retrial and back to classical, a suitable cost function in N is constructed. Some numerical results are provided.

Chapter 2.

$M|G|1$ retrial queue with orbital search

In this chapter, we investigate a single server queue with linear retrial policy, where the server can go in search of customers immediately after each service completion.

The inter-retrial times can be modelled according to different disciplines depending on each particular application. In telephone systems the repeated attempts are made individually by each blocked customer following an exponential law of rate μ. This is the so-called *classical retrial policy* whose rate is $j\mu$, when the orbit size is $j \geq 0$. In contrast, there are other types of queueing situations in which the intervals separating successive repeated attempts are independent of the number of customers in orbit. This second possibility is the *constant retrial policy*, i.e. the retrial rate is $\alpha\ (1 - \delta_{j0})$ where δ_{j0} denotes Kronecker function. It can be motivated in the context of the $CSMA/CD$ (Carrier Sense Multiple Access with Collision Detection) protocol [25]. Artalejo and Gomez-Corral [8] unify both policies by defining the *linear retrial policy* with rate $\alpha\ (1 - \delta_{j0}) + j\mu$. The following application motivates the analysis of the model considered here.

Repair service with search of customers : The job-shop keeps a register of customers who are forced to leave the system since they encountered a busy server at the time of arrival. On completing a service the server decides to have the next service started immediately by picking up an unsatisfied (orbital) customer with probability p_j. The search time is assumed to be negligible. The probability for not going for the search of customers is $q_j = 1 - p_j$. If the server does not pick up the next customer to be served from the orbit then there is a competition between primary and orbital customers for getting into the counter for the next service.

Thus the present work includes classical queue when $p_j \equiv 1,\ j \neq 0$ and the classical retrial queue when $p_j \equiv 0$, as particular cases. It should be noted that the service time distribution, the number of customers in the orbit and the retrial policy in queueing terminology correspond to the repair time, blocked demands and customer's/server's search mechanism, respectively, in the above example.

This chapter is organized as follows: In section 2.1 we describe the mathematical model and study the stability condition. In section 2.2, we concentrate on the case of exponential service times to obtain explicit expressions of the limiting probabilities, factorial moments and various other performance characteristics in the steady state. $M|G|1$ case is analysed in section 2.3 using two different ways. We obtain the limiting probabilities using the supplementary-variable technique and also develop a stable recursive scheme for the computation of the limiting probabilities. In section 2.4, we list some system performance measures and in 2.5, the effect of p on these measures are analysed by illustrative numerical examples.

2.1 The mathematical model

We consider a single server queueing system to which primary customers arrive according to a Poisson stream of rate λ. Any customer who, upon arrival, finds the server busy immediately leaves the service area and joins the orbit. The interval between two successive repeated attempts is exponentially distributed with rate

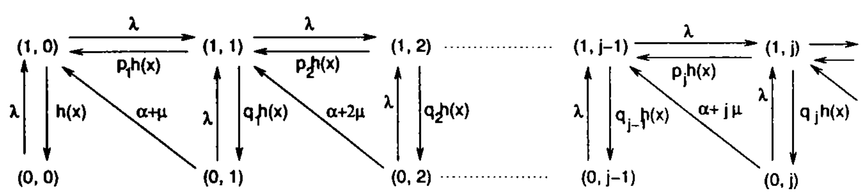


Figure 2.1: State space and transitions

$\alpha(1-\delta_{j0})+j\mu$, given that the number of customers in orbit is j. The service times are independent with distribution function $B(x)$ $(B(0)=0)$.

Let $\beta(s)=\int_0^\infty e^{-sx}dB(x)$ be the Laplace-Stieltjes transform of $B(x)$, $\beta_k=(-1)^k\beta^{(k)}(0)$ be the kth moment of the service time about the origin, $\rho=\lambda\beta$, the system load due to primary arrivals, $h(r)=\frac{B'(x)}{1-B(x)}$ be the instantaneous service intensity given that the elapsed service time is equal to x, $k(z)=\beta(\lambda-\lambda z)$. It can be shown that $k(z)=\sum_{n=0}^{\infty}k_n z^n$, where $k_n=\int_0^n \frac{(\lambda x)^n}{n!}e^{-\lambda x}dB(x)$. Let η_n be the time at which the nth service completion occurs. Immediately after this, the server goes for a search of customers in the orbit with a probability p_j $(p_0=0)$ which depends on the number of customers j in orbit. With probability $q_j=1-p_j$ the server remains free. In the latter case the event to follow depends on the competition between a primary arrival of rate λ and the flow of repeated attempts of rate $\alpha(1-\delta_{j0})+j\mu$. The search time is assumed to be negligible. The flow of primary arrivals, the intervals between repeated attempts, and service times are assumed to be mutually independent.

Let $N(t)$ be the number of customers in orbit and $C(t)$ be the state of the server at time t. We have $C(t)$ equal to 1 or 0 according to whether the server is busy or free. Note that the state space of the process $X(t)=(C(t),N(t))$ is $S=\{0,1\}\times\mathbb{N}$. The transitions among states are shown in Illustration 1 for the case of exponential service times with rate ν.

We now study necessary and sufficient conditions for the system to be stable. Observe that the sequence $N_n=N(\eta_n+)$ forms a Markov chain which is the embedded Markov renewal process of the continuous time Markov process

$(X(t), \xi(t))$, where $\xi(t)$ represents the elapsed service time of the customer being served ($\xi(t)$ is 0 if $C(t) = 0$).

According to the intuitive expectations the system approaches the standard $M/G/1$ queue when $\mu > 0$ and $N(t) = j$ is large, and when $\lim_{j\to\infty} p_j = 1$. In those cases, we expect that $\rho = \lambda\beta_1 < 1$ would be the stability condition. This is proved in the following.

Proposition 2.1. *Let us assume that* $\lim_{j\to\infty} p_j$ *exists. If* $\mu > 0$, *then* $\{N_n\}_{n=1}^{\infty}$ *is positive recurrent if and only if* $\rho < 1$.

Proof. Firstly, we observe that $\{N_n\}_{n=1}^{\infty}$ satisfies the state equation

$$N_n = N_{n-1} - B_n + V_n,$$

where V_n is the number of customers arriving during the nth service time and, $B_n = 1$ if the nth customer in service proceeds from the orbit and $B_n = 0$ otherwise.

Note that $\{N_n\}_{n=1}^{\infty}$ is irreducible and aperiodic so to investigate the positive recurrence we shall use Foster's criterion which states than an irreducible and aperiodic Markov chain is positive recurrent if there exist a non-negative function $f(j)$, $j \in \mathbb{N}$, and $\varepsilon > 0$ such that the mean drift $\varphi_j = E\left[f(N_{n+1}) - f(N_n) \mid N_n = j\right]$ is finite for all $j \in \mathbb{N}$, and $\varphi_j \leq -\varepsilon$ except perhaps for a finite number.

By choosing the test function $f(j) = j$, we obtain

$$\varphi_j = E\left[N_{n+1} - N_n \,|N_n = j\right] = E\left[-B_{n+1} + V_{n+1} \,|N_n = j\right]$$

$$= \begin{cases} \rho, & \text{if } j = 0, \\ \rho - \left(p_j + q_j \frac{\alpha + j\mu}{\lambda + \alpha + j\mu}\right), & \text{if } j \geq 1. \end{cases}$$

Then, we have that $\lim_{j\to\infty} \varphi_j = \rho - 1$. Thus, $\rho < 1$ is sufficient for the positive recurrence.

To study non-ergodicity we employ the Theorem 1 in Sennott et al. [67] which states that $\{N_n\}_{n=1}^{\infty}$ is non-ergodic if Kaplan's condition is fulfilled, $\varphi_j < \infty$, for all $j \in \mathbb{N}$, and there exists and index j_0 such that $\varphi_j \geq 0$, for $j \geq j_0$. If $\rho \geq 1$, it is obvious that $\varphi_j \geq 0$, for $j \geq 1$. Furthermore, Kaplan's condition is satisfied because there exists an index k such that $p_{ij} = 0$, for $j < i - k$, $i > 0$, where $P = (p_{ij})$ is the one-step transition matrix associated to $\{N_n\}_{n=1}^{\infty}$. This completes the proof. □

Proposition 2.2. *If* $\lim_{j\to\infty} p_j = 1$ *then* $\{N_n\}_{n=1}^{\infty}$ *is positive recurrent if and only if* $\rho < 1$.

Proof. We easily find that $\lim_{j\to\infty} \varphi_j = \rho - 1$ so Foster's criterion guarantees again that $\rho < 1$ is sufficient for the positive recurrence. The necessity follows from the argument given in Proposition 2.1. □

We also analyze the case $\mu = 0$ and $p_j = r$, for $j \geq 1$.

Proposition 2.3. *If* $\mu = 0$, $\alpha > 0$ *and* $p_j = r$, *for* $j \geq 1$, *then* $\{N_n\}_{n=1}^{\infty}$ *is positive recurrent if and only if* $\beta = \frac{\rho(\lambda+\alpha)}{r\lambda+\alpha} < 1$.

Proof. The mean drifts are given by

$$\varphi_j = \rho - \frac{r\lambda + \alpha}{\lambda + \alpha}, \; j \geq 1.$$

Then the proof follows the lines of the previous propositions. □

Finally, taking into account that the arrival input is a Poisson process and Burke's theorem [28], pp.187–188, it follows that the limiting probabilities

$$P_{ij} = \lim_{t\to\infty} P\left\{(C(t), N(t)) = (i,j)\right\}, \; (i,j) \in S,$$

exist and are positive under the same conditions of the embedded chain $\{N_n\}_{n=1}^{\infty}$.

2.2 The $M|M|1$ case

Through this section we consider $B(t) = 1-e^{-\nu t}, t > 0$, then the process $X(t)$ becomes an irreducible continuous time Markov chain. We assume that the positive recurrence conditions investigated in Section 2.1 are fulfilled. The consideration of exponential service times allows to express the main performance characteristics in terms of hypergeometric functions.

The set of statistical equilibrium equations for the probabilities P_{0j} and P_{1j} is

$$(\lambda + \alpha(1 - \delta_{j0}) + j\mu)P_{0j} = q_j \nu P_{1j},\ j \geq 0 \tag{2.1}$$

$$(\lambda + \nu)P_{1j} = \lambda P_{1,j-1} + \lambda P_{0j} + [\alpha + (j+1)\mu]P_{0,j+1} + \nu p_{j+1} P_{1,j+1},\ j \geq 0 \tag{2.2}$$

Using equation (2.1), eliminate the probabilities P_{1j} from the equation (2.2). After some algebra on the resulting equation we get:

$$\nu q_{j-1} q_j (\alpha + (j+1)\mu + \lambda p_{j+1}) P_{0,j+1} - \lambda q_{j-1} q_{j+1} (\lambda + \alpha + j\mu) P_{0j}$$
$$= \nu q_{j-1} q_{j+1} (\alpha + j\mu + \lambda p_j) P_{0j} - \lambda q_j q_{j+1} (\lambda + \alpha(1 - \delta_{j-1,0}) + (j-1)\mu) P_{0,j-1}$$

This implies that

$$\nu q_{j-1} q_{j+1} (\alpha + j\mu + \lambda p_j) P_{0j} - \lambda q_j q_{j+1} (\lambda + \alpha(1 - \delta_{j-1,0}) + (j-1)\mu) P_{0,j-1} = 0$$

Thus

$$P_{0j} = \frac{\lambda q_j (\lambda + \alpha(1 - \delta_{j-1,0}) + (j-1)\mu)}{v q_{j-1} (\alpha + j\mu + \lambda p_j)} P_{0,j-1}.$$

Solving recursively, we find that

$$P_{0j} = P_{00} q_j \rho^j \prod_{k=0}^{j-1} \frac{\lambda + \alpha(1 - \delta_{k0}) + k\mu}{p_{k+1}\lambda + \alpha + (k+1)\mu},\ j \geq 1, \tag{2.3}$$

$$P_{1j} = P_{00}\rho^{j+1} \prod_{k=1}^{j} \frac{\lambda + \alpha + k\mu}{p_k\lambda + \alpha + k\mu},\ j \geq 0, \tag{2.4}$$

$$P_{00}^{-1} = \sum_{j=0}^{\infty} \rho^{j+1} \left(1 + \frac{q_j\nu}{\lambda + \alpha(1 - \delta_{j0}) + j\mu}\right) \prod_{k=1}^{j} \frac{\lambda + \alpha + k\mu}{p_k\lambda + \alpha + k\mu}. \tag{2.5}$$

It seems impossible to express formulas (2.3)-(2.5) in terms of any known function, indeed in the case of *geometric search* (i.e. $p_j = 1 - p^j$, $p \in [0,1]$, $j \geq 1$). Hereafter we assume the case of *constant search* $p_j = p$, $p \in [0,1]$, $j \geq 1$, to get some nice closed-form expressions. First, we introduce some notation. Let F be the hypergeometric series given by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1, & \text{if } k = 0, \\ x(x+1)...(x+k-1), & \text{if } k \geq 1. \end{cases}$$

We also introduce the partial generating functions

$$P_i(z) = \sum_{j=0}^{\infty} z^j P_{ij},\ i \in \{0, 1\},\ |z| \leq 1,$$

and the partial factorial moments M_k^i defined by

$$M_0^i = \sum_{j=0}^{\infty} P_{ij},\ i \in \{0, 1\},$$

$$M_k^i = \sum_{j=k}^{\infty} j(j-1)...(j-k+1)P_{ij},\ i \in \{0,1\},\ k \geq 1.$$

Theorem 2.4. *Let us assume that $\{X(t);\ t \geq 0\}$ is positive recurrent, then*
(i) The limiting probabilities $\{P_{ij}\}_{(i,j)\in S}$ are given by

$$P_{0j} = P_{00}\frac{(1-p)\lambda}{\lambda+\alpha}\rho^j \frac{\left(\frac{\lambda+\alpha}{\mu}\right)_j}{\left(\frac{p\lambda+\alpha}{\mu}+1\right)_j},\ j \geq 1, \tag{2.6}$$

$$P_{1j} = P_{00}\rho^{j+1}\frac{\left(\frac{\lambda+\alpha}{\mu}+1\right)_j}{\left(\frac{p\lambda+\alpha}{\mu}+1\right)_j},\ j \geq 0, \tag{2.7}$$

$$P_{00}^{-1} = F\left(1, \frac{\lambda+\alpha}{\mu}+1; \frac{p\lambda+\alpha}{\mu}+1; \rho\right). \tag{2.8}$$

(ii) The partial generating functions $P_i(z)$, $0 \leq i \leq 1$, are given by

$$P_0(z) = P_{00}(1-\rho z)F\left(1, \frac{\lambda+\alpha}{\mu}+1; \frac{p\lambda+\alpha}{\mu}+1; \rho z\right), \tag{2.9}$$

$$P_1(z) = P_{00}\rho F\left(1, \frac{\lambda+\alpha}{\mu}+1; \frac{p\lambda+\alpha}{\mu}+1; \rho z\right). \tag{2.10}$$

(iii) The partial factorial moments M_k^i, $i \in \{0,1\}$, $k \geq 0$, are given by

$$M_0^0 = 1-\rho,$$

$$M_k^0 = P_{00}k!\frac{(1-p)\lambda}{\lambda+\alpha}\rho^k \frac{(\frac{\lambda+\alpha}{\mu})_k}{(\frac{p\lambda+\alpha}{\mu}+1)_k}F(k+1, \frac{\lambda+\alpha}{\mu}+k; \frac{p\lambda+\alpha}{\mu}+k+1; \rho),\ k \geq 1,$$

$$M_0^1 = \rho,$$

$$M_k^1 = P_{00}k!\rho^{k+1}\frac{(\frac{\lambda+\alpha}{\mu}+1)_k}{(\frac{p\lambda+\alpha}{\mu}+1)_k}F(k+1, \frac{\lambda+\alpha}{\mu}+k+1; \frac{p\lambda+\alpha}{\mu}+k+1; \rho),\ k \geq 1.$$

Proof. For the case $p_j = p$, $j \geq 1$, formulas (2.3)–(2.5) reduce to (2.6)–(2.8). By taking generating functions, we obtain (2.10) and

$$P_0(z) = \frac{P_{00}}{\lambda+\alpha}\left(p\lambda+\alpha+(1-p)\lambda F\left(1,\frac{\lambda+\alpha}{\mu};\frac{p\lambda+\alpha}{\mu}+1;\rho z\right)\right). \quad (2.11)$$

After some rearrangement (2.11) yields the alternative expression (2.9).

The key to computing the partial factorial moments is the following identity

$$P_i(1+z) = \sum_{k=0}^{\infty} M_k^i \frac{z^k}{k!}, \; i \in \{0,1\}.$$

After expanding $(1+z)^j$ as $\sum_{k=0}^{j}\binom{j}{k}z^k$, we can obtain M_k^i by a direct identification of the coefficients of the series $P_i(1+z)$. □

For the sake of completeness, we next give the expressions corresponding to the classical and constant retrial policies.

Corollory 2.5. (Classical retrial policy) *Let us consider that* $\alpha = 0$ *and* $\mu > 0$, *then*

(i) The limiting probabilities are given by

$$P_{0j} = P_{00}(1-p)\rho^j \frac{\left(\frac{\lambda}{\mu}\right)_j}{\left(\frac{p\lambda}{\mu}+1\right)_j}, \; j \geq 1,$$

$$P_{1j} = P_{00}\rho^{j+1}\frac{\left(\frac{\lambda}{\mu}+1\right)_j}{\left(\frac{p\lambda}{\mu}+1\right)_j}, \; j \geq 0,$$

$$P_{00}^{-1} = F\left(1,\frac{\lambda}{\mu}+1;\frac{p\lambda}{\mu}+1;\rho\right).$$

(ii) The partial generating functions are given by

$$P_0(z) = P_{00}(1-\rho z)F\left(1, \frac{\lambda}{\mu}+1; \frac{p\lambda}{\mu}+1; \rho z\right),$$

$$P_1(z) = P_{00}\rho F\left(1, \frac{\lambda}{\mu}+1; \frac{p\lambda}{\mu}+1; \rho z\right).$$

(iii) The partial factorial moments are given by

$$M_0^0 = 1-\rho,$$

$$M_k^0 = P_{00}k!(1-p)\rho^k \frac{\left(\frac{\lambda}{\mu}\right)_k}{\left(\frac{p\lambda}{\mu}+1\right)_k} F\left(k+1, \frac{\lambda}{\mu}+k; \frac{p\lambda}{\mu}+k+1; \rho\right), \ k \geq 1,$$

$$M_0^1 = \rho,$$

$$M_k^1 = P_{00}k!\rho^{k+1} \frac{\left(\frac{\lambda}{\mu}+1\right)_k}{\left(\frac{p\lambda}{\mu}+1\right)_k} F\left(k+1, \frac{\lambda}{\mu}+k+1; \frac{p\lambda}{\mu}+k+1; \rho\right), \ k \geq 1.$$

Corollory 2.6. (Constant retrial policy) *Let us consider that* $\alpha > 0$ *and* $\mu = 0$, *then*

(i) The limiting probabilities are given by

$$P_{0j} = P_{00}\frac{(1-p)\lambda}{\lambda+\alpha}\beta^j, \ j \geq 1,$$

$$P_{1j} = P_{00}\rho\beta^j, \ j \geq 0,$$

$$P_{00} = 1-\beta.$$

(ii) The partial generating functions are given by

$$P_0(z) = \frac{(1-\rho z)(1-\beta)}{1-\beta z},$$

$$P_1(z) = \frac{\rho(1-\beta)}{1-\beta z}.$$

(iii) The partial factorial moments are given by

$$M_0^0 = 1-\rho,$$

$$M_k^0 = k!\frac{(1-p)\lambda}{\lambda+\alpha}\left(\frac{\beta}{1-\beta}\right)^k, \ k \geq 1,$$

$$M_0^1 = \rho,$$

$$M_k^1 = k!\rho\left(\frac{\beta}{1-\beta}\right)^k, \ k \geq 1.$$

For the choices $p_j = 1, j \geq 1$, and $p_j = 0, j \geq 1$, we can deduce the performance characteristics of the standard $M|M|1$ queue and the $M|M|1$ retrial queue.

2.3 The $M|G|1$ case

Consider now the case of a general distribution function $B(x)$ of the service times. We analyze this case in two different ways.

(a) Supplementary variable method

For simplicity, in this method, we put $\alpha = 0$ (ie. we consider the classical retrial policy) and $p_j = p, p \in [0,1], j \geq 1$.

Theorem 2.7. *If $\rho < 1$ and the system is in the steady state, then the joint distribution of the server state and queue (orbit) length*

$$P_{0n} = P\{C(t) = 0, N(t) = n\}$$

$$P_{1n}(x) = \frac{d}{dx}P\{C(t) = 1, \xi(t) < x, N(t) = n\}$$

has partial generating functions.

$$\begin{aligned} P_0(z) &= \sum_{n=0}^{\infty} z^n P_{0n} \\ &= \frac{\lambda p P_{00}}{\mu}(z)^{\frac{-\lambda p}{\mu}} \int_0^z (t)^{\frac{\lambda p}{\mu}-1} s(z,t)dt \end{aligned} \tag{2.12}$$

$$\begin{aligned} P_1(z,x) &= \sum_{n=0}^{\infty} z^n P_{1n}(x) \\ &= \frac{\lambda(1-z)}{[\beta(\lambda-\lambda z)-z]} P_0(z)[1-B(x)]e^{-\lambda(1-z)x} \end{aligned} \tag{2.13}$$

where

$$s(t_1\ t_2) = \exp\{\frac{\lambda(1-p)}{\mu}\int_{t_1}^{t_2}[\frac{1-\beta(\lambda-\lambda u)}{u-\beta(\lambda-\lambda u)}]du\} \tag{2.14}$$

$$P_{00} = \frac{\mu(1-\lambda\beta_1)s(0,1)}{\lambda p \int_0^1 (t)^{\frac{\lambda p}{\mu}-1} s(0,t)dt} \tag{2.15}$$

If, in the case $C(t) = 1$, *we neglect the elapsed service time* $\xi(t)$, *then for the probabilities* $P_{1n} = P[C(t) = 1,\ N(t) = n]$,

$$\begin{aligned} P_1(z) &= \sum_{n=0}^{\infty} z^n P_{1n} \\ &= \frac{1-\beta(\lambda-\lambda z)}{\beta(\lambda-\lambda z)-z} P_0(z) \end{aligned} \tag{2.16}$$

Proof. The set of statistical equilibrium equations are obtained as :

$$\begin{aligned} (\lambda + n\mu)P_{0n} &= [1-(1-\delta_{n0})p]\int_0^{\infty} P_{1n}(x)h(x)dx \\ P'_{1n}(x) &= -(\lambda + h(x))P_{1n}(x) + \lambda P_{1,n-1}(x) \\ P_{1n}(0) &= \lambda P_{0n} + (n+1)\mu P_{0,n+1} + p\int_0^{\infty} P_{1,n+1}(x)h(x)dx \end{aligned}$$

For generating functions $P_0(z)$ and $P_1(z,x)$ these equations are transformed to:

$$\lambda P_0(z) + \mu z P_0'(Z) = (1-p)\int_0^\infty P_1(z,x)h(x)dx + \lambda p P_{00} \quad (2.17)$$

$$\frac{\partial P_1(z,x)}{\partial x} = -(\lambda - \lambda z + h(x))P_1(z,x) \quad (2.18)$$

$$(z - p\beta(\lambda - \lambda z))P_1(z,0) + \lambda p P_{00} = \mu z P_0'(z) + \lambda z P_0(z) \quad (2.19)$$

Solving (2.18) yields,

$$P_1(z,x) = P_1(z,0)(1 - B(x))e^{-\lambda(1-z)x} \quad (2.20)$$

Combining (2.17), (2.19) and (2.20) and after some algebra we get

$$\mu z(z - \beta(\lambda - \lambda z))P_0'(z) + (\lambda z - ((1-p)\lambda z + \lambda p)\beta(\lambda - \lambda z))P_0(z)$$
$$= \lambda p P_{00}(z - \beta(\lambda - \lambda z)) \quad (2.21)$$

Coefficient of $P_0'(z)$ has two zeros $z_1 = 0$ and $z_2 = 1$. Choose an arbitrary point $a \in (0,1)$. Solving (2.21) for $z \in (0,a]$ we get

$$P_0(z) = [(\frac{z}{a})^{\frac{\lambda p}{\mu}} s(a,z)]^{-1}\{P_0(a) + \frac{\lambda p P_{00}}{\mu(a)^{\frac{\lambda p}{\mu}}}\int_a^z (t)^{\frac{\lambda p}{\mu}-1} s(a,t)dt\}$$

As $z \to 0+$, $P_0(0) < \infty$ and $(\frac{z}{a})^{\frac{-\lambda p}{\mu}}$ diverges. Thus,

$$P_0(a) = \frac{\lambda p P_{00}}{\mu(a)^{\frac{\lambda p}{\mu}}}\int_0^a (t)^{\frac{\lambda p}{\mu}-1} s(a,t)dt \quad (2.22)$$

On the other hand, solving (2.21) for $z \in [a,1)$, and taking limit as $z \to 1-$, we get,

$$P_0(a) = \frac{P_0(1)s(a,1)}{(a)^{\frac{\lambda p}{\mu}}} - \frac{\lambda p P_{00}}{\mu(a)^{\frac{\lambda p}{\mu}}}\int_a^1 (t)^{\frac{\lambda p}{\mu}-1} s(a,t)dt \quad (2.23)$$

For obtaining relation (2.23), it should be noted that

$$\lim_{z\to 1-}[\frac{1-\beta(\lambda-\lambda z)}{\beta(\lambda-\lambda z)-z}]=\frac{\lambda\beta_1}{1-\lambda\beta_1}<\infty$$

Equating (2.22) and (2.23) we get

$$P_0(1)=\frac{\lambda p P_{00}}{\mu s(0,1)}\int_0^1 (t)^{\frac{\lambda p}{\mu}-1}s(0,t)dt \tag{2.24}$$

Then we can rewrite the solution of (2.21) as (2.12).

Combining the equation (2.19), (2.20) and (2.21) we get (2.13).
Since $P_1(z)=\int_0^\infty P_1(z,x)dx$, we obtain (2.16).

Now applying the normalizing condition $P_0(1)+P_1(1)=1$, we get

$$P_0(1)=1-\lambda\beta_1=1-\rho \tag{2.25}$$

Using (2.24) and (2.25), we obtain the expression for P_{00} as in (2.15). □

Corollory 2.8. *The partial factorial moments M_k^i, $i\in\{0,1\}$ $k\in\{0,1\}$ are given by*

$$\begin{aligned}
M_0^0 &= 1-\rho\\
M_0^1 &= \rho\\
M_1^0 &= \frac{\lambda p P_{00}}{\mu}+\frac{\lambda(\rho-p)}{\mu}\\
M_1^1 &= \frac{\lambda^2}{(1-\rho)\mu}\{\mu\beta_2+\lambda\beta_1^2-p\beta_1(1-P_{00})\}
\end{aligned}$$

(b) An algorithmic solution.

Our next objective is to develop a stable recursive scheme for the computation of the limiting probabilities P_{ij}. The derivation is based on a versatile regenerative approach [69], pp. 266–268, which was also useful to compute the limiting distribution of other retrial queues [11, 12, 48].

We can assume a more general model description where the arrival rate λ_{ij} depends on the system state and the retrial rate γ_j depends on the number of customers in orbit. Let a regeneration cycle be the elapsed time between two consecutive visits of the process $X(t)$ to the state $(0,0)$. We define some random variables:

T = the length of a regeneration cycle,

T_{ij} = the amount of time in a cycle during which $X(t) = (i,j)$, $(i,j) \in E$,

N_{ij} = the number of service completions in a cycle leaving the system at the state (i,j), $(i,j) \in E$.

From the theory of regenerative processes, we can express the limiting probabilities as

$$P_{ij} = \frac{E[T_{ij}]}{E[T]}, \ (i,j) \in E, \tag{2.26}$$

where $E = \{0,1\} \times \{0,\ldots,K\}$ and K denotes the orbit capacity.

We now consider the balance equations

$$(\lambda_{0j} + \gamma_j) E[T_{0j}] = E[N_{0j}], \ 0 \le j \le K, \tag{2.27}$$

$$\gamma_j E[T_{0j}] + E[N_{1,j-1}] = \lambda_{1,j-1} E[T_{1,j-1}], \ 1 \le j \le K. \tag{2.28}$$

Equations (2.27) and (2.28) can be obtained by equating the flow rate into and the flow rate out of $(0,j)$ and $\{(i,k) \mid i \in \{0,1\}, 0 \le k \le j-1\}$, respectively.

By combining (2.27) and (2.28), we get

$$\lambda_{0j} E[T_{0j}] + \lambda_{1,j-1} E[T_{1,j-1}] = E[N_{0j}] + E[N_{1,j-1}], \ 1 \le j \le K. \tag{2.29}$$

Now an appeal to Wald's theorem yields

$$E[T_{1j}] = \sum_{k=0}^{K} E[N_k] (q_k A_{kj} + p_k B_{kj}), \; 0 \le j \le K, \qquad (2.30)$$

where $N_k = N_{0k} + (1 - \delta_{k0}) N_{1,k-1}$, and the auxiliary quantities A_{kj} and B_{kj} are defined as

A_{kj} = the expected amount of time that during a service time j customers are in orbit given that at the previous service completion the server did not search customers in the orbit and the system state was $(0, k)$,

B_{kj} = the expected amount of time that during a service time j customers are in orbit given that at the previous service completion the server went for a search of a customer in the orbit, so the system state was $(1, k-1)$.

From (2.29) and (2.30), we find that

$$\begin{aligned} E[T_{1j}] = A_{0j} + &\sum_{k=1}^{\min(j+1,K)} q_k A_{kj} (\lambda_{0k} E[T_{0k}] + \lambda_{1,k-1} E[T_{1,k-1}]) \\ &+ \sum_{k=1}^{j+1} p_k B_{kj} (\lambda_{0k} E[T_{0k}] + \lambda_{1,k-1} E[T_{1,k-1}]), \; 0 \le j \le K. \qquad (2.31) \end{aligned}$$

We now observe that N_{0j} and $N_{1,j-1}$ are Bernoulli trials with success probability q_j and p_j, respectively. Thus, we have

$$\begin{aligned} E[N_{0j}] &= q_j E[N_j], \; 0 \le j \le K, \\ E[N_{1j}] &= p_j E[N_j], \; 1 \le j \le K. \qquad (2.32) \end{aligned}$$

By combining (2.27), (2.29) and (2.32), we obtain

$$(p_j \lambda_{0j} + \gamma_j) E[T_{0j}] = (1 - \delta_{j0}) q_j \lambda_{1,j-1} E[T_{1,j-1}], \; 0 \le j \le K. \qquad (2.33)$$

By inserting (2.33) in (2.31) we find that

$$E[T_{1j}] = A_{0j} + \sum_{k=1}^{\min(j+1,K)} \frac{\lambda_{1,k-1}(\lambda_{0k}+\gamma_k)}{p_k\lambda_{0k}+\gamma_k} q_k A_{kj} E[T_{1,k-1}] + \sum_{k=1}^{j+1} \frac{\lambda_{1,k-1}(\lambda_{0k}+\gamma_k)}{p_k\lambda_{0k}+\gamma_k} p_k B_{kj} E[T_{1,k-1}],\ 0 \le j \le K. \quad (2.34)$$

Dividing both sides of (2.33) and (2.34) by $E[T]$ and using (2.26) and the fact that $E[T] = (\lambda_{00}P_{00})^{-1}$, we get

$$P_{0j} = \frac{q_j\lambda_{1,j-1}}{p_j\lambda_{0j}+\gamma_j} P_{1,j-1},\ 1 \le j \le K, \quad (2.35)$$

$$P_{1j} = \lambda_{00}A_{0j}P_{00} + \sum_{k=1}^{\min(j+1,K)} \frac{\lambda_{1,k-1}(\lambda_{0k}+\gamma_k)}{p_k\lambda_{0k}+\gamma_k} q_k A_{kj} P_{1,k-1} + \sum_{k=1}^{j+1} \frac{\lambda_{1,k-1}(\lambda_{0k}+\gamma_k)}{p_k\lambda_{0k}+\gamma_k} p_k B_{kj} P_{1,k-1},\ 0 \le j \le K. \quad (2.36)$$

The above formulas (2.35) and (2.36) provide a stable recursive scheme for computing $\{P_{0j}\}_{j=1}^{\infty}$ and $\{P_{1j}\}_{j=0}^{\infty}$ in terms of P_{00}. Letting $\lim_{j\to\infty} p_j = 0$ and $\gamma_j = j\mu$ in (2.35) and (2.36), we get the formulas given by De Kok [48] for the $M/G/1$ queue with classical retrial policy.

It remains to specify how to determine the quantities A_{kj} and B_{kj}. This can be done with the help of a third auxiliary quantity C_{kj} and the following relationships

$$A_{kj} = \frac{\gamma_k}{\lambda_{0k}+\gamma_k} C_{k-1,j} + \frac{\lambda_{0k}}{\lambda_{0k}+\gamma_k} C_{kj},\ 0 \le k \le j \le K,$$

$$A_{j+1,j} = \frac{\gamma_{j+1}}{\lambda_{0,j+1}+\gamma_{j+1}} C_{jj},\ 0 \le j \le K-1,$$

$$B_{kj} = C_{k-1,j},\ 1 \le k \le K,\ k-1 \le j \le K,$$

where C_{kj}, $0 \leq k \leq j \leq K$, is defined as

C_{kj} = the expected amount of time that during a service time j customers are in orbit given that immediately after the beginning of the service k customers were in orbit.

Then, if $\lambda_{ij} = \lambda$, $(i,j) \in E$, we have

$$C_{kj} = \begin{cases} \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{j-k}}{(j-k)!}(1 - B(t))dt, & \text{for } K = \infty \text{ or } j < K < \infty, \\ \int_0^\infty e^{-\lambda t}(1 - B(t)) \sum\limits_{n=K-k}^{\infty} \frac{(\lambda t)^n}{n!} dt, & \text{for } j = K < \infty. \end{cases} \tag{2.37}$$

Let us verify the validity of C_{kj} in the case that $K = \infty$ or $j < K < \infty$. Note that the infinitesimal interval $(t, t + \Delta t)$ contributes to C_{kj} if the service time has not expired before time t (with probability $1 - B(t)$) and $j - k$ primary customers arrive in $(0, t]$ (with probability $e^{-\lambda t}\,(\lambda t)^{j-k}/(j-k)!$). The case $j = K < \infty$ follows a similar argument noting that there must be at least $K - k$ arrivals in the interval $(0, t]$.

