RELIABILITY MODELLING AND ANALYSIS USING QUANTILE FUNCTIONS

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CERTIFICATE

This is to certify that the thesis entitled **" RELIABILITY MODELLING AND ANALYSIS USING QUANTILE FUNCTIONS "** is a bonafide record of work done by Dileepkumar M under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

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This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Cochin- 22 **Dileepkumar M**

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Dileepkumar M

To My Teachers..

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The term reliability has been formulated as the science of predicting, estimating or optimizing the probability of survival, the mean life or more generally the life distribution of components or systems. Most of the statistical models used in the context of reliability studies are developed in terms of the distribution function $F(x)$ or its related concepts. An equivalent and an alternative approach is by means of using the quantile function defined by

$$Q\,(u) = F^{-1}\,(x) = \inf\{x : F\,(x) \geq u\},\ \ 0 \leq u \leq 1.$$

Both $F(x)$ and $Q(u)$ contain the same information about the random mechanism of the subject with different implications. Although both convey the same information about the underlying distribution, most of the popular statistical concepts and methodologies in the current literature is based on distribution functions. The quantile-based studies are carried out mostly when the traditional approach is either difficult or fails to provide the desired results. Further, the quantile function has several interesting characteristics that are not shared by the distribution function. For example, the sum of two quantile functions is again a quantile function.

Historically, many researchers have pointed out the potential of quantiles in data analysis even before the nineteenth century. Quetelet [121] initiated the use of quantiles in statistical analysis by introducing the concept of inter-quantile range for measuring the data variability. Hastings et al. [56] introduced a class of distributions through the quantile function which does not have a closed form expression for its distribution function. This was a major development in portraying quantile functions to represent distributions. The class of distributions by Hastings et al. [56] was later refined by Tukey [152] to form a symmetric distribution, called Tukey lambda distribution. This model was generalized in different ways referred to as the lambda distributions. These include various forms of quantile functions discussed in Ramberg and Schmeiser [123], Ramberg [122], Ramberg et al. [125], and Freimer et al. [40]. Govindarajulu [45] introduced and then studied by Nair et al. [104], a new quantile function by taking the weighted sum of quantile functions of two power distributions. Gilchrist [42] presented the power-Pareto distribution by taking the product of power and Pareto quantile functions. van Staden and Loots [153] developed a four parameter distribution, using a weighted sum of generalized Pareto and its reflection quantile functions. Sankaran et al. [136] developed a new quantile function based on the sum of quantile functions of generalized Pareto and Weibull distributions. The density function or distribution function for these models are not available in closed forms except for certain special cases. The great advantage of these models is that the simple forms of the quantile functions make it extremely straightforward to simulate random values, which is useful in inference problems. Another milestone in the development of quantile functions is the seminal paper by Parzen [117], in which he established the description of a distribution in terms of the quantile function and its role in data modelling. Gilchrist [42] presented several properties of quantile function which are useful for the construction of new flexible distribution models based on quantile functions to analyse various types of statistical data sets. He also discussed different estimation methods and goodness of fit tests for the quantile function models. The estimates based on quantiles are more accurate and robust in the presence of outliers when heavy tailed distributions are employed for the analysis of various lifetime data sets. In the simulation studies, quantile functions are used for generating random samples.

Researchers such as Parzen [117], Freimer et al. [40] and Gilchrist [42] have pointed out the scope of using quantile functions in reliability analysis. Nair and Sankaran [97] elaborately elucidated the importance of quantile functions in reliability analysis. They presented quantile-based definitions for various reliability measures such as the distribution function, hazard rate, mean residual life, percentiles, higher moments of residual life, etc. Several characterization results and the interrelationships between the quantile-based reliability measures are also presented. A detailed review on the properties and applications of quantile functions in reliability theory are available in Nair et al. [105]. Recently, Nair et al. [107] and Nair et al. [106] have developed quantile-based definitions of the well-known proportional hazards model and proportional reversed hazards model respectively and presented various properties and applications in the context of reliability analysis. They also discussed the advantages of these models over the existing ones defined in terms of the distribution functions.

We now give a brief review of the background concepts that are used in the subsequent chapters. First, we present the definition of a quantile function and list its important properties. Then the quantile forms of various measures such as moments, percentiles, etc. are presented. The quantile-based reliability concepts such as residual life, hazard quantile function, mean residual quantile function etc. are given subsequently. We then discuss some most celebrated quantile function models in the literature such as lambda distributions by Tukey [152], Ramberg and Schmeiser [124], Freimer et al. [40], the Govindarajalu distribution by Govindarajulu [45], the Power-Pareto distribution by Gilchrist [42], the Jones class of distributions by Jones [66] and the model proposed by van Staden and Loots [153].

1.1 Basic concepts

In this section, we define basic concepts and properties of the quantile function, which are useful for forthcoming chapters.

1.1.1 Quantile function

For a real valued random variable X with right continuous distribution function $F(x)$, the quantile function $Q(u)$ is defined as

$$Q(u) = F^{-1}(x) = \inf\{x : F(x) \geq u\}, \;\; 0 \leq u \leq 1. \tag{1.1.1}$$

For $-\infty < x < \infty$ and $0 < u < 1$, we have $F(x) \geq u$ if and only if $Q(u) \leq x$. Thus if there exists an x such that $F(x) = u$, then $F(Q(u)) = u$ and $Q(u)$ is the smallest value of x satisfying $F(x) = u$. For a continuous random variable X, (1.1.1) reduces to

$$Q(u) = \inf\{x : F(x) = u\},$$

and when $F(x)$ is strictly increasing too, $Q(u)$ is the unique value of x such that $F(x) = u$. In this case, we can easily formulate the quantile function by solving $F(x) = u$ for x in terms of u. Note that the quantile function $Q(u)$ characterizes the underlying distribution. The important properties of the quantile functions are given below:

1. $Q(u)$ is non-decreasing in u, where $u \in (0, 1)$.
2. $Q(F(x)) \leq x$ for all $x \in (-\infty, \infty)$ for which $0 < F(x) < 1$ and $F(Q(u)) \geq u$ for any $u \in (0, 1)$.

3. $Q(u)$ is left continuous, i.e., $Q(u^-) = Q(u)$. Further, $Q(u^+) = \inf\{x : F(x) > u\}$ so that $Q(u)$ has limits from above.

4. Any jumps of $F(x)$ are flat points of $Q(u)$ and flat points of $F(x)$ are jumps of $Q(u)$.

5. For a uniform random variable U over $(0, 1)$, we get

$$P\{Q(u) \leq x\} = P\{U \leq F(x)\} = F(x). \tag{1.1.2}$$

Thus $Q(u)$ and X are identically distributed for the standard uniform distribution.

6. The mixture of two quantile functions $Q_1(u)$ and $Q_2(u)$ given by, $Q(u) = \alpha Q_1(u) + (1-\alpha)Q_2(u),\ 0 \leq u \leq 1$ is also a quantile function, which lies between $Q_1(u)$ and $Q_2(u)$.

7. Sum of two quantile functions is again a quantile function. For example, Govindarajulu [45] constructed a new quantile function by taking the weighted sum of quantile functions of two power distributions.

8. The product of two positive quantile functions is again a quantile function. Gilchrist [42] formed the power-Pareto quantile function by considering the product of quantile functions of the power and Pareto distributions.

9. Suppose $L(x)$ is a non-decreasing function, then $L(Q(u))$ is also a quantile function. Further, if $L(x)$ is non-decreasing over the interval $(0, 1)$ with the property that $L(0) = 0$ and $L(1) = 1$, then $Q(L(u)),\ u \in (0, 1)$ is also a quantile function with the same support as that of $Q(u)$. Gilchrist [42] studied various applications of these transformations in the context of statistical data modelling.

Some of the aforementioned properties (properties 6 to 9) of quantile functions are not true in the case of distribution functions. These suggest the use of quantile functions in modelling and analysis of various types of statistical data sets. For more properties and applications of quantile functions, we refer to Gilchrist [42] and Nair et al. [105].

1.1.2 Quantile density function

Throughout the study, we consider X as a non-negative continuous random variable with distribution function $F(x)$ and quantile function $Q(u)$. Let $f\left(x\right)$ is the probability density function of X. Then the function $f\left(Q\left(u\right)\right)$ is called the density quantile function and the derivative of $Q\left(u\right)$, given by

$$q\left(u\right) = Q'\left(u\right),$$

is called quantile density function of X. From the identity, $F\left(Q\left(u\right)\right) = u$, upon differentiating with respect to u, we get

$$q\left(u\right) f\left(Q\left(u\right)\right) = 1. \tag{1.1.3}$$

In words, $f(Q(u))$ and $q(u)$ are reciprocals of each other. This justifies calling them by names which are the reverse of each other.

1.1.3 Quantile form of some general concepts

In this section, we present quantile-based definitions of some important general concepts. These concepts are useful for the discussions in the sequel. A detailed survey of various quantile-based concepts are given in Nair et al. [105].

1.1.3.1 Percentiles

A percentile (or a centile) is a measure used to indicate the value below which a certain percentage of observations in a group of observations falls. There are different measures based on percentiles, which are used for identifying various distributional properties. Some of the most commonly used measures based on percentiles are listed below.

An important measure of location, the median is defined by

$$M = Q\left(0.5\right). \tag{1.1.4}$$

The inter-quartile range, which is used as a measure of dispersion has the form

$$IQR = Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right), \tag{1.1.5}$$

The Galton's coefficient of skewness (also known as Bowley's coefficient of skewness (Bowley [19])) denoted by S is given by

$$S = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2M}{IQR}, \tag{1.1.6}$$

and the Moor's coefficient of kurtosis (Moors [92]), T is

$$T = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}. \tag{1.1.7}$$

It is to be noted that, $-1 \leq S \leq +1$ and the extreme positive skewness occurs when $Q(\frac{1}{4}) \to M$ and the extreme negative skewness is attained when $Q(\frac{3}{4}) \to M$. For a symmetric distribution, we get $M = \frac{Q(\frac{1}{4})+Q(\frac{3}{4})}{2}$ and hence $S = 0$. As a measure of kurtosis,

T is justified on the grounds that the differences $Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)$ and $Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)$ become large (small).

1.1.3.2 Order statistics

Let $X_1, X_2, ..., X_n$ denotes the lifetimes of n identical units in a life testing experiment. Suppose $F(x)$ is the common distribution function. The random variables $X_{(1)},\ X_{(2)}, ...,\ X_{(n)}$ correspond to the ordered sample values are referred to as the order statistics, where $X_{(1)} = \min\limits_{1 \le i \le n} X_i$ and $X_{(n)} = \max\limits_{1 \le i \le n} X_i$. The distribution of rth order statistic is given by

$$F_r\left(x\right) = P\{X_{(r)} \ \le x\} = \sum_{k=r}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \left(F\left(x\right)\right)^k \left(1 - F\left(x\right)\right)^{n-k}. \tag{1.1.8}$$

In particular, the distributions of $X_{(1)}$ and $X_{(n)}$ are

$$F_1\left(x\right) = 1 - \left(1 - F\left(x\right)\right)^n \quad \text{and} \quad F_n\left(x\right) = \left(F\left(x\right)\right)^n,$$

respectively. To derive the quantile form of the distribution of order statistics, we use the definitions of the beta function,

$$B\left(m, n\right) = \int_0^1 t^{m-1}\left(1-t\right)^{n-1} dt\,, \quad m, n > 0$$

and the incomplete beta function ratio,

$$I_x\left(m, n\right) = \frac{B_x\left(m, n\right)}{B\left(m, n\right)},$$

where, $B_x(m,n) = \int_0^x t^{m-1}(1-t)^{n-1}\,dt$. The incomplete beta function and the upper tail of the binomial distribution are connected through the relation

$$\sum_{k=r}^{n} \begin{pmatrix} n \\ r \end{pmatrix} p^k (1-p)^{n-k} = I_p\left(r,\ n-r+1\right). \tag{1.1.9}$$

Denote $u_r = F_r(x)$ and $F(x) = u$. Now from (1.1.8) and (1.1.9), we have $u_r = I_u(r,\ n-r+1)$. Using this relation, the quantile function of the rth order statistic is obtained as

$$Q_r(u_r) = Q\left({I_{u_r}}^{-1}\left(r,\ n-r+1\right)\right), \tag{1.1.10}$$

where I^{-1} represents the inverse of the incomplete beta function ratio I. The order statistics $X_{(1)}$ and $X_{(n)}$ have simple forms for their quantile functions given by

$$Q_1(u_1) = Q(1-(1-u_1)^{\frac{1}{n}}) \text{ and } Q_n(u_n) = Q(u_n^{\frac{1}{n}}). \tag{1.1.11}$$

1.1.3.3 Residual life

The concept of residual life has received several applications in reliability theory. It is the remaining life of an item after it has attained age t, say. The residual life associated with a life time random variable X is the random variable $X_t = (X-t|X>t)$. The survival function of X_t is defined by

$$\bar{F}_t(x) = P\{X_t > x\} = \frac{\bar{F}(x+t)}{\overline{F}(t)}, \tag{1.1.12}$$

where $\bar{F}(x) = P\{X > x\} = 1 - F(x)$. This implies

$$F_t(x) = \frac{F(x+t) - F(t)}{1 - F(t)}. \tag{1.1.13}$$

Suppose $F(t) = u_0,\ F(x+t) = v$ and $F_t(x) = u$. Then, we have $x+t = Q(v)$ and $x = Q_1(u)$, say. This gives $Q_1(u) = Q(v) - Q(u_0)$ and from (1.1.13), we obtain $u(1-u_0) = v - u_0$ or $v = u_0 + (1-u_0)u$. Thus the quantile function of X_t becomes

$$Q_1(u) = Q(u_0 + (1-u_0)u) - Q(u_0). \tag{1.1.14}$$

1.1.3.4 Gini's mean difference

Gini's mean difference is one of the popular measures used in the context of econometrics, which is defined by

$$\begin{aligned} \Delta &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\ f(x)\ f(y)\ dx\,dy \\ &= 2\int_{-\infty}^{\infty} F(x)(1-F(x))\ dx. \end{aligned} \tag{1.1.15}$$

By setting $F(x) = u$, we get

$$\Delta = 2\int_0^1 u(1-u)\ q(u)\ du \tag{1.1.16}$$

$$= 2\int_0^1 (2u-1)\ Q(u)\ du. \tag{1.1.17}$$

The expression (1.1.17) is obtained by integrating (1.1.16) by parts. One can use (1.1.16) or (1.1.17) depending on whether $q(u)$ or $Q(u)$ is known.

1.1.3.5 Moments

The rth ordinary moment is defined as

$$\mu_r^{'} = E\left(X^r\right) = \int_0^{\infty} x^r \, f\left(x\right) \; dx.$$

An equivalent expression in terms of the quantile function is

$$\mu_r^{'} = \int_0^1 \left(Q\left(u\right)\right)^r \; du. \tag{1.1.18}$$

In particular, for $r = 1$, $E(X)$ has the form

$$\mu = \int_0^1 Q\left(u\right) \; du = \int_0^1 \left(1-u\right) \; q\left(u\right) \; du.$$

Other measures based on moments to describe spread, skewness and kurtosis in terms of quantile function are respectively given by

$$\begin{aligned}
\sigma^2 &= E\left(X-\mu\right)^2 = \int_0^1 \left(Q\left(u\right)-\mu\right)^2 \; du \,(\text{variance}),\\
\mu_3 &= E\left(X-\mu\right)^3 = \int_0^1 \left(Q\left(u\right)-\mu\right)^3 \; du,\\
\text{and,}\quad \mu_4 &= E\left(X-\mu\right)^4 = \int_0^1 \left(Q\left(u\right)-\mu\right)^4 \; du.
\end{aligned}$$

1.1.3.6 L-moments

Sections 1.1.3.5 and 1.1.3.1 have considered the moments and percentiles that are capable for summarizing probability distributions. In this section, we present the L-moments, which are the competing alternatives to the ordinary moments. Hosking [60] provided a

unified theory and a systematic study on L-moments. The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics.
The rth L-moment of the random variable X is defined as

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \left(\begin{array}{c} r-1 \\ k \end{array} \right) E\left(X_{r-k:r}\right), \quad r = 1, 2, ... \tag{1.1.19}$$

By inserting the expression of $E\left(X_{r:n}\right)$, we get

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \left(\begin{array}{c} r-1 \\ k \end{array} \right) \frac{r!}{k!\,(r-k-1)!} \int_0^1 u^{r-k-1} (1-u)^k Q(u)\, du.$$

An alternative form for the above expression is given in Jones [65]. In particular, the first four L-moments are

$$L_1 = \int_0^1 Q(u)\, du = \int_0^1 (1-u)\, q(u)\, du = \mu \tag{1.1.20}$$

$$L_2 = \int_0^1 (2u-1)\, Q(u)\, du = \int_0^1 \left(u-u^2\right) q(u)\, du \tag{1.1.21}$$

$$L_3 = \int_0^1 \left(6u^2-6u+1\right) Q(u)\, du = \int_0^1 \left(3u^2-2u^3-u\right) q(u)\, du \tag{1.1.22}$$

and $$L_4 = \int_0^1 \left(20u^3-30u^2+12u-1\right) Q(u)\, du = \int_0^1 \left(u-6u^2+10u^3-5u^4\right) q(u)\, du. \tag{1.1.23}$$

The L-moments have several theoretical advantages over ordinary moments. They are able to characterize a wider range of distributions and are more robust to the effects of outliers in the data. The L-moments exist whenever $E(X)$ is finite, whereas for many distributions additional restrictions are required for the ordinary moments to be finite. Sillito [144], Greenwood et al. [46], Hosking [59], Hosking [58] and Hosking and Wallis [61] have made detailed studies on various properties and applications of L-moments in summarizing and characterizing probability distributions, different estimation techniques based on L-moments and the comparison between the ordinary moments and L-moments in analysing various measures of distributional shapes. As in the case of ordinary moments, the L-moments are useful to summarize the probability distributions, to identify the distributions and to fit models to data. A distribution may be specified in terms of its L-moments, even if some of its ordinary moments do not exist (Hosking [59]). Hosking [58] showed that the L-moments are more preferable than the ordinary moments to provide summary measures of distributional shape. Various measures based L-moments have generally lower sampling variance and robust against outliers.

For a non-degenerate random variable X, with $E(X) < \infty$, consider the ratios, $\tau_r = \frac{L_r}{L_2}$, $r = 3, 4, ...$ We can observe that, $|\tau_r| < 1$ for $r \geq 3$. Using these $L-$moment ratios, Hosking [60] defined an alternative measure of the coefficient of variation called the L-coefficient of variation, given by

$$\tau_2 = \frac{L_2}{L_1}. \tag{1.1.24}$$

Note that $0 < \tau_2 \leq 1$, when X is non-negative. This is due to the fact that when X is non-negative $L_1 \geq 0,\ L_2 \geq 0$ and

$$L_2 = \int_0^1 u\,(1-u)\,q\,(u)\,du \ \leq \ \int_0^1 (1-u)\,q\,(u)\,du \ = \ L_1.$$

To measure the skewness and kurtosis, the $L-$coefficient of skewness (τ_3) and $L-$coefficient of kurtosis (τ_4) are defined as

$$\tau_3 = \frac{L_3}{L_2}, \tag{1.1.25}$$

and

$$\tau_4 = \frac{L_4}{L_2}. \tag{1.1.26}$$

The range of τ_3 is $(-1,\, 1)$ and that of τ_4 is $\frac{1}{4}\,(5\,\tau_3^2 - 1) \;\leq\; \tau_4 \;<\; 1$. These results are proved in Hosking [58] and Jones [65]. Nair and Vineshkumar [100] presented various properties of the first two L-moments of residual life and their relevance in different aspects of reliability analysis as well as in economics. The second L-moment of residual life was found to be a better measure of variability when compared to variance residual quantile function.

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the population. Suppose $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the ordered sample observations. Then the rth sample L-moment is defined by

$$l_r = \sum_{k=0}^{r-1} p_{r-1,k}\; b_k, \tag{1.1.27}$$

where

$$p_{r-1,k} = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k}$$

and

$$b_k = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \frac{(i-1)(i-2)...(i-k)}{(n-1)(n-2)...(n-k)} X_{(i)}, \qquad \text{for } \kappa = 0, 1, 2, ...n-1. \tag{1.1.28}$$

Hosking [60] derived the asymptotic properties of the L-moments. The following theorem provides asymptotic normality of the sample L-moments.

Theorem 1.1.1. (Hosking [60]) Let X be a real valued random variable with quantile function $Q(u, \theta)$, where θ is a vector of m parameters, L-moments L_r and finite variance. Let $l_r,\; r = 1, 2, 3, ...m$ be sample L-moments calculated from a random sample of size n drawn from the distribution of X. Then $\sqrt{n}(l_r - L_r)$, $r = 1, 2, ..., m$, converges to the multivariate normal distribution $N(0, \Lambda)$, where the elements $\Lambda r, s$ $(r, s = 1, 2, 3, ..., m)$

of Λ are given by

$$\Lambda_{r,s} = \iint\limits_{0<u<v<1} \{P^*_{r-1}(u)\, P^*_{s-1}(v) + P^*_{s-1}(u)\, P^*_{r-1}(v)\}\, u(1-v)q(u)q(v)dudv, \quad (1.1.29)$$

with

$$P^*_r(x) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\binom{r+k}{k}x^k,$$

where $P^*_r(x)$ is the rth order shifted Legendre polynomial.

1.2 Quantile based reliability concepts

There exist several quantile function models which do not have a closed form expression for the corresponding distribution function. The equivalent definitions of various reliability measures in terms of quantile functions are necessary for the analysis of lifetime data using quantile functions. Nair and Sankaran [97] presented basic reliability measures in terms of quantile functions and established their inter-relationships. In this section, we give a brief review of the basic concepts and results of quantile-based reliability measures which are of use in the sequel and are referred to in the text. We assume that the support of X is $(0, \infty)$ and the distribution function $F\,(x)$ is continuous and strictly increasing.

1.2.1 Hazard quantile function

One of the basic concepts in reliability theory is the hazard rate (failure rate). The hazard rate function of a random variable X, denoted by $h(x)$ is defined as

$$h(x) = \lim_{\Delta t \to 0} \frac{P[x \leq X < x + \Delta x | X > x]}{\Delta x}. \tag{1.2.1}$$

$h(x)$ gives the instantaneous rate of failure of X in a small interval $(x,\, x + \Delta x)$ given the survival of the unit at time x. For an absolutely continuous random variable X with probability density function $f(x)$, (1.2.1) reduces to

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{d}{dx}\left[-\log \bar{F}(x)\right]. \tag{1.2.2}$$

Since $F(0) = 0$, by integrating (1.2.2) over $(0, x)$, we get

$$\bar{F}(x) = \exp\left[-\int_0^x h(t)dt\right]. \tag{1.2.3}$$

Thus $h(x)$ characterizes the underlying distribution through the identity (1.2.3). For more properties and applications of $h(x)$, one could refer to Lai and Xie [78].

By setting $F(x) = u$ in (1.2.2), Nair and Sankaran [97] defined the hazard quantile function, which is the quantile version of the hazard rate $h(x)$. The hazard quantile function, denoted by $H(u)$ is defined as

$$H(u) \; = \; h(Q(u)) = [(1-u)q(u)]^{-1}. \tag{1.2.4}$$

Now $H(u)$ can be interpreted as the conditional instantaneous rate of failure of a unit in the next small interval of time given the survival of the unit until $100(1-u)$ % point of the distribution. Note that $H(u)$ uniquely determines the distribution using the identity,

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{1.2.5}$$

Thus $H(u)$ uniquely determines $F(x)$. Moreover, Nair and Sankaran [97] showed that the monotonicity properties of $h(x)$ and $H(u)$ are same. Thus the failure pattern of a unit or a system can be analysed through its hazard quantile function.

1.2.2 Mean residual quantile function

Mean residual life (MRL) is a well-known measure, which has been widely used for modelling lifetime data in reliability and survival analysis. For a non-negative random variable X, the mean residual life function is defined as

$$m(x) = E(X-x|X>x) = \frac{1}{1-F(x)} \int_x^\infty (1-F(t))dt. \tag{1.2.6}$$

Note that $m(x)$ represents the average lifetime remaining for a component, which has already survived up to time x. In terms of the density function, (1.2.6) can be written as

$$m(x) = \frac{1}{\bar{F}(x)} \int_x^\infty (x-t)f(t)dt. \tag{1.2.7}$$

On differentiating (1.2.7) with respect to x, we get

$$h(x) = \frac{1+m'(x)}{m(x)}. \tag{1.2.8}$$

Now from (1.2.3), we have

$$\bar{F}(x) = \frac{m(0)}{m(x)} \exp\left\{ -\int_0^x \frac{dx}{m(x)} \right\}. \tag{1.2.9}$$

From (1.2.9), we see that $m(x)$ uniquely determines the distribution of X. Basic properties and applications of $m(x)$ are studied by Guess and Proschan [48] and Nanda et al. [110]. The mean residual life has been extensively discussed in reliability theory by various authors such as Bryson and Siddiqui [21], Barlow and Proschan [11] and Muth [96]. The properties including the limiting distribution have been studied by Meilijson [88], Balkema and De Haan [8] and Bradley and Gupta [20]. A smooth estimator of the mean residual life is given by Chaubey and Sen [24]. Chen and Cheng [25] and Nanda et al. [110] studied the proportional mean residual life model for the analysis of survival data. Abouammoh and El-Neweihi [1], Gupta and Kirmani [49], Abu-Youssef [2] and Ahmad and Mugdadi [4] have discussed various properties of the class of distributions generated by monotonic mean residual life function. For more details on $m(x)$, we refer to Sankaran and Sunoj [133], Gupta and Kirmani [50], Nair and Sankaran [98] and Nanda [108].

Nair and Sankaran [97] introduced the quantile version of mean residual life function, called mean residual quantile function. The mean residual quantile function denoted by $M(u)$ is obtained by letting $F(x) = u$ has the form

$$M(u) = m(Q(u)) = (1-u)^{-1} \int_u^1 (Q(p) - Q(u))\, dp. \tag{1.2.10}$$

In terms of the quantile density function, (1.2.10) can be expressed as

$$M(u) = (1-u)^{-1} \int_u^1 (1-p)\; q(p)\, dp. \tag{1.2.11}$$

Using (1.2.4), we have

$$M(u) = (1-u)^{-1} \int_u^1 (H(p))^{-1}\, dp. \tag{1.2.12}$$

Differentiating (1.2.12), we get

$$(H(u))^{-1} = M(u) - (1-u)\, M'(u). \tag{1.2.13}$$

Thus (1.2.5) becomes,

$$Q(u) = \mu - M(u) + \int_0^u (1-p)^{-1} M(p)\, dp, \tag{1.2.14}$$

where $\mu = M(0) < \infty$. Thus $M(u)$ uniquely determines $Q(u)$. $M(u)$ is interpreted as the mean remaining life of a unit beyond the $100(1-u)\%$ of the distribution. The exponential distribution with mean λ is characterized by the constant mean residual quantile function.

1.2.3 Residual variance quantile function

The variance residual life function of the random variable X with $E(X) < \infty$ is defined as

$$\begin{aligned} \sigma^2(x) &= V(X - x | X > x) \\ &= \frac{1}{\bar{F}(x)} \int_x^\infty (t-x)^2 f(t) dt - m^2(x). \end{aligned} \tag{1.2.15}$$

Researchers such as Launer [81], Gupta et al. [53] and Gupta and Kirmani [51] studied various important reliability properties of $\sigma^2(x)$. Unlike mean residual life and hazard rate functions, there does not exist an explicit formula to determine $F(x)$ from $\sigma^2(x)$. This brings in the importance of characterizing various families of distributions based on $\sigma^2(x)$.

