Fluid Mechanics

A GEOMETRIC APPROACH TO FLUID DYNAMICS IN A SPACE TIME MANIFOLD

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Susan Mathew Panakkal

(Reg no. 3500)

Department of Mathematics

Cochin University of Science and Technology

Kochi - 682 022, Kerala, India

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A GEOMETRIC APPROACH TO FLUID DYNAMICS IN A SPACE TIME MANIFOLD

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Author:
Susan Mathew Panakkal
Research Scholar, Department of Mathematics
Cochin University of Science and Technology
Kochi - 682 022, Kerala, India
Email: susanmathewpanakkal@gmail.com

Supervisor:
Dr. M. Jathavedan
Emeritus Professor
Department of Computer Applications
Cochin University of Science and Technology
Kochi - 682 022, Kerala, India.
Email: mjvedan@gmail.com

March 2019

Dr. M. Jathavedan
Emeritus Professor
Department of Computer Applications
Cochin University of Science and Technology
Kochi - 682 022, Kerala, India.

7th March, 2019

Certificate

Certified that the work presented in this thesis entitled "**A GEOMETRIC APPROACH TO FLUID DYNAMICS IN A SPACE TIME MANIFOLD"** is based on the authentic record of research carried out by Mrs. Susan Mathew Panakkal under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted for the award of any degree.

Dr. M. Jathavedan
(Supervising Guide)

Phone : 9496448160. Email: mjvedan@gmail.com

Department of Mathematics
Cochin University of Science and Technology
Cochin - 682 022

Certificate

Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis entitled "**A GEOMETRIC APPROACH TO FLUID DYNAMICS IN A SPACE TIME MANIFOLD**".

Dr. M. Jathavedan
(Supervising Guide)
Emeritus Professor
Department of Computer Applications
Cochin University of Science & Technology
Cochin - 682 022

Cochin
7^{th} March, 2019

7^{th} March, 2019

Declaration

I, SUSAN MATHEW PANAKKAL, hereby declare that the work presented in this thesis entitled "**A GEOMETRIC APPROACH TO FLUID DYNAMICS IN A SPACE TIME MANIFOLD** "is based on the original research work carried out by me under the supervision and guidance of Dr. M. Jathavedan, Emeritus Professor, Department of Computer Applications, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted previously for the award of any degree.

Susan Mathew Panakkal
Research Scholar (Reg. No. 3500)
Department of Mathematics
Cochin University of Science & Technology

Cochin-22
7^{th} March, 2019

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Susan Mathew Panakkal

“To God be all Glory”.

A GEOMETRIC APPROACH TO FLUID DYNAMICS IN A SPACE TIME MANIFOLD.

Contents

List of Notations

v	velocity vector in three dimensional space
p	pressure
ρ	mass density
φ	potential for the body forces
ν	kinematic viscosity
ε	Bernoulli function
w	voticity vector in three dimensions
M	manifold
α^k	k-form
D^k	k-dimensional chain
C^k	k-dimensional cycle
ψ_t	flow map
$\mathcal{L}_\xi v$	Lie derivative of v with respect to ξ
$\mathbb{R}^n$	Euclidean n dimensional space
$\mathcal{V}$	velocity four vector (f, v)

ϕ	$f + \varepsilon$
ξ	flow field
σ	velocity one form in $\mathbb{R}^4$
$\tilde{\sigma}$	generalized velocity one form in $\mathbb{R}^4$
$\hat{v}$	velocity one form in $\mathbb{R}^3$
ω	vorticity two form in $\mathbb{R}^4$
$\tilde{\omega}$	generalized vorticity two form in$\mathbb{R}^4$
L	$\partial_t v + \nabla \varepsilon$
$d\mathbf{x}$	line element
dV	volume element
ds	surface element
S	specific entropy
Π	fluid maxwell matrix tensor
F	vorticity matrix tensor
T	fluid dynamic stress energy matrix tensor
W	matrix tensor associated with vorticity
h	helicity density
η	helicity three form in $\mathbb{R}^4$
$\tilde{\eta}$	generalized helicity three form in $\mathbb{R}^4$
$\hat{h}$	helicity three form in $\mathbb{R}^3$
$\mathcal{H}$	helicity in $\mathbb{R}^4$
$\mathbb{H}$	helicity in $\mathbb{R}^3$
κ	parity four form
$\tilde{\kappa}$	generalized parity four form
τ	thermasy

I	specific enthalpy
T	absolute temperature
q	hydrodynamic charge density
J	hydrodynamic current vector in $\mathbb{R}^3$
$\mathcal{J}$	hydrodynamic current density four vector (q, J) in $\mathbb{R}^4$
$\mathcal{G}_4$	geometric algebra of Euclidean space $\mathbb{R}^4$
$\mathbf{V}$	velocity four vector in $\mathcal{G}_4$
$\boldsymbol{\nabla}$	vector derivative in $\mathcal{G}_4$
$\boldsymbol{W}$	vorticity bivector in $\mathcal{G}_4$
$\mathcal{P}$	fluid dynamical Poynting vector
e_i	basis vectors in $\mathbb{R}^4$

Chapter 1

Introduction

Laws of conservation and the invariance of physical quantities are fundamental in analyzing and interpreting fluid flows. Every fluid mechanical system is assumed to obey the laws of : (a) conservation of mass, (b) conservation of energy and (c) conservation of momentum. Equation of continuity expresses the law of conservation of mass, while the Euler's equation represents the conservation of momentum for ideal fluids and the Navier-Stokes' equation for the case of viscous fluids. The first law of thermodynamics represents the law of conservation of energy. Helicity, which is interpreted as a measure of knottedness of vortex lines is another invariant of Euler's equations.

Mathematical frameworks for representing dynamical systems include vector calculus, differential forms, dyadics, spinors, tensors, quaternions, and geometric/Clifford algebras. Vector notation is the most commonly used tool to describe physical systems, but it is restricted to the three dimensional space. In higher dimensions advanced analytical tools like that of differential forms, geometric algebra, tensors and spinors play a significant role in complex fields of special relativity, quantum mechanics and string theory. Differential forms provide visualization and geometrical insight into dynamical systems and also balance the notion of concreteness and mathematical abstractness. Differential forms are not to be considered as a replacement for vector calculus, but can aid in extracting additional information.

Differential forms have extensively been used in the field of electromagnetism over the years. One of the early literature in the treatment of differential forms in the theory of electromagnetism can be seen in the work of Flanders (1963). Two contrasting interpretations of the electric field are, as a force on a test charge or as a change in energy experienced by the test charge as it moves through the electric field. The former concept is usually represented in the vector form, while the latter is better represented through the calculus of exterior forms. Both the interpretations do not contradict each other but are complementary in describing fundamental physical properties of a system. The application of differential forms can also be seen in network theory, transformation optics and in the theory of inverse scattering, while discrete differential forms are used in numerical analysis based on finite element methods (Warnick and Russer, 2014).

In non-relativistic classical mechanics time is treated as a universal quantity uniform throughout space and assumes that space is Euclidean. Poincare introduced the innovative concept of the four dimensional space time and defined four-vectors like the four-position, four-velocity, and four-force. The beginning of the twentieth century saw the development of the concepts of Poincare's four dimensional vector space, Minkowski's matrix calculus and space time geometry and Sommerfeld's four-vector algebra, which paved the way for four dimensional physics. Later on Drobot and Rybarski (1958) introduced and defined four-vectors in a Euclidean four dimensional space and developed a new variational principle for barotropic flows. George Mathew and M. J. Vedan (1988, 1989, 1991) introduced four-vectors and extended this variational principle to non-barotropic flows and they derived the conservation laws using Noether's theorems. The same four-vector fields were used by Geetha S., Thomas Joseph and M. J. Vedan (1995) to derive the law of conservation of potential vorticity as an application of Noether's second theorem. Geetha S. and M. J. Vedan (1994) extended this to the study of stability of flows using Arnold's method.

Differential forms and geometric algebra play a major role in merging the physical, mathematical and geometrical ideas of fluid mechanics. The concept of differential forms originated in the work of Grassmann. Differential forms establish a direct connection to geometrical images and provide additional physical insight into field theories in classical physics.

A modern geometrical analysis of fluid dynamics using the theory of differential forms in a four dimensional manifold can be seen in Fecko (2013). Fecko introduced the concept of absolute and relative integral invariance into the realms of fluid dynamics, but the studies were restricted to ideal barotropic flows. Analysis of the theory of integral invariants in the field of mechanics can also be found in Arnold (1989), Gantmacher (1975) and Lam (2014).

The concept of invariance of flows over manifolds has been described using the notion of Lie derivatives of differential forms. Subin and Vedan (2004) have investigated the invariant topological properties of flows in the four dimensional Euclidean space time manifold using differential forms. An in-depth analysis of the dynamics of fluid flows in higher dimensions using topological methods can be seen in the works of Arnold and Khesin (1999).

The theory of integral invariants was first formally introduced by Poincare and later conceptually developed by Cartan (1922). For an integral invariant, the chains considered for integration need not be a collection of equal time points. In this sense, Cartan's concept of integral invariance is broader than Poincare's concept (Kiehn, 1975). Studies pertaining to four dimensional classical flows using the language of differential forms can also be seen in the works of Scofield and Huq (2010, 2014), Kiehn (1975, 2001), Gumral (2016) and Shashikanth (2012).

Almost all classical fluid dynamical theories deal with ideal barotropic flows. A flow is said to be non-barotropic when the pressure does not depend on density alone but also on another quantity such as the specific entropy or in certain cases the chemical composition (Akgun et.al, 2013). In the studies of radiative envelopes of massive stars, interiors of degenerate stars, plasma physics and atmospheric dynamics, flow is considered to be non-barotropic. Hence fluid models must be extended to include non-barotropic flows, where plasma flows, realistic magnetohydrodynamics and propagation of shock waves are taken into consideration.

We consider non-barotropic flows for which $p = p(\rho, S)$, where p is the pressure, ρ is the mass density and S is the specific entropy. Kelvin's circulation theorem, Helmholtz' vorticity theorems and the conservation laws for helicity and potential vorticity hold for barotropic ideal flows but not for non-barotropic flows.

Eckart in 1960 introduced a new quantity $\oint \tau S \cdot dl$, called the thermodynamic circulation, where τ is the thermasy, defined as the time integral of the temperature T (Eckart 1960, Schutz and Sorkin 1977). This scalar field was introduced by Helmholtz and subsequently used in relativistic thermodynamics and classical continuum thermodynamics. The relation $\tau = \int_0^t kTdt$ was formulated by D.Van Dantzig also (Preston, 2016). Eckart established the generalization of Kelvin's circulation theorem by showing that

$$\frac{D}{Dt}\oint_C (v - \tau\nabla S)\cdot dl = 0,$$

where C is a closed curve moving with the fluid. Based on the work of Eckart, Mobbs (1981) defined non-barotropic flows for which $\nabla\tau \times \nabla S \neq 0$. Mobbs obtained the generalizations of the Helmholtz' vorticity theorems by considering tubes of generalized vorticity $(w - (\nabla\tau \times \nabla S))$ and established the invariance of the generalized helicity and generalized potential vorticity for three dimensional flows. The conservation laws for non-barotropic flows using variational principles of hydromechanics have been established by George Mathew and M. J. Vedan (1991). For three dimensional non-barotropic gases, non-local conservation laws for fluid helicity and cross helicity, using Clebsch variables were derived by Webb et.al (2014) and the conservation for the cross helicity for non-barotropic magneto hydrodynamic flows is shown by Yahalom (2017). In our work we have extended the study of non-barotropic flows to a four dimensional space time manifold using the calculus of differential forms.

Apart from the calculus of exterior forms, we have tried to explore fluid dynamical concepts using the language of geometric algebra also. William Clifford introduced geometric algebra during the second half of 19^{th} century by combining the algebraic ideas of Hamilton and Grassmann. Geometric algebra was structured by combining the outer and inner product of elements of a linear space into a new product called the geometric product. The inclusion of geometrical concepts into abstract Clifford algebra has enriched geometric algebra and has made it evolve into a powerful mathematical theory. Detailed description of the subject of geometric algebra can be seen in the works of Hestenes (1966, 1999), Doran, A. Lasenby and J. Lasenby (2003) and Bromborsky (2010). Geometric algebra provides a compact and intuitive description in the fields of classical mechanics, quantum mechanics, relativity and electromagnetic theory. Geometric algebra is also being successfully applied as a computational tool in robotics and computer graphics. As geometric algebra combines the properties of both the inner and the outer product, additional information can be extracted while executing the geometric product. We find that certain properties and results pertaining to the field of fluid dynamics follow from the geometric properties encoded in the geometric product.

Geometric algebra has been successfully applied to a variety of fields in classical mechanics, but the applications of geometric algebra in the field of fluid dynamics have been restricted to specialized problems like that of fluid flows with variable viscosity in quaternionic settings, elliptic boundary value problems and visualization of vector fields in the study of gas dynamics and combustion. A geometric algebraic approach to fluid dynamics to derive Kelvin's circulation theorems and Helmholtz' vorticity theorems can be seen in the work of Cibura and Hildenbrand (2008).

1.1 Preliminaries

1.1.1 Euler and Navier-Stokes' Flows in $\mathbb{R}^3$

Let $v = (v_1, v_2, v_3)$ be a divergence free velocity field in $\mathbb{R}^3$. The dynamics of an ideal incompressible flow with constant density is described by the Euler equation,

$$\partial_t v + (v \cdot \nabla)v = -\frac{\nabla p}{\rho} - \nabla \varphi \tag{1.1}$$

and that of viscous incompressible fluids are described by the Navier-Stokes' equation,

$$\partial_t v + (v \cdot \nabla)v = -\frac{\nabla p}{\rho} - \nabla\varphi + \nu\nabla^2 v, \tag{1.2}$$

where p is the pressure, ρ is the mass density, φ is the potential for the volume force field and ν is the kinematic viscosity (Batchelor, 1967). The identity $(v \cdot \nabla)v = \nabla\dfrac{v^2}{2} - v \times (\nabla \times v)$ is used to convert the above equations into the form

$$\partial_t v - v \times (\nabla \times v) = -\nabla\varepsilon \tag{1.3}$$

and

$$\partial_t v - v \times (\nabla \times v) = -\nabla\varepsilon + \nu\nabla^2 v \tag{1.4}$$

respectively, where $\varepsilon = \frac{v^2}{2} + \frac{p}{\rho} + \varphi$ is the Bernoulli function.
In terms of the divergence free vorticity field $w = \nabla \times v$, the vorticity equation for the inviscid incompressible flow is given by

$$\partial_t w - \nabla \times (v \times w) = 0 \tag{1.5}$$

and for viscous flow

$$\partial_t w - \nabla \times (v \times w) = \nu\nabla^2 w \tag{1.6}$$

Taking the dot product of (1.3) and (1.4) with the velocity, we get

$$\partial_t(\frac{v^2}{2}) + v \cdot \nabla\varepsilon = 0.$$

and

$$\partial_t(\frac{v^2}{2}) + v \cdot \nabla\varepsilon = v \cdot \nu\nabla^2 v$$

It follows that,

$$\partial_t(\varepsilon) + v \cdot \nabla \varepsilon = 0, \tag{1.7}$$

for ideal flows if, $\dfrac{p}{\rho}$ and φ do not depend on time. In such cases, the Bernoulli function is a conserved quantity along the trajectories of v (Gumral, 2016).

1.1.2 Vector Fields and Flows

A fundamental object in a dynamical system is a manifold. A manifold M is an n-dimensional space that is locally isomorphic to an n-dimensional Euclidean space $\mathbb{R}^n$. The evolution of a fluid is described by a family of maps ψ_t. Here ψ_t is defined as a parametric map $\psi : I \times \mathbb{R} \to I$, such that $\psi(x,t) = \psi_t(x)$, where I is a domain of fluid. Given such a flow map ψ_t, we can find a velocity field v given by

$$\dot{\psi}_t(x) = \frac{\partial \psi}{\partial t}(x,t) = v(\psi_t(x), t)$$

To each vector field v, we can find a flow ψ_t having v as its velocity field. The integral curve of the system of differential equations $\dfrac{dx_j}{dt} = v_j$ is called the flow generated by the vector field v.

Let C be a smooth curve on M. The flow map $\psi_t : M \to M$, carries a fluid particle at a position x at time $t = 0$ to the position $\psi_t(x)$ at time $t > 0$. Each point of x of the curve C has a tangent vector of the form $\dot{\psi}_t(x) = v|_x = v_i(x)\dfrac{\partial}{\partial x_i}$ (Kambe, 2004) (here and throughout the thesis Einstein's summation convention is used). The vector space of all tangent vectors to all possible curves passing through a given point x is the tangent space to M at x, denoted as TM_x. The collection of all tangent spaces at all points of M is the tangent bundle TM.

Definition 1 (Exterior Forms)**.** *An exterior form α^k of degree k is a function of k vectors in R^n which is k-linear and antisymmetric. That is*

$$\alpha^k : R^n \times R^n \times \cdots \times R^n \to R$$

satisfies

$$\alpha^k(ax_1 + by_1, x_2, \cdots x_k) = a\alpha^k(x_1, x_2 \cdots, x_k) + b\alpha^k(y_1, x_2 \cdots, x_k)$$

and

$$\alpha^k(x_{i_1}, x_{i_2} \cdots, x_{i_k}) = (-1)^\nu \alpha^k(x_1, \cdots, x_k),$$

where

$$\nu = \begin{cases} 0 & \text{if the permutation} \quad i_1, i_2, \cdots i_k \quad \text{is even} \\ 1 & \text{if the permutation} \quad i_1, i_2, \cdots i_k \quad \text{is odd.} \end{cases}$$

The set of all k-forms is a vector space under addition and scalar multiplication defined by

$$(\alpha_1^k + \alpha_2^k)v = \alpha_1^k(v) + \alpha_2^k(v)$$

$$(\lambda\alpha^k)v = \lambda\alpha^k(v),$$

where $v = \{v_1, v_2, \cdots, v_k\}$, $v_i \in \mathbb{R}^n$.

Definition 2 (Exterior multiplication or wedge product)**.** *Exterior multiplication of a k-form α^k with an l-form α^l on R^n is defined to be an exterior $(k+l)$-form such that*

$$\alpha^k \wedge \alpha^l(x_1, \cdots x_{k+l}) = \sum (-1)^n \alpha^k(x_{i_1}, \cdots, x_{i_k})\alpha^l(x_{j_1}, \cdots, x_{j_l}),$$

where $i_1 < \cdots < i_k$ and $j_1 < \cdots, < j_l$ is a permutation of the numbers $(1, 2, \cdots, k+l)$ and $n = 0$ if permutation is even and $n = 1$ otherwise.

The exterior multiplication satisfies the properties of being skew-symmetric, associative and distributive as follows:

$$\alpha^k \wedge \alpha^l = (-1)^{kl} \alpha^l \wedge \alpha^k$$
$$(\alpha^k \wedge \alpha^l) \wedge \alpha^m = \alpha^k \wedge (\alpha^l \wedge \alpha^m)$$
$$(a\alpha_1^k + b\alpha_2^k) \wedge \alpha_l = a\alpha_1^k \wedge \alpha^l + b\alpha_2^k \wedge \alpha^l.$$

Definition 3 (Interior product)**.** *If α^k is a k-form and v be a vector field, then the interior product of v with α^k is defined as*

$$(i_v \alpha^k)(v_1, v_2, \cdots v_{k-1}) = \alpha^k(v, v_1, \cdots, v_{k-1}),$$

for every set of vectors $v_1, \cdots, v_{k-1}$. For basis elements, the interior product is given by

$$i_{\partial_{x_l}}(dx_{j_1} \wedge \cdots \wedge dx_{j_k}) = \begin{cases} (-1)^{m-1} dx_{j_1} \wedge \cdots \wedge dx_{j_{m-1}} \wedge dx_{x_{m+1}} \wedge \cdots \wedge dx_{j_k} \\ \qquad\qquad\qquad\qquad\qquad \text{if} \quad l = j_m \\ 0 \quad \text{if} \quad l \neq j_m \quad \text{for all} \quad m. \end{cases}$$

In local coordinates a k-form at a point on the manifold is spanned by

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where I ranges over all strictly increasing i's such that $1 \leq i_1 < i_2 \cdots < i_k \leq n$.