The integrals in (2.37) can be reduced to a finite sum for many practical service distributions. For example, if $B(t) = 1 - e^{-\nu t}$, $t > 0$, then we find that

$$C_{kj} = \begin{cases} \frac{1}{\lambda+\nu}\left(\frac{\lambda}{\lambda+\nu}\right)^{j-k}, & \text{for } K = \infty \text{ or } j < K < \infty, \\ \frac{1}{\nu}\left(\frac{\lambda}{\lambda+\nu}\right)^{K-k}, & \text{for } j = K < \infty. \end{cases}$$

Finally, we observe that the probability P_{00} remains to be specified. A first possibility is to assume $K = \infty$ and to determine P_{00} with the help of the partial generating function $P_0(z) = \sum_{j=0}^{\infty} z^j P_{0j}$ by setting $P_0(0) = P_{00}$. On the other hand, P_{00} can be approximated by using the normalizing condition $\sum_{(i,j)\in E} P_{ij} = 1$. This second possibility implies in practice the assumption $K < \infty$.

Note that, the closed form solution obtained in the $M|M|1$ case can be deduced from $M|G|1$ case using the above method. If suffices to consider $B(t) = 1 - e^{\nu t}$, $t > 0$. Let us consider again the original model described in Section 2, i.e.

$K = \infty$, $\lambda_{ij} = \lambda$ and $\gamma_j = \alpha(1 - \delta_{j0}) + j\mu$. After some algebraic manipulations, formulas (2.35) and (2.36) reduce to the following

$$P_{0j} = \frac{q_j \lambda}{p_j \lambda + \alpha + j\mu} P_{1,j-1},\ j \geq 1, \tag{2.38}$$

$$P_{1j} = \lambda \sum_{k=0}^{j} a_{j-k} \left(P_{0k} + P_{1k} \right),\ j \geq 0, \tag{2.39}$$

where

$$a_k = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} (1 - B(t)) dt = \frac{1}{\lambda + \nu} \left(\frac{\lambda}{\lambda + \nu} \right)^k,\ k \geq 0.$$

From (2.39) we obtain by induction that

$$\lambda \left(P_{0j} + (1 - \delta_{j0}) P_{1,j-1} \right) = \nu P_{1j},\ j \geq 0. \tag{2.40}$$

Using (2.38) and (2.40) we find that

$$\left(\lambda + \alpha (1 - \delta_{j0}) + j\mu \right) P_{0j} = q_j \nu P_{1j},\ j \geq 0. \tag{2.41}$$

Now, combining (2.40) and (2.41) we get

$$\lambda P_{1j} = p_{j+1} \nu P_{1,j+1} + \left(\alpha + (j+1)\mu \right) P_{0,j+1},\ j \geq 0. \tag{2.42}$$

Equations (2.41) and (2.42) play the same role than (2.27) and (2.28) and, consequently, they can be viewed as balance equations that equate the flow rate into and the flow rate out of $(0, j)$ and the jth orbit level, respectively.

Solving recursively (2.41) and (2.42) we get (2.3) and (2.4).

2.4 System performance Measures

1. The probability mass function of the server state.

$$Pr[\text{ server is idle }] = M_0^0 = 1 - \rho$$
$$Pr[\text{ server is busy }] = M_0^1 = \rho$$

   In particular, the blocking probability (ie, the probability that an arriving customer is blocked)= $M_0' = \rho$.

2. The probability mass function of the number of customers in the orbit

   Pr [there are i customers in the orbit] = $P_{0i} + P_{1i}$

3. Expected number of customers in the orbit

   The mean EN number of customers in the orbit = $M_1^0 + M_1^1$

4. The busy period

   The expected busy period EL (ie, the period that starts at the epoch when an arriving customer finds an empty system and ends at the departure epoch at which the system is empty again),

$$= \frac{1}{\lambda}(\frac{1 - P_{00}}{P_{00}})$$

   The above formula can be obtained by the theory of regenerative process which allow us to express the limiting probabilities $P_{ij} = \frac{E[T_{ij}]}{\lambda^{-1}+E[L]}$.

5. The overall rate of retrials.

   The overall rate μ_1^* of trials at which the orbiting customers request service is given by

$$\mu_1^* = \mu \sum_{i=1}^{\infty} i(P_{0i} + P_{1i}) = \mu EN$$

6. The rate at which the orbiting customers successfully reach the server is given by

$$\mu_2^* = \mu \sum_{i=1}^{\infty} iP_{0i} = \mu M_1^0$$

7. The fraction of successful rate of retrials is given by

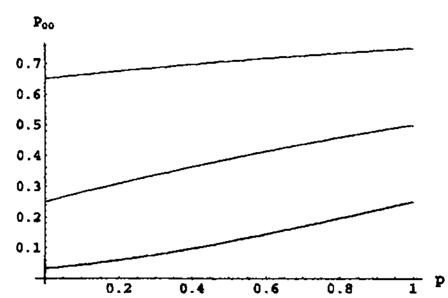
$$\frac{\mu_2^*}{\mu_1^*} = \frac{M_1^0}{EN}$$

2.5 Numerical examples

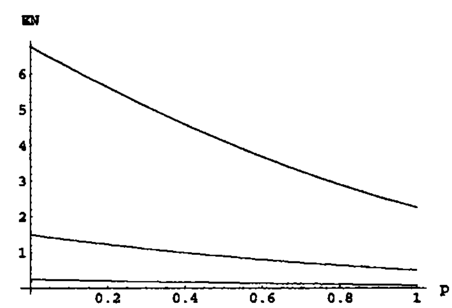
In this section, we present some illustrative numerical examples that qualitatively describe the queueing model under study. These examples are generated using MATHEMATICA [72].

In section 1, we mentioned that the model with constant search can be viewed as a versatile mechanism which allows us to consider simultaneously both the standard and the retrial queue. To illustrate this fact, we now display the probability of an empty system P_{00}, the mean number of customers in the orbit EN, and the expected busy period EL as a function of the recovery probability p. We consider the case $\alpha = 0$, $\mu = 0.5$ (classical retrial policy). Without loss of generality we normalize the service rate to be $\nu = 1$ so that $\rho = \lambda = 0.25, 0.5, 0.75$. Fig 2.2(a) shows that P_{00} increases from its values for the $M|M|1$ retrial queues ($p = 0$) to the corresponding one for the standard $M|M|1$ queues ($p = 1$). In, Fig 2.2(b) and 2.2 (c), it is noted that EN and EL decrease from thier values for the $M|M|1$ retrial queue to standard $M|M|1$ queue. In all the figures we observe that the differences between the standard and the retrial queue become more apparent for increasing values of ρ.

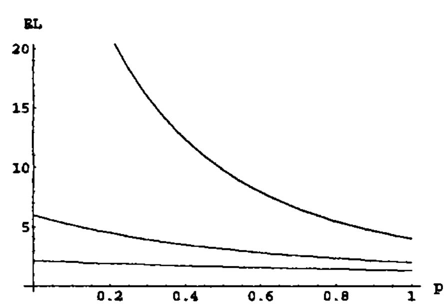
The second set of figures 2.3(a), (b), and (c) display the combined effect of p and ρ on P_{00}, EN, and EL respectively.



(a) P_{00} versus p.

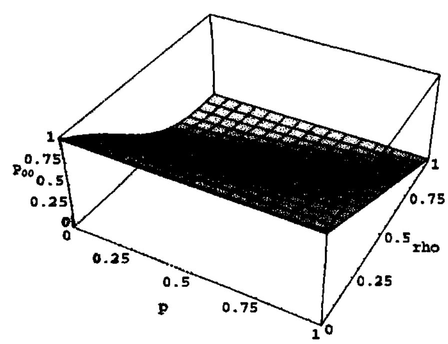


(b) EN versus p.

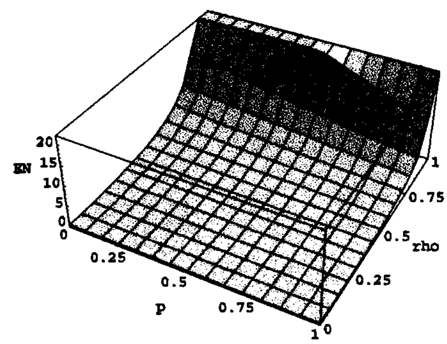


(c) EL versus p.

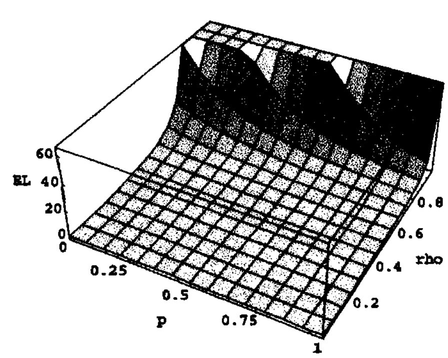
Figure 2.2: Red $\rho = 0.25$, Green $\rho = 0.5$, Blue $\rho = 0.75$.
$M|M|1$; $\alpha = 0$, $\mu = 0.5$, $\nu = 1$



(a) P_{00} versus (p, ρ).



(b) EN versus (p, ρ).



(c) EL versus (p, ρ).

Figure 2.3: $M|M|1$; $\alpha = 0$, $\mu = 0.5$, $\nu = 1$

Chapter 3.

$M|G|1$ retrial queue with nonpersistent customers and orbital search

In this chapter, we extend the search mechanism to a structurally complex single server retrial queueing model with nonpersistent customers. We consider a retrial queueing model with orbital search in which a calling subscriber after some unsuccessful retrials gives up further repetitions and abandons the system. Let H_i be the probability that after the ith attempt fails, a customer will make the $(i + 1)$th one. The set of probabilities $\{H_i : i \geq 1\}$ is called the persistence function. We assume that the probability of a call reinitiating after failure of a repeated attempt, does not depend on the number of previous attempts (ie, $H_2 = H_3 = \cdots$). Statistical measurements in telephone networks show that this is a realistic assumption in application to such networks. One of the important features of the model under consideration is that, for many problems, the cases $H_2 < 1$ and $H_2 = 1$ yield essentially different solutions. The case $H_2 = 1$ can be analysed in full detail while the case $H_2 < 1$ is far more complicated and closed form solution is available only in the case of exponential service time distribution.

This chapter is organised as follows. In section 3.1, we describe the mathematical model. For the case $H_2 = 1$, the model is analysed in full detail in section 3.2. In section 3.2.1, stability condition is derived and in 3.2.2,the limiting distribution of the system state is obtained based on the supplementary-variable technique. The structure of the busy period and its analysis in terms of Laplace transforms have been discussed in 3.2.3. In section 3.2.4, we provide a direct method of calculation for the first and second moments of the busy period. In section 3.3, the case $H_2 < 1$ is considered and the closed form solution is obtained for the exponential service time distribution in terms of hypergeometric series. In section 3.4, we present some numerical examples to illustrate the effect of the parameters on the system performance

3.1 The mathematical model

We consider a single server queueing system to which primary customer arrive according to a Poisson stream of rate λ. If the server is busy at the time of arrival of a primary customer, then with probability $1-H_1$ the customer leaves the system without service and with probability $H_1 > 0$ forms a source of repeated calls. Every such source produces a Poisson process of repeated calls with rate $j\mu$, when there are j customers in the orbit. If the repeated call finds the server free, it is served and leaves the system permanently. Otherwise, ie, if the server is occupied at the time of arrival of a repeated call, with probability $1 - H_2$, the source leaves the system without service and with probability H_2, it goes back to the orbit and retries for service. The service times are independent with common probability function $B(x)$ ($B(0) = 0$). Let $\beta(s) = \int_0^\infty e^{-sx}dB(x)$ be the Laplace-Stieltjes transform of the service time distribution $B(x)$, $\beta_k = (-1)^k\beta^{(k)}(0)$ be the kth moment of the service time about the origin, $h(x) = \frac{B'(x)}{1-B(x)}$ be the instantaneous service intensity given that the elapsed service time is equal to x. Let η_n be the time at which the nth service completion occurs. Immediately after this service completion, the server goes for search of customers in the orbit with some known probability p. With probability $1-p$ the server remains idle. In the latter case, the event to follow depends on the competition between a primary arrival and retrial attempt. The search time is assumed to be negligible. The flow of primary arrivals, retrial of customers and service times are assumed to be mutually independent.

At time t, let $N(t)$ be the number of customers in the orbit and $C(t)$ be the state of the server. ($C(t) = 1$ or 0 according as the server is busy or free). The state space of the process $X(t) = (C(t), N(t))$ is $S = \{0, 1\} \times \mathbb{N}$. The transitions among states are illustrated in Figure 3.1

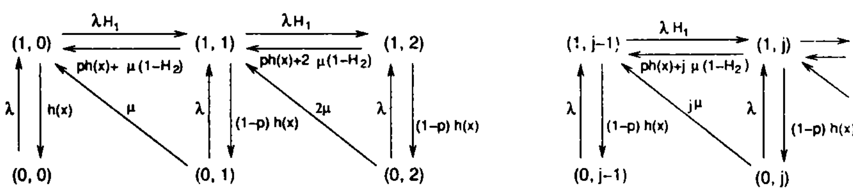


Figure 3.1:

3.2 The case $H_2 = 1$

3.2.1 Stability condition

We now study the necessary and sufficient condition for the system to be stable. Let $N_i = N(\eta_i)$ be the number of customers in the orbit at the time η_i of the ith service completion.

Theorem 3.1. *$\{N_i,\ i \in \mathbb{N}\}$ is positive recurrent if and only if $\rho H_1 < 1$, where $\rho = \lambda\beta_1$.*

Proof. $\{N_i,\ i \in \mathbb{N}\}$ satisfies the equation

$$N_i = N_{i-1} - B_i + V_i$$

where V_i is the number of customers arriving during the ith service time, with

$$B_i = \begin{cases} 1 & \text{if the } i\text{th customer served proceeds from the orbit} \\ 0 & \text{otherwise} \end{cases}$$

The conditional distribution of the Bernoulli random variable B_i is given by

$$P\{B_i = 1/N_{i-1} = n\} = (1-p)\frac{n\mu}{(\lambda + n\mu)} + p \quad n \neq 0,$$
$$P\{B_i = 0/N_{i-1} = n\} = (1-p)\frac{\lambda}{(\lambda + n\mu)}$$

The random variable V_i has distribution

$$k_n = P\{V_i = n\} = \int_0^\infty \frac{(\lambda H_1 x)^n}{n!} e^{-\lambda H_1 x} dB(x)$$

with generating function

$$\sum_{n=0}^{\infty} k_n z^n = \beta(\lambda H_1 - \lambda H_1 z) \equiv k(z)$$

and mean value

$$EV_i = \sum_{n=0}^{\infty} n k_n = \lambda H_1 \beta_1 = \rho H_1$$

Thus the sequence of random variables N_i forms a Markov chain. It is not difficult to see that $\{N_i;\ i \in \mathbb{N}\}$ is irreducible and aperiodic. To prove ergodicity, we shall use Foster's criterion, which states that an irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(s)$, $s \in \mathbb{N}$, and $\epsilon > 0$ such that the mean drift.

$$\phi_s = E[f(N_{i+1}) - f(N_i)/N_i = s]$$

is finite for all $s \in \mathbb{N}$ and $\phi_s \leq -\epsilon$ for all $s \in \mathbb{N}$, except perhaps a finite number. In our case, we take the function $f(s) = s$. We then obtain

$$\phi_s = \begin{cases} \rho H_1; & \text{if } s = 0 \\ \frac{-(1-p)s\mu}{\lambda+s\mu} - p + \rho H_1; & \text{if } s \geq 1 \end{cases}$$

Clearly, if $\rho H_1 < 1$, then we have $\lim_{s\to\infty} \phi_s < 0$. Therefore, the embedded Markov chain $\{N_i,\ i \in \mathbb{N}\}$ is ergodic.

To prove the necessity of the condition, we use theorem 3.1 in Sennott et. al. [67], which states that $\{N_i,\ i \in \mathbb{N}\}$ is non-ergodic if it satisfies the Kaplan's condition, $\phi_j < +\infty$ $(j \geq 0)$ and there is a j_0 such that $\phi_j \geq 0$, $(j \geq j_0)$. Further more, Kaplan's condition is satisfied because there exists an index k such that $P_{ij} = 0$ for $j < i - k$; $i > 0$; where $P = \{P_{ij}\}_{i,j\in\mathbb{N}}$ is the one-step transition matrix associated with $\{N_i;\ i \in \mathbb{N}\}$. This completes the proof. □

3.2.2 Analysis of the steady state probabilities

In this section we study the steady state distribution of our queueing system.

Theorem 3.2. *If $\rho H_1 = \lambda H_1 \beta_1 < 1$ and the system is in the steady state, then the joint distribution of the server state and queue length*

$$P_{0n} = P\{C(t) = 0,\ N(t) = n\}$$
$$P_{1n}(x) = \frac{d}{dx} P\{C(t) = 1, \xi(t) < x,\ N(t) = n\}$$

has partial generating functions

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{0n} = \frac{\lambda p P_{00}}{\mu} (z)^{\frac{-\lambda p}{\mu}} \int_0^z t^{\frac{\lambda p}{\mu}-1} r(z,t)dt \tag{3.1}$$

$$P_1(z,x) = \sum_{n=0}^{\infty} z^n P_{1n}(x) = \frac{\lambda(1-z)}{[\beta(\lambda H_1 - \lambda H_1 z) - z]} P_0(z)(1 - B(x))e^{-\lambda H_1(1-z)x} \tag{3.2}$$

where

$$r(t_1,t_2) = \exp\{\frac{\lambda(1-p)}{\mu}\int_{t_1}^{t_2}\frac{1-\beta(\lambda H_1-\lambda H_1 u)}{u-\beta(\lambda H_1-\lambda H_1 u)}du\} \tag{3.3}$$

$$P_{00} = \frac{\mu(1-\lambda H_1\beta_1)r(0,1)}{\lambda p(1+\lambda\beta_1-\lambda H_1\beta_1)\int_0^1 (t)^{\frac{\lambda p}{\mu}-1} r(0,t)dt} \tag{3.4}$$

If in the case $C(t) = 1$ *we neglect the elapsed service time* $\xi(t)$, *then for the probabilities* $P_{1n} = P\{C(t)=1,\ N(t)=n\}$,

$$P_1(z) = \sum_{n=0}^{\infty} z^n P_{1n} = \frac{1-\beta(\lambda H_1-\lambda H_1 z)}{H_1(\beta(\lambda H_1-\lambda H_1 z)-z)}P_0(z) \tag{3.5}$$

Proof. The set of statistical equilibrium equations are obtained as

$$(\lambda+n\mu)P_{0n} = [1-(1-\delta_{n0})p]\int_0^\infty P_{1n}(x)h(x)dx$$

$$P'_{1n}(x) = -(\lambda H_1+h(x))P_{1n}(x)+\lambda H_1 P_{1,n-1}(x)$$

$$P_{1n}(0) = \lambda P_{0n}+(n+1)\mu P_{0,n+1}+p\int_0^\infty P_{1,n+1}(x)h(x)dx$$

For generating function $P_0(z)$ and $P_1(z,x)$ these equations are transformed to:

$$\lambda P_0(z)+\mu z P_0'(z) = (1-p)\int_0^\infty P_1(z,x)h(x)dx+\lambda p P_{00} \tag{3.6}$$

$$\frac{\partial P_1(z,x)}{\partial x} = -(\lambda H_1(1-z)+h(x))P_1(z,x) \tag{3.7}$$

$$(z-p\beta(\lambda H_1-\lambda H_1 z))P_1(z,0)+\lambda p P_{00} = \mu z p_0'(z)+\lambda z P_0(z) \tag{3.8}$$

Solving (3.7) yields

$$P_1(z,x) = P_1(z,0)(1-B(x))e^{-\lambda H_1(1-z)x} \tag{3.9}$$

Combining (3.6), (3.8) and (3.9), and after some algebra we get

$$\mu z(z-\beta(\lambda H_1-\lambda H_1 z))P_0'(z)+(\lambda z-((1-p)\lambda z+\lambda p)\beta(\lambda H_1-\lambda H_1 z))P_0(z)$$
$$=\lambda p P_{00}(z-\beta(\lambda H_1-\lambda H_1 z)) \quad (3.10)$$

Coefficient of $P_0'(z)$ has two zeros $z_1 = 0$ and $z_2 = 1$. Choose an arbitrary point $a \in (0,1)$. Solving (3.10) for $z \in (0,a]$, we get

$$P_0(z) = [(\frac{z}{a})^{\frac{\lambda p}{\mu}} r(a,z)]^{-1}\{P_0(a)+\frac{\lambda p P_{00}}{\mu(a)^{\frac{\lambda p}{\mu}}}\int_a^z (t)^{\frac{\lambda p}{\mu}-1} r(a,t)dt$$

As $z \to 0+$, $P_0(0) < +\infty$ and $(\frac{z}{a})^{\frac{-\lambda p}{\mu}}$ diverges. Thus we get

$$P_0(a) = \frac{\lambda p P_{00}}{\mu(a)^{\frac{\lambda p}{\mu}}}\int_0^a (t)^{\frac{\lambda p}{\mu}-1} r(a,t)dt \quad (3.11)$$

On the otherhand, solving (3.10) for $z \in [a,1)$, and taking limit as $z \to 1-$; we get

$$P_0(a) = \frac{P_0(1)r(a,1)}{(a)^{\frac{\lambda p}{\mu}}} - \frac{\lambda p P_{00}}{\mu(a)^{\frac{\lambda p}{\mu}}}\int_a^1 (t)^{\frac{\lambda p}{\mu}-1} r(a,t)dt \quad (3.12)$$

For obtaining the relation (3.12) it should be noted that

$$\lim_{z\to 1-}[\frac{1-\beta(\lambda H_1-\lambda H_1 z)}{\beta(\lambda H_1-\lambda H_1 z)-z}] = \frac{+\lambda H_1\beta_1}{1-\lambda H_1\beta_1} < \infty$$

Equating (3.11) and (3.12) we get

$$P_0(1) = \frac{\lambda p P_{00}}{\mu r(0,1)}\int_0^1 (t)^{\frac{\lambda p}{\mu}-1} r(0,t)dt \quad (3.13)$$

Then we can rewrite solution of (3.10), as (3.1). Combining the equations (3.8)–(3.10), we get (3.2). Since $P_1(z) = \int_0^\infty P_1(z,x)dx$, we obtain (3.5). Now applying the normalizing condition $P_0(1) + P_1(1) = 1$, we get

$$P_0(1) = \frac{1 - \lambda H_1\beta_1}{1 + \lambda\beta_1 - \lambda H_1\beta_1} \tag{3.14}$$

Using (3.13) and (3.14) we obtain (3.4). □

Corollory 3.3. *The partial factorial moments M_k^i, for $i \in \{0, 1\}, k \in \{0, 1\}$ are given by*

$$\begin{aligned}
M_0^0 &= \frac{1 - \lambda H_1\beta_1}{1 + \lambda\beta_1 - \lambda H_1\beta_1}, \\
M_0^1 &= \frac{\lambda\beta_1}{1 + \lambda\beta_1 - \lambda H_1\beta_1} \\
M_1^0 &= \frac{\lambda p P_{00}}{\mu} + \frac{\lambda(\lambda H_1\beta_1 - p)}{\mu(1 + \lambda\beta_1 - \lambda H_1\beta_1)} \\
M_1^1 &= \frac{\lambda^2}{(1 - \lambda H_1\beta_1)}\left\{\frac{\mu H_1\beta_2 + \lambda H_1\beta_1^2 - p\beta_1}{\mu(1 + \lambda\beta_1 - \lambda H_1\beta_1)} + \frac{p\beta_1 P_{00}}{\mu}\right\}
\end{aligned}$$

3.2.3 Analysis of the busy period

The busy period is defined as the period starting at an epoch when an arriving customer finds an empty system and ending at the next departure epoch at which the system is empty. Without loss of generality we may assume that at time $t = 0$ the system is empty. i.e, $C(0) = N(0) = 0$, and one primary customer just arrives at time $t = 0$. Then a system busy period starts and ends at the first departure epoch at which the process $\{(C(t), N(t)),\ t \geq 0\}$ return to the state $(0, 0)$ for the first time.

Let $P_0(t)$ be the distribution function of the busy period L and $\pi_0^*(s) = E[e^{-sL}]$ be the Laplace-Stieltjes transform of L. Let us define the transient taboo probabilities of process $X(t) = \{(C(t), N(t), \xi(t)),\ t \geq 0\}$ as follows:

$$P_0(t) = P\{C(t) = 0,\ N(t) = 0\}, \quad t \geq 0 \tag{3.15}$$

$$P_n(t) = P\{L > t,\ C(t) = 0,\ N(t) = n\}, \quad t \geq 0,\ n \geq 1 \tag{3.16}$$

$$Q_n(x,t) = P\{L > t,\ C(t) = 1,\ N(t) = n,\ x \leq \xi(t) < x + dx\}$$
$$t \geq 0,\ x \geq 0,\ n \geq 0 \tag{3.17}$$

and, the boundary conditions are

$$P_0(0) = 0,\ Q_n(x,0) = \delta_{n0}\delta(x), \quad n \geq 0 \tag{3.18}$$

where δ_{n0} and $\delta(x)$ are Kronecker and Dirac functions, respectively.

With the help of supplementary variable, we obtain the Kolmogorov equations that govern the dynamics of the system behaviour as:

$$\frac{d}{dt}P_0(t) = \int_0^\infty Q_0(x,t)h(x)dx \tag{3.19}$$

$$\frac{d}{dt}P_n(t) = -(\lambda + n\mu)P_n(t) + (1-p)\int_0^\infty Q_n(x,t)h(x)dx, \quad n \geq 1 \tag{3.20}$$

$$\frac{\partial Q_n(x,t)}{\partial t} + \frac{\partial Q_n(x,t)}{\partial t} = -(\lambda H_1 + h(x))Q_n(x,t)$$
$$+ \lambda H_1(1-\delta_{n0})Q_{n-1}(x,t), \quad n \geq 0 \tag{3.21}$$

$$Q_n(0,t) = \lambda(1-\delta_{n0})P_n(t) + (n+1)\mu P_{n+1}(t)$$
$$+ p\int_0^\infty Q_{n+1}(x,t)h(x)dx, \quad n \geq 0 \tag{3.22}$$

To solve the above equations, we now introduce Laplace transforms and generating functions:

$$P_n^*(s) = \int_0^\infty e^{-st}P_n(t)dt, \quad n \geq 1$$

$$
\begin{aligned}
Q_n^*(x,s) &= \int_0^\infty e^{-st} Q_n(x,t)dt, \quad n \geq 0 \\
P(s,z) &= \sum_{n=1}^{\infty} z^n P_n^*(s) \\
Q(x,s,z) &= \sum_{n=0}^{\infty} z^n Q_n^*(x,s)
\end{aligned} \tag{3.23}
$$

Then equations (3.19)–(3.22) become

$$\pi_0^*(s) = \int_0^\infty Q_0^*(x,s)h(x)dx \tag{3.24}$$

$$\mu z \frac{\partial P(s,z)}{\partial z} + (s+\lambda)P(s,z) = (1-p)\int_0^\infty (Q(x,s,z) - Q_0^*(x,s))h(x)dx \tag{3.25}$$

$$\frac{\partial Q(x,s,z)}{\partial x} + (s + \lambda H_1(1-z) + h(x))Q(x,s,z) = \delta(x) \tag{3.26}$$

$$Q(0,s,z) = \frac{\mu}{(1-p)}\frac{\partial P(s,z)}{\partial z} + \frac{p(s+\lambda) + \lambda(1-p)z}{(1-p)z}P(s,z) \tag{3.27}$$

Solving (3.25), we get:

$$Q(x,s,z) = (1 + Q(0,s,z))(1 - B(x))e^{-(s+\lambda H_1(1-z))x} \tag{3.28}$$

Combining (3.23), (3.24), (3.26) and (3.27), we get:

$$
\begin{aligned}
&\mu z(z - \beta(s + \lambda H_1(1-z)))\frac{\partial P(s,z)}{\partial z} \\
&\quad + ((s+\lambda)z - ((1-p)\lambda z + (s+\lambda)p)\beta(s+\lambda H_1(1-z)))P(s,z) \\
&\quad\quad = (1-p)z(\beta(s+\lambda H_1(1-z)) - \pi_0^*(s))
\end{aligned} \tag{3.29}
$$

Let $g(z,s,x) = z - \beta(s + \lambda H_1(1-z))$ for Re $(s) > 0$, $|z| \leq 1$ and $|x| \leq 1$. Then for each fixed value of (s,x), g has a unique zero $\pi_\infty^*(s/\lambda H_1)$ in the unit disc $|z| \leq 1$, where $\pi^*_\infty(s|\lambda H_1)$ is the Laplace transform of the length of the busy period in the standard $M|G|1|\infty$ queue (without retrials) with persistence function H_1. Thus, coefficient of $\frac{\partial P(s,z)}{\partial z}$ has two zeros $z_1 = 0$ and $z_2 = \pi^*_\infty(s/\lambda H_1)$.

Choose an arbitrary number $\bar{z} \in (z_1, z_2)$. Solve (3.29) for $z \in [\bar{z}, z_2]$;

$$P(s,z) = \left[(\frac{z}{\bar{z}})^{\frac{(s+\lambda)}{\mu}p} e^{\frac{\lambda(1-p)(z-\bar{z})}{\mu}} r^*(\bar{z},z)\right]^{-1} \left\{P(s,\bar{z}) + \int_{\bar{z}}^{z} Q(u)(\frac{u}{\bar{z}})^{\frac{(s+\lambda)p}{\mu}} e^{\frac{\lambda(1-p)(u-\bar{z})}{\mu}} r^*(\bar{z},u)du\right\}$$

where

$$Q(u) = \frac{(1-p)u\beta(s+\lambda H_1(1-u)) - (1-p)u\pi_0^*(s)}{\mu u(u-\beta(s+\lambda H_1(1-u)))}$$

and

$$r^*(t_1,t_2) = \exp\{\frac{(1-p)}{\mu}\int_{t_1}^{t_2} \frac{s+\lambda(1-u)}{u-\beta(s+\lambda H_1(1-u))}du\}$$

As $z \to z_2$, $[r^*(\bar{z},z)]^{-1}$ diverges and $P(s,z_2)$ is finite. Thus,

$$P(s,\bar{z}) = \int_{z_2}^{\bar{z}} Q(u)(\frac{u}{\bar{z}})^{\frac{(s+\lambda)p}{\mu}} e^{\frac{\lambda(1-p)(u-\bar{z})}{\mu}} r^*(\bar{z},u)du$$

In the next step, solve (3.29) for $z \in (z_1, \bar{z}]$ and taking limit as $z \to 0+$, we get:

$$P(s,\bar{z}) = \int_{z_1}^{\bar{z}} Q(u)(\frac{u}{\bar{z}})^{\frac{(s+\lambda)p}{\mu}} e^{\frac{\lambda(1-p)(u-\bar{z})}{\mu}} r^*(\bar{z},u)du$$

Equating the above expressions for $P(s,\bar{z})$ and after some algebra, we get:

$$\pi_0^*(s) = \frac{\int_0^{\pi^*_\infty(s/\lambda H_1)} \left\{\frac{u^{\frac{(s+\lambda)p}{\mu}} e^{\frac{\lambda(1-p)u}{\mu}} \beta(s+\lambda H_1(1-u))r^*(0,u)}{u-\beta(s+\lambda H_1(1-u))}\right\}}{\int_0^{\pi^*_\infty(s/\lambda H_1)} \left\{\frac{u^{\frac{(s+\lambda)p}{\mu}} e^{\frac{\lambda(1-p)u}{\mu}} r^*(0,u)}{u-\beta(s+\lambda H_1(1-u))}\right\}} \tag{3.30}$$

Note that the domain of $\pi_0^*(s)$ is $s > 0$. Taking limit as $s \to 0$, we get undetermined expressions. These expressions appears difficult to be solved using L'Hospital rule.

3.2.4 Calculation of the first and second moment of the busy period

In the last section, even though we have obtained the expression for $E[e^{-sL}]$, the first moments of L can not be obtained from it. A direct way to find the first moment of $E(L)$ of the busy period is provided by the theory of regenerative process as follows: The limiting probabilities P_{ij} can be expressed as

$$P_{ij} = \frac{E[T_{ij}]}{\lambda^{-1} + E(L)} \tag{3.31}$$

where T_{ij} is the amount of time in a regenerative cycle during which the system is in the state (i, j). Since $E(T_{00}) = \frac{1}{\lambda}$, we get $E(L) = \lambda^{-1}(P_{00}^{-1} - 1)$.

In this section we obtain a closed form expression for $E(L^2)$ using the approach adopted in [13].

First we define

$$a(z) = P(0, z) = \sum_{n=1}^{\infty} z^n \int_0^{\infty} P_n(t)dt \tag{3.32}$$

$$b(z) = \int_0^{\infty} Q(x, 0, z)dx = \sum_{n=0}^{\infty} z^n \int_0^{\infty} \int_0^{\infty} Q_n(x, t)dtdx \tag{3.33}$$

Then

$$\begin{aligned} a(1) &= \int_0^{\infty} P[L > t, C(t) = 0]dt \\ \text{and} \quad b(1) &= \int_0^{\infty} P[L > t, C(t) = 1]dt \end{aligned} \tag{3.34}$$

Putting $s = 0$ in (3.29), we get,

$$\mu z(z - \beta(\lambda H_1 - \lambda H_1 z))a'(z) + (\lambda z - ((1-p)\lambda z + \lambda p)\beta(\lambda H_1 - \lambda H_1 z))a(z)$$
$$= (1-p)z[\beta(\lambda H_1 - \lambda H_1 z) - 1] \tag{3.35}$$

Solution of the above differential equation is obtained by employing the same technique that we have used for solving (3.10): Then

$$a(z) = \frac{-1}{\lambda} + \frac{p}{\mu}(z)^{\frac{-\lambda p}{\mu}} \int_0^z (t)^{\frac{\lambda p}{\mu}-1} r(z,t)dt \tag{3.36}$$

where $r(z,t)$ is as in (3.3) and

$$a(1) = \frac{-1}{\lambda} + \frac{p}{\mu r(0,1)} \int_0^1 (u)^{\frac{\lambda p}{\mu}-1} r(0,u)du \tag{3.37}$$

On the otherhand, putting $s = 0$ in (3.26), we get

$$Q(x,0,z) = (1 + Q(0,0,z))(1 - B(x))e^{-\lambda H_1(1-z)x}$$

so that,

$$b(z) = (1 + Q(0,0,z))\frac{(1 - \beta(\lambda H_1 - \lambda H_1 z))}{\lambda H_1 - \lambda H_1 z} \tag{3.38}$$

Putting $s = 0$ in (3.27), and as $z \to 1$, we get

$$Q(0,0,1) = \frac{\lambda}{1 - \lambda H_1 \beta_1}(a(1) + H_1 \beta_1) \tag{3.39}$$

Taking limit as $z \to 1$ in (3.38) we get another expression for $Q(0,0,1)$ and equating the above two expressions for $Q(0,0,1)$, we get:

$$b(1) = \frac{(1 - \lambda a(1))}{1 - \lambda H_1 \beta_1}\beta_1 \tag{3.40}$$

Now,

$$E[L] = a(1) + b(1) = \frac{\beta_1}{1 - \lambda H_1 \beta_1} + \frac{(1 + \lambda\beta_1 - \lambda H_1 \beta_1)}{1 - \lambda H_1 \beta_1} a(1) \qquad (3.41)$$

where $a(1)$ is given by (3.37).