Nair and Sudheesh [99] presented some interesting characterization results in this direction.

The quantile form of the variance residual life function is known as the residual variance quantile function, which is defined as

$$\begin{aligned} V(u) &= \sigma^2(Q(u)) \\ &= \frac{1}{1-u}\int_u^1 Q^2(p)\,dp - (M(u)+Q(u))^2\,. \end{aligned} \tag{1.2.16}$$

Nair and Sankaran [97] showed that,

$$V(u) = \frac{1}{1-u}\int_u^1 M^2(p)\,dp \quad \text{or} \quad M^2(u) = V(u) - (1-u)V'(u). \tag{1.2.17}$$

It follows from (1.2.17) that $V(u)$ and $M(u)$ determines each other uniquely. Thus $V(u)$ characterizes the underlying distribution. This result highlights the importance of residual variance quantile function over the variance residual life function defined in (1.2.15). For details, we refer to Nair et al. [105].

1.2.4 Percentile residual quantile function

For $\alpha \in (0,1)$, the αth percentile residual life is the αth percentile of the residual life distribution of X. It is denoted by $P^*_\alpha(x)$ and is defined by

$$P^*_\alpha(x) = F_x^{-1}(\alpha) = F^{-1}\left(1-(1-\alpha)\,\bar{F}(x)\right) - x. \tag{1.2.18}$$

$P^*_\alpha(x)$ can be interpreted as the age that will be survived, on the average, by $100(1-\alpha)\%$ of units that have lived beyond age x. For various properties and applications of $P^*_\alpha(x)$, one could refer to Launer [82], Schmittlein and Morrison [138] and Gupta and Langford [52].

Now by taking $F(x) = u$, in (1.2.18), the percentile residual quantile function denoted by $P_\alpha(u)$ is defined as

$$P_\alpha(u) = P_\alpha^*(Q(u)) = Q[1-(1-\alpha)(1-u)] - Q(u). \tag{1.2.19}$$

Nair et al. [105] have discussed various reliability properties and applications of $P_\alpha(u)$.

1.2.5 Quantile based reliability concepts in reversed time

There exist several important reliability concepts defined in reversed time. These concepts are useful to analyse the failure pattern of the lifetime random variable X under the condition $X \leq x$. In this section, we present the quantile form of such measures and its importance in reliability analysis. The definitions are listed below.

1.2.5.1 Reversed hazard quantile function

The reversed hazard rate denoted by $\lambda(x)$ is defined by

$$\lambda(x) = \frac{f(x)}{F(x)},$$

which was introduced by Keilson and Sumita [70] and then it has been widely used in several contexts such as stochastic ordering, estimation and modelling of left censored data and characterization of probability distributions. The quantile form of the reversed hazard rate $\lambda(x)$ denoted by $\Lambda(u)$ is called the reversed hazard quantile function. Nair and

Sankaran [97] defined $\Lambda(u)$ as

$$\Lambda\left(u\right)=\lambda\left(Q\left(u\right)\right)=\left(u\,q\left(u\right)\right)^{-1}. \tag{1.2.20}$$

$\Lambda\left(u\right)$ determines the distribution through the formula,

$$Q\left(u\right)=\int_{0}^{u}\left(P\Lambda\left(p\right)\right)^{-1}dp, \tag{1.2.21}$$

and the identity connecting $\Lambda\left(u\right)$ and $H\left(u\right)$ is

$$H\left(u\right)=\left(1-u\right)^{-1}u\,\Lambda\left(u\right).$$

More properties and interrelations between $\Lambda\left(u\right)$ with other quantile-based measures are given in Nair et al. [105].

1.2.5.2 Reversed mean residual quantile function

The reversed mean residual life function has the expression,

$$r(x)=\frac{1}{F(x)}\int_{0}^{x}F(t)dt.$$

Researchers such as Finkelstein [39], Nanda et al. [109] and Li and Garrido [85] studied various properties and applications of $r(x)$. The reversed mean residual quantile function denoted by $R\left(u\right)$ is the quantile version of $r(x)$ and has the form

$$\begin{aligned}R\left(u\right)&=r\left(Q\left(u\right)\right)\\&=u^{-1}\int_{0}^{u}\left(Q\left(u\right)-Q\left(p\right)\right)dp\end{aligned}$$

$$= u^{-1} \int_0^u p\, q\,(p)\, dp. \tag{1.2.22}$$

The reversed mean residual life has been studied by many researchers such as Finkelstein [39], Nanda et al. [109] and Li and Garrido [85]. Important reliability properties and applications of the reversed mean residual quantile function can be found in Nair et al. [105].

1.2.6 Total time on test transform

The total time on test transform is a widely accepted statistical tool, which has many applications in reliability analysis (see Lai and Xie [78]). This was first studied in the early 1970s, see for example Barlow et al. [9], Barlow and Doksum [10]. When several units are tested for studying their life lengths, some of the units would fail while others may survive the test period. The sum of all observed and incomplete life lengths is the total time on test statistic. As the number of units on test tends to infinity, the limit of this statistic is called the total time on test transform (TTT). The TTT is basically a quantile-based concept, although it is defined in terms of $F(x)$ in the literature.

For a non-negative random variable X, the TTT transform is defined as

$$H_F^{-1}(u) = \int_0^{F^{-1}(u)} \bar{F}(t)\, dt. \tag{1.2.23}$$

The quantile-based TTT introduced by Nair et al. [102] has the form

$$T(u) = \int_0^u (1-p) q(p) dp. \tag{1.2.24}$$

On differentiating (1.2.24), we get $T'(u) = (1-u)q(u)$ and hence the quantile function can be uniquely determined from $T(u)$ as

$$Q(u) = \int_0^u \frac{T'(p)}{1-p} dp. \tag{1.2.25}$$

Nair et al. [102] found various properties and applications of $T(u)$ in the context of reliability analysis.

1.2.7 Parzen score function

The Parzen score function defined by Parzen [117] is given by

$$J(u) = \frac{q'(u)}{q^2(u)}. \tag{1.2.26}$$

Parzen [117] discussed the role of $J(u)$ in statistical data modelling using quantile functions. This measure can be viewed as an equivalent representation (if we set $x = Q(u)$) of the function $\eta(x) = \frac{-f'(x)}{f(x)}$, introduced and studied by Glaser [43]. Nair et al. [103] presented various reliability properties of $J(u)$, which are useful in the context of the lifetime data modelling. From (1.2.26), we get

$$q(u) = \left(\int_u^1 J(p)\, dp \right)^{-1}, \tag{1.2.27}$$

Thus $J(u)$ uniquely determines the distribution. Nair et al. [103] introduced some new methods using Parzen score function for constructing new flexible quantile functions with monotonic as well as non-monotonic hazard quantile functions. A detailed review of this topic is available in Nair et al. [105].

1.3 Quantile function models

We already mentioned that the quantile functions are efficient alternative to distribution functions for modelling different types of lifetime data. In this section, we present some quantile function models available in the literature, that can be utilized for the analysis of different types of lifetime data sets.

1.3.1 Lambda family of distributions

Tukey [152] introduced a simple one parameter quantile function known as the Tukey lambda distribution. This is the basic model from which several extensions and generalizations were done by many researchers in the literature. The quantile function of the model is given by

$$Q(u) = \begin{cases} \frac{u^{\lambda}-(1-u)^{\lambda}}{\lambda}, & \lambda \neq 0 \\ \frac{\log(u)}{1-u}, & \lambda = 0. \end{cases} \tag{1.3.1}$$

We can see that the quantile function (1.3.1) is a linear combination of the Pareto and power quantile functions. The quantile density function and hazard quantile function of the model are

$$q(u) = (1-u)^{\lambda-1} + u\lambda - 1,$$

and

$$H(u) = u\left(u(1-u)^{\lambda} + u^{\lambda}(1-u)\right)^{-1}.$$

respectively. The mean residual quantile function takes the form

$$M(u) = \left(u^{\lambda}(\lambda(u-1)-1) + \lambda((1-u)^{\lambda} - u(1-u)^{\lambda}) + 1\right)\left(\lambda(\lambda+1)(1-u)\right)^{-1}.$$

Researchers like Joiner and Rosenblatt [64] and Shapiro and Wilk [143] studied some asymmetric versions of the Tukey lambda distribution. All such versions are subsumed in the generalized lambda distribution family introduced by Ramberg and Schmeiser [123] with quantile function,

$$Q(u) = \lambda_1 + \frac{1}{\lambda_2}\left(u^{\lambda_3} - (1-u)^{\lambda_4}\right), \quad 0 \le u \le 1,$$

where λ_1 is the location parameter, λ_2 is the scale parameter, while λ_3 and λ_4 determines the shape. This model is widely accepted for modelling and analysis of various types of lifetime data due to its versality and special properties. Hazard quantile function has the form

$$H(u) = \frac{\lambda_2}{(1-u)\left(\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}\right)},$$

and mean residual quantile function becomes

$$M(u) = \frac{1}{\lambda_2(1-u)}\left(\frac{1-u^{\lambda 3+1}}{\lambda_3+1} - (1-u)u^{\lambda_3} + \frac{\lambda_4(1-u)^{\lambda_4+1}}{\lambda_4+1}\right).$$

1.3.2 van Staden and Loots model

Consider the quantile function of the generalized Pareto model,

$$Q_1(u) = \begin{cases} \frac{-1}{\lambda_4}\left((1-u)^{\lambda_4} - 1\right), & \lambda_4 \ne 0 \\ -\ln(1-u) & , \lambda_4 = 0 \end{cases} \tag{1.3.2}$$

and its reflection quantile function,

$$Q_2(u) = -Q_1(1-u) = \begin{cases} \frac{1}{\lambda_4}\left(u^{\lambda_4}-1\right), & \lambda_4 \neq 0 \\ \log u & , \lambda_4 = 0. \end{cases} \tag{1.3.3}$$

van Staden and Loots [153] introduced a new quantile function by taking the weighted sum of the quantile functions (1.3.2) and (1.3.3) with respective weights $1-\lambda_3$ and λ_3, where $0 \leqslant \lambda_3 \leqslant 1$, with additional location and scale parameters λ_1 and λ_2 respectively. Thus, the new quantile function is of the form

$$Q\left(u\right) = \lambda_1 + \lambda_2\left[\left(1-\lambda_3\right)\frac{u^{\lambda_4}-1}{\lambda_4} - \lambda_3\frac{\left(1-u\right)^{\lambda_4}-1}{\lambda_4}\right], \ \lambda_2 > 0. \tag{1.3.4}$$

Some well-known distributions such as exponential, logistic and uniform are special cases of this model. The hazard quantile function of (1.3.4) is

$$H(u) = \frac{1}{\lambda_2(1-u)\left(\left(1-\lambda_3\right)u^{\lambda_4-1} + \lambda_3(1-u)^{\lambda_4-1}\right)},$$

and the mean residual quantile function is obtained as

$$M(u) = \lambda_2\left(\frac{\left(1-\lambda_3\right)\left(\frac{1-u^{\lambda_4+1}}{(\lambda_4+1)(1-u)} - u^{\lambda_4}\right)}{\lambda_4} - \frac{\lambda_3(1-u)^{\lambda_4}}{\lambda_4+1}\right).$$

1.3.3 Power-Pareto distribution

Gilchrist [42] introduced the power-Pareto distribution with quantile function

$$Q(u) = \frac{Cu^{\lambda_1}}{\left(1-u\right)^{\lambda_2}}, \ C, \lambda_1, \lambda_2 > 0. \tag{1.3.5}$$

The model (1.3.5) is the product of quantile functions of the power and Pareto distributions with respective quantile functions $Q_1(u) = \alpha u^{\lambda_1}$ and $Q_2(u) = \sigma(1-u)^{-\lambda_2}$, with $C = \alpha\,\sigma$ and $\alpha,\ \lambda_1,\ \lambda_2 > 0$. Researchers such as Gilchrist [42], Hankin and Lee [55] and Nair et al. [105] have presented various properties and characterizations of this quantile function. The hazard quantile function and mean residual quantile function are respectively given by

$$H(u) = \frac{(1-u)^{\lambda_2}}{C\,u^{\lambda_1 - 1}\left(\lambda_1(1-u) + \lambda_2 u\right)},$$

and

$$M(u) = \frac{1}{1-u} B_u(1-\lambda_{2,} 1+\lambda_1) - \frac{C u^{\lambda_1}}{(1-u)^{\lambda_2}}.$$

1.3.4 Govindarajulu distribution

The earliest attempt for modelling failure time data using quantile function which does not have a closed form expression for its distribution function was done by Govindarajulu [45]. He developed the quantile function,

$$Q(u) = \theta + \sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right), \qquad \theta,\ \sigma,\ \beta > 0. \tag{1.3.6}$$

This model was applied to a real life data on the failure times of a set of 20 refrigerators which were ran to destruction under advanced stress conditions. Since the range of the distribution is $(\theta, \theta + \sigma)$, for lifetime studies, we can set θ to be zero, and hence (1.3.6) reduces to

$$Q(u) = \sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right), \qquad \theta,\ \sigma,\ \beta > 0. \tag{1.3.7}$$

Later, Nair et al. [104] have established some important reliability properties and characterizations of the Govindarajalu distribution. The hazard quantile function and mean residual quantile function are obtained by

$$H(u) = \left[\sigma\beta(\beta+1)u^{\beta-1}(1-u)^2\right]^{-1}, \text{ and}$$

$$M(u) = \left[2-(\beta+1)(\beta+2)u^{\beta}+\beta(\beta+2)u^{\beta+1}-\beta(\beta+1)u^{\beta+2}\right]\times[(\beta+2)(1-u)]^{-1}\sigma$$

respectively.

1.3.5 Jones distribution

Jones [66] introduced a new class of distributions using the functional relation between $f(x)$ and $F(x)$ given by

$$f(x) = (F(x))^{\alpha}(1-F(x))^{\beta},\ \alpha,\ \beta \in \mathbb{R} \tag{1.3.8}$$

where α and β are real constants. An equivalent form of (1.3.8) in terms of the quantile density function is

$$q(u) = u^{-\alpha}(1-u)^{-\beta},\ \ 0<u<1. \tag{1.3.9}$$

Hence the quantile function is obtained as

$$Q(u) = B_u(1-\alpha,\, 1-\beta),\ 0<u<1,\ \alpha,\beta<1, \tag{1.3.10}$$

where $B_u(\cdot,\cdot)$ represents the incomplete beta function. The hazard quantile function and the mean residual quantile function are respectively given by

$$H(u) = u^{\alpha}(1-u)^{\beta-1} \quad \text{and} \quad M(u) = \frac{\frac{\Gamma(1-\alpha)\Gamma(2-\beta)}{\Gamma(-\alpha-\beta+3)} - B_u(1-\alpha,\, 2-\beta)}{1-u}. \tag{1.3.11}$$

1.4 Censoring

In reliability studies, the lifetime data sets are mainly of two types, one where the lifetime is known exactly and the other where it is not. In the latter case, it is only known to have a certain amount of time when the event of interest did not occur. Data of this kind are called censored data. Different types of censoring are possible in life testing experiments and this depends on the criteria, which is used to conclude the experiment.

1.4.1 Type I censoring

Censoring which occurs as a function of time is known as type I censoring. Here we record the failure times of items which are failed prior to a particular point of time. Data of this type are called right censored. Data can also be left censored. This types of censoring arise when it is only known that the failures have occurred prior to some time. As compared to left censoring, right censoring is most common in reliability studies.

1.4.2 Type II censoring

Type I censoring may provide a limited amount of information if the period of study is too short. An alternative is to continue observation until a fundamental number $r(< n)$ of failures have occurred. This also results in incomplete data, since the lifetimes of the remaining $n - r$ units are not observed. This type of censoring is called type II censoring.

1.4.3 Independent random censoring

In this case, we assume that each individual has a lifetime X and a censoring time C, where X and C are mutually independent continuous random variables. Here we observe the variable (T, δ), where $T = \min(X, C)$ and the censoring indicator δ is defined as, δ = 1 if $T \leq C$ and 0 otherwise.

There are other kinds of censoring schemes such as interval and middle censoring. For more details, one could refer to Klein and Moeschberger [72].

1.5 Q-Q plot

The quantile-quantile (Q-Q) plot is an efficient graphical tool to assess if a data set plausibly came from the chosen candidate distribution. Let $X_{(r)}$ denotes the rth ordered observation in a data of size n, when the observations are arranged in ascending order of magnitude. Then Q-Q plot is the scatter plot obtained by plotting $(Q(u_r),\ X_{(r)})$ for $r = 1, 2, ..., n$. In the ideal case, the graph should show a straight line that bisects the axes of coordinates. Since the data are random, the fitted values lying approximately along the aforementioned line can be taken as an indication of a satisfactory model.

1.6 Chi-square test for goodness of fit

When two Q-Q plots looks very similar, it is useful to have some numerical measure to conclude the analysis. Though a number of methods are available for this purpose, one that is relevant to modelling with quantiles is the chi-square goodness of fit test. We first divide the interval of $u \in (0,1)$ into m equal parts using $u_i = \frac{i}{m}$, $i = 1, 2, ..., m-1$; $u_0 = 0$, $u_m = 1$. Let f_i be the number of observations in the data lying in the interval $(\hat{Q}(u_{i-1}), \hat{Q}(u_i))$, which is the observed frequency (O_i) in the ith interval. The expected value (E_i) of f_i is $\frac{n}{m}$ for all i. Using this fact, we can construct the test statistic, $\chi^2 = \sum_{i=1}^{m} \frac{(O_i - E_i)^2}{E_i} = \sum_{i=1}^{m} \frac{\left(f_i - \frac{n}{m}\right)^2}{\frac{n}{m}}$, which approximately follows the chi-square distribution with $m-1$ degrees of freedom. If the data under consideration is very different from the fitted quantile function, the value of χ^2 will be larger than indicated by a χ^2 distribution.

Throughout the thesis, the terms increasing and decreasing are used in a wide sense, that is, a function f is increasing (decreasing) if $f(x) \leq (\geq)\, f(y)$ for all $x \leq y$. Whenever we use a derivative, an expectation, or a conditional random variable, we are tacitly assuming that it exists.

1.7 Objectives and present study

We have already seen that quantile functions have several advantages over the distribution functions, so that we can efficiently employ quantile functions for the analysis of various types of lifetime data. The literature provides a number of quantile functions, which are useful in the context of lifetime studies. One of the main objectives of the present work is to introduce new flexible lifetime models based on quantile functions with no closed form distribution functions. We will investigate important ageing properties and stochastic orders of the proposed models. Also, we examine the adequacy of these models to represent real life situations in the light of various data sets. Another objective of our study is to develop quantile-based definitions of some important reliability concepts such as relevation transform and proportional odds model and study the relevance of these representations in reliability analysis. In addition, we introduce the concept of cause specific hazard quantile function and develop a kernel-based non-parametric estimator for the same, which is useful for the analysis of failure pattern in competing risks set up.

The present work is organized into nine chapters. After the present introductory chapter, which focuses attention on a brief review of the basic quantile-based reliability concepts and review of literature, Chapter 2 proposes a new quantile function called half logistic-exponential geometric quantile function. The new quantile function is the sum of quantile functions of the half logistic and exponential geometric distributions. Various distributional properties and reliability characteristics are discussed. The estimation of parameters of the model using the method of L-moments is studied. The proposed model is applied to a real life data.

As pointed out earlier, the product of two positive quantile functions is again a quantile function. Using this property, in Chapter 3, we introduce a new family of distributions by taking the product of quantile functions of the Pareto and Weibull distributions. Important reliability characteristics and distributional properties are discussed. The practical utility of the model is established with the help of two real life data sets.

In Chapter 4, we introduce a new method for constructing more flexible families of distributions using the properties of quantile functions. We derive a new class of distributions, which is an extension of the class of distributions with linear mean residual quantile function. Distributional properties and reliability measures are presented. The estimation of parameters of the model using the method of percentiles is studied. Application of the proposed model in lifetime data analysis is ascertained by fitting the model to a real data.

An important measure used in quantile-based reliability analysis to measure the expected remaining life beyond $100(1-u)\%$ point of the distribution is the mean residual quantile function. This can advantageously used as a measure to characterize any probability distribution. In Chapter 5, we propose a new class of distributions with quadratic mean residual quantile function. Various distributional properties along with reliability characteristics are discussed. Characterizations of the class of distributions using different quantile-based reliability measures are presented. The estimation of parameters of the model using the method of L-moments is studied. The proposed quantile function is applied to a real life data and compared the performance with other competing alternative models.

Even though a lot of work has been carried out on the well-known proportional odds model in the classical frame work, to the best of our knowledge, the approach based on quantile function is new. Chapter 6 presents a quantile-based definition for the proportional odds model. We discuss important reliability properties of the model using quantile functions. Various ageing properties of the model are derived. A generalization for the class of distributions with homo-graphic hazard quantile function is provided and the practical application of this model is illustrated with two real life data sets.

In survival studies, it is common that the failure of subjects may be attributed to more than one cause. Competing risks models are usually employed to analyze such type of data. Two frameworks are often employed to deal with standard competing risks data such as cumulative incidence function formulations and cause specific hazard formulations. In Chapter 7, we discuss modelling and analysis of competing risks data using quantile functions. We introduce and study the cause specific hazard quantile function. Moreover, we present new competing risks models using various functional forms for the cause specific hazard quantile functions. A non-parametric estimator of the cause specific hazard quantile function is also derived. Asymptotic properties of the estimators are studied. Simulation studies are carried out to assess the performance of the estimators. We apply the proposed procedure to two real life data sets.

Relevation transform introduced by Krakowski [74] is extensively studied in the literature. Chapter 8 presents a quantile-based definition of the relevation transform and study its properties in the context of lifetime data analysis. We study the reliability properties of a special case of relevation transform namely proportional hazards relevation transform. Various stochastic orders and ageing concepts are discussed. A new lifetime distribution called proportional hazards relevated Weibull (PHRW) is introduced and discussed its applications with two real data sets. We also give important special cases of relevation transform in the context of proportional hazards and equilibrium models in terms of quantile functions.

Finally, Chapter 9 summarizes the major conclusions of the thesis and present some proposals for future research.

Chapter 2

Half Logistic-Exponential Geometric Quantile Function

2.1 Introduction

A major objective of the quantile-based reliability analysis is to develop new quantile functions that are useful for modelling and analysis of lifetime data sets. In the present chapter, we introduce and study a new flexible family of distributions defined in terms of a quantile function, which does not have a closed form expression for its distribution function. Quantile functions have several properties that are not shared by distribution functions. For example, the sum of two quantile functions is again a quantile function. Further, the product of two positive quantile functions is again a quantile function in the non-negative setup. The aim of the present chapter is to introduce a new quantile function which is derived by taking the sum of quantile functions of the half logistic and the exponential geometric distributions. The proposed class gives a wide variety of distributional shapes for various choices of the parameters.

The chapter is organized as follows. In Section 2.2, we present a family of distributions and

Results in this chapter have been published as entitled "A new class quantile functions useful in reliability analysis" in the "Journal of Statistical Theory and Practice" (See Sankaran and Kumar M [131].)

study its basic properties. Section 2.3 presents some well-known distributions which are either a member of the proposed class of distributions or obtained by applying some suitable transformations on the proposed quantile function. The distributional properties such as measures of location and scale, L-moments are given in Section 2.4. In Section 2.5, we present various reliability characteristics of the class. Section 2.6 focuses on the inference procedures associated with the parameters of the model. We then provide application of the class of distributions in a real life situation. Finally, Section 2.7 provides major conclusions of this chapter.

2.2 Half logistic - exponential geometric (HLEG) quantile function

Balakrishnan [7] considered the folded form of the standard logistic distribution and referred it as the half logistic distribution. The survival function and quantile function of this distribution are respectively given by

$$\bar{G}_1(x) = 2\left(1 + e^{\frac{x}{\beta}}\right)^{-1}, \quad \beta \geq 0, \tag{2.2.1}$$

and

$$Q_1(u) = \beta \log\left(\frac{1+u}{1-u}\right), \quad \beta \geq 0. \tag{2.2.2}$$

The model (2.2.1) is a possible lifetime model, which has several characterizations based on the recurrence relations for the single and product moments of its order statistics.

Adamidis and Loukas [3] introduced the exponential geometric (EG) distribution with applications to reliability modelling in the context of decreasing failure rate data. The survival function and quantile function of the EG distribution are given by,

$$\bar{G}_2(x) = 1 - G_2(x) = (1-p)e^{-\frac{1}{\alpha}x}(1 - p\,e^{-\frac{1}{\alpha}x})^{-1}, \quad \alpha > 0 \text{ and } 0 \leq p < 1. \qquad (2.2.3)$$

and

$$Q_2(u) = \alpha \log\left(\frac{1-pu}{1-u}\right), \quad \alpha > 0 \text{ and } 0 \leq p < 1. \qquad (2.2.4)$$

We now introduce a new class of distributions through the quantile function

$$\begin{aligned} Q(u) &= Q_1(u) + Q_2(u) \\ &= \alpha \log\left(\frac{1-pu}{1-u}\right) + \beta \log\left(\frac{u+1}{1-u}\right), \quad 0 \leq p \leq 1, \alpha \geq 0, \beta \geq 0. \end{aligned} \qquad (2.2.5)$$

Thus $Q(u)$ is the sum of (2.2.2) and (2.2.4) with extended parameter space. The support of the proposed class of distributions (2.2.5) is $(0, \infty)$. The quantile density function is obtained as

$$q(u) = \frac{2\beta + \alpha((1-p))(u+1) - 2\beta pu}{(u^2-1)(pu-1)}. \qquad (2.2.6)$$

The quantile function (2.2.5) represents a family of distributions with neither the density nor the distribution function available in closed form. However, these can be calculated by numerical inversion of the quantile function. For the proposed class of distributions, the density function $f(x)$ can be written in terms of the distribution function as

$$f(x) = \frac{(1-pF(x))(1-(F(x)^2)}{\alpha(1-p)(1+F(x)) + 2(1-pF(x))\beta}. \qquad (2.2.7)$$

For all values of the parameters, the density is strictly decreasing in x and it tends to zero as $x \to \infty$. Plots of the density function for different combinations of parameters are shown in Figure 2.1. The mode of the distribution is at zero and the modal value is $\frac{1}{2\beta+\alpha(1-p)}$.

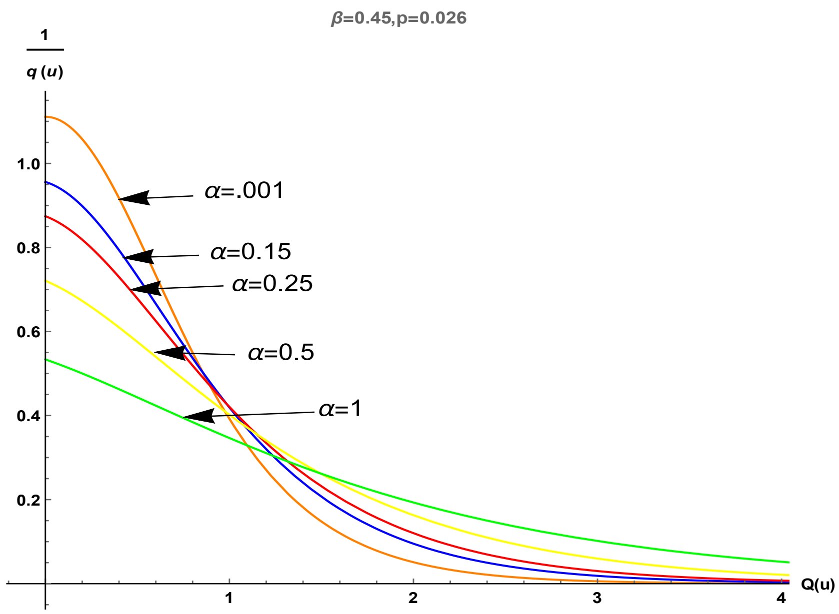


Figure 2.1: Plots of the density function for different values of parameters.