An exterior k-form on M is expressed as

$$\alpha^k = f_I(x) dx_I,$$

where f_I's are real valued functions. The exterior operator d takes k-forms to $(k+1)$-forms.

Definition 4 (Exterior Derivative)**.** *The exterior derivative of a k-form $\alpha^k = f_I(x) dx_I$ is the $(k+1)$-form*

$$d\alpha^k = \partial_{x_j} f_I dx_j \wedge dx_I.$$

Exterior differentiation has the following properties:

1. $d(a\alpha^k + b\alpha^l) = ad\alpha^k + bd\alpha^l$,
2. $d(\alpha^k \wedge \alpha^l) = d\alpha^k \wedge \alpha^l + (-1^k)\alpha^k \wedge d\alpha^l$,
3. $d(d\alpha^k) = d^2(\alpha^k) = 0$.

1.1.3 Differential Forms

A differential k-form α^k at a point x of the manifold M is an exterior k-form on the tangent space TM_x to M at x. 0-forms are smooth real valued functions on M. If $(x_1, \cdots, x_n)$ is the local coordinate system, then the tangent space TM_x has a basis $\{\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}\}$. The dual space of the tangent space TM_x, termed as the cotangent space T^*M_x has the dual basis $\{dx_1, dx_2, \cdots, dx_n\}$. The space of all 1-forms (covectors) at a point x is the cotangent space and a 1-form has the general expression

$$\alpha^1 = f_1(x)dx_1 + f_2(x)dx_2 + \cdots + f_n(x)dx_n,$$

where the coefficient function f_i's are smooth.

Let $y = y_i\partial_{x_i}$ and $z = z_i\partial_{x_i}$ be vectors in TM_x. An inner product on TM_x is given by

$$\langle y, z\rangle = g_{ij}y_iz_j,$$

where g_{ij} is the metric tensor. If the metric tensor g_{ij} is the unit matrix such that $g_{ij} = \delta_{ij}$, (δ_{ij} is the Kronecker delta) then the metric tensor is said Euclidean. To a vector $a = a_i\partial_{x_i}$ we can associate a covector or a 1-form α^1 through the Euclidean metric tensor.

1.1.4 Lie Derivative

The Lie derivative measures the infinitesimal change of physical objects expressed as functions ,vector fields, tensors or differential forms when acted upon by the flow. The Lie derivative of a scalar function f with respect to a flow induced by the vector field v is the directional derivative $df(v)$. The Lie derivative of a vector field w with respect to another vector field v is the Lie bracket of v and w expressed as

$$\mathcal{L}_v w = [v, w].$$

If $v = v_1\partial_{x_1} + \cdots + v_n\partial_{x_n}$ be a vector field on the manifold M and $w = w_1\partial_{x_1} + \cdots + w_n\partial_{x_n}$, then

$$\mathcal{L}_v w = (v_i\partial_{x_i} w_j - w_i\partial_{x_i} v_j)\partial_{x_j}.$$

The Lie derivative of a differential form α^k with respect to a vector field v is given by the Cartan's formula

$$\mathcal{L}_v \alpha^k = d i_v \alpha^k + i_v d\alpha^k.$$

The Lie derivative satisfies the following properties.

$$\begin{aligned}
&\mathcal{L}_v d\alpha^k = d\mathcal{L}_v \alpha^k \\
&\mathcal{L}_v i_v \alpha^k = i_v \mathcal{L}_v \alpha^k \\
&i_v df = \mathcal{L}_v f \\
&\mathcal{L}_v(\alpha^k \wedge \beta^k) = (\mathcal{L}_v \alpha^k) \wedge \beta^k + \alpha \wedge \mathcal{L}_v \beta^k
\end{aligned}$$

Chain: In practice, often we encounter the contours of integration, consisting of several pieces transversed in either direction more than once. The analogous concept in higher manifolds is called a chain.

Stokes' Theorem: For a continuously differentiable k-form α^k on a manifold M, Stokes' theorem states that

$$\int_{D^{(k+1)}} d\alpha^k = \int_{\partial D^{(k+1)}} \alpha^k,$$

where D^{k+1} is a $(k+1)$-dimensional chain in M with boundary $\partial D^{(k+1)}$.
Hodge Star Operator: Hodge star operator, denoted by $*$, defined on an n-dimensional manifold equipped with an Euclidean metric, is a linear map from k-forms to $n-k$ forms given by,

$$*(dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_k}) = (-1)^{\nu} dx_{i_{k+1}} \wedge dx_{i_{k+1}} \cdots \wedge dx_{i_n},$$

where $\nu = 0$ if $\{i_1, i_2, \cdots, i_n\}$ is an even permutation of $\{1, 2, \cdots, n\}$ and $\nu = 1$ otherwise and $i_{k+1} < \cdots < i_n$.

1.1.5 Invariants in Fluid Dynamics

Laws of conservation and the invariance of physical quantities are fundamental in analyzing and interpreting fluid dynamical theories. Physical quantities are treated as geometric objects like scalars, vectors, differential forms or as tensors. Differential geometry based on Cartan's calculus of differential forms is considered to be the natural language of field theory in higher dimensions and is often used by mathematicians and physicists to study and analyze invariant properties of physical quantities along the motion of a fluid. Here we review the theory of invariance of physical quantities expressed as differential forms.

1.1.6 Local Invariants in $\mathbb{R}^3$

Tur and Yanovsky (1993) have proposed a general geometric method of derivation of invariants in hydrodynamic dissipation less media and defined four types of local invariants in the three dimensional space $\mathbb{R}^3$. The invariants thus defined using the language of exterior forms, are universal and independent of the type of hydrodynamic models considered. According to Tur and Yanovsky, in a three dimensional space, exterior forms of degree $p \leq 3$ $(p = 0, 1, 2, 3)$ lead to the existence of four types of local invariants: Lagrangian invariants associated with 0-forms, **S**-invariants associated with one-forms, frozenness invariants associated with two-forms, and density invariants associated with three-forms. A point in $\mathbb{R}^3$ is taken as (x, y, z) or equivalently (x_1, x_2, x_3).

The differential k-form α^k is said to be an invariant if it satisfies the condition

$$\partial_t \alpha^k + \mathcal{L}_v \alpha^k = 0, \tag{1.8}$$

where v is the flow field and $\mathcal{L}_v \alpha^k$ is the Lie derivative of the differential form α^k with respect to the flow v.

1. Lagrangian invariants: 0-forms or scalar functions of the form $\alpha^0 = a(t, x, y, z)$ satisfying condition (1.8) result in the equation

   $$\partial_t a + v_k \partial_{x_k} a = 0. \tag{1.9}$$

   or in vector form,

   $$\partial_t a + (v \cdot \nabla) a = 0. \tag{1.10}$$

   Such functions are called Lagrangian invariants. The physical meaning of Lagrangian invariants reduces to their advection by the flow.

2. **S**-invariants: Condition (1.8) applied to one-forms $\alpha^1 = a_1 dx + a_2 dy + a_3 dz$, results in

   $$(\partial_t a_i + v_k \partial_{x_k} a_i + a_k \partial_{x_i} v_k) dx_i = 0, \tag{1.11}$$

   which in vector form gives

   $$\partial_t a + (v \cdot \nabla) a + (a \cdot \nabla) v + a \times (\nabla \times v) = 0, \tag{1.12}$$

   where $a = (a_1, a_2, a_3)$.

$a_1dx + a_2dy + a_3dz = 0$ defines a plane orthogonal to the vector field a at each position (x, y, z) and the invariant one-form defines a local field of planes frozen into the flow medium. Invariant one-forms determine **S**-invariants. Here the surfaces orthogonal to the vector field a is thus said to be frozen into the flow.

3. Frozen-in invariants associated with two-forms : A two-form can be represented as $\alpha^2 = A_1dy \wedge dz + A_2dz \wedge dx + A_3dx \wedge dy$. Condition (1.8) of invariance leads to the equation

$$(\partial_t A_i + v_m \partial_{x_m} A_i + A_i \partial_{x_m} v_m + \epsilon_{ijk} \tilde{A_m} \partial_{x_m} v_i) dx_j \wedge dx_k = 0, \quad (1.13)$$

where ϵ_{ijk} is the permutation tensor or the Levi- Civita symbol of three dimensions and $\tilde{A_m} = (-1)^{m+1} A_m$. The above equation in vector form (1.13) becomes

$$\partial_t A + (v \cdot \nabla)A + A(\nabla \cdot v) - (A \cdot \nabla)v = 0, \quad (1.14)$$

where $A = (A_1, A_2, A_3)$. Such two-forms define frozen-in invariants.

4. Density invariants: For a three-form $\alpha^3 = a(t, x, y, z)dx \wedge dy \wedge dz$ (1.8) yields the equation,

$$(\partial_t a + \partial_{x_i}(av_i))dx \wedge dy \wedge dz = 0 \quad (1.15)$$

and in vector form this becomes,

$$\partial_t a + \nabla \cdot (av) = 0. \quad (1.16)$$

Physically such forms are associated with the non-destructibility of mass, hence the name density invariant.

1.1.7 Integral Invariants

It was Poincare who characterized the theory of integral invariants. An integral invariant is used to denote an integral that, when taken over an arbitrary set of simultaneous (i.e., ones that correspond to the same value of t) points, does not change value when one displaces the points of that set along the corresponding trajectories up to another arbitrary instant t' (Cartan, 1922). The physical quantity under the integral sign in an integral invariant is expressed as an exterior form. In this thesis we discuss the basic Poincare integral invariants and Cartan's generalization to an extended phase space.

It should be noted that interpretation of the problem may be reversed; i.e., rather than being given the vector field and searching for those invariant objects, it is possible to solve for those vector fields that leave an object invariant. This procedure is the basis of Cartans analysis of Hamiltonian dynamics; Cartan assumes that the closed integral of action is an absolute invariant of some vector field v in state space, and solves for this vector field. The vector field turns out to be unique, and its components are given by Hamilton's equations (Kiehn, 1975).

Absolute and Relative Integral Invariance

Consider an n-dimensional manifold M on which a velocity vector field v is defined. The structure (M, ψ_t) is called a phase space where ψ_t is the flow generated by v. Let α^k be a k-form defined on M and D^k be a k-dimensional surface or a k-chain on M. Due to the flow ψ_t, the k-dimensional surfaces flow with a velocity v and as a result D^k is mapped to $\psi_t(D^k)$. For any k-chain D^k, if

$$\int_{D^k} \alpha^k = \int_{\psi_t(D^k)} \alpha^k,$$

then α^k is said to be an integral invariant with respect to the field v. The condition for integral invariance is given by

$$\int_{D^k} \mathcal{L}_v \alpha^k = 0, \tag{1.17}$$

where $\mathcal{L}_v \alpha^k$ is the Lie derivative of the differential form α with respect to the flow induced by v. For the statement (1.17) to be true, $\mathcal{L}_v \alpha^k$ must vanish, i.e., $\mathcal{L}_v \alpha^k = 0$ for every k-dimensional chain D^k (Fecko, 2013). Vanishing of the Lie derivative implies that the corresponding physical quantity remains invariant along the integral curves of the vector field that constitutes the flow. This leads to the consideration of two types of integral invariants: absolute and relative invariants.

Definition 5. *A k-form α^k on a phase space is called an absolute integral invariant of the phase flow ψ_t if*

$$\int_{\psi_t(D^k)} \alpha^k = \int_{D^k} \alpha^k$$

for every k-dimensional region or a k-chain.

Definition 6. *A k-form α^k on a phase space is called a relative integral invariant of the phase flow ψ_t if*

$$\oint_{\psi_t(C^k)} \alpha^k = \oint_{C^k} \alpha^k$$

for every closed k-dimensional region (a region without boundary, also called a k-cycle).

A chain is called a cycle when its boundary is zero. i.e, a chain C^k is

a cycle if $\partial C^k = 0$. k-cycles form a sub class of k-chains. If the integral $\oint_{C^k} \mathcal{L}_v \alpha^k = 0$ in equation (1.17) is to be satisfied over each cycle, then it is enough that $\mathcal{L}_v \alpha^k$ be exact (de Rham theorem). i.e, $\mathcal{L}_v \alpha^k = d\bar{\beta}$ for some form $\bar{\beta}$, where the integration is taken over a cycle C^k.
Thus, $\int_{D^k} \alpha^k$ is an absolute integral invariant if $\mathcal{L}_v \alpha = 0$, for every chain D^k and $\oint_{C^k} \alpha^k$ is a relative integral invariant if $\mathcal{L}_v \alpha = d\bar{\beta}$, for every cycle C^k in M.

If α^k provides a relative integral invariant, by Cartan's formula $\mathcal{L}_v \alpha^k = i_v d\alpha^k + d i_v \alpha^k$, it holds that

$$i_v d\alpha^k = d\beta,$$

where $\beta = \bar{\beta} - i_v \alpha^k$. Then v, α^k, and β generate a series of relative integral invariants

$$\oint_{C^k} \alpha^k, \quad \oint_{C^k} \alpha^k \wedge d\alpha^k, \quad \cdots \oint_{C^k} \alpha \wedge (d\alpha)^k, \quad k = 0, 1, 2 \cdots \tag{1.18}$$

where C^k's are cycles of appropriate dimensions (Fecko, 2013).

Proposition 1. *α^k is a relative integral invariant of a flow ψ_t of a phase space if and only if its exterior differential $d\alpha^k$ is an absolute integral invariant of ψ_t.*

Proof. Let a k-form α^k on M be a relative integral invariant of the flow ψ_t. Let C^k be a k-dimensional chain on M. Then,

$$\int_{C^{k+1}} d\alpha^k = \int_{\partial C^{k+1}} \alpha^k = \int_{\psi_t(\partial C^{k+1})} \alpha^k = \int_{\partial \psi_t(C^{k+1})} \alpha^k = \int_{\psi_t(C^{k+1})} d\alpha^k$$

□

Also, $\mathcal{L}_v\alpha^k = d\bar{\beta} \Rightarrow \mathcal{L}_v d\alpha^k = 0$. i.e.,

$$\oint_{C^k} \alpha^k \text{ is a relative integral invariant} \Leftrightarrow \int_{D^{k+1}} d\alpha^k \text{ is an absolute invariant,} \tag{1.19}$$

where $C^k = \partial D^{k+1}$. If $d\alpha^k = \omega$, it follows that,

$$\int_{D^k} \omega, \quad \int_{D^k} \omega \wedge \omega, \quad \cdots \int_{D^k} \omega \wedge \omega^k, \quad k = 0, 1, 2 \cdots \tag{1.20}$$

is a series of absolute integral invariants, where D^k's are chains of appropriate dimensions.

Absolute and Relative Integral Invariants in $\mathbb{R}^4$

We consider Cartan's generalization of integral invariance of objects in the extended phase space.

Consider the Euclidean space-time manifold $M = \mathbb{R}^4 \cong T \times \mathbb{R}^3$, where $T \cong \mathbb{R}$ is the time and $\mathbb{R}^3$ is the Euclidean space. Let $\xi = \partial_t + v$, where v has time dependent components, be a flow field defined on the manifold $\mathbb{R}^4$. A fluid motion is usually represented by a diffeomorphism ϕ_t, the flow induced by ξ. Then $(\mathbb{R}^4, \phi_t)$ is termed as the extended phase space.

Any general k-form σ^k on $\mathbb{R}^4$ can be represented in the form $\sigma^k = \hat{\alpha}^k + dt \wedge \hat{\beta}^{k-1}$ where $\hat{\alpha}^k$ is a k-form and $\hat{\beta}$ is a $(k-1)$-form and both are possibly time dependent spatial forms ($\hat{\alpha}^k$ and $\hat{\beta}^{k-1}$ need not be pull backs of forms from $\mathbb{R}^3$ onto $T \times \mathbb{R}^3$ as described in Poincare's theory). The conditions for integral invariance in the extended phase space are

$$\oint_{C^k} \sigma^k \text{ is a relative invariant } \Leftrightarrow \mathcal{L}_{\partial_t}\hat{\alpha}^k + i_v \hat{d}\hat{\alpha}^k = \hat{d}\hat{\beta}^{k-1} \Leftrightarrow i_\xi d\sigma_k = 0, \tag{1.21}$$

where C^k is a cycle and

$$\int_{D^k} \sigma^k \text{ is an absolute invariant } \Leftrightarrow \mathcal{L}_{\partial_t}\hat{\alpha}^k + \hat{\mathcal{L}}_v \hat{\alpha}^k = 0 \Leftrightarrow i_\xi d\sigma^k = 0, \tag{1.22}$$

where D^k is an arbitrary chain in $\mathbb{R}^4$ and $\sigma^k = \hat{\alpha}^k - dt \wedge i_v \hat{\alpha}^k$ and $\hat{\mathcal{L}}_v = i_v \hat{d} + \hat{d} i_v$ is the spatial Lie derivative (Fecko, 2013).

Let C_0^k and C_1^k be k-cycles such that the integral curves of the vector field $\xi = \partial_t + v$, connects points on a k-cycle C_0^k to points lying on C_1^k. The family of these integral curves form a $(k+1)$ dimensional surface (a tubular structure), with cycles C_0^k and C_1^k as boundaries. These cycles may contain points lying on coordinates of different time [12].

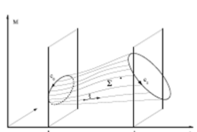

Figure 1.1:
Cycles C_0^k and C_1^k do not lie on hyperplanes of constant time in $\mathbb{R}^4$ in general.

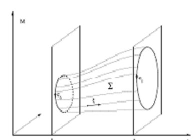

Figure 1.2:
Cycles C_0^k and C_1^k lie on hyperplanes of constant time in $\mathbb{R}^4$(special case).
(Figures taken from Fecko, 2013)

Let $P(t)$ be a three dimensional hypersurface in $\mathbb{R}^4$ for a fixed t. From the relation $P(t) = \psi_t(P(0))$, we can consider geometric objects (chains) co-moving along with the fluid. For relativistic flows objects lying on a fixed time hypersurface $P(t_1)$ need not necessarily lie on fixed time hypersurface $P(t_2)$. By generalizing the concept of integral invariants from phase space to extended phase space, we are able to consider chains lying at different time coordinates instead of chains lying on hypersurfaces of fixed time, irrespective of the type of flow.

1.1.8 Geometric Algebra

Geometric algebra combines the languages of vector algebra, complex algebra and matrix algebra to form a unified framework. A detailed study of geometric algebra and its applications in classical mechanics can be seen in the works of Hestenes (1999) and Doran et.al (2003). In mathematical literature geometric algebra is referred to as Clifford algebra. In chapter 4 of this thesis we have preferred to use the terminology of geometric algebra as given in Doran et.al (2003).

Let V be a finite dimensional vector space over $\mathbb{R}$. V together with a geometric product is called a geometric algebra. For a,$b \in V$, the geometric product is defined as

$$ab = a \cdot b + a \wedge b,$$

where $a \cdot b$ is the inner product and $a \wedge b$ is termed as outer product (exterior product). Thus the geometric product is the sum of a scalar product and an outer product and hence are termed as multivectors. Thus the elements of a geometric algebra $\mathcal{G}$ are multivectors. The outer product of two vectors a and b ($a \wedge b$) is defined to be a bivector and is taken as the signed area of a parallelogram with a and b as its sides. The outer product of three vectors $a \wedge b \wedge c$ (trivector) is taken as an oriented volume of a parallelepiped with the three vectors forming its sides. The outer product of n vectors is visualized to be an n dimensional parallelogram with a magnitude and an orientation associated to it.

The outer product of k vectors is called a blade of grade-k. Scalars are grade-0 blades, linear combination of vectors are grade-1 blades, linear combination of bivectors are grade-2 blades and so on. For an n dimensional vector space, grade-n blades are called pseudoscalars and grade-(n-1) blades are called pseudovectors. Multivectors are a linear combination of blades of different grades. The sum and products of multivectors are unique, so that $\mathcal{G}$ is algebraically closed.