Now, inorder to compute $E[L^2]$, we define

$$\psi(s, z) = \frac{\partial P(s, z)}{\partial s} \text{ and } c(z) = \psi(0, z) \qquad (3.42)$$

Differentiating (3.29) with respect to s, and setting $s = 0$, we obtain after some algebra

$$c'(z) + \frac{\lambda z - ((1-p)\lambda z + \lambda p)\beta(\lambda H_1 - \lambda H_1 z)}{\mu z(z - \beta(\lambda H_1 - \lambda H_1 z))} c(z) = g(z) \qquad (3.43)$$

where

$$\begin{aligned} g(z) = {} & \frac{1}{\mu z(z - \beta(\lambda H_1 - \lambda H_1 z))} \Big\{ (1-p) z \beta'(\lambda H_1 - \lambda H_1 z) + \frac{(1-p) z \beta_1}{1-\rho} \\ & - \frac{(1-p) z (1 - \beta(\lambda H_1 - \lambda H_1 z)) \beta'(\lambda H_1 - \lambda H_1 z)}{z - \beta(\lambda H_1 - \lambda H_1 z)} + \frac{(1-p) z (1 + \lambda\beta_1 - \rho)}{1-\rho} a(1) \\ & - \Big(z - ((1-p)\lambda z + \lambda p)\beta'(\lambda H_1 - \lambda H_1 z) - p\beta(\lambda H_1 - \lambda H_1 z) \\ & + \frac{(\lambda z - ((1-p)\lambda z + \lambda p)\beta(\lambda H_1 - \lambda H_1 z))\beta'(\lambda H_1 - \lambda H_1 z)}{z - \beta(\lambda H_1 - \lambda H_1 z)} \Big) a(z) \Big\} \end{aligned} \qquad (3.44)$$

Solving (3.43), we get:

$$c(z) = (z^{\frac{\lambda p}{\mu}} r(1, z))^{-1} \{ c(1) + \int_1^z t^{\frac{\lambda p}{\mu}} g(t) r(1, t) dt \} \qquad (3.45)$$

and

$$c(1) = \frac{1}{r(0, 1)} \int_0^1 g(t) t^{\frac{\lambda p}{\mu}} r(0, t) dt \qquad (3.46)$$

Now we consider the case $C(t) = 1$ and define

$$\psi_1(x, s, z) = \frac{\partial Q(x, s, z)}{\partial z} \text{ and } d(z) = \int_0^\infty \psi_1(x, 0, z)dx \tag{3.47}$$

Differentiating (3.26) with respect to s and setting $s = 0$, yields:

$$\frac{\partial \psi_1(x, 0, z)}{\partial x} = -Q(x, 0, z) - (\lambda H_1 - \lambda H_1 z + h(x))\psi_1(x, 0, z) \tag{3.48}$$

Solving (3.48) and after some algebra, we get:

$$\psi_1(x, 0, z) = (1 - B(x))e^{-(\lambda H_1 - \lambda H_1 z)x}\{\psi_1(0, 0, z) - x(1 + Q(0, 0, z))\}$$

Integrating the above expression with respect to x we find that

$$d(z) = \psi_1(0, 0, z)\frac{1 - \beta(\lambda H_1 - \lambda H_1 z)}{\lambda H_1 - \lambda H_1 z} - (1 + Q(0, 0, z))\int_0^\infty e^{-(\lambda H_1 - \lambda H_1 z)x}x(1 - B(x))dx \tag{3.49}$$

Thus

$$d(1) = \beta_1\psi_1(0, 0, 1) - \frac{\beta_2}{2}(1 + Q(0, 0, 1)) \tag{3.50}$$

$Q(0, 0, 1)$ is given by (3.39). To obtain $\psi_1(0, 0, 1)$, differentiate (3.27) with respect to s and set $(s, z) = (0, 1)$.
This yields:

$$\psi_1(0, 0, 1) = \frac{1}{(1 - p)}(\lambda c(1) + pa(1) + \mu c'(1)) \tag{3.51}$$

From (3.43)

$$c'(1) = g(1) + \frac{\lambda}{\mu}\frac{\lambda H_1\beta_1}{(1 - \lambda H_1\beta_1)}c(1) \tag{3.52}$$

Thus,

$$\psi_1(0, 0, 1) = \frac{1}{(1 - p)}(pa(1) + \mu g(1) + \frac{\lambda}{1 - \lambda H_1\beta_1}c(1)) \tag{3.53}$$

Using (3.39), (3.50) and (3.53) we get

$$d(1) = \frac{-\beta_2}{2} + (\frac{p\beta_1}{1-p} - \frac{\lambda\beta_2}{2(1-p)})a(1) - \frac{\mu\beta_2}{2(1-p)}a'(1) + \frac{\lambda\beta_1}{(1-p)(1-\rho H_1)}c(1) + \frac{\mu\beta_1}{(1-p)}g(1) \quad (3.54)$$

From (3.35)

$$a'(1) = \frac{(1-p)\rho H_1}{\mu(1-\rho H_1)} - \frac{\lambda(p-\rho H_1)}{\mu(1-\rho H_1)}a(1) \quad (3.55)$$

From (3.44),

$$\begin{aligned} g(1) = \frac{1}{2\mu(1-\rho H_1)^2}\Big\{ &((\lambda H_1)^2(1-p)\beta_2 - 2\lambda(1-p)\beta_1 - 2(1-\rho H_1)(1-p\rho H_1) \\ &- 2\lambda^2(1-p)H_1\beta_2)a(1) - 2(1-p)(1+\lambda\beta_1-\rho H_1)a'(1) \\ &- 2\lambda H_1(1-p)(1-\rho H_1)(\beta_1^2+\beta_2) \\ &+ (1-p)(E[L]-\beta_1)(2(1-\rho H_1)-(\lambda H_1)^2\beta_2)\Big\} \end{aligned} \quad (3.56)$$

Now

$$E[L^2] = -2(C(1)+d(1)) = 2\int_0^\infty t\,P\{L>t\}dt \quad (3.57)$$

Using (3.41), (3.55), (3.56) and (3.57) we get:

$$\begin{aligned} E[L^2] = \frac{1}{\mu(1-\rho H_1)^3}\Big\{ &[\lambda\mu\beta_2 - 2\lambda\beta_1(p-\rho H_1)(1+\lambda\beta_1-\rho H_1)]a(1) \\ &+ 2\beta_1\rho H_1(1-p)(1-\rho H_1) + 2\beta_1^2\rho H_1[\lambda(1-p) - \mu\rho H_1(1-\rho H_1)] \\ &+ \mu\beta_2[1-3(\rho H_1)^2(1-\rho H_1)]\Big\} - 2[1+\frac{\lambda\beta_1}{(1-p)(1-\rho H_1)}]c(1) \end{aligned} \quad (3.58)$$

where $a(1)$ and $c(1)$ are given by (3.37) and (3.46).

3.3 The care $H_2 < 1$

The care $H_2 < 1$ is far more complicated and closed form solution is available only in the case of exponential service time distribution. We obtain the limiting distribution of the system state and factorial moments in terms of hypergeometric series. We also obtain some important performance measures in the steady state and their numerical results for some examples.

3.3.1 Limiting distribution of the system state

First we introduce some notation.

$$F_j^i(x) = {}_2F_2[i, i + \frac{\lambda}{\mu};\ A + B + j, A - B + j;\ x]$$

where ${}_2F_2$ is the hyper geometric series given by

$${}_2F_2[a, b;\ c, d;\ z] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} \frac{z^k}{k!}$$

Theorem 3.4. *For an $M|M|1$ retrial queue with nonpersistent customer and orbital search, in the steady state, the limiting probabilities $\{P_{ij}\}_{(i,j)\in S}$ are given by*

$$P_{0n} = P_{00}(\lambda H_1)^n (1-p) \prod_{i=0}^{n-1} \{\frac{\lambda + i\mu}{\lambda\nu p + (i+1)\mu[\nu + (1 - H_2)(\lambda + (i+1)\mu)]}\} \tag{3.59}$$

$$P_{1n} = P_{00}(\frac{\lambda}{\nu})(\lambda H_1)^n \prod_{i=0}^{n-1} \{\frac{\lambda + (i+1)\mu}{\lambda\nu p + (i+1)\mu[\nu + (1 - H_2)(\lambda + (i+1)\mu)]}\} \tag{3.60}$$

where P_{00} is given by

$$P_{00}^{-1} = 1 + \frac{(1-p)\lambda^2 H_1}{\lambda\nu p + \mu[\nu + (1-H_2)(\lambda+\mu)]} F_1^1(\gamma) + \frac{\lambda}{\nu} F_0^1(\gamma) \tag{3.61}$$

Corresponding partial generating functions $P_0(z)$ and $P_1(z)$ are

$$P_0(z) = P_{00}\{1 + \frac{(1-p)\lambda^2 H_1 z}{\lambda\nu p + \mu[\nu + (1-H_2)(\lambda+\mu)]} F_1^1(\gamma z)\} \tag{3.62}$$

and

$$P_1(z) = P_{00}\frac{\lambda}{\nu} F_0^1(\gamma z) \tag{3.63}$$

where $A = 1 + \frac{\lambda}{2\mu} + \frac{\nu}{2\mu(1-H_2)}$; $B = \frac{\sqrt{(\lambda+\nu-\lambda H_2)^2 - 4\lambda\nu p(1-H_2)}}{2\mu(1-H_2)}$ *and* $\gamma = \frac{\lambda H_1}{\mu(1-H_2)}$.

Proof. Since the service time is exponentially distributed, the process $X(t) = (C(t), N(t))$ is a Markov process with state space $\{0,1\} \times \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of non-negative integers.

The system of statistical equilibrium equations for the probabilities P_{0n} and P_{1n} is

$$(\lambda + n\mu)P_{0n} = \nu[1 - p + p\delta_{n0}]P_{1n} \tag{3.64}$$

$$\begin{aligned}[\lambda H_1 + \nu + n\mu(1-H_2)]P_{1n} &= \lambda P_{0n} + (n+1)\mu P_{0,n+1} + \lambda H_1 P_{1,n-1} \\ &\quad + [\nu p + (n+1)\mu(1-H_2)]P_{1,n+1}\end{aligned} \tag{3.65}$$

Eliminate probabilities P_{1n} with the help of equation (3.64) and rewrite the resulting equation as

$$\begin{aligned}&[\lambda\nu p + (n+1)\mu(\nu + (1-H_2)(\lambda + (n+1)\mu))]P_{0,n+1} - \lambda H_1(\lambda + n\mu)P_{0n} \\ &\quad = [\lambda\nu p + n\mu(\nu + (1-H_2)(\lambda + n\mu))]P_{0n} - \lambda H_1(\lambda + (n-1)\mu)P_{0,n-1}\end{aligned}$$

This implies that

$$[\lambda\nu p + n\mu(\nu + (1 - H_2)(\lambda + n\mu))]P_{0n} - \lambda H_1(\lambda + (n-1)\mu)P_{0,n-1} = 0$$

Thus

$$P_{0n} = \frac{\lambda H_1(\lambda + (n-1)\mu)}{\lambda\nu p + n\mu(\nu + (1 - H_2)(\lambda + n\mu))}P_{0,n-1}$$

Then we get (3.59).

Using (3.59) and (3.64), we obtain (3.60). Using equation (3.59) and (3.60), we get the expression for $P_0(z)$ and $P_1(z)$ as in (3.62) and (3.63).

Applying the normalizing condition $P_0(1) + P_1(1) = 1$, we get the probability P_{00} as in (3.61) □

Corollary 3.5. *The patial factorial moments M_k^i, defined by $M_0^i = \sum_{j=0}^{\infty} P_{ij}$, $M_k^i = \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)P_{ij}$; $i \in \{0,1\}$, $k \geq 1$, are given by*

$$M_0^0 = P_{00}\{1 + \frac{(1-p)\lambda^2 H_1}{\lambda\nu p + \mu[\nu + (1 - H_2)(\lambda + \mu)]}F_1^1(\gamma)\}$$

$$M_k^0 = \frac{P_{00}(1-p)\lambda^2 H_1\gamma^{k-1}}{(\lambda\nu p + \mu[\nu + (1 - H_2)(\lambda + \mu)])}\frac{(1)_{k-1}(1 - \frac{\lambda}{\mu})_{k-1}}{(A + B + 1)_{k-1}(A - B + 1)_{k-1}}$$

$$\{kF_k^k(\gamma) + \frac{k(k + \frac{\lambda}{\mu})\gamma}{(A + B + k)(A - B + k)}F_{k+1}^{k+1}(\gamma)\}$$

$$M_0^1 = P_{00}\frac{\lambda}{\nu}F_0^1(\gamma)$$

$$M_k^1 = P_{00}\frac{\lambda}{\nu}\frac{(1)_k(1 + \frac{\lambda}{\mu})_k}{(A + B)_k(A - B)_k}\gamma^k F_k^{k+1}(\gamma)$$

Proof. To get the above expresions, we use the following well-known relations for the hypergeometric series

$$\frac{d}{dz}{}_2F_2[a, b;\ c, d;\ kz] = k\frac{ab}{cd}{}_2F_2[a+1, b+1;\ c+1, d+1;\ kz]$$

and

$$\frac{d^n}{dz^n} {}_2F_2[a, b;\ c, d;\ kz] = \frac{k^n (a)_n (b)_n}{(c)_n (d)_n} {}_2F_2[a+n, b+n;\ c+n, d+n;\ kz]$$

□

3.4 Numerical examples

In this section we analyse the effect of the parameters p, ρ, H_1 and H_2 on P_{00}, EN and EL through tables and graphs.

In table 3.1, we consider the case $H_2 = 1$ (ie. $M|G|1$ case) and an $M|E_3|1$ retrial queue with $\mu = 0.5$, $\nu = 1$ and $H_1 = 0.6$. P_{00}, EN and EL are evaluated for three different values of ρH_1 (0.25, 0.5 and 0.75). As expected, P_{00} increases while EN and EL decrease with the increase of p. We use the equations (3.4), Corollory 3.3 and (3.41) for the evaluation of there values.

We plot P_{00} and EN for the case $H_2 < 1$ (ie. $M|M|1$ case) in the figures 3.2–3.5. We considers the case $\mu = 0.5$ and normalize the service rate $\nu = 1$ so that $\rho = \lambda = 0.4$, 0.7 and 1 in fig. 3.2. It is found that P_{00} increases and EN decreases their values for the $M|M|1$ retrial queues with non-persistent customers $(p = 0)$ to the corresponding one for the standard $M|M|1$ queues with non-persistent customers $(p = 1)$. The combined effect of p and ρ on P_{00} and EN for three sets of values of (H_1, H_2) are displayed in fig 3.3. In fig 3.4, the combined effect of p and H_1 on P_{00} and EN for fixed $H_2 = 0.7$ for three different values of ρ are analysed.

By a similar way, P_{00} and EN versus p and H_2 for fixed $H_1 = 0.6$ for three different values of ρ are displayed in fig. 3.5.

p	P_{00}	EN	EL
0.1	0.597463	0.240102	4.85096
0.2	0.603311	0.229387	4.73414
0.3	0.608935	0.219148	4.62392
0.4	0.614344	0.209358	4.51981
0.5	0.61955	0.199991	4.42135
0.6	0.624561	0.191024	4.32809
0.7	0.629389	0.182434	4.23966
0.8	0.634042	0.174202	4.15571
0.9	0.638529	0.166306	4.07592
1.0	0.642857	0.15873	4

$\rho H_1 = 0.25$

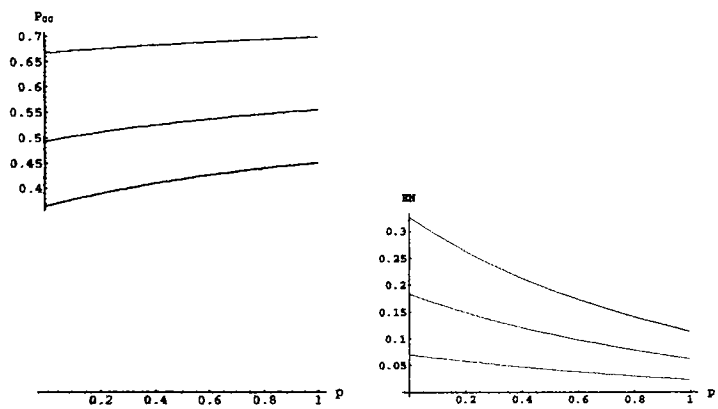
p	P_{00}	EN	EL
0.1	0.260504	1.31637	10.2194
0.2	0.274927	1.24823	9.4944
0.3	0.288951	1.18398	8.85884
0.4	0.302558	1.12374	8.29854
0.5	0.315733	1.06721	7.80204
0.6	0.328468	1.01419	7.35998
0.7	0.340759	0.964491	6.96464
0.8	0.352609	0.917907	6.60961
0.9	0.36402	0.874249	6.28956
1.0	0.375	0.833333	6

$\rho H_1 = 0.5$

p	P_{00}	EN	EL
0.1	.052253	5.52613	43.5304
0.2	0.062323	5.22899	36.1093
0.3	0.073295	4.94328	30.3443
0.4	0.085087	4.67017	25.8065
0.5	0.097596	4.41066	22.1913
0.6	0.110705	4.16545	19.2792
0.7	0.124292	3.93502	16.9094
0.8	0.138229	3.71958	14.9624
0.9	0.152394	3.51911	13.3487
1.0	0.166667	3.33333	12

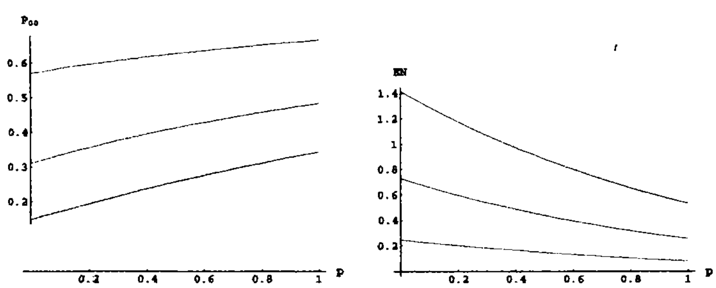
$\rho H_1 = 0.75$

Table 3.1 : $M|E_3|1$; $\mu = 0.5$, $\nu = 1$, $H_1 = 0.6$.



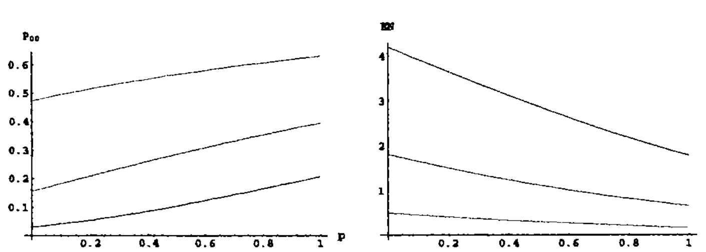
(a) P_{00} versus p (b) EN versus p

$H_1 = 0.25, H_2 = 0.3$



(c) P_{00} versus p (d) EN versus p

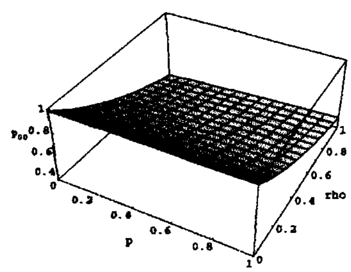
$H_1 = 0.6, H_2 = 0.7$



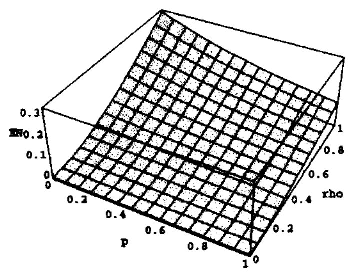
(e) P_{00} versus p (f) EN versus p

$H_1 = 0.85, H_2 = 0.9$

Figure 3.2: $M|M|1$: $\mu = 0.5, \nu = 1$. Red $\rho = 0.4$, Green $\rho = 0.7$, Blue $\rho = 1$

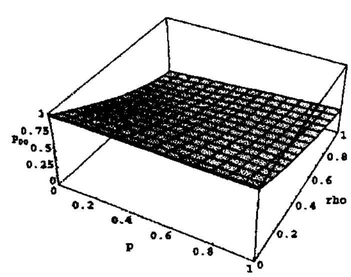


(a) P_{00} versus (p, ρ)

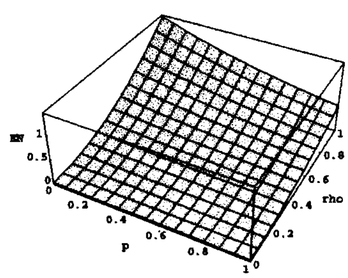


(b) EN versus (p, ρ)

$H_1 = 0.25, H_2 = 0.3$

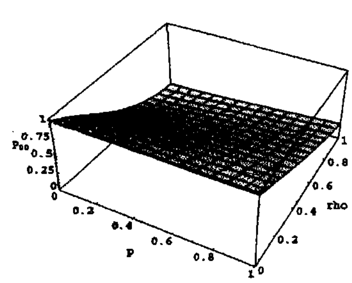


(c) P_{00} versus (p, ρ)

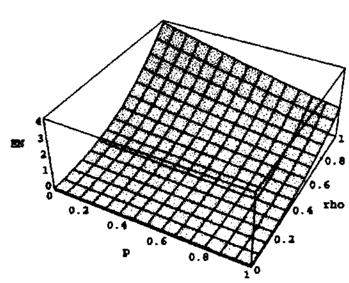


(d) EN versus (p, ρ)

$H_1 = 0.6, H_2 = 0.7$



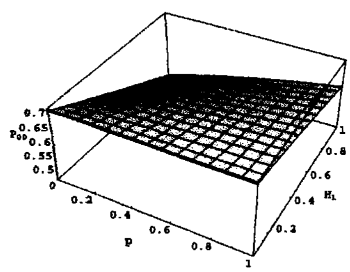
(e) P_{00} versus (p, ρ)

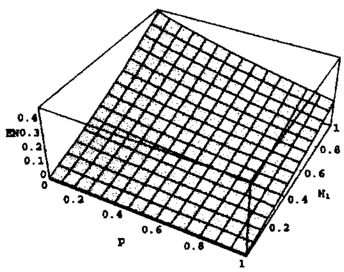


(f) EN versus (p, ρ)

$H_1 = 0.85, H_2 = 0.9$

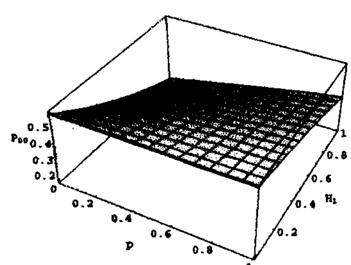
Figure 3.3: $M|M|1 : \mu = 0.5, \nu = 1.$

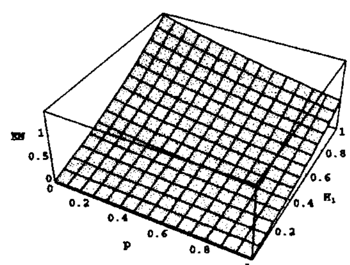




(a) P_{00} versus (p, H_1) (b) EN versus (p, H_1)

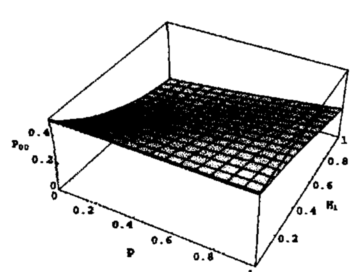
$\rho = 0.4, H_2 = 0.7$

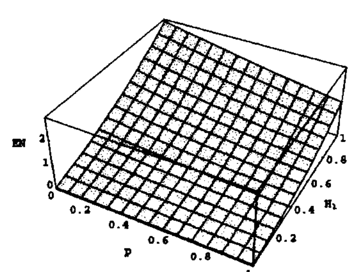




(c) P_{00} versus (p, H_1) (d) EN versus (p, H_1)

$\rho = 0.7, H_2 = 0.7$

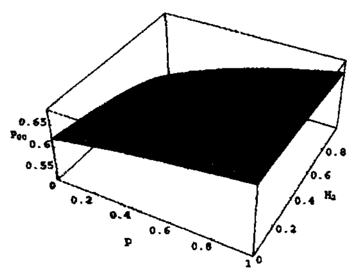




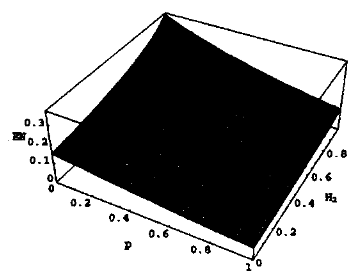
(e) P_{00} versus (p, H_1) (f) EN versus (p, H_1)

$\rho = 1, H_2 = 0.7$

Figure 3.4: $M|M|1 : \mu = 0.5, \nu = 1.$

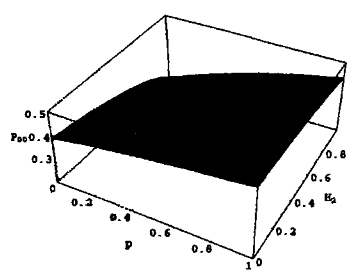


(a) P_{00} versus (p, H_2)

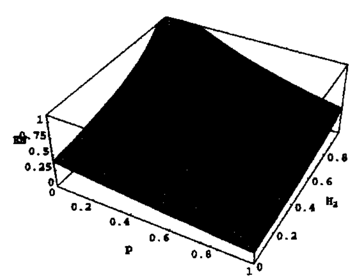


(b) EN versus (p, H_2)

$\rho = 0.4, H_1 = 0.6$

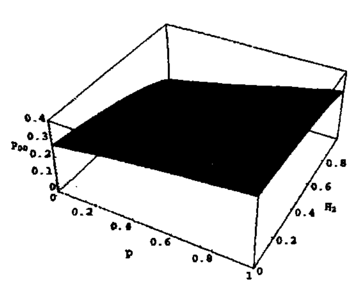


(c) P_{00} versus (p, H_2)

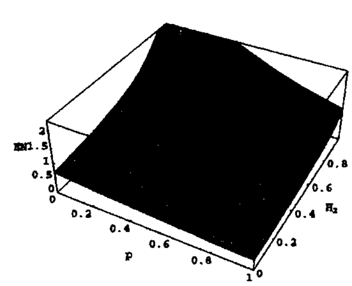


(d) EN versus (p, H_2)

$\rho = 0.7, H_1 = 0.6$



(e) P_{00} versus (p, H_2)



(f) EN versus (p, H_2)

$\rho = 1, H_1 = 0.6$

Figure 3.5: $M|M|1 : \mu = 0.5, \nu = 1$.

Chapter 4.

$MAP|M|c$ retrial queue with orbital search

In this chapter, we extend the search mechanism to a multi-server retrial queueing model. In a multiserver retrial queue, if an arriving primary customer finds some server free, it immediately occupies such a server and leaves the system after service. Otherwise, if all servers are engaged, it produces a source of repeated attempts called orbit. Every such source, after some delay, produces repeated attempts until after one or more attempts it finds a free server, in which case the source is eliminated and the customer receives service and then leaves the system.

The analysis of multi-server retrial queues is essentially more difficult than that one of single server models. Except for a few special cases explicit results for multi-server retrial queues are very rare. Hence, numerical investigation to bring out the qualitative behaviour of multi-server retrial queues is very important. By the implementation of approximations and truncated models, numerical investigation is carried out. Very briefly important truncation methods are described below.

Direct truncation method

In this method, the orbit size is restricted to M such that any arriving customer finding the orbit full is considered lost. The value of M can be chosen so that the loss probability is small. Due to the intrinsic nature of the system of equations obtained, the only choice available for studying M is through numerical analysis. While a number of approaches is available for determining the cut-off point, M, the one that seems to perform well (w.r.t. approximating the system performance measures) is to increase M until the largest individual change in the elements of the steady state probability vector for two successive values, is less than ϵ, a predetermined infinitesimal value.

Generalized truncation method

Falin [35] introduced the generalized truncation wherein the infinite system (which can not be solved directly) is truncated with the help of another infinite, but solvable system. The fact that the original infinite system is approximated by another infinite one provides a much better accuracy. In this procedure the retrial rate is linear ($j\ \mu$, for $j \leq M$) up to a level, say M, beyond which it is taken as infinity (thus reducing the system to a classical queue when orbital size is very large). Falin's generalized truncation has a pitfall in that for a low retrial rate the cut-off point for truncation becomes very high. Artalejo and Pozo [14] further modified the truncation due to Falin for the multi-server retrial queue.

Neuts-Rao truncation

For a multi-server retrial queue in which customers enter into orbit before getting service, Neuts and Rao [62] suggested an approximation with the help of the model where the retrial rate stays constant when the number of customers in the orbit exceeds some level. Their justification that the probability of a successful retrial request progressively decreases as the number of customers in the orbit increases, is applicable to our model and hence we expect this method to work better in terms of convergence and computational effort. Here retrial rate is taken to be $j\,\mu$ for $j \leq N$, when there are j customers in the orbit and $N\,\mu$ for $j \geq N$. In this method, the process becomes level independent beyond the truncation level.

Method by Bright & Taylor

Bright and Taylor [17] proposed approximate solutions to level-dependent quasi-birth-and-death processes. Their algorithm requires computation of several matrices obtained as solutions to quadratic equations. A considerable amount of quantities need to be evaluated and the truncation process is based on the tail probabilities. As mentioned in Neuts and Rao, this may pose some problem especially when θ is very small.

4.1 The mathematical model

We consider a multi-server retrial queueing system in which customers arrive according to a Markovian arrival process (MAP). The MAP, a special class of tractable Markov renewal process, is a rich class of point processes that includes many well-known processes such as Poisson, PH-renewal processes, and Markov-modulated Poisson process. One of the most significant features of the MAP is the underlying Markovian structure and fits ideally in the context of matrix-analytic solutions to stochastic models. As is well known, Poisson processes are the simplest and most tractable ones used extensively in stochastic modeling. The idea of the MAP is to significantly generalize the Poisson processes and still keep the tractability for modelling purposes. Furthermore, in many practical applications, notably in communications engineering, production and manufacturing engineering, the arrivals do not usually form a renewal process. So, MAP is a convenient tool to model both renewal and non-renewal arrivals. While MAP is defined for both discrete and continuous times, here we will need only the continuous time case.

The MAP in continuous time is described as follows. Let the underlying Markov chain be irreducible and let $Q = (q_{i,j})$ be the generator of this Markov chain. At the end of a sojourn time in state i, that is exponentially distributed with parameter λ_i, one of the following two events could occur: with probability p_{ij} the transition corresponds to an arrival, and the underlying Markov chain is in state j with $1 \leq i, j \leq m$; with probability $p_{i,j}(0)$ the transition corresponds to no arrival and the state of the Markov chain is j, $j \neq i$. Note that the Markov chain can go from state i to state i only through an arrival. Also, we have

$$\sum_{j=1}^{m} p_{i,j}(1) + \sum_{j=1, j\neq i}^{m} p_{i,j}(0) = 1, 1 \leq i \leq m.$$

Define matrices $D_k = (d_{ij}(k))$ for $k = 0, 1$ such that $d_{ii}(0) = -\lambda_i$, $1 \leq i, j \leq m$;, $d_{ij}(0) = \lambda_i p_{ij}(0)$, for $j \neq i$, $1 \leq i$, $j \leq m$, and $d_{ij}(1) = \lambda_i p_{ij}(1)$. By assuming D_0 to be a nonsingular matrix, the interarrival times will be finite with probability one and the arrival process does not terminate. Hence, we see that D_0 is a stable matrix. The generator Q is then given by $Q^* = D_0 + D_1$.

Thus, D_0 governs the transitions corresponding to **no** arrival and D_1 governs those corresponding to an arrival. The point process described by the MAP is a special class of semi-Markov processes with transition probability matrix given by

$$\int_0^x e^{D_0 t} dt D_1 = [I - e^{D_0 x}](-D_0)^{-1} D_1, x \geq 0.$$

Let π be the stationary probability vector of the Markov process with generator Q^*. That is, π is the unique (positive) probability vector satisfying.

$$\pi Q^* = 0, \pi e = 1.$$

Let α be the initial probability vector of the underlying Markov chain governing the MAP. Then, by choosing α appropriately we can model the time origin to be (a) an arbitrary arrival point; (b) the end of an interval during which there are at least k arrivals; (c) the point at which the system is in specific state such as the busy period ends or busy period begins. The most interesting case is the one where we get the stationary version of the MAP by $\alpha = \pi$. The constant $\lambda = \pi D_1 \mathbf{e}$, referred to as the fundamental rate gives the expected number of arrivals per unit of time in the stationary version of the MAP.

Often, in model comparisons, it is convenient to select the time scale of the MAP so that λ has a certain value. That is accomplished, in the continuous MAP case, by multiplying the coefficient matrices $D_k, k = 0, 1$, by the appropriate common constant. For further details on MAP and their usefulness in Stochastic modeling, we refer to Lucantoni [55], Neuts [58] and for a review and recent work on MAP we refer to Chakravarthy [22].