2.3 Members of the family

The proposed family of distributions (2.2.5) includes several well-known distributions for various values of the parameters. Further, we can derive certain popular distributions from the proposed model by making use of various transformations described in Gilchrist [42].

Case 1. $\boldsymbol{\beta = 0, p = 0}$ **and** $\boldsymbol{\alpha > 0}$.

The quantile function of the proposed class of distributions reduces to the quantile function,

$$Q(u) = \alpha(-\log(1-u)), \tag{2.3.1}$$

which is the exponential distribution with mean α. We can apply the power transformation of the form $T(x) = x^K$ on (2.3.1) to form the Weibull distribution with parameters α and K.

Case 2. $\boldsymbol{\alpha = \beta}$ **and** $\boldsymbol{p = 1}$.

The quantile function of the proposed class of distributions becomes,

$$Q(u) = \alpha \log\left(\frac{1+u}{1-u}\right), \tag{2.3.2}$$

which corresponds to the half-logistic distribution. Moreover, this belongs to the class of distributions with linear hazard quantile functions introduced and studied by Midhu et al. [91], with quantile function,

$$Q(u) = \frac{1}{a(1+\theta)} \log\left(\frac{1+\theta u}{1-u}\right), \tag{2.3.3}$$

with $\theta = 1$ and $a = \frac{1}{2\alpha}$.

Case 3. $\boldsymbol{\beta = 0, \alpha > 0}$ **and** $\boldsymbol{0 < p < 1}$.

The quantile function of the proposed class of distributions reduces to the quantile function of the exponential geometric distribution,

$$Q(u) = \alpha \log\left(\frac{1-pu}{1-u}\right). \tag{2.3.4}$$

This also belongs to the class of distributions (2.3.3), with parameters $\theta = -p, (-1 < \theta < 0)$ and $a = \frac{1}{\alpha(1-p)}$.

Case 4. $\boldsymbol{p = 0, \alpha > 0}$ **and** $\boldsymbol{\beta > 0}$.

The quantile function is obtained as

$$Q(u) = \frac{(A-B)\log(1+Au) - A(B+1)\log(1-u)}{A(A+1)K}, \tag{2.3.5}$$

where $K = \frac{1}{\alpha+2\beta}$, $A = 1$ and $B = \frac{\alpha}{\alpha+2\beta}$. The quantile function (2.3.5) corresponds to the family of distributions with homographic hazard quantile function given in Sankaran et al. [135].

In the construction of our family, the sum of two quantile functions are involved. In the following theorems, we derive the random variable associated with the proposed quantile function (2.2.5).

Theorem 2.3.1. If $Z \sim HL(\beta)$, then the random variable $X = Z+\alpha\log\left(\frac{(1+p)+(1-p)exp\left(\frac{Z}{\beta}\right)}{2}\right)$ has $HLEG(\alpha,\beta,p)$ distribution.

Proof. Consider two random variables S and T with quantile functions $Q_S(u)$ and $Q_T(u)$ and distribution functions $F_S(x)$ and $F_T(x)$ respectively.
Now, suppose $Q^*(u)$ is defined by,

$$Q^*(u) = Q_S(u) + Q_T(u).$$

Then the random variable corresponds to the quantile function $Q^*(u)$ is $S + Q_T(F_S(S))$ or $T + Q_S(F_T(T))$(Sankaran et al. [136]).
Now take $Y \sim EG(\alpha, p)$ and $Z \sim HL(\beta)$, then we have $Z+Q_Y(F_Z(Z))$ has $HLEG(\alpha,\beta,p)$ distribution.
Since $Q_Y(u) = \alpha\log\left(\frac{1-pu}{1-u}\right)$ and $F_Z(Z) = 1-2\left(1+exp\left(\frac{Z}{\beta}\right)\right)^{-1}$, we get

$$Z + Q_Y(F_Z(Z)) = Z + \alpha\log\left(\frac{(1+p)+(1-p)exp\left(\frac{Z}{\beta}\right)}{2}\right), \tag{2.3.6}$$

which completes the proof. ■

Theorem 2.3.2. If $Y \sim EG(\alpha, p)$, then the random variable $X = Y+\beta\log\left(\frac{2\left(1-pexp\left(-\frac{Y}{\alpha}\right)\right)}{(1-p)exp\left(-\frac{Y}{\alpha}\right)}\right)$ has $HLEG(\alpha,\beta,p)$ distribution.

Proof. The proof follows along the same lines as in Theorem 2.3.1, once we note that if $Y \sim EG(\alpha, p)$ and $Z \sim HL(\beta)$, then $Y + Q_Z(F_Y(Y))$ has $HLEG(\alpha, \beta, p)$ distribution, and therefore the details are omitted. ■

2.4 Distributional characteristics

The quantile-based measures of distributional characteristics for location, dispersion, skewness and kurtosis are popular in statistical analysis. These measures are also useful for estimating parameters of the model by matching population characteristics with corresponding sample characteristics. For the model (2.2.5), basic descriptive measures such as median (M), inter-quartile-range (IQR), Galton's coefficient of skewness (S) and Moor's coefficient of kurtosis (T) are obtained as

$$M = \alpha \log(2-p) + \beta \log(3),$$

$$IQR = \alpha \log\left(\frac{12-9\,p}{4-p}\right) + \beta \log\left(\frac{21}{5}\right),$$

$$S = \frac{1.43\beta - 2(1.09\beta + \alpha \log(2.(1-0.5p))) - \alpha \log(1.33(1-0.25p)) + \alpha \log(4.(1-0.75p))}{1.43\beta - \alpha \log(1.33(1-0.25p)) + \alpha \log(4.(1-0.75p))},$$

and

$$T = \frac{\alpha(-0.69 \log(1.14-0.14p) + 0.7 \log(1.6-0.6p) - 0.7 \log(2.67-1.7p) + 0.7 \log(8-7p)) + 1.24\beta}{-0.7\alpha \log(1.34-0.34p) + 0.7\alpha \log(4-3p) + \beta}.$$

The rth L moment is given by

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-1-k} \begin{pmatrix} r-1 \\ k \end{pmatrix} \begin{pmatrix} r-1+k \\ k \end{pmatrix} u^k Q(u)du.$$

For the model (2.2.5), first four L-moments are

$$L_1 = \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p}, \tag{2.4.1}$$

$$L_2 = \alpha + 2\beta - \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p^2} - \frac{\alpha}{p}, \tag{2.4.2}$$

$$L_3 = -4\beta + \beta\log(64) - \frac{\alpha(p-2)(p-1)\log(1-p)}{p^3} + \frac{2\alpha(p-1)}{p^2}, \tag{2.4.3}$$

and

$$L_4 = \frac{p\Big(4\beta p^3(23-33\log(2))+\alpha(p-1)((p-15)p+30)\Big)+6\alpha(p-1)((p-5)p+5)\log(1-p)}{6p^4}. \tag{2.4.4}$$

The L-coefficient of variation (τ_2), L-coefficient of skewness (τ_3) and L-coefficient of kurtosis (τ_4) have the following forms;

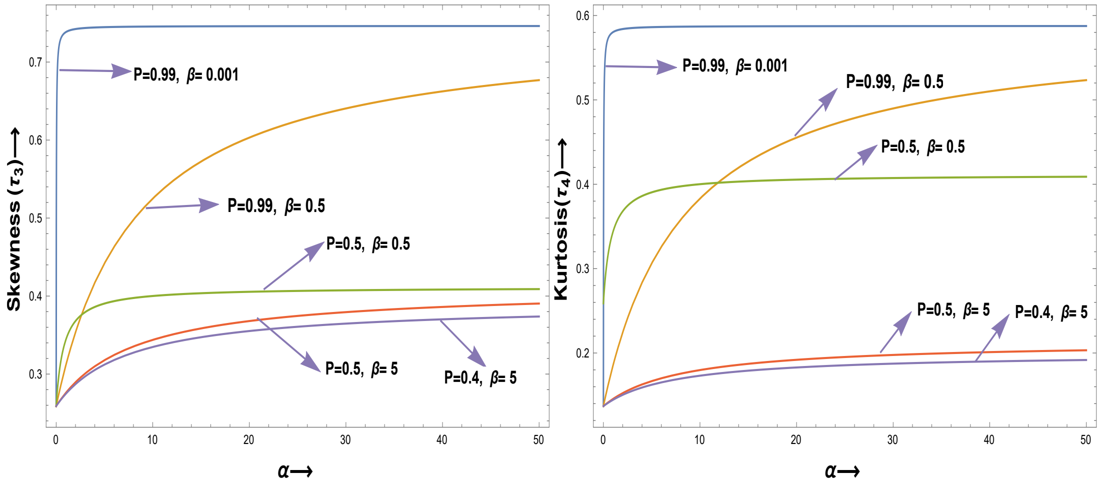
$$\tau_2 = \frac{L_2}{L_1} = \frac{\alpha + 2\beta - \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p^2} - \frac{\alpha}{p}}{\beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p}},$$

$$\tau_3 = \frac{L_3}{L_2} = \frac{-4\beta + \beta\log(64) - \frac{\alpha(p-2)(p-1)\log(1-p)}{p^3} + \frac{2\alpha(p-1)}{p^2}}{\alpha + 2\beta - \beta\log(4) + \frac{\alpha(p-1)\log(1-p)}{p^2} - \frac{\alpha}{p}},$$

and

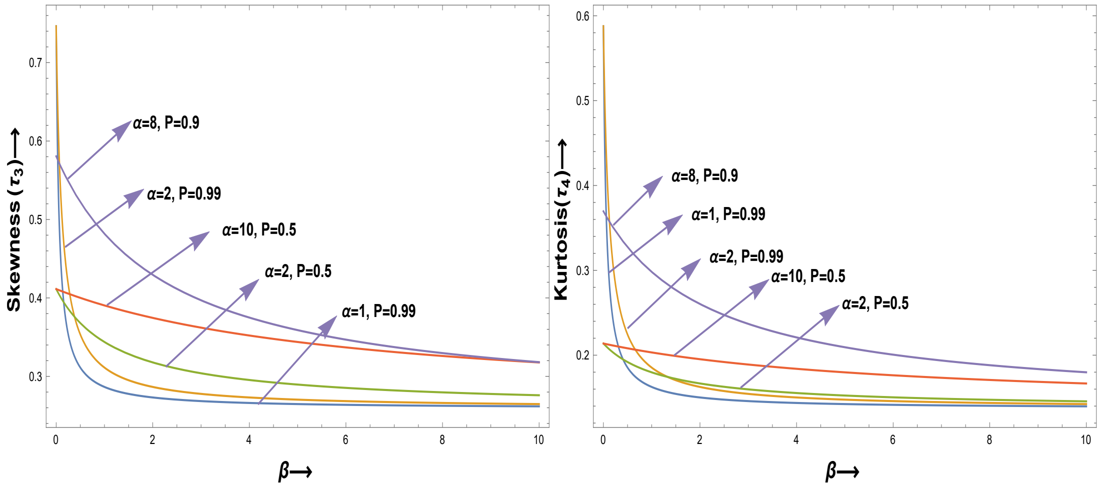
$$\tau_4 = \frac{L_4}{L_2} = \frac{p\,(4\beta p^3(23-33\log(2)) + \alpha(p-1)((p-15)p+30)) + 6\alpha(p-1)((p-5)p+5)\log(1-p)}{6p^2(p(\alpha(p-1) - \beta p(\log(4)-2)) + \alpha(p-1)\log(1-p))}.$$

Figures 2.2, 2.3 and 2.4 present skewness (τ_3) and kurtosis (τ_4) measures for various parameter values. We can show that τ_3 lies in (0.25, 1) and τ_4 lies in (0.12, 0.67) using numerical optimization techniques. Thus, the proposed class of distributions (2.2.5) consists only positively skewed distributions. The curves of τ_3 and τ_4 increase with α for fixed β and p, decrease with β for fixed α and p, and first increase and then decrease with p for fixed α and β.



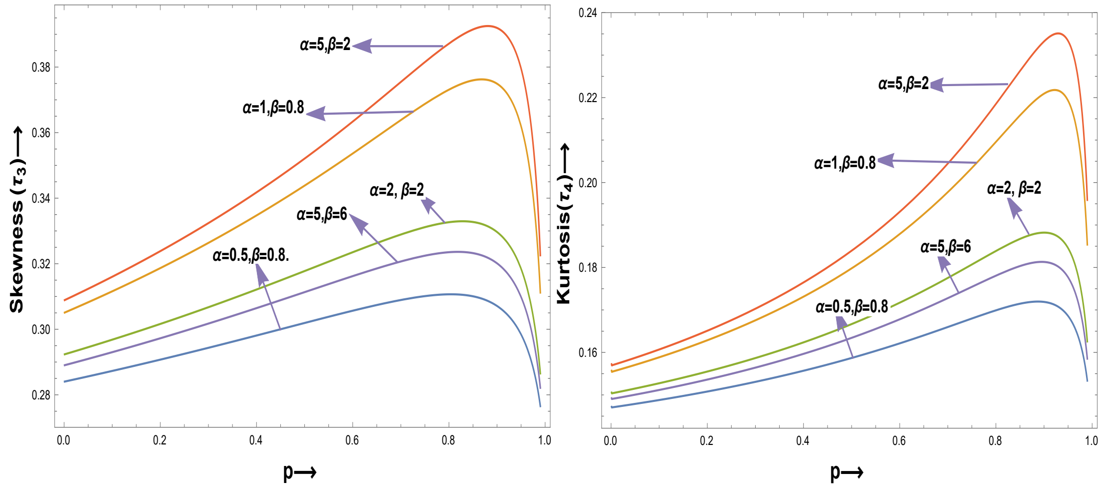
(a) Skewness(τ_3) (b) Kurtosis (τ_4)

Figure 2.2: Skewness and kurtosis of the $HLEG(\alpha, \beta, p)$ distribution for selected values of β and p as a function of the parameter α.



(a) Skewness(τ_3) (b) Kurtosis (τ_4)

Figure 2.3: Skewness and kurtosis of the $HLEG(\alpha, \beta, p)$ distribution for selected values of α and p as a function of the parameter β.



(a) Skewness(τ_3) (b) Kurtosis (τ_4)

Figure 2.4: Skewness and kurtosis of the $HLEG(\alpha, \beta, p)$ distribution for selected values of α and β as a function of the parameter p.

2.4.1 Order statistics

If $X_{r:n}$ is the rth order statistic in a random sample of size n, then the density function of $X_{r:n}$ can be written as

$$f_r(x) = \frac{1}{B(r, n-r+1)} f(x) F^{r-1}(x)(1 - F(x))^{n-r}.$$

From (2.2.7), we have

$$f_r(x) = \frac{1}{B(r, n-r+1)} \frac{(1 - F(x))^{n-r}(1 - pF(x))(1 - (F(x)^2)(F(x))^{r-1}}{\alpha(1-p)(1 + F(x)) + 2(1 - pF(x))\beta}.$$

Hence

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^\infty x \frac{(1 - F(x))^{n-r}(1 - pF(x))(1 - (F(x)^2)(F(x))^{r-1}}{\alpha(1-p)(1 + F(x)) + 2(1 - pF(x))\beta} dx.$$

In quantile terms, we have

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^1 Q(u) \frac{(1-u)^{n-r}(1-p\,u)(1-u^2)u^{r-1}}{\alpha(1-p)(1+u) + 2(1-p\,+u)\beta} dx.$$

The quantile function of the first order statistics $X_{1:n}$ has the form,

$$\begin{aligned} Q_{(1)}(u) &= Q(1-(1-u)^{\frac{1}{n}}) \\ &= \alpha \log\left(p - (p-1)(1-u)^{-1/n}\right) + \beta \log\left(2(1-u)^{-1/n} - 1\right), \end{aligned}$$

and nth order statistic $X_{n:n}$ has the quantile function,

$$\begin{aligned} Q_{(n)}(u) &= Q(u^{\frac{1}{n}}) \\ &= \alpha \log\left(\frac{1-pu^{1/n}}{1-u^{1/n}}\right) + \beta \log\left(\frac{u^{1/n}+1}{1-u^{1/n}}\right). \end{aligned}$$

2.5 Reliability properties

Since the proposed quantile function is the sum of exponential geometric and half logistic quantile functions, we get the relation

$$(1-u)q(u) = (1-u)q_1(u) + (1-u)q_2(u). \tag{2.5.1}$$

where $q_1(u)$ and $q_2(u)$ are the quantile density functions of exponential geometric and half logistic distributions respectively. Now from (1.2.4) and (2.5.1)

$$\frac{1}{H(u)} = \frac{1}{H_1(u)} + \frac{1}{H_2(u)}, \tag{2.5.2}$$

where $H_1(u)$, $H_2(u)$ and $H(u)$ are the hazard quantile functions of the exponential geometric, half logistic and the proposed class of distributions respectively. From (2.5.2), we can observe that the hazard quantile function of the proposed class of distributions (2.2.5) is proportional to the harmonic average of the hazard quantile functions of exponential geometric and half logistic distributions. For the class of distributions (2.2.5), we have

$$H(u) = \frac{(u+1)(pu-1)}{\alpha(p-1)(u+1) + 2\beta(pu-1)}. \tag{2.5.3}$$

The shape of the hazard function is determined by the derivative of $H(u)$, given by

$$H^{'}(u) = \frac{\alpha p(p-1)(u+1)^2 + 2\beta(pu-1)^2}{(\alpha(p-1)(u+1) + 2\beta(pu-1))^2}.$$

Since $(\alpha(p-1)(u+1) + 2\beta(pu-1))^2 > 0$ for all values of the parameters, the sign of $H'(u)$ depends only on the term,

$$K(u) = \alpha p(p-1)(u+1)^2 + 2\beta(pu-1)^2. \tag{2.5.4}$$

The hazard quantile function accommodates increasing, decreasing, linear and upside-down bathtub shapes for different choices of parameters. Plots of hazard quantile function for different values of parameters are given in Figure 2.5. Now we consider the following cases.

Case 1. $\boldsymbol{p = 0, \alpha > 0}$ **and** $\boldsymbol{\beta > 0}$.

We obtain $K(u) = 2\beta$. The first term in $K(u)$ is zero and the second term is positive, so that $K(u) > 0$ for all $0 < u < 1$ and the distribution have increasing hazard rate (IHR).

Case 2. $\boldsymbol{p = 1, \alpha > 0}$ **and** $\boldsymbol{\beta > 0}$.

We have $K(u) = 2\beta(u-1)^2$. The first term in $K(u)$ is zero and the second term is positive, so that $K(u) > 0$ for all $0 < u < 1$ and hence the distribution distribution is IHR.

Case 3. $\boldsymbol{p = 0}$, $\boldsymbol{\beta = 0}$ **and** $\boldsymbol{\alpha > 0}$.

This case leads to, $H(u) = \frac{1}{\alpha}$, a constant. Thus the distribution is exponential.

Case 4. $\boldsymbol{0 < p < 1, \alpha > 0}$ **and** $\boldsymbol{\beta > \frac{2\alpha p}{(1-p)}}$.

Now X is IHR if and only if $K(u) > 0$ for all $u \in (0,1)$. This holds if and only if,

$$p(p-1)\alpha(1+u)^2 > -2\beta(pu-1)^2,$$

which gives,

$$\frac{2\beta}{\alpha p(1-p)} > \frac{(1+u)^2}{(pu-1)^2}. \tag{2.5.5}$$

Since $(1+u)^2 > (pu-1)^2$, for all $0 < u < 1$ and $0 < p < 1$, we have the right side of (2.5.5) is increasing in u and attains its maximum when $u = 1$. Now for $u = 1$, the inequality (2.5.5) reduces to $\beta > \frac{2\alpha p}{(1-p)}$, thus it is clear that $H(u)$ is increasing in this case.

Case 5. $\boldsymbol{0 < p < 1, \alpha > 0}$ **and** $\boldsymbol{0 < \beta < \frac{\alpha p(1-p)}{2}}$.

Similar to Case 4, we can show that $H(u)$ have a decreasing hazard rate (DHR) if and only if,

$$p(p-1)\alpha(1+u)^2 < -2\beta(pu-1)^2$$

or

$$\frac{2\beta}{\alpha p(1-p)} < \frac{(1+u)^2}{(pu-1)^2}. \tag{2.5.6}$$

Since right side of (2.5.6) is increasing in u and attains its minimum when $u = 0$, the above inequality reduces to $\beta < \frac{\alpha p(1-p}{2}$. Thus the distribution is DHR.

Case 6. $\boldsymbol{0 < p < 1, \alpha > 0}$ and $\boldsymbol{\frac{\alpha p(1-p}{2} < \beta < \frac{2\alpha p}{1-p}}$.

The first term of $K(u)$ is negative and second term is positive, so that $K(u)$ attains a zero in this case. This, in turn, gives $H'(u) = 0$ suggesting the possibility for non-monotonic hazard quantile function. Let u_0 be the solution of the equation $K(u) = 0$. From (2.5.4), we have u_0 is the solution corresponding to the quadratic equation,

$$u^2(\alpha p(p-1) + 2\beta p^2) + u(2\alpha p(p-1) - 4p\beta) + (\alpha p(p-1) + 2\beta) = 0,$$

which provides,

$$u_0 = \frac{-\alpha p^2 - \sqrt{2}\sqrt{-\alpha\beta p^4 - \alpha\beta p^3 + \alpha\beta p^2 + \alpha\beta p} + \alpha p + 2\beta p}{\alpha p^2 + 2\beta p^2 - \alpha p}. \quad (2.5.7)$$

For further inference, we note that the second derivative of $H(u)$ has the form

$$H''(u) = \frac{4\alpha\beta(1-p)(p+1)^2}{(\alpha(p-1)(u+1) + 2\beta(pu-1))^3}.$$

For the change point u_0 obtained in (2.5.7), we get

$$H''(u_0) = -\frac{\sqrt{2}p^2}{\sqrt{\alpha\beta(1-p)p(p+1)^2}}. \quad (2.5.8)$$

Since $H''(u_0) < 0$, we have $H(u)$ attains a maximum at u_0. Hence X has an upside-down bathtub-shaped hazard quantile function (see Nair et al. [105]).

The ageing pattern of $H(u)$ for various parameter values are summarized in Table 2.1. We can easily show the following lemma, which is useful for finding bounds of $H(u)$.

Lemma 2.5.1. The limits of $HLEG(\alpha, \beta, p)$ hazard quantile function are

Table 2.1: Ageing behaviour of the hazard quantile function for different regions of parameter space.

Sl.No	Parameter Region	Shape of the hazard quantile function
1	$p = 0, \alpha > 0$ and $\beta > 0$	IHR
2	$p = 1, \alpha > 0$ and $\beta > 0$	IHR
3	$p = 0$, $\alpha > 0$ and $\beta = 0$	Constant
4	$0 < p < 1, \alpha > 0$ and $\beta > \frac{2\alpha p}{(1-p)}$	IHR
5	$0 < p < 1, \alpha > 0$ and $0 < \beta < \frac{\alpha p(1-p)}{2}$	DHR
6	$0 < p < 1, \alpha > 0$ and $\frac{\alpha p(1-p)}{2} < \beta < \frac{2\alpha p}{1-p}$	upside-down bathtub
7	$\alpha = 0$, $\beta > 0$ and $0 < p < 1$	IHR
8	$\beta = 0$, $\alpha > 0$ and $0 < p < 1$	DHR

$$\lim_{u \to 0} H(u) = \frac{1}{\alpha(1-p) + 2\beta} \quad \text{and} \quad \lim_{u \to 1} H(u) = \frac{1}{\alpha + \beta}, \tag{2.5.9}$$

where $\alpha > 0,\ \beta > 0$ and $0 < p < 1$.

Proof. It is straight forward to show the results in (2.5.9) by taking the corresponding limits of the hazard quantile function (2.5.3). ■

Theorem 2.5.1. If $X \sim HLEG(\alpha, \beta, 1)$, then the two limits of hazard quantile function are independent of the parameter α, given by;

(i) $\lim_{u \to 1} H(u) = 2 \lim_{u \to 0} H(u)$

and

(ii) $\frac{1}{2\beta} < H(u) < \frac{1}{\beta}$, for all $0 < u < 1$ and $\beta > 0$.

Proof.

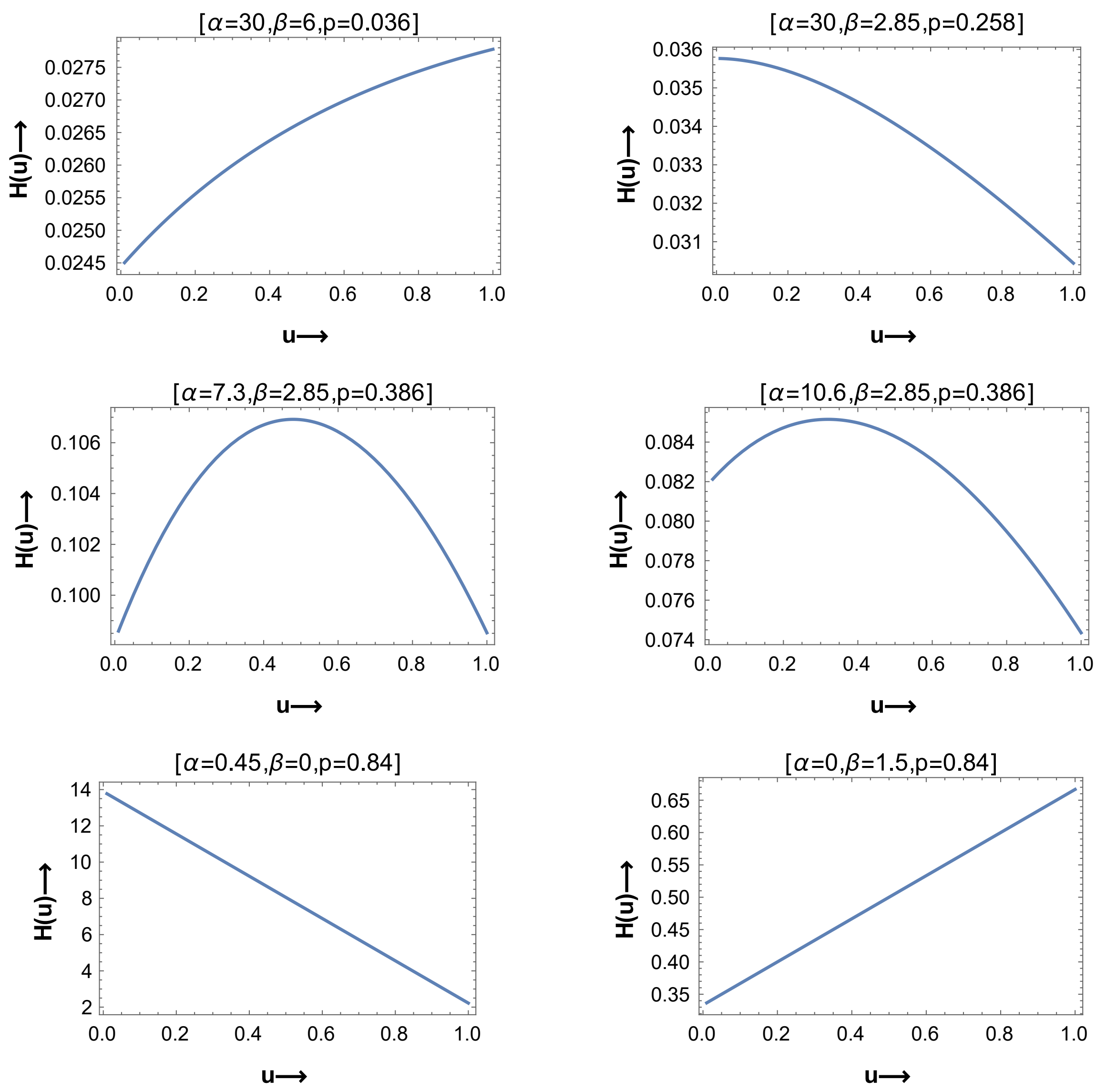


Figure 2.5: Plots of hazard quantile function for different values of parameters.