If $A, B, \cdots, C$ are used to denote multivectors, then the geometric product of multivectors satisfies the property of being associative and distributive over addition. Thus the the sum and the geometric product of multivectors satisfies the following axioms:
Addition is commutative : $A + B = B + A$.
Addition and multiplication are associative:

$$A + (B + C) = (A + B) + C,$$

$$A(BC) = (AB)C.$$

Multiplication is distributive over addition

$$A(B + C) = AB + AC,$$

$$(A + B)C = AC + BC.$$

Unique additive and multiplicative identities (0 and 1 respectively) exists such that:

$$A + 0 = A \text{ and } 1A = A.$$

There exists a unique additive inverse $(-A)$ for every multivector A such that:

$$A + (-A) = 0.$$

The space of multivectors form a vector space over $\mathbb{R}$. Multivectors are classified into different grades. Geometric algebra of a finite dimensional vector space is an associative algebra.

The introduction of geometric algebra into classical mechanics was done by D. Hestenes through his work 'Space Time Algebra′ in 1966. Thereon through a series of papers Hestenes advocated the applications of geometric algebra in wide range of fields of quantum mechanics, electrodynamics, relativistic physics, Lie group theory, projective and conformal geometry.

Space Time Algebra

Space time algebra (STA) is the geometric algebra of space time. Space time algebra is commonly referred by physicists as the geometric algebra of the Minkowski space time, where as the geometric algebra $G(4,0)$ or $\mathcal{G}_4$ of the Euclidean four dimensional space R^4 and the geometric algebra $\mathcal{G}(1,3)$ of Minkowski space time $R^{1,3}$ are found to be algebraically isomorphic (Sobczyk, 2017). Here we consider the geometric algebra of the Euclidean space time manifold.

Let $\{e_0, e_1, e_2, e_3\}$ be a set of orthonormal vectors tangent to a point lying in the Euclidean space time manifold. e_o is taken to be a time like vector and e_1, e_2, e_3 are space like vectors. The e_i's satisfy the relations $e_i^2 = 1$, $e_i \cdot e_j = \delta_{ij}$, where i and j take values from 0 to 3. The six bivectors $e_i \wedge e_j$ satisfy $(e_i \wedge e_j)^2 = -1$ and they generate rotations in a plane. The four trivectors satisfy $e_i e_j e_k = \epsilon_{ijkl} I e_l$, where I is a grade-4 pseudoscalar defined as $I = e_0 e_1 e_2 e_3$ and can be considered as the unit four dimensional oriented volume element. The trivector or otherwise termed as a pseudovector can be considered as a three dimensional unit volume. Thus the set $\{e_i\}$ generates a basis consisting of 16 terms (1 scalar, 4 vectors, 6 bivectors, 4 trivectors and 1 pseudoscalar) for the geometric algebra $\mathcal{G}_4$ of the four dimensional Euclidean space R^4.

A general element A of $\mathcal{G}_4$ can be expressed as

$$A = A_S + A_V + A_B + A_T + A_P,$$

where A_S denotes the scalar part, A_V denotes the vector part, A_B denotes the bivector part, A_T denotes the trivector part and A_P denotes the pseudoscalar part of the multivector A.

$\tilde{A}$ denotes the reversion of A in which the order of the vector products in A is reversed. Therefore, $\tilde{A} = A_S + A_V - A_B - A_T + A_P$ (Hestenes,2016).

SUMMARY OF THE THESIS

In this thesis we have discussed the properties of fluid dynamical objects in a Euclidean space time manifold using the languages of differential forms and geometric algebra.

In the introductory chapter we have discussed the preliminary theories required for the thesis. A brief survey on the available literature based on related works is also discussed. A brief synopsis of the work done follows.

In chapter 2 we discuss the theory of integral invariants using the notion of Lie derivatives of differential forms in the frame work of classical fluid flows governed by Euler and Navier-Stokes' equations in the Euclidean space time manifold. Integral invariance of physical objects of incompressible ideal flows are analyzed using the concepts of relative and absolute invariance of forms in a four dimensional Euclidean space time manifold. Corresponding to exterior forms of degrees $k = 0, 1, 2, 3$ and 4, we obtain five types of local invariants. The expressions for the rate of change of circulation, vorticity flux, helicity and parity in the case of three and four dimensional ideal and Navier-Stokes' flows are also obtained. Fluid Maxwell's equations are obtained not by analogy but as mathematical consequences derived from the properties and operations of the differential forms. Analogous equations for the Poynting theorem and Lorentz force in electromagnetic theory are also derived in fluid dynamics. Analogous to the electromagnetic stress-energy tensor a fluid dynamic stress-energy tensor is also obtained.

In chapter 3 we consider non-barotropic flows in a Euclidean four dimensional space time manifold. Integral invariants of non-barotropic perfect and viscous flows are studied using the concepts of relative and absolute invariance of forms. The four dimensional expressions for the rate of change of the generalized circulation, generalized vorticity flux, generalized helicity and generalized parity in the case of ideal and viscous non-barotropic flows are thereby obtained.

In chapter 4 we try to apply the geometric algebraic methods to four dimensional flows in the Euclidean space time manifold. We find that certain properties and results pertaining to the field of fluid dynamics follow from the geometric properties encoded in the geometric product. On applying the geometric product to physical objects we are able to extract additional information and quantities which may need further investigation. The fluid dynamic stress-energy tensor and the Poynting theorem is re-derived using the language of geometric algebra.

In chapter 5, we conclude the thesis, compiling the results obtained in the previous chapters.

Chapter 2

Local and Integral Invariants in a Space Time Manifold

Based on Grassman's geometric algebraic concept of exterior multiplication, Cartan developed the theory of exterior differential systems. The theory of integral invariants was first formally introduced by Poincare and later conceptually developed by Cartan (1922). Cartan considered the invariance of objects in the extended phase space (where time is added to the ordinary space coordinates). For an integral invariant, the integration chain need not be a collection of equal time points; in this sense, Cartan's concept of integral invariance is broader than Poincare's concept, which was confined to equal time point sets (Kiehn, 1975). Moving from Poincare's version of integral invariance to Cartan's version, one is able to consider time-dependent situations rather than the time-independent ones (Fecko, 2013). Apart from mechanics, integral invariants are applied in the fields of differential equations, image and geometry processing, shape matching etc.

In this chapter we discuss the theory of integral invariants of differential forms in the frame work of classical fluid flows governed by Euler and Navier-Stokes' equations in the Euclidean space time manifold. Integral invariance of fluid dynamical objects are analyzed using the concepts of relative and absolute invariance of forms.

In the next section we extend the conditions of local invariance of differential forms as derived by Tur and Yanovsky to the four dimensional Euclidean space time manifold.

2.1 Local Invariants in $\mathbb{R}^4$

We consider the Euclidean space time $\mathbb{R}^4 \cong T \times \mathbb{R}^3$, where $T \cong \mathbb{R}$ is the time and $\mathbb{R}^3$ is the Euclidean space. A point in space-time is represented by a set of coordinates (t, x, y, z) or equivalently (x_0, x_1, x_2, x_3). We extend the definitions of the four types of local invariants in a three dimensional space as discussed in section (1.1.6) to a four dimensional space. As in section (1.1.6) we are able to define five types of local invariants in a four dimensional space time manifold instead of four in the three dimensional space. Here corresponding to the exterior forms of degree $k \leq 4$ $(k = 0, 1, 2, 3, 4)$ there exists five types of local invariants.

The differential k-form α^k is said to be an invariant if it satisfies the condition

$$\mathcal{L}_\varsigma \alpha^k = 0, \tag{2.1}$$

where $\varsigma = v_0\partial_{x_0} + v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3}$ or $\varsigma = v_i\partial_{x_i}$ is the flow field $(i = 0, 1, 2, 3)$.

1. Invariants associated with 0-forms: 0-forms or scalar functions of the form $\alpha^0 = a(x_0, x_1, x_2, x_3)$ satisfying (2.1) result in the equation

$$v_i\partial_{x_i} a = 0. \tag{2.2}$$

For the case where the flow field is of the form $\xi = \partial_t + v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3}$ or $\xi = \partial_t + v$. ($v_0 = 1, x_0 = t$), the above equation takes the form,

$$\partial_t a + v_k \partial_{x_k} a = 0, \tag{2.3}$$

($k = 1, 2, 3$) or in vector form,

$$\partial_t a + (v \cdot \nabla) a = 0, \tag{2.4}$$

which is the standard form of a Lagrangian invariant ($v = (v_1, v_2, v_3)$). The specific entropy S and the Ertel's invariant $\varpi = \frac{w \cdot \nabla S}{\rho}$ for a compressible adiabatic fluid are Lagrangian invariants. The magnetic helicity h_m and cross helicity h_c are Lagrangian invariants in magnetohydrodynamic flows of incompressible ideal fluids.

2. Invariants associated with one-forms or **S**-invariants: One-forms can be represented as $\alpha^1 = a_0 dx_0 + a_1 dx_1 + a_2 dx_2 + a_3 dx_4$. Then (2.1) results in

$$(v_k \partial_{x_k} a_i + a_k \partial_{x_i} v_k) dx_i = 0. \tag{2.5}$$

For the flow ξ, the above equation takes the form

$$(\partial_t a_i + v_k \partial_{x_k} a_i + a_k \partial_{x_i} v_k) dx_i = 0, \tag{2.6}$$

where i takes the value $0, 1, 2, 3$ and summation over $k = 1, 2, 3$. Comparing with equation (1.11), it can be observed that the above equation is the analogous form of an **S**-invariant in four dimensions.

Let $v = (v_1, v_2, v_3)$ be a divergence-free velocity field in $\mathbb{R}^3$ satisfying Euler's or Navier-Stokes' equations as in (1.1) or (1.2). Let $f : \mathbb{R}^4 \to \mathbb{R}$. The velocity four-vector field can be represented as $\mathcal{V} = (f, v)$ in $\mathbb{R}^4$. Equipping $\mathbb{R}^4$ with the Euclidean metric, the one-form or the covector field associated with the velocity $\mathcal{V}$ can be defined as

$$\sigma = fdt + v_1 dx + v_2 dy + v_3 dz \text{ or } \sigma = fdt + \hat{v}. \qquad (2.7)$$

It will be shown later (see equation (2.14)) for an ideal incompressible flow and $f = -\varepsilon$,

$$\mathcal{L}_\xi \sigma = d(\frac{v^2}{2} - (\frac{p}{\rho} + \varphi)) = d\beta_1.$$

The transformation $\sigma^{'} = \sigma + d\theta$, where the scalar function θ is chosen such that $\mathcal{L}_\xi \theta = -\beta_1$, leads to an **S**-invariant in $\mathbb{R}^4$.

In general, if a is a Lagrangian invariant, then da is an **S**-invariant. Thus the exterior differentials of the Lagrangian invariants, such as dS, $d\varpi$, dh_m, dh_c are examples of **S**-invariants.

3. Invariants associated with two-forms: A two-form can be represented as

$$\alpha^2 = b_{01} dx_0 \wedge dx_1 + b_{02} dx_0 \wedge dx_2 + b_{03} dx_0 \wedge dx_3 + b_{12} dx_1 \wedge dx_2 + b_{13} dx_1 \wedge dx_3 + b_{23} dx_2 \wedge dx_3,$$

with the convention that $b_{ij} = -b_{ji}$. Condition (2.1) of invariance of forms results in

$$(v_k \partial_{x_k} b_{ij} + [b_{kj} \partial_{x_i} v_k]_{k \neq j} + [b_{ik} \partial_{x_j} v_k]_{k \neq i}) dx_i \wedge dx_j = 0. \qquad (2.8)$$

For the velocity one-form σ, the vorticity two-form $\omega = d\sigma$ for ideal fluids and the Faraday two-form for electrically conducting fluids are invariants associated with two-forms. If α_1 and α_2 are invariant forms, then $\alpha_1 \wedge \alpha_2$ is also an invariant form. Hence the pair-wise exterior products of all the invariant one-forms lead to additional invariant two-forms.

4. Invariants associated with three-forms: A three-form can be represented as $\alpha^3 = c_0 dx_1 \wedge dx_2 \wedge dx_3 + c_1 dx_0 \wedge dx_2 \wedge dx_3 + c_2 dx_0 \wedge dx_1 \wedge dx_3 + c_3 dx_0 \wedge dx_1 \wedge dx_2$ or $\alpha^3 = c_i dx_j \wedge dx_k \wedge dx_l$. On applying (2.1) we get

$$(v_m \partial_{x_m} c_i + c_i \partial_{x_m} v_m + \epsilon_{ijkl} \tilde{c_m} \partial_{x_m} v_i) dx_j \wedge dx_k \wedge dx_l = 0, \qquad (2.9)$$

where ϵ_{ijkl} is the Levi- Civita symbol of four dimensions and $\tilde{c_m} = (-1)^{m+1} c_m$. The above equation takes the analogous form of the frozenness invariants associated with the three-forms in $\mathbb{R}^3$. The exterior differential of two-forms and exterior products of two, one and zero forms lead to invariants of the above form. $\dfrac{dS \wedge d\varpi \wedge q}{\rho}$ can be considered as the four dimensional analogue to the invariant in the Hollman model (Tur and Yanovsky, 1993).

From the velocity one-form σ and the vorticity two-form ω, the helicity three-form is defined as $\eta = \sigma \wedge \omega$, It will shown later that (from equation (2.48)) $\mathcal{L}_\xi \eta = d((\frac{v^2}{2} - (P + \varphi))\omega) = d\beta_2$. The transformation $\eta' = \eta + d\theta'$, where the two-form θ' is so chosen that $\mathcal{L}_\xi \theta' = -\beta_2$ leads to an invariant three-form.

5. Invariants associated with four-forms: A four-form can be represented as $\alpha^4 = g(x_0, x_1, x_2, x_3) dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$. On applying (2.1) we get,

$$(\partial_{x_k} g v_k) dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 = 0. \qquad (2.10)$$

For the flow ξ, the above equation takes the form

$$(\partial_t g + \partial_{x_k}(g v_k)) dt \wedge dx \wedge dy \wedge dz = 0, \qquad (2.11)$$

which is analogous to the density invariants associated with three-forms in $\mathbb{R}^3$.

The invariance of density four-form $\rho dt \wedge dx \wedge dy \wedge dz$, leads to the mass conservation law for ideal flows without sources and sinks and the invariance of $\rho S dt \wedge dx \wedge dy \wedge dz$ leads to the entropy conservation law. The invariance of the four-form of parity $\kappa = \omega \wedge \omega$ leads to the Ertel's theorem of potential vorticity (see equation (2.69).

Comparing the invariant relations in the three and four dimensional cases, we find that the invariant relation associated with the two-forms in $\mathbb{R}^4$ has no counterpart in the three dimensional manifold and its geometric significance is to be further investigated.

2.2 Integral Invariants in $\mathbb{R}^4$

In the next sections we apply the concepts of absolute and relative invariance of forms as described in section (1.1.7) to classical flows in a four dimensional manifold and obtain some results.

2.2.1 Integral Invariance Associated with the One-Form of Velocity

Let $v = (v_1, v_2, v_3)$ be a divergence-free velocity field in $\mathbb{R}^3$ satisfying Euler's or Navier-Stokes' equations as in (1.1) or (1.2). Associated with the four-velocity vector field $\mathcal{V} = (f, v)$ in $\mathbb{R}^4$ the velocity one-form is expressed as,

$$\sigma = f dt + v_1 dx + v_2 dy + v_3 dz \text{ or } \sigma = f dt + \hat{v}.$$

If the four-velocity $\mathcal{V}$ is also divergence free then we have,

$$\partial_t f + v \cdot \nabla f = 0, \tag{2.12}$$

so that $\mathcal{V}$ satisfies the Euler's equation (Shashikanth, 2012). The velocity one-form σ satisfies the divergence free condition $\eth\sigma = 0$ where $\eth = -*d*$ is the co-differential operator and $*$ is the Hodge star dual operator with respect to the Euclidean metric.

For an incompressible fluid, $\frac{dp}{\rho}$ is exact, say dP. In particular, choose $f = -\varepsilon$, the Bernoulli function, then the velocity one-form is expressed as

$$\sigma = -\varepsilon dt + v_1 dx + v_2 dy + v_3 dz \text{ or } \sigma = -\varepsilon dt + \hat{v}. \tag{2.13}$$

Let $\Gamma = \int_{D^1} \sigma$, where D^1 is an arbitrary 1 - chain.
Then,

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int_{D^1} \sigma = \int_{D^1} \mathcal{L}_\xi \sigma.$$

For the flow field $\xi = \partial_t + v$,

$$\mathcal{L}_\xi \sigma = d(\frac{v^2}{2} - (P + \varphi)) = d\bar{\beta}, \tag{2.14}$$

for incompressible ideal flows, Hence $\mathcal{L}_\xi \sigma$ is exact for ideal flows. By applying Stokes' theorem we have,

$$\frac{d\Gamma}{dt} = \int_{D^1} d\bar{\beta} = \int_{\partial D^1} (\frac{v^2}{2} - (P + \varphi)).$$

$\frac{d\Gamma}{dt}$ vanishes when $\partial D^1 = 0$. *i.e*, when D^1 is a 1 - cycle C^1. Thus Γ is an invariant over arbitrary 1 - cycles C^1 in $\mathbb{R}^4$. Thus $\oint_{C^1} \sigma$ represents a relative integral invariant. If the domain of integration lies on the hyper surface of constant time, then $\oint_{C^1} \sigma$ is the circulation and its invariance implies the Kelvin's circulation theorem. But the invariance of $\oint_{C^1} \sigma$ still remains valid even if the cycle C^1 lies in different coordinates of time.

From the condition $\mathcal{L}_{\partial_t}\hat{\alpha} + i_v \hat{d}\hat{\alpha} = \hat{d}\hat{\beta}$ of relative invariance of the velocity one form σ, we get $\mathcal{L}_{\partial_t}\hat{v} + i_v \hat{d}\hat{v} = -\hat{d}\varepsilon$, which in vector form yields the Euler's equation (1.3).

For viscous flows, using equation (1.4), we get

$$\mathcal{L}_\xi \sigma = \nu(v \cdot \nabla \times w)dt - \nu(\nabla \times w) \cdot d\mathbf{x} + d(\frac{v^2}{2} - (P + \varphi)).$$

Then,

$$\frac{d\Gamma}{dt} = \int_{D^1} \mathcal{L}_\xi \sigma = \int_{D^1} \nu(v \cdot (\nabla \times w))dt - \nu(\nabla \times w) \cdot d\mathbf{x} + d(\frac{v^2}{2} - (P + \varphi)). \tag{2.15}$$

If D^1 lies on fixed coordinates of time the above equation reduces to

$$\frac{d\Gamma}{dt} = \int_{D^1} (-\nu(\nabla \times w) + \nabla(\frac{v^2}{2} - (P + \varphi))) \cdot d\mathbf{x}.$$

Also,

$$\frac{d\Gamma}{dt} = \oint_{C^1} -\nu(\nabla \times w) \cdot d\mathbf{x},$$

over 1 - cycles. The above expression evaluates the rate of change of circulation for viscous incompressible fluids for three dimensional flows and equation (2.15) gives the four dimensional expression to evaluate the rate of change of circulation.

Circulation plays an important role in biomechanics. The emergence of the study of four dimensional flows in the field of biofluids and magnetic resonance imaging is reflected in recent studies (Francois et.al (2013), Kamphius et.al (2013), Markl et.al (2016)). Equation (2.15) can be considered as the general expression to evaluate the rate of change of circulation of viscous fluids for four dimensional flows.