For use in sequel, let $\mathbf{e}(r)$, $\mathbf{e}_j(r)$ and I_r denote, respectively, the (column) vector of dimension r consisting of 1's, column vector of dimension r with 1 in the j^{th} position and 0 elsewhere, and an identity matrix of dimension r. The notation $\otimes$ will stand for the Kronecker product of two matrices. Thus, if A is a matrix of order $m \times n$ and if B is a matrix of order $p \times q$, then $A \otimes B$ will denote a matrix of order $mp \times nq$ whose $(i, j)^{th}$ block matrix is given by $a_{ij}B$. For more details on Kronecker products, we refer the reader to Bellman [16].

The customers from the orbit generate the retrial flow whose intensity is equal to $j\,\theta$, when j customers are present in the orbit. Service times are assumed to be exponentially distributed with rate μ. Additionally, we assume that each server can go in search of customers immediately after a service completion with a known probability p. The search time is assumed to be negligible. The probability for not going for the search of customers (equivalently, the server remains idle) is $q = 1 - p$. If the server does not pick up the next customer to be served from the orbit then there is a competition between primary and orbital customers for getting into the server for the next service.

4.2 Steady state analysis of the model at an arbitrary epoch

Let $N_1(t), N_2(t)$, and $J(t)$ denote, respectively, the number of customers in the orbit, the number of busy servers, and the phase of the arrival process at time t. The triplets $\{(N_1(t), N_2(t), J(t)), t \geq 0\}$ form a continuous-time Markov chain on the state space $\{(i, j, k) : i \geq 0, 0 \leq j \leq c, 1 \leq k \leq m\}$. Let $\boldsymbol{i}, i \geq 0$ denote the set of states $\{(i, j, k) : 0 \leq j \leq c, 1 \leq k \leq m\}$. Enumerating the states of the continuous time Markov chain in lexicographic order, the generator of the Markov chain is of the form:

$$Q = \begin{pmatrix} A_{10} & A_0 & 0 & 0 & 0 & \cdots \\ A_{21} & A_{11} & A_0 & 0 & 0 & \cdots \\ 0 & A_{22} & A_{12} & A_0 & 0 & \cdots \\ 0 & 0 & A_{23} & A_{13} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \tag{4.1}$$

where the entries are given by

$$A_{10} = \begin{pmatrix} D_0 & D_1 & 0 & 0 & \cdots & 0 & 0 \\ \mu I & D_0 - \mu I & D_1 & 0 & & 0 & 0 \\ 0 & 2\mu I & D_0 - 2\mu I & D_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & D_0 - (c-1)\mu I & D_1 \\ 0 & 0 & 0 & 0 & & c\mu I & D_0 - c\mu I \end{pmatrix}, \tag{4.2}$$

$$A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & D_1 \end{pmatrix}, \tag{4.3}$$

and for $i \geq 1$,

$$A_{1i} = \begin{pmatrix} D_0 - i\theta I & D_1 & 0 & 0 & 0 & 0 \\ \mu(1-p)I & D_0 - (i\theta + \mu)I & D_1 & 0 & 0 & 0 \\ 0 & 2\mu(1-p)I & D_0 - (i\theta + 2\mu)I & D_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & D_0 - (i\theta + (c-1)\mu)I & D_1 \\ 0 & 0 & 0 & 0 & c\mu(1-p)I & D_0 - c\mu I \end{pmatrix} \tag{4.4}$$

$$A_{2i} = \begin{pmatrix} 0 & i\theta I & 0 & \cdots & 0 & 0 & \\ 0 & p\mu I & i\theta I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & 0 & & (c-1)p\mu I & i\theta I & \\ 0 & 0 & 0 & & 0 & c\mu p I & \end{pmatrix}. \tag{4.5}$$

Let $\rho = \frac{\lambda}{c\mu}$. Then the queuing system under study is stable if and only if $\rho < 1$ (see, e.g., Falin [34]). Let $\mathbf{x}$, partitioned as $\mathbf{x} = (\mathbf{x}(0), \mathbf{x}(1), \mathbf{x}(2), \cdots)$, denote the steady-state probability vector of Q. That is,$\mathbf{x}$ satisfies

$$\mathbf{x}Q = \mathbf{0}, \mathbf{x}\mathbf{e} = 1. \tag{4.6}$$

In this paper we consider the direct truncation method and the Neuts-Rao truncation to solve (4.6). First we consider the direct truncation method.

4.2.1 Steady-state probability vector using direct truncation

Let M denote the cut-off point for this truncation method. In this case, the generator $\tilde{Q}$ as given in (4.1) is modified as follows.

$$\hat{Q} = \begin{pmatrix} A_{10} & A_0 & 0 & 0 & 0 & & & \\ A_{21} & A_{11} & A_0 & 0 & 0 & & & \\ 0 & A_{22} & A_{12} & A_0 & 0 & & & \\ 0 & 0 & A_{23} & A_{13} & A_0 & & & \\ & & & \ddots & \ddots & & & \\ & & & & & A_{2,M-1} & A_{1,M-1} & A_0 \\ & & & & & & A_{2,M} & A_0 + A_{1,M} \end{pmatrix}. \quad (4.7)$$

The steady-state probability vector, $\mathbf{x}^{(M)}$, is now partitioned as $\mathbf{x}^{(M)} = (\mathbf{x}(0), \mathbf{x}(1), \cdots \mathbf{x}(M))$ and is solved using (block) Gauss-Seidel procedure. Since there is no clear cut choice for M, we start the iterative process by taking $M = 1$ and increase it until the individual elements of $\mathbf{x}$ do not change significantly. That is, if M^* denote the truncation point, then $\max_{0 \leq i \leq M^*-1} ||\mathbf{x}^{M^*}(i) - \mathbf{x}^{M^*-1}(i)||_\infty < \epsilon$, where ϵ is an infinitesimal quantity and the infinity norm is defined by $||\mathbf{x}||_\infty = \max_{1 \leq i \leq n} |x_i|$.

4.2.2 Steady-state probability vector using Neuts-Rao method

Let N denote the cut-off point for this truncation method. In this case, the generator in (1) is modified as

$$\bar{Q} = \begin{pmatrix} A_{10} & A_0 & 0 & 0 & 0 & & & & \\ A_{21} & A_{11} & A_0 & 0 & 0 & & & & \\ 0 & A_{22} & A_{12} & A_0 & 0 & & & & \\ 0 & 0 & A_{23} & A_{13} & A_0 & & & & \\ & & & \ddots & \ddots & & & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (4.8)$$

where $A_1 = A_{1N}$ and $A_2 = A_{2N}$.

Let π denote the steady state probability vector of the generator $A = A_0 + A_1 + A_2$. The vector, π, partitioned as $\pi = (\pi_0, \cdots, \pi_c)$ is computed by solving the following equations.

$$\pi_0 = (1-p)\mu\pi_1(N\theta I - D_0)^{-1},$$
$$\pi_i = [\pi_{i-1}(D_1 + N\theta I) + (i+1)(1-p)\mu\pi_{i+1}](N\theta I + i(1-p)\mu I - D_0)^{-1}, 1 \le i \le c-1$$
$$\pi_c = \pi_{c-1}(D_1 + N\theta I)(c\mu(1-p) - D)^{-1};$$

subject to $\sum_{i=0}^{c} \pi_i = 1$.

Note that the above equations are well suited for numerical evaluation using (block) Gauss-Seidel iterative method.

The steady-state probability vector, $\mathbf{x}$, *for the modified generator exists if and only if the following holds good.*

$$\pi A_0 e < \pi A_2 e. \quad (4.9)$$

It is easy to verify that the condition (4.9) reduces to

$$\pi_c D_1 e < N\theta \sum_{j=0}^{c-1} \pi_j e + p\mu \sum_{j=1}^{c} j\pi_j e. \tag{4.10}$$

Under the stability condition given in (4.10) the steady-state probability vector x is given by [58]

$$\mathbf{x}(i + N - 1) = \mathbf{x}(N - 1)R^i, i \geq 0, \tag{4.11}$$

where the matrix R satisfies the matrix quadratic equation:

$$R^2 A_2 + RA_1 + A_0 = 0, \tag{4.12}$$

and the vectors $\mathbf{x}(0), \cdots, \mathbf{x}(1)$ are obtained by solving

$$\begin{aligned} &\mathbf{x}(0)A_{10} + \mathbf{x}(1)A_{21} = 0. \\ &\mathbf{x}(i-1)A_0 + \mathbf{x}(i)A_{1i} + \mathbf{x}(i+1)A_{2,i+1} = 0, 1 \leq i \leq N-1, \\ &\mathbf{x}(N-2)A_0 + \mathbf{x}(N-1)[A_{1,N-1} + RA_2] = 0. \end{aligned} \tag{4.13}$$

subject to the normalizing condition

$$\sum_{i=0}^{N-2} \mathbf{x}(i) + \mathbf{x}(N-1)(I - R)^{-1}\mathbf{e} = 1 \tag{4.14}$$

The computation of the R matrix and the vector $\mathbf{x}$ can be carried out by exploiting the special structure of the coefficient matrices. The details are given in the next section.

4.3 Algorithmic analysis

In this section we present efficient algorithmic procedures for computing the R matrix and the vector $\mathbf{x}$, which are the main ingredients for discussing the qualitative behavior of the model under study.

4.3.1 Computation of the matrix R

Due to the special structure of the coefficient matrices appearing in (4.12), the matrix R of dimension $(c+1)m$ can be efficiently computed as follows. Since $RA_2\mathbf{e} = \mathbf{e}_{c+1}(c+1) \otimes D_1\mathbf{e}$, R is of the form

$$R = \begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ R_0 & R_1 & & R_c \end{pmatrix}, \tag{4.15}$$

where the matrices R_j, for $0 \le j \le c$ are of dimension $(c+1)m$. In terms of these R_j matrices, equation (4.12) is rewritten for immediate numerical evaluation as

$$\begin{aligned} R_0 &= \mu(1-p)R_1(N\theta I - D_0)^{-1}, \\ R_i &= \big[N\theta R_c R_{i-1} + ip\mu R_c R_i + R_{i-1}D_1 \\ &\quad + (i+1)\mu(1-p)R_{i+1}\big](i\mu I + N\theta I - D_0)^{-1}, \le i \le c-1, \\ R_c &= \big[N\theta R_c R_{c-1} + cp\mu R_c^2 + R_{c-1}D_1 + D_1\big](c\mu I - D_0)^{-1}. \end{aligned} \tag{4.16}$$

4.3.1.1 Choice of N

Choosing an appropriate value for N in the truncation process is crucial in using matrix-geometric approximation for the queuing model under study. In this section to emphasize the dependence of R matrix on N, we will write $R(N)$ when referring to R. First note that from (4.16), as $N \to \infty$, $R_i, 0 \le i \le c-1 \to 0$, and R_c satisfies

$$c\mu R_c^2 + R_c(D_0 - c\mu I) + D_1 = 0. \tag{4.17}$$

Note that the rate matrix satisfying (4.17) is that of the classical $MAP/M/c$ queue.

Let $\eta(N)$ denote the spectral radius of the matrix $R(N)$ and let η be the spectral radius of the rate matrix satisfying (4.17). Observe that as $N \to \infty$, $\eta(N) \to \eta$. Since the steady-state probability vectors, $\mathbf{x}(i), i \geq 0$, depend largely on $\eta(N)$, to minimize the effect of the approximation caused by the truncation, we would like to choose N such that it satisfies (a) thc stability condition (4.10) and (b) at least one of the following two conditions (when all parameters of the model are fixed).

1. $\eta(N)$ is very close to η. That is, for a given $\epsilon_1 > 0$, $|\eta(N) - \eta| < \epsilon_1$.
2. Change in $\eta(N)$ due to a marginal increase in N is sufficiently small. That is, for a given $\epsilon_2 > 0$, $|\eta(N+1) - \eta(N)| < \epsilon_2$.

In the sequel, such an N will be denoted by N^*. For large values of θ the above criteria yield very small values for N^*. Hence, we put a minimum threshold, say, 10 for such cases.

Evaluation of $R(N)$: Equations in (4.16) are well suited for (block)Gauss-Seidel iterative procedure to evaluate $R(N)$. Note that this procedure is numerically stable due to the fact that the quantities involved in the evaluation are all nonnegative. When the maximum absolute (componentwise) difference between two successive iterates is less than a predetermined infinitesimal value, say, ϵ_3, the process is terminated. Once $R(N)$ is evaluated, the spectral radius $\eta(N)$ can be computed using, say, Elsner's algorithm. However, one can compute $\eta(N)$ without explicitly evaluating $R(N)$. For details on this we refer to [59][pp38-40].

4.3.2 Computation of the vector x

Due to the special structure of the matrix R as given in (4.14), the vector, $\mathbf{x}$, can be computed very efficiently. First, we partition $\mathbf{x}$ as

$$\mathbf{x}(i) = (\mathbf{y}_0(i), \cdots, \mathbf{y}_c(i)), \; i \geq 0,$$

where the vectors $\mathbf{y}_j(i))$ are of dimension m.

First note that from equation (4.11), we have

$$\mathbf{y}_j(i) = \mathbf{y}_c(N-1)R_c^{i-N}R_j, \; 0 \leq j \leq c, \; i \geq N. \tag{4.18}$$

The equations in (4.13) can be rewritten in terms of vectors of dimension m that are well suited for numerical implementation. For example, the first equation in (4.13) reduces to

$$\begin{aligned}
\mathbf{y}_0(0) &= \mu\mathbf{y}_1(0)(-D_0)^{-1}, \\
\mathbf{y}_j(0) &= \left[\mathbf{y}_{j-1}(0)D_1 + (j+1)\mu\mathbf{y}_{j+1}(0) + \theta\mathbf{y}_{j-1}(1) + jp\mu\mathbf{y}_j(1)\right](j\mu I - D_0)^{-1}, \\
&\qquad 1 \leq j \leq c-1, \\
\mathbf{y}_c(0) &= \left[\mathbf{y}_{c-1}(0)D_1 + \theta\mathbf{y}_{c-1}(1) + cp\mu\mathbf{y}_c(1)\right](c\mu I - D_0)^{-1}.
\end{aligned}$$

The other equations are similarly written and the details are omitted.

4.4 System performance measures

In this section we will list some important performance measures along with their formulas. These measures are used to bring out the qualitative behavior of the queueing model under study.

1. The Probability Mass Function of the Number of Customers in Orbit. The probability that there are i customers in orbit is given by

$$a_i = \mathbf{x}(i)\mathbf{e} = \begin{cases} \sum_{j=0}^{c} \mathbf{y}_j\mathbf{e}, 0 \leq i \leq N-1, \\ \mathbf{y}_c(N-1)R_c^{i-N} \sum_{j=0}^{c} R_j\mathbf{e}, i \geq N. \end{cases} \quad (4.19)$$

2. The Mean Number of Customers in Orbit. The mean, MNO, number of customers in orbit is given by

$$\begin{aligned} MNO = \sum_{i=0}^{\infty} i\mathbf{x}(i)\mathbf{e} &= \sum_{i=0}^{N-1} i \sum_{j=0}^{c} \mathbf{y}_j(i)\mathbf{e} \\ &+ \mathbf{y}_c(N-1)(I-R_c)^{-1}[(N-1)I + (I-R_c)^{-1}] \sum_{j=0}^{c} R_j\mathbf{e}. \end{aligned} \quad (4.20)$$

3. The Probability Mass Function of the Number of Busy Servers. The probability that j servers are busy is given by

$$b_j = \sum_{i=0}^{\infty} \mathbf{y}_j(i)\mathbf{e} = \sum_{i=0}^{N-1} \mathbf{y}_j(i)\mathbf{e} + \mathbf{y}_j(N-1)(I-R_c)^{-1}R_j\mathbf{e}, 0 \leq j \leq c. \quad (4.21)$$

4. The Overall Rate of Retrials. The overall rate of trials at which the orbiting customers request service is given by

$$\theta_1^* = \theta \sum_{i=1}^{\infty} i\mathbf{x}(i)\mathbf{e} = \theta\mu_{orbit}. \quad (4.22)$$

5. The Successful Rate of Retrials. The rate at which the orbiting customers successfully reach a free server is given by

$$\theta_2^* = \theta \left[\sum_{i=1}^{\infty} i \sum_{j=0}^{c-1} \mathbf{y}_j(i)\mathbf{e} \right]. \tag{4.23}$$

6. The Fraction of Successful Rate of Retrials. The fraction, FSR, of successful rate of retrials is given by $\frac{\theta_2^*}{\theta_1^*}$.

7. The Blocking Probability. The probability, PBK, that an orbiting customer is blocked is given by b_c which is given in equation (4.21) by putting $j = c$.

8. The Probability of an Immediate Access. The probability, P_{access}, that an arriving customer will enter into service immediately is given by

$$P_{\text{access}} = \frac{1}{\lambda} \sum_{i=0}^{\infty} \sum_{j=0}^{c-1} y_j(i) D_1 e. \tag{4.24}$$

9. The Probability of an Immediate Access with at least one customer waiting in the orbit. The probability. PES, that an arriving customer will enter into service immediately with at least one customer waiting in the orbit is given by

$$PES = \frac{1}{\lambda} \sum_{i=1}^{\infty} \sum_{j=0}^{c-1} y_j(i) D_1 e. \tag{4.25}$$

4.5 Numerical examples

In order to test the feasibility of the algorithms proposed in this paper, a Fortran code was developed and tested on a large number of examples. The correctness and the accuracy of the code are verified by a number of accuracy checks. For example, we obtained the numerical solution for the Poisson arrivals in its simple form. Next, we implemented the general algorithm, but using the following MAP representation: Let D_0 be an irreducible, stable matrix with eigenvalue of maximum real part $-\delta < 0$. Let α denote the corresponding left eigenvector, normalized by αe=1. Taking $D_1 = -D_0 e\alpha$, the MAP representation reduces to the Poisson arrival process with intensity rate δ. The general algorithm does not utilize this fact in any manner, but the numerical results agreed very much. Also when $N \to \infty$ or $p \to 1$, the current model reduces to the classical $MAP/M/c$ queue. In this section we discuss some interesting numerical examples that qualitatively describe the performance of the queuing model under study. For the arrival process, we consider five Markovian arrival processes with representation D_0 and D_1 given as follows.

1. Erlang (ERL):

$$D_0 = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix} \quad D_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

2. Exponential (EXP):

$$D_0 = (-1), \quad D_1 = (1)$$

3. Hyperexponential (HEX):

$$D_0 = \begin{pmatrix} -1.9 & 0 \\ 0 & -0.19 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1.71 & 0.19 \\ 0.171 & 0.019 \end{pmatrix}$$

4. MAP with negative correlation (MNC):

$$D_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0.00000 \\ 0.00000 & -1.00222 & 0.00000 \\ 0.00000 & 0.00000 & -225.75000 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.00000 & 0.00000 & 0.00000 \\ 0.01002 & 0.00000 & 0.99220 \\ 223.49250 & 0.00000 & 2.25750 \end{pmatrix}$$

5. MAP with positive correlation (MPC):

$$D_0 = \begin{pmatrix} -1.00222 & 1.00222 & 0.00000 \\ 0.00000 & -1.00222 & 0.00000 \\ 0.00000 & 0.00000 & -225.75000 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.00000 & 0.00000 & 0.00000 \\ 0.99220 & 0.00000 & 0.01002 \\ 2.25750 & 0.00000 & 223.49250 \end{pmatrix}$$

All these five MAP processes are normalized so that $\lambda = 1$. However, these are qualitatively different in that they have different variance and correlation structure. The first three arrival processes correspond to renewal processes and so the correlation is 0. The arrival process labelled MNC has correlated arrivals with a correlation value of -0.48891, and the arrivals corresponding to the process labelled MPC has a positive correlation with a value of 0.48891. The standard deviations of these five arrival processes are, respectively, 0.70711, 1.0, 2.24472, 1.40952 and 1.40952.

In all our examples, we fix $\lambda = 1$. The service rate μ is chosen so as to arrive at a specific value of $\rho = \frac{\lambda}{c\mu}$ given a value for c. We take $\epsilon = 10^{-6}$ in the direct truncation method; $\epsilon = 10^{-7}$ in the Gauss-Seidel iterative procedure when computing the rate matrix as well as the steady state probability vector. We take $\epsilon_1 = \epsilon_2 = 10^{-4}$ in the matrix-geometric truncation procedure. All other parameters such as p, ρ, c, θ are varied.

Discussion on M^* and N^*

Here we illustrate the effect of the parameters p, ρ, θ, and the type of arrival process on M^* and N^*. The values of M^* and N^* are given in Tables 1A (for the case when $c = 1, 2$) and 1B (when $c = 4, 8$) below. We notice the following interesting observations from these tables.

- Generally speaking the truncation cut-off point appears to be smaller under matrix-geometric approximation compared to direct truncation whenever (a) the system load (i.e., ρ) is large; or (b) the retrial rate (θ) is large; or (c) the search probability (p) is reasonably large, for all five arrival processes under study and for all values of c. This is to be expected since under these scenarios, the probability of seeing an idle server will be relatively small; thus increasing the size of the system of equations and the number of iterations required to achieve the desired convergence.

- The truncation cut-off point appears to be small for the direct truncation method whenever (a) ρ is relatively small; or (b) θ is small, for the first four arrival process. In this case the size of the system of equations is relatively small; however the rate matrix convergence requires a larger cut-off value.

- With regards to the fifth arrival process that is labelled as MPC, some important observations need to be made when comparing the two truncation methods. First note that this arrival process has variation that is smaller than that of the hyperexponential; however the interarrival times here are positively correlated. Secondly, the rate matrix $R(N)$ and its spectral radius converge very slowly to the rate matrix and the corresponding spectral radius of the classical $MAP/M/c$ queue as $N \to \infty$. Hence, with the stopping criteria that is considered here, the truncation cut-off point for the MPC arrival process is significantly less in the matrix-geometric approximation. As we will see later this has a tremendous effect on the system performance measures. It is worth mentioning that this type of behavior is not seen with the hyperexponential arrivals even though it has the largest variation among the five arrival processes considered. Hence, (positive) correlation appears to play a major role.

- As is to expected, N^* appears to decrease as θ decreases. This is true for all values of p, ρ, c, and for all five arrival processes. In all cases, the rate of decrease appears to be large when p is small and ρ is large.

- As ρ increases, M^* as well as N^* appear to increase for all arrival processes. Similarly, these two quantities appear to decrease when c is varied by fixing all other parameters.

- It is interesting to note that M^* is significantly large for MPC compared to that of MNC. Note that both these two arrival processes have the same mean and variance, but have different correlation structure. This indicates the vital role played by correlation, which is largely ignored in the literature.

Discussion on system performance measures

The effect of the parameters p, ρ, θ, c, and the type of arrival process on the four selected system performance measures: (a) the fraction of successful rate of retrials; (b) the mean number of customers in the orbit; (c) the probability that an arriving customer will enter into service when at least one customer is in the orbit; and (d) the probability that an orbiting customer is blocked, for a wide range of parameter values is analyzed. These measures are given in Tables 2A through 5B. In interpreting these tables, we need to keep in mind the differences in the cut-off values (M^* in the case of direct truncation and N^* in the case of matrix-geometric approximation) for different arrival processes and for various values of other parameters. Examining these tables we observe the following salient points.

- The matrix-geometric approximation seems to perform extremely well in all except the following cases: (a) For the first four arrival processes when ρ is large and when θ is small; (b) For the fifth arrival process (MPC) when θ is small irrespective of the values of ρ and c. These can be explained as follows. In case (a), the number of customers in the orbit grows rapidly and hence limiting the retrial rate to be a fixed quantity may significantly affect the system dynamics. In case (b) as mentioned before, the rate matrix, $R(N)$ converges very slowly reflecting in very small cut-off values, N^*, compared to M^*. This has a drastic effect in the dynamics of the system. However, the system performance measures are greatly improved when $N = M^*$ as seen from Table 6.

- With respect to the performance measure, FSR, the fraction of successful rate of retrials, we notice the following.

  1. FSR appears to decrease whenever either p increases or θ increases or ρ increases when all other parameters are fixed. This is as expected since the servers will be made more busy under these cases resulting in a fewer successful rate for orbiting customers.

  2. As c increases (by fixing all other parameters), FSR appears to increase for all arrival processes.

- With respect to the performance measure, MNO, the mean number of customers in the orbit, we observe that whenever p increases (or θ increases or c increases) by fixing all other parameters, MNO appears to decrease for the first four arrival processes. With respect to MPC arrivals, the above phenomenon doesn't appear to hold good. This indicates that we need to increase the cut-off points, M^* in the case of direct truncation and N^* in the case of matrix-geometric approximation. For example, compare the entries corresponding to the case when $c = 1$ and $\rho = 0.9$ in Tables 3A and 6.

- It is interesting to note that the performance measure, the probability PES, that an arriving customer will enter into service immediately with at least one customer waiting in the orbit, appears to be not significantly affected by increasing N from N^* to M^* in the case of MPC arrivals for $c = 1, 2$. When $c = 4, 8$, there appears to be a significant difference which tend to increase as ρ increases.(see Tables 4A, 4B and 6).

- We notice that when $c = 1$, the blocking probability appears not to depend on either p or θ for all five arrival processes. However, as c increases this measure changes depending on the arrival process, p, and θ among other parameters.

- As is to be expected, the system performance measures of the model under study approach to the corresponding measures in the classical $MAP/M/c$ queue whenever (a) $p \to 1$; or (b) $N \to \infty$; or (c) $\theta \to \infty$. The measures for classical $MAP/M/c$ are listed in Table 7 for comparison purposes. The convergence rate is faster for the matrix-geometric approximation compared to the direct truncation method. This is not surprising since the steady state probability vector in the classical $MAP/M/c$ queue is of matrix-geometric type.

- We have also noticed that the mean number of busy servers is independent of the values of θ, p and the truncation methods. This mean is given by $\frac{\lambda}{\mu}$.

Table 4.1A: Values of M^* and N^*

				Direct Truncation (M^*)					Matrix-geometric Approximation(N^*)				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
1	0.1	0.1	0.00	11	11	16	19	420	53	101	167	131	25
			0.50	11	11	11	12	388	35	67	109	95	10
			0.90	11	11	11	11	430	11	24	37	45	11
			0.99	11	11	11	11	433	10	10	10	14	10
		1	0.00	11	11	11	11	447	16	33	54	37	11
			0.50	11	11	11	11	562	11	23	38	27	11
			0.90	11	11	11	11	430	11	11	16	13	10
			0.99	11	11	11	11	518	10	10	10	11	10
		100.00	0.00	11	11	11	11	493	10	10	11	10	10
			0.50	11	11	11	11	527	10	10	11	10	10
			0.90	11	11	11	11	545	10	10	10	10	10
			0.99	11	11	11	11	548	10	10	10	10	10
	0.5	0.1	0.00	29	38	70	46	919	213	225	248	233	32
			0.50	19	25	47	27	833	146	154	171	163	20
			0.90	13	19	36	20	883	60	63	68	71	11
			0.99	13	17	31	17	903	10	10	10	20	10
		1.00	0.00	15	21	38	20	564	67	72	83	73	11
			0.50	15	21	40	20	944	47	51	58	51	11
			0.90	13	20	34	19	935	21	22	26	10	10
			0.99	13	18	33	18	945	10	10	10	11	10
		100.00	0.00	13	17	33	17	909	11	11	11	10	10
			0.50	13	17	36	17	927	11	11	11	11	10
			0.90	13	18	33	17	946	10	10	10	10	10
			0.99	13	17	34	17	932	10	10	10	10	10
	0.9	0.1	0.00	204	259	384	313	1885	341	301	179	305	96
			0.50	197	231	384	253	2091	236	208	125	213	45
			0.90	78	142	343	149	2222	98	87	52	92	11
			0.99	70	100	254	90	2223	10	10	10	25	11
		1.00	0.00	77	120	267	105	1638	108	96	60	96	14
			0.50	69	87	264	97	2277	76	68	43	68	11
			0.90	78	100	255	96	2297	34	30	19	31	11
			0.99	66	89	239	83	2298	10	10	10	11	10
		100.00	0.00	68	91	233	84	2299	12	11	11	11	10
			0.50	64	85	233	83	2361	11	11	11	11	10
			0.90	65	86	400	84	2410	11	10	10	10	10
			0.99	65	88	359	84	2393	10	10	10	10	10
2	0.1	0.1	0.00	11	11	11	11	509	17	47	92	67	23
			0.50	11	11	11	11	374	11	22	48	38	14
			0.90	11	11	11	11	503	11	11	11	11	11
			0.99	11	11	11	11	537	10	10	10	11	10
		1.00	0.00	11	11	11	11	518	11	20	35	23	11
			0.50	11	11	11	11	520	11	12	23	16	11
			0.90	11	11	11	11	522	10	11	11	11	10
			0.99	11	11	11	11	524	10	10	10	10	10
		100.00	0.00	11	11	11	11	524	10	10	11	10	10
			0.50	11	11	11	11	526	10	10	10	10	10
			0.90	11	11	11	11	528	10	10	10	10	10
			0.99	11	11	11	11	545	10	10	10	10	10
	0.5	0.1	0.00	19	25	49	27	1254	145	155	176	163	25
			0.50	14	20	51	21	926	97	104	118	112	10
			0.90	12	17	41	17	922	34	37	10	45	11
			0.99	12	16	32	17	919	10	10	10	11	10
		1.00	0.00	16	20	36	22	826	47	51	59	51	11
			0.50	12	16	32	18	922	33	35	41	36	11
			0.90	12	16	30	17	927	14	15	18	16	10
			0.99	12	16	32	16	923	10	10	10	11	10
		100.00	0.00	11	16	34	16	917	11	11	11	10	10
			0.50	12	16	31	16	929	10	11	11	11	10
			0.90	12	16	32	16	922	10	10	10	10	10
			0.99	11	16	34	16	920	10	10	10	10	10
	0.9	0.1	0.00	137	196	309	168	1940	240	213	130	217	51
			0.50	128	165	309	163	2039	164	146	89	150	22
			0.90	119	120	295	109	2087	66	59	35	63	11
			0.99	65	89	246	78	2089	10	10	10	16	10
		1.00	0.00	81	104	244	95	1608	76	68	43	68	11
			0.50	63	85	243	87	2145	54	48	30	48	11
			0.90	69	93	243	82	2147	10	21	14	22	10
			0.99	64	87	238	80	2149	10	10	10	11	10
		100.00	0.00	64	87	232	80	2150	11	11	11	10	10
			0.50	63	86	232	78	2174	11	11	10	11	10
			0.90	64	87	310	78	2201	10	10	10	10	10
			0.99	64	86	234	78	2203	10	10	10	10	10

Table 4.1B: Values of M^* and N^*

				Direct Truncation (M^*)					Matrix-geometric Approximation(N^*)				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
4	0.1	0.1	0.00	11	11	11	11	368	11	13	41	21	23
			0.50	11	11	11	11	348	11	11	11	11	13
			0.90	11	11	11	11	395	10	11	11	11	11
			0.99	11	11	11	11	404	10	10	10	10	10
		1.00	0.00	11	11	11	11	510	11	11	22	14	11
			0.50	11	11	11	11	516	11	11	13	11	10
			0.90	11	11	11	11	519	10	11	11	11	10
			0.99	11	11	11	11	522	10	10	10	10	10
		100.00	0.00	11	11	11	11	522	10	10	10	10	10
			0.50	11	11	11	11	525	10	10	10	10	10
			0.90	11	11	11	11	528	10	10	10	10	10
			0.99	11	11	11	11	537	10	10	10	10	10
	0.5	0.1	0.00	14	19	38	19	1254	98	106	124	114	21
			0.50	11	17	40	17	865	62	68	80	76	12
			0.90	11	16	32	15	856	16	20	24	27	11
			0.99	11	15	30	14	852	10	10	10	11	10
		1.00	0.00	11	21	32	17	732	33	36	42	36	11
			0.50	11	16	28	15	863	23	25	29	26	11
			0.90	11	16	30	14	855	11	11	12	11	10
			0.99	11	15	32	15	853	10	10	10	11	10
		100.00	0.00	11	15	31	14	846	11	11	11	10	10
			0.50	11	15	31	14	861	10	10	10	10	10
			0.90	11	15	29	15	855	10	10	10	10	10
			0.99	11	15	31	14	854	10	10	10	10	10
	0.9	0.1	0.00	98	193	260	240	1806	169	150	94	155	28
			0.50	91	110	266	131	2008	114	101	62	106	11
			0.90	81	106	262	96	2010	43	39	24	43	11
			0.99	63	86	240	78	2012	10	10	10	11	10
		1.00	0.00	74	96	236	87	1323	54	48	31	49	11
			0.50	62	84	236	81	2045	38	34	22	34	11
			0.90	66	87	240	79	2047	17	15	11	16	10
			0.99	63	85	236	77	2049	10	10	10	11	10
		100.00	0.00	63	85	231	77	2050	11	11	10	10	10
			0.50	62	83	231	76	2056	11	10	10	11	10
			0.90	62	84	283	76	2067	10	10	10	10	10
			0.99	62	108	258	82	2069	10	10	10	10	10
8	0.1	0.1	0.00	11	11	11	11	349	11	11	11	11	23
			0.50	11	11	11	11	319	11	11	11	11	13
			0.90	11	11	11	11	374	10	11	11	11	10
			0.99	11	11	11	11	379	10	10	10	10	10
		1.00	0.00	11	11	11	11	498	11	11	13	11	11
			0.50	11	11	11	11	500	10	11	11	11	10
			0.90	11	11	11	11	502	10	10	11	10	10
			0.99	11	11	11	11	500	10	10	10	10	10
		100.00	0.00	11	11	11	11	500	10	10	10	10	10
			0.50	11	11	11	11	506	10	10	10	10	10
			0.90	11	11	11	11	511	10	10	10	10	10
			0.99	11	11	11	11	506	10	10	10	10	10
	0.5	0.1	0.00	11	15	31	15	675	65	72	86	79	18
			0.50	11	17	30	17	725	39	44	53	50	11
			0.90	11	13	27	12	806	11	11	13	15	10
			0.99	11	13	26	13	803	10	10	10	11	10
		1.00	0.00	11	16	29	12	790	23	25	30	26	11
			0.50	11	14	26	13	808	16	17	20	18	10
			0.90	11	14	26	12	805	11	11	11	11	10
			0.99	11	13	28	12	801	10	10	10	10	10
		100.00	0.00	11	13	27	12	802	10	10	10	10	10
			0.50	11	13	26	13	812	10	10	10	10	10
			0.90	11	13	27	13	804	10	10	10	10	10
			0.99	11	13	26	12	802	10	10	10	10	10
	0.9	0.1	0.00	74	150	233	137	1786	118	106	68	110	19
			0.50	68	85	241	103	1854	78	70	44	74	11
			0.90	63	83	241	85	1857	27	25	16	29	11
			0.99	60	82	235	76	1859	10	10	10	11	10
		1.00	0.00	75	90	231	83	1150	39	34	22	35	11
			0.50	60	82	231	77	1891	27	24	10	25	10
			0.90	60	81	235	76	1893	12	11	11	11	10
			0.99	60	148	233	75	1895	10	10	10	11	10
		100.00	0.00	60	85	229	75	1896	11	11	10	10	10
			0.50	60	83	229	74	1898	10	10	10	10	10
			0.90	59	82	256	74	1901	10	10	10	10	10
			0.99	62	84	243	74	1903	10	10	10	10	10