(i) The proof is direct once we note that,

$$\lim_{u\to 0} H(u) = \frac{1}{2\beta} \quad \text{and} \quad \lim_{u\to 1} H(u) = \frac{1}{\beta}. \tag{2.5.10}$$

(ii) From Table 2.1, $H(u)$ is IHR for $p = 1$, $\alpha > 0$ and $\beta > 0$. Thus lower and upper bounds for $H(u)$ exist when u approaches 0 and 1 respectively.

Now from (2.5.10), we get

$$\frac{1}{2\beta} \; < \; H(u) \; < \; \frac{1}{\beta} \quad \text{for all } 0 < u < 1 \text{ and } \beta > 0.$$

This completes the proof. ■

Theorem 2.5.2. If $X \sim HLEG(\alpha, \beta, 0)$. Then the bounds of $H(u)$ are given by

$$\frac{1}{\alpha + 2\beta} \; < \; H(u) \; < \; \frac{1}{\alpha + \beta}, \quad \text{for all } \; 0 < u < 1 \text{ and } \beta > 0.$$

Proof. The proof is similar to that of Theorem 2.5.1 once we note that,

$$\lim_{u \to 0} H(u) = \frac{1}{\alpha + 2\beta} \;\text{ and }\; \lim_{u \to 1} H(u) = \frac{1}{\alpha + \beta}, \tag{2.5.11}$$

and $H(u)$ is increasing for $p = 0, \alpha > 0$ and $\beta > 0$. ■

Theorem 2.5.3. Let $X \sim HLEG(\alpha, \beta, p)$. Then the hazard quantile function satisfies the following;

(i) If $\beta > \frac{2\alpha p}{(1-p)}$ then $\frac{1}{\alpha(1-p)+2\beta} \; < \; H(u) \; < \; \frac{1}{\alpha+\beta}$

(ii) If $0 < \beta < \frac{\alpha p(1-p)}{2}$ then $\frac{1}{\alpha+\beta} \; < \; H(u) \; < \; \frac{1}{\alpha(1-p)+2\beta}$.

Proof.

From Table 2.1, note that X is IHR when $\beta > \frac{2\alpha p}{(1-p)}$. Now from Lemma 2.5.1, we get,

$$\frac{1}{\alpha(1-p) + 2\beta} \; < \; H(u) \; < \; \frac{1}{\alpha + \beta}. \tag{2.5.12}$$

To prove (ii), note that X is DHR for $0 < \beta < \frac{\alpha p(1-p)}{2}$. Since $H(u)$ is decreasing over u, boundary values are reversed. This completes the proof. ■

For the class of distributions (2.2.5), $M(u)$ has the form

$$M(u) = \frac{\beta\log(4) + \frac{\alpha(p-1)\log\left(\frac{p-1}{pu-1}\right)}{p} - 2\beta\log(u+1)}{1-u}. \tag{2.5.13}$$

It is well-known that increasing (decreasing) failure rate implies decreasing (increasing) mean residual life (See Lai and Xie [78]). The ageing behaviour of the class of distributions (2.2.5) based on mean residual quantile function can be defined from Table 2.1. There exists closed form expressions of the hazard quantile function and mean residual quantile function defined in reverse time (see Nair and Sankaran [97]) for the proposed class of distributions (2.2.5).

The quantile-based total time on test transform (TTT) introduced in Nair et al. [102] is obtained as

$$T(u) = \int_0^u (1-p)\,q(p)dp = \frac{\alpha(p-1)\log(1-pu)}{p} + 2\beta\log(u+1). \tag{2.5.14}$$

Nair et al. [103] studied various properties and applications of the Parzens score function $J(u)$ in the context of lifetime data analysis. For the class of distributions (2.2.5), $J(u)$ is obtained as

$$J(u) = \frac{q^{'}(u)}{q^2(u)} = \frac{\alpha(p-1)(u+1)^2(p(2u-1)-1) + 4\beta u(pu-1)^2}{(\alpha(p-1)(u+1) + 2\beta(pu-1))^2}. \tag{2.5.15}$$

It is of great importance to characterize life distributions by the relationships among reliability concepts. In the same spirit, we prove the following characterization theorem.

Theorem 2.5.4. A non-negative continuous random variable X follows;

(a) HLEG$(u;\alpha,0,p)$ if and only if any one of the following properties hold.

(i) $H(u) = A_1 - A_2\,u, \quad 0 < A_2 < A_1$

(ii) $J(u) = H(u) + C\,(1-u), \;\; C > 0$

(iii) $T(u) = \frac{-1}{A_2}\log\left(\frac{H(u)}{A_1}\right)$

and

(b) HLEG$(u; 0, \beta, p)$ if and only if any one of the following properties hold.

(i) $H(u) = K(1+u)$, $K > 0$

(ii) $J(u) = 2K\,u$

(iii) $T(u) = \frac{1}{K}\log(K^K H(u))$

where $A_1,\ A_2,\ C$ and K are constants.

Proof. We prove the result for (a). The proof for (b) is similar.

(a) Suppose $H(u) = A_1 - A_2\,u$ is true. Then the corresponding quantile function is obtained as

$$Q(u) = \frac{\log\left(\frac{A_1 - A_2\,u}{A_1(1-u)}\right)}{A_1 - A_2}, \tag{2.5.16}$$

which is equivalent to HLEG$(u; \alpha, 0, p)$, with $\alpha = \frac{1}{A_1 - A_2} > 0$ and $0 < p = \frac{A_2}{A_1} < 1$. Conversely for $\beta = 0$, the expression of $H(u)$ reduces to $H(u) = \frac{1}{\alpha\,(1-p)} - \frac{p}{\alpha\,(1-p)}$, which is in the required form (i).

When $J(u) = H(u) + C\,(1-u)$ is true, from Nair and Sankaran [97], we have the identity, $(1-u)H'(u) = H(u) - J(u)$, which gives

$$(1-u)H'(u) = C\,(u-1). \tag{2.5.17}$$

The solution of the ordinary differential equation (2.5.17) is

$$H(u) = D - C\,u, \quad C > 0, D - C > 0,$$

which satisfies (i), so the proof is completed.

Suppose $T(u) = \frac{-1}{A_2}\log\left(\frac{H(u)}{A_1}\right)$ is true. Differentiating this with respect to u, we get

$$T'(u) = \frac{-H'(u)}{A_2 H(u)}. \tag{2.5.18}$$

Differentiating (2.5.14) with respect to u, we get,

$$T'(u) = (1-u)q(u) = \frac{1}{H(u)}. \tag{2.5.19}$$

From (2.5.18) and (2.5.19), we get $H'(u) = -A_2$, which leads to (i). Conversely for the class of distributions HLEG$(u; \alpha, 0, p)$ we obtain,

$$T(u) = \frac{\alpha(p-1)\log(1-pu)}{p},$$

or

$$T(u) = \frac{-1}{A_2}\log\left(\frac{H(u)}{A_1}\right),$$

where $A_1 = \frac{1}{\alpha(1-p)}$ and $A_2 = \frac{p}{\alpha(1-p)}$.

This completes the proof. ■

2.6 An application

There are different methods for estimating the parameters of a quantile function. The method of minimum absolute deviation, method of least squares, method of maximum likelihood, method of percentiles and the method of L-moments are commonly employed for this purpose. In the first three methods, the estimates are found by optimizing non-linear function of parameters. This will lead to complex expressions as pointed out in Gilchrist [42] and Hosking [59]. To estimate the parameters of (2.2.5), we use the method of L-moments. We equate sample L-moments to corresponding population L-moments. Let $X_1, X_2, ..., X_n$ be a random sample of size n from the population with quantile function (2.2.5), then the first three sample L-moments are given by

$$\begin{aligned}
l_1 &= \left(\frac{1}{n}\right)\sum_{i=1}^{n} X_{(i)} \\
l_2 &= \left(\frac{1}{2}\right)\binom{n}{2}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{1}-\binom{n-i}{1}\right)X_{(i)} \\
l_3 &= \left(\frac{1}{3}\right)\binom{n}{3}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{2}-2\binom{i-1}{1}\binom{n-i}{1}+\binom{n-i}{2}\right)X_{(i)}
\end{aligned}$$

where $X_{(i)}$ is the ith order statistic.

For estimating the parameters α, β and σ, we equate first three sample L-moments to population L-moments given in Section 2.4. The parameters are obtained by solving the equations

$$l_r = L_r; \quad r = 1, 2, 3. \tag{2.6.1}$$

Since L_1 is the mean of the distribution, mean survival time is estimated as l_1. Similarly estimate of variance is obtained as $\hat{V}(x) = \int_0^1 (\hat{Q}(u))^2 du - l_1^2$, which can be evaluated with the help of numerical integration techniques. Hosking [60] has shown that the L-moment estimates are asymptotically normal and consistent. Since the set of equations (2.6.1) are non-linear in α, β and p, asymptotic variances of the L-moment estimates are difficult to compute. The bootstrap method is commonly used to obtain the asymptotic variance of the estimates (see Efron and Tibshirani [37]).

To illustrate the application of the proposed class of distributions, we consider a real data set reported in Zimmer et al. [157]. The data consist of time to first failure of 20 electric carts. We estimate the parameters using the method of L-moments. The sample L-moments are obtained as

$$l_1 = 12.66 \qquad l_2 = 5.91 \quad \text{and} \quad l_3 = 1.57. \tag{2.6.2}$$

We then equate these values to the corresponding population L-moments given in (2.4.1), (2.4.2) and (2.4.3), so that we have three non-linear equations. The Newton-Raphson method is used to find the solutions of these equations. Least square method of estimation for quantile functions given in Öztürk and Dale [115] was employed for fixing the initial estimates for the Newton-Raphson iterative procedure. The estimates of the parameters are obtained as

$$\hat{\alpha} = 8.518 \qquad \hat{\beta} = 1.209 \quad \text{and} \quad \hat{p} = 0.329. \tag{2.6.3}$$

The standard errors of the estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{p}$ are obtained as 0.035, 0.141 and 0.021 respectively. To examine the adequacy of the model, we use the Q-Q plot, which is given in Figure 2.6. The Q-Q plot shows that the proposed model provides a good fit for the given data. We also carry out the chi-square goodness of fit test. The chi-square test statistic value is 0.210, giving p-value 0.647 with one degree of freedom. This also indicates the adequacy of proposed model for the given data set. We also compute the estimate of $H(u)$ by substituting the parameter values (2.6.3) in (2.5.3), which is given in Figure 2.7. Note that the estimate $\hat{H}(u)$ is decreasing in u, which is consistent with our claim in Table 2.1.

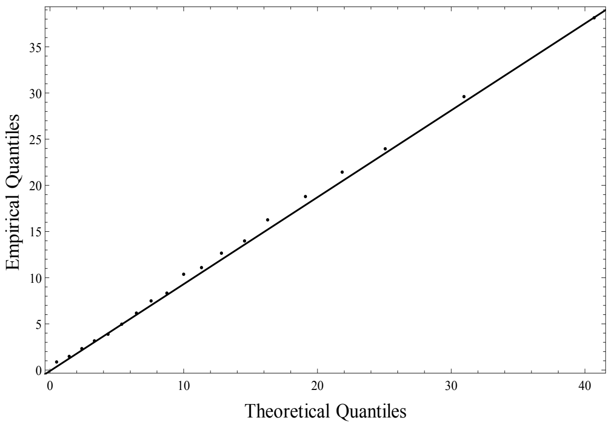


Figure 2.6: Q-Q plot for the electric cart data set.

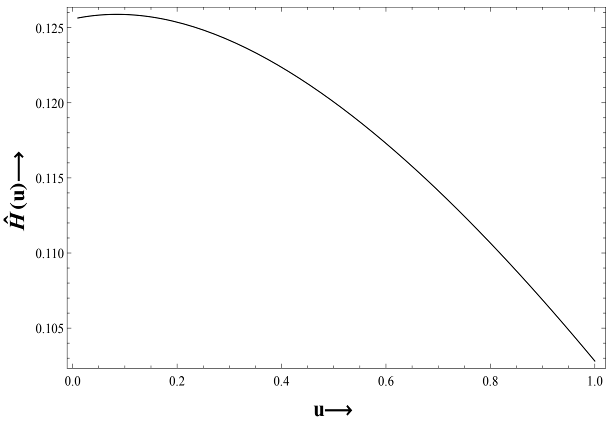


Figure 2.7: Plot of $\hat{H}(u)$ for the electric cart data set.

We can also infer that the data has a decreasing failure pattern.

2.7 Conclusion

In this chapter, we have introduced a class of distributions (2.2.5), which is the sum of quantile functions of the half logistic and exponential geometric quantile functions. Various reliability properties were studied. We have identified several well-known distributions which are members of the proposed class of distributions. The estimation of parameters using the method of L-moments was studied and discussed the estimation procedure with the aid of a real data. The proposed model has several advantages over the existing quantile function models. The analysis of hazard quantile function over the whole parameter space can be done without using numerical methods. The model is useful for fitting different types of lifetime data due to the flexible behaviour of hazard quantile function. Unlike generalized lambda distribution and generalized Tukey lambda distribution, the estimation of parameters does not involve any computational difficulties.

Chapter 3

Pareto-Weibull Quantile Function

3.1 Introduction

Quantile function has several properties that are not shared by the distribution function. For example, the sum of two quantile functions is a quantile function and the product of two quantile functions is also a quantile function in the non-negative set up. In Chapter 2, we studied the properties and applications of a new quantile function, which is formed by taking the sum of quantile functions of half logistic and exponential geometrical distributions. The aim of the present chapter is to construct a new flexible quantile function by considering the product of quantile functions of two lifetime distributions.

The Weibull distribution is popular due to its wide use to model various types of failure time data. Several generalizations of the Weibull distribution were developed in the literature to add more flexibility. These generalizations include the generalized Weibull distribution by Mudholkar and Kollia [93], the exponentiated-Weibull distribution by Mudholkar et al. [94], and the beta-Weibull distribution by Famoye et al. [38]. For various properties of Weibull and related distributions, we may refer to Johnson et al. [63] and Murthy et al. [95]. Pareto distribution is also an important distribution used in the context of reliability

Results in this chapter have been published as entitled "Pareto-Weibull quantile function" in the "Journal of Applied Probability and Statistics" (See Sankaran and Kumar M [132]).

studies for modelling decreasing failure time data. The Pareto family of distributions are commonly used to model a wide variety of heavy tailed social and economic data. A comprehensive review on properties and generalizations of the Pareto distribution is available in Arnold [6].

We propose a new class of distributions defined by a quantile function by taking the product of quantile functions of Weibull and Pareto distributions. The proposed class gives a wide variety of distributional shapes for various choices of the parameters.

The rest of the chapter is organized into six different sections. In Section 3.2, we present a class of distributions and study its basic properties. Section 3.3 presents some well-known distributions which are either a member of the proposed class of distributions or obtained by applying some suitable transformations on the proposed quantile function. The distributional properties such as measures of location and scale, L-moments etc., are given in Section 3.4. In Section 3.5, we present various reliability characteristics of the class. Section 3.6 focuses on inference procedures. We then provide applications of this class of distributions in two real life situations. Finally, Section 3.7 provides major conclusions of the study.

3.2 A class of distributions

Motivated by the property that the product of two positive quantile functions is again a quantile function, we introduce a class of distributions defined by

$$Q(u) = \sigma(1-u)^{-\alpha}(-\log(1-u))^{\beta}, \;\; 0 < u < 1, \;\; \alpha, \beta \geq 0, \, \sigma > 0. \tag{3.2.1}$$

The proposed quantile function $Q(\cdot)$ is the product of two quantile functions $Q_1(\cdot)$ and $Q_2(\cdot)$, where $Q_1(u) = \sigma_1(1-u)^{-\alpha}$ is the quantile function of the Pareto distribution and $Q_2(u) = \sigma_2(-\log(1-u))^{\beta}$ is the quantile function of the Weibull distribution. The support of the distribution is $(0,\infty)$. The quantile density function is obtained as

$$q(u) = \sigma(1-u)^{-\alpha-1}(-\log(1-u))^{\beta-1}(\beta - \alpha\log(1-u)). \tag{3.2.2}$$

The quantile function (3.2.1) represents a family of distributions with no closed form expression for its density or distribution function. However, these can be calculated by numerical inversion of the quantile function. The family of distributions (3.2.1) accommodates a

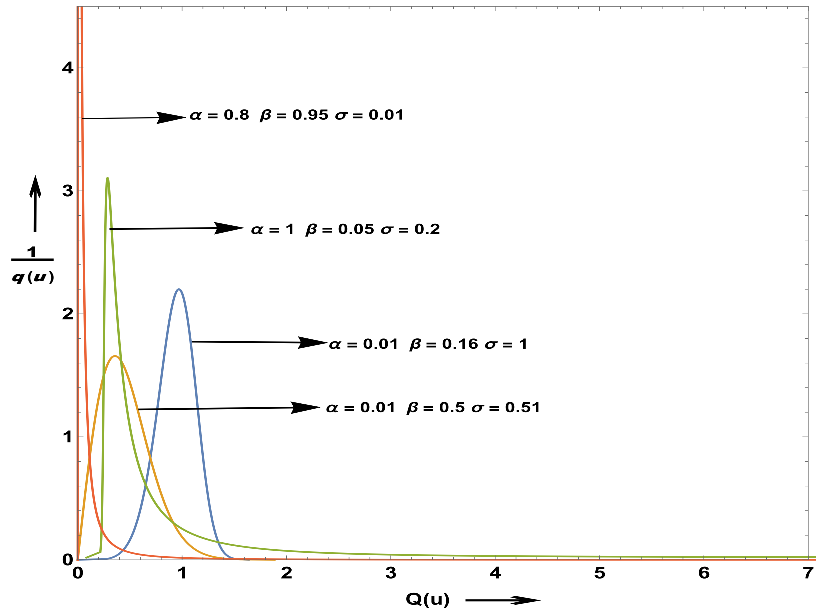


Figure 3.1: Plots of the density function for different values of parameters.

variety of shapes for its probability density function. Plots of the density function for different parameter values are given in Figure 3.1. It is seen that for appropriate choices of the parameter values the family includes uni-modal, positive and negatively skewed members. For the proposed class of distributions, the density function $f(\cdot)$ can be written in terms of the distribution function and the survival function as

$$f(x) = \frac{(1-F(x))^{1+\alpha}(-\log(1-F(x))^{1-\beta}}{\sigma(\beta-\alpha\log(1-F(x))}. \tag{3.2.3}$$

The family of distributions (3.2.3) is a rich family, which contains several class of distributions corresponding to different types of distribution functions in the expression of $f(x)$. The derivative of $f(x)$ is obtained as

$$\begin{aligned} f'(x) = & \frac{(1-F(x))^{1+\alpha}(-\log(1-F(x))^{1-\beta}}{\sigma(\beta-\alpha\log(1-F(x))}f(x) \\ & \left[-(1+\alpha)-\frac{(1-\beta)}{\log(1-F(x)}-\frac{\alpha}{(\beta-\alpha\log(1-F(x))}\right]. \end{aligned} \tag{3.2.4}$$

Thus for $\beta > 1$, the density function is always decreasing and when $0 < \beta \leq 1$, the density function is uni-modal with mode $x_0 = Q(u_0)$, where

$$u_0 = 1 - e^{\left(\frac{\beta+2\alpha\beta-\sqrt{4\alpha\beta+4\alpha^2\beta+\beta^2}}{2\alpha(1+\alpha)}\right)}. \tag{3.2.5}$$

3.3 Members of the family

The family of distributions (3.2.1) includes some well-known distributions as its special cases. For $\alpha = 0$, we get the Weibull distribution with quantile function $Q(u) = \sigma(-\log(1-u))^{\beta}$, which contains the exponential distribution with mean σ for $\beta = 1$ and the Rayleigh distribution when $\beta = 2$. When $\beta = 0$, the quantile function (3.2.1) reduces to $Q(u) = \sigma(1-u)^{-\alpha}$, which is the quantile function of the Pareto distribution.

By making use of various transformations on quantile functions described in Gilchrist [42], we can derive certain popular distributions from the proposed model (3.2.1). By applying logarithmic transformation on (3.2.1), with $\beta = 0$, we get

$$Q_1(u) = \log(Q(u)) \; = \; \log(\sigma) - \alpha \log(1-u) = \frac{1}{\log(C)} - \alpha \log(1-u), \tag{3.3.1}$$

with $C = e^{(\log(\sigma))^{-1}}$, which leads to the Gompertz distribution.

Consider the power u-transformation of the form $T(u) = u^{\frac{1}{\theta}}$ with $\beta = 1, \alpha = 0$ and $\theta > 0$, we have

$$Q_2(u) = Q(T(u)) = -\sigma \log(1 - u^{\frac{1}{\theta}}). \tag{3.3.2}$$

This is the quantile function of the generalized exponential distribution.

Using the reciprocal transformation, putting $\beta = 0$ the quantile function became

$$Q_3(u) = \frac{1}{Q(1-u)} = ku^{\alpha}. \tag{3.3.3}$$

with $k = \frac{1}{\sigma}$. This is the quantile function of the power distribution.

For $\beta = 0$, the centering transformation gives the Pareto distribution of II kind with quantile function,

$$Q_4(u) = 1 - Q(u) = \frac{(1-(1-u)^{\alpha})}{(1-u)^{\alpha}}. \tag{3.3.4}$$

Raising to a power η, the above quantile function became

$$Q_5(u) = (Q_4(u))^{\eta} = ((1-u)^{-\alpha} - 1)^{\eta}, \tag{3.3.5}$$

which is the quantile function of the Burr XII distribution.

For the formulation of our model (3.2.1), the product of two quantile functions are involved. However, to the best of our knowledge, the random variable associated with the quantile function of the product is not explicitly derived in the literature. The following theorem helps to find the random variable associated with the product of two quantile functions.

Theorem 3.3.1. Let X and Y be two random variables with strictly increasing distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$ and quantile functions $Q_X(\cdot)$ and $Q_Y(\cdot)$ respectively. Then $Q_X(\cdot) \times Q_Y(\cdot)$ is the quantile function of the random variable $Z = XQ_Y(F_X(X))$.

Proof. Let Z be a continuous random variable with quantile function $Q_Z(\cdot)$, where $Q_Z(u) = Q_X(u) \times Q_Y(u)$. If $Q_X(u) = t \quad$ then $\quad u = F_X(t)$. Thus,

$$Q_Z(u) = tQ_Y(F_X(t)) \text{ for all } u \in (0,1).$$

$$\text{Or,} \qquad u = F_Z(tQ_Y(F_X(t))).$$

$$\text{Finally,} \quad F_Z(z) = F_Z(tQ_Y(F_X(t))), \text{where } Q_Z(u) = z. \tag{3.3.6}$$

From (3.3.6), since z and t are the realizations of Z and X respectively, we get $Z = XQ_Y(F_X(X))$. On similar lines, if we choose $Q_Y(u) = t$, we can show that $Z = YQ_X(F_Y(Y))$. ■

Remark 3.3.1. When X follows Weibull distribution variable with parameters σ_1 and β and Y follows Pareto distribution with parameters σ_2 and α, then the random variable $Z = \sigma_2 X \left(e^{\alpha \left(\frac{X}{\sigma 1}\right)^{\frac{1}{\beta}}} \right)$ has quantile function of the form (3.2.1).