2.2.2 Integral Invariance Associated with the Two-Form of Vorticity

From the velocity one-form $\sigma = f dt + \hat{v}$, the vorticity two-form is defined as

$$\begin{aligned}\omega &= d\sigma \\ &= w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy \\ &+ (\partial_t v_1 - \partial_x f) dt \wedge dx + (\partial_t v_2 - \partial_y f) dt \wedge dy + (\partial_t v_3 - \partial_z f) dt \wedge dz \\ &= \hat{\Omega} + dt \wedge \hat{\mathcal{E}},\end{aligned} \tag{2.16}$$

where

$$\hat{\Omega} = w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy = w \cdot ds$$

and

$$\hat{\mathcal{E}} = (\partial_t v_1 - \partial_x f) dx + (\partial_t v_2 - \partial_y f) dy + (\partial_t v - \partial_z f) dz.$$

$w = \nabla \times v = (w_1, w_2, w_3)$ is the vorticity vector in $\mathbb{R}^3$. The vorticity two-form is expressed as

$$\omega = w \cdot ds + ((v \times w) - \nabla \phi) dt \wedge d\mathbf{x}, \tag{2.17}$$

in the space of solutions of Euler equations and for viscous fluids,

$$\omega = w \cdot ds + ((v \times w) - \nabla \phi + \nu \nabla^2 v) dt \wedge d\mathbf{x}, \tag{2.18}$$

where $\phi = f + \varepsilon$. If $f = -\varepsilon$, the above equations reduce to

$$\omega = w \cdot ds + (v \times w) dt \wedge d\mathbf{x} \tag{2.19}$$

and

$$\omega = w \cdot ds + ((v \times w) + \nu \nabla^2 v) dt \wedge d\mathbf{x}. \tag{2.20}$$

In $\mathbb{R}^4$, the vorticity two-form can also be represented as (L, w), where $L = \partial_t v + \nabla\varepsilon = (L_1, L_2, L_3)$.

$L = v \times w$, the negative of the Lamb vector for ideal fluids and $L = (v \times w) + \nu\nabla^2 v$ for viscous fluids.

Let $W = \int_{D^2} \omega$, then

$$W = \int_{D^2} d\sigma = \int_{\partial D^2} \sigma.$$

Thus $W = \Gamma$ when $\partial D^2 = C^1$.

$$\frac{dW}{dt} = \int_{D^2} \mathcal{L}_\xi \omega = \int_{D^2} \mathcal{L}_\xi d\sigma = \int_{D^2} d\mathcal{L}_\xi \sigma = \int_{\partial D^2} \mathcal{L}_\xi \sigma. \tag{2.21}$$

Thus W is invariant over 2 - cycles for which $\partial D^2 = 0$ or when $\mathcal{L}_\xi \sigma = 0$ over ∂D^2.

For ideal flows, $\mathcal{L}_\xi \sigma$ is exact. Then,

$$\frac{dW}{dt} = 0.$$

Hence W represents an integral invariant.

Now, $\oint_{C^1} \sigma$ is a relative invariant for ideal flows. From equation (1.19) it follows that $\int_{D^2} \omega$ is an absolute integral invariant, when $C^1 = \partial D^2$.

We also have $\hat{\mathcal{E}} = -i_v \hat{\Omega}$. Thus ω can be represented in the form $\hat{\Omega} - dt \wedge i_v \hat{\Omega}$. The condition

$$\mathcal{L}_{\partial_t} \hat{\Omega} + \hat{\mathcal{L}}_v \hat{\Omega} = 0,$$

for absolute invariance of forms as in equation (1.22), in vector form results in the vorticity equation (1.5). The condition $\mathcal{L}_\xi \omega = 0$ for the absolute invariance of two form ω results in the following vector equations,

$$\partial_t((\partial_t v + \nabla\varepsilon) + w \times v) + \nabla((\partial_t v + \nabla\varepsilon) \cdot v) = 0$$

and

$$\nabla \times ((\partial_t v + \nabla\varepsilon) + w \times v) = 0.$$

In $\mathbb{R}^3$, vortex lines are curves tangent to the vorticity vector field. But vorticity vector fields do not exist in $\mathbb{R}^4$. Because of the two-form nature of ω there are no vortex lines in $\mathbb{R}^4$, instead ω can be considered as vortex surfaces (Shashikanth, 2012).

The absolute invariance of the two-form ω implies that $\int_{D_1^2} \omega = \int_{D_2^2} \omega$, where D_1^2 and $D_2^2 = \varphi_t(D_1^2)$ are two dimensional surfaces or 2-chains (not necessarily lying in fixed hypersurfaces of constant time) encircling vortex surfaces in $\mathbb{R}^4$.

If D^2 is a two-chain lying on the hypersurface of constant time, then $\int_{D^2} \omega$ reduces to $\int_{D^2} \omega \cdot ds$, conventionally termed as the net vorticity flux, is an invariant for ideal three dimensional flows.

For non-barotropic fluids, $\mathcal{L}_\xi \omega = \dfrac{d\rho \wedge dp}{\rho^2} \neq 0$ and hence the two-form ω fails to satisfy the invariance condition (Fecko, 2013). But it will be shown that the generalized vorticity two-form $\tilde{\omega}$ is an absolute integral invariant (see chapter 3).

For viscous fluids, using equation (2.21)

$$\begin{aligned}\frac{dW}{dt} &= \int_{D^2} -\nu[(\nabla \times (\nabla \times w)) \cdot ds + dt \wedge (\partial_t(\nabla \times w) + \nabla((\nabla \times w) \cdot v) \cdot d\mathbf{x}] \\ &= \int_{\partial D^2} \nu(v \cdot (\nabla \times w))dt - \nu(\nabla \times w) \cdot d\mathbf{x} + d(\frac{v^2}{2} - (P + \varphi)).\end{aligned} \tag{2.22}$$

If D^2 lies on fixed coordinates of time then,

$$\begin{aligned}\frac{dW}{dt} &= \int_{D^2} -\nu(\nabla \times (\nabla \times w)) \cdot ds \\ &= \int_{\partial D^2} [-\nu(\nabla \times w) + \nabla(\frac{v^2}{2} - (P + \varphi))] \cdot d\mathbf{x}. \end{aligned} \tag{2.23}$$

The expression (2.23) can be considered as the rate of change of vorticity flux for viscous fluids in three dimensions. Equation (2.22) can be used to evaluate the vorticity flux for four dimensional flows.

2.2.3 Vorticity Stress Tensor Analogous to Magnetic Stress Tensor

An analogy between Maxwell stress tensor in electromagnetic theory and a tensor influenced by vorticity in incompressible inviscid fluids was explored by M.J. Vedan et.al (2015).

For an incompressible three dimensional flow of an ideal fluid, the vorticity field is frozen-in and satisfies the vorticity equation (1.5). Analogous to the magnetic energy, the energy associated with the vorticity can be defined as $\frac{|w|^2}{2} = \frac{w \cdot w}{2}$.

The rate of change of this energy can be expressed as $\partial_t \left(\frac{|w|^2}{2} \right)$. Suppose that the vorticity is confined to the sub domain of the fluid, then on integrating the above equation over a volume V, we get,

$$\frac{dM}{dt} = \int_V \partial_t \left(\frac{|w|^2}{2} \right) = -\int_V v \cdot P dV, \tag{2.24}$$

where $P = (\nabla \times w) \times w$ and

$$M = \int_V (\frac{w^2}{2}) dV,$$

is the total enstrophy. Then the i^{th} component of P, i.e., $P_i = \partial_{x_j}\Pi_{ij}$, where $\Pi_{ij} = w_i w_j - \dfrac{|w|^2}{2}\delta_{ij}$, is the ij^{th} entry of the matrix $\Pi = [\Pi_{ij}]$

$$\Pi = \begin{bmatrix} w_1^2 - \dfrac{|w|^2}{2} & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 - \dfrac{|w|^2}{2} & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 - \dfrac{|w|^2}{2} \end{bmatrix} \tag{2.25}$$

The vorticity stress tensor Π thus obtained is associated to the enstrophy in the same way as the magnetic stress tensor is associated to the magnetic energy. The normal components of this tensor represents the tension in the vortices and the tangential components represent the shearing forces on the plane to which w is normal i.e., the plane associated with w as a bivector.

Π can be diagonalized to obtain

$$\Pi' = \begin{bmatrix} \frac{|w|^2}{2} & 0 & 0 \\ 0 & -\frac{|w|^2}{2} & 0 \\ 0 & 0 & -\frac{|w|^2}{2} \end{bmatrix} \tag{2.26}$$

Thus the stress tensor constitutes a tension $\dfrac{|w|^2}{2}$ along a vortex surface and an equal pressure normal to it. The expressions of the vorticity stress tensor can be compared with that of the magnetic stress tensor discussed by Stierstadt and Liu (2014). In this work they present a number of applications of the stress tensor in modern devices and ferrofluids. Similar results can be applied to the vorticity stress tensor also.

The vorticity two-form in a three dimensional manifold is expressed as $\omega = w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy$. Thus we have

$$W = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix}$$

as the tensor associated with the vorticity two-form. The dual of the vorticity two-form ω is a one form $*\omega = w_1 dx + w_2 dy + w_3 dz$, where $*$ denotes the Hodge dual operator. Let W^* denote the (3×1) column matrix associated with $*\omega$. The vorticity stress tensor can be derived from W by

$$\Pi = \frac{1}{2}(WW + W^* W^{*T}), \tag{2.27}$$

where W^{*T} is the transpose of W^*. The two-form of vorticity

$$\omega = L_1 dt \wedge dx + L_2 dt \wedge dy + L_3 dt \wedge dz + w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy$$

given in equation (2.19) in the four dimensional case can be associated to a vorticity field tensor F in $\mathbb{R}^4$. This tensor can be expressed as

$$F = \begin{bmatrix} 0 & L_1 & L_2 & L_3 \\ -L_1 & 0 & w_3 & -w_2 \\ -L_2 & -w_3 & 0 & w_1 \\ -L_3 & w_2 & -w_1 & 0 \end{bmatrix}.$$

Let ∇' denote the matrix $\begin{bmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{bmatrix}$, then

$\nabla' F = \begin{bmatrix} -q & -J_1 & -J_2 & -J_3 \end{bmatrix}$ yields the following equations

$$\begin{aligned} -\partial_x L_1 - \partial_y L_2 - \partial_z L_3 \quad &= -q \\ \partial_t L_1 - \qquad \partial_y w_3 + \partial_z w_2 \quad &= -J_1 \\ \partial_t L_2 + \partial_x w_3 - \qquad \partial_z w_1 \quad &= -J_2 \\ \partial_t L_3 - \partial_x w_2 + \partial_y w_1 \qquad &= -J_3, \end{aligned}$$

which can be expressed as,

$$\nabla \cdot L = q \tag{2.28}$$

and

$$-\partial_t L + \nabla \times w = J. \tag{2.29}$$

Here $q = \nabla^2 \varepsilon$ and $J = -\partial_t^2 v - \nabla \partial_t \varepsilon + \nabla \times w$.

The dual of the two-form ω is given by

$$*\omega = w_1 dt \wedge dx + w_2 dt \wedge dy + w_3 dt \wedge dz + L_1 dy \wedge dz + L_2 dz \wedge dx + L_3 dx \wedge dy,$$

where $*$ is the Hodge dual operator. $*\omega$ can be associated to the tensor F^* given by

$$F^* = \begin{bmatrix} 0 & w_1 & w_2 & w_3 \\ -w_1 & 0 & L_3 & -L_2 \\ -w_2 & -L_3 & 0 & L_1 \\ -w_3 & L_2 & -L_1 & 0 \end{bmatrix}.$$

Thus $\nabla' F^*$ yields the equations,

$$\begin{aligned} -\partial_x w_1 - \partial_y w_2 - \partial_z w_3 \quad &= 0 \\ \partial_t w_1 - \qquad \partial_y L_3 + \partial_z L_2 \quad &= 0 \\ \partial_t w_2 + \partial_x L_3 - \qquad \partial_z L_1 \quad &= 0 \\ \partial_t w_3 - \partial_x L_2 + \partial_y L_1 \qquad &= 0, \end{aligned}$$

which in vector notation can be expressed as

$$\nabla \cdot w = 0 \tag{2.30}$$

and

$$\partial_t w - \nabla \times L = 0. \tag{2.31}$$

Thus we have $\nabla' F^* = 0$. The equations (2.28), (2.29), (2.30) and (2.31) together form the equations analogous to Maxwell's equations in electromagnetic theory.

Also we have $\det F = \det F^* = (L \cdot w)^2$. Here F is analogous to the electromagnetic tensor $\mathcal{F}$ in electromagnetic theory.

2.2.4 Maxwell's Equations in Fluid Dynamics

Maxwell's equations in electromagnetic theory were formulated in 1865. The formulations of equations in fluid dynamics analogous to Maxwell's equations can be seen in Marmanis (1998), Kambe (2010) and Shridhar (1998).

Equations (2.28), (2.29), (2.30) and (2.31) together form a set of fluid Maxwell's equations

$$\nabla \cdot w = 0, \tag{2.32}$$

$$\nabla \times L = \partial_t w, \tag{2.33}$$

$$\nabla \cdot L = q, \tag{2.34}$$

$$\nabla \times w = J + \partial_t L. \tag{2.35}$$

Here $q = \nabla^2 \varepsilon$ is termed as hydrodynamic charge density and J, is termed as hydrodynamic current vector. Marmanis recognized the vorticity and the Lamb vector as the kernel of a dynamical theory of turbulence and introduced the concept of turbulent charge and turbulent current in the study of metafluid dynamics. A study of the effectiveness of the Lamb vector and the hydrodynamic charge density in analyzing the dynamics of coherent structures of turbulent flow can be seen in Rousseaux et.al (2007). The first two equations in the above set can be considered as the conservation and evolution equations for the vorticity and the last two equations as the conservation and the evolution of the Lamb vector. We have

$$d\omega = (\nabla \cdot w)dx \wedge dy \wedge dz + (\partial_t w - \nabla \times L)dt \wedge ds$$

and

$$d(*\omega) = (\nabla \cdot L)dx \wedge dy \wedge dz + (\partial_t L - \nabla \times w)dt \wedge ds.$$

$$\text{Or,} \qquad * \, d(*\omega) = -(\nabla \cdot L)dt + (\partial_t L - \nabla \times w) \cdot d\boldsymbol{x}.$$

Thus the above set of four fluid Maxwell's equations can be derived from

$$d\omega = 0 \qquad \text{and} \qquad * \, d(*\omega) = (-q, -J).$$

Also the four vector $\mathcal{J} = (q, J)$ satisfies the conservation equation

$$\partial_t q + \nabla \cdot J = 0.$$

As the vorticity stress tensor Π was constructed from the tensor associated with vorticity W in equation (2.27), we can construct a tensor T associated to F such that

$$T = \frac{1}{2}(FF + F^* F^{*T}). \tag{2.36}$$

Thus,

$$T = \begin{bmatrix} \frac{1}{2}(-|L|^2+|w|^2) & -(L\times w)_1 & -(L\times w)_2 & -(L\times w)_3 \\ -(L\times w)_1 & \pi_{11} & \pi_{12} & \pi_{13} \\ -(L\times w)_2 & \pi_{21} & \pi_{22} & \pi_{23} \\ -(L\times w)_3 & \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix} \tag{2.37}$$

where $\pi_{ij} = -(L_iL_j - \frac{1}{2}|L|^2\delta_{ij}) + (w_iw_j - \frac{1}{2}|w|^2\delta_{ij})$. Also T is traceless. The four dimensional fluid dynamic stress-energy tensor T is analogous to the electromagnetic stress-energy tensor $\mathcal{T}$ in electromagnetic theory. As $\mathcal{T}$ is associated to the Maxwell stress tensor, T is associated to the vorticity stress tensor given in equation (2.25).

The electromagnetic tensor $\mathcal{F}$ is a second-rank antisymmetric tensor field that describes the electromagnetic field and the forces that would act on a charged test particle at a given location with a given velocity. The electromagnetic stress-energy tensor $\mathcal{T}$ is a symmetric second rank tensor which describes the energy and momentum carried by the electromagnetic field.

The term $-(L\times w) = \mathcal{P}$ can be seen as the Poynting vector, which describes the energy flux. The term $\frac{1}{2}(-|L|^2+|w|^2) = \tilde{\boldsymbol{\xi}}$ represents the fluid dynamic energy density. From the fluid dynamic stress energy tensor T, we can derive the four dimensional energy conservation laws. Evaluating $\nabla' T$, we get

$$\partial_t \xi + \nabla\cdot\mathcal{P} = J\cdot L. \tag{2.38}$$

$$\begin{aligned} \partial_t\mathcal{P}_1 + \partial_i\pi_{1i} &= (J\times w)_1 \\ \partial_t\mathcal{P}_2 + \partial_i\pi_{2i} &= (J\times w)_2 \\ \partial_t\mathcal{P}_3 + \partial_i\pi_{3i} &= (J\times w)_3. \end{aligned} \tag{2.39}$$

Thus we have $\nabla' T = \begin{bmatrix} J\cdot L & (J\times w)_1 & (J\times w)_2 & (J\times w)_3 \end{bmatrix}$.

The equation (2.38) is the Poynting theorem and it describes the transfer of the energy momentum or the energy dissipation rate. The term $J \cdot L$ describes the rate at which the energy density is produced.

The set of three equations in (2.39) together can be expressed as

$$\partial_t \mathcal{P} + \nabla \cdot \pi = J \times w, \tag{2.40}$$

which can be considered as the fluid dynamical Lorentz force law. If $\mathcal{J} = \begin{bmatrix} q & J_1 & J_2 & J_3 \end{bmatrix}^T$, then

$$F\mathcal{J} = \begin{bmatrix} J \cdot L & -qL_1 + (J \times w)_1 & -qL_2 + (J \times w)_2 & -qL_1 + (J \times w)_3 \end{bmatrix}^T.$$

The four-vector of the form $f_l = (J \cdot L, -qL + (J \times w))$ can be considered as the four dimensional force analogous to the Lorentz force.

Thus we have derived the fluid dynamical analogs of the Poynting theorem and the Lorentz force. These results can be applied to analyze the stress-energy propagation and dissipation based on Navier-Stokes' theory in Euclidean space time. Scofield and Huq (2014) have derived similar results in the context of geometrodynamical theory of fluids in Minkowski space time.

2.2.5 Integral Invariance Associated with the Three-Form of Helicity

Helicity is the natural tendency of flows to form vortices or coherent structures and is defined as a measure of linkage, knottedness or intertwining of vortex lines or tubes in the flow (Moffat 1969). Helicity is one of the tools used in fluid topology to measure the topological change of flows and to study the development and decay of turbulence in fluid flows.

Total helicity is conserved for ideal fluids, but how helicity changes in real fluids in the presence of viscosity is still being investigated. Scheeler et.al (2017) describes the measurement of total helicity in a real fluid by using a set of hydrofoils to track linking, twisting, and writhing. They show that twisting dissipates total helicity, whereas writhing and linking conserve it. This provides a fundamental insight into tornado genesis, atmospheric flows, and the formation of turbulence.

The conventional helicity of a fluid flow confined to a bounded or unbounded domain in $\mathbb{R}^3$ is defined as

$$\mathbb{H} = \int_D v \cdot w dV.$$

The quantity $h(x,t) = v \cdot w$ is the helicity density of the flow. Helicity density is a measure of how much the velocity and the vorticity are not orthogonal and is a conserved quantity for perfect fluid flows (Moffat, (1969)).