Table 4.2A: Fraction of successful rate of retrials

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
1	0.1	0.1	0.00	0.893	0.891	0.876	0.895	0.482	0.893	0.891	0.876	0.895	0.765
			0.50	0.811	0.815	0.797	0.867	0.314	0.811	0.815	0.797	0.867	0.759
			0.90	0.462	0.471	0.444	0.614	0.081	0.462	0.471	0.444	0.614	0.367
			0.99	0.079	0.082	0.074	0.140	0.009	0.000	0.000	0.000	0.140	0.057
		1	0.00	0.835	0.818	0.770	0.843	0.092	0.835	0.818	0.770	0.843	0.457
			0.50	0.719	0.697	0.636	0.755	0.046	0.719	0.697	0.636	0.755	0.296
			0.90	0.339	0.317	0.263	0.405	0.010	0.339	0.317	0.263	0.405	0.084
			0.99	0.049	0.045	0.035	0.065	0.001	0.000	0.000	0.000	0.065	0.009
		100.00	0.00	0.088	0.082	0.075	0.087	0.001	0.088	0.082	0.075	0.087	0.010
			0.50	0.046	0.043	0.039	0.045	0.001	0.046	0.043	0.039	0.045	0.005
			0.90	0.010	0.009	0.008	0.009	0.000	0.010	0.009	0.008	0.009	0.001
			0.99	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.001	0.000
	0.5	0.1	0.00	0.483	0.476	0.430	0.481	0.116	0.483	0.476	0.430	0.481	0.349
			0.50	0.391	0.399	0.352	0.441	0.059	0.391	0.399	0.352	0.441	0.281
			0.90	0.122	0.133	0.122	0.186	0.012	0.122	0.133	0.122	0.186	0.105
			0.99	0.014	0.015	0.014	0.024	0.001	0.000	0.000	0.000	0.024	0.012
		1.00	0.00	0.366	0.333	0.223	0.354	0.015	0.366	0.333	0.224	0.354	0.148
			0.50	0.243	0.219	0.140	0.249	0.007	0.243	0.219	0.140	0.249	0.080
			0.90	0.064	0.057	0.035	0.071	0.001	0.064	0.057	0.035	0.071	0.019
			0.99	0.007	0.006	0.004	0.008	0.000	0.000	0.000	0.000	0.008	0.002
		100.00	0.00	0.012	0.010	0.005	0.010	0.000	0.012	0.010	0.006	0.010	0.002
			0.50	0.006	0.005	0.003	0.005	0.000	0.006	0.005	0.003	0.005	0.001
			0.90	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	0.1	0.00	0.093	0.090	0.079	0.091	0.023	0.093	0.091	0.078	0.090	0.099
			0.50	0.086	0.082	0.061	0.083	0.010	0.086	0.083	0.063	0.084	0.099
			0.90	0.039	0.037	0.022	0.042	0.002	0.039	0.037	0.025	0.042	0.061
			0.99	0.005	0.004	0.003	0.006	0.000	0.000	0.000	0.000	0.006	0.006
		1.00	0.00	0.060	0.053	0.027	0.054	0.003	0.060	0.052	0.029	0.053	0.071
			0.50	0.040	0.034	0.015	0.035	0.001	0.040	0.034	0.018	0.035	0.046
			0.90	0.011	0.009	0.003	0.009	0.000	0.011	0.009	0.006	0.009	0.010
			0.99	0.001	0.001	0.000	0.001	0.000	0.000	0.000	0.000	0.001	0.001
		100.00	0.00	0.001	0.001	0.000	0.001	0.000	0.002	0.001	0.001	0.001	0.001
			0.50	0.001	0.001	0.000	0.001	0.000	0.001	0.001	0.001	0.001	0.001
			0.90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.1	0.1	0.00	0.981	0.972	0.959	0.944	0.547	0.981	0.972	0.959	0.944	0.901
			0.50	0.821	0.818	0.798	0.830	0.354	0.821	0.818	0.798	0.830	0.847
			0.90	0.214	0.217	0.207	0.269	0.080	0.214	0.217	0.207	0.269	0.422
			0.99	0.018	0.018	0.017	0.022	0.008	0.000	0.000	0.000	0.022	0.057
		1.00	0.00	0.896	0.881	0.854	0.874	0.093	0.896	0.881	0.854	0.874	0.542
			0.50	0.678	0.658	0.615	0.685	0.047	0.678	0.658	0.615	0.685	0.335
			0.90	0.165	0.159	0.145	0.182	0.010	0.165	0.159	0.145	0.182	0.087
			0.99	0.016	0.015	0.014	0.017	0.001	0.000	0.000	0.000	0.017	0.009
		100.00	0.00	0.088	0.082	0.076	0.079	0.001	0.088	0.082	0.076	0.079	0.010
			0.50	0.045	0.042	0.039	0.040	0.001	0.045	0.042	0.039	0.040	0.005
			0.90	0.009	0.009	0.008	0.008	0.000	0.009	0.009	0.008	0.008	0.001
			0.99	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.001	0.000
	0.5	0.1	0.00	0.677	0.656	0.576	0.651	0.112	0.677	0.656	0.576	0.651	0.466
			0.50	0.450	0.461	0.415	0.504	0.059	0.450	0.461	0.415	0.504	0.457
			0.90	0.100	0.109	0.104	0.141	0.012	0.100	0.109	0.106	0.141	0.111
			0.99	0.010	0.011	0.010	0.015	0.001	0.000	0.000	0.000	0.015	0.012
		1.00	0.00	0.444	0.400	0.268	0.420	0.014	0.444	0.400	0.268	0.420	0.158
			0.50	0.253	0.229	0.151	0.253	0.007	0.253	0.229	0.151	0.253	0.083
			0.90	0.055	0.050	0.033	0.058	0.001	0.055	0.050	0.033	0.058	0.019
			0.99	0.006	0.005	0.003	0.006	0.000	0.000	0.000	0.000	0.006	0.002
		100.00	0.00	0.012	0.010	0.006	0.010	0.000	0.012	0.010	0.006	0.010	0.002
			0.50	0.006	0.005	0.003	0.005	0.000	0.006	0.005	0.003	0.005	0.001
			0.90	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	0.1	0.00	0.161	0.153	0.120	0.157	0.023	0.161	0.155	0.120	0.155	0.178
			0.50	0.135	0.127	0.084	0.132	0.011	0.135	0.127	0.087	0.132	0.181
			0.90	0.041	0.040	0.023	0.047	0.002	0.042	0.040	0.030	0.047	0.063
			0.99	0.004	0.004	0.003	0.005	0.000	0.000	0.000	0.000	0.006	0.007
		1.00	0.00	0.080	0.068	0.031	0.069	0.002	0.080	0.068	0.036	0.069	0.092
			0.50	0.047	0.039	0.017	0.041	0.001	0.047	0.039	0.023	0.041	0.047
			0.90	0.011	0.009	0.004	0.009	0.000	0.013	0.009	0.008	0.010	0.010
			0.99	0.001	0.001	0.000	0.001	0.000	0.000	0.000	0.000	0.001	0.001
		100.00	0.00	0.001	0.001	0.000	0.001	0.000	0.002	0.001	0.001	0.001	0.001
			0.50	0.001	0.001	0.000	0.001	0.000	0.001	0.001	0.001	0.001	0.001
			0.90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 4.2B: Fraction of successful rate of retrials

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
4	0.1	0.1	0.00	0.989	0.988	0.984	0.988	0.627	0.989	0.988	0.984	0.988	0.968
			0.50	0.697	0.703	0.693	0.741	0.385	0.701	0.703	0.693	0.741	0.940
			0.90	0.129	0.129	0.127	0.144	0.084	0.128	0.129	0.127	0.144	0.463
			0.99	0.013	0.013	0.012	0.014	0.008	0.000	0.000	0.000	0.014	0.057
		1.00	0.00	0.903	0.895	0.881	0.901	0.095	0.903	0.895	0.881	0.901	0.615
			0.50	0.571	0.561	0.538	0.589	0.047	0.571	0.561	0.538	0.589	0.393
			0.90	0.113	0.112	0.107	0.122	0.010	0.113	0.112	0.107	0.122	0.089
			0.99	0.011	0.011	0.011	0.012	0.001	0.000	0.000	0.000	0.012	0.009
		100.00	0.00	0.088	0.082	0.076	0.082	0.001	0.088	0.082	0.076	0.082	0.010
			0.50	0.045	0.041	0.038	0.041	0.001	0.045	0.041	0.038	0.041	0.005
			0.90	0.009	0.008	0.008	0.008	0.000	0.009	0.008	0.008	0.008	0.001
			0.99	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.001	0.000
	0.5	0.1	0.00	0.818	0.791	0.689	0.795	0.115	0.818	0.791	0.689	0.795	0.579
			0.50	0.453	0.468	0.432	0.515	0.060	0.453	0.468	0.432	0.515	0.469
			0.90	0.089	0.097	0.094	0.122	0.012	0.089	0.097	0.094	0.122	0.114
			0.99	0.009	0.010	0.009	0.013	0.001	0.000	0.000	0.000	0.013	0.012
		1.00	0.00	0.484	0.438	0.297	0.462	0.014	0.484	0.438	0.297	0.462	0.165
			0.50	0.250	0.228	0.154	0.251	0.007	0.250	0.228	0.154	0.251	0.084
			0.90	0.051	0.046	0.032	0.053	0.001	0.051	0.046	0.032	0.053	0.019
			0.99	0.005	0.005	0.003	0.005	0.000	0.000	0.000	0.000	0.005	0.002
		100.00	0.00	0.012	0.010	0.006	0.010	0.000	0.012	0.010	0.006	0.010	0.002
			0.50	0.006	0.005	0.003	0.005	0.000	0.006	0.005	0.003	0.005	0.001
			0.90	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	0.1	0.00	0.252	0.234	0.163	0.236	0.025	0.251	0.236	0.166	0.240	0.300
			0.50	0.177	0.166	0.101	0.177	0.011	0.178	0.166	0.110	0.178	0.301
			0.90	0.043	0.041	0.024	0.049	0.002	0.043	0.042	0.037	0.050	0.064
			0.99	0.004	0.004	0.003	0.005	0.000	0.000	0.000	0.000	0.007	0.007
		1.00	0.00	0.095	0.078	0.033	0.081	0.003	0.095	0.078	0.044	0.081	0.094
			0.50	0.051	0.042	0.017	0.044	0.001	0.051	0.042	0.028	0.044	0.048
			0.90	0.011	0.009	0.004	0.009	0.000	0.011	0.010	0.010	0.011	0.011
			0.99	0.001	0.001	0.000	0.001	0.000	0.000	0.000	0.000	0.001	0.001
		100.00	0.00	0.001	0.001	0.000	0.001	0.000	0.002	0.001	0.001	0.001	0.001
			0.50	0.001	0.001	0.000	0.001	0.000	0.001	0.001	0.001	0.001	0.001
			0.90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.1	0.1	0.00	1.000	0.990	0.988	0.990	0.672	1.000	0.990	0.988	0.990	0.986
			0.50	0.985	0.593	0.571	0.601	0.406	0.992	0.601	0.572	0.601	0.971
			0.90	0.399	0.147	0.114	0.129	0.084	0.110	0.111	0.111	0.121	0.515
			0.99	0.011	0.013	0.011	0.014	0.009	0.000	0.000	0.000	0.012	0.057
		1.00	0.00	0.663	0.896	0.888	0.903	0.096	0.975	0.899	0.888	0.903	0.675
			0.50	0.558	0.631	0.483	0.516	0.048	0.507	0.495	0.483	0.516	0.417
			0.90	0.113	0.126	0.096	0.110	0.010	0.099	0.099	0.096	0.106	0.090
			0.99	0.010	0.011	0.010	0.012	0.001	0.000	0.000	0.000	0.011	0.009
		100.00	0.00	0.084	0.082	0.076	0.082	0.001	0.088	0.082	0.076	0.082	0.010
			0.50	0.046	0.041	0.038	0.041	0.001	0.044	0.041	0.038	0.041	0.005
			0.90	0.009	0.009	0.008	0.008	0.000	0.009	0.008	0.008	0.008	0.001
			0.99	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.001	0.000
	0.5	0.1	0.00	0.883	0.862	0.764	0.872	0.125	0.883	0.862	0.764	0.871	0.691
			0.50	0.439	0.455	0.427	0.501	0.061	0.439	0.455	0.426	0.501	0.530
			0.90	0.084	0.091	0.090	0.113	0.012	0.084	0.091	0.090	0.113	0.122
			0.99	0.008	0.009	0.009	0.012	0.001	0.000	0.000	0.000	0.012	0.012
		1.00	0.00	0.502	0.457	0.314	0.482	0.014	0.502	0.457	0.314	0.481	0.168
			0.50	0.247	0.226	0.156	0.248	0.007	0.247	0.226	0.156	0.248	0.093
			0.90	0.049	0.045	0.031	0.051	0.001	0.049	0.045	0.032	0.051	0.019
			0.99	0.005	0.004	0.003	0.005	0.000	0.000	0.000	0.000	0.005	0.002
		100.00	0.00	0.012	0.010	0.006	0.010	0.000	0.012	0.010	0.006	0.010	0.002
			0.50	0.006	0.005	0.003	0.005	0.000	0.006	0.005	0.003	0.005	0.001
			0.90	0.001	0.001	0.001	0.001	0.000	0.001	0.001	0.001	0.001	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	0.1	0.00	0.345	0.315	0.199	0.324	0.026	0.342	0.315	0.210	0.325	0.417
			0.50	0.203	0.190	0.112	0.208	0.011	0.203	0.191	0.132	0.209	0.315
			0.90	0.043	0.042	0.024	0.050	0.002	0.044	0.044	0.047	0.052	0.065
			0.99	0.004	0.004	0.003	0.005	0.000	0.000	0.000	0.000	0.007	0.007
		1.00	0.00	0.104	0.084	0.035	0.088	0.003	0.104	0.085	0.056	0.089	0.095
			0.50	0.053	0.043	0.017	0.045	0.001	0.054	0.045	0.052	0.047	0.052
			0.90	0.011	0.009	0.004	0.009	0.000	0.012	0.012	0.010	0.012	0.011
			0.99	0.001	0.001	0.000	0.001	0.000	0.000	0.000	0.000	0.001	0.001
		100.00	0.00	0.001	0.001	0.000	0.001	0.000	0.002	0.001	0.001	0.001	0.001
			0.50	0.001	0.001	0.000	0.001	0.000	0.001	0.001	0.001	0.001	0.001
			0.90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
			0.99	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 4.3A: Mean Number of Customers in Orbit

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
1	0.1	0.1	0.00	0.35	1.12	1.83	5.68	10.40	0.35	1.12	1.83	5.68	35.12
			0.50	0.03	0.12	0.21	1.22	7.28	0.03	0.12	0.21	1.22	33.77
			0.90	0.01	0.02	0.04	0.16	5.38	0.01	0.02	0.04	0.16	8.89
			0.99	0.00	0.01	0.02	0.06	4.96	0.00	0.01	0.02	0.06	5.62
		1	0.00	0.04	0.12	0.22	0.60	5.51	0.04	0.12	0.22	0.60	10.72
			0.50	0.01	0.05	0.08	0.28	5.44	0.01	0.05	0.08	0.28	7.85
			0.90	0.01	0.02	0.03	0.09	4.97	0.01	0.02	0.03	0.09	5.81
			0.99	0.00	0.01	0.02	0.06	5.10	0.00	0.01	0.02	0.06	5.32
		100.00	0.00	0.00	0.01	0.02	0.06	5.06	0.00	0.01	0.02	0.06	5.33
			0.50	0.00	0.01	0.02	0.05	5.12	0.00	0.01	0.02	0.05	5.30
			0.90	0.00	0.01	0.02	0.05	5.13	0.00	0.01	0.02	0.05	5.27
			0.99	0.00	0.01	0.02	0.05	5.13	0.00	0.01	0.02	0.05	5.27
	0.5	0.1	0.00	8.56	10.50	13.99	14.91	61.29	8.56	10.50	14.00	14.92	163.98
			0.50	1.41	2.48	5.52	5.00	54.56	1.41	2.48	5.52	5.00	112.70
			0.90	0.41	0.68	1.78	1.01	49.85	0.41	0.68	1.78	1.01	62.62
			0.99	0.32	0.52	1.34	0.67	48.82	0.31	0.50	1.29	0.67	50.67
		1.00	0.00	1.11	1.50	2.95	1.99	45.82	1.11	1.50	2.95	1.99	70.16
			0.50	0.59	0.89	2.04	1.21	49.28	0.59	0.89	2.04	1.21	58.80
			0.90	0.35	0.56	1.43	0.74	49.04	0.35	0.56	1.43	0.74	51.33
			0.99	0.31	0.51	1.31	0.65	48.90	0.31	0.50	1.29	0.65	49.61
		100.00	0.00	0.32	0.51	1.32	0.65	18.72	0.32	0.51	1.32	0.65	49.62
			0.50	0.31	0.51	1.30	0.65	48.97	0.31	0.51	1.30	0.65	49.52
			0.90	0.31	0.50	1.30	0.64	48.98	0.31	0.50	1.30	0.64	49.44
			0.99	0.31	0.50	1.29	0.64	48.85	0.31	0.50	1.29	0.64	49.42
	0.9	0.1	0.00	96.30	99.99	115.76	103.74	397.71	96.64	99.22	124.93	105.04	33948.41
			0.50	45.10	48.67	68.80	53.41	436.41	44.81	48.39	70.62	53.16	47076.72
			0.90	9.67	13.01	32.39	14.66	421.23	9.68	12.85	33.03	14.40	1165.34
			0.99	6.24	8.52	25.31	8.84	417.62	5.93	8.10	24.49	8.83	480.73
		1.00	0.00	14.68	17.11	34.66	17.60	293.19	14.83	17.15	35.80	17.70	1547.24
			0.50	9.96	12.32	29.76	12.80	409.31	9.95	12.31	30.55	12.76	830.96
			0.90	6.66	8.89	25.65	9.18	422.33	6.63	8.88	26.26	9.16	496.89
			0.99	6.01	8.19	24.62	8.40	421.62	5.93	8.10	24.49	8.42	454.32
		100.00	0.00	6.01	8.19	24.57	8.40	416.45	6.03	8.22	24.78	8.43	454.58
			0.50	5.98	8.16	24.61	8.38	427.67	5.98	8.16	24.63	8.38	452.04
			0.90	5.95	8.12	26.74	8.34	428.63	5.94	8.11	24.52	8.33	450.03
			0.99	5.94	8.11	25.03	8.32	426.20	5.93	8.10	24.49	8.32	449.58
2	0.1	0.1	0.00	0.03	0.17	0.45	0.36	8.87	0.03	0.17	0.45	0.36	29.49
			0.50	0.00	0.01	0.03	0.03	6.23	0.00	0.01	0.03	0.03	19.77
			0.90	0.00	0.00	0.01	0.01	5.29	0.00	0.00	0.01	0.01	7.34
			0.99	0.00	0.00	0.01	0.00	5.09	0.00	0.00	0.01	0.00	5.39
		1.00	0.00	0.00	0.02	0.05	0.04	5.36	0.00	0.02	0.05	0.04	8.62
			0.50	0.00	0.01	0.02	0.01	5.20	0.00	0.01	0.02	0.01	6.70
			0.90	0.00	0.00	0.01	0.01	5.08	0.00	0.00	0.01	0.01	5.50
			0.99	0.00	0.00	0.01	0.00	5.06	0.00	0.00	0.01	0.00	5.24
		100.00	0.00	0.00	0.00	0.01	0.00	5.05	0.00	0.00	0.01	0.00	5.25
			0.50	0.00	0.00	0.01	0.00	5.07	0.00	0.00	0.01	0.00	5.23
			0.90	0.00	0.00	0.01	0.00	5.07	0.00	0.00	0.01	0.00	5.22
			0.99	0.00	0.00	0.01	0.00	5.08	0.00	0.00	0.01	0.00	5.22
	0.5	0.1	0.00	3.08	4.50	7.84	6.27	58.62	3.08	4.50	7.84	6.28	117.66
			0.50	0.50	0.94	2.86	1.36	52.01	0.50	0.94	2.86	1.36	113.01
			0.90	0.22	0.39	1.25	0.47	49.22	0.22	0.39	1.26	0.47	55.66
			0.99	0.19	0.34	1.08	0.39	48.58	0.19	0.33	1.06	0.39	49.75
		1.00	0.00	0.49	0.77	1.93	0.94	49.43	0.49	0.77	1.93	0.94	59.32
			0.50	0.29	0.49	1.42	0.59	48.87	0.29	0.49	1.42	0.59	53.79
			0.90	0.20	0.36	1.12	0.41	48.60	0.20	0.36	1.12	0.41	50.09
			0.99	0.19	0.34	1.07	0.38	48.52	0.19	0.33	1.06	0.38	49.23
		100.00	0.00	0.19	0.34	1.07	0.38	48.46	0.19	0.34	1.07	0.38	49.23
			0.50	0.19	0.34	1.07	0.38	48.56	0.19	0.34	1.07	0.38	49.18
			0.90	0.19	0.33	1.06	0.38	48.60	0.19	0.33	1.06	0.38	49.14
			0.99	0.19	0.33	1.06	0.38	48.52	0.19	0.33	1.06	0.38	49.13
	0.9	0.1	0.00	49.46	53.53	71.37	56.56	401.68	49.62	52.95	74.80	57.32	13870.33
			0.50	21.78	25.77	46.09	29.20	414.44	21.65	25.62	47.07	29.08	32951.12
			0.90	7.28	9.89	27.90	10.64	413.37	7.19	9.81	28.67	10.51	666.79
			0.99	5.72	7.88	24.35	8.04	410.62	5.56	7.67	23.93	8.06	465.08
		1.00	0.00	9.98	12.28	29.27	12.66	382.05	10.02	12.31	30.33	12.70	862.35
			0.50	7.56	9.82	26.70	10.11	401.97	7.55	9.81	27.51	10.08	591.13
			0.90	5.94	8.09	24.57	8.25	415.72	5.99	8.09	25.03	8.24	473.96
			0.99	5.61	7.72	24.00	7.83	414.65	5.56	7.67	23.93	7.85	451.35
		100.00	0.00	5.61	7.72	23.95	7.83	408.87	5.62	7.73	24.07	7.86	451.48
			0.50	5.60	7.71	24.02	7.83	419.90	5.59	7.70	24.00	7.83	450.22
			0.90	5.58	7.69	24.04	7.81	419.42	5.57	7.68	23.94	7.80	449.22
			0.99	5.57	7.68	23.92	7.79	417.44	5.56	7.67	23.93	7.80	449.00

Table 4.3B: Mean Number of Customers in Orbit

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
4	0.1	0.1	0.00	0.00	0.01	0.04	0.02	7.37	0.00	0.01	0.04	0.02	25.82
			0.50	0.00	0.00	0.00	0.00	5.45	0.00	0.00	0.00	0.00	17.88
			0.90	0.00	0.00	0.00	0.00	4.78	0.00	0.00	0.00	0.00	6.34
			0.99	0.00	0.00	0.00	0.00	4.70	0.00	0.00	0.00	0.00	5.21
		1.00	0.00	0.00	0.00	0.01	0.00	5.12	0.00	0.00	0.01	0.00	7.28
			0.50	0.00	0.00	0.00	0.00	5.03	0.00	0.00	0.00	0.00	6.06
			0.90	0.00	0.00	0.00	0.00	4.97	0.00	0.00	0.00	0.00	5.26
			0.99	0.00	0.00	0.00	0.00	4.96	0.00	0.00	0.00	0.00	5.13
		100.00	0.00	0.00	0.00	0.00	0.00	4.96	0.00	0.00	0.00	0.00	5.13
			0.50	0.00	0.00	0.00	0.00	4.97	0.00	0.00	0.00	0.00	5.13
			0.90	0.00	0.00	0.00	0.00	4.97	0.00	0.00	0.00	0.00	5.12
			0.99	0.00	0.00	0.00	0.00	4.97	0.00	0.00	0.00	0.00	5.12
	0.5	0.1	0.00	0.92	1.65	4.26	2.12	54.56	0.92	1.65	4.26	2.12	91.67
			0.50	0.16	0.35	1.47	0.44	49.64	0.16	0.35	1.47	0.44	73.98
			0.90	0.09	0.19	0.84	0.22	48.10	0.09	0.19	0.84	0.22	51.88
			0.99	0.08	0.18	0.77	0.20	47.73	0.08	0.17	0.76	0.20	48.93
		1.00	0.00	0.18	0.34	1.19	0.40	47.53	0.18	0.34	1.19	0.40	53.72
			0.50	0.11	0.23	0.93	0.27	47.95	0.11	0.23	0.93	0.27	50.95
			0.90	0.09	0.18	0.79	0.21	47.81	0.09	0.18	0.79	0.21	49.10
			0.99	0.08	0.18	0.77	0.19	47.70	0.08	0.17	0.76	0.19	48.67
		100.00	0.00	0.08	0.18	0.77	0.20	47.62	0.08	0.18	0.77	0.20	48.67
			0.50	0.08	0.18	0.77	0.19	48.07	0.08	0.18	0.77	0.19	48.65
			0.90	0.08	0.17	0.77	0.19	47.78	0.08	0.17	0.77	0.19	48.63
			0.99	0.08	0.17	0.76	0.19	47.71	0.08	0.17	0.76	0.19	48.62
	0.9	0.1	0.00	26.02	30.00	48.07	33.15	314.58	26.12	29.68	50.35	32.61	8219.68
			0.50	11.56	14.97	34.28	16.81	402.16	11.50	14.91	35.26	16.71	7985.05
			0.90	5.87	8.15	25.08	8.53	409.14	5.83	8.11	25.82	8.46	539.22
			0.99	5.15	7.20	23.26	7.29	406.28	5.07	7.09	23.04	7.33	455.93
		1.00	0.00	7.33	9.50	25.89	9.74	346.26	7.35	9.52	26.83	9.77	599.23
			0.50	6.09	8.21	24.54	8.39	395.44	6.08	8.20	25.20	8.38	511.10
			0.90	5.27	7.31	23.42	7.42	409.19	5.26	7.32	23.70	7.42	460.21
			0.99	5.10	7.12	23.08	7.19	408.11	5.07	7.09	23.04	7.20	449.18
		100.00	0.00	5.09	7.11	23.04	7.19	401.88	5.10	7.12	23.12	7.21	449.24
			0.50	5.09	7.12	23.12	7.20	413.25	5.08	7.11	23.08	7.19	448.61
			0.90	5.08	7.10	24.29	7.19	412.00	5.07	7.09	23.05	7.18	448.11
			0.99	5.08	7.10	23.24	7.18	409.65	5.07	7.09	23.04	7.18	448.00
8	0.1	0.1	0.00	0.00	0.00	0.00	0.00	6.55	0.00	0.00	0.00	0.00	24.03
			0.50	0.00	0.00	0.00	0.00	4.77	0.00	0.00	0.00	0.00	15.97
			0.90	0.00	0.00	0.00	0.00	4.47	0.00	0.00	0.00	0.00	5.68
			0.99	0.00	0.00	0.00	0.00	4.43	0.00	0.00	0.00	0.00	4.97
		1.00	0.00	0.00	0.00	0.00	0.00	4.85	0.00	0.00	0.00	0.00	6.30
			0.50	0.00	0.00	0.00	0.00	4.79	0.00	0.00	0.00	0.00	5.45
			0.90	0.00	0.00	0.00	0.00	4.75	0.00	0.00	0.00	0.00	5.00
			0.99	0.00	0.00	0.00	0.00	4.74	0.00	0.00	0.00	0.00	4.94
		100.00	0.00	0.00	0.00	0.00	0.00	4.74	0.00	0.00	0.00	0.00	4.94
			0.50	0.00	0.00	0.00	0.00	4.75	0.00	0.00	0.00	0.00	4.93
			0.90	0.00	0.00	0.00	0.00	4.76	0.00	0.00	0.00	0.00	4.93
			0.99	0.00	0.00	0.00	0.00	4.75	0.00	0.00	0.00	0.00	4.93
	0.5	0.1	0.00	0.18	0.44	1.96	0.53	47.82	0.18	0.44	1.96	0.53	77.34
			0.50	0.04	0.10	0.69	0.12	47.22	0.04	0.10	0.69	0.12	62.30
			0.90	0.02	0.06	0.47	0.07	46.63	0.02	0.06	0.47	0.07	49.39
			0.99	0.02	0.06	0.45	0.07	46.44	0.02	0.06	0.44	0.07	47.81
		1.00	0.00	0.04	0.10	0.62	0.12	46.99	0.04	0.10	0.62	0.12	50.22
			0.50	0.03	0.07	0.51	0.08	46.60	0.03	0.07	0.51	0.08	48.94
			0.90	0.02	0.06	0.45	0.07	46.53	0.02	0.06	0.45	0.07	47.89
			0.99	0.02	0.06	0.44	0.07	46.42	0.02	0.06	0.44	0.07	47.67
		100.00	0.00	0.02	0.06	0.44	0.07	46.38	0.02	0.06	0.45	0.07	47.68
			0.50	0.02	0.06	0.44	0.07	46.54	0.02	0.06	0.44	0.07	47.66
			0.90	0.02	0.06	0.44	0.07	46.50	0.02	0.06	0.44	0.07	47.65
			0.99	0.02	0.06	0.44	0.07	46.42	0.02	0.06	0.44	0.07	47.65
	0.9	0.1	0.00	14.16	17.64	35.45	19.26	332.43	14.40	17.63	37.13	19.24	2113.37
			0.50	7.19	9.87	27.50	10.73	386.63	7.17	9.85	28.43	10.65	951.28
			0.90	4.81	6.83	22.79	7.04	395.51	4.79	6.83	23.44	7.01	488.39
			0.99	4.47	6.37	21.83	6.43	393.53	4.42	6.31	21.71	6.45	450.13
		1.00	0.00	5.63	7.65	23.24	7.74	318.89	5.60	7.61	24.06	7.77	512.94
			0.50	4.96	6.91	22.54	7.02	384.68	4.94	6.91	23.67	7.02	479.23
			0.90	4.53	6.43	21.95	6.51	397.34	4.53	6.45	22.04	6.52	452.23
			0.99	4.44	6.44	21.74	6.38	396.14	4.42	6.31	21.71	6.39	446.78
		100.00	0.00	4.43	6.32	21.70	6.37	389.50	4.44	6.33	21.75	6.39	446.81
			0.50	4.44	6.33	21.79	6.39	400.52	4.43	6.32	21.73	6.39	446.50
			0.90	4.43	6.33	22.40	6.38	399.39	4.43	6.32	21.72	6.38	446.25
			0.99	4.43	6.32	21.82	6.36	397.08	4.42	6.31	21.71	6.38	446.19

Table 4.4A: P(an arrival enters into service immediately with at least one customer in orbit)