3.4 Distributional characteristics

For the model (3.2.1), we have

$$\text{Median} = Q\left(\frac{1}{2}\right) = 0.5^{-\alpha}\sigma(-\log(0.5))^{\beta}.$$

The inter-quartile-range, IQR is obtained as

$$
\begin{aligned}
IQR &= Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) \\
&= \sigma\left(\left(\frac{1}{4}\right)^{-\alpha}\left(-\log\left(\frac{1}{4}\right)\right)^{\beta} - \left(\frac{3}{4}\right)^{-\alpha}\left(-\log\left(\frac{3}{4}\right)\right)^{\beta}\right).
\end{aligned}
$$

The Galton's coefficient of skewness, S has the expression

$$
\begin{aligned}
S &= \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2M}{IQR} \\
&= \frac{-2^{1-\alpha}\left(-\log\left(\frac{1}{2}\right)\right)^{\beta} + 3^{-\alpha}\left(-\log\left(\frac{3}{4}\right)\right)^{\beta} + \left(-\log\left(\frac{1}{4}\right)\right)^{-\beta}}{\left(-\log\left(\frac{1}{4}\right)\right)^{-\beta} - 3^{-\alpha}\left(-\log\left(\frac{3}{4}\right)\right)^{\beta}},
\end{aligned}
$$

and the Moor's coefficient of kurtosis, T is

$$
\begin{aligned}
T &= \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{IQR}\frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2M}{IQR} \\
&= \frac{-3^{-\alpha}\left(-\log\left(\frac{3}{8}\right)\right)^{\beta} + 5^{-\alpha}\left(-\log\left(\frac{5}{8}\right)\right)^{\beta} - 7^{-\alpha}\left(-\log\left(\frac{7}{8}\right)\right)^{\beta} + \left(-\log\left(\frac{1}{8}\right)\right)^{\beta}}{2^{-\alpha}\left(-\log\left(\frac{1}{4}\right)\right)^{\beta} - 6^{-\alpha}\left(-\log\left(\frac{3}{4}\right)\right)^{\beta}}.
\end{aligned}
$$

The first $L-$moment L_1, which is the mean of the distribution is obtained as

$$
L_1 = \int_0^1 Q(u)\,du = \sigma(1-\alpha)^{-(\beta+1)}\Gamma(\beta+1). \tag{3.4.1}
$$

The second $L-$moment of the family has the form,

$$
\begin{aligned}
L_2 &= \int_0^1 (2u-1)Q(u)\,du \\
&= \sigma\left((1-\alpha)^{-(\beta+1)} - 2(2-\alpha)^{-(\beta+1)}\right)\Gamma(\beta+1),
\end{aligned} \tag{3.4.2}
$$

which is twice the mean differences of the population. The third and the fourth L-moments are obtained by

$$\begin{aligned} L_3 &= \int_0^1 \left(6u^2 - 6u + 1\right) Q(u)\, du \\ &= \sigma\left((1-\alpha)^{-(\beta+1)} - 6(2-\alpha)^{-(\beta+1)} + 2(3-\alpha)^{-(\beta+1)}\right)\Gamma(\beta+1), \end{aligned} \tag{3.4.3}$$

and

$$\begin{aligned} L_4 &= \int_0^1 \left(20u^3 - 30u^2 + 12u - 1\right) Q(u)\, du \\ &= \sigma\left((1-\alpha)^{-(\beta+1)} - 12(2-\alpha)^{-(\beta+1)} + 30(3-\alpha)^{-(\beta+1)}\right. \\ &\qquad \left. - 20(4-\alpha)^{-(\beta+1)}\right)\Gamma(\beta+1). \end{aligned} \tag{3.4.4}$$

respectively. The L-coefficient of variation, analogous to the coefficient of variation based on ordinary moments is given by

$$\begin{aligned} \tau_2 &= \frac{L_2}{L_1} \\ &= \left((1-\alpha)^{-\beta-1} - 2(2-\alpha)^{-\beta-1}\right)(1-\alpha)^{\beta+1} \\ &= 1 - 2\left(\frac{\eta}{1+\eta}\right)^{\beta+1}, \end{aligned}$$

where $\eta = 1-\alpha$. Thus τ_2 lies in $(0,1)$, attains maximum when β tends to infinity and its minimum when both α and β tend to zero. To measure the skewness of the distribution, we use the L-coefficient of skewness defined by

$$\begin{aligned} \tau_3 &= \frac{L_3}{L_2} \\ &= \frac{(1-\alpha)^{-(\beta+1)} - 6(2-\alpha)^{-(\beta+1)} + 2(3-\alpha)^{-(\beta+1)}}{(1-\alpha)^{-(\beta+1)} - 2(2-\alpha)^{-(\beta+1)}} \end{aligned}$$

$$= \frac{\eta^{-(\beta+1)} - 6(\eta+1)^{-(\beta+1)} + 2(\eta+2)^{-(\beta+1)}}{\eta^{-(\beta+1)} - 2(\eta+1)^{-(\beta+1)}}.$$

L-coefficient of kurtosis for the family (3.2.1) is obtained as

$$\begin{aligned}
\tau_4 &= \frac{L_4}{L_2} \\
&= \frac{(1-\alpha)^{-(\beta+1)} - 12(2-\alpha)^{-(\beta+1)} + 30(3-\alpha)^{-(\beta+1)} - 20(4-\alpha)^{-(\beta+1)}}{(1-\alpha)^{-(\beta+1)} - 2(2-\alpha)^{-(\beta+1)}} \\
&= \frac{\eta^{-(\beta+1)} - 12(\eta+1)^{-(\beta+1)} + 30(\eta+2)^{-(\beta+1)} - 20(\eta+3)^{-(\beta+1)}}{\eta^{-(\beta+1)} - 2(\eta+1)^{-(\beta+1)}}.
\end{aligned}$$

3.5 Some approximations

In this section, we establish the relationship of the proposed model (3.2.1) with some familiar standard distributions through approximations. The advantage of this approximation is justified from the practical and analytical points of view. In statistical data modelling, usually we choose one among the candidate distributions, estimate the parameters of the model and then carry out proper goodness of fit test. If the choice is not appropriate, we repeat the procedure with another model, sometimes with a different strategy for estimation and model adequacy test. If we have a quantile function for which the approximation to different types of distributions is available, we can utilize the distributional properties and inferential aspects of this quantile function as an alternative to other distributions in the context of lifetime data analysis.

We first consider the class of distributions defined by Jones [66], with quantile density function

$$q(u) = Ku^{-\alpha_1}(1-u)^{-\beta_1}, \quad K > 0,\ \alpha_1,\ \beta_1 \text{ are real.} \tag{3.5.1}$$

To approximate (3.5.1) with (3.2.1) we need to identify the parameters α, β and σ for which the approximation is appropriate. Equating first three L-moments of both quantile functions and solving for α, β and σ, we get the parameters values of the quantile function (3.2.1). For illustration, we consider (3.5.1) with α_1 = 0.12, β_1 = 1.2, and K = 6. The corresponding parameter values of the quantile function (3.2.1) are α = 0.129, β = 0.875 and σ = 6.769. We plot the probability density function of the two models in Figure 3.2, which shows that our model is a good approximation to Jones distribution. In Figure 3.2 the solid line and the broken line represents the probability density functions of Jones model and the corresponding approximation respectively. On similar lines, the proposed model

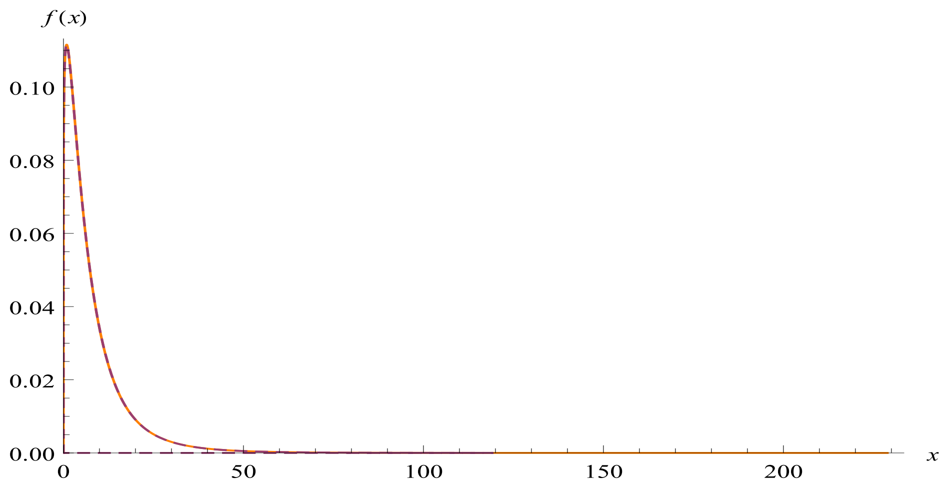


Figure 3.2: Probability density functions of Jones distribution and its approximation.

can be approximated to the gamma distribution with density function,

$$f(x) = \frac{\beta_2^{-\alpha_2} x^{\alpha_2 - 1} e^{-\frac{x}{\beta_2}}}{\Gamma(\alpha_2)}, \quad x > 0. \tag{3.5.2}$$

When α_2 = 2.5 and β_2 = 1.5, the values of parameters in (3.2.1) are α = 0.004 , β = 0.593 and σ = 4.172. We plot the density function of two models in Figure 3.3, which shows that our model is a reasonable approximation to the gamma distribution. In Figure 3.3, solid line represents density function of the gamma model and broken line denotes the density function of the approximation. Other lifetime models such as inverse Gaussian, Log

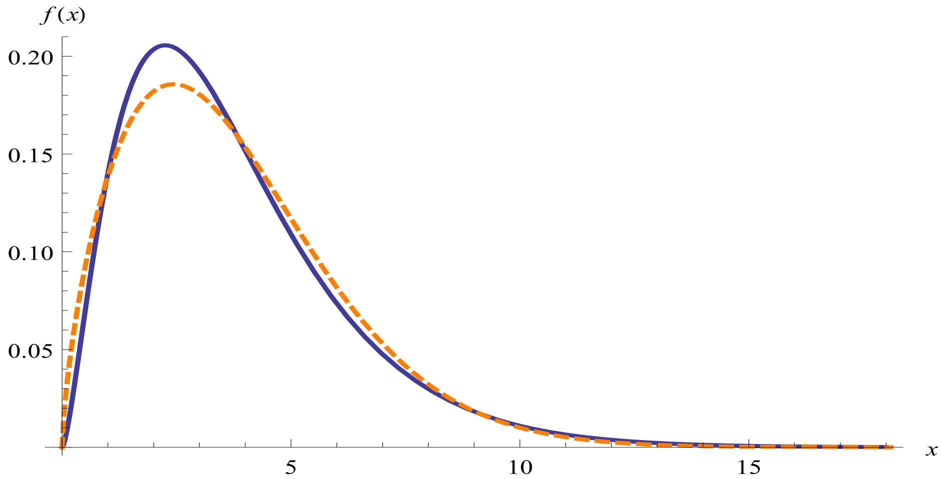


Figure 3.3: Probability density functions of gamma distribution and its approximation.

normal, etc., can also be approximated by the proposed class of distributions. Since these distributions does not have closed form distribution functions, one need to depend on numerical methods for simulating observations. But it is much easier to generate observations from the approximated form of $Q(u)$.

3.6 Reliability properties

For the class of distributions (3.2.1), the hazard quantile function is obtained as

$$H(u) = \frac{(1-u)^{\alpha}(-\log(1-u))^{1-\beta}}{\sigma(\beta - \alpha\log(1-u))}. \tag{3.6.1}$$

The identity (3.6.1) can also be written as

$$H(u) = \left(-Q(u)\left(\alpha + \frac{\beta}{\log(1-u)}\right)\right)^{-1}, \tag{3.6.2}$$

where $Q(u)$ is the quantile function defined in (3.2.1). The shape of the hazard function is determined by the derivative of $H(u)$, which is obtained as

$$H'(u) = \frac{(1-u)^{\alpha-1}(-\log(1-u))^{-\beta}\left(2\alpha\beta\log(1-u)) - (\beta-1)\beta - \alpha^2\log^2(1-u)\right)}{\sigma(\beta-\alpha\log(1-u))^2}.$$

Since $(1-u)^{\alpha-1} > 0$ and $\sigma(\beta-\alpha\log(1-u))^2 > 0$ for all values of the parameters, the sign of $H'(u)$ depends only on the function

$$K(y) = \alpha\,\mathrm{y}(2\beta-\alpha\mathrm{y}) - \beta(\beta-1), \tag{3.6.3}$$

where $y = log(1-u) < 0$. The parameter α does not affect the sign of the two terms in $K(y)$. Now, we have six different cases.

Case 1: $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\beta > 1}$.

The first term in $K(y)$ is 0 and the second term is negative so that $K(y) < 0$ and the distribution is DHR.

Case 2: $\boldsymbol{\alpha > 0}$ and $\boldsymbol{\beta > 1}$.

Both terms in $K(y)$ are negative. Then $K(y) < 0$ and the distribution is DHR.

Case 3: $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\beta = 1}$.

From (3.6.1), we get, $H(u) = \frac{1}{\sigma},$ a constant, so the distribution is exponential.

Case 4: $\boldsymbol{\alpha > 0}$ and $\boldsymbol{\beta = 1}$.

The second term in $K(y)$ is 0 and the first term is negative. Then $K(y) < 0$ and distribution is DHR.

Case 5: $\boldsymbol{\alpha = 0}$ and $\boldsymbol{0 < \beta < 1}$.

The first term in $K(y)$ is 0 and the second term is positive, which gives $K(y) > 0$

and distribution is IHR.

Case 6: $\boldsymbol{\alpha > 0}$ and $\boldsymbol{0 < \beta < 1}$.

In this case, the first term in $K(y)$ is negative and the second term is positive. Thus $K(y)$ can be zero. The first derivative of $H(u)$ is obtained as

$$H'(u) = \frac{e^{(\alpha-1)y}(-y)^{-\beta}K(y)}{\sigma(\beta-\alpha y)^2}. \tag{3.6.4}$$

Further, the second derivative of $H(u)$ is given by,

$$H''(u) = \frac{dH'(u)}{dy}\,\frac{1}{u-1}.$$

The sign of $H''(u)$ depends only on

$$-\frac{e^{(\alpha-2)y}(-y)^{-\beta}}{\sigma(\beta-\alpha y)^2}[K'(y)+(\alpha-1)K(y)-\frac{\beta}{y}\,K(y)+\frac{2\alpha}{(\beta-\alpha y)}K(y)].$$

Let y_0 be the solution of the equation $K(y)=0$. From (3.6.4), we have $H'(u_0)=0$, where $u_0 = 1-e^{y_0}$. Then the sign of $H''(u_0)$ depends on $K'(y_0)$. We have

$$K'(y) = 2\alpha\beta - 2\alpha^2 y.$$

Since $y < 0$ for all $u \in (0,1)$ and the parameters α and β are non-negative, we get $K'(y_0) \geq 0$ and $H''(u_0) \leq 0$. So that $H(u)$ attains a maximum at $y_0 = \frac{\beta-\sqrt{\beta}}{\alpha}$. Hence X has an upside-down bathtub-shaped hazard quantile function with change point $u_0 = 1-e^{\frac{\beta-\sqrt{\beta}}{\alpha}}$. It may be noticed that monotonicity of the conventional hazard rate is equivalent to that of the hazard quantile function.

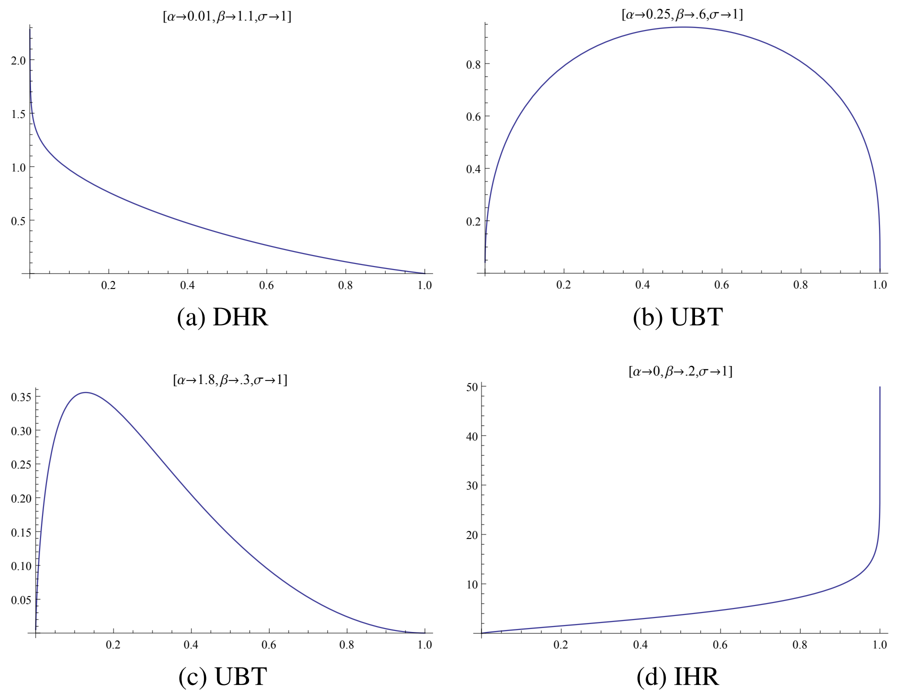


Figure 3.4: Plots of $H(u)$.

Summarizing the analysis, the family (3.2.1) possesses, IHR for $\alpha = 0$, $0 < \beta < 1$, DHR when $\alpha \geq 0$, $\beta > 1$ and UBT if $\alpha > 0$, $0 < \beta < 1$. Figure 3.4 gives plots of $H(u)$ for the different values of parameters. For the class of distributions (3.2.1), mean residual quantile function $M(u)$ has the form

$$M(u) = \frac{\sigma(1-\alpha)^{-(\beta+1)}\Gamma[\beta+1,(\alpha-1)\log(1-u)]}{1-u} - \sigma(1-u)^{-\alpha}(-\log(1-u))^{\beta}.$$

The hazard quantile function and mean residual quantile function defined in reverse time (see Nair and Sankaran [97]) have the respective expressions,

$$\Lambda(u) = [u\mathrm{q(u)}]^{-1} = \frac{(1-u)^{\alpha+1}(-\log(1-u))^{1-\beta}}{\sigma u(\beta - \alpha\log(1-u))},$$

$$
\begin{aligned}
R(u) &= \left(-\frac{1}{u}\right)\int_0^u p\ q(p)dp \\
&= \left(-\frac{1}{u}\right)\Bigg(\sigma(1-\alpha)^{-\beta-1}\Gamma(\beta+1)+\frac{\sigma(1-u)^{-\alpha}(-\log(1-u))^{\beta}}{\alpha-1} \\
&\qquad (1-\alpha u)((\alpha-1)\log(1-u))^{\beta}+\beta(1-u)^{\alpha}\Gamma(\beta,(\alpha-1)\log(1-u)) \\
&\qquad\quad ((\alpha-1)\log(1-u))^{-\beta}\Bigg).
\end{aligned}
$$

The total time on test (TTT) is a useful concept in reliability theory. For the class of distributions (3.2.1), we obtain $T(u)$ as

$$
\begin{aligned}
T(u) &= \sigma(1-\alpha)^{-\beta-1}\Gamma(\beta+1)+\left(\frac{\sigma(1-u)^{-\alpha}(-\log(1-u))^{\beta}((\alpha-1)\log(1-u))^{-\beta}}{\alpha-1}\right) \\
&\quad (\beta(1-u)^{\alpha}\Gamma(\beta,(\alpha-1)\log(1-u))-\alpha(u-1)((\alpha-1)\log(1-u))^{\beta}).
\end{aligned} \tag{3.6.5}
$$

The first $L-$moment of the residual life random variable $X_t = X|X > t$ is the vitality function studied by Kupka and Loo [75]. In the quantile set up which is given by

$$
\begin{aligned}
\alpha_1(u) &= \frac{1}{1-u}\int_u^1 Q(p)dp \\
&= \frac{\sigma(1-\alpha)^{-\beta-1}\Gamma(\beta+1,(\alpha-1)\log(1-u))}{1-u}.
\end{aligned}
$$

The second $L-$moment of residual life (Nair and Vineshkumar [100]) is given by

$$
\begin{aligned}
\alpha_2(u) &= \frac{1}{(1-u)^2}\int_u^1 Q(p)(2p-u-1)dp \\
&= \frac{\sigma(1-\alpha)^{-\beta-1}(2-\alpha)^{-\beta-1}}{1-u}\Bigg[(\alpha-2)(u-1)(2-\alpha)^{\beta} \\
&\qquad \Gamma(\beta+1,(\alpha-1)\log(1-u))-2(1-\alpha)^{\beta+1} \\
&\qquad\qquad \Gamma(\beta+1,(\alpha-2)\log(1-u))\Bigg].
\end{aligned}
$$

Residual life can be also explained in terms of percentiles as percentile residual life function, defined by

$$\begin{aligned} P_{\alpha}(u) &= Q(1-(1-\alpha)(1-u)) - Q(u) \\ &= \sigma\left(((\alpha-1)(u-1))^{-\alpha}(-\log((\alpha-1)(u-1)))^{\beta} - (1-u)^{-\alpha}(-\log(1-u))^{\beta}\right). \end{aligned}$$

and the reversed percentile residual life function is (Nair and Vineshkumar [101])

$$\begin{aligned} R_{\alpha}(u) &= Q(u) - Q((1-\alpha)u) \\ &= \sigma(1-u)^{-\alpha}(-\log(1-u))^{\beta} - \sigma((\alpha-1)u+1)^{-\alpha}(-\log((\alpha-1)u+1))^{\beta}. \end{aligned}$$

The class of distributions defined in (3.2.1) can be characterized by the relationship between $J(u)$ and $H(u)$ as

$$J(u) = H(u)\left[1+\alpha+\frac{1-\beta}{log(1-u)}+\frac{\alpha}{\beta-\alpha log(1-u)}\right].$$

3.7 Applications

To estimate the parameters of (3.2.1), we use the method of L-moments. We equate sample L-moments to corresponding population L-moments. The sample L-moments are given by

$$l_1 = \left(\frac{1}{n}\right)\sum_{i=1}^{n} X_{(i)} \tag{3.7.1}$$

$$l_2 = \left(\frac{1}{2}\right)\binom{n}{2}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{1}-\binom{n-i}{1}\right)X_{(i)} \tag{3.7.2}$$

$$l_3 = \left(\frac{1}{3}\right)\binom{n}{3}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{2}-2\binom{i-1}{1}\binom{n-i}{1}+\binom{n-i}{2}\right)X_{(i)}$$

where $X_{(i)}$ is the ith order statistic. For estimating the parameters α, β and σ, we equate first three sample L-moments to population L-moments given in Section 3.5. The parameters are obtained by solving the equations,

$$l_r = L_r; \quad r = 1, 2, 3. \tag{3.7.3}$$

The set of equations (3.7.3) are non-linear in α, β and σ.

To illustrate the application of the proposed class of distributions. We consider two real life data sets. We first consider a real data set reported in Von Alven [154] and Chhikara and Folks [26]. The data consists of active repair times (in hours) for an airborne communication transceiver. We estimate the parameters using the method of L-moments and the sample L-moments are obtained as

$$l_1 = 3.607, \qquad l_2 = 2.116 \quad \text{and} \quad l_3 = 1.102. \tag{3.7.4}$$

We now equate these values to the corresponding population L-moments given in (3.4.1), (3.4.2) and (3.4.3), so that we have three non-linear equations. Newton-Raphson method is used to find the solution of the resulting system of non-linear equations. The values of the parameters which minimizes the residual sum of squares are used as the initial values for the Newton-Raphson procedure. The estimates of the parameters are obtained as

$$\hat{\alpha} = 0.302, \qquad \hat{\beta} = 0.773 \quad \text{and} \quad \hat{\sigma} = 2.061. \tag{3.7.5}$$

To examine the adequacy of the model, two goodness of fit techniques are employed. The first one is the Q-Q plot, which is given in Figure 3.5. The Q-Q plot reveals the physical closeness of the model. We also carry out the chi-square goodness of fit test. The chi-square

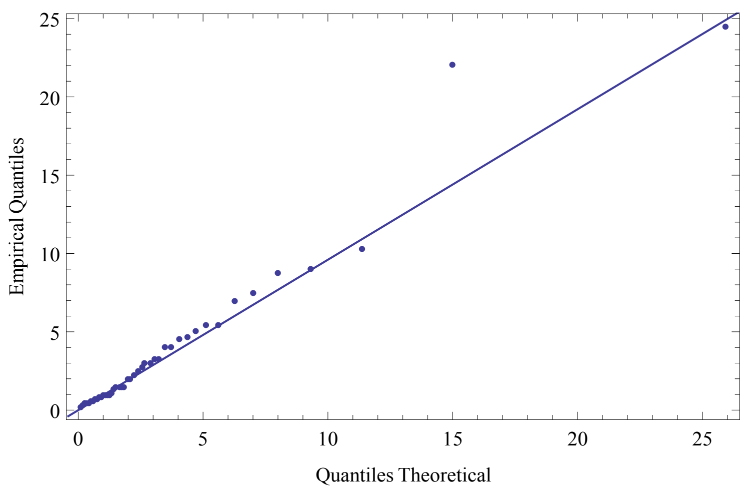


Figure 3.5: Q-Q plot for the repair time data.

value is 1.022 with p-value 0.79, which does not reject the model.

Now, consider another set of data reported in Arnold [6] on the lifetime tournament earnings through 1980 of all those professional golfers whose lifetime tournament earnings exceed \$700,000 during that period. We estimate the parameters using the method of L-moments. The sample L-moments are obtained as

$$l_1 = 1168.12 \qquad l_2 = 253.314 \quad \text{and} \quad l_3 = 103.538. \tag{3.7.6}$$

Now by equating these to the corresponding population L-moments and solving the equations numerically, the estimates obtained are

$$\hat{\alpha} = 0.291 \qquad \hat{\beta} = 0.065 \quad \text{and} \quad \hat{\sigma} = 837.795. \tag{3.7.7}$$

To check the goodness of fit, we use Q-Q plot and chi-square goodness of fit test. Figure 3.6 presents Q-Q plot, which shows that most of the data points are close to the straight line. This means that the quantile function (3.2.1) gives a reasonable fit for the data. The

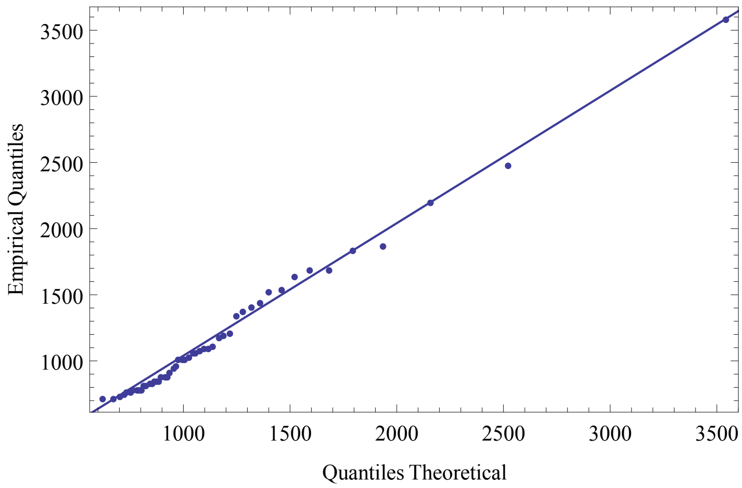


Figure 3.6: Q-Q plot for the income data.

chi-square value is 1.52 with p-value 0.68, which does not reject the model.

3.8 Conclusion

We have introduced a class of distributions (3.2.1), which is the product of quantile functions of the Pareto and Weibull distributions. The random variable associated with the quantile function was identified. Several well-known distributions which are either the members of the proposed class of distributions or obtained through some suitable transformations on the quantile function (3.2.1) were presented. The estimation of parameters of the model using L-moments was studied and discussed the estimation procedure with the aid of two real data sets. The proposed model has several advantages over the existing quantile function models. The analysis of shape of hazard quantile function over the whole parameter space can be done without using numerical methods. The proposed model was applied to two real life situations to illustrate its utility.

Chapter 4

A Class of Quantile Functions with Applications to Reliability Analysis

4.1 Introduction

As previously mentioned, quantile functions have several distinct properties that are not shared with the distribution functions. This makes quantile-based analysis more attractive in certain practical situations. There exist many simple quantile functions, that serve well in empirical model building, for which distribution functions are not in tractable forms. In the present chapter, we provide a new method of constructing flexible quantile functions based on the probability integral transform method given in Lai [77]. We then develop a new class of distributions using the proposed method. The proposed class gives a wide variety of distributions with different shapes for various choices of the parameters.

The summary of the chapter is as follows. In Section 4.2, we present a general method of transformation and a particular case for the construction of new quantile functions. Section 4.3 presents a new family of distributions using the proposed method of transformation. Various distributional properties and L-moments are given in Section 4.4. Section 4.5 presents important reliability properties of the class. Section 4.6 studies inference proce-

Results in this chapter have been communicated to an international journal.

dures of the model. We then provide an application of this class of distributions in a real life situation. Finally, Section 4.7 provides major conclusions of the study.