From the velocity one-form σ and the vorticity two-form ω, we can define,

$$\begin{aligned}\eta &= \sigma \wedge \omega \\ &= v \cdot w dx \wedge dy \wedge dz + h_1 dt \wedge dy \wedge dz + h_2 dt \wedge dz \wedge dx + h_3 dt \wedge dx \wedge dy \\ &= v \cdot w dx \wedge dy \wedge dz + dt \wedge ((\partial_t v - \nabla f) \times v + fw) ds \qquad (2.41) \\ &= \hat{h} - dt \wedge \hat{H},\end{aligned}$$

as the three-form associated with helicity. Here $h = v \cdot w$ and

$$-H = (h_1, h_2, h_3) = (\partial_t v - \nabla f) \times v + fw.$$

Substituting for the term $\partial_t v$ from equation (1.3) we get,

$$\eta = v \cdot w dx \wedge dy \wedge dz + dt \wedge (((v \times w) - \nabla \phi) \times v + fw) \cdot ds, \qquad (2.42)$$

in the space of solutions of Euler equations and substituting for $\partial_t v$ from equation (1.4),

$$\eta = v \cdot w dx \wedge dy \wedge dz + dt \wedge (((v \times w) - \nabla \phi + \nu \nabla^2 v) \times v + f w) \cdot ds, \quad (2.43)$$

for viscous fluids. In particular if we choose $f = -\varepsilon$, then

$$\eta = v \cdot w dx \wedge dy \wedge dz + dt \wedge ((\partial_t v + \nabla \varepsilon) \times v - \varepsilon w) \cdot ds. \quad (2.44)$$

From equation (2.42) for Euler flows we have,

$$\eta = v \cdot w dx \wedge dy \wedge dz + dt \wedge ((v \times w) \times v - \varepsilon w) \cdot ds, \quad (2.45)$$

and from (2.43) for viscous flows we have

$$\eta = v \cdot w dx \wedge dy \wedge dz + dt \wedge (((v \times w) + \nu \nabla^2 v) \times v - \varepsilon w) ds. \quad (2.46)$$

In the Euclidean space time $\mathbb{R}^4$, we define helicity $\mathcal{H}$ as

$$\mathcal{H} = \int_{D^3} \eta = \int_{D^3} \sigma \wedge \omega,$$

where D^3 is a three dimensional volume co-moving with the fluid and $D^3(t) = \psi_t(D^3(0))$. The integrand three-form η contains space time coupled terms and the three dimensional chain D^3 may contain points lying on different coordinates of time. The rate of change of helicity is given by

$$\frac{d\mathcal{H}}{dt} = \frac{d}{dt} \int_{D^3} \eta = \int_{D^3} \mathcal{L}_\xi \eta.$$

By Cartan's formula we have,

$$\mathcal{L}_\xi \eta = i_\xi d\eta + d(i_\xi \eta). \quad (2.47)$$

We also have the property,

$$i_\xi \eta = i_\xi(\sigma \wedge \omega) = (i_\xi \sigma) \wedge \omega + (-1)\sigma \wedge (i_\xi \omega).$$

Now,

$$i_\xi d\eta = (\partial_t h + \nabla \cdot H)dx \wedge dy \wedge dz - v(\partial_t h + \nabla \cdot H)dt \wedge ds.$$

From the expression which will be later derived in equation (2.65) we have,

$$i_\xi d\eta = -2(\nabla\phi) \cdot w dx \wedge dy \wedge dz + 2v((\nabla\phi) \cdot w)dt \wedge ds$$

and

$$\begin{aligned} d(i_\xi \eta) = {} & \nabla \cdot (\frac{v^2}{2} - (P + \varphi))w dx \wedge dy \wedge dz \\ & + [(\partial_t(\frac{v^2}{2} - (P + \varphi)))w \\ & - (\nabla(\frac{v^2}{2} - (P + \varphi))) \times (v \times w - \nabla\phi)]dt \wedge ds \end{aligned}$$

for ideal flows.

Choose $f = -\varepsilon$. Then for ideal flows $i_\xi d\eta = 0$. Substituting in Cartan's formula (2.47) we get,

$$\begin{aligned} \mathcal{L}_\xi \eta = {} & \nabla \cdot (\frac{v^2}{2} - (P + \varphi))w dx \wedge dy \wedge dz \\ & + [(\partial_t(\frac{v^2}{2} - (P + \varphi)))w \\ & - (\nabla(\frac{v^2}{2} - (P + \varphi))) \times (v \times w - \nabla\phi)]dt \wedge ds \\ & = d((\frac{v^2}{2} - (P + \varphi))\omega). \end{aligned} \tag{2.48}$$

From the expression which will later be derived (in equation 2.66), for viscous flows we have,

$$i_\xi d\eta = 2(-\nabla\phi+\nu\nabla^2 v)\cdot w dx\wedge dy\wedge dz - 2v((-\nabla\phi+\nu\nabla^2 v)\cdot w)dt\wedge ds \quad (2.49)$$

and

$$\begin{aligned} d(i_\xi\eta) = {} & [(\nabla\cdot(\frac{v^2}{2}-(P+\varphi))w) - \nu(\nabla\cdot(\nabla\times w)\times v)]dx\wedge dy\wedge dz \\ & + [(\partial_t(\frac{v^2}{2}-(P+\varphi))w \\ & - (\nabla(\frac{v^2}{2}-(P+\varphi)))\times(v\times w-\nabla\phi+\nu\nabla^2 v)) \\ & - \nu((\partial_t(\nabla\times w)\times v)+\nabla\times(\varepsilon(\nabla\times w)+(v\cdot\nabla\times w)v))]dt\wedge ds. \end{aligned} \quad (2.50)$$

Combining the above two expressions and substituting in equation (2.47) we get,

$$\begin{aligned} \mathcal{L}_\xi\eta = {} & [2(-\nabla\phi+\nu\nabla^2 v)\cdot w + \nabla\cdot(\frac{v^2}{2}-(P+\varphi))w \\ & - \nu(\nabla\cdot(\nabla\times w)\times v)]dx\wedge dy\wedge dz \\ & [-2v((-\nabla\phi+\nu\nabla^2 v)\cdot w) + (\partial_t(\frac{v^2}{2}-(P+\varphi))w \\ & - (\nabla(\frac{v^2}{2}-(P+\varphi)))\times(v\times w-\nabla\phi+\nu\nabla^2 v)) \\ & - \nu((\partial_t(\nabla\times w)\times v)+\nabla\times(\varepsilon(\nabla\times w)+(v\cdot\nabla\times w)v))]dt\wedge ds, \end{aligned} \quad (2.51)$$

for viscous flows.

Choose $f = -\varepsilon$, then the above equations reduce to

$$\begin{aligned}\mathcal{L}_\xi \eta = {} & [2(\nu\nabla^2 v)\cdot w + \nabla\cdot(\frac{v^2}{2} - (P+\varphi))w \\ & - \nu(\nabla\cdot(\nabla\times w)\times v)]dx\wedge dy\wedge dz \\ & [-2v((\nu\nabla^2 v)\cdot w) + (\partial_t(\frac{v^2}{2} - (P+\varphi))w \\ & - (\nabla(\frac{v^2}{2} - (P+\varphi)))\times(v\times w + \nu\nabla^2 v)) \\ & - \nu((\partial_t(\nabla\times w)\times v) + \nabla\times(\varepsilon(\nabla\times w) + (v\cdot\nabla\times w)v))]dt\wedge ds,\end{aligned} \tag{2.52}$$

for viscous flows.

For an ideal flow, consider cycles C^3 such that $\partial C^3 = 0$. We have seen that $\oint_{C^1}\sigma$ represents a relative integral invariant and hence from (1.18), $\mathcal{H} = \int_{C^3}\sigma\wedge d\sigma = \int_{C^3}\eta$ must also be a relative integral invariant in $\mathbb{R}^4$.

From equation (2.48) for ideal flows $\mathcal{L}_\xi\eta$ is exact, i.e,

$$\mathcal{L}_\xi\eta = d((\frac{v^2}{2} - (P+\varphi))\omega) = d(\tilde{\beta}).$$

The exactness of $\mathcal{L}_\xi\eta$ can also be derived from equation (2.14) as,

$$\begin{aligned}\mathcal{L}_\xi\eta &= \mathcal{L}_\xi(\sigma\wedge d\sigma) \\ &= (\mathcal{L}_\xi\sigma)\wedge d\sigma \\ &= d(\frac{v^2}{2} - (P+\varphi))\wedge d\sigma \\ &= d((\frac{v^2}{2} - (P+\varphi))\omega).\end{aligned} \tag{2.53}$$

Also,

$$\frac{d\mathcal{H}}{dt} = \int_{C^3} d((\frac{v^2}{2} - (P+\varphi))\omega) = \oint_{\partial C^3}(\frac{v^2}{2} - (P+\varphi))\omega = 0. \tag{2.54}$$

Thus for an ideal fluid, the helicity $\mathcal{H}$ is conserved over a three dimensional chain whose boundary vanishes.
Also note that, from (2.48)

$$\begin{aligned}\frac{d\mathcal{H}}{dt} &= \int_{C^3} d((\frac{v^2}{2} - (P+\varphi))\omega) \\ &= \int_{C^3} (\nabla \cdot (\frac{v^2}{2} - (P+\varphi))w)dx \wedge dy \wedge dz \\ &+ [\partial_t(\frac{v^2}{2} - (P+\varphi))w - (\nabla(\frac{v^2}{2} - (P+\varphi))) \times (v \times w)]dt \wedge ds.\end{aligned} \tag{2.55}$$

For cycles lying on hypersurfaces of fixed time, the rate of change of helicity reduces to

$$\begin{aligned}\frac{d\mathcal{H}}{dt} &= \int_{C^3} (\nabla \cdot (\frac{v^2}{2} - (P+\varphi))w)dx \wedge dy \wedge dz \\ &= \int_{C^3} (w \cdot \nabla)(\frac{v^2}{2} - (P+\varphi))dx \wedge dy \wedge dz,\end{aligned} \tag{2.56}$$

since $\nabla \cdot w = 0$.
The rate of change of helicity as derived by Moffat [1969] [34] is,

$$\begin{aligned}\frac{d\mathcal{H}}{dt} &= \int_V (w \cdot \nabla)(\frac{v^2}{2} - (P+\varphi))dV \\ &= \int_S (\mathbf{n} \cdot w)(\frac{v^2}{2} - (P+\varphi))ds \\ &= 0,\end{aligned} \tag{2.57}$$

over surfaces for which $\mathbf{n} \cdot w = 0$.
The equation (2.56) derived as a particular case in the four dimensional space using the language of differential forms is exactly the expression derived by Moffat to establish the invariance of helicity for ideal flows

in three dimensions. Thus from (2.54) and (2.56) we find that helicity is conserved and hence is an invariant. The equation (2.55) can be considered as the general expression for the rate of change of helicity for ideal flows in the four dimensional space time manifold. Thus (2.54) establishes the invariance of helicity in a more general four dimensional space time manifold irrespective of the fact that the three dimensional cycles C^3 over which the integrals are considered, lie on different or fixed co-ordinates of time.

The rate of change of helicity for Navier-Stokes' viscous flow in $\mathbb{R}^3$ as derived by Moffat is

$$\frac{d\mathbb{H}}{dt} = -2\nu \int_V w \cdot (\nabla \times w) dV. \tag{2.58}$$

From (2.52) the rate of change of helicity for viscous flow is,

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \int_{C^3} \mathcal{L}_\xi \eta \\ &= \int_{C^3} [2(\nu\nabla^2 v) \cdot w + \nabla \cdot (\frac{v^2}{2} - (P+\varphi))w \\ &- \nu(\nabla \cdot (\nabla \times w) \times v)]dx \wedge dy \wedge dz \\ &\quad [-2v((\nu\nabla^2 v) \cdot w) + (\partial_t(\frac{v^2}{2} - (P+\varphi))w \\ &- (\nabla(\frac{v^2}{2} - (P+\varphi))) \times (v \times w + \nu\nabla^2 v)) \\ &- \nu((\partial_t(\nabla \times w) \times v) + \nabla \times (\varepsilon(\nabla \times w) + (v \cdot \nabla \times w)v))]dt \wedge ds. \end{aligned} \tag{2.59}$$

As a particular case, if we consider cycles C^3 lying on fixed hyper surfaces of time, the above equations reduce to

$$\frac{d\mathcal{H}}{dt} = \int_{C^3} [(2\nu\nabla^2 v \cdot w + \nabla \cdot (\frac{v^2}{2} - (P + \varphi))w)$$
$$- \nu(\nabla \cdot (\nabla \times w) \times v)] dx \wedge dy \wedge dz$$
$$= -2\nu \int_{C^3} w.(\nabla \times w) dx \wedge dy \wedge dz, \tag{2.60}$$

since $\nabla^2 v = -(\nabla \times w)$ and using the equations (2.56) and (2.57). The above equation is exactly the expression derived by Moffat as in (2.58). Thus (2.59) can be considered as the general expression for the rate of change of helicity in the four dimensional space time manifold.

The condition (1.21) for the relative invariance of forms

$$\mathcal{L}_{\partial_t} \hat{h} + i_v \hat{d}\hat{h} = \hat{d}\hat{H},$$

in vector form gives the equation

$$\partial_t h + \nabla \cdot H = 0, \tag{2.61}$$

which is the helicity evolution equation for perfect fluids.

As mentioned above, helicity measures the linking and knotting of vortex lines composing a flow. Recent research reveals the predominant role of knots and links the in fields of plasma motion, biological and DNA structures, liquid crystals, optics and electromagnetics. Hence the question whether these conservation laws extend to real, dissipative flows becomes relevant. The conservation of helicity in the presence of viscosity is therefore important to understand the fundamental dynamics of real fluids. By considering the four dimensional Euclidean space time instead of the conventional three dimensional space, we are able to construct the two-form vorticity (2.20) and the three-form helicity (2.46) including the terms involving viscosity, in the space of solutions of Navier-Stokes' equations.

Recent research in the medical field shows that four dimensional flows are used in new imaging techniques. Vorticity, circulation and helicity are some of the haemodynamic quantities which are analyzed during medical diagnosis. The above results may find applications in medical and other relevant fields.

To every vorticity surface on which the boundary condition $w{\cdot}n = 0$ is satisfied, there corresponds a helicity invariant. If the vortex lines of flow lie on a family of (nested) surfaces, then there is a corresponding family of helicity invariants. If $w{\cdot}n \neq 0$ on the boundary of the domain, then the total helicity of the flow is not invariant. Helicity is then either created or destroyed in a boundary of the domain. Viscosity is responsible for the reconnection of vortex lines and hence for the evolution of helicity.

Helicity in three dimensions is an invariant of the flow. However, in four dimensional Minkowski space time helicity need not be an invariant. The non conservation of helicity in four dimensional spaces is attributed to the fact that 1-cycles do not link in four dimensional space. But it is observed that for ideal barotropic flows, vortex lines link in spatial or temporal cross sections of space time and hence linking number is conserved (Yoshida et.al, 2014).

The Hodge dual of η is a one-form.

$$*\eta = -hdt + -H \cdot dx.$$

The four-vector $(-h, -H)$ associated with the one-form is termed as topological torsion tensor (Kiehn, 2001). The first component h is the helicity density as defined by Moffatt and the latter component form a helicity or torsion current vector $-H$. If the four components of the torsion tensor vanish over a domain, then the flow is Frobenius integrable and can never be chaotic. The Frobenius integrability condition for the velocity one-form σ is that $\sigma \wedge d\sigma = 0$. Thus helicity is produced when the velocity field is not Frobenius integrable.

The integral of the torsion density

$$H_G = \frac{1}{16\pi^2} \int_{C^3} \eta,$$

is a kind of a Hopf invariant in algebraic topology. Since η is a relative integral invariant, the three dimensional cycle C^3 can be taken as a boundary of a four dimensional manifold ∂C^4. Also by Stokes' theorem $\int_{\partial C^4} \eta = \int_{C^4} d\eta$. Thus the integral $\oint_{\partial C^4} \eta$ vanishes if η is closed ($d\eta = 0$) or η is exact ($\eta = d\beta$) or if the boundary ∂C^4 vanishes (for example, if the boundary is of the form of a sphere or a torus)(Scofield and Huq, 2010).

In $\mathbb{R}^4$, vortex filaments link, because vorticities are two-forms and the corresponding two chains link in four dimension. Thus helicity measures the linking number of vortex filaments that are proper-time cross-sections of the vorticity two chains.

2.2.6 Integral Invariance Associated with the Four-Form of Parity

From the vorticity two-form ω given in equation (2.16), we define the four-form parity

$$\kappa = \omega \wedge \omega = 2(\partial_t v - \nabla f) \cdot w dt \wedge dx \wedge dy \wedge dz.$$

Thus for ideal Euler flows we have

$$\begin{aligned} \kappa &= 2((v \times w) - \nabla\phi) \cdot w dt \wedge dx \wedge dy \wedge dz \\ &= -2(\nabla\phi) \cdot w dt \wedge dx \wedge dy \wedge dz \end{aligned} \tag{2.62}$$

and for viscous flows,

$$\begin{aligned}\kappa &= 2((v \times w) - \nabla\phi + \nu\nabla^2 v) \cdot w dt \wedge dx \wedge dy \wedge dz \\ &= 2(-\nabla\phi + \nu\nabla^2 v) \cdot w dt \wedge dx \wedge dy \wedge dz. \end{aligned} \tag{2.63}$$

If we choose $f = -\varepsilon$, then the four-form κ vanishes for ideal flows. For viscous flows we have

$$\begin{aligned}\kappa &= 2\nu\nabla^2 v \cdot w dt \wedge dx \wedge dy \wedge dz \\ &= -2\nu(\nabla \times w) \cdot w dt \wedge dx \wedge dy \wedge dz. \end{aligned} \tag{2.64}$$

Clearly κ is exact. For,

$$\kappa = \omega \wedge \omega = (d\sigma) \wedge \omega = d(\sigma \wedge \omega) = d\eta.$$

Also we have, $d\eta = (\partial_t h + \nabla \cdot H) dt \wedge dx \wedge dy \wedge dz$.
Thus, $d\eta = \kappa \Rightarrow$

$$\partial_t h + \nabla \cdot H = -2(\nabla\phi) \cdot w \tag{2.65}$$

for Euler flows and

$$\partial_t h + \nabla \cdot H = 2(-\nabla\phi + \nu\nabla^2 v) \cdot w \tag{2.66}$$

for viscous flows.

If $f = -\varepsilon$, the above equations reduce to

$$\partial_t h + \nabla \cdot H = 0. \tag{2.67}$$

which is the helicity evolution equation for ideal flows as derived in equation (2.61) and for viscous flows we have

$$\begin{aligned}\partial_t h + \nabla \cdot H &= 2\nu(\nabla^2 v) \cdot w \\ &= -2\nu(\nabla \times w) \cdot w. \end{aligned} \tag{2.68}$$

The above equation (2.68) can be considered as the helicity evolution equation for viscous flows (Gumral, 2016). This also shows that helicity can be conserved when $(\nabla \times w) \cdot w$ vanishes even in the presence of viscosity.

Define $K = \int_{D^4} \kappa$. Then by Stokes' theorem we get,

$$\frac{dK}{dt} = \frac{d}{dt}\int_{D^4} \kappa = \int_{D^4} \mathcal{L}_\xi \kappa = \int_{D^4} \mathcal{L}_\xi (d\eta) = \int_{D^4} d(\mathcal{L}_\xi \eta) = \int_{\partial D^4} \mathcal{L}_\xi \eta.$$

This shows that K is an integral invariant over a four dimensional cycle for which $\partial D^4 = 0$ or if $\mathcal{L}_\xi \eta = 0$ over ∂D^4.

The condition for the absolute invariance of the four form is given by $\mathcal{L}_\xi \kappa = 0$. Thus for an inviscid flow, $\mathcal{L}_\xi \kappa = 0$. The condition for the absolute invariance of the four-form κ leads to the equation

$$\partial_t (w \cdot \nabla\phi) + \nabla \cdot (w \cdot \nabla\phi)v = 0.$$

For an incompressible ideal fluid, we obtain

$$\partial_t (w \cdot \nabla\phi) + v \cdot \nabla(w \cdot \nabla\phi) = 0.$$

i.e,

$$\frac{D}{Dt}(w \cdot \nabla\phi) = 0, \qquad (2.69)$$

where $\frac{D}{Dt}$ is the total derivative. From the relations (1.7) and (2.12), ϕ is a material conserved quantity and hence $w \cdot \nabla\phi$ is also a conserved quantity. Thus by using the absolute invariance condition of the four form κ we are able to establish the Ertel's theorem for the potential vorticity $w \cdot \nabla\phi$. Arnold and Khesin (1999) calls the term $w \cdot \nabla\phi$ as the vorticity function.

In particular, if we choose $f = S - \varepsilon$, where S is the specific entropy, then $\phi = S$ and $\kappa = -2(w \cdot \nabla S) dt \wedge dx \wedge dy \wedge dz$. Hence (2.69) represents the equation of Ertel's invariant.