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
1	0.1	0.1	0.00	0.991	0.931	0.864	0.493	0.498	0.991	0.931	0.864	0.493	0.497
			0.50	0.998	0.982	0.949	0.693	0.491	0.998	0.982	0.949	0.693	0.491
			0.90	0.999	0.989	0.968	0.937	0.484	0.999	0.989	0.968	0.937	0.484
			0.99	0.999	0.990	0.971	0.976	0.482	0.999	0.990	0.971	0.976	0.481
		1	0.00	0.998	0.981	0.940	0.812	0.496	0.998	0.981	0.940	0.812	0.499
			0.50	0.999	0.987	0.961	0.905	0.489	0.999	0.987	0.961	0.905	0.490
			0.90	0.999	0.990	0.970	0.966	0.483	0.999	0.990	0.970	0.966	0.483
			0.99	0.999	0.990	0.971	0.979	0.481	0.999	0.990	0.971	0.979	0.481
		100.00	0.00	0.999	0.990	0.971	0.979	0.483	0.999	0.990	0.971	0.979	0.485
			0.50	0.999	0.990	0.971	0.980	0.482	0.999	0.990	0.971	0.980	0.483
			0.90	0.999	0.990	0.971	0.980	0.481	0.999	0.990	0.971	0.980	0.482
			0.99	0.999	0.990	0.971	0.980	0.481	0.999	0.990	0.971	0.980	0.481
	0.5	0.1	0.00	0.587	0.500	0.400	0.282	0.299	0.587	0.500	0.400	0.282	0.291
			0.50	0.742	0.599	0.387	0.344	0.323	0.742	0.599	0.387	0.344	0.314
			0.90	0.840	0.727	0.455	0.636	0.331	0.840	0.727	0.455	0.636	0.329
			0.99	0.853	0.748	0.477	0.702	0.332	0.854	0.750	0.480	0.702	0.332
		1.00	0.00	0.730	0.625	0.411	0.449	0.331	0.730	0.625	0.410	0.449	0.324
			0.50	0.806	0.695	0.437	0.578	0.331	0.806	0.695	0.437	0.578	0.328
			0.90	0.846	0.740	0.470	0.683	0.332	0.846	0.740	0.470	0.683	0.332
			0.99	0.853	0.749	0.479	0.706	0.333	0.854	0.750	0.480	0.706	0.332
		100.00	0.00	0.853	0.748	0.478	0.706	0.333	0.853	0.748	0.478	0.706	0.333
			0.50	0.853	0.749	0.479	0.707	0.333	0.853	0.749	0.479	0.707	0.333
			0.90	0.854	0.750	0.480	0.708	0.332	0.854	0.750	0.480	0.708	0.332
			0.99	0.854	0.750	0.480	0.709	0.333	0.854	0.750	0.480	0.709	0.332
	0.9	0.1	0.00	0.107	0.099	0.090	0.055	0.066	0.107	0.099	0.089	0.054	0.055
			0.50	0.114	0.100	0.079	0.056	0.072	0.114	0.100	0.080	0.056	0.055
			0.90	0.193	0.147	0.068	0.107	0.081	0.193	0.147	0.068	0.107	0.069
			0.99	0.241	0.185	0.069	0.166	0.083	0.246	0.190	0.070	0.166	0.081
		1.00	0.00	0.120	0.109	0.075	0.067	0.090	0.119	0.109	0.075	0.067	0.062
			0.50	0.160	0.135	0.071	0.098	0.082	0.160	0.135	0.072	0.098	0.070
			0.90	0.226	0.176	0.070	0.154	0.083	0.226	0.176	0.070	0.155	0.080
			0.99	0.244	0.189	0.070	0.172	0.083	0.246	0.190	0.070	0.172	0.082
		100.00	0.00	0.243	0.188	0.070	0.172	0.084	0.243	0.188	0.070	0.172	0.082
			0.50	0.244	0.189	0.070	0.173	0.083	0.244	0.189	0.070	0.173	0.082
			0.90	0.246	0.190	0.069	0.174	0.083	0.246	0.190	0.070	0.174	0.082
			0.99	0.246	0.190	0.069	0.174	0.083	0.246	0.190	0.070	0.174	0.082
2	0.1	0.1	0.00	1.000	0.996	0.980	0.987	0.515	1.000	0.996	0.980	0.987	0.521
			0.50	1.000	0.998	0.991	0.994	0.504	1.000	0.998	0.991	0.994	0.511
			0.90	1.000	0.998	0.992	0.995	0.491	1.000	0.998	0.992	0.995	0.495
			0.99	1.000	0.998	0.992	0.995	0.487	1.000	0.998	0.992	0.995	0.487
		1.00	0.00	1.000	0.998	0.990	0.994	0.504	1.000	0.998	0.990	0.994	0.513
			0.50	1.000	0.998	0.991	0.995	0.496	1.000	0.998	0.991	0.995	0.500
			0.90	1.000	0.998	0.992	0.995	0.488	1.000	0.998	0.992	0.995	0.489
			0.99	1.000	0.998	0.992	0.995	0.486	1.000	0.998	0.992	0.995	0.486
		100.00	0.00	1.000	0.998	0.992	0.995	0.488	1.000	0.998	0.992	0.995	0.490
			0.50	1.000	0.998	0.992	0.995	0.487	1.000	0.998	0.992	0.995	0.488
			0.90	1.000	0.998	0.992	0.995	0.486	1.000	0.998	0.992	0.995	0.487
			0.99	1.000	0.998	0.992	0.995	0.486	1.000	0.998	0.992	0.995	0.486
	0.5	0.1	0.00	0.805	0.713	0.554	0.595	0.354	0.805	0.713	0.554	0.595	0.372
			0.50	0.895	0.803	0.558	0.727	0.351	0.895	0.803	0.558	0.727	0.376
			0.90	0.910	0.831	0.574	0.800	0.343	0.910	0.831	0.574	0.800	0.350
			0.99	0.912	0.833	0.574	0.810	0.341	0.912	0.833	0.574	0.810	0.341
		1.00	0.00	0.886	0.802	0.563	0.736	0.344	0.886	0.802	0.563	0.736	0.353
			0.50	0.903	0.822	0.570	0.780	0.343	0.903	0.822	0.570	0.780	0.347
			0.90	0.911	0.832	0.573	0.806	0.341	0.911	0.832	0.573	0.806	0.342
			0.99	0.912	0.833	0.574	0.811	0.340	0.912	0.833	0.574	0.811	0.340
		100.00	0.00	0.911	0.833	0.573	0.811	0.341	0.911	0.833	0.573	0.811	0.341
			0.50	0.912	0.833	0.573	0.811	0.340	0.912	0.833	0.573	0.811	0.341
			0.90	0.912	0.833	0.574	0.812	0.340	0.912	0.833	0.574	0.812	0.340
			0.99	0.912	0.833	0.574	0.812	0.340	0.912	0.833	0.574	0.812	0.340
	0.9	0.1	0.00	0.203	0.180	0.147	0.112	0.092	0.203	0.180	0.146	0.111	0.100
			0.50	0.219	0.183	0.124	0.120	0.093	0.220	0.184	0.124	0.120	0.104
			0.90	0.278	0.220	0.098	0.189	0.090	0.278	0.221	0.100	0.190	0.096
			0.99	0.291	0.231	0.092	0.220	0.089	0.293	0.233	0.091	0.220	0.088
		1.00	0.00	0.223	0.193	0.111	0.145	0.091	0.223	0.193	0.112	0.145	0.094
			0.50	0.256	0.211	0.102	0.178	0.091	0.256	0.211	0.103	0.178	0.091
			0.90	0.285	0.228	0.093	0.213	0.089	0.285	0.228	0.095	0.213	0.088
			0.99	0.292	0.232	0.091	0.222	0.089	0.293	0.233	0.091	0.222	0.088
		100.00	0.00	0.291	0.232	0.091	0.222	0.090	0.291	0.232	0.091	0.222	0.088
			0.50	0.292	0.232	0.091	0.222	0.088	0.292	0.232	0.091	0.223	0.088
			0.90	0.292	0.233	0.091	0.223	0.088	0.292	0.233	0.091	0.223	0.088
			0.99	0.293	0.233	0.091	0.223	0.089	0.293	0.233	0.091	0.223	0.088

Table 4.4B: P(an arrival enters into service immediately with at least one customer in orbit)

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
4	0.1	0.1	0.00	1.000	1.000	0.999	1.000	0.530	1.000	1.000	0.999	1.000	0.537
			0.50	1.000	1.000	0.999	1.000	0.517	1.000	1.000	0.999	1.000	0.526
			0.90	1.000	1.000	0.999	1.000	0.502	1.000	1.000	0.999	1.000	0.508
			0.99	1.000	1.000	0.999	1.000	0.497	1.000	1.000	0.999	1.000	0.497
		1.00	0.00	1.000	1.000	0.999	1.000	0.515	1.000	1.000	0.999	1.000	0.527
			0.50	1.000	1.000	0.999	1.000	0.506	1.000	1.000	0.999	1.000	0.513
			0.90	1.000	1.000	0.999	1.000	0.498	1.000	1.000	0.999	1.000	0.500
			0.99	1.000	1.000	0.999	1.000	0.496	1.000	1.000	0.999	1.000	0.496
		100.00	0.00	1.000	1.000	0.999	1.000	0.497	1.000	1.000	0.999	1.000	0.500
			0.50	1.000	1.000	0.999	1.000	0.497	1.000	1.000	0.999	1.000	0.498
			0.90	1.000	1.000	0.999	1.000	0.496	1.000	1.000	0.999	1.000	0.496
			0.99	1.000	1.000	0.999	1.000	0.496	1.000	1.000	0.999	1.000	0.496
	0.5	0.1	0.00	0.947	0.892	0.721	0.853	0.381	0.947	0.892	0.720	0.853	0.425
			0.50	0.963	0.917	0.720	0.896	0.368	0.963	0.917	0.720	0.896	0.411
			0.90	0.962	0.914	0.700	0.903	0.353	0.962	0.914	0.700	0.903	0.364
			0.99	0.961	0.913	0.693	0.904	0.349	0.961	0.913	0.693	0.904	0.350
		1.00	0.00	0.960	0.913	0.713	0.891	0.356	0.960	0.913	0.713	0.891	0.372
			0.50	0.961	0.914	0.705	0.899	0.353	0.961	0.914	0.705	0.899	0.360
			0.90	0.961	0.913	0.696	0.903	0.349	0.961	0.913	0.696	0.903	0.351
			0.99	0.961	0.913	0.693	0.904	0.349	0.961	0.913	0.693	0.904	0.349
		100.00	0.00	0.961	0.913	0.693	0.904	0.349	0.961	0.913	0.693	0.904	0.349
			0.50	0.961	0.913	0.693	0.904	0.349	0.961	0.913	0.693	0.904	0.349
			0.90	0.961	0.913	0.693	0.904	0.349	0.961	0.913	0.693	0.904	0.348
			0.99	0.961	0.913	0.693	0.904	0.349	0.961	0.913	0.693	0.904	0.348
	0.9	0.1	0.00	0.346	0.300	0.216	0.219	0.120	0.345	0.301	0.216	0.219	0.171
			0.50	0.364	0.303	0.180	0.240	0.106	0.364	0.304	0.182	0.240	0.177
			0.90	0.361	0.297	0.138	0.274	0.096	0.362	0.297	0.140	0.275	0.112
			0.99	0.356	0.292	0.126	0.283	0.095	0.355	0.291	0.125	0.283	0.095
		1.00	0.00	0.342	0.291	0.156	0.247	0.101	0.341	0.291	0.158	0.247	0.117
			0.50	0.352	0.293	0.141	0.265	0.097	0.352	0.293	0.144	0.265	0.105
			0.90	0.355	0.292	0.128	0.280	0.094	0.355	0.292	0.130	0.281	0.096
			0.99	0.355	0.291	0.125	0.284	0.094	0.355	0.291	0.125	0.284	0.093
		100.00	0.00	0.355	0.291	0.125	0.284	0.095	0.355	0.291	0.125	0.284	0.093
			0.50	0.355	0.291	0.125	0.284	0.094	0.355	0.291	0.125	0.284	0.093
			0.90	0.355	0.291	0.124	0.284	0.094	0.355	0.291	0.125	0.284	0.093
			0.99	0.355	0.291	0.124	0.284	0.094	0.355	0.291	0.125	0.284	0.093
8	0.1	0.1	0.00	1.000	1.000	1.000	1.000	0.550	1.000	1.000	1.000	1.000	0.555
			0.50	1.000	1.000	1.000	1.000	0.537	1.000	1.000	1.000	1.000	0.546
			0.90	1.000	1.000	1.000	1.000	0.521	1.000	1.000	1.000	1.000	0.528
			0.99	1.000	1.000	1.000	1.000	0.516	1.000	1.000	1.000	1.000	0.516
		1.00	0.00	1.000	1.000	1.000	1.000	0.534	1.000	1.000	1.000	1.000	0.548
			0.50	1.000	1.000	1.000	1.000	0.525	1.000	1.000	1.000	1.000	0.533
			0.90	1.000	1.000	1.000	1.000	0.517	1.000	1.000	1.000	1.000	0.518
			0.99	1.000	1.000	1.000	1.000	0.515	1.000	1.000	1.000	1.000	0.515
		100.00	0.00	1.000	1.000	1.000	1.000	0.516	1.000	1.000	1.000	1.000	0.519
			0.50	1.000	1.000	1.000	1.000	0.515	1.000	1.000	1.000	1.000	0.516
			0.90	1.000	1.000	1.000	1.000	0.515	1.000	1.000	1.000	1.000	0.515
			0.99	1.000	1.000	1.000	1.000	0.515	1.000	1.000	1.000	1.000	0.514
	0.5	0.1	0.00	0.992	0.975	0.867	0.968	0.403	0.992	0.975	0.867	0.968	0.467
			0.50	0.992	0.975	0.852	0.970	0.385	0.992	0.975	0.852	0.970	0.443
			0.90	0.991	0.971	0.828	0.968	0.367	0.991	0.971	0.828	0.968	0.382
			0.99	0.990	0.971	0.823	0.968	0.363	0.990	0.971	0.822	0.968	0.364
		1.00	0.00	0.991	0.973	0.845	0.968	0.370	0.991	0.973	0.845	0.968	0.390
			0.50	0.991	0.972	0.835	0.968	0.367	0.991	0.972	0.835	0.968	0.377
			0.90	0.990	0.971	0.825	0.968	0.363	0.990	0.971	0.825	0.968	0.365
			0.99	0.990	0.971	0.822	0.968	0.362	0.990	0.971	0.822	0.968	0.362
		100.00	0.00	0.990	0.971	0.823	0.968	0.362	0.990	0.971	0.823	0.968	0.363
			0.50	0.990	0.971	0.822	0.968	0.362	0.990	0.971	0.822	0.968	0.362
			0.90	0.990	0.971	0.822	0.968	0.362	0.990	0.971	0.822	0.968	0.362
			0.99	0.990	0.971	0.822	0.968	0.362	0.990	0.971	0.822	0.968	0.362
	0.9	0.1	0.00	0.513	0.447	0.297	0.377	0.135	0.511	0.447	0.299	0.377	0.245
			0.50	0.501	0.429	0.248	0.381	0.115	0.501	0.429	0.252	0.381	0.214
			0.90	0.453	0.383	0.191	0.367	0.103	0.453	0.384	0.196	0.368	0.123
			0.99	0.439	0.370	0.177	0.364	0.101	0.438	0.369	0.175	0.365	0.101
		1.00	0.00	0.459	0.394	0.213	0.358	0.110	0.459	0.395	0.219	0.358	0.131
			0.50	0.451	0.384	0.195	0.361	0.104	0.452	0.384	0.207	0.362	0.116
			0.90	0.441	0.372	0.179	0.363	0.100	0.441	0.372	0.181	0.364	0.102
			0.99	0.438	0.368	0.175	0.364	0.100	0.438	0.369	0.175	0.364	0.098
		100.00	0.00	0.438	0.369	0.176	0.364	0.102	0.438	0.369	0.176	0.364	0.099
			0.50	0.437	0.369	0.175	0.364	0.100	0.438	0.369	0.176	0.364	0.098
			0.90	0.437	0.369	0.174	0.364	0.100	0.438	0.369	0.175	0.364	0.098
			0.99	0.437	0.369	0.175	0.364	0.100	0.438	0.369	0.175	0.364	0.098

Table 4.5A: P(an orbiting customer is blocked)

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
1	0.1	0.1	0.00	0.100	0.100	0.100	0.100	0.099	0.100	0.100	0.100	0.100	0.100
			0.50	0.100	0.100	0.100	0.100	0.099	0.100	0.100	0.100	0.100	0.100
			0.90	0.100	0.100	0.100	0.100	0.099	0.100	0.100	0.100	0.100	0.100
			0.99	0.100	0.100	0.100	0.100	0.099	0.100	0.100	0.100	0.100	0.100
		1	0.00	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
			0.50	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
			0.90	0.100	0.100	0.100	0.100	0.099	0.100	0.100	0.100	0.100	0.100
			0.99	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
		100.00	0.00	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
			0.50	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
			0.90	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
			0.99	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100	0.100
	0.5	0.1	0.00	0.500	0.500	0.500	0.500	0.498	0.500	0.500	0.500	0.500	0.500
			0.50	0.500	0.500	0.500	0.500	0.499	0.500	0.500	0.500	0.500	0.500
			0.90	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
			0.99	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
		1.00	0.00	0.500	0.500	0.500	0.500	0.496	0.500	0.500	0.500	0.500	0.500
			0.50	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
			0.90	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
			0.99	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
		100.00	0.00	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
			0.50	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
			0.90	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
			0.99	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
	0.9	0.1	0.00	0.901	0.901	0.900	0.901	0.888	0.901	0.901	0.902	0.902	0.900
			0.50	0.901	0.900	0.901	0.901	0.897	0.900	0.900	0.901	0.901	0.900
			0.90	0.900	0.900	0.900	0.900	0.899	0.900	0.900	0.900	0.900	0.900
			0.99	0.900	0.900	0.900	0.900	0.899	0.900	0.900	0.900	0.900	0.900
		1.00	0.00	0.900	0.900	0.900	0.900	0.882	0.900	0.900	0.900	0.900	0.900
			0.50	0.900	0.900	0.900	0.900	0.897	0.900	0.900	0.900	0.900	0.900
			0.90	0.900	0.900	0.900	0.900	0.899	0.900	0.900	0.900	0.900	0.900
			0.99	0.900	0.900	0.900	0.900	0.899	0.900	0.900	0.900	0.900	0.900
		100.00	0.00	0.900	0.900	0.900	0.900	0.898	0.900	0.900	0.900	0.900	0.900
			0.50	0.900	0.900	0.900	0.900	0.900	0.900	0.900	0.900	0.900	0.900
			0.90	0.900	0.900	0.901	0.900	0.900	0.900	0.900	0.900	0.900	0.900
			0.99	0.900	0.900	0.900	0.900	0.899	0.900	0.900	0.900	0.900	0.900
2	0.1	0.1	0.00	0.009	0.017	0.025	0.049	0.034	0.009	0.017	0.025	0.049	0.024
			0.50	0.009	0.018	0.027	0.050	0.039	0.009	0.018	0.027	0.050	0.027
			0.90	0.009	0.018	0.028	0.051	0.049	0.009	0.018	0.028	0.051	0.043
			0.99	0.009	0.018	0.029	0.051	0.053	0.009	0.018	0.029	0.051	0.052
		1.00	0.00	0.009	0.017	0.026	0.049	0.048	0.009	0.017	0.026	0.049	0.039
			0.50	0.009	0.018	0.027	0.050	0.050	0.009	0.018	0.027	0.050	0.045
			0.90	0.009	0.018	0.028	0.051	0.053	0.009	0.018	0.028	0.051	0.051
			0.99	0.009	0.018	0.029	0.051	0.054	0.009	0.018	0.029	0.051	0.053
		100.00	0.00	0.009	0.018	0.029	0.051	0.053	0.009	0.018	0.029	0.051	0.053
			0.50	0.009	0.018	0.029	0.051	0.054	0.009	0.018	0.029	0.051	0.054
			0.90	0.009	0.018	0.029	0.051	0.054	0.009	0.018	0.029	0.051	0.054
			0.99	0.009	0.018	0.029	0.051	0.054	0.009	0.018	0.029	0.051	0.054
	0.5	0.1	0.00	0.280	0.295	0.325	0.314	0.354	0.280	0.295	0.325	0.314	0.320
			0.50	0.286	0.307	0.356	0.334	0.363	0.286	0.307	0.356	0.334	0.323
			0.90	0.299	0.327	0.397	0.364	0.378	0.299	0.327	0.397	0.364	0.368
			0.99	0.302	0.333	0.409	0.373	0.382	0.302	0.333	0.410	0.373	0.381
		1.00	0.00	0.286	0.308	0.368	0.335	0.376	0.286	0.308	0.368	0.335	0.360
			0.50	0.293	0.319	0.386	0.352	0.378	0.292	0.319	0.386	0.352	0.370
			0.90	0.300	0.330	0.405	0.369	0.381	0.300	0.330	0.405	0.369	0.380
			0.99	0.302	0.333	0.409	0.374	0.382	0.302	0.333	0.410	0.374	0.382
		100.00	0.00	0.302	0.333	0.409	0.374	0.382	0.302	0.333	0.409	0.374	0.382
			0.50	0.302	0.333	0.409	0.374	0.382	0.302	0.333	0.409	0.374	0.382
			0.90	0.302	0.333	0.410	0.374	0.382	0.302	0.333	0.410	0.374	0.383
			0.99	0.302	0.333	0.410	0.374	0.382	0.302	0.333	0.410	0.374	0.383
	0.9	0.1	0.00	0.819	0.820	0.824	0.819	0.839	0.819	0.820	0.826	0.821	0.818
			0.50	0.820	0.822	0.835	0.822	0.848	0.820	0.821	0.834	0.822	0.817
			0.90	0.836	0.842	0.865	0.845	0.859	0.836	0.842	0.863	0.844	0.846
			0.99	0.844	0.852	0.878	0.859	0.861	0.845	0.853	0.879	0.859	0.861
		1.00	0.00	0.826	0.831	0.855	0.832	0.854	0.826	0.831	0.855	0.833	0.839
			0.50	0.832	0.838	0.865	0.842	0.857	0.832	0.838	0.864	0.842	0.850
			0.90	0.842	0.849	0.876	0.856	0.861	0.841	0.849	0.875	0.856	0.860
			0.99	0.844	0.852	0.879	0.860	0.861	0.845	0.853	0.879	0.860	0.863
		100.00	0.00	0.844	0.852	0.879	0.860	0.860	0.844	0.852	0.879	0.860	0.863
			0.50	0.844	0.852	0.879	0.860	0.862	0.844	0.852	0.879	0.860	0.863
			0.90	0.845	0.853	0.880	0.861	0.862	0.844	0.853	0.879	0.861	0.863
			0.99	0.845	0.853	0.879	0.861	0.862	0.845	0.853	0.879	0.861	0.863

Table 4.5B: P(an orbiting customer is blocked)

				Direct Truncation					Matrix-geometric Approximation				
c	ρ	θ	p	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
4	0.1	0.1	0.00	0.000	0.001	0.003	0.002	0.020	0.000	0.001	0.003	0.002	0.007
			0.50	0.000	0.001	0.003	0.002	0.028	0.000	0.001	0.003	0.002	0.009
			0.90	0.000	0.001	0.003	0.002	0.044	0.000	0.001	0.003	0.002	0.034
			0.99	0.000	0.001	0.003	0.002	0.050	0.000	0.001	0.003	0.002	0.049
		1.00	0.00	0.000	0.001	0.003	0.002	0.042	0.000	0.001	0.003	0.002	0.027
			0.50	0.000	0.001	0.003	0.002	0.046	0.000	0.001	0.003	0.002	0.036
			0.90	0.000	0.001	0.003	0.002	0.050	0.000	0.001	0.003	0.002	0.048
			0.99	0.000	0.001	0.003	0.002	0.051	0.000	0.001	0.003	0.002	0.051
		100.00	0.00	0.000	0.001	0.003	0.002	0.051	0.000	0.001	0.003	0.002	0.051
			0.50	0.000	0.001	0.003	0.002	0.051	0.000	0.001	0.003	0.002	0.051
			0.90	0.000	0.001	0.003	0.002	0.051	0.000	0.001	0.003	0.002	0.051
			0.99	0.000	0.001	0.003	0.002	0.051	0.000	0.001	0.003	0.002	0.051
	0.5	0.1	0.00	0.108	0.131	0.194	0.139	0.280	0.108	0.131	0.194	0.139	0.201
			0.50	0.120	0.149	0.235	0.157	0.302	0.120	0.149	0.235	0.157	0.230
			0.90	0.131	0.169	0.282	0.181	0.328	0.131	0.169	0.282	0.181	0.311
			0.99	0.133	0.173	0.294	0.188	0.335	0.134	0.174	0.295	0.188	0.333
		1.00	0.00	0.118	0.148	0.245	0.158	0.325	0.118	0.148	0.245	0.158	0.297
			0.50	0.126	0.160	0.268	0.171	0.330	0.126	0.160	0.268	0.171	0.316
			0.90	0.132	0.171	0.290	0.185	0.334	0.132	0.171	0.290	0.185	0.332
			0.99	0.133	0.174	0.295	0.188	0.335	0.134	0.174	0.295	0.188	0.336
		100.00	0.00	0.133	0.173	0.294	0.188	0.335	0.133	0.173	0.294	0.188	0.336
			0.50	0.133	0.174	0.295	0.188	0.336	0.133	0.174	0.295	0.188	0.336
			0.90	0.134	0.174	0.295	0.189	0.336	0.134	0.174	0.295	0.189	0.336
			0.99	0.134	0.174	0.295	0.189	0.336	0.134	0.174	0.295	0.189	0.336
	0.9	0.1	0.00	0.694	0.700	0.729	0.701	0.780	0.694	0.700	0.731	0.700	0.692
			0.50	0.708	0.716	0.762	0.713	0.810	0.708	0.716	0.761	0.713	0.692
			0.90	0.755	0.769	0.824	0.769	0.828	0.754	0.769	0.821	0.768	0.803
			0.99	0.768	0.786	0.844	0.790	0.831	0.770	0.788	0.847	0.790	0.831
		1.00	0.00	0.728	0.743	0.806	0.744	0.818	0.728	0.743	0.804	0.745	0.788
			0.50	0.745	0.762	0.825	0.764	0.825	0.745	0.762	0.822	0.764	0.810
			0.90	0.764	0.782	0.842	0.786	0.831	0.764	0.782	0.840	0.786	0.829
			0.99	0.769	0.787	0.846	0.792	0.831	0.770	0.788	0.847	0.792	0.834
		100.00	0.00	0.769	0.787	0.846	0.792	0.829	0.769	0.787	0.846	0.792	0.834
			0.50	0.769	0.787	0.847	0.793	0.832	0.769	0.787	0.846	0.793	0.834
			0.90	0.770	0.788	0.848	0.793	0.832	0.770	0.788	0.847	0.793	0.834
			0.99	0.770	0.788	0.847	0.793	0.832	0.770	0.788	0.847	0.793	0.834
8	0.1	0.1	0.00	0.000	0.000	0.000	0.000	0.014	0.000	0.000	0.000	0.000	0.003
			0.50	0.000	0.000	0.000	0.000	0.023	0.000	0.000	0.000	0.000	0.004
			0.90	0.000	0.000	0.000	0.000	0.041	0.000	0.000	0.000	0.000	0.027
			0.99	0.000	0.000	0.000	0.000	0.048	0.000	0.000	0.000	0.000	0.047
		1.00	0.00	0.000	0.000	0.000	0.000	0.039	0.000	0.000	0.000	0.000	0.020
			0.50	0.000	0.000	0.000	0.000	0.043	0.000	0.000	0.000	0.000	0.031
			0.90	0.000	0.000	0.000	0.000	0.048	0.000	0.000	0.000	0.000	0.046
			0.99	0.000	0.000	0.000	0.000	0.049	0.000	0.000	0.000	0.000	0.049
		100.00	0.00	0.000	0.000	0.000	0.000	0.049	0.000	0.000	0.000	0.000	0.049
			0.50	0.000	0.000	0.000	0.000	0.049	0.000	0.000	0.000	0.000	0.049
			0.90	0.000	0.000	0.000	0.000	0.049	0.000	0.000	0.000	0.000	0.050
			0.99	0.000	0.000	0.000	0.000	0.049	0.000	0.000	0.000	0.000	0.050
	0.5	0.1	0.00	0.024	0.038	0.096	0.040	0.248	0.024	0.038	0.096	0.040	0.130
			0.50	0.029	0.048	0.128	0.050	0.282	0.029	0.048	0.128	0.050	0.178
			0.90	0.033	0.057	0.162	0.061	0.314	0.033	0.057	0.162	0.061	0.291
			0.99	0.034	0.059	0.170	0.063	0.322	0.034	0.059	0.171	0.063	0.321
		1.00	0.00	0.028	0.047	0.134	0.050	0.311	0.028	0.047	0.134	0.050	0.276
			0.50	0.031	0.053	0.151	0.056	0.316	0.031	0.053	0.151	0.056	0.297
			0.90	0.033	0.058	0.167	0.062	0.322	0.033	0.058	0.167	0.062	0.319
			0.99	0.034	0.059	0.171	0.063	0.323	0.034	0.059	0.171	0.063	0.323
		100.00	0.00	0.033	0.059	0.170	0.063	0.323	0.033	0.059	0.170	0.063	0.323
			0.50	0.034	0.059	0.171	0.063	0.323	0.034	0.059	0.171	0.063	0.324
			0.90	0.034	0.059	0.171	0.064	0.323	0.034	0.059	0.171	0.064	0.324
			0.99	0.034	0.059	0.171	0.064	0.323	0.034	0.059	0.171	0.064	0.324
	0.9	0.1	0.00	0.537	0.555	0.632	0.552	0.750	0.540	0.555	0.631	0.553	0.556
			0.50	0.584	0.602	0.689	0.593	0.786	0.584	0.602	0.685	0.593	0.620
			0.90	0.653	0.679	0.771	0.677	0.809	0.653	0.678	0.764	0.676	0.777
			0.99	0.670	0.699	0.795	0.702	0.813	0.672	0.702	0.798	0.702	0.813
		1.00	0.00	0.614	0.643	0.749	0.642	0.796	0.614	0.642	0.743	0.642	0.757
			0.50	0.640	0.669	0.772	0.670	0.807	0.640	0.669	0.760	0.670	0.784
			0.90	0.665	0.695	0.793	0.698	0.813	0.665	0.694	0.790	0.697	0.811
			0.99	0.671	0.701	0.798	0.705	0.814	0.672	0.702	0.798	0.705	0.817
		100.00	0.00	0.670	0.700	0.797	0.704	0.811	0.670	0.700	0.797	0.704	0.817
			0.50	0.671	0.701	0.798	0.705	0.815	0.671	0.701	0.798	0.705	0.817
			0.90	0.672	0.702	0.799	0.706	0.814	0.671	0.701	0.798	0.705	0.818
			0.99	0.672	0.702	0.798	0.705	0.814	0.672	0.702	0.798	0.705	0.818

Table 4.6: Performance Measures for MPC arrival process at $N = M^*$

			$c=1$				$c=2$			
ρ	θ	p	FSR	MNO	PES	PBK	FSR	MNO	PES	PBK
0.1	0.1	0.00	0.467	10.97	0.497	0.100	0.538	9.11	0.515	0.034
		0.50	0.298	7.91	0.490	0.100	0.333	6.87	0.503	0.040
		0.90	0.077	5.78	0.483	0.100	0.078	5.48	0.490	0.050
		0.99	0.008	5.33	0.481	0.100	0.008	5.24	0.487	0.053
	1	0.00	0.088	5.87	0.496	0.100	0.090	5.55	0.504	0.048
		0.50	0.046	5.56	0.489	0.100	0.046	5.38	0.495	0.051
		0.90	0.009	5.34	0.483	0.100	0.009	5.25	0.488	0.053
		0.99	0.001	5.28	0.481	0.100	0.001	5.22	0.486	0.054
	100.00	0.00	0.001	5.27	0.483	0.100	0.001	5.23	0.488	0.054
		0.50	0.001	5.28	0.482	0.100	0.001	5.23	0.487	0.054
		0.90	0.000	5.27	0.481	0.100	0.000	5.22	0.486	0.054
		0.99	0.000	5.27	0.481	0.100	0.000	5.22	0.486	0.054
0.5	0.1	0.00	0.112	63.89	0.298	0.500	0.115	56.74	0.355	0.352
		0.50	0.058	55.89	0.323	0.500	0.058	52.50	0.351	0.363
		0.90	0.012	50.67	0.331	0.500	0.012	49.78	0.343	0.378
		0.99	0.001	49.57	0.332	0.500	0.001	49.23	0.340	0.382
	1.00	0.00	0.014	50.86	0.329	0.500	0.013	49.90	0.345	0.375
		0.50	0.007	50.16	0.331	0.500	0.007	49.53	0.343	0.378
		0.90	0.001	49.59	0.332	0.500	0.001	49.24	0.341	0.382
		0.99	0.000	49.46	0.332	0.500	0.000	49.18	0.340	0.383
	100.00	0.00	0.000	49.43	0.333	0.500	0.000	49.75	0.339	0.384
		0.50	0.000	49.47	0.332	0.500	0.000	49.19	0.340	0.383
		0.90	0.000	49.46	0.332	0.500	0.000	49.17	0.340	0.383
		0.99	0.000	49.45	0.332	0.500	0.000	49.17	0.340	0.383
0.9	0.1	0.00	0.015	677.09	0.049	0.917	0.017	565.46	0.081	0.860
		0.50	0.009	517.75	0.068	0.903	0.009	492.02	0.088	0.857
		0.90	0.002	468.46	0.079	0.901	0.002	465.67	0.087	0.863
		0.99	0.000	458.82	0.081	0.901	0.000	460.30	0.087	0.865
	1.00	0.00	0.001	804.63	0.038	0.950	0.001	959.62	0.028	0.956
		0.50	0.001	469.16	0.078	0.902	0.001	469.25	0.086	0.864
		0.90	0.000	460.43	0.081	0.901	0.000	462.10	0.086	0.865
		0.99	0.000	458.59	0.081	0.901	0.000	460.58	0.086	0.865
	100.00	0.00	0.000	449.63	0.082	0.900	0.000	1480.42	0.009	0.986
		0.50	0.000	463.67	0.081	0.902	0.000	466.62	0.086	0.866
		0.90	0.000	460.32	0.081	0.901	0.000	462.25	0.086	0.865
		0.99	0.000	459.28	0.081	0.901	0.000	461.18	0.086	0.865
			$c=4$				$c=8$			
0.1	0.1	0.00	0.595	8.02	0.530	0.021	0.633	7.24	0.549	0.016
		0.50	0.356	6.20	0.516	0.030	0.368	5.65	0.536	0.025
		0.90	0.078	5.28	0.501	0.045	0.078	5.03	0.520	0.042
		0.99	0.008	5.15	0.496	0.051	0.008	4.96	0.515	0.049
	1	0.00	0.092	5.31	0.515	0.043	0.093	5.06	0.534	0.039
		0.50	0.046	5.21	0.506	0.046	0.046	4.99	0.524	0.044
		0.90	0.009	5.14	0.498	0.050	0.009	4.95	0.516	0.048
		0.99	0.001	5.13	0.496	0.051	0.001	4.94	0.515	0.050
	100.00	0.00	0.001	5.13	0.497	0.051	0.001	4.94	0.516	0.049
		0.50	0.001	5.13	0.497	0.051	0.001	4.94	0.515	0.050
		0.90	0.000	5.12	0.496	0.052	0.000	4.94	0.514	0.050
		0.99	0.000	5.12	0.496	0.052	0.000	4.94	0.514	0.050
0.5	0.1	0.00	0.118	52.87	0.382	0.278	0.120	50.06	0.403	0.249
		0.50	0.059	50.43	0.368	0.302	0.059	48.65	0.385	0.282
		0.90	0.012	48.99	0.353	0.329	0.012	47.88	0.367	0.315
		0.99	0.001	48.70	0.349	0.336	0.001	47.73	0.362	0.323
	1.00	0.00	0.013	49.06	0.356	0.325	0.013	47.97	0.370	0.311
		0.50	0.007	48.87	0.352	0.330	0.007	47.84	0.366	0.317
		0.90	0.001	48.71	0.349	0.335	0.001	47.74	0.362	0.323
		0.99	0.000	48.67	0.348	0.336	0.000	47.71	0.362	0.324
	100.00	0.00	0.000	50.21	0.346	0.341	0.000	50.38	0.357	0.332
		0.50	0.000	48.70	0.348	0.336	0.000	47.75	0.362	0.324
		0.90	0.000	48.68	0.348	0.336	0.000	47.72	0.361	0.324
		0.99	0.000	48.67	0.348	0.336	0.000	47.71	0.361	0.324
0.9	0.1	0.00	0.018	504.92	0.104	0.810	0.018	491.99	0.117	0.784
		0.50	0.009	481.25	0.100	0.821	0.009	475.77	0.108	0.801
		0.90	0.002	464.85	0.093	0.833	0.002	464.18	0.098	0.817
		0.99	0.000	461.48	0.091	0.837	0.000	461.73	0.096	0.821
	1.00	0.00	0.002	460.00	0.094	0.831	0.002	456.40	0.100	0.814
		0.50	0.001	469.89	0.092	0.835	0.001	470.72	0.097	0.820
		0.90	0.000	463.27	0.091	0.837	0.000	463.85	0.096	0.821
		0.99	0.000	461.83	0.091	0.837	0.000	462.34	0.096	0.822
	100.00	0.00	0.000	1440.81	0.009	0.984	0.000	1358.39	0.010	0.981
		0.50	0.000	468.67	0.090	0.838	0.000	470.21	0.095	0.824
		0.90	0.000	463.33	0.091	0.837	0.000	463.89	0.096	0.822
		0.99	0.000	462.11	0.091	0.837	0.000	462.48	0.096	0.822

Table 4.7: Performance Measures for $MAP/M/c$ Queue)

		Mean Number in the Queue					P(arrival finds all servers busy)				
c	ρ	ERL	EXP	HEX	MNC	MPC	ERL	EXP	HEX	MNC	MPC
1	0.10	0.00	0.01	0.02	0.05	5.27	0.100	0.100	0.100	0.100	0.100
	0.50	0.31	0.50	1.29	0.64	49.42	0.500	0.500	0.500	0.500	0.500
	0.90	5.93	8.10	24.49	8.32	449.53	0.900	0.900	0.900	0.900	0.900
	0.99	73.25	97.81	295.40	98.49	4953.38	0.990	0.990	0.990	0.990	0.990
2	0.10	0.00	0.00	0.01	0.00	5.22	0.009	0.018	0.029	0.051	0.054
	0.50	0.19	0.33	1.06	0.38	49.13	0.302	0.333	0.410	0.375	0.383
	0.90	5.56	7.67	23.93	7.79	448.97	0.845	0.853	0.879	0.861	0.863
	0.99	72.82	97.32	294.76	97.90	4952.75	0.984	0.985	0.988	0.986	0.986
4	0.10	0.00	0.00	0.00	0.00	5.12	0.000	0.001	0.003	0.002	0.052
	0.50	0.08	0.17	0.76	0.19	48.62	0.134	0.174	0.295	0.189	0.336
	0.90	5.07	7.09	23.04	7.18	447.99	0.770	0.788	0.847	0.793	0.834
	0.99	72.20	96.62	293.73	97.16	4951.65	0.976	0.978	0.984	0.979	0.983
8	0.10	0.00	0.00	0.00	0.00	4.93	0.000	0.000	0.000	0.000	0.050
	0.50	0.02	0.06	0.44	0.06	47.65	0.034	0.059	0.171	0.064	0.324
	0.90	4.42	6.31	21.71	6.38	446.18	0.672	0.702	0.798	0.705	0.818
	0.99	71.34	95.62	292.13	96.14	4949.64	0.964	0.968	0.979	0.968	0.980

Chapter 5.