4.2 Method for constructing new quantile functions

Let $G_1(\cdot)$ and $G_2(\cdot)$ be the cumulative distribution functions of two continuous lifetime distributions with the latter having support on the unit interval. Then a new class of distribution is defined by the cumulative distribution function

$$F(t) = G_2(G_1(t)), \tag{4.2.1}$$

which was discussed in Lai [77]. We now consider an equivalent transformation method in the context of quantile functions. Let $Q(u), Q_1(u)$ and $Q_2(u)$ be the quantile functions corresponding to the cumulative distribution functions $F(t), G_1(t)$ and $G_2(t)$ respectively. Suppose $F(t) = u$, then $G_2(G_1(t)) = u$, which gives $G_1(t) = Q_2(u)$ and $t = Q_1(Q_2(u))$. Thus we have

$$Q(u) = Q_1(Q_2(u)). \tag{4.2.2}$$

In general, $Q_1(Q_2(u))$ is a quantile function for any two choices of quantile functions $Q_1(u)$ and $Q_2(u)$. When $Q_2(u)$ is a unit support quantile function, we get $Q(0) = Q_1(0)$ and $Q(1) = Q_1(1)$, and hence the support of $Q(u)$ and $Q_1(u)$ are same. This will be useful for the construction of flexible lifetime models from the existing non-negative distributions. Thus, we consider $Q_2(u)$ as a unit support quantile function. Using the relation (4.2.2), we can expand existing families of distributions defined in terms of quantile functions by adding suitable parameters. We can also apply all the standard transformation methods given in Gilchrist [42] to add more flexibility. For example, if X has the quantile function (4.2.2), then $\frac{1}{X}$ has the quantile function

$$Q^*(u) = \frac{1}{Q(1-u)} = \frac{1}{Q_1(Q_2(1-u))}. \tag{4.2.3}$$

The quantile density function $q(u)$ for the model (4.2.2) is obtained as

$$q(u) = \frac{d}{du} Q_1(Q_2(u)) = q_1(Q_2(u))q_2(u), \tag{4.2.4}$$

where $q_1(u)$ and $q_2(u)$ are the quantile density functions corresponding to $Q_1(u)$ and $Q_2(u)$. The density quantile function are related through the expression

$$f(Q(u)) = f_1(Q_1(Q_2(u)))f_2(Q_2(u)). \tag{4.2.5}$$

For the class of distributions (4.2.2), the hazard quantile function is of the form

$$\begin{aligned} H(u) &= ((1-u)q_1(Q_2(u))q_2(u))^{-1} \\ &= H_1(Q_2(u))\frac{1-Q_2(u)}{(1-u)q_2(u)} && (4.2.6) \\ &= (1-Q_2(u))H_1(Q_2(u))H_2(u), && (4.2.7) \end{aligned}$$

where $H_j(u)$ is the hazard quantile function corresponding to $Q_j(u),\ j=1,2$. Note that the monotonic properties of hazard rate and hazard quantile function are identical.

We now consider some special cases of (4.2.2) by properly choosing the unit support quantile function. When $Q_2(u) = 1-(1-u)^{\frac{1}{\theta}}$, we have

$$Q(u) = Q_1(1-(1-u)^{\frac{1}{\theta}}).$$

This leads to the well-known proportional hazards model with baseline quantile function $Q_1(u)$, which yields $1-F(x) = (1-G_1(x))^{\theta}$ (see Nair et al. [105]).

If we substitute $Q_2(u) = u^{\frac{1}{\theta}}$, we get

$$Q(u) = Q_1(u^{\frac{1}{\theta}}),$$

which provides proportional reversed hazards model with $F(x) = (G_1(x))^{\theta}$.

For $Q_2(u) = \frac{u}{\beta+(1-\beta)u}$ with $\beta > 0$, we obtain

$$\begin{aligned} Q(u) &= Q_1(Q_2(u)) \\ &= Q_1\left(\frac{u}{\beta+(1-\beta)u}\right), \end{aligned} \tag{4.2.8}$$

where $Q_1(u)$ is the baseline quantile function. This is the quantile version of the Marshall and Olkin [87] model with distribution function

$$1-F(x) = \frac{\alpha(1-G(x))}{1-(1-\alpha)(1-G(x))}, \ \alpha > 0, \tag{4.2.9}$$

with $\alpha = \frac{1}{\beta}$. Now for the model (4.2.8), the quantile density function $q(u)$ has the form

$$\begin{aligned} q(u) &= \frac{d}{du}\left(Q_1(Q_2(u))\right) \\ &= \left(\frac{\beta}{(\beta+(1-\beta)u)^2}\right) q_1\left(\frac{u}{\beta+(1-\beta)u}\right). \end{aligned} \tag{4.2.10}$$

Since $Q_2(u) = \frac{u}{\beta+(1-\beta)u}$, $q_2(u) = \frac{d}{du}Q_2(u) = \frac{\beta}{(\beta+(1-\beta)u)^2}$ and $1-Q_2(u) = \frac{\beta(1-u)}{\beta+(1-\beta)u}$, we get

$$\frac{1-Q_2(u)}{(1-u)q_2(u)} = \beta+(1-\beta)u. \tag{4.2.11}$$

Now from (4.2.6), the relation between the hazard function of $Q(u)$ and the baseline hazard quantile function is obtained as

$$H(u) = (\beta + (1-\beta)u)\, H_1\left(\frac{u}{\beta + (1-\beta)u}\right). \tag{4.2.12}$$

In the next proposition, we provide bounds for $H(u)$ in terms of the baseline hazard quantile function $H_1(u)$.

Theorem 4.2.1. Let X be a random variable with quantile function $Q(u)$ as described in (4.2.8). Then $H(u)$ satisfies the boundary condition,

$$\text{Minimum}\{\beta, 1\} H_1\left(\frac{u}{\beta + (1-\beta)u}\right) \leq H(u) \leq \text{Maximum}\{\beta, 1\}\, H_1\left(\frac{u}{\beta + (1-\beta)u}\right) \tag{4.2.13}$$

Proof. We have the function $\beta + (1-\beta)u$ satisfies the relation,

$$\text{Minimum}\{\beta, 1\} \leq \beta + (1-\beta)u \leq \text{Maximum}\{\beta, 1\}. \tag{4.2.14}$$

Now using this relation in (4.2.12), we get the inequality (4.2.13), which completes the proof. ∎

Remark 4.2.1. From (4.2.12), it follows that $\frac{H(u)}{H_1\left(\frac{u}{\beta+(1-\beta)u}\right)}$ is increasing in u for $0 < \beta \leq 1$ and decreasing in u for $\beta \geq 1$.

4.3 A new class of distributions

We now construct a new class of distributions, which is an extended version of the class of distributions with linear mean residual quantile function defined in Midhu et al. [89]. The quantile function of the class of distributions with linear mean residual quantile function is given by

$$Q_1(u) = (-(\alpha+\delta)\log(1-u) - 2\alpha u)\ , \quad -\delta < \alpha < \delta,\ \delta > 0,\ \sigma > 0. \tag{4.3.1}$$

Note that the class of distributions with linear mean residual quantile function have no closed form expression for it's distribution function. Now from (4.2.8) and (4.3.1), we present a class of distributions with quantile function,

$$Q(u) = \left((\alpha+\delta)\log\left(1+\frac{u}{\beta(1-u)}\right) - \frac{2\alpha u}{\beta+(1-\beta)u}\right) \quad \beta > 0. \tag{4.3.2}$$

By re-parametrizing (4.3.2), using $\sigma = \alpha + \delta$ and $\gamma = \frac{-2\alpha}{\alpha+\delta}$, the quantile function (4.3.2) becomes

$$Q(u) = \sigma\left(\log\left(1+\frac{u}{\beta(1-u)}\right) + \frac{\gamma u}{\beta+(1-\beta)u}\right) \qquad \sigma > 0, \beta > 0 \ \text{ and } \ \gamma > -1. \tag{4.3.3}$$

Since the support of the distribution is $(Q(0), Q(1))$ = $(0, \infty)$, this can be efficiently employed for modelling various types of lifetime data. The quantile density function $q(u)$ is given by

$$q(u) = \frac{\sigma(u(\beta\gamma+\beta-1) - \beta(\gamma+1))}{(u-1)(\beta-\beta u+u)^2}. \tag{4.3.4}$$

The distribution function or density function of the random variable X corresponding to (4.3.3) cannot be expressed in closed form by solving $Q(u) = x$ and has to be evaluated numerically. However, we can represent (4.3.4) in terms of the density function $f(x)$ and the distribution function $F(x)$ as

$$f(x) = \frac{(F(x)-1)(\beta-\beta F(x)+F(x))^2}{\sigma(F(x)(\beta\gamma+\beta-1) - \beta(\gamma+1)}. \tag{4.3.5}$$

Plots of the density function for different combinations of parameters is shown in Figure 4.1.

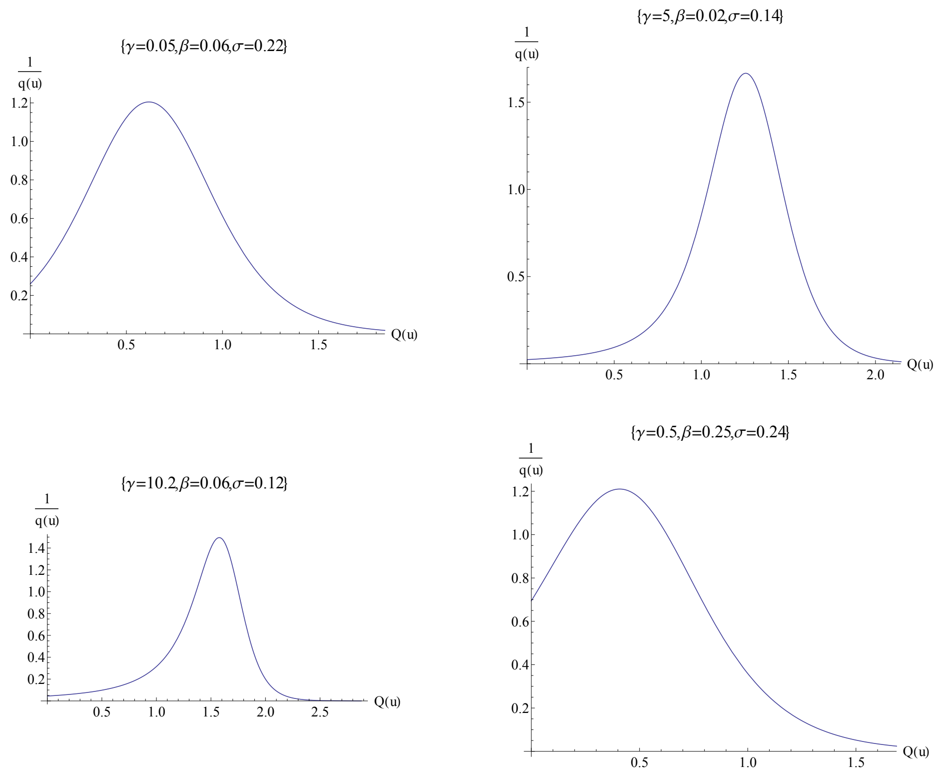


Figure 4.1: Plots of density function for different values of parameters.

For $\gamma = 0$, we can observe that the model (4.3.3) is essentially a particular case of the class of distributions with linear hazard quantile function considered in Midhu et al. [90] with quantile function

$$Q(u) = \frac{1}{a+b} \log\left(\frac{a+b\,u}{a-a\,u}\right), \;\; a>0, \; a+b>0, \tag{4.3.6}$$

with $a = \beta$ and $b = 1-\beta$. Further, this particular case includes several well-known families of distributions for various values of the parameter β. These consists of the exponential distribution with mean σ for $\beta = 1$ and the extended Weibull distribution, defined by

Marshall and Olkin [87] with quantile function,

$$Q(u) = \sigma \log\left(\frac{\theta + (1-\theta)(1-u)}{1-u}\right) \tag{4.3.7}$$

with $\theta = \frac{1}{\beta} > 0$ for $\beta > 0$. In particular, for $\beta = \frac{1}{2}$, (4.3.7) reduces to

$$Q(u) = -\sigma \log\left(\frac{1+u}{1-u}\right),$$

which is the half-logistic distribution.

In addition, when $\beta > 1$, the quantile function (4.3.7) can be expressed as

$$Q(u) = \sigma \log\left(\frac{1-Pu}{1-u}\right), \tag{4.3.8}$$

with $0 < P = \frac{\beta-1}{\beta} < 1$, which is the exponential geometric distribution defined by Adamidis and Loukas [3].

4.4 Distributional characteristics

The ordinary moments are given by $E(X^r) = \int_0^1 (Q(p))^r dp$ and in particular the mean of the distribution has the form

$$\mu = \frac{\sigma(-\beta\gamma + (\beta\gamma + \beta - 1)\log(\beta) + \gamma)}{(\beta-1)^2}.$$

The quantile-based measures of the distributional characteristics like location, dispersion, skewness and kurtosis are popular in statistical analysis. These measures are also useful in inference problems. For the class of distributions (4.3.3), we have

$$\text{Median} = Q\left(\frac{1}{2}\right) = \sigma\left(\frac{\gamma}{\beta+1} + \log\left(\frac{1}{\beta}+1\right)\right).$$

The inter-quartile range is obtained as

$$\begin{aligned} IQR &= Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) \\ &= \sigma\left(\frac{8\beta\gamma}{3\beta^2+10\beta+3} + \log\left(\frac{\beta+3}{\beta}\right) - \log\left(\frac{1}{3\beta}+1\right)\right). \end{aligned}$$

Galton's coefficient of skewness,

$$\begin{aligned} S &= \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2M}{IQR} \\ &= \frac{\frac{4(\beta-1)\beta\gamma}{(\beta+1)(\beta+3)(3\beta+1)} - 2\log\left(\frac{1}{\beta}+1\right) + \log\left(\frac{\beta+3}{\beta}\right) + \log\left(\frac{1}{3\beta}+1\right)}{\frac{8\beta\gamma}{3\beta^2+10\beta+3} + \log\left(\frac{\beta+3}{\beta}\right) - \log\left(\frac{1}{3\beta}+1\right)}. \end{aligned}$$

and Moor's coefficient of kurtosis,

$$\begin{aligned} T &= \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \\ &= \frac{\left(-\frac{5}{3\beta+5} + \frac{3}{5\beta+3} + \frac{1}{-7\beta-1} + \frac{7}{\beta+7}\right)\gamma + \log\left(\frac{72}{35\beta+5} + \frac{56}{5(3\beta+5)} + 1\right)}{\frac{8\beta\gamma}{3\beta^2+10\beta+3} + \log\left(\frac{\beta+3}{\beta}\right) - \log\left(\frac{1}{3\beta}+1\right)}. \end{aligned}$$

For the class of distributions (4.3.3), we obtain the second L-moment as

$$\begin{aligned} L_2 &= \int_0^1 (2u-1)Q(u)\,du \\ &= \frac{\sigma(\beta(\beta\gamma+\beta+\gamma-1)\log(\beta) - (\beta-1)(2\beta\gamma+\beta-1))}{(\beta-1)^3}, \end{aligned} \tag{4.4.1}$$

which is twice the mean differences of the population. The third and fourth L- moments are

$$L_3 = \int_0^1 \left(6u^2 - 6u + 1\right) Q(u)\, du$$
$$= \frac{\beta\sigma\left((\beta^2(\gamma+1) + 4\beta\gamma + \gamma - 1)\log(\beta) - (\beta-1)(\beta(3\gamma+2) + 3\gamma - 2)\right)}{(\beta-1)^4}, \quad (4.4.2)$$

and $L_4 = \int_0^1 \left(20u^3 - 30u^2 + 12u - 1\right) Q(u)\, du$

$$= \frac{\sigma(6\beta(\beta(\beta((\beta+9)\gamma+\beta+2)+9\gamma-2)+\gamma-1)\log(\beta)))}{6(\beta-1)^5}$$
$$- \frac{((\beta-1)(\beta(\beta(2\beta(11\gamma+8)+76\gamma-3)+22\gamma-12)-1))}{6(\beta-1)^5} \quad (4.4.3)$$

respectively. The L-coefficient of variation, L-coefficient of skewness and L-coefficient of kurtosis are

$$\tau_2 = \frac{L_2}{L_1} = \frac{\beta(\beta\gamma+\beta+\gamma-1)\log(\beta) - (\beta-1)(2\beta\gamma+\beta-1)}{(\beta-1)(-\beta\gamma+(\beta\gamma+\beta-1)\log(\beta)+\gamma)}, \quad (4.4.4)$$

$$\tau_3 = \frac{L_3}{L_2}$$
$$= \frac{\beta(\beta((\beta+4)\gamma+\beta)+\gamma-1)\log(\beta) - (\beta-1)\beta(\beta(3\gamma+2)+3\gamma-2)}{(\beta-1)(\beta(\beta\gamma+\beta+\gamma-1)\log(\beta) - (\beta-1)(2\beta\gamma+\beta-1))}, \quad (4.4.5)$$

and

$$\tau_4 = \frac{L_4}{L_2}$$
$$= \frac{(\beta-1)(\alpha(\beta(\beta(28\beta+155)+56)+1)+\beta(\beta(3-16\beta)+12)\delta+\delta)}{6(\beta-1)^2((\beta-1)(3\alpha\beta+\alpha-\beta\delta+\delta)-\beta\log(\beta)(\alpha(\beta+3)-\beta\delta+\delta))}$$
$$- \frac{6\beta\log(\beta)(\alpha(\beta(\beta(\beta+16)+20)+3)-\beta(\beta(\beta+2)-2)\delta+\delta)}{6(\beta-1)^2((\beta-1)(3\alpha\beta+\alpha-\beta\delta+\delta)-\beta\log(\beta)(\alpha(\beta+3)-\beta\delta+\delta))}, \quad (4.4.6)$$

respectively. We can show that τ_3 lies in (-1, 1) and τ_4 lies in (-0.25, 1) using numerical techniques.

4.4.1 Order statistics

The density function of the rth order statistic $X_{r:n}$ is of the form

$$f_r(x) = \frac{1}{B(r, n-r+1)} f(x) F^{r-1}(x)(1-F(x))^{n-r}.$$

For the proposed model, it reduces to

$$f_r(x) = \frac{1}{B(r, n-r+1)} \frac{(1-F(x))^{n-r}(\beta F(x) - \beta - F(x))^2 (F(x))^{r-1}}{\sigma(F(x)(\beta\gamma + \beta - 1) - \beta(\gamma + 1))}.$$

This gives

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^\infty x \frac{(1-F(x))^{n-r}(\beta F(x) - \beta - F(x))^2 (F(x))^{r-1}}{\sigma(F(x)(\beta\gamma + \beta - 1) - \beta(\gamma + 1))} dx.$$

In terms of quantile functions, we have

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^1 Q(u) \frac{(1-u)^{n-r}(\beta u - \beta - u)^2 u^{r-1}}{\sigma u(\beta\gamma + \beta - 1) - \beta(\gamma + 1))} dx.$$

The first order statistic $X_{1:n}$ has quantile function,

$$\begin{aligned} Q_{(1)}(u) &= Q(1-(1-u)^{\frac{1}{n}}) \\ &= \sigma \left(\frac{\gamma - \gamma(1-u)^{1/n}}{(\beta - 1)(1-u)^{1/n} + 1} + \log \left(\frac{(1-u)^{-1/n} - 1}{\beta} + 1 \right) \right), \end{aligned}$$

and the nth order statistic $X_{n:n}$ has quantile function

$$\begin{aligned} Q_{(n)}(u) &= Q(u^{\frac{1}{n}}) \\ &= \sigma\left(\frac{\gamma u^{1/n}}{\beta-(\beta-1)u^{1/n}} + \log\left(\frac{1}{\beta u^{-1/n}-\beta}+1\right)\right). \end{aligned}$$

The above expressions of order statistics are useful when studying the series and parallel systems in reliability theory (see Lai and Xie, 2006).

4.5 Reliability properties

Hazard quantile function is one of the basic concept employed for the modelling and analysis of lifetime data in quantile set up. For the class of distributions (4.3.3), the hazard quantile function has the expression

$$H(u) = \frac{(\beta-\beta u+u)^2}{\sigma(\beta\gamma+\beta-u(\beta\gamma+\beta-1))}. \tag{4.5.1}$$

Special Cases:

(a). For $\gamma=0$, we get linear hazard quantile function $H(u)=A+Bu$, where $A=\frac{\beta}{\sigma}$ and $B=\frac{(1-\beta)}{\sigma}$. This was studied by Midhu et al. [91]. In this case, we have $H(u)$ is IHR for $\beta<1$, DHR for $\beta>1$ and constant for $\beta=1$. Thus for $\beta=1$ and $\gamma=0$, the underlying distribution is exponential.

(b). When $\beta=0$, the hazard quantile function is of the form $H(u)=Ku$, where $K=\frac{1}{\sigma}$. We can observe that $H(u)$ is linear in this case also.

(c). When $\beta=1$, hazard quantile function takes the form $H(u)=\frac{1}{C-Du}$, where $C=\sigma(1+\gamma)$ and $D=\sigma\gamma$. Some special cases of Rescaled beta, power and generalized

Weibull distributions have hazard quantile function of this form.

The shape of the hazard quantile function is determined by the derivative of $H(u)$, which is obtained as

$$H'(u) = \frac{(\beta-(\beta-1)u)((\beta-\beta(\beta+(\beta-2)\gamma))+(\beta-1)u(\beta\gamma+\beta-1))}{\sigma(\beta\gamma+\beta-u(\beta\gamma+\beta-1))^2}.$$

Since $\beta-(\beta-1)u>0$ and $\sigma(\beta\gamma+\beta-u(\beta\gamma+\beta-1))^2>0$ for all values of the parameters and $0<u<1$, the sign of $H'(u)$ depends only on

$$K(u) = \beta((2-\beta)\gamma+(1-\beta))+(\beta-1)u(\beta(\gamma+1)-1).$$

By thoroughly analysing the function $K(u)$, we observe that the hazard quantile function accommodates increasing, decreasing, linear and bathtub shapes for different choices of parameters β and γ. Plots of hazard quantile function for different values of parameters are given in Figure 4.2. The hazard quantile function have both monotonic as well as non-monotonic members for various parameter values. We consider 9 different regions of the parameter space and analyse the behaviour of $H(u)$ in each of them separately. The shape of $H(u)$ is determined based on the sign of $K(u)$. The ageing pattern of $H(u)$ for various parameter values are given in Table 4.1. The hazard quantile function has bathtub shape for cases 5, 7 and 8 with a change point

$$u_0 = \frac{\beta^2(1-\gamma)-\beta(1-2\gamma)}{\beta^2(\gamma+1)-\beta(\gamma+2)+1}.$$

For the class of distributions (4.3.3), $M(u)$ has the form

$$M(u) = \frac{\sigma\left(-\frac{(\beta-1)\beta\gamma}{\beta-\beta u+u}-\frac{(\beta\gamma+\beta-1)\log(\beta-\beta u+u)}{u-1}\right)}{(\beta-1)^2}. \tag{4.5.2}$$

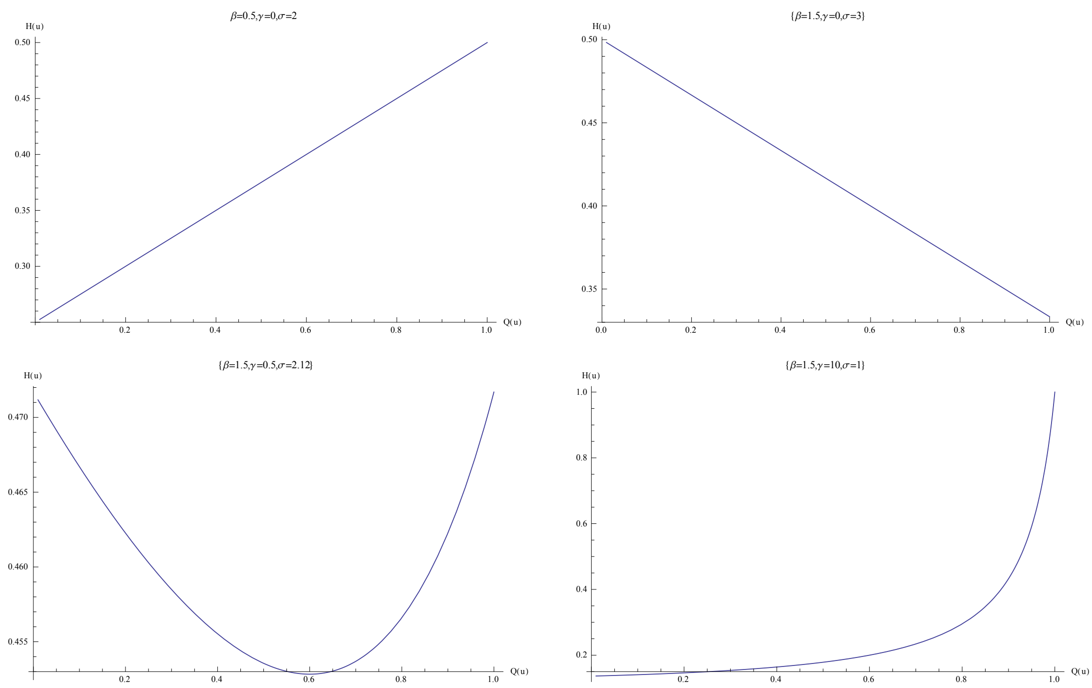


Figure 4.2: Plots of the hazard function for different values of parameters.

It is well-known that increasing (decreasing) failure rate implies decreasing (increasing) mean residual life (see Lai and Xie [78]). The ageing behaviour of the class of distributions (4.3.3) based on mean residual quantile function can be defined from Table 4.1. There exists closed form expressions of the hazard quantile function and mean residual quantile function defined in reverse time (see Nair and Sankaran, 2009) for the proposed class of distributions (4.3.3).

The total time on test transform has the expression

$$\begin{aligned} T(u) &= \int_0^u (1-p)q(p)dp. \\ &= \frac{\sigma\left((\beta\gamma+\beta-1)\log\left(\frac{\beta}{\beta-\beta u+u}\right)-\frac{(\beta-1)\gamma u}{\beta-\beta u+u}\right)}{(\beta-1)^2}. \end{aligned} \tag{4.5.3}$$

Table 4.1: Shape of the hazard quantile function for different regions of the parameter space.

Sl.No	Parameter Region	Shape of the hazard quantile function
1	$\beta \neq 1,\, \gamma = 0$	DHR
2	$\beta = 1,\, \gamma = 0$	Constant
3	$1 < \beta < 2,\, -1 < \gamma < 0$	DHR
4	$1 < \beta < 2,\, \gamma > \frac{\beta-1}{2-\beta}$	IHR
5	$1 < \beta < 2,\, 0 < \gamma < \frac{\beta-1}{2-\beta}$	Bathtub
6	$0 < \beta < 1,\, \gamma > \frac{1-\beta}{\beta-2}$	IHR
7	$0 < \beta < 1,\, -1 < \gamma < \frac{1-\beta}{\beta-2}$	Bathtub
8	$\beta > 2,\, \gamma > \frac{\beta-1}{\beta}$	Bathtub
9	$\beta > 2,\, -1 < \gamma < \frac{\beta-1}{\beta}$	DHR

Residual life can be also explained in terms of percentiles as the percentile residual life function. From (4.3.3) we have

$$\begin{aligned} P_\alpha(u) &= Q(1-(1-\alpha)(1-u)) - Q(u) \\ &= \sigma\left(\frac{\alpha\beta\gamma(u-1)}{(\beta(u-1)-u)(\alpha+(\alpha-1)\beta(u-1)-\alpha u+u)}\right) \\ &\quad + \sigma\left(\log\left(\frac{\alpha-\alpha u+u}{(\alpha-1)\beta(u-1)}+1\right) - \log\left(\frac{u}{\beta-\beta u}+1\right)\right), \end{aligned} \tag{4.5.4}$$

and the reversed percentile residual life function,

$$\begin{aligned} R_\alpha(u) &= Q(1-(1-\alpha)(1-u)) \\ &= \sigma\left(\frac{\alpha\beta\gamma(u-1)}{(\beta(u-1)-u)(\alpha+(\alpha-1)\beta(u-1)-\alpha u+u)}\right) \\ &\quad + \sigma\log\left(\frac{\alpha-\alpha u+u}{(\alpha-1)\beta(u-1)}+1\right) - \log\left(\frac{u}{\beta-\beta u}+1\right). \end{aligned} \tag{4.5.5}$$

The Parzens score function defined in Nair et al. [103] is obtained as

$$J(u) = \frac{q'(u)}{q^2(u)}$$

$$= \frac{((\beta-1)u-\beta)(((\beta-1)u-\beta)(-(-2\beta+2(\beta-1)u+1)) - 2(\beta-1)\beta\gamma(u-1)^2)}{(\beta-1)^3(u-1)^2}. \tag{4.5.6}$$

It is important to give characterizations for life distributions by the relationships among reliability concepts. In the same spirit, we present the following characterization theorems.