For viscous flows we have,

$$\begin{aligned}\frac{dK}{dt} &= \int_{D^4} \mathcal{L}_\xi \kappa, \\ &= 2\int_{D^4} [\partial_t (w \cdot (-\nabla\phi + \nu\nabla^2 v)) \\ &+ \nabla \cdot ((w \cdot (-\nabla\phi + \nu\nabla^2 v))v)] dt \wedge dx \wedge dy \wedge dz. \end{aligned} \tag{2.70}$$

In particular, if $f = -\varepsilon$ then, the equation (2.70) reduces to,

$$\begin{aligned}\frac{dK}{dt} &= \int_{D^4} \mathcal{L}_\xi \kappa, \\ &= 2\int_{D^4} [\partial_t (w \cdot (\nu\nabla^2 v)) + \nabla \cdot ((w \cdot (\nu\nabla^2 v))v)] dt \wedge dx \wedge dy \wedge dz \\ &= -2\nu \int_{D^4} [\partial_t (w \cdot (\nabla \times w)) + \nabla \cdot ((w \cdot (\nabla \times w))v] dt \wedge dx \wedge dy \wedge dz. \\ &= \int_{\partial D^4} (2\nu\nabla^2 v \cdot w + \nabla \cdot (\frac{v^2}{2} - (P + \varphi))w) dx \wedge dy \wedge dz \\ &\quad [-((2v(\nu\nabla^2 v) \cdot w) + (\partial_t (\frac{v^2}{2} - (P + \varphi))w \\ &\qquad - \nabla \times (v \times w + \nu\nabla^2 v)))] dt \wedge ds. \end{aligned} \tag{2.71}$$

For incompressible fluids,

$$\frac{dK}{dt} = -2\nu \int_{D^4} \left(\frac{D(w \cdot (\nabla \times w))}{Dt} \right) dt \wedge dx \wedge dy \wedge dz. \tag{2.72}$$

The scalar function $w \cdot (\nabla \times w)$ is called the vortical helicity density. Thus equation (2.72) evaluates the rate of change of the integral of parity.

The non vanishing of the integral of the parity four-form plays an important role in the studies of turbulent flows. $\kappa \neq 0$ is a necessary condition for vector fields to describe turbulent evolution. For ideal fluids where f is chosen to be $-\varepsilon$, κ vanishes. If the velocity one-form is Frobenius integrable, i.e $\sigma \wedge d\sigma = 0$, then both η and κ vanish. If $\kappa = 0$, then the associated flow field is reversible and not turbulent. Most of the known closed solutions to dynamical systems have domains for which $\eta = 0$ and $\kappa = 0$ (Kiehn, 2001).

Chapter 3

Integral Invariants for Non-barotropic flows

In this chapter we analyze the properties of non-barotropic ideal and viscous flows using the algebra of differential forms in a four dimensional space time manifold. We have applied the concepts of absolute and relative integral invariance to non-barotropic flows.

3.1 Non-barotropic Flows in $\mathbb{R}^3$

Let $v = (v_1, v_2, v_3)$ be a velocity field in $\mathbb{R}^3$. The equations describing non-barotropic fluids are given by

1. Equation of continuity:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0. \tag{3.1}$$

2. Conservation of momentum:

$$\partial_t v + (v \cdot \nabla) v = -\nabla (I + \varphi) + T \nabla S, \tag{3.2}$$

for perfect fluids and

$$\partial_t v + (v \cdot \nabla) v = -\nabla(I + \varphi) + T\nabla S + \nu \nabla^2 v + \frac{\nu}{3}\nabla(\nabla \cdot v), \quad (3.3)$$

for viscous fluids (Vazsonyi, 1945).

3. Conservation of entropy:

$$\partial_t S + (v \cdot \nabla) S = 0. \quad (3.4)$$

Here we have used the relation

$$-\nabla I + T\nabla S = -\nabla p/\rho, \quad (3.5)$$

where p is the pressure, I is the specific enthalpy, ρ is the mass density, φ is the potential for the volume force field and ν is the kinematic viscosity.

The identity $(v \cdot \nabla)v = \nabla \frac{v^2}{2} - v \times (\nabla \times v)$ is used to convert the equations (3.2) and (3.3) into the form

$$\partial_t v - v \times (\nabla \times v) = -\nabla \frac{v^2}{2} - \nabla(I + \varphi) + T\nabla S \quad (3.6)$$

and

$$\partial_t v - v \times (\nabla \times v) = -\nabla \frac{v^2}{2} - \nabla(I + \varphi) + T\nabla S + \nu \nabla^2 v + \frac{\nu}{3}\nabla(\nabla \cdot v) \quad (3.7)$$

for ideal and viscous flows respectively.

Defining $\tilde{v} = v - \tau \nabla S$, where τ is the thermasy and using the relation

$$\frac{D(\tau \nabla S)}{Dt} = T\nabla S - \tau[(\nabla S \cdot \nabla)v + (\nabla S \times w)],$$

as given in (Mobbs, 1981), equation (3.6) takes the form,

$$\begin{aligned} \partial_t \tilde{v} - v \times \tilde{w} &= -\nabla(\tfrac{v^2}{2} + I + \varphi - v \cdot \tau \nabla S) \\ &= \nabla \tilde{f}, \end{aligned} \tag{3.8}$$

for non-barotropic perfect flows and equation (3.7) takes the form

$$\begin{aligned} \partial_t \tilde{v} - v \times \tilde{w} &= -\nabla(\tfrac{v^2}{2} + I + \varphi - v \cdot \tau \nabla S) + \nu \nabla^2 v + \tfrac{\nu}{3} \nabla(\nabla \cdot v) \\ &= \nabla \tilde{f} + \nu \nabla^2 v + \tfrac{\nu}{3} \nabla(\nabla \cdot v) \end{aligned} \tag{3.9}$$

for viscous flows, where $\tilde{f} = -(\frac{v^2}{2} + I + \varphi - v \cdot \tau \nabla S)$.
Following Mobbs, we define the generalized vorticity as $\tilde{w} = (w - \nabla \tau \times \nabla S) = \nabla \times \tilde{v}$. The generalized vorticity equation for the non-barotropic inviscid flow is given by

$$\partial_t \tilde{w} - \nabla \times (v \times \tilde{w}) = 0, \tag{3.10}$$

$$\partial_t (\frac{\tilde{w}}{\rho}) + (v \cdot \nabla)(\frac{\tilde{w}}{\rho}) = (\frac{\tilde{w}}{\rho} \cdot \nabla) v, \tag{3.11}$$

and for non-barotropic viscous flow,

$$\partial_t \tilde{w} - \nabla \times (v \times \tilde{w}) = \nu \nabla^2 w \tag{3.12}$$

$$\partial_t (\frac{\tilde{w}}{\rho}) + (v \cdot \nabla)(\frac{\tilde{w}}{\rho}) = (\frac{\tilde{w}}{\rho} \cdot \nabla) v + \nu \nabla^2 w, \tag{3.13}$$

which can be derived from the Vazsonyi's vorticity equation for non-barotropic flows.

In the next sections we apply the concepts of absolute and relative integral invariance of forms to non-barotropic flows in the space time manifold and obtain some results pertaining to it. In classical mechanics, time is independent of the frame of reference and is uniform throughout space. Fluid dynamics is a non-relativistic field theory, so we consider the Euclidean four dimensional space time manifold.

3.1.1 One-Form of Generalized Velocity

The velocity field of a fluid in $\mathbb{R}^4$ can be represented as a four-vector $\mathcal{V} = (f, v)$. The one-form or the covector field associated with the velocity $\mathcal{V}$ can be defined as

$$\sigma = f dt + v_1 dx + v_2 dy + v_3 dz. \tag{3.14}$$

Let the flow field be $\xi = \partial_t + v_i \partial_{x_i}$.

For a non-barotropic fluid, we let $f = \tilde{f} = -(\frac{v^2}{2} + I + \varphi - v \cdot \tau \nabla S)$. Then the generalized velocity one-form can be expressed as

$$\tilde{\sigma} = \tilde{f} dt + \tilde{v}_1 dx + \tilde{v}_2 dy + \tilde{v}_3 dz. \tag{3.15}$$

Let $\tilde{\Gamma} = \int_{D^1} \tilde{\sigma}$, where D^1 is an arbitrary 1 - chain. Then,

$$\frac{d\tilde{\Gamma}}{dt} = \frac{d}{dt} \int_{D^1} \tilde{\sigma} = \int_{D^1} \mathcal{L}_\xi \tilde{\sigma}. \tag{3.16}$$

For non-barotropic perfect flows,

$$\mathcal{L}_\xi \tilde{\sigma} = d(\frac{v^2}{2} - (I + \varphi)) = d\tilde{\beta}_1. \tag{3.17}$$

Hence we obtain that $\mathcal{L}_\xi \tilde{\sigma}$ is exact for perfect fluids. By applying Stokes' theorem we have,

$$\frac{d\tilde{\Gamma}}{dt} = \int_{D^1} d\tilde{\beta}_1 = \int_{\partial D^1} (\frac{v^2}{2} - (I + \varphi)). \tag{3.18}$$

Thus, $\frac{d\tilde{\Gamma}}{dt}$ vanishes when $\partial D^1 = 0$. *i.e*, when D^1 is a 1 - cycle C^1 and $\tilde{\Gamma}$ is invariant over arbitrary 1 - cycles C^1 in $\mathbb{R}^4$. Thus $\tilde{\sigma}$ represents a relative integral invariant and

$$\frac{D}{Dt}\oint_{C^1}\tilde{\sigma}=0.$$

If the domain of integration lies on the hyper surface of constant time, then $\oint_{C^1}\tilde{\sigma}$ evaluates the generalized circulation and its invariance implies the generalized Kelvin's circulation theorem in $\mathbb{R}^3$.

By taking into consideration the four dimensional space time Euclidean manifold we are able to consider chains and cycles lying on different coordinates of time (not necessarily lying on hypersurfaces of constant time) and the invariance of $\oint_{C^1}\tilde{\sigma}$ and thereby the generalized circulation theorem remains valid even in such cases also (Fecko, 2013).

From the condition $\mathcal{L}_{\partial_t}\hat{\alpha}+i_v\hat{d}\hat{\alpha}=\hat{d}\hat{\beta}$ of relative invariance of the velocity one-form $\tilde{\sigma}$, we get $\mathcal{L}_{\partial_t}\tilde{v}+i_v\hat{d}\tilde{v}=-\hat{d}\tilde{f}$, which in vector form yields the equation (3.6).
For viscous flows, using (3.7), we get

$$\begin{aligned}\mathcal{L}_\xi\tilde{\sigma}=&(v\cdot(\nu(\nabla\times w)-\frac{\nu}{3}\nabla(\nabla\cdot v)))dt\\&+(-\nu(\nabla\times w)+\frac{\nu}{3}\nabla(\nabla\cdot v))\cdot d\mathbf{x}+d(\tilde{\beta}_1).\end{aligned}$$

Then,

$$\begin{aligned}\frac{d\tilde{\Gamma}}{dt}&=\int_{D^1}\mathcal{L}_\xi\tilde{\sigma},\\&=\int_{D^1}(v\cdot(\nu(\nabla\times w)-\frac{\nu}{3}\nabla(\nabla\cdot v)))dt\\&+(-\nu(\nabla\times w)+\frac{\nu}{3}\nabla(\nabla\cdot v))\cdot d\mathbf{x}+d(\tilde{\beta}_1).\end{aligned}\tag{3.19}$$

If D^1 lies on fixed coordinates of time the above equation reduces to

$$\frac{d\tilde{\Gamma}}{dt}=\int_{D^1}(-\nu(\nabla\times w)+\frac{\nu}{3}\nabla(\nabla\cdot v)+\nabla(\tilde{\beta}_1))\cdot d\mathbf{x}.$$

Also,

$$\frac{d\tilde{\Gamma}}{dt} = \oint_{C^1} -\nu(\nabla \times w) \cdot d\mathbf{x},$$

over 1 - cycles. The above expression evaluates the rate of change of generalized circulation for viscous non-barotropic fluids for three dimensional flows and equation (3.19) evaluates the rate of change of circulation for viscous non-barotropic flows in the four dimensional manifold.

3.1.2 Two-Form of Generalized Vorticity

From the velocity one-form, $\tilde{\sigma} = \tilde{f}dt + \tilde{v}_1 dx + \tilde{v}_2 dy + \tilde{v}_3 dz$, the vorticity two-form can be defined as

$$\begin{aligned} \tilde{\omega} &= d\tilde{\sigma}, \\ &= \tilde{\Omega} + dt \wedge \tilde{\mathcal{E}}, \end{aligned} \tag{3.20}$$

where $\tilde{\Omega} = \tilde{w}_1 dy \wedge dz + \tilde{w}_2 dz \wedge dx + \tilde{w}_3 dx \wedge dy = \tilde{w} \cdot ds$,
$\tilde{\mathcal{E}} = (\partial_t \tilde{v}_1 - \partial_x \tilde{f})dx + (\partial_t \tilde{v}_2 - \partial_y \tilde{f})dy + (\partial_t \tilde{v}_3 - \partial_z \tilde{f})dz = (\partial_t \tilde{v} - \nabla \tilde{f}) \cdot d\mathbf{x}$
and $\tilde{w} = \nabla \times \tilde{v} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ is the generalized vorticity vector in $\mathbb{R}^3$.

From equations (3.6) and (3.7), for perfect fluids, the generalized vorticity two-form can be expressed in the form

$$\tilde{\omega} = \tilde{w} \cdot ds + (v \times \tilde{w})dt \wedge d\mathbf{x}, \tag{3.21}$$

and for viscous non-barotropic fluids as,

$$\tilde{\omega} = \tilde{w} \cdot ds + ((v \times \tilde{w}) + \nu \nabla^2 v + \frac{\nu}{3} \nabla(\nabla \cdot v))dt \wedge d\mathbf{x} \tag{3.22}$$

In $\mathbb{R}^4$, the generalized vorticity vector can be represented in the form $\tilde{\mathcal{W}} = (\tilde{E}, \tilde{w})$, where $\tilde{E} = \partial_t \tilde{v} - \nabla \tilde{f}$.
Let $\tilde{W} = \int_{D^2} \tilde{\omega}$, then $\tilde{W} = \int_{D^2} d\tilde{\sigma} = \int_{\partial D^2} \tilde{\sigma}$.

Thus $\tilde{W} = \tilde{\Gamma}$ over boundaries of 2 - chains (ie, when $D^1 = \partial D^2$).

Then,

$$\frac{d\tilde{W}}{dt} = \int_{D^2} \mathcal{L}_\xi \tilde{\omega} = \int_{D^2} \mathcal{L}_\xi d\tilde{\sigma} = \int_{D^2} d\mathcal{L}_\xi \tilde{\sigma} = \int_{\partial D^2} \mathcal{L}_\xi \tilde{\sigma}. \qquad (3.23)$$

Thus in general $\tilde{W}$ is an invariant over 2 - cycles for which $\partial D^2 = 0$ or when $\mathcal{L}_\xi \tilde{\sigma} = 0$ over ∂D^2.

From equation (3.17), $\mathcal{L}_\xi \tilde{\sigma}$ is exact for perfect flows, which implies

$$d\tilde{W}/dt = 0.$$

Hence $\tilde{W}$ represents an integral invariant. Since $\tilde{\sigma}$ is a relative integral invariant for ideal flows over C^1, it follows from equation (1.19) that $\tilde{\omega}$ represents an absolute integral invariant over two-chains D^2, where $C^1 = \partial D^2$.

For ideal flows we have, $\tilde{\mathcal{E}} = -i_v \tilde{\Omega}$. Thus $\tilde{\omega}$ can be represented in the form $\tilde{\Omega} - dt \wedge i_v \tilde{\Omega}$. The condition (1.22)

$$\mathcal{L}_{\partial_t} \tilde{\Omega} + \hat{\mathcal{L}}_v \tilde{\Omega} = 0$$

for absolute invariance of forms in vector form results in the vorticity equation (3.10). Thus the absolute integral invariance of the generalized vorticity two-form implies the generalizations of the Helmholtz vorticity theorems to non-barotropic flows in a four dimensional manifold.

If D^2 is a two-chain lying on the hypersurface of constant time, $\int_{D^2} \tilde{\omega}$ reduces to $\int_{D^2} \tilde{w} \cdot ds$. Thus the invariance of $\int_{D^2} \tilde{w} \cdot ds$ implies the invariance of net flux of the generalized vorticity in R^3 in an ideal non-barotropic flow.

For viscous fluids, using equation (3.23)

$$
\begin{aligned}
\frac{d\tilde{W}}{dt} &= -\nu \int_{D^2} [\nabla \times ((\nabla \times w) - \frac{1}{3}\nabla(\nabla \cdot v))] \cdot ds \\
&\quad + [\partial_t((\nabla \times w) - \frac{1}{3}\nabla(\nabla \cdot v)) + \nabla(((\nabla \times w) - \frac{1}{3}\nabla(\nabla \cdot v)) \cdot v)]dt \wedge d\mathbf{x}, \\
&= \int_{\partial D^2} (v \cdot (\nu(\nabla \times w) - \frac{\nu}{3}\nabla(\nabla \cdot v)))dt + (-\nu(\nabla \times w) \\
&\qquad + \frac{\nu}{3}\nabla(\nabla \cdot v)) \cdot d\mathbf{x} + d(\tilde{\beta}_1). \qquad (3.24)
\end{aligned}
$$

If D^2 lies on fixed coordinates of time then,

$$
\begin{aligned}
\frac{d\tilde{W}}{dt} &= \int_{D^2} -\nu[\nabla \times ((\nabla \times w) - \frac{1}{3}\nabla(\nabla \cdot v))] \cdot ds, \\
&= \int_{\partial D^2} [-\nu(\nabla \times w) + \frac{\nu}{3}\nabla(\nabla \cdot v) + \nabla(\tilde{\beta}_1)] \cdot d\mathbf{x}. \qquad (3.25)
\end{aligned}
$$

Equation (3.25) can be considered as the rate of change of flux of the generalized vorticity for non-barotropic viscous fluids in three dimensions and equation (3.24) the corresponding expression in a four dimensional manifold.

3.1.3 Three-Form of Generalized Helicity

In classical theory helicity is defined as a measure of linkage, knottedness or intertwining of vortex lines or tubes in the flow. Moreau proved the conservation of helicity for ideal barotropic fluids while Woltjer proved the conservation of magnetic helicity for perfect magnetohydrodynamics and Moffat gave a topological interpretation to the two analogous results (Moffat, 2018). Here we show the invariance of the generalized helicity for non-barotropic flows in a four dimensional space time manifold and also derive an expression to evaluate the dissipation rate of helicity for viscous non-barotropic flows.

From the velocity one-form $\tilde{\sigma}$ given in equation (3.15) and the vorticity two-form $\tilde{\omega}$ given in (3.20), the generalized helicity three-form can be defined as

$$\begin{aligned}\tilde{\eta} &= \tilde{\sigma} \wedge \tilde{\omega}, \\ &= \tilde{v} \cdot \tilde{w} dx \wedge dy \wedge dz + \tilde{h}_1 dt \wedge dy \wedge dz + \tilde{h}_2 dt \wedge dz \wedge dx + \tilde{h}_3 dt \wedge dx \wedge dy, \\ &= \tilde{v} \cdot \tilde{w} dx \wedge dy \wedge dz + dt \wedge ((\partial_t \tilde{v} - \nabla \tilde{f}) \times \tilde{v} + \tilde{f}\tilde{w}) \cdot dS, \qquad (3.26) \\ &= \tilde{h} - dt \wedge \tilde{H}\end{aligned}$$

where $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) = -\vec{H} = (\partial_t \tilde{v} - \nabla \tilde{f}) \times \tilde{v} + \tilde{f}\tilde{w}$, and $\tilde{H} = \vec{H} \cdot ds$.

The Hodge dual $*\tilde{\eta}$ is a one-form and the helicity four-vector associated to the one-form can be expressed as $(-h, -\vec{H})$.