$M|PH|1$ retrial queue with service interruption and orbital search

Queueing system with service interruption have been introduced by White and Christie [73] who considered the problem as a pre-emptive priority system. Different types of interruptions have been extensively studied by many researchers. Service interruptions can be viewed as a special type of breakdown of the server in which the server is restarted instantaneously. See Aissani [1, 2], Aissani and Artalejo [3], Artalejo [4] and references therein for queueing system with breakdown. Queues with service interruption also fall into the category of queues with feedback (See Choi and Kulkarni [24]) and queues with disaster to the unit undergoing service (See A. Krishnamoorthy and P. V. Ushakumari [50]). Artalejo and Gomez-Correl [10] considered a retrial queueing system with two types of service interruptions. In their model, depending on the type of the interruption that the unit has encountered, it may rejoin the orbit for another attempt or leave the system for ever.

In this chapter, we consider a single server retrial queue in which the server is subject to service interruptions with auto repeat facilities and equipped with the 'mechanism of search of customers from the orbit' as we have introduced in chapter 2. The customer whose service is interrupted rejoins the orbit with some known probability q or leaves the system with the complimentary probability $(1 - q)$. At the departure epoch, (ie, the epoch at which the server becomes free either by a successful completion of a service or by a service interruption), the server goes for search of customers in the orbit with some known probability p or remains idle with the complimentary probability $1 - p$.

There are lots of real life situations which fit to our model. For example, consider the transmission of messages in fascimile 'networks' having the autorepeat facilities. If the transmission is not successfully completed for some reasons such as a power failure or a transmission error, then the message leaves the server and joins the buffer with some known probability and leaves the system with the complimentary probability. Immediately after a successful transmission or an interruption, instead of staying idle, the server goes for search of customers in the buffer with a known probability. By the introduction of the 'search mechanism', the idle time of the server is considerably reduced and there by attaining the optimum utilization of the server facility.

5.1 The mathematical model

We consider a single server retrial queueing system at which primary customers arrive according to a Poisson process with rate λ. The retrial is assumed to be exponential with rate $j\mu$, when there as j customers in the orbit (ie, the classical retrial policy). The service is interrupted at an exponential rate σ. The interrupted customer goes back to the orbit with a known probability q or leaves the system with the complimentary probability $(1-q)$. At the epoch when the server becomes free, (either by a service completion or by a service interruption) it goes for search of customers in the orbit with some known probability p or remains idle with the probability $1-p$. The search time is assumed to be negligible. The service time is assume to follow a phase-type distribution (PH- distribution) with representation

(β, S) of order m.

PH-distributions and 'PH-renewal processes' were introduce by Neuts [59]. They have gained widespread attention in the area of stochastic modelling, particularly in queueing theory. A phase-type distribution on $[0, \infty)$ of order m is defined as the absorption-time distribution in a finite state Markov process with m transient states and one absorbing state as follows: Consider a Markov process on the states $\{0, 1, \dots m\}$ with infinitesimal generator

$$Q = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix} \tag{5.1}$$

where the $m \times m$ matrix S satisfies $S_{ii} < 0$, for $1 \leq i \leq m$, and $S_{ij} \geq 0$ for $i \neq j$. Also $Se + S^0 = 0$ and the initial probability vector of Q is given by (β, β_0), with $\beta e + \beta_0 = 1$, where e is a column vector of 1's in all its m positions. It can be shown that the states $1, \dots m$ are transient if and only if the matrix S is non-singular. Also, the probability distribution $H(\cdot)$ of the time until absorption in the state 0, corresponding to the initial probability vector (β, β_0), is given by

$$H(x) = 1 - \beta \exp(Sx)e, \quad \text{for } x \geq 0 \tag{5.2}$$

The probability distribution $H(\cdot)$ on $[0, \alpha)$ is a PH-distribution if and only if it is the distribution of the time until absorption in a finite Markov chain of the type defined in (5.1). Note that the distribution $H(\cdot)$ has a jump of height β_0 at $x = 0$ and its density $H'(x)$ on $(0, \infty)$ is given

$$H'(x) = \beta \exp(Sx)S^0 \tag{5.3}$$

The Laplace-Stieltjes transform $f(s)$ of $H(\cdot)$ is given by

$$f(s) = \beta_0 + \beta(sI - S)^{-1}S^0 \quad \text{for Re } s \geq 0 \tag{5.4}$$

The non-central moments μ_i' of $H(\cdot)$ are all finite and is give by

$$\mu_i' = (-1)^i \, i!(\beta S^{-i} - e), \text{ for } i \geq 0 \tag{5.5}$$

Discrete PH-distributions are defined by considering an $(m+1)$- state Markov chain P of the form

$$P = \begin{bmatrix} S & S^0 \\ 0 & 1 \end{bmatrix}$$

where S is a substochastic matrix, such that $I - S$ is non-singular . The initial probability vector is (β, β_0). The probability density $\{p_k\}$ of phase type is given by

$$p_0 = \beta_0$$
$$p_k = \beta S^{k-1} S^0; \; k \geq 1$$

Its probability generating function $P(z) = \beta_0 + z\beta(I - zS)^{-1}S^0$ and the factorial moments are given by

$$P^{(k)}(1) = k!\beta S^{k-1}(I - S)^{-k}e$$

5.2 Algorithmic solution

Our model may be studied as a level dependent quasi birth-and-death process (LDQBD) with the state space given by

$$E = \{j, \bar{j}; \; j \geq 0\}, \text{ where } j = \{(j,0); \; j \geq 0\} \text{ and}$$
$$\bar{j} = \{(j,1,k); \; j \geq 0, \; 1 \leq k \leq m\}.$$

The states $j = (j, 0)$, $j \geq 0$ correspond to the idle server; with j customers in the orbit; the states $\bar{j} = (j, 1, k)$, $j \geq 0$, $1 \leq k \leq m$ correspond to the busy server with the service process in the phase k, and j customers in the orbit.
The generator Q is given by

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & 0 & \dots \\ A_{21} & A_{11} & A_0 & 0 & 0 & 0 & \\ 0 & A_{22} & A_{12} & A_0 & 0 & 0 & \\ 0 & 0 & A_{23} & A_{13} & A_0 & 0 & \\ \dots & & & & & & \end{bmatrix}$$

Where

$$B_0 = \begin{bmatrix} -\lambda & \lambda\beta \\ \sigma(1-q)e + S^0 & S - (\lambda+\sigma)I + \sigma pqe\beta \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & 0 \\ \sigma q(1-p)e & \lambda I \end{bmatrix}$$

$$A_{2i} = \begin{bmatrix} 0 & i\mu\beta \\ 0 & S^0 p\beta + \sigma p(1-q)e\beta \end{bmatrix}; \quad i \geq 1$$

and

$$A_{1i} = \begin{bmatrix} -(\lambda+i\mu) & \lambda\beta \\ S^0(1-p) + \sigma(1-q)(1-p)e & S - (\lambda+\sigma)I + \sigma pqe\beta \end{bmatrix}; \quad i \geq 1$$

Let $x = (x(0), x(1), \dots)$ be the steady state probability vector associated with Q. That is, when the queue is stable,x is the unique solution to

$$xQ = 0 \text{ and } xe = 1 \tag{5.6}$$

Even though Q is highly structured, x cannot be expressed in a tractable analytical form. So we propose an algorithmic solution based on the Neuts-Rao truncation approach as described in section 2 of chapter 4. Recall that, if the number of customers in the orbit is small, the likelihood of an idle server and therefore that of a retrial request being successful is not small. When the number of customers in the orbit is sufficiently large, a majority of the retrial requests fail to find a free server and do not result in a change of state. Therefore, if the number of customers in the orbit who can generate retrial requests is restricted to an appropriately chosen number N, then the effect on the system dynamics and the equilibrium probability vector is minimal. That approximation modifies the infinitesimal generator Q to that given below:

$$\bar{Q} = \begin{bmatrix} B_0 & A_0 & & & & & & & \\ A_{21} & A_{11} & A_0 & & & & & & \\ & A_{22} & A_{12} & A_0 & & & & & \\ \cdots & & & & & & & & \\ & & & A_{2,N-1} & A_{1,N-1} & A_0 & \cdots & \cdots & \\ & & & & A_2 & A_1 & A_0 & \cdots & \\ & & & & & A_2 & A_1 & A_0 & \cdots \\ & & & & & & & & \\ \cdots & & & & & & & & \end{bmatrix}$$

where $A_1 = A_{1N}$, and $A_2 = A_{2N}$.

5.2.1 Stability condition

The steady state probability vector exists if and only if

$$\pi A_0 e < \pi A_2 e. \tag{5.7}$$

where π is the invariant probability vector of the matrix $A = A_0 + A_1 + A_2$ given by

$$A = \begin{bmatrix} -(\lambda + N\mu) & (\lambda + N\mu)\beta \\ (1-p)S^0 + \sigma(1-p)e & S^0 p\beta + S + \sigma p e\beta - \sigma I \end{bmatrix} \quad (5.8)$$

Let $\pi = (\pi_0, \bar{\pi}_1)$ where π_0 is a scalar and $\bar{\pi}_1$ is a vector of order m. $\pi A = 0$, subject to $\pi_0 + \bar{\pi}_1 e = 1$ implies

$$\bar{\pi}_1 = (\lambda + N\mu)\pi_0\beta[\sigma I - pS^0\beta - S - p\sigma e\beta]^{-1}$$

Then from (5.7), after some algebra we get the stability condition as

$$(\lambda + n\mu)\beta[\sigma I - pS^0\beta - S - p\sigma e\beta]^{-1}[(\lambda + \sigma(q-p))e - ps^0] < N\mu$$

5.2.2 Computation of the vector x

Because of the special structure of $\bar{Q}$, x can be expressed as

$$x(i + N - 1) = x(N-1)R^i; \quad i \geq 0 \quad (5.9)$$

where the matrix R is the unique non-negative solution with spectral radius less than 1 of the equation

$$R^2 A_2 + RA_1 + A_0 = 0 \quad (5.10)$$

The vectors $x(0), \ldots x(N-1)$ can be obtained by solving the following equations.

$$\begin{aligned} x(0)B_0 + x(1)A_{21} &= 0 \\ x(i-1)A_0 + x(i)A_{1i} + x(i+1)A_{2,i+1} &= 0, \quad 1 \leq i \leq N-1 \quad (5.11) \\ x(N-2)A_0 + x(N-1)[A_{1,N-1} + RA_2] &= 0 \end{aligned}$$

Subject to the normalizing condition

$$\sum_{i=0}^{N-2} x(i) + x(N-1)(I-R)^{-1}e = 1 \tag{5.12}$$

Due to the special structure of A_0, the matrix R can be computed as follows: Rewrite the matrix R as

$$R = \begin{bmatrix} 0 & 0 \\ R_0 & R_1 \end{bmatrix} \tag{5.13}$$

where R_0 is a column vector of order m in and R_1 is a square matrix of order m.

Then (5.10) yields

$$-(\lambda + N\mu)R_0 + (1-p)R_1S^0 + \sigma(1-p)(1-q)R_1e + \sigma q(1-p)e = 0$$

and

$$N\mu R_1R_0\beta + pR_1^2S^0\beta + \sigma(1-q)pR_1^2e\beta + \lambda R_0\beta + R_1[S-(\lambda+\sigma)I + \sigma qpe\beta] + \lambda I = 0$$

Thus we obtain R_0 and R_1 as

$$R_0 = \frac{(1-p)}{(\lambda + N\mu)}\{R_1[S^0 + \sigma(1-q)e] + \sigma qe\}$$

and

$$R_1 = \{N\mu R_1R_0\beta + pR_1^2S^0\beta + \sigma(1-q)pR_1^2e\beta + \lambda R_0\beta + \lambda I\}$$
$$\{(\lambda+\sigma)I - S - \sigma qpe\beta\}^{-1} \tag{5.14}$$

Partition the components $x(i)$ of the vector x as $x(i) = (x_i(0), x_i(1))$, $i \geq 0$. Where $x_i(0)$ is a scalar and $x_i(1)$ is vector of order m.

From (5.9), we get

$$(x_{i+N-1}(0), x_{i+N-1}(1)) = (x_{N-1}(0), x_{N-1}(1)) \begin{bmatrix} 0 & 0 \\ R_1^{i-1}R_0 & R_1^i \end{bmatrix}$$

which yields

$$x_{i+N-1}(j) = x_{N-1}(1)R_1^{i-1}R_j, \quad i \geq 1,\, j \in \{0,1\} \tag{5.15}$$

Now from, (5.11),

$$x(0) = x(1)A_{21}(-B_0)^{-1} = x(1)A_{21}(-A_0')^{-1}$$

and

$$\begin{aligned} x(1) &= -x(2)A_{22}[A_{11} + A_{21}(-A_0')^{-1}A_0]^{-1} \\ &= x(2)A_{22}(-A_1')^{-1} \end{aligned}$$

In general,

$$x(i) = x(i+1)A_{2,i+1}(-A_i')^{-1}; \quad 0 \leq i \leq N-1 \tag{5.16}$$

where

$$A_i' = \begin{cases} B_0 & i = 0 \\ A_{1i} + A_{2i}(-A_{i-1}')^{-1}A_0; & 1 \leq i \leq N \end{cases} \tag{5.17}$$

Now, by applying block Gaussian elimination, the partitioned subvector $(x(N), x(N+1), \ldots)$ corresponding to non-boundary states, satisfies the relation.

$$(x(N), x(N+1)\ldots) \begin{bmatrix} A_N' & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ \cdots & & & \end{bmatrix} = 0 \tag{5.18}$$

Let

$$\delta = \sum_{i=N}^{\infty} x(i)e \tag{5.19}$$

$$\text{and} \quad y(i) = \delta^{-1}x(N+i), \quad i \geq 0 \tag{5.20}$$

From (5.18), we get

$$x(N)A'_N + x(N+1)A_2 = 0$$

$$x(N+i) = x(N+i-1)R, \quad i \geq 1$$

which implies

$$y(0)A'_N + y(1)A_2 = 0$$

and

$$y(i) = y(i-1)R, \quad i \geq 1$$

.

Since $\sum_{i=0}^{\infty} y(i)e = 1$, we get

$$y(0)(I-R)^{-1}e = 1$$

Thus, $x(i) = \delta y(0)R^{i-N}, \quad i \geq N$.
Again by (5.16), we get

$$x(i) = \delta y(0)\prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1}, \quad 0 \leq i \leq N-1$$

Therefore,

$$x(i) = \begin{cases} \delta y(0)\prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1}, & 0 \leq i \leq N-1 \\ \delta y(0)R^{i-N} & i \geq N \end{cases} \tag{5.21}$$

where $y(0)$ is the unique solution of the system

$$\begin{aligned} y(0)(A'_N + RA_2) &= 0 \\ y(0)(I - R)^{-1}e &= 1 \end{aligned} \tag{5.22}$$

Now $xe = 1$ implies

$$\delta y(0) \sum_{i=0}^{N-1} \prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1}e + \delta y(0) \sum_{i=N}^{\infty} R^{i-N}e = 1$$

Now by using the second equation in (5.22) we get

$$\delta = [1 + y(0) \sum_{i=0}^{N-1} \prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1}e]^{-1} \tag{5.23}$$

5.3 Other system characteristics

1. Probability mass function of the number of customers in the orbit.
   Pr [i customers in the orbit] is given by

$$\begin{aligned} a_i &= x(i)e \\ &= \begin{cases} \delta y(0) \prod_{j=i}^{N} A_{2j}(-A'_{j-1})^{-1}e, & 0 \leq i \leq N-1 \\ \delta y(0) R^{i-N}e, & i \geq N \end{cases} \end{aligned}$$

2. Expected number of customers in the orbit is given by

$$\begin{aligned} EN &= \sum_{i=1}^{\infty} ix(i)e \\ &= \delta y(0)\{\sum_{i=0}^{N-1} i \prod_{j=i}^{N} A_{2j}(-A'_{j-1})e + R(I - R)^{-2}e + N(I - R)^{-1}e\} \end{aligned}$$

3. Probability mass function of the server state

$$
\begin{aligned}
P_0 = \Pr[\text{ server is idle }] &= \sum_{i=0}^{\infty} x_i(0) \\
&= \sum_{i=0}^{N-1} x_i(0) + x_{N-1}(1)(I - R_1)^{-1}R_0 \\
P_1 = \Pr[\text{ server is busy }] &= \sum_{i=0}^{\infty} x_i(1)e \\
&= \sum_{i=0}^{N-1} x_i(1)e + x_{N-1}(1)(I - R_1)^{-1}R_1 e
\end{aligned}
$$

4. The mean time spent by an arbitrary customer in the orbit $W_q = \frac{1}{\lambda}EN$.

5. The overall rate of retrials

$$
\mu_1^* = \sum_{i=1}^{\infty} i\mu x(i)e = \mu \cdot EN
$$

6. The successful rate of retrials

$$
\begin{aligned}
\mu_2^* &= \sum_{i=1}^{\infty} i\mu x_i(0) \\
&= \mu\{\sum_{i=1}^{N-1} i x_i(0) + x_{N-1}(1)[R_1(I - R_1)^{-2} + N(I - R_1)^{-1}]R_0\}
\end{aligned}
$$

7. Expected number of busy servers

$$
EC = p_1
$$

8. Expected number of customers in the system

$$ES = EN + EC$$

9. The mean time spent by an arbitrary customer in the system

$$W_s = \frac{1}{\lambda} ES.$$

Chapter 6.

Excursion between classical and retrial queue

In this chapter, we consider an excursion between classical queue and the retrial queue in the following way : Note that when probability of search for a customer after each service completion is one, retrial queue behaves like the classical queue, except for the order in which customers are served. For the present model as long as the number of customers in the orbit remains less than or equal to N, the server, immediately on a service completion picks up the next customer from the orbit with probability 1 and starts service. The search time is assumed to be negligible. When the orbit size reaches $N + 1$, no more search is made for customer until it comes down to N at a service completion epoch. Thus during the period of no search, customers from orbit have to make trials on their own. Hence the present problem deals with a back and forth movement between classical queue and retrial queue. The motivation behind this model is that when orbit size increases, retrial rate (linear) also correspondingly increases thereby reducing the idle time of the server between services. Thus for large N, the retrial part behaves very close to classical queue whereas for small values of N, servers idle time between two consecutive services will be large if the system is in the retrial setup and so to eliminate this, customer search is made as long as orbit has $\leq N$ customers. It may be noted that the model under investigation is classical retrial queue for $N = 0$ and it behaves almost like the classical queue for large N. By assigning costs to customer search and cost for switching to retrial and back to classical we construct a suitable cost function in N. Its behaviour is very much like a convex function, though we are not able to prove it analytically presumably because of the presence of hyper geometric functions involving N in the numerator and denominator of the expression, together with terms in ρ^N (ρ = traffic intensity), for the total expected cost per unit time measured over a cycle.

In the next section we obtain explicit expressions for the system state probabilities. We also derive several system performance measures. In the last section we examine the cost function. Some numerical illustrations are provided in the concluding section.

6.1 Mathematical modelling

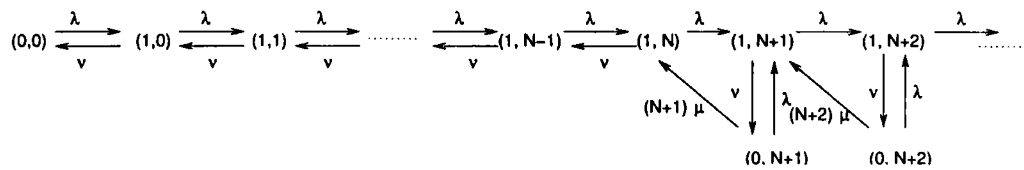
We assume that arrival of primary customers is governed by a Poisson process of intensity λ. Service times are exponentially distributed with parameter ν. Inter-retrial times are exponentially distributed with parameter $j\mu$, where j is the number of customers in the orbit.

Let $C(t) = \begin{cases} 1 & \text{if server busy at time } t, \\ 0 & \text{otherwise} \end{cases}$

and $N(t)$ = Number of customers in orbit at time t. Thus $\{(C(t), N(t)),\ t \geq 0\}$ is a Markov chain on $\{0, 1\} \times Z_+$.

By our assumption whenever orbit size is $\leq N$ at a service completion epoch $p_{0j}(t) = 0$ for $j \leq N$, since immediately on service completion the server picks up a customer from the orbit with probability 1 and the time for this procedure is negligible. Note that the equilibrium condition for the present model is the same as that for the classical retrial queue, namely that the traffic intensity $\rho\ (= \lambda/\nu) < 1$.

A diagrammatic representation of the state transitions is as follows :



The system of statistical equilibrium equations for the probabilities p_{0n} and p_{1n} is

$$(\lambda+\nu)p_{10} = \lambda p_{00} + \nu p_{11} \tag{6.1}$$

$$(\lambda+\nu)p_{1i} = \lambda p_{1,i-1} + \nu p_{1,i+1}\ ; \qquad 1 \le i \le N-1 \tag{6.2}$$

$$(\lambda+\nu)p_{1N} = \lambda p_{1,N-1} + (N+1)\mu p_{0,N+1} \tag{6.3}$$

$$(\lambda+\nu)p_{1i} = \lambda p_{1,i-1} + (i+1)\mu p_{0,i+1} + \lambda p_{0i}\ ; \qquad i \ge N+1 \tag{6.4}$$

$$\lambda p_{00} = \nu p_{10} \tag{6.5}$$

$$[\lambda + i\mu]p_{0i} = \nu p_{1,i}\ ; \qquad i \ge N+1. \tag{6.6}$$

With the help of equations (6.1), (6.2) and (6.6) eliminate probabilities p_{0i}, we get

$$\nu p_{1i} - \lambda p_{1,i-1} = 0\ ; \qquad 1 \le i \le N \tag{6.7}$$

$$ie, \qquad p_{1i} = \rho^{i+1} p_{00}\ ; \qquad 0 \le i \le N \tag{6.8}$$

Using (6.4) and (6.6),

$$(i+1)\mu\nu p_{0,i+1} - \lambda(\lambda+i\mu)p_{0i} = (i+2)\mu\nu p_{0,i+2} - \lambda(\lambda+(i+1)\mu)p_{0,i+1}\ ; \quad i \ge N+1 \tag{6.9}$$

$$ie, \qquad (i+1)\mu\nu p_{0,i+1} - \lambda(\lambda+i\mu)p_{0i} = 0\ ; \qquad i \ge N+1 \tag{6.10}$$

$$ie, \qquad p_{0,i+1} = \frac{\lambda(\lambda+i\mu)}{(i+1)\mu\nu} p_{0i}\ ; \qquad i \ge N+1 \tag{6.11}$$

and by using (6.3),

$$(N+1)\mu\nu p_{0,N+1} - \lambda\nu p_{1N} = 0 \tag{6.12}$$

Thus

$$p_{0,N+1} = \frac{\lambda\rho^{N+1}}{(N+1)\mu} p_{00} \tag{6.13}$$

$$p_{0i} = \frac{\lambda\rho^i}{\mu^{i-N}(N+1)} \prod_{j=N+2}^{i} [\frac{\lambda+(j-1)\mu}{j}]\, p_{00}; \qquad i \geq N+2 \tag{6.14}$$

Using (6.6)

$$p_{1i} = \frac{\rho^{N+i+1}}{\mu^i} \prod_{j=1}^{i} [\frac{\lambda+(N+j)\mu}{(N+j)}] p_{00}\ ; \qquad i \geq N+1 \tag{6.15}$$

(6.8), (6.13), (6.14) and (6.15) give the probabilities p_{0n} and p_{1n} in terms of p_{00}.

The probability p_{00} can be obtained with the help of normalizing condition $\sum_{i=0}^{\infty} p_{0i} + \sum_{i=0}^{\infty} p_{1i} = 1$ as follows :

First, we introduce some notation. Let ${}_2F_1$ be the hypergeometric series given by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{x^k}{k!}$$

where $(x)_k$ is the Pochhammer symbol defined by,

$$(x)_k = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1)\cdots(x+k-1) & \text{if } k \geq 1 \end{cases}$$

Now,

$$\sum_{i=0}^{\infty} p_{0i} = p_{00}\{\frac{\lambda\rho^{N+1}}{(N+1)\mu} + \sum_{i=N+2}^{\infty} \frac{\lambda\rho^i}{\mu^{i-N}(N+1)} \prod_{j=N+2}^{i} [\frac{\lambda+(j-1)\mu}{j}]\},$$

which reduces to

$$p_{00}\{1+\frac{\lambda\nu}{(N+1)\mu}\rho^{N+1}{}_2F_1(1,\frac{\lambda}{\mu}+N+1;\ N+2;\ \rho)\}$$

and

$$\sum_{i=0}^{\infty}p_{1i}=p_{00}\{\sum_{i=0}^{N}\rho^{i+1}+\sum_{i=N+1}^{\infty}\frac{\rho^{N+i+1}}{\mu^i}\prod_{j=1}^{i}[\frac{\lambda+(N+j)\mu}{(N+j)}]\}$$

which reduces to

$$p_{00}\{\frac{\rho(1-\rho^N)}{1-\rho}+\rho^{N+1}{}_2F_1(1,\frac{\lambda}{\mu}+N+1;\ N+1;\ \rho)\}.$$

Thus

$$p_{00}^{-1}=1+\frac{\rho(1-\rho^N)}{1-\rho}-\nu(\rho)^N+\rho^N(\rho+\nu(1-\rho))\,{}_2F_1(1,\frac{\lambda}{\mu}+N+1;\ N+1;\ \rho) \tag{6.16}$$

We get the corresponding partial generating functions as :

$$\begin{aligned}p_0(z)&=\sum_{i=0}^{\infty}z^ip_{0i}\\&=p_{00}\{1-\nu(\rho_z)^N+(1-\rho_z)\nu(\rho_z)^N{}_2F_1(1,\frac{\lambda}{\mu}+N+1;\ N+1;\ \rho_z)\}\end{aligned} \tag{6.17}$$

and

$$p_1(z)=\rho p_{00}\{\frac{1-(\rho_z)^N}{1-\rho_z}+(\rho_z)^N{}_2F_1(1,\frac{\lambda}{\mu}+N+1;\ N+1;\ \rho_z)\} \tag{6.18}$$

The generating function of the stationary distribution of the number of customers in the orbit $N(t)$ is $p(z)=p_0(z)+p_1(z)$. Thus expected number $E(N(t))$ of customers in the orbit is given by $p_0'(1)+p_1'(1)$.

After some simplification we get

$$E(N(t)) = \frac{\rho^2 - \rho^N[\rho^2 + N(1-\rho)(\rho + \nu(1-\rho))]}{(1-\rho)^2}$$
$$+ \rho^N\left[N\nu(1-\rho) + \rho(N-\nu)\right]{}_2F_1(1, \frac{\lambda}{\mu} + N + 1;\ N+1\ ;\rho)$$
$$+ \frac{\rho^{N+1}}{(N+1)\mu}[\lambda + (N+1)\mu]\left[\rho + \nu(1-\rho)\right]{}_2F_1(2, \frac{\lambda}{\mu} + N + 2; N+2, \rho) \tag{6.19}$$

B= Blocking probability = Pr [server is busy]= $p_1(1)$

$$= \rho p_{00}\left\{\frac{1-\rho^N}{1-\rho} + \rho^N{}_2F_1(1, \frac{\lambda}{\mu} + N + 1;\ N+1;\ \rho)\right\} \tag{6.20}$$

6.2 Control problem

In this section we construct a cost function in N by assigning a fixed cost for picking up a customer for service at a service completion epoch, provided number of customers in the orbit is $\leq N$. A fixed cost is assigned to each switching to retrial setup and back to system with search for customers.

Let $\mathcal{T}_x^y$ denote the first time that the process starting at x reaches y if $E_i =$ E[time to reach $(0,0)$ starting with $(0,0)$ | $(2i)$ switchings take place].
and $p_i =$ Pr $[(2i)$ switchings$]$ then expected cycle length $E(CL) = \sum_{k=0}^{\infty} E_k p_k$.
Now, $p_0 =$ Pr[0 switching] = Pr[$\mathcal{T}_{(1,0)}^{(0,0)} < \mathcal{T}_{1,0}^{1,N+1}$]

$$p_0 = \frac{(1-\rho^{N+1})}{1-\rho^{N+2}} \tag{6.21}$$

$$p_1 = \Pr[2 \text{ switchings}] = \Pr[1 \text{ right switching and a left switching}]$$
$$= \Pr[\, \mathcal{T}_{(1,0)}^{(1,N+1)} < \mathcal{T}_{(1,0)}^{(0,0)}] = \frac{\rho^{N+1}(1-\rho)}{1-\rho^{N+2}}. \quad (6.22)$$

Now $p_k = \Pr[(2k) \text{ switchings}] = \Pr[k \text{ right switchings}]$

$$= \Pr[\, \mathcal{T}_{(1,0)}^{(1,N+1)} < \mathcal{T}_{(1,0)}^{(0,0)}]\,[P[\, \mathcal{T}_{(1,N-1)}^{(1,N+1)} < \mathcal{T}_{(1,N-1)}^{(0,0)}]]^{k-1} = \frac{p^{N+1}(1-p)}{1-p^{N+2}}[\frac{p^2(1-p^N)}{(1-p^{N+2})}]^k \quad (6.23)$$

Case 1: $E_0 = E[CL|0$ switching take place]. Define T_{1i} : Time to reach $(1, i-1)$ from $(1, i)$ and T_{10} : Time to reach $(0, 0)$ from $(1, 0)$.
Then

$$E[T_{1i}] = (\frac{\nu}{\lambda+\nu})(\frac{1}{\lambda+\nu}) + (\frac{\lambda}{\lambda+\nu})\{\frac{1}{\lambda+\nu} + E[T_{1,i+1}] + E[T_{1i}]\};\ 0 \le i \le N-1$$

$$\text{ie,} \qquad E[T_{1,i}] = \frac{1}{\nu} + \rho E[T_{1,i+1}]; 0 \le i \le N-1 \quad (6.24)$$

$$\text{and} \qquad E[T_{1N}] = \frac{1}{\nu} \quad (6.25)$$

Starting with (6.25), and proceeding recursively, we get

$$E[T_{1,0}] = \frac{(1-\rho^{N+1})}{\rho(1-\rho)} \quad (6.26)$$

$$\text{Thus} \qquad E_0 = \frac{1}{\lambda} + \frac{(1-\rho^{N+1})}{\rho(1-\rho)} \quad (6.27)$$

Case 2: $k \ge 1$; E_k.

We proceed in 3 steps. In step 1, the system moves from $(1, 0)$ to $(1, N+1)$ (ie. one switching). In step 2, it moves from $(1, N+1)$ to $(1, N-1)$, k times and from $(1, N-1)$ to $(1, N+1)$, $(k-1)$ times. Finally in step 3, it moves from $(1, N-1)$ to $(0, 0)$ without any switching.