Theorem 4.5.1. A random variable X belongs to the family of distributions (4.3.3) if and only if $J(u)$ and $H(u)$ satisfies the relation

$$\frac{J(u)}{H(u)} = 2 + \frac{2}{A+A'u} + \frac{1}{B+B'u}, \tag{4.5.7}$$

where A and B are real constants, with $A' = (1-A)$ and $B' = (1-B)$.

Proof. For the class of distributions (4.3.3), we have from (4.5.6) and (4.5.1),

$$\frac{J(u)}{H(u)} = 2 - \frac{2}{\beta-\beta u+u} + \frac{1}{\beta\gamma+\beta-u(\beta\gamma+\beta-1)},$$

which gives

$$\frac{J(u)}{H(u)} = 2 + \frac{2}{A+(1-A)u} + \frac{1}{B+(1-B)u}, \tag{4.5.8}$$

where $A = \beta$ and $B = \beta(1+\gamma)$.

Then (4.5.8) gives,

$$\frac{J(u)}{H(u)} = 2 + \frac{2}{A+A'u} + \frac{1}{B+B'u} \tag{4.5.9}$$

where $A' = (1-A)$ and $B' = (1-B)$.

To prove the converse part, we use the identity given by Nair et al. [103],

$$J(u) = H(u) - (1-u)H'(u). \tag{4.5.10}$$

This can be written as

$$\frac{H'(u)}{H(u)} = \left(\frac{1}{1-u}\right)\left(1 - \frac{J(u)}{H(u)}\right),$$

or

$$\frac{d}{du}\log(H(u)) = \left(\frac{1}{1-u}\right)\left(1 - \frac{J(u)}{H(u)}\right).$$

Thus, we have

$$H(u) = exp\left(\int \frac{1}{1-u}\left(1 - \frac{J(u)}{H(u)}\right)du\right). \tag{4.5.11}$$

From (4.5.7) and (4.5.11), we get

$$H(u) = \frac{(A + (1-A)u)^2}{C(B + (1-B)u)}, \tag{4.5.12}$$

where C is the integrating constant. This identity is the same as (4.5.1) with $C = \sigma, A = \beta$ and $B = \beta(1+\gamma)$. Since $H(u)$ uniquely determines $Q(u)$, X belongs to the proposed class of distribution, which completes the proof. ∎

Theorem 4.5.2. The distribution of the random variable X belongs to the family (4.3.3), if and only if,

$$M(u) - \frac{\beta}{u}T(u) = \left[\frac{A}{u(1-u)}\log\left(\beta^u\left(1 + \frac{Cu}{\beta}\right)^{\beta + Cu}\right)\right] \tag{4.5.13}$$

where A and C real are constants.

Proof. For the class of distributions (4.3.3), from (4.5.2) and (4.5.3), we have

$$\begin{aligned}(1-u)\,M(u) - \left(\frac{(1-u)\beta}{u}\right)T(u) = \frac{A}{u}\big((u-1)\beta\log\left(\frac{\beta}{u+\beta-u\beta}\right) \\ + u\log(u+\beta-u\beta)\big),\end{aligned} \tag{4.5.14}$$

where $A = \frac{(\beta+\beta\gamma-1)\sigma}{u(\beta-1)^2}$.

This can also be expressed as

$$(1-u)\,M(u) - \left(\frac{(1-u)\beta}{u}\right)T(u) = \frac{A}{u}\log\left[\beta^u\left(\frac{\beta+Cu}{\beta}\right)^{\beta+Cu}\right] \tag{4.5.15}$$

where $C = 1-\beta$. This gives

$$(1-u)\,M(u) - \left(\frac{(1-u)\beta}{u}\right)T(u) = \frac{A}{u}\log\left[\beta^u\left(\frac{\beta+Cu)}{\beta}\right)^{\beta+Cu}\right],$$

which implies

$$M(u) - \frac{\beta}{u}T(u) = \left[\frac{A}{u(1-u)}\log\left(\beta^u\left(1+\frac{C}{\beta}\right)^{\beta+Cu}\right)\right].$$

Now to proof the converse part, we have the identity,

$$T(u) = \mu - (1-u)M(u). \tag{4.5.16}$$

From (4.5.16) and (4.5.13), we get $M(u)$ of the form (4.5.2), which completes the proof.

■

4.6 An application

Since the $L-$moments of the proposed model are not in a simple form, to estimate the parameters of (4.3.3), we employ the method of percentiles illustrated in Karian and Dudewicz [69]. We consider the three population percentiles for the proposed class of distributions

(4.3.3) as

$$\rho_1 = Q(0.5) = \sigma\left(\frac{0.5\gamma}{0.5(1-\beta)+\beta} + \log\left(\frac{1}{\beta}+1\right)\right),$$

$$\rho_2 = Q(0.9) - Q(0.1)$$
$$= \sigma\left(\frac{8.89\beta\gamma}{\beta(\beta+9.11)+1} - \log\left(\frac{\beta+0.111}{\beta}\right) + \log\left(\frac{\beta+9}{\beta}\right)\right)$$

and

$$\rho_3 = \frac{Q(0.5) - Q(0.1)}{Q(0.9) - Q(0.5)}$$
$$= \frac{\frac{0.889\beta\gamma}{\beta(\beta+1.11)+0.111} - \log\left(\frac{\beta+0.111}{\beta}\right) + \log\left(\frac{\beta+1}{\beta}\right)}{\frac{8\beta\gamma}{\beta(\beta+10)+9} - \log\left(\frac{\beta+1}{\beta}\right) + \log\left(\frac{\beta+9}{\beta}\right)}. \tag{4.6.1}$$

For a given data $X_1,\, X_2,\, ..., X_n$, let $\tilde{\pi}_p$ denote the $(100p)$th percentile of the data. We compute $\tilde{\pi}_p$ by first writing $(n+1)p$ as $r + \frac{a}{b}$, where r is a positive integer and $\frac{a}{b}$ is a fraction in the interval $[0, 1]$. Let $X_{(1)},\, X_{(2)}, ...,\, X_{(n)}$ be the order statistics of the data, then $\tilde{\pi}_p$ is defined as

$$\tilde{\pi}_p = x_{(r)} + \frac{a}{b}(X_{(r+1)} - X_{(r)}). \tag{4.6.2}$$

Karian and Dudewicz [69] employed this definition of percentiles for estimating parameters of lambda distribution. We now consider the three sample percentiles $\tilde{\rho}_1, \tilde{\rho}_2,$ and $\tilde{\rho}_3$ defined by

$$\tilde{\rho}_1 = \tilde{\pi}_{0.5},\ \ \tilde{\rho}_2 = \tilde{\pi}_{0.9} - \tilde{\pi}_{0.1}\ \text{ and }\ \tilde{\rho}_3 = \frac{\tilde{\pi}_{0.5} - \tilde{\pi}_{0.1}}{\tilde{\pi}_{0.9} - \tilde{\pi}_{0.5}}.$$

The estimates of the parameters β, γ and σ are obtained by equating the sample percentiles to the corresponding population percentiles, $\tilde{\rho}_i = \rho_i, (i = 1, 2, 3)$.

To illustrate the application of the proposed class of distributions, we consider a real data reported in Smith and Naylor [145]. The data are the strengths of 1.5cm glass fibres, measured at the National Physical Laboratory, England. We estimate the parameters using the method of percentiles, the sample percentiles are obtained as

$$\tilde{\rho}_1 = 1.59 \qquad \tilde{\rho}_2 = 0.858 \quad \text{and} \quad \tilde{\rho}_3 = 2.545.$$

We then equate these values to the corresponding population percentiles given in (4.6.1), so that we have three non-linear equations, Newton-Raphson method is used to find the solutions of these equations. The value of the parameters which minimizes the residual sum of squares is used as the initial values for the Newton-Raphson procedure. The estimates of the parameters are

$$\hat{\beta} = 0.072 \qquad \hat{\gamma} = 20.918 \quad \text{and} \quad \hat{\sigma} = 0.0711.$$

Plot of $\hat{H}(u)$ is increasing in u as shown in Figure 4.3. This data was analysed earlier by Barreto-Souza et al. [12]. They employed beta Frechet (BF), exponentiated Frechet (EF) and Frechet distributions for modelling this data. Figure 4.4 gives the density functions of BF, EF and Frechet distributions with the histogram of the data. Figure 4.5 presents

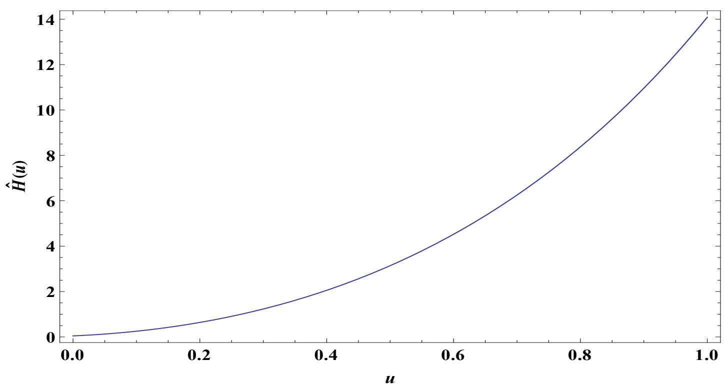


Figure 4.3: Estimate of the hazard quantile function.

the fitted density function of the proposed model for the data. This shows that our model gives a better fit than the other three models to the data set. To examine the adequacy of

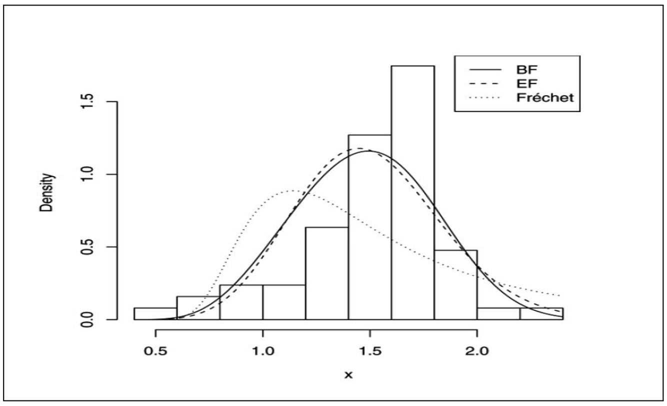


Figure 4.4: The densities of the BF, EF and Frechet distributions for the glass fibres data.

the model, two goodness of fit techniques are employed. The first one is the Q-Q plot, which is given in Figure (4.6). The Q-Q plot ensures the physical closeness of the model to the data. We also carry out the chi-square goodness of fit test. The chi-square value is 1.84 with $p-$value 0.87. This indicates that the proposed model is a reasonable one for the given data set. P-values corresponding to BF, EF and Frechet distributions are 0.65, 0.54 and 0.38 respectively.

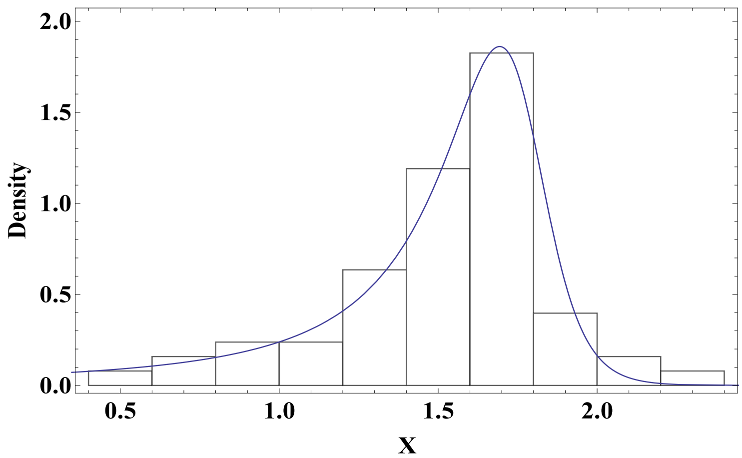


Figure 4.5: The density of the proposed model for the glass fibres data.

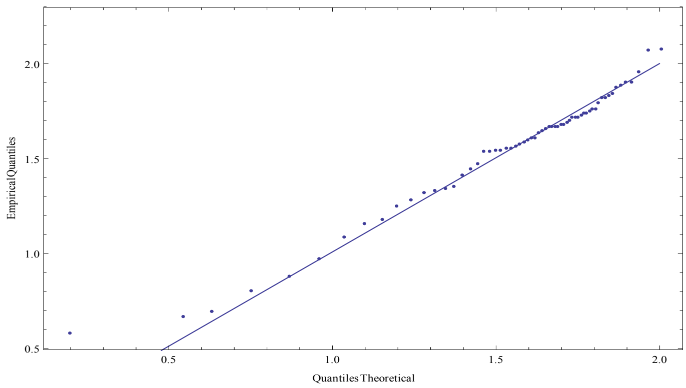


Figure 4.6: Q-Q plot for the glass fibres data.

4.7 Conclusion

We introduced a new method for creating new quantile function models from the existing ones. A general method for the construction of new quantile function from a baseline quantile function was introduced. We derived a new class of distributions using the proposed method by choosing the class of distributions with linear mean residual quantile function as the baseline quantile function. Several well-known distributions are members of the proposed class of distributions. The proposed class of distributions provides a general and flexible framework for the analysis of lifetime data. The estimation of parameters of the model using the method of percentiles was studied. The model was applied to a real data on the strengths of glass fibres. The analysis of shape of hazard quantile function over the whole parameter space can be done without using numerical methods. The quantile function is useful for modelling various kinds of lifetime data due to the flexible behaviour of hazard quantile function.

Chapter 5

A Class of Distributions with Quadratic Mean Residual Quantile Function

5.1 Introduction

The residual life plays a vital role in life testing experiments. It represents the lifetime remaining to a unit after it has attained age t. For a non-negative random variable X with cumulative distribution function $F(x)$, the survival function and the quantile function of the residual life random variable X_t are derived in Section 1.1.3.3. The expected value of X_t is called the mean residual life, which is given in (1.2.7).

Nair and Sankaran [97] defined the mean residual quantile function $M(u)$, which is the quantile version of the mean residual life as defined in Section 1.2.2. Various properties and applications of $M(u)$ are extensively discussed in Nair et al. [105]. Both mean residual quantile function and mean residual life function contains the same information about the underlying lifetime distribution, but their interpretations are different from one another. Midhu et al. [89] proposed a class of distributions with linear mean residual quantile function. Thomas et al. [151] introduced a class of distributions with inverse linear mean resid-

Results in this chapter have been published as entitled "A class of distributions with the quadratic mean residual quantile function" in the Journal of "Communication in Statistics Theory and Methods" (See Sankaran and Dileep Kumar [128]).

ual quantile function. Motivated by this, in the present chapter, we introduce a new class of distributions with quadratic mean residual quantile function. The proposed model can be viewed as a generalization of the class of distributions with linear mean residual quantile function.

The rest of the Chapter is organized as follows. In Section 5.2, we present a new family of distributions with quadratic mean residual quantile function. The distributional properties such as measures of location and scale, and L-moments are given in Section 5.3. Section 5.4 presents various reliability characteristics and some important characterizations of the class. Section 5.5 focuses on the inference procedures of the parameters of the model. We provide an application of this class of distributions in a real life situation in Section 5.6. Finally, Section 5.7 summarizes major conclusions of the study.

5.2 A class of distributions with quadratic mean residual quantile function

We consider a class of distributions with quadratic mean residual quantile function given by $M(u) = A + Bu + Cu^2$. Using (1.2.14), we obtain the corresponding quantile function as

$$Q(u) = -(A + B + C)\log(1 - u) - (2B + C)u - \frac{3C}{2}u^2. \qquad (5.2.1)$$

By re-parametrizing (5.2.1), using $\alpha = A + B + C$, $\beta = A - B$ and $\gamma = A - B - 3C$, the quantile function $Q(u)$ becomes,

$$Q(u) = -\alpha\log(1 - u) + (\beta - \alpha)u + \frac{(\gamma - \beta)}{2}u^2, \quad \alpha \geq 0,\, \beta \geq 0, \gamma \geq 0. \qquad (5.2.2)$$

The support of the proposed class of distributions (5.2.2) is $(0, \infty)$. Since $Q(u)$ is always differentiable for $0 < u < 1$, the quantile density function $q(u)$ is given by

$$q(u) = \beta + u\left(\gamma - \beta + \frac{\alpha}{1-u}\right). \tag{5.2.3}$$

The family of distributions (5.2.3) satisfies the identifiability property. This means that for any two parameter combinations say, $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$, we can easily prove that, $q(\alpha_1, \beta_1, \gamma_1) = q(\alpha_2, \beta_2, \gamma_2)$ if and only if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$. Note that the class of distributions (5.2.2) does not have closed form expression for its distribution function and has to be evaluated numerically. However, we can represent (5.2.3) in terms of the density function $f(x)$ and the distribution function $F(x)$ as

$$f(x) = \frac{1}{q(F(x))} = \left(\beta + F(x)\left(\gamma - \beta + \frac{\alpha}{1-F(x)}\right)\right)^{-1}. \tag{5.2.4}$$

$f(x)$ includes increasing, decreasing and uni-modal bathtub shapes for various choices of the parameters. From (5.2.4), we obtain the derivative of $f(Q(u))$ with respect to u as

$$f'(Q(u)) = \frac{(u-1)^2(\beta-\gamma) - \alpha}{\left(\beta + u^2(\beta-\gamma) + u(\alpha - 2\beta + \gamma)\right)^2}. \tag{5.2.5}$$

Since $\left(\beta + u^2(\beta-\gamma) + u(\alpha - 2\beta + \gamma)\right)^2 \geq 0$ for all $0 \leq u \leq 1$, the sign of $f'(Q(u))$ depends only on the function $V(u) = (u-1)^2(\beta-\gamma) - \alpha$. Now, we consider the following cases;

Case 1. When $\alpha = 0$ and $\beta > \gamma$, we have $V(u) > 0$ for all $0 < u < 1$ and the distribution has increasing probability density function.

Case 2. For $\beta < \gamma$ and $\alpha > 0$, the first and the second terms in $V(u)$ are negative, so that $V(u) < 0$ for all $0 < u < 1$. Thus the distribution has decreasing probability density function.

Case 3. When $\beta > \gamma$ and $\alpha > (\beta - \gamma)$, the first term in $V(u)$ is positive and the second term negative. Since $\alpha > (\beta - \gamma)$, we have $V(u) < 0$ for all $0 < u < 1$ and the distribution has decreasing probability density function.

Case 4. For $\beta > \gamma$ and $0 < \alpha < (\beta - \gamma)$, we obtain the first term in $V(u)$ as positive and the second term as negative. Since $0 < \alpha < (\beta - \gamma)$, we have $V(u)$ can attain a zero at $u_0 = 1 - \frac{\alpha}{\beta - \gamma}$. We can also see that u_0 is a point of maxima, since $V''(u_0) < 0$. Thus the density function $f(x)$ is unimodal and the mode of the distribution is obtained as $x_0 = Q(u_0)$.

Plots of the density function for various combinations of the parameters are shown in Figure 5.1.

5.2.1 Special cases

When $\alpha = \beta = \gamma$, the distribution is exponential with,

$$Q(u) = -\alpha \log(1 - u), \quad \alpha > 0,$$

and when $\alpha = 0$ and $\beta = \gamma$, the distribution is uniform $(0, \beta)$ with

$$Q(u) = \beta u, \quad \beta > 0.$$

For $\alpha = \beta = 0$, we have

$$Q(u) = \frac{\gamma}{2} u^2, \quad \gamma > 0,$$

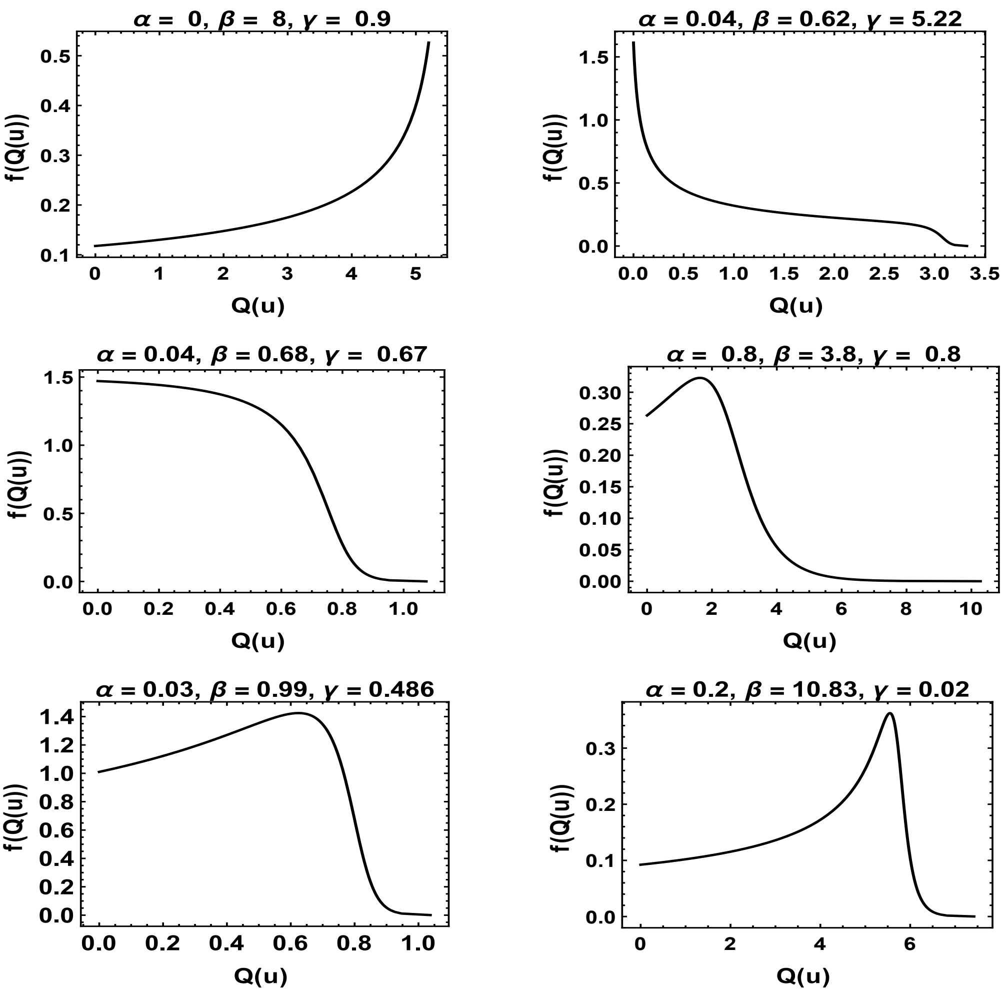


Figure 5.1: Plots of density function for different values of parameters.

which is the quantile function of a power distribution with parameters $\frac{\gamma}{2}$ and 2 respectively. When $\beta = \gamma$, the proposed class of distributions reduces to the class of distributions with linear mean residual quantile function with,

$$Q(u) = -\alpha \log(1-u) + (\beta - \alpha)u. \tag{5.2.6}$$

Midhu et al. [89] studied various properties and applications of (5.2.6). Note that the quantile function (5.2.6) includes only decreasing probability density functions.

5.3 Distributional properties

The quantile-based measures of the distributional characteristics of location, dispersion, skewness and kurtosis are popular in statistical analysis, which are also useful for estimating parameters of the model by matching population and sample characteristics.

5.3.1 Measures of location, spread and shape

For the model (5.2.2), we get the population median M as

$$M = Q\left(\frac{1}{2}\right) = \frac{1}{8}\left(\alpha(\log(256) - 4) + 3\beta + \gamma\right). \tag{5.3.1}$$

The inter-quartile-range, IQR is obtained as

$$\begin{aligned} IQR &= Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) \\ &= \frac{1}{4}(\alpha(\log(81) - 2) + \beta + \gamma). \end{aligned} \tag{5.3.2}$$

Galton's coefficient of skewness (also known as Bowley's coefficient of skewness (Bowley [19])), S is given by

$$\begin{aligned} S &= \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2M}{IQR} \\ &= \frac{16\alpha \log\left(\frac{4}{3}\right) - \beta + \gamma}{4(\alpha(\log(81) - 2) + \beta + \gamma)}, \end{aligned} \tag{5.3.3}$$

and the Moor's coefficient of kurtosis(Moors [92])

$$\begin{aligned} T &= \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \\ &= \frac{\alpha\left(\log\left(\frac{194481}{625}\right) - 2\right) + \beta + \gamma}{\alpha(\log(81) - 2) + \beta + \gamma}. \end{aligned} \tag{5.3.4}$$

5.3.2 L-moments

The rth $L-$moment in terms of the quantile function is given by

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-1-k} \begin{pmatrix} r-1 \\ k \end{pmatrix} \begin{pmatrix} r-1+k \\ k \end{pmatrix} u^k Q(u) du. \tag{5.3.5}$$

The first $L-$moment, L_1 is the mean of the distribution

$$L_1 = \int_0^1 Q(u)\,du = \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{6}. \tag{5.3.6}$$

The second L-moment for the family (5.2.2) is obtained as

$$L_2 = \int_0^1 (2u-1)Q(u)\,du = \frac{\alpha}{3} + \frac{\beta}{12} + \frac{\gamma}{12}, \tag{5.3.7}$$

which is twice the mean differences of the population.