From equation (3.6) we get,

$$\tilde{\eta} = \tilde{v} \cdot \tilde{w} dx \wedge dy \wedge dz + dt \wedge ((v \times \tilde{w}) \times \tilde{v} + \tilde{f}\tilde{w}) \cdot ds, \qquad (3.27)$$

for ideal non-barotropic flows and from equation (3.7),

$$\tilde{\eta} = \tilde{v} \cdot \tilde{w} dx \wedge dy \wedge dz + dt \wedge (((v \times \tilde{w}) + \nu \nabla^2 v + \frac{\nu}{3} \nabla(\nabla \cdot v)) \times \tilde{v} + \tilde{f}\tilde{w}) \cdot ds, \qquad (3.28)$$

for viscous fluids.

In the Euclidean space time $\mathbb{R}^4$, we define the generalized helicity $\tilde{\mathcal{H}}$ as

$$\tilde{\mathcal{H}} = \int_{D^3} \tilde{\eta} = \int_{D^3} \tilde{\sigma} \wedge \tilde{\omega},$$

where D^3 is a three dimensional chain co-moving with the fluid. The rate of change of the generalized helicity is given by

$$\frac{d\tilde{\mathcal{H}}}{dt} = \frac{d}{dt} \int_{D^3} \tilde{\eta} = \int_{D^3} \mathcal{L}_\xi \tilde{\eta}.$$

$$\mathcal{L}_\xi \tilde{\eta} = \mathcal{L}_\xi(\tilde{\sigma} \wedge \tilde{\omega}) = (\mathcal{L}_\xi \tilde{\sigma}) \wedge \tilde{\omega} = (d\tilde{\beta}_1) \wedge \tilde{\omega},$$
$$= (\nabla \cdot (\tilde{\beta}_1 \tilde{w}))dx \wedge dy \wedge dz + ((\partial_t(\tilde{\beta}_1))\tilde{w} - (\nabla(\tilde{\beta}_1)) \times (v \times \tilde{w}))dt \wedge ds,$$
$$= d((\tilde{\beta}_1)\tilde{\omega}) = d(\tilde{\beta}_2) \tag{3.29}$$

and

$$\begin{aligned}\mathcal{L}_\xi \tilde{\eta} = {} & (2(\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla \cdot v)) \cdot \tilde{w} + \nabla \cdot (\tilde{\beta}_1)\tilde{w} \\ & + \nu(\nabla \cdot (\nabla^2 v + \frac{1}{3}\nabla(\nabla \cdot v)) \times v))dx \wedge dy \wedge dz \\ & [-2v((\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla \cdot v)) \cdot \tilde{w}) + (\partial_t(\tilde{\beta}_1)\tilde{w} \\ & - \nabla(\tilde{\beta}_1) \times (v \times \tilde{w} + \nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla \cdot v))) \\ & + \nu((\partial_t(\nabla^2 v + \frac{1}{3}\nabla(\nabla \cdot v)) \times v) \\ & + \nabla \times (\tilde{f}(\nabla^2 v + \frac{1}{3}\nabla(\nabla \cdot v)) + (v \cdot (\nabla^2 v + \frac{1}{3}\nabla(\nabla \cdot v)))v))]dt \wedge ds\end{aligned} \tag{3.30}$$

for ideal and viscous non-barotropic flows respectively.

For an ideal flow, we have seen that $\tilde{\sigma}$ represents a relative integral invariant and hence from equation (1.18), $\tilde{\sigma} \wedge d\tilde{\sigma} = \tilde{\eta}$ will also represent a relative integral invariant in $\mathbb{R}^4$ and also from equation (3.29) we see that for ideal non-barotropic flow $\mathcal{L}_\xi \tilde{\eta}$ is exact. Thus,

$$\frac{d\tilde{\mathcal{H}}}{dt} = \int_{D^3} d((\tilde{\beta}_1)\tilde{\omega}) = \int_{\partial D^3} (\tilde{\beta}_1)\tilde{\omega}. \tag{3.31}$$

If the three dimensional chain D^3 is a cycle such that $\partial D^3 = 0$, then $\frac{d\tilde{\mathcal{H}}}{dt} = 0$. Thus for a perfect non-barotropic flow, the generalized helicity $\tilde{\mathcal{H}}$ is conserved over a three dimensional cycle C^3 whose boundary vanishes.

Thus we have,

$$\frac{d\tilde{\mathcal{H}}}{dt} = \oint_{\partial C^3} (\frac{v^2}{2} - (I + \varphi))\tilde{\omega} = 0, \tag{3.32}$$

where C^3 is three-cycle such that $\partial C^3 = 0$. Also note that, from equation (3.29)

$$\begin{aligned} \frac{d\tilde{\mathcal{H}}}{dt} &= \int_{D^3} d((\tilde{\beta}_1)\tilde{\omega}), \\ &= \int_{D^3} (\nabla \cdot (\tilde{\beta}_1 \tilde{w}))dx \wedge dy \wedge dz \\ &+ [(\partial_t(\tilde{\beta}_1))\tilde{w} - (\nabla(\frac{v^2}{2} - (I + \varphi))) \times (v \times \tilde{w})]dt \wedge ds. \end{aligned} \tag{3.33}$$

For three-cycles lying on hypersurfaces of fixed time, the rate of change of helicity reduces to

$$\begin{aligned} \frac{d\tilde{\mathcal{H}}}{dt} = 0 &= \int_{C^3} (\nabla \cdot (\frac{v^2}{2} - (I + \varphi))\tilde{w})dx \wedge dy \wedge dz, \\ &= \int_{C^3} (\tilde{w} \cdot \nabla)(\frac{v^2}{2} - (I + \varphi))dx \wedge dy, \wedge dz, \end{aligned} \tag{3.34}$$

since $\nabla \cdot \tilde{w} = 0$.

The equation (3.33) can be considered to be the general expression to evaluate the rate of change of generalized helicity for ideal non-barotropic inviscid flows in the four dimensional space-time manifold. Equation (3.34) derived as a particular case of (3.33) is analogous to the expression derived by Moffat to establish the invariance of helicity for ideal incompressible flows in three dimensions.

The rate of change of helicity for classical viscous flows derived by Moffat for three dimensional flows is

$$\frac{d\mathcal{H}}{dt} = -2\nu \int_V w \cdot (\nabla \times w)dV, \tag{3.35}$$

where $\mathcal{H}$ is the fluid helicity defined for classical barotropic flows. From equation (3.30), the rate of change of the generalized helicity for viscous flow is,

$$\begin{aligned}
\frac{d\tilde{\mathcal{H}}}{dt} &= \int_{D^3} \mathcal{L}_\xi \tilde{\eta}, \\
&= \int_{D^3} (2(\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))\cdot\tilde{w} + \nabla\cdot(\tilde{\beta}_1\tilde{w}) \\
&+ \nu(\nabla\cdot(\nabla^2 v + \frac{1}{3}\nabla(\nabla\cdot v))\times v))dx\wedge dy\wedge dz \\
&[-2v((\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))\cdot\tilde{w}) + (\partial_t(\tilde{\beta}_1)\tilde{w} \\
&- \nabla(\tilde{\beta}_1)\times(v\times\tilde{w} + \nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))) \\
&+ \nu((\partial_t(\nabla^2 v + \frac{1}{3}\nabla(\nabla\cdot v))\times v) \\
&+ \nabla\times(\tilde{f}(\nabla^2 v + \frac{1}{3}\nabla(\nabla\cdot v)) + (v\cdot(\nabla^2 v + \frac{1}{3}\nabla(\nabla\cdot v)))v))]dt\wedge ds.
\end{aligned} \tag{3.36}$$

As a particular case if we consider chains D^3 lying on fixed hyper surfaces of time, the above equations reduce to

$$\begin{aligned}
\frac{d\tilde{\mathcal{H}}}{dt} &= \int_{D^3} (2(\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))\cdot\tilde{w} + \nabla\cdot(\tilde{\beta}_1\tilde{w}) \\
&+ \nu(\nabla\cdot(\nabla^2 v + \frac{1}{3}\nabla(\nabla\cdot v))\times v))dx\wedge dy\wedge dz, \\
&= \int_{D^3} (-2\nu(\nabla\times w) + \frac{2\nu}{3}\nabla(\nabla\cdot v))\cdot\tilde{w}dx\wedge dy\wedge dz \\
&+ \int_{D^3} \nabla\cdot(\tilde{\beta}_1\tilde{w} + (\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))\times v)dx\wedge dy\wedge dz.
\end{aligned}$$

If we consider cycles C^3, the second term on the right hand side above vanishes so that

$$\frac{d\tilde{\mathcal{H}}}{dt} = \int_{C^3} (-2\nu(\nabla \times w) + \frac{2\nu}{3}\nabla(\nabla \cdot v)) \cdot \tilde{w} dx \wedge dy \wedge dz. \qquad (3.37)$$

For the incompressible case we have

$$\frac{d\tilde{\mathcal{H}}}{dt} = -2\nu \int_{C^3} (\nabla \times w) \cdot \tilde{w} dx \wedge dy \wedge dz. \qquad (3.38)$$

The above equation is analogous to the expression derived by Moffat for barotropic flows as in equation (3.35). Thus equation (3.36) can be considered as the general expression to evaluate the rate of change of generalized helicity for non-barotropic viscous flows in the four dimensional space time manifold.

The condition (1.21) for the relative invariance of forms

$$\mathcal{L}_{\partial_t} \tilde{h} + i_v \hat{d}\tilde{h} = \hat{d}\vec{H}$$

in vector form gives the equation

$$\partial_t \tilde{h} + \nabla \cdot \vec{H} = 0, \qquad (3.39)$$

which is the evolution equation of the generalized helicity or the conservation law of the helicity four-vector $(-h, -\vec{H})$ for non-barotropic ideal flows.

3.1.4 Four-form of Generalized Parity

From the generalized vorticity two-form $\tilde{\omega}$ given in (3.20) a four-form which can be termed 'the generalized parity' can be defined as

$$\tilde{\kappa} = \tilde{\omega} \wedge \tilde{\omega} = (2(\partial_t \tilde{v} - \nabla \tilde{f}) \cdot \tilde{w}) dt \wedge dx \wedge dy \wedge dz.$$

For ideal non-barotropic flows we have

$$\tilde{\kappa} \quad = (2(v \times \tilde{w}) \cdot \tilde{w}) dt \wedge dx \wedge dy \wedge dz = 0. \tag{3.40}$$

For viscous flows,

$$\begin{aligned} \tilde{\kappa} \quad &= (2((v \times \tilde{w}) + \nu \nabla^2 v + \tfrac{\nu}{3}(\nabla(\nabla \cdot v)) \cdot \tilde{w}) dt \wedge dx \wedge dy \wedge dz \\ &= 2(\nu \nabla^2 v + \tfrac{\nu}{3}(\nabla(\nabla \cdot v)) \cdot \tilde{w} dt \wedge dx \wedge dy \wedge dz. \end{aligned} \tag{3.41}$$

Clearly $\tilde{\kappa}$ is exact, for

$$\tilde{\kappa} = \tilde{\omega} \wedge \tilde{\omega} = (d\tilde{\sigma}) \wedge \tilde{\omega} = d(\tilde{\sigma} \wedge \tilde{\omega}) = d\tilde{\eta}.$$

Also we have, $d\tilde{\eta} = (\partial_t \tilde{h} + \nabla \cdot \vec{H}) dt \wedge dx \wedge dy \wedge dz$.
Thus, $d\tilde{\eta} = \tilde{\kappa} \Rightarrow$

$$\partial_t \tilde{h} + \nabla \cdot \vec{H} = 0, \tag{3.42}$$

which is the helicity evolution equation for ideal non-barotropic flows as derived in equation (3.39).

For viscous non-barotropic flows we have

$$\partial_t \tilde{h} + \nabla \cdot \vec{H} = 2(\nu(\nabla^2 v) + \frac{\nu}{3} \nabla(\nabla \cdot v)) \cdot \tilde{w}. \tag{3.43}$$

In the case of incompressible flows, we have

$$\partial_t \tilde{h} + \nabla \cdot \vec{H} = -2\nu(\nabla \times w) \cdot \tilde{w}. \tag{3.44}$$

The above equation (3.44) can be termed as the evolution equation of the generalized helicity for viscous non-barotropic flows. For flows for which the condition $(\nabla \times w) \cdot \tilde{w} = 0$ is satisfied, the generalized helicity is conserved, even for viscous non-barotropic case.

Also, we have

$$\mathcal{L}_\xi \tilde{\kappa} = \mathcal{L}_\xi (d\tilde{\eta}) = d(\mathcal{L}_\xi \tilde{\eta}).$$

As described in section (1.1.7) the Lie derivative of four-form of the generalized parity being exact, $\oint_{C^4}\tilde{\kappa}$ shows the property of being a relative integral invariant for viscous flows too.

Define $\tilde{K} = \int_{D^4}\tilde{\kappa}$. Then by Stokes' theorem we get,

$$\frac{d\tilde{K}}{dt} = \frac{d}{dt}\int_{D^4}\tilde{\kappa} = \int_{D^4}\mathcal{L}_\xi\tilde{\kappa} = \int_{\partial D^4}\mathcal{L}_\xi\tilde{\eta}. \tag{3.45}$$

This shows that $\tilde{K}$ is an invariant over a four dimensional cycles for which $\partial D^4 = 0$ or if $\mathcal{L}_\xi\tilde{\eta} = 0$ over ∂D^4 . Thus $\tilde{K} = \oint_{C^4}\tilde{\kappa}$ is conserved for viscous non-barotropic flows.

For viscous non-barotropic flows, we have

$$\begin{aligned}
\frac{d\tilde{K}}{dt} &= \int_{D^4}\mathcal{L}_\xi\tilde{\kappa},\\
&= 2\int_{D^4}[\partial_t(\tilde{w}\cdot(\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v)))\\
&+ \nabla\cdot(\tilde{w}\cdot(\nu\nabla^2 v + \frac{\nu}{3}(\nabla(\nabla\cdot v)))v)]dt\wedge dx\wedge dy\wedge dz,\\
&= -2\nu\int_{D^4}[\partial_t(\tilde{w}\cdot((\nabla\times w) - \frac{1}{3}\nabla(\nabla\cdot v)))\\
&+ \nabla\cdot(\tilde{w}\cdot((\nabla\times w) - \frac{1}{3}\nabla(\nabla\cdot v)))v]dt\wedge dx\wedge dy\wedge dz,\\
&= \int_{\partial D^4}(2(\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))\cdot\tilde{w} + \nabla\cdot\tilde{\beta}_1\tilde{w})dx\wedge dy\wedge dz\\
&- ((2v(\nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))\cdot\tilde{w})\\
&+ (\partial_t\tilde{\beta}_1\tilde{w} - \nabla\tilde{\beta}_1\times(v\times\tilde{w} + \nu\nabla^2 v + \frac{\nu}{3}\nabla(\nabla\cdot v))))dt\wedge ds.
\end{aligned} \tag{3.46}$$

For incompressible flows the above equation becomes,

$$\frac{d\tilde{K}}{dt} = -2\nu\int_{D^4}\left(\frac{D(\tilde{w}\cdot(\nabla\times w))}{Dt}\right)dt\wedge dx\wedge dy\wedge dz. \tag{3.47}$$

For cycles C^4 such that $\partial C^4 = 0$, by equation (3.45),

$$\frac{d\tilde{K}}{dt} = \oint_{C^4} \mathcal{L}_\xi \tilde{\kappa} = 0. \tag{3.48}$$

Thus we have

$$\oint_{C^4} \left(\frac{D(\tilde{w} \cdot (\nabla \times w))}{Dt} \right) dt \wedge dx \wedge dy \wedge dz = 0 \tag{3.49}$$

in this case. The scalar function $\tilde{w} \cdot (\nabla \times w)$ can be termed as the generalized vortical helicity density. Thus equation (3.47) evaluates the rate of change of the integral of the generalized parity. The above equation (3.49) implies the invariance of the generalized vortical helicity for viscous non-barotropic flows.

Chapter 4

A Geometric Algebraic Approach to Fluid Dynamics

A methodical approach to fluid dynamical theories using the calculus of geometric algebra is still an open task. In this chapter we have used the language of geometric algebra to analyze the properties of fluid flows.

4.1 Geometric Algebra in Fluid Dynamics

The four-velocity field of a four dimensional fluid flow $\mathcal{V} = (f, v)$, can be expressed as a vector in $\mathcal{G}_4$ as

$$\boldsymbol{V} = fe_0 + v_1e_1 + v_2e_2 + v_3e_3,$$

along with the incompressibility condition or the divergence free condition

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \partial_t f + \partial_x v_1 + \partial_y v_2 + \partial_z v_3 = 0,$$

where $\boldsymbol{\nabla}$ denotes the vector derivative,

$$\boldsymbol{\nabla} = e_i\partial_i = e_0\frac{\partial}{\partial t} + e_i\frac{\partial}{\partial x_i} = e_0\frac{\partial}{\partial t} + \nabla.$$

4.1.1 Bivector Associated with Vorticity

For the given velocity $\boldsymbol{V}$, the derivative of the velocity can be expressed as a bivector

$$\begin{aligned}\boldsymbol{W} &= \boldsymbol{\nabla}\boldsymbol{V} = \boldsymbol{\nabla}\cdot\boldsymbol{V} + \boldsymbol{\nabla}\wedge\boldsymbol{V} = \boldsymbol{\nabla}\wedge\boldsymbol{V},\\ &= (\partial_t v_1 - \partial_x f)e_0\wedge e_1 + (\partial_t v_2 - \partial_y f)e_0\wedge e_2 + (\partial_t v_3 - \partial_z f)e_0\wedge e_3\\ &\quad + w_1 e_2\wedge e_3 + w_2 e_3\wedge e_1 + w_3 e_1\wedge e_2,\end{aligned}$$

where $w = (w_1, w_2, w_3)$ is the three dimensional vorticity vector.

Here $\boldsymbol{W}$ is a grade-2 blade associated with the vorticity. Hence vorticity field takes the form of oriented surfaces. As in the previous chapters, if we take $f = -\varepsilon$, where ε is the Bernoulli energy function then,

$$\begin{aligned}\boldsymbol{W} = &(\partial_t v_1 + \partial_x\varepsilon)e_0\wedge e_1 + (\partial_t v_2 + \partial_y\varepsilon)e_0\wedge e_2 + (\partial_t v_3 + \partial_z\varepsilon)e_0\wedge e_3\\ &+ w_1 e_2\wedge e_3 + w_2 e_3\wedge e_1 + w_3 e_1\wedge e_2.\end{aligned} \tag{4.1}$$

Let $L = \partial_t v + \nabla\varepsilon$. Then from Euler's equation we have, $L = v\times w$ for ideal fluids and from Navier-Stokes' equation we have $L = v\times w + \nu\nabla^2 v$, for viscous flows. Thus for ideal and viscous flows, we express $\boldsymbol{W}$ as,

$$\begin{aligned}\boldsymbol{W} = &L_1 e_0\wedge e_1 + L_2 e_0\wedge e_2 + L_3 e_0\wedge e_3\\ &+ w_1 e_2\wedge e_3 + w_2 e_3\wedge e_1 + w_3 e_1\wedge e_2.\end{aligned}$$

Let

$$\boldsymbol{F} = \boldsymbol{\nabla}\boldsymbol{W} = \boldsymbol{\nabla}\cdot\boldsymbol{W} + \boldsymbol{\nabla}\wedge\boldsymbol{W}$$

Now,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{W} &= -(\nabla\cdot L)e_0 + (\partial_t L_1 - (\nabla\times w)_1)e_1 + (\partial_t L_2 - (\nabla\times w)_2)e_2 \\ &\qquad + (\partial_t L_3 - (\nabla\times w)_3)e_3 \\ &= -qe_0 - J_1e_1 - J_2e_2 - J_3e_3 \\ &= (-q,-J) = -\mathcal{J}, \end{aligned} \tag{4.2}$$

where q and J are the hydrodynamic charge density and the hydrodynamic current as defined in section (2.2.4).

$$\begin{aligned}\boldsymbol{\nabla}\wedge\boldsymbol{W} &= (\partial_t w_1 - (\nabla\times L)_1)e_0\wedge e_2\wedge e_3 + (\partial_t w_2 - (\nabla\times L)_2)e_0\wedge e_3\wedge e_1 \\ &\qquad + (\partial_t w_3 - (\nabla\times L)_3)e_0\wedge e_1\wedge e_2 + (\nabla\cdot w)e_1\wedge e_2\wedge e_3, \\ &= 0,\end{aligned}$$

for ideal and viscous flows.