Thus

$$E_k = \mathcal{T}^{(1,0)}_{(0,0)} + \mathcal{T}^{(1,N+1)}_{(1,0)} + k\,\mathcal{T}^{(1,N-1)}_{(1,N+1)} + (k-1)\,\mathcal{T}^{(1,N+1)}_{(1,N-1)} + \mathcal{T}^{(0,0)}_{(1,N-1)} \quad (6.28)$$

Step 1: Define T'_{1i}: Time to reach $(1, i+1)$ from $(1, i); 0 \leq i \leq N$. Then

$$E[T'_{1i}] = (\frac{\lambda}{\lambda+\nu})(\frac{1}{\lambda+\nu}) + (\frac{\nu}{\lambda+\nu})\{\frac{1}{\lambda+\nu} + E[T'_{1,i-1}] + E[T'_{1,i}]\}$$

Then we get

$$E[T'_{1i}] = \frac{1}{\nu}\frac{1-\rho^{i+1}}{\rho^{i+1}(1-\rho)} \quad (6.29)$$

Thus

$$\mathcal{T}^{(1,N+1)}_{(1,0)} = E[T'_{1,0}] + E[T'_{1,1}] + \cdots + E[T'_{1,N}]$$
$$= \frac{1}{\nu(1-\rho)}\{\frac{1-\rho^{N+2}}{\rho^{N+2}(1-\rho)} - (N+1)\} \quad (6.30)$$

Step 2: Define ${}_sT_{1,i}$: time to reach $(0, i)$ from $(1, i); i \geq N+1$ and ${}_sT'_{0,i}$: time to reach $(1, i-1)$ from $(0, i); i \geq N+1$. Then

$$E[{}_sT_{1,i}] = (\frac{\nu}{\lambda+\nu})(\frac{1}{\lambda+\nu}) + (\frac{\lambda}{\lambda+\nu})\{\frac{1}{\lambda+\nu} + E[{}_sT_{1,i+1}] + E[{}_sT'_{0,i+1}] + E[{}_sT_{1,i}]\} \quad (6.31)$$

Thus,

$$E[{}_sT_{1,i}] = (\frac{1}{\nu})_2F_1(1, \frac{\lambda}{\mu}+i+1; i+1; \rho) + (\frac{1}{\nu})(\frac{\lambda}{(i+1)\mu})_2F_1(1, \frac{\lambda}{\mu}+i+1; i+2; \rho) \quad (6.32)$$

and

$$E[{}_sT'_{0,i}] = \frac{1}{i\mu} + (\frac{1}{\nu})(\frac{\lambda}{i\mu})_2F_1(1, \frac{\lambda}{\mu}+i+1; i+1; \rho) + (\frac{1}{\nu})\frac{\lambda^2}{i(i+1)\mu^2}{}_2F_1(1, \frac{\lambda}{\mu}+i+1; i+2; \rho) \quad (6.33)$$

Now,

$$\mathcal{T}^{(1,N-1)}_{(1,N)} = (\frac{1}{\lambda+\nu})(\frac{\nu}{\lambda+\nu}) + \frac{\lambda}{(\lambda+\nu)}\{\frac{1}{\lambda+\nu} + \mathcal{T}^{(1,N)}_{(1,N+1)} + \mathcal{T}^{(1,N-1)}_{(1,N)}\}$$

$$\text{ie,} \quad \mathcal{T}^{(1,N-1)}_{(1,N)} = \frac{1}{\nu} + \rho\,\mathcal{T}^{(1,N)}_{(1,N+1)}. \tag{6.34}$$

$$\text{But} \quad \mathcal{T}^{(1,N-1)}_{(1,N+1)} = \mathcal{T}^{(1,N)}_{(1,N+1)} + \mathcal{T}^{(1,N-1)}_{(1,N)} \tag{6.35}$$

Using (6.34) and (6.35),

$$\mathcal{T}^{(1,N-1)}_{(1,N+1)} = \frac{1}{\nu} + (1+\rho)\,\mathcal{T}^{(1,N)}_{(1,N+1)} = \frac{1}{\nu} + (1+\rho)\{E[{}_sT_{1,N+1}] + E[{}_sT'_{0,N+1}]\}$$

$$\mathcal{T}^{(1,N-1)}_{(1,N+1)} = (\frac{1}{\nu}) + \frac{(1+\rho)}{(N+1)\mu} + \frac{(1+\rho)[\lambda+(N+1)\mu]}{\nu(N+1)(N+2)\mu^2}$$
$$[\lambda {}_2F_1(1, \frac{\lambda}{\mu}+N+2; N+3; \rho) + (N+2)\mu {}_2F_1(1, \frac{\lambda}{\mu}+N+2; N+2; \rho)] \tag{6.36}$$

Now

$$\mathcal{T}^{(1,N+1)}_{(1,N-1)} = E[T_{1,N-1}] + E[T_{1N}] = \frac{1-\rho^N}{\nu\rho^N(1-\rho)} + \frac{1-\rho^{N+1}}{\nu\rho^{N+1}(1-\rho)} \tag{6.37}$$

Step 3: Define $T''_{1,i}$: Time to reach $(1, i-1)$ from $(1, i); i \leq N$. Then

$$E[T''_{1N}] = \frac{1}{\nu} \tag{6.38}$$

and

$$E[T''_{1,i}] = (\frac{\nu}{\lambda+\nu})(\frac{1}{\lambda+\nu}) + (\frac{\lambda}{\lambda+\nu})\{(\frac{1}{\lambda+\nu}) + E[T''_{1,i+1}] + E[T''_{1,i}]\};\ i \leq N-1$$

ie,

$$E[T''_{1,i}] = \frac{1}{\nu} + \rho E[T''_{1,i+1}]; \qquad i, \leq N-1 \tag{6.39}$$

Starting from (6.38) and proceeding recursively, we get

$$E[T''_{1,i}] = \frac{1-\rho^{N+1-i}}{\nu(1-\rho)}; \qquad i \leq N \tag{6.40}$$

Thus

$$T^{(0,0)}_{(1,N-1)} = E[T''_{1,N-1}] + \cdots + E[T''_{1,0}] = \frac{1}{\nu(1-\rho)}\{(N+1) - \frac{\rho(1-\rho^{N+1})}{(1-\rho)}\} \tag{6.41}$$

After some simplification we get

$$E(CL) = \sum_{k=0}^{\infty} E_K \rho_k = \{\frac{1}{\lambda} + (\frac{1}{\nu})(\frac{1-\rho^{N+1}}{1-\rho})\}\{\frac{1-\rho^{N+1}}{1-\rho^{N+2}}\}$$
$$+ \{\frac{1}{\lambda} + \frac{1}{\nu(1-\rho)}[\frac{1-\rho^{N+2}}{\rho^{N+2}(1-\rho)} - \frac{\rho(1-\rho^{N+1})}{(1-\rho)}]\}\frac{\rho^{N+1}(1-\rho)}{(1-\rho^2)}$$
$$+ \{\frac{1}{\nu} + \frac{(1+\rho)}{(N+1)\mu} + \frac{(1+\rho)[\lambda + (N+1)\mu]}{\nu(N+1)(N+2)\mu^2}$$

$$[\lambda_2 F_1(1, \frac{\lambda}{\mu} + N + 2; N+3; \rho) + (N+2)\mu_2 F_1(1, \frac{\lambda}{\mu} + N + 2; N+2; \rho)]\}$$
$$\frac{\rho^{N+1}(1-\rho)(1-\rho^{N+2})}{(1-\rho^2)^2}$$
$$+ \{\frac{(1-\rho^N)}{\nu\rho^N(1-\rho)} + \frac{1-\rho^{N+1}}{\nu\rho^{N+1}(1-\rho)}\}\frac{\rho^{N+3}(1-\rho)(1-\rho^N)}{(1-\rho^2)^2} \tag{6.42}$$

Let $E[SW]$ denote the expected number of switchings in a cycle.
Then $E[SW] = \sum_{k=1}^{\infty}(2k)p_k$

$$= 2\sum_{k=1}^{\infty} k\frac{\rho^{N+1}(1-\rho)}{(1-\rho^{N+2})}[\frac{\rho^2(1-\rho^N)}{(1-\rho^{N+2})}]^{k-1} = 2\frac{\rho^{N+1}(1-\rho)}{(1-\rho^{N+2})}[\frac{1-\rho^{N+2}}{1-\rho^2}]^2 \quad (6.43)$$

Let $E[SR]$ denote the expected number of searches in a cycle.

$$\text{Then } E[SR] = \sum_{k=0}^{\infty} E[\text{ No of searches } |(2k) \text{ switchings in a cycle }]$$

$$\cdot Pr[(2k) \text{ switching }].$$

$$= \frac{\mathcal{T}_{(1,0)}^{(0,0)}}{\nu}p_0 + \sum_{K=1}^{\infty}[\frac{(\mathcal{T}_{(1,0)}^{(1,N+1)} - \frac{1}{\lambda})}{\nu} + (k-1)\frac{(\mathcal{T}_{(1,N-1)}^{(1,N+1)} - \frac{1}{\lambda})}{\nu} + k.+\frac{\mathcal{T}_{(1,N-1)}^{(0,0)}}{\nu}]p_k$$

$$= \frac{(1-\rho^{N+1})^2}{\nu\rho(1-\rho)(1-\rho^{N+2})} + \frac{\rho^{N+1}(1-\rho)}{(1-\rho^2)}\{\frac{1-\rho^{N+2}}{\nu^2\rho^{N+2}(1-\rho)^2} - \frac{1}{\lambda\nu} - \frac{\rho(1-\rho^{N+1})}{\nu(1-\rho)^2}\}$$

$$+ \frac{\rho^{N+3}(1-\rho)(1-\rho^N)}{(1-\rho^2)^2}\{\frac{1-\rho^N}{\nu^2\rho^2(1-\rho)} + \frac{1-\rho^{N+1}}{\nu^2\rho^{N+1}(1-\rho)} - \frac{1}{\lambda\nu}\}$$

$$+ \frac{\rho^{N+1}(1-\rho)(1-\rho^{N+2})}{(1-\rho^2)^2} \quad (6.44)$$

Let C_w be the cost for one switching and C_r be the cost for one search.

$$\text{Thus Cost function, } CF = \frac{E[SW]C_w + E[SR]C_r}{E(CL)} \quad (6.45)$$

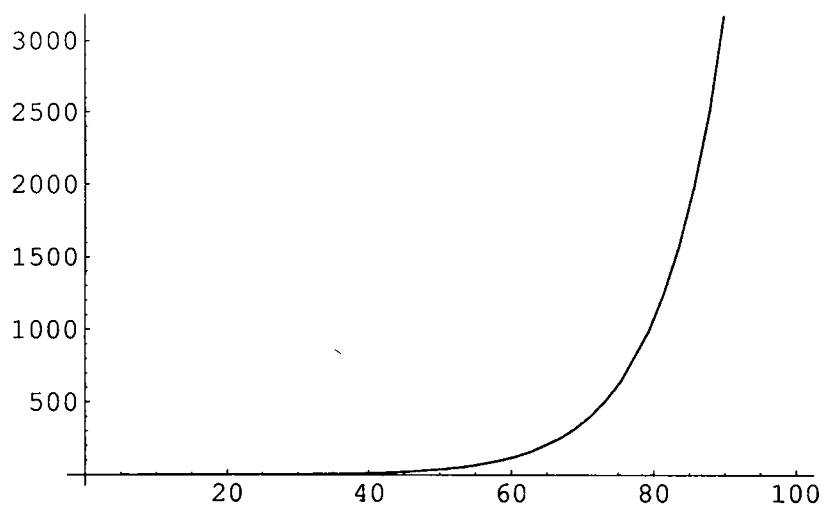
Though we are not able to prove analytically that the expected cost per unit time is convex in N, all computations that we have made indicate a strong tendency of this function towards convexity. Some typical illustrations are given in the graphs and the tables. The convexity of the cost function is to be expected as the number searches and hence search cost increases with increasing N, at the expected number of switchings per cycle decreases as shown in the tables.

$\lambda = 9; \mu = 5; \nu = 10$					
N	$E(N(t))$	B	$E(CL)$	$E[SR]$	E[SW]
1	20.8300	0.48575	35.480	0.1850	1.21612
11	19.3821	0.67900	4.29840	1.0258	1.16698
21	13.1294	0.83765	1.20961	1.6894	0.49722
31	9.8515	0.88517	0.37565	1.6952	0.18435
41	8.6918	0.89640	0.12127	2.0667	0.06561
51	8.3018	0.89909	0.04004	2.1027	0.02304
61	8.1696	0.89976	0.01341	2.1154	0.00805
71	8.1242	0.89993	0.00454	2.1198	0.00281
81	8.1084	0.89998	0.00154	2.1214	0.00098
91	8.1029	0.89999	0.00052	2.1219	0.00034
101	8.1010	0.89999	0.00018	2.1221	0.00011

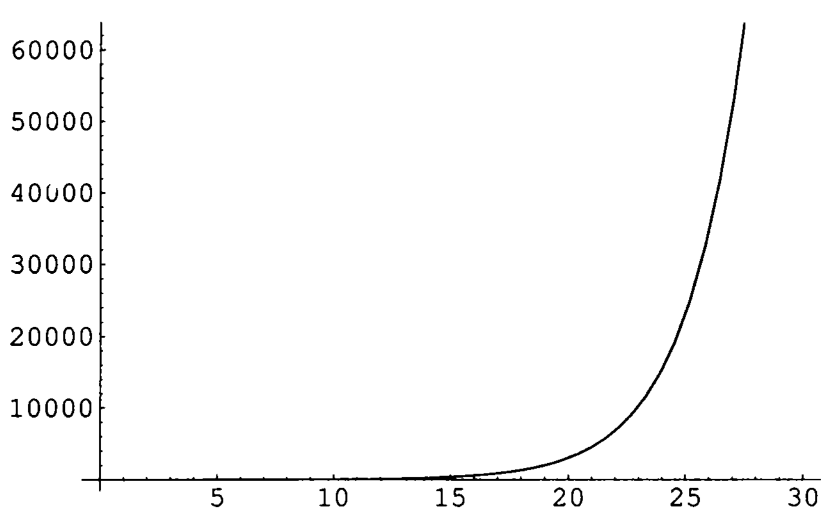
TABLE 6.1.

$\lambda = 2; \mu = 3; \nu = 3$					
N	$E(N(t))$	B	$E(CL)$	$E[SR]$	E[SW]
1	2.6409	0.6536	1.6837	1.3178	0.6755
11	1.3891	0.6687	0.0347	2.6388	0.0165
21	1.3350	0.6667	0.0005	2.6662	0.0002
31	1.33363	0.6667	0.00001	2.66668	5.007×10^{-6}
41	1.3336	0.6667	1.73×10^{-7}	2.6666	8.68×10^{-8}
51	1.3336	0.6667	3.004×10^{-9}	2.6666	1.50×10^{-9}
61	1.3336	0.6667	5.19×10^{-11}	2.6666	2.61×10^{-11}
71	1.3336	0.6667	5.11×10^{-13}	2.6666	4.52×10^{-13}
81	1.3336	0.6667	8.81×10^{-15}	2.6666	7.85×10^{-15}
91	1.3336	0.6667	1.52×10^{-16}	2.6666	1.36×10^{-16}
101	1.3336	0.6667	2.65×10^{-18}	2.6666	2.36×10^{-18}

TABLE 6.2.



(a) $\lambda = 9$; $\mu = 5$, $\nu = 10$



(b) $\lambda = 2$; $\mu = 3$, $\nu = 3$

Figure 6.1: Graph of cost function CF (N along X-axis and cost along Y-axis.)

Bibliography

[1] A. Aissani, Unreliable queueing with repeatede orders, Microelectronics and Reliability 33 ,(1993)

[2] A. Aissani, A retrial queueing with redundancy and unreliable server, Queueing systems 17, (1994)

[3] A. Aissani and J. R. Artalejo, On the sungle server retrial queue subject to breakdowns, Queueing systems 30 (1998)

[4] J. R. Artalejo, New results in retrial queueing systems with breakdown of the servers, Statistica Neerlandica, 48(1), 1994.

[5] J. R. Artalejo, Accessible bibliography on retrial queues, *Mathematical and Computer Modelling* , **30**, (1999).

[6] J. R. Artalejo, A classified bibliography of research on retrial queues: Progress in 1990-1999, *Top* **7**, (1999).

[7] J. R. Artalejo and G. I. Falin, Standard and retrial queueing systems: A comparative analysis, *Revista Matematica Complutense*, **15** (2002) (to appear).

[8] J. R. Artalejo and A. Gomez-Corral, Steady state solution of a single-server queue with linear repeated requests, *Journal of Applied Probability* **34**, (1997).

[9] J. R. Artalejo and A. Gomez-Corral, Analysis of a stochastic clearing Systems with repeated attempts, Commun. Statist-Stochastic Models, 14(3), (1998).

[10] J. R. Artalejo and A. Gomez-Corral, Unreliable retrial queues due to service interruptions arising from facsimile networks, Belgian Journal of Operations Research, Statistics and Computer Science, Vol 38(1) 1998.

[11] J. R. Artalejo and A. Gomez-Corral, On a single server queue with negative arrivals and request repeated, *Journal of Applied Probability* **36**, (1999).

[12] J.R. Artalejo and M.J. Lopez-Herrero, On the single server retrial queue with balking, *Infor* **38**, (2000).

[13] J. R. Artalejo and M. J. Lopez-Herrero, On the busy period of the $M|G|1$ retrial queue, Naval Research Logistics, Vol 47 (2000).

[14] J. R. Artalejo and M. Pozo, Numerical calculation of the stationary distribution of the main multi-server retrial queue, Annals of O.R., 2000.

[15] Sren Asmussen, Phase-type distributions and related point processes: Fitting and recent advances, Eds. S. R. Chakravarthy and A. S. Alfa, Matrix-analytic methods in stochastic models, Marcel Dekker, Inc. (1997)

[16] R. E. Bellman, Introduction to Matrix Analysis, McGraw Hill, New York, 1960.

[17] L. W. Bright and P. G. Taylor, Calculating the equilibrium distribution in level dependent quasi-birth-and-death process, Comm. Stast. Stochastic Models, 11, (1995).

[18] S. R. Chakravarthy, Reliability analysis of a parallal system with exponential lifetimes and phase-type repairs, OR Spectrum 5, (1983).

[19] S. R. Chakravarthy, Analysis of a finite $MAP|G|1$ queue with group services, Queueing Systems, 13, (1993).

[20] S. R. Chakravarthy and A. S. Alfa, A multi-server queue with Markovian arrivals and group services with thresholds, Naval Research Logistics, Vol 40, (1993).

[21] S. R. Chakravarthy, Analysis of the $MAP|PH|1|K$ queue with service control, Applied stochastic Models and Data Analysis, Vol 12, (1996).

[22] S. R. Chakravarthy, The batch Markovian arrival process: A review and future work. Advances in Probability Theory and Stochastic Processes. Eds., A. Krishnamoorthy, et al., Notable Publications Inc., NJ, (2000).

[23] S. R. Chakravarthy and A. S. Alfa, A multi-server queue with BMAP arrivals and group services, Queueing Systems, (2002).

[24] B. D. Choi amd V. G. Kulkarnni, Feedback retrial queueing systems in: queueing and related models, Eds. U. N. Bhat and I. V. Basawa, Oxford Science Publications (1992).

[25] B. D. Choi, Y. W. Shin and W. C. Ahn, Retrial queues with collision arising from unslotted $CSMA/CD$ protocol, *Queueing Systems* **11**, (1992).

[26] J. W. Cohen, Basic problems of telephone traffic theory and the influence of repeated calls, Philips Telecommunication Review, 18, No. 2, 1957.

[27] D. R. Cox and H. D. Miller, The Theory of Stochastic Processes, Chapman & Hall, 1965.

[28] R. B. Cooper, *Introduction to Queueing Theory*, Elsevier, Amsterdam (1981).

[29] A. N. Dudin and V. I. Klimenock, Queueing system $BMAP|G|1$ with repeated calls, Mathematical and Computer Modelling, 30 (1999).

[30] A. N. Dudin and V. I. Klimenock, Multidimentional quasi-toeplitz Markov chains, Journel of Applied Mathematics and Stochastic Analysis, 12 (1999).

[31] A. N. Dudin and V. I. Klimenock, $BMAP|SM|1$ model with Markov modulated retrials, TOP, 7 (1999).

[32] G. I. Falin, Aggregate arrival of customers in one-line system with repeated calls, Ukrainian Math. J., 28, No. 4, (1976).

[33] G. I. Falin, Waiting time in a single-channel queueing system with repeated calls, Moscow University Computational Mathematics and Cybernetics, No.4, (1977).

[34] G. I. Falin, On sufficient conditions for erogicity of multichannel queueing systems with repeated calls, Adv. Appl. Probab. 16, (1984).

[35] G. I. Falin, Multi-channel queueing systems with repeated calls under high intensity of repetition, Jl. Inf. Proces. and Cybernetics, 1, (1987).

[36] G. I. Falin, A survey of retrial queues, Queueing Systems, 7 (1990).

[37] G. I. Falin and J. G. C. Templeton, *Retrial Queues*, Chapman and Hall, London (1997).

[38] D. Gross and C. M. Harris, Fundamentals of queueing theory, John Wiley and Sons Inc. 1985.

[39] Hideaki Takagi, Queueing Analysis, Vol 1, North Holland, 1991.

[40] Hideaki Takagi, Queueing Analysis, Vol 2, North Holland, 1993.

[41] Hideaki Takagi, Queueing Analysis, Vol 3, North Holland, 1993.

[42] G. L. Jonın and J. J. Sedol., Telephone systems with repeated calls, 6th Int. Teletraffic Congress, Munich, (1970), 435, 5.

[43] J. Keilson, J. Cozzolino and H. Young, A service system with unfilled requests repeated, Operations Research, 16, (1968).

[44] D. G. Kendall, Stochastic processes occuring in the theory of queues and their analysis by the methord of the imbedded Markov chain, Ann. math. Statist., 24 (1953).

[45] L. Kleinrock, *Queueing Systems, Volume 1: Theory*, Wiley, New York (1975).

[46] L. Kosten, On the influence of repeated calls in the theory of probabilities of blocking, De Ingenieur, 59, (1947).

[47] A. G. de Kok, Computational methods for single server systems with repeated attempts, Repost No. 89, Interfaculteit der Actuariele Wetenschappen en Econometric, Amsterdam, (1982).

[48] A. G. de Kok, Algorithmic methods for single server systems with repeated attempts, *Statistica Neerlandica* **38**, (1984).

[49] A. Krishnamoorthy and T. G. Deepak, Modified N-policy for $M|G|1$ queues, Computers & Operations Research, 29, (2002).

[50] A. Krishnamoorthy and P. V. Ushakumari, $M|G|1$ queue with disaster to the unit in service, presented to the I International Workshop in Retrial Queues held at Madrid, Sept. 1998.

[51] A. Krishnamoorthy and P.V. Ushakumari, Reliability of a k-out-of-n system with repair and retrial of failed units, *Top* **7**, (1999).

[52] A. Krishnamoorthy and P. V. Ushakumari, $GI|M|1|1$ queue with finite retrials and finite orbits, Stochastic Analysis and Applications, vol 20, (2002).

[53] V. G. Kulkarnni and B. D. Choi, Retrial queues with server subject to breakdown and repairs, Queueing systems, (7) 1990.

[54] G. Latouche and V. Ramaswami, Introduction to matrix analytic methods in stochastic modelling, American Statustical Association and the Society for Industrial and Applied Mathematics, 1999

[55] D. M. Lucantoni, New results on the single server queue with a batch Markovian arrival process, Stochastic Models, 7. (1991).

[56] M.F. Neuts and M.F. Ramalhoto, A service model in which the server is required to search for customers, *Journal of Applied Probability* **21**, (1984).

[57] M. F. Nuets and S. Chakravarthy, A single server queue with platooned arrivals and phase type services.

[58] M. F. Neuts, A versatile Markovian point process, J. Appl. Prob., 16, (1979).

[59] M. F. Neuts, Matrix-geometric solutions in stochastic models: An algorithmic approach, The Johns Hopkins University Press, Baltimore, MD (1981).

[60] M. F. Neuts, Structured stochastic matrices of M/G/1 type and their applications, Marcel Dekker Inc., New York, (1989).

[61] M. F. Neuts, Modeling based on the Markovian arrival process, IEICE Trans. Commun. E75-B, 12, (1992).

[62] M. F. Neuts and B. M. Rao, Numerical investigation of a multiserver retrial model, Queueing systems, 7 (1990).

[63] V. Ramaswami, The $N|G|1$ queue and its detailed analysis, Adv. in Appl. Probab., 12, (1980)

[64] V. Ramaswami and G. Latouche, A general class of Markov process with explicit matrix-geometric solutions, OR Spectrum, 8, (1986).

[65] V. Ramaswami and P. G. Taylor, Some Properties of the rate operators in level dependent quasi-birth-and-death processes with a countable number of phases, Comm. Statist., Stochastic Models, 12, (1996).

[66] T. L. Saaty, Elements of queueing theory with applications, New York, McGrew-Hill.

[67] L.I. Sennott, P.A. Humblet and R.L. Tweedie, Mean drifts and the non-ergodicity of Markov chains, *Operations Research* **31**, (1983).

[68] H. M. Srivastava and B. R. K. Kashyap, Special functions in queueing theory, Academic Press, (1982).

[69] H.C. Tijms, *Stochastic Models: An Algorithmic Approach*, Wiley, Chichester (1994).

[70] J. Wang, J. Cao and Q. Li, Reliability analysis of the retrial queue with server breakdowns and repairs, *Queueing Systems* **38**, (2001).

[71] R. I. Wilkinson, Theories for toll traffic engineering in the USA, Bell. Syst. Techn. J., 35, No.2, (1956).

[72] S. Wolfram, The Mathmatica, Fourth Edn, Mathematica version 4.

[73] H. C. White and L. S. Christie, Queueing with preemptine priorities or with breakdown, Operations Research, 6, 1958.

[74] T. Yang and J. G. C. Templeton, A survey on retrial queues, Queueing systems, 2, (1987).

Topic of research

STOCHASTIC MODELLING ANALYSIS AND APPLICATIONS

SYNOPSIS

of the thesis entitled

RETRIAL QUEUES WITH ORBITAL SEARCH

to be submitted for the degree of

Doctor of Philosophy

to the

Cochin University of Science and Technology

by

VARGHESE C. JOSHUA

Varghese C. Joshua
Research Scholar (Reg.No.1467)
Dept. of Mathematics
Cochin University of
Science & Technology
Kochi – 682022.

Dr. A. Krashnamoorthy
Professor
Dept. of Mathematics
Cochin University of
Science & Technology
Kochi – 682022.

RETRIAL QUEUES WITH ORBITAL SEARCH

INTRODUCTION:

Queueing theory was developed to model that predict behavior of systems which attempt to provide service for randomly arising demands. Thus, a queueing system can be described as customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served. The earliest problems studied were those of telephone traffic congestion. The pioneer investigator was the Danish mathematician A.K.Erlang, who, in 1909 published "The Theory of Probabilities and Telephone Conversations". Work on the application of the theory of telephone systems continued after Erlang. In 1927, Molina published his "Application of the Theory of probability to telephone Trunking Problems', which was followed by Thornton Fry's "Probability and its Engineering Uses" which expanded much of Erlang's earlier work. In the early 1930's Felix Pollaczek did some further pioneering work on Poisson input, arbitrary output, and single and multiple-channel problems. The early work in queueing theory picked up momentum rather slowly. Telephone systems remained the principal application of the theory through about 1950. But the trend began to change in the 1950's and of late numerous other applications have been found and there has been much work in the area. Some of the major applications of the theory are landing of aircrafts, loading and unloading of ships, taxi services, machine repair, inventory control, telecommunication networks, computer networks and computer systems.

Queueing system in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called retrial queues or queues with repeated attempts. Thus, retrial queues are characterized by the following feature: a customer arriving when all servers accessible for him are busy leaves the service area but after some random time repeats his demand. Retrial queue is a type of network with reservicing after blocking. Thus, this network contains two nodes: the main node where blocking is possible and a delay node for repeated attempts. As for other networks with blocking, the investigation of such systems presents great analytical difficulties. In contrast, there are a great number of numerical and approximation methods. Nevertheless, the main features of the theory of retrial queueing systems as an independent part of queueing theory are quite clearly drawn. In particular, the nature of results obtained, methods of analysis and areas of applications allow us to divide retrial queues into three large groups in a natural way: single-channel systems, multi-channel fully available systems, and structurally complex systems. The first mathematical results about retrial queues were published in the 1950's and at present, a full bibliography consists of several hundred items. For a systematic account of the fundamental methods and results on this topic, we refer to the monograph by Falin and Templeton [4], two extensive survey articles due to Yang and Templeton [7], Falin [5] and the bibliographical information in J. R.Artalejo [1] and [2].

Retrial queues considered by researchers so far have the characteristic that each service is preceded and followed by an idle period which is terminated either by the arrival of a customer from the orbit or by a primary customer. Even if there are some customers in the system who want to get service they cannot occupy the server immediately, because of their ignorance of the server state. Therefore, after the completion of each service, next customer enters service only after some time interval during which the server is free while there may be waiting customers in the orbit. All material of the proposed thesis are concerned with retrial queueing models in which, even without a waiting room, each service completion epoch need not necessarily be followed by an idle time. This is achieved as follows: immediately on a service completion, the server picks up a customer from the orbit with probability p_j, when there are j customers in the orbit (it is assumed that the server is aware of the orbital status, for example there is a register with him of customers in orbit, whereas the orbital customers are ignorant of the server status). With probability $1-p_j$ no search is made on a service completion epoch and in this case, as in the classical retrial queue, a competition takes place in between primary and secondary (orbital) customers for service. Thus, if search is made, a service is followed by another service and if no search is made, a service is followed by an idle time.

Our study has two main objectives. The first one is to introduce orbital search in retrial queueing models which allows to minimize the idle time of the server. If the holding costs and cost of using the search of customers will be introduced, the results we obtained can be used for the optimal tuning of the parameters of the search mechanism. The second objective is to provide insight of the link between the corresponding retrial queue and the classical queue. To this end, we observe that when the search probability $p_j = 1$ for all j, the model reduces to the classical queue and when $p_j = 0$ for all j, the model becomes the retrial queue..

Summary of the Thesis

In chapter1, an introduction is given to the functioning and analysis of various retrial queueing models.

In chapter 2, we concentrate on the performance evaluation of a single-server retrial queue with orbital search as follows: we consider a single-server queueing system to which primary customers arrive according to a Poisson stream of rate λ. If the server is free at the time of arrival of a primary customer, the arriving customer begins to be served immediately and leaves the system after service completion. Otherwise, if the server is busy, the arriving customer becomes a source of repeated calls. Every such source produces a Poisson process of repeated calls with intensity μ. The service times are independent with common probability function B(x), Immediately after completion of each service, the server goes for search of customers in the orbit with probability p_j and remains idle with probability $1-p_j$, when there are j customers in the orbit. The stability condition of the system state is investigated. Explicit expressions of the limiting probabilities and their moments are obtained for the case of exponential service times.

In chapter 3, a single-server retrial queueing model with impatient customers and orbital search is considered. If the server is busy at the time of arrival of a primary call then with probability $1-H_1$ it leaves the system without service and with probability $H_1 > 0$, forms a source of repeated calls. Similarly, if the server is occupied at the time of arrival of a repeated call, with probability $1-H_2$ it leaves the system without service and with probability H_2, it goes back to the orbit. All other assumptions and notations introduced in chapter 2 hold in this chapter as well. An important feature of the model under consideration is that for many problems, the cases $H_2 < 1$ and $H_2 = 1$ yield essentially different solutions. In the case $H_2 = 1$, the model is analyzed in full detail using supplementary-variable method. Stability condition is obtained. The joint distribution of the server state and the orbit length in steady state is studied. The structure of the busy period and its analysis in terms of Laplace transforms have been discussed. This chapter also provides a direct method of evaluation for the first and second moments of the busy period.

The case $H_2 < 1$ is far more complicated and so closed form solution is obtained only in the case of exponential service time distribution. In the general case, a complete closed form solution seems impossible.

In chapter 4, We consider an M/PH/1 retrial queue with disaster to the unit in service and orbital search. In addition to the assumptions of the model considered in chapter 2, the unit undergoing service is subject to disasters; (or equivalently, the server is subject to interruptions during service time). Disasters occur according to a Poisson process with rate σ. Here service times are assumed to be of phase type distribution which has a number of interesting properties. PH distributions and PH renewal processes were introduced by M. F. Neuts [6]. The class of PH-distributions includes many well-known distributions such as generalized Erlang, hyper exponential etc., as special cases and has a number of interesting closure properties.

In chapter 5, we consider a multi- server retrial queueing model (MAP/M/c) with search of customers from the orbit. The Markovian arrival processes (MAP), a special class of tractable Markov renewal process, is a rich class of point process that includes many well-known process, and Markov-modulated Poisson process. One of the most significant features of the MAP is the underlying Markovisan structure and fits ideally in the context of matrix-analytic solutions to stochastic models. Matrix- analytic methods were first introduced by M. F. Neuts [6]. The idea of the MAP is to significantly generalize the Poisson process and still keep the tractability for modeling purposes. In many practical applications, notably in communication engineering, production and manufacturing engineering, the arrivals do not usually form a renewal process. MAP is a convenient tool to model both renewal and non-renewal arrivals. The steady-state analysis of the model using direct-truncation and matrix-analytic approximation are performed. Efficient algorithms for computing various steady-state performance measures and illustrative numerical examples are presented.

In chapter 6, we consider an excursion between classical and the retrial queue in the following way: As long as the number of customers in the orbit remains less than or equal to N, the server immediately on a service completion, picks up the next customer from the orbit with probability 1 and starts service. When the orbit size reaches N+1, no more search is made for customers until the orbit size comes down to N at a service completion epoch. Thus, the present model deals with a back and forth movement between classical queue and retrial queue. The motivation behind this model is that, when orbit size increases, retrial rate also correspondingly increases thereby reducing the idle time of the server between services. By assigning costs to customer search and cost for switching to retrial and back to classical, a suitable cost function in N is constructed. Some numerical results are provided.

References:

[1] J. R. Artalejo, Accessible bibliography on Retrial queues, Mathematical and Computer Modelling 30, 1-6, (1999).

[2] J. R. Artalejo, A classified bibliography of research on retrial queues: Progress in 1990-1999, Top 7, 187-211, (1999).

[3] Donald Gross and Carl M. Harris, Fundamentals of Queueing Theory, John Wiley & Sons (1985).

[4] G. I. Falin and J. G. C. Templeton, Retrial Queues, Chapman and Hall, London, (1997).

[5] Gennadij Falin, A Survey of Retrial Queues, Queueing Systems, 7, 127-168, (1990).

[6] Neuts, M.F., Matrix-Geometric Solution in Stochastic Models: An Algorithmic Approach, The John Hopkins University Press, Baltimore, MD, (1981).

[7] Tao Yang and J. G .C. Templeton, A Survey On Retrial Queues, Queueing Systems 2, 201-233, (1987).