The third and the fourth L-moments of the proposed family are

$$L_3 = \frac{\alpha}{6} - \frac{\beta}{60} + \frac{\gamma}{60}, \tag{5.3.8}$$

and

$$L_4 = \frac{\alpha}{12}. \tag{5.3.9}$$

The L-coefficient of variation τ_2, analogous to the coefficient of variation based on ordinary moments has the form

$$\tau_2 = \frac{L_2}{L_1} = \frac{4\alpha + \beta + \gamma}{6\alpha + 4\beta + 2\gamma}. \tag{5.3.10}$$

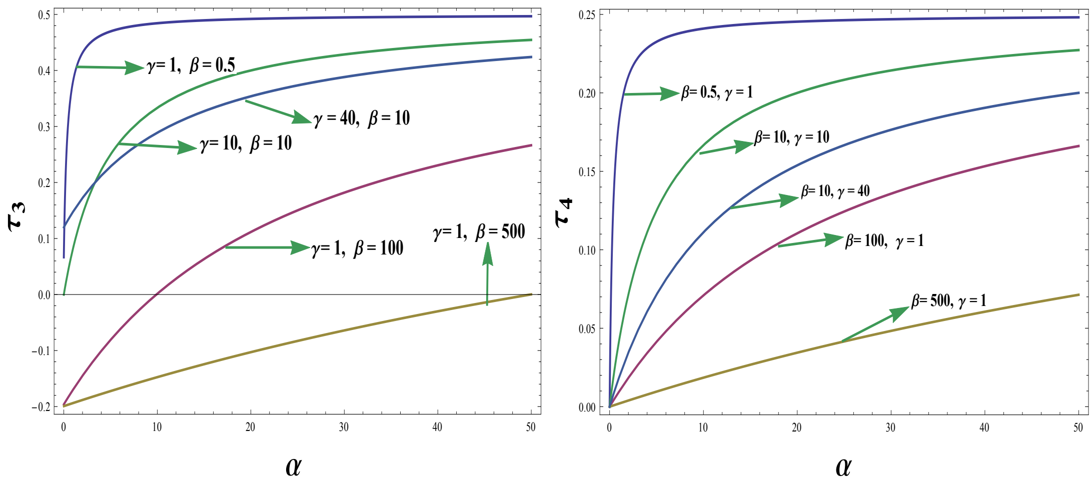
To measure the skewness of the distribution in the quantile set up, the L-coefficient of skewness τ_3 is given by

$$\tau_3 = \frac{L_3}{L_2} = \frac{10\alpha - \beta + \gamma}{5(4\alpha + \beta + \gamma)}. \tag{5.3.11}$$

The L-coefficient of kurtosis for the family (2.6) is obtained as

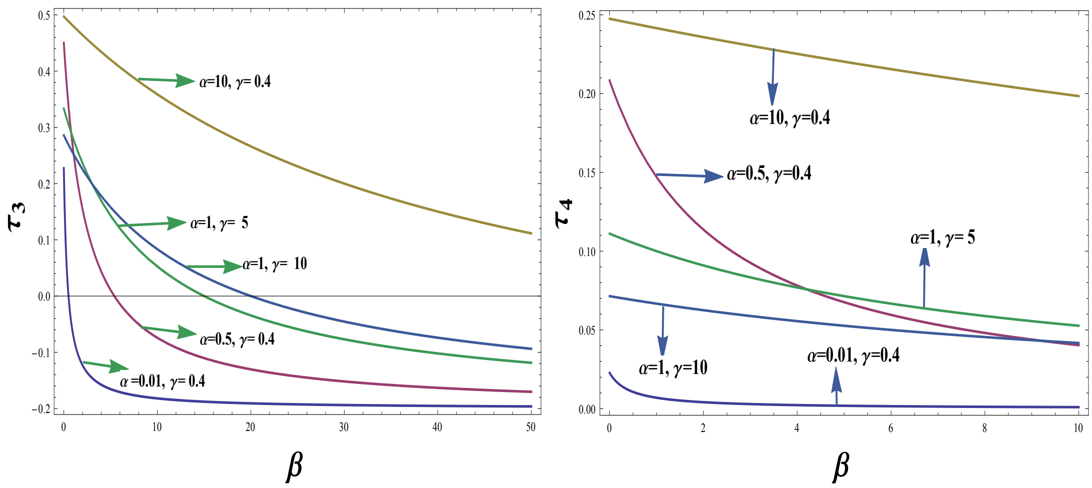
$$\tau_4 = \frac{L_4}{L_2} = \frac{\alpha}{4\alpha + \beta + \gamma}. \tag{5.3.12}$$

Figures 5.2, 5.3 and 5.4 present L-coefficient of skewness (τ_3) and L-coefficient of kurtosis (τ_4) measures for various parameter values. We can show that τ_3 lies in (-0.2, 1) and τ_4 lies in (0, 0.25), using numerical optimization techniques. The values of τ_3 and τ_4 increases with α for fixed β and γ, decreases with β for fixed α and γ. The curve of τ_4 decreases with γ for fixed values of α and β and increases with γ for fixed α and β when $\beta > \alpha$ and decreases when $\beta < \alpha$. Thus the proposed class of distributions (5.2.2) consists of both positively and negatively skewed distributions.



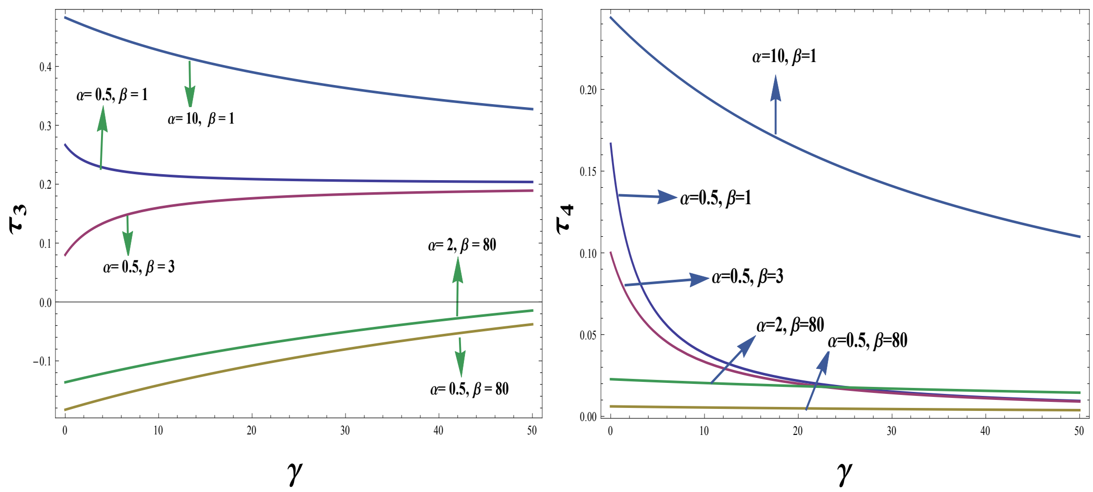
(a) τ_3 (b) τ_4

Figure 5.2: L-coefficients of Skewness and kurtosis for selected values of β and γ as a function of the parameter α.



(a) τ_3 (b) τ_4

Figure 5.3: L-coefficients of Skewness and kurtosis for selected values of α and γ as a function of the parameter β.



(a) τ_3 (b) τ_4

Figure 5.4: L-coefficients of Skewness and kurtosis for selected values of α and β as a function of the parameter γ.

5.3.3 Order statistics

If $X_{r:n}$ is the rth order statistic in a random sample of size n, then the density function of $X_{r:n}$ can be written as

$$f_r(x) = \frac{1}{B(r, n-r+1)} f(x) F^{r-1}(x)(1-F(x))^{n-r}.$$

For the model (5.2.4), we have

$$f_r(x) = \frac{1}{B(r, n-r+1)} \frac{(1-F(x))^{n-r+1}(F(x))^{r-1}}{(1-F(x))(\gamma F(x) + (1-F(x)\beta) + \alpha F(x)}.$$

Hence,

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^\infty x \frac{(1-F(x))^{n-r+1}(F(x))^{r-1}}{(1-F(x))(\gamma F(x) + (1-F(x)\beta) + \alpha F(x)} dx.$$

In quantile terms, we have,

$$\begin{aligned}
E(X_{r:n}) &= \frac{1}{B(r,n-r+1)}\int_0^1 Q(u)u^{r-1}(1-u)^{n-r}du \\
&= \alpha(H_n - H_{n-r}) + \frac{r(\gamma - 2\alpha(n+2) + \beta(2n-r+3) + \gamma r)}{2(n+1)(n+2)},
\end{aligned}$$

where H_n is the nth harmonic number given by $H_n = \sum_{i=1}^{n} \frac{1}{i}$. In particular,

$$E(X_{1:n}) = \frac{\alpha + \beta + \gamma + \frac{2\alpha}{n} + \beta n}{(n+1)(n+2)},$$

and

$$E(X_{n:n}) = \alpha H_n + \frac{n(\gamma - 2\alpha(n+2) + \beta(n+3) + \gamma n)}{2(n+1)(n+2)}.$$

For the class of distributions (5.2.2), the first order statistic $X_{1:n}$ has quantile function,

$$\begin{aligned}
Q_{(1)}(u) &= Q(1-(1-u)^{\frac{1}{n}}) \\
&= -\frac{1}{2}\left((1-u)^{1/n} - 1\right)\left(-2\alpha + \beta + \gamma + (\beta - \gamma)(1-u)^{1/n}\right) - \alpha \log\left((1-u)^{1/n}\right),
\end{aligned}$$

and the quantile function of the nth order statistic is

$$\begin{aligned}
Q_{(n)}(u) &= Q(u^{\frac{1}{n}}) \\
&= (\beta - \alpha)u^{1/n} - \alpha \log\left(1 - u^{1/n}\right) + \frac{1}{2}(\gamma - \beta)u^{2/n}.
\end{aligned}$$

The expressions of the order statistics $X_{1:n}$ and $X_{n:n}$ are useful when studying the series and parallel systems in reliability theory (see Lai and Xie [78]).

5.3.4 Reliability measures

For the class of distributions (5.2.2), the hazard quantile function is obtained as

$$H(u) = \frac{1}{\beta + u(\alpha + \gamma + \beta(u-2) - \gamma u)}. \tag{5.3.13}$$

The hazard quantile function accommodates increasing, decreasing, linear, bathtub and upside-down bathtub shapes for different choices of the parameters with $H(0) = \frac{1}{\alpha(1-p)+2\beta}$ and $H(1) = \frac{1}{\alpha+\beta}$. The shape of the hazard function is determined by the derivative of $H(u)$, which is obtained as

$$H^{'}(u) = \frac{u(\gamma - \beta) + \beta - \left(\frac{\alpha+\gamma}{2}\right)}{(\beta + u(\alpha + \gamma + \beta(u-2) - \gamma u))^2}. \tag{5.3.14}$$

Since $(\beta + u(\alpha + \gamma + \beta(u-2) - \gamma u))^2 > 0$ for all values of the parameters, the sign of $H'(u)$ depends only on

$$K(u) = u(\gamma - \beta) + \beta - \left(\frac{\alpha+\gamma}{2}\right). \tag{5.3.15}$$

Now, we consider the following cases,

Case 1: $\boldsymbol{\alpha = \beta = \gamma}$.

$$H(u) = \frac{1}{\gamma}, \quad \text{a constant. Thus the distribution is exponential.}$$

Case 2: $\boldsymbol{\beta = \gamma > \alpha}$.

The first term in $K(u)$ is zero and the second term is positive, so that $K(u) > 0$ for all $0 < u < 1$ and hence the distribution is IHR.

Case 3: $\boldsymbol{\beta = \gamma < \alpha}$.

The first term in $K(u)$ is zero and the second term is negative, which gives $K(u) < 0$ for all $0 < u < 1$ and the distribution has decreasing failure rate.

Case 4: $\boldsymbol{\gamma < \beta = \frac{\gamma+\alpha}{2}}$.

The first term in $K(u)$ is negative and the second term is zero, so that $K(u) < 0$ for all $0 < u < 1$ and the distribution is DHR.

Case 5: $\boldsymbol{\gamma > \beta = \frac{\gamma+\alpha}{2}}$.

The first term in $K(u)$ is positive and the second term is zero, which implies $K(u) > 0$ for all $0 < u < 1$. Thus the distribution is IHR.

Case 6: $\boldsymbol{\frac{\alpha+\gamma}{2} < \beta < \gamma}$.

Both terms of $K(u)$ are positive and hence the distribution is IHR.

Case 7: $\boldsymbol{\gamma < \beta < \frac{\alpha+\gamma}{2}}$.

Both terms of $K(u)$ are negative and hence the distribution is DHR.

Case 8: $\boldsymbol{\gamma < \beta, \beta > \frac{\alpha+\gamma}{2}}$ **and** $\boldsymbol{\alpha < \gamma}$.

Since $\alpha < \gamma$, we have $\beta - \gamma < \beta - \frac{\alpha+\gamma}{2}$. Now, in this case, it is clear that $-\left(\beta - \frac{\alpha+\gamma}{2}\right) < \gamma - \beta < 0$, which gives $K(u) > 0$ for all $0 < u < 1$ and the distribution is IHR.

Case 9: $\boldsymbol{\gamma < \beta, \beta > \frac{\alpha+\gamma}{2}}$ **and** $\boldsymbol{\alpha > \gamma}$.

We have $0 < \beta - \frac{\alpha+\gamma}{2} < \beta - \gamma$, which is obtained from the condition $\alpha > \gamma$. The first term of $K(u)$ is negative and the second term is positive so that $K(u)$ can attain a zero in this case. This in turn gives the possibility for $H(u)$ to be non-monotone. Let u_0 be the solution of the equation $K(u) = 0$. From (5.3.15), we have u_0 is the solution corresponding to the linear equation,

$$u(\gamma-\beta)+\beta-\left(\frac{\alpha+\gamma}{2}\right)=0, \tag{5.3.16}$$

which provides,

$$u_0=\frac{\alpha+\gamma-2\beta}{2(\gamma-\beta)}. \tag{5.3.17}$$

The second derivative of $H(u)$ is

$$H''(u)=\frac{2\Big(\alpha^2+\alpha(\beta(3u-4)+\gamma(2-3u))+3\beta^2(u-1)^2-3\beta\gamma(u-1)(2u-1)+\gamma^2(3(u-1)u+1)\Big)}{(\beta+u(\alpha+\gamma+\beta(u-2)-\gamma u))^3}. \tag{5.3.18}$$

For the change point u_0 obtained in (5.3.17), we obtain

$$H''(u_0)=-\frac{32(\beta-\gamma)^3}{\left(\alpha^2+2\alpha(\gamma-2\beta)+\gamma^2\right)^2}. \tag{5.3.19}$$

Since $H''(u_0)<0$, $H(u)$ attains a maximum at u_0. Hence X has an upside-down bathtub-shaped hazard quantile function (see Nair et al. [105]).

Case 10: $\boldsymbol{\gamma>\beta}$, $\boldsymbol{\beta<\frac{\alpha+\gamma}{2}}$ **and** $\boldsymbol{\alpha>\gamma}$.

Since $\alpha>\gamma$, we obtain $0<\gamma-\beta<\frac{\alpha+\gamma}{2}-\beta$. This leads to $K(u)>0$ for all $0<u<1$ and the distribution is IHR.

Case 11: $\boldsymbol{\gamma>\beta}$, $\boldsymbol{\beta<\frac{\alpha+\gamma}{2}}$ **and** $\boldsymbol{\alpha<\gamma}$.

First term of $K(u)$ is positive and the second term is negative so that $K(u)$ can attain a zero in this case. Now as similar to Case 9, $H(u)$ is non-monotone with change point, say $u_1\in(0,1)$, obtained as

$$u_1=\frac{\alpha+\gamma-2\beta}{2(\gamma-\beta)}.$$

The second derivative of $H(u)$ at u_1 is obtained as

$$H''(u_1) = -\frac{32(\beta-\gamma)^3}{(\alpha^2+2\alpha(\gamma-2\beta)+\gamma^2)^2}.$$

Since $H''(u_1) > 0$, $H(u)$ attains a minimum at u_1. Thus X has a bathtub shaped hazard quantile function.

The ageing pattern of $H(u)$ for various parameter values are summarised in Table 5.1. Plots

Table 5.1: Shape of the hazard quantile function for different regions of the parameter space.

Case	Parameter Region	Shape of the hazard quantile function
1	$\alpha=\beta=\gamma$	Constant
2	$\beta=\gamma>\alpha$	IHR
3	$\beta=\gamma<\alpha$	DHR
4	$\gamma<\beta=\frac{\gamma+\alpha}{2}$	DHR
5	$\gamma>\beta=\frac{\gamma+\alpha}{2}$	IHR
6	$\gamma>\beta, \beta>\frac{\alpha+\gamma}{2}$	IHR
7	$\gamma<\beta, \beta<\frac{\alpha+\gamma}{2}$	DHR
8	$\gamma<\beta, \beta>\frac{\alpha+\gamma}{2}, \alpha<\gamma$	IHR
9	$\gamma<\beta, \beta>\frac{\alpha+\gamma}{2}, \alpha>\gamma$	Upside-down bathtub
10	$\gamma>\beta, \beta<\frac{\alpha+\gamma}{2}, \alpha>\gamma$	IHR
11	$\gamma>\beta, \beta<\frac{\alpha+\gamma}{2}, \alpha<\gamma$	Bathtub

of the hazard quantile function for different values of the parameters are given in Figure 5.5. The lower and upper limit's for the hazard quantile function of the class of distributions (5.2.2) are independent of the parameter γ, and has the form,

$$\lim_{u\to 0} H(u) = \frac{1}{\beta} \quad \text{and} \quad \lim_{u\to 1} H(u) = \frac{1}{\alpha}. \tag{5.3.20}$$

For the class of distributions (5.2.2), the total time on test transform $T(u)$ is obtained as

$$T(u) = \frac{1}{6}u\left(6\beta+2u^2(\beta-\gamma)+3u(\alpha-2\beta+\gamma)\right). \tag{5.3.21}$$

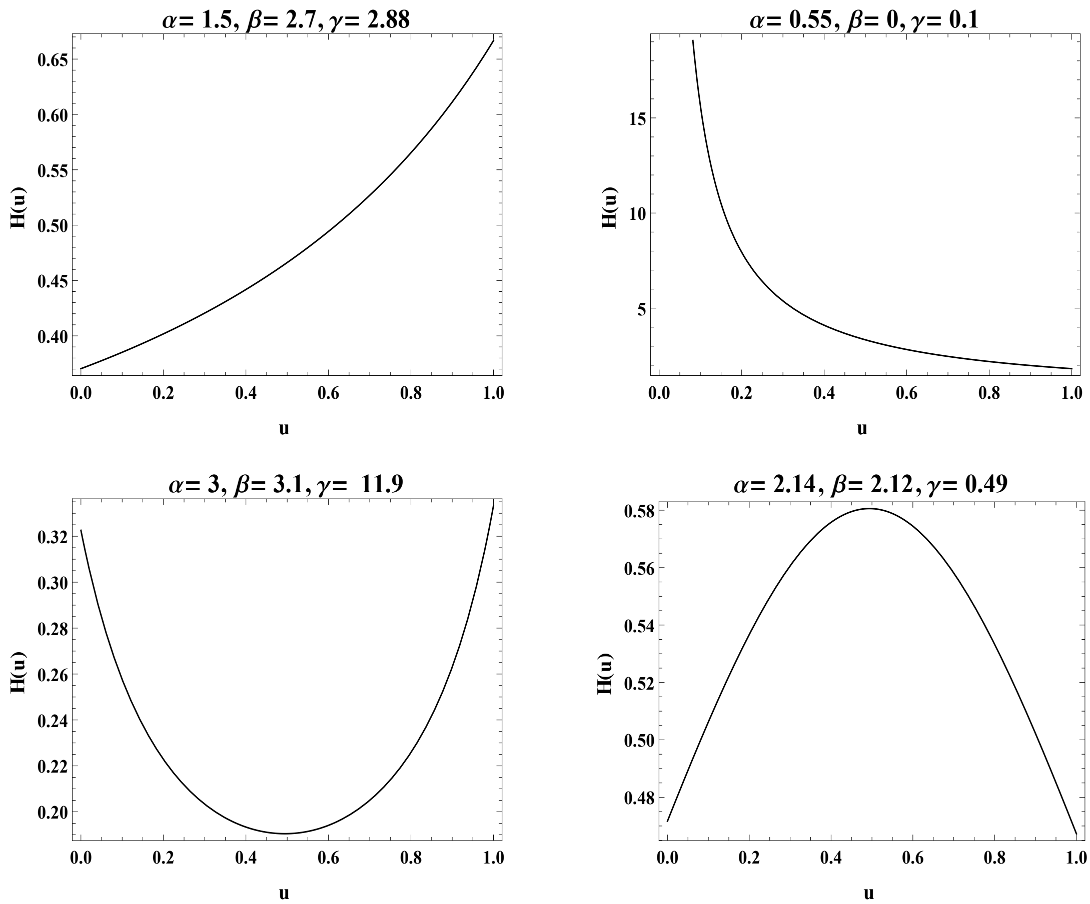


Figure 5.5: Plots of the hazard quantile function for different values of parameters.

The quantile form of the first L-moment of residual life given by Nair and Vineshkumar [100] has the form

$$\begin{aligned} \alpha_1(u) &= \frac{1}{1-u}\int_u^1 Q(p)dp \\ &= \frac{1}{6}\left(2\beta+\gamma-\beta u^2+\gamma u^2-3\alpha(u-1)-6\alpha\log(1-u)+2\beta u+\gamma u\right). \end{aligned} \quad (5.3.22)$$

The second L-moment of residual life (Nair and Vineshkumar [100]) is obtained as

$$\alpha_2(u) = \frac{1}{(1-u)^2}\int_u^1 (2p-u-1)Q(p)dp$$

$$= \frac{\alpha}{3} + \frac{\beta}{12} + \frac{\gamma}{12} + u^2\left(\frac{\beta}{12} - \frac{\gamma}{12}\right) + u\left(\frac{\alpha}{6} - \frac{\beta}{6}\right). \tag{5.3.23}$$

In reversed time, the first L-moment of $X|X \leq t$ of (5.2.2) is obtained as

$$\theta_1(u) = \frac{u(u(3\beta + \beta(-u) + \gamma u) - 3\alpha(u-2)) - 6\alpha(u-1)\log(1-u)}{6u}. \tag{5.3.24}$$

The second L-moment of $X|X \leq t$ is

$$\theta_2(u) = \frac{-\beta u^4 + \gamma u^4 - 2\alpha u^3 + 2\beta u^3 - 6\alpha u^2 + 12\alpha u - 12\alpha u\log(1-u) + 12\alpha\log(1-u)}{12u^2}. \tag{5.3.25}$$

From Nair and Vineshkumar [100], we have the identity,

$$R(u) = u\theta_1'(u) = u\theta_2'(u) + 2\theta_2(u),$$

where $R(u)$ is the reversed mean residual quantile function.

Parzen's score function, defined in Parzen [117] and Nair et al. [103] is

$$\begin{aligned} J(u) &= -\frac{d}{du}\left(\frac{1}{q(u)}\right) = \frac{q'(u)}{q^2(u)} \\ &= \frac{\alpha - (u-1)^2(\beta-\gamma)}{(\beta + u(\alpha + \gamma + \beta(u-2) - \gamma u))^2}. \end{aligned} \tag{5.3.26}$$

5.3.5 Characterizations

We now present some useful characterizations of the proposed family of distributions using various quantile-based reliability measures.

Theorem 5.3.1. A random variable X has the quadratic mean residual quantile function if and only if the hazard quantile function $H(u)$ is a reciprocal quadratic function of the form,

$$H(u) = \frac{1}{K_1 + K_2 u + K_3 u^2}, \tag{5.3.27}$$

where $K_1 > 0$, K_2, K_3 are real constants.

Proof. Suppose that (5.3.27) holds. From Nair et al. [105], we have

$$M(u) = \frac{1}{1-u} \int_u^1 \frac{1}{H(p)} dp. \tag{5.3.28}$$

Substituting (5.3.27) in (5.3.28) and integrating we get

$$M(u) = \zeta_1 + \zeta_2 u + \zeta_3 u^2,$$

where $\zeta_1 = K_1 + \frac{K_2}{2} + \frac{K_3}{3}$, $\zeta_2 = \frac{K_2}{2} + \frac{K_3}{3}$ and $\zeta_3 = \frac{K_3}{2}$.
Conversely, for the class of distributions (5.2.2), we obtain

$$H(u) = \frac{1}{\beta(1 + u^2 - 2u) + \gamma\, u(1-u) + \alpha\, u},$$

or

$$H(u) = \frac{1}{\beta + u^2(\beta - \gamma) + u(\alpha - 2\beta + \gamma)}, \tag{5.3.29}$$

which is reciprocal quadratic in u. The rest of the proof follows from (5.3.29). ■

Theorem 5.3.2. A random variable X has the quadratic mean residual quantile function if and only if the hazard quantile function satisfies the first order homogeneous ordinary differential equation,

$$H'(u) + H(u)\left(\frac{B + 2Cu}{A + Bu + Cu^2}\right) = 0, \tag{5.3.30}$$

where $A > 0$, B and C are real constants. Assume that $H(u)$ is twice differentiable with respect to u.

Proof. For the class of distributions (5.2.2), $H(u)$ satisfies the first order homogeneous ordinary differential equation,

$$H'(u)(\beta + u(\alpha + \gamma + \beta(u-2) - \gamma u)) + H(u)(\alpha + \gamma + 2\beta(u-1) - 2\gamma u) = 0.$$

After simplification we obtain,

$$H'(u) + H(u)\left(\frac{B + 2Cu}{A + Bu + Cu^2}\right) = 0,$$

where $A = \beta, B = \alpha + \gamma - 2\beta$ and $C = \beta - \gamma$.

Conversely, when $H(u)$ is twice differentiable and satisfies the differential equation (5.3.30), we rewrite the differential equation as

$$\frac{d}{du}\left[H(u)(A + Bu + Cu^2)\right] = 0. \tag{5.3.31}$$

On integrating (5.3.31), we obtain $H(u)$ as

$$H(u) = \frac{K}{A + Bu + Cu^2}, \tag{5.3.32}$$

where K is the integrating constant. From (5.3.32), we have $H(u)$ is reciprocal quadratic in u, which completes the proof. ∎

Theorem 5.3.3. A random variable X has the quadratic mean residual quantile function if and only if

$$T(u) - uM(u) = (C_1 + C_2 u)u, \tag{5.3.33}$$

where C_1, C_2 are real constants.

Proof. Suppose (5.3.33) is true. From Nair et al. [105], we have the identity,

$$T(u) = \mu - (1-u)M(u). \tag{5.3.34}$$

From (5.3.33) and (5.3.34), we obtain

$$M(u) = \mu - C_1 u - C_2 u^2, \text{ which is quadratic in } u$$

Conversely, for the class of distributions (5.2.2), we have,

$$T(u) = \frac{1}{6} u \left(6\beta + 2u^2(\beta - \gamma) + 3u(\alpha - 2\beta + \gamma)\right). \tag{5.3.35}$$

Now, (5.3.35) can be written as

$$T(u) - uM(u) = (C_1 + C_2 u)u,$$

where $C_1 = \frac{2\beta}{3} - \frac{\gamma}{6} - \frac{\alpha}{2}$ and $C_2 = \frac{\gamma - \beta}{3}$. This completes the proof. ■

Theorem 5.3.4. Let X be a random variable having the quantile function $Q(u)$. Then X belongs to the class of distributions with quadratic mean residual quantile function if and only if

$$\frac{H(u)}{J(u)} = \frac{A + Bu + Cu^2}{(A+B) + Cu(2-u)}, \tag{5.3.36}$$

where A, B and C are real constants.

Proof. From Nair et al. [103], we have,

$$(1-u)H'(u) = H(u) - J(u). \tag{5.3.37}$$

From (5.3.37) and Theorem 5.3.2, we obtain,

$$
\begin{aligned}
& H'(u) + H(u)\left(\frac{B+2Cu}{A+Bu+Cu^2}\right) = 0. \\
\Leftrightarrow & \left(\frac{H(u)-J(u)}{1-u}\right) + H(u)\left(\frac{B+2Cu}{A+Bu+Cu^2}\right) = 0 \\
\Leftrightarrow & H(u)\left(1+\frac{(1-u)(B+2Cu)}{A+Bu+Cu^2}\right) - J(u) = 0 \\
\Leftrightarrow & H(u)\left(\frac{(A+B)+Cu(2-u)}{A+Bu+Cu^2}\right) - J(u) = 0 \\
\Leftrightarrow & \frac{H(u)}{J(u)} = \frac{A+Bu+Cu^2}{(A+B)+Cu(2-u)}.
\end{aligned}
$$

This completes the proof. ■