Here $\boldsymbol{\nabla}\cdot\boldsymbol{W} = (-q,-J) = -\mathcal{J}$ is a four-vector and $\boldsymbol{\nabla}\wedge\boldsymbol{W} = 0$ is the trivector part of $\boldsymbol{F}$. The above 4-vector $\mathcal{J}$ satisfies the conservation equation

$$\partial_t q + \nabla\cdot J = 0$$

for both ideal and viscous flows. Thus, the governing equations of the vorticity and the Lamb vector as derived by (Marmanis, 1998) or otherwise termed as the fluid Maxwell equations (as in 2.2.4) can be extracted from a single equation

$$\boldsymbol{F} = \boldsymbol{\nabla}\boldsymbol{W} = -\mathcal{J}.$$

Since $\boldsymbol{W} = \boldsymbol{\nabla}\boldsymbol{V}$,

$$\boldsymbol{F} = \boldsymbol{\nabla}^2\boldsymbol{V} = -\mathcal{J}$$

can be considered as the wave equation for $\boldsymbol{V}$.

4.1.2 Fluid Dynamic Stress-Energy Tensor

Define $T_i = \frac{1}{2}\tilde{\boldsymbol{W}} e_i \boldsymbol{W}$, where $\tilde{\boldsymbol{W}}$ is the reverse of $\boldsymbol{W}$. Since $\boldsymbol{W}$ is a bivector, we have $\tilde{\boldsymbol{W}} = -\boldsymbol{W}$. Thus we get,

$$
\begin{aligned}
T_0 &= \frac{1}{2}(-|L|^2 + |w|^2)e_0 + (w \times L)_1 e_1 + (w \times L)_2 e_2 + (w \times L)_3 e_3, \\
T_1 &= (w \times L)_1 e_0 + \frac{1}{2}[-(L_1^2 - L_2^2 - L_3^2) + (w_1^2 - w_2^2 - w_3^2)]e_1 + (-L_1L_2 + w_1w_2)e_2 \\
&\qquad + (-L_1L_3 + w_1w_3)e_3, \\
T_2 &= (w \times L)_2 e_0 + (-L_1L_2 + w_1w_2)e_1 + \frac{1}{2}[-(L_2^2 - L_1^2 - L_3^2) + (w_2^2 - w_1^2 - w_3^2)]e_2 \\
&\qquad + (-L_2L_3 + w_2w_3)e_3
\end{aligned}
$$

and

$$
\begin{aligned}
T_3 &= (w \times L)_3 e_0 + (-L_1L_3 + w_1w_3)e_1 + (-L_2L_3 + w_2w_3)e_2 \\
&\qquad + \frac{1}{2}[-(L_3^2 - L_1^2 - L_2^2) + (w_3^2 - w_1^2 - w_2^2)]e_3.
\end{aligned}
$$

Here $T_0 = \frac{1}{2}(-|L|^2 + |w|^2)e_0 + \mathcal{P}_1 e_1 + \mathcal{P}_2 e_2 + \mathcal{P}_3 e_3$ is the four dimensional Poynting vector where $\mathcal{P} = (w \times L) = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$.

The elements $T = [T_{ij}]$ in the fluid dynamic stress-energy tensor as given in equation (2.37) can be obtained from the scalar part of the geometric product $T_i e_j$ or we have $T_{ij} = \frac{1}{2}(\tilde{\boldsymbol{W}} e_i \boldsymbol{W} e_j)_S$, where T_{ij} is the ij^{th} entry of the stress-energy tensor T

Consider

$$\boldsymbol{W}\mathcal{J} = \boldsymbol{W} \cdot \mathcal{J} + \boldsymbol{W} \wedge \mathcal{J}. \tag{4.3}$$

$$
\begin{aligned}
\boldsymbol{W} \cdot \mathcal{J} &= (L_1J_1 + L_2J_2 + L_3J_3)e_0 \\
&\qquad + (-qL_1 + (J \times w)_1)e_1 + (-qL_2 + (J \times w)_2)e_2 \\
&\qquad\qquad + (-qL_3 + (J \times w)_3)e_3, \\
&= [L \cdot J, -qL + J \times w]
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
\boldsymbol{W} \wedge \mathcal{J} = &(w_1 J_1 + w_2 J_2 + w_3 J_3) e_1 e_2 e_3 \\
&+ (q w_1 + (L \times J)_1) e_0 e_2 e_3 + (q w_2 + (L \times J)_2) e_0 e_3 e_1 \\
&\quad + (q w_3 + (L \times J)_3) e_0 e_1 e_2, \\
&= I[-w \cdot J, -(-qw + J \times L)]. \qquad (4.5)
\end{aligned}
$$

Therefore,

$$\frac{1}{2}(\boldsymbol{W}\mathcal{J} + \mathcal{J}\boldsymbol{W}) = \boldsymbol{W} \cdot \mathcal{J} = [L \cdot J, -qL + J \times w]$$

can be considered as the four dimensional force analogous to the Lorentz force in fluid dynamics. Hence the vector part of the geometric product $\boldsymbol{W}\mathcal{J}$ yields the fluid dynamic Lorentz force. Here we have considered only the vector part of the product $\boldsymbol{W}\mathcal{J}$. The physical and the geometric significance of the trivector part which was obtained along with the vector part needs to be further investigated.

4.1.3 Multivector Associated with Helicity

The geometric product of velocity vector $\boldsymbol{V}$ and the vorticity bivector $\boldsymbol{W}$ gives a multivector which can be associated with the helicity. Thus,

$$\boldsymbol{H} = \boldsymbol{V}\boldsymbol{W} = \boldsymbol{V} \cdot \boldsymbol{W} + \boldsymbol{V} \wedge \boldsymbol{W}$$

$$
\begin{aligned}
\boldsymbol{V} \cdot \boldsymbol{W} = &-(v \cdot L) e_0 + (-\varepsilon L_1 + (w \times v)_1) e_1 \\
&\quad + (-\varepsilon L_2 + (w \times v)_2) e_2 + (-\varepsilon L_3 + (w \times v)_3) e_3, \\
&= [-(v \cdot L), -\varepsilon L + (w \times v)], \qquad (4.6) \\
&= (-h', -H'). \qquad (4.7)
\end{aligned}
$$

The quantity $-h' = -(v \cdot L)$ vanishes for ideal fluids, but it not for viscous fluids. Hence for ideal fluids $\boldsymbol{V} \cdot \boldsymbol{W}$ is a usual vector in the three dimensional space. Here $-H' = -\varepsilon L + (w \times v) = (-H_1', -H_2', -H_3')$. Now,

$$\begin{aligned}
\boldsymbol{V} \wedge \boldsymbol{W} &= (v \cdot w)e_1e_2e_3 + (-\varepsilon w_1 + (L \times v)_1)e_0e_2e_3 \\
&\qquad + (-\varepsilon w_2 + (L \times v)_2)e_0e_3e_1 + (-\varepsilon w_3 + (L \times v)_3)e_0e_1e_2, \\
&= I[-(v \cdot w), -\varepsilon w + (L \times v)], \\
&= I(-h, -H), \qquad (4.8)
\end{aligned}$$

where $h = v \cdot w$ is the helicity density and $-H = -\varepsilon w + (L \times v) = (-H_1, -H_2, -H_3)$.

Therefore $\boldsymbol{H}$ is a sum of a vector and a grade-3 blade. The dual of the trivector part $\boldsymbol{H}_T$ of $\boldsymbol{H}$ gives a four-vector $(-h, -H)$ which can be associated to charge density four-vector. It has been shown in section (2.2.5) that the dual of $\boldsymbol{H}_T$ satisfies the conservation equation for helicity,

$$\partial_t h + \nabla \cdot H = 0,$$

for ideal fluids.

The trivector part of $\boldsymbol{H}$ can be associated with the helicity three-form seen in chapter 2. The vector part or the scalar product obtained while executing the geometric product contains additional terms. The physical and geometrical relevance of these terms need to be further studied.

4.1.4 Multivector Associated with Parity

Evaluating $\boldsymbol{W}^2 = \boldsymbol{W}\boldsymbol{W}$ we get,

$$\boldsymbol{W}^2 = \boldsymbol{W} \cdot \boldsymbol{W} + \boldsymbol{W} \wedge \boldsymbol{W}.$$

Here the scalar part is given by

$$\boldsymbol{W}\cdot\boldsymbol{W} = -(L_1^2+L_2^2+L_3^3+w_1^2+w_2^2+w_3^2) = -(|L|^2+|w|^2),$$

and the pseudoscalar part is evaluated as

$$\boldsymbol{W}\wedge\boldsymbol{W} = 2(L\cdot w)e_0e_1e_2e_3.$$

Thus,

$$\boldsymbol{W}\boldsymbol{W} = -(|L|^2+|w|^2)+2(L\cdot w)e_0e_1e_2e_3 \tag{4.9}$$

It has been shown in section(2.2.6) that the quantity $L\cdot w = 0$ for ideal fluids and $L\cdot w = -\nu(\nabla\times w)\cdot w$ for viscous fluids.

Evaluating $\boldsymbol{\nabla H}$, we get

$$\boldsymbol{\nabla H} = \boldsymbol{\nabla}\cdot\boldsymbol{H}+\boldsymbol{\nabla}\wedge\boldsymbol{H}. \tag{4.10}$$

Here,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{H} &= [\partial_t(-h')+\nabla\cdot(-H')]\\ &+[(\nabla\times(-H))_1e_0e_1+(\nabla\times(-H))_2e_0e_2+(\nabla\times(-H))_3e_0e_3\\ &+(\partial_t(-H_1)+\partial_xh)e_2e_3+(\partial_t(-H_2)+\partial_yh)e_3e_1\\ &+(\partial_t(-H_3)+\partial_zh)e_1e_2]\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\nabla}\wedge\boldsymbol{H} &= [(\partial_t(-H_1^{'})+\partial_xh^{'})e_0e_1+(\partial_t(-H_2^{'})+\partial_yh^{'})e_0e_2\\ &+(\partial_t(-H_3^{'})+\partial_zh^{'})e_0e_3\\ &+(\nabla\times(-H^{'}))_1e_2e_3+(\nabla\times(-H^{'}))_2e_3e_1+(\nabla\times(-H^{'}))_3e_1e_2]\\ &+[(\partial_th+\nabla\cdot H)e_0e_1e_2e_3].\end{aligned}$$

Therefore $\boldsymbol{\nabla H}$ is a sum of a scalar, a bivector and a pseudoscalar.

Since $\boldsymbol{H} = \boldsymbol{V}\boldsymbol{W}$, we can evaluate

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{H} = \boldsymbol{\nabla}(\boldsymbol{V}\boldsymbol{W}) &= (\boldsymbol{\nabla}\boldsymbol{V})\boldsymbol{W} + \boldsymbol{V}\boldsymbol{\nabla}\boldsymbol{W} \\ &= \boldsymbol{W}\boldsymbol{W} + \boldsymbol{V}(-\mathcal{J}) \end{aligned} \tag{4.11}$$

By comparing the scalar parts from equations (4.9), (4.10) and (4.11) we get,

$$-(\partial_t(h') + \nabla \cdot (H')) = -(|L|^2 + |w|^2) + (\varepsilon q - v \cdot J). \tag{4.12}$$

The above equation describes the evolution equation of the four-vector $(-h', -H')$. By comparing the pseudoscalar parts equations (4.9), (4.10) and (4.11) we get,

$$\partial_t h + \nabla \cdot H = 2(L \cdot w). \tag{4.13}$$

For ideal fluids, $(L \cdot w) = 0$ and for viscous fluids, we have

$$\partial_t h + \nabla \cdot H = -2\nu(\nabla \times w) \cdot w,$$

which is the evolution equation for helicity for viscous flows. Thus we have re-derived the helicity conservation law for ideal fluids and the helicity evolution equation for viscous fluids using geometric calculus.

In the same manner we can compare the bivector parts from equations (4.9), (4.10) and (4.11) and extract additional quantities whose physical and geometrical aspects can be investigated further.

Chapter 5

Conclusion and Scope For Further Research

Towards 1980's a feeling had arisen that the conventional analytical tools are not sufficient for newly arising problems in fluid mechanics. A new direction in this regard was given in the symposium of International Union for Theoretical and Applied Mechanics (IUTAM) held at Cambridge during August 1989. In the proceedings of the symposium, the editors Moffat and Tsinober say; "the topic of this meeting was chosen in response to the developing interest in aspects of fluid mechanics and of magnetohydrodynamics, that can be properly described as topological, rather than exclusively analytical in character"(Topological fluid dynamics: Proceedings of the IUTAM symposium, Cambridge University Press, 1990). Other important works on topological fluid dynamics are 'Topological aspects of the dynamics of fluids and plasmas' by Moffatt, H. K., Zaslavsky, G. M., Comte, P., Tabor, M. (2013), 'Topological methods in hydrodynamics' by Arnold V. I. and Khesin B.A. (1999) and 'Lectures on topological fluid mechanics', Editors: De witt Sumners, Ricca et al. (2009).

Albert Einstein developed the special theory of relativity in 1905. Ever since, time is considered to be the fourth dimension of the physical world. Even then higher dimensional physics was considered to be mathematical rather than being experimental. Recently two teams of scientists from USA, Germany, Italy, Israel and Switzerland were able to successfully illustrate the four dimensional quantum Hall effect through an experiment in a laboratory (Zilbergerg et.al, 2018). Thus a time has reached when physicists can now experimentally investigate the phenomena occurring in four or higher dimensions. Scientists were able to use a gas of ultracold atoms to visualize the dynamics of the quantum Hall effect, which was predicted to occur in four dimensions, paving the way for higher dimensional experimental physics.

The analytical methods used in classical fluid mechanics are differential and integral equations, variational calculus and integral transforms. However a paper which did not get much attention was that of Drobot and Rybarski (1958). In their work, they used group theory in the form of Noether's theorem to derive invariants of barotropic flows. Here for the first time they used a four dimensional space time manifold as the domain of flow. A follow up for this can be seen in the works of M.J. Vedan and his group of students. George Mathew (1988, 1989, 1991), Thomas Joseph (1996) generalized this work in the case of non-barotropic flows while Geetha (1994) used this method to study stability of flows. Instead of the method of calculus of variations used in the above works, Subin (2004, 2006) used differential forms based on the work of Tur and Yanovsky (1993).

Recent research in the field of medical sciences reveal that four dimensional flows provide breakthrough advancements in new imaging techniques. Four dimensional flows (time resolved phase contrast magnetic resonance imaging (MRI) with three dimensional velocity) are effectively used in cardiovascular and abdominal anatomy. Four dimensional flows yield precise and advanced visualization and quantification of complex blood flow patterns. Haemodynamic parameters like that of velocity, kinetic energy, pressure gradients, wall shear stress, vorticity and helicity of pulmonary arterial blood flow can be analyzed using four dimensional imaging tools (Garcia et.al (2019)).

The four dimensional expressions in chapter 2, which we have derived by exploring the mathematical concepts of differential forms and integral invariance, may find applications in a wide variety of fields. The expressions obtained to evaluate the rate of circulation, vorticity flux, rate of change of helicity and parity for viscous fluids in four dimensions can be utilized in engineering, medical, physical and theoretical problems. It should be noted that, the fluid Maxwell's equations, Poynting theorem and the hydrodynamic stress-energy tensor were thereby obtained as consequences of applying mathematical concepts and not by mere analogical methods.

In chapter 3, we extend the above studies to non-barotropic flows in a four dimensional space time Euclidean manifold. The four dimensional expressions to evaluate the rate of change of the generalized circulation, generalized vorticity flux, generalized helicity and generalized parity in the case of ideal as well as viscous non-barotropic flows are thereby obtained. The expressions thus derived, when restricted to hyper surfaces of constant time, yield results analogous to expressions obtained in three dimensional barotropic flows.

The same method of study can be adopted into other models of fluid flows like that of magneto hydrodynamics, two-fluid models and so on.

A later development is the application of geometric algebra in fluid dynamics (Cibura and Hildenbrand, 2008). The introduction of geometric algebra in classical mechanics was done by Hestenes (1966).

Hestenes has described the development of the concept of numbers from that of the Greeks to the present one of multivectors which involves scalars, vectors, bivectors, trivectors and pseudoscalars. Classical fluid mechanics considers vorticity as a vector. The bivector character of vorticity comes into picture when vorticity is expressed in the language of differential forms or geometric algebra. Thus vorticity is a bivector and so has a plane associated with it. This is the basis of our definition of vorticity stress tensor in chapter 2, analogous to Maxwell stress tensor in electromagnetic theory. Quaternions form a subalgebra of the geometric algebra $\mathcal{G}_3$. Therefore the bivector character of vorticity can also be exploited in the form of a quaternion. Quaternions are finding applications in computer graphics and robotics where rotations play an important role. The main advantage of quaternions is in the illustrations of rotations of a sphere in space time manifold. This concept can be made use in vortex dynamics. As geometric algebra integrates the concept of vectors, complex numbers, tensors, spinors and quaternions into a coherent mathematical language, it not only retains the advantages of each special algebras, but it also gives more insight into the problems of fluid dynamics.

The introduction of geometric algebra in fluid dynamics has been restricted to some specialized problems. The complete potential of the geometric product is yet to be explored in the field of fluid dynamics. In chapter 4, we have adopted a geometric algebraic approach to fluid flows in the four dimensional Euclidean space time manifold rather than the conventional Minkowski space time manifold used elsewhere. The introduction of a four dimensional velocity vector with the Bernoulli energy function as the time component into geometric algebra is a novel approach. We find that the most common physical quantities like that of vorticity, helicity and parity appear as multivectors, i.e., as sum of scalars, vectors, bivectors, trivectors and pseudoscalars, thereby equipping them with additional geometric and physical properties. As mentioned in sections (4.1.2), (4.1.3) and (4.1.4), the expressions obtained while executing the geometric product for the vector derivative of the vorticity and helicity contain additional terms which need to be mathematically, physically and experimentally investigated, paving the way for future research.

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- Vedan M.J., Rajeshwari Devi M.B., Susan M.P., Maxwell Stress Tensor In Hydro-dynamics, IOSR Journal of Mathematics, 2015 , Volume 11, Issue 1 Ver. V, 58-60 .
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Papers Presented

- Presented a paper : Vorticity and Stress Tensor, National Conference on Fluid Mechanics at SSN College of Engineering, Chennai, 27-28 October, 2017.
- Presented a poster : Integral invariants of flows in a Four Dimensional Manifold at the Indian Women and Mathematics: Regional Workshop held at Cochin University of Science and Technology, Cochin, 2-3 January, 2018.

- Presented a paper : Local invariants of flows in a four dimensional manifold, at the International Conference in Science, Technology, Engineering and Social Sciences at Kuriakose Elias College, Mannanam, Kottayam, January 18 - 19, 2018.

Papers Communicated

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<u>**CURRICULUM VITAE**</u>

Name :	Susan Mathew Panakkal
Official Address :	Assistant Professor, Department of Mathematics, St. Teresa's College, Cochin, Kerala, India – 682 011.
Permanent Address :	S2 Sunshine Walkups, Tagore Nagar, Kadavanthra, Ernakulam, Kerala, India – 682020.
Email :	susanmathewpanakkal@gmail.com

Qualifications :	**B.Sc.** (Mathematics), 2001. Calicut University, Thenhipalam, Kerala 673635. **M.Sc.** (Mathematics), 2003. Calicut University, Thenhipalam, Kerala 673635. **B.Ed** (Mathematics), 2005. Calicut University, Thenhipalam, Kerala 673635. **M.Phil** (Mathematics), 2008. University of Kerala, Thiruvananthapuram, Kerala.
Research Interest :	Fluid Mechanics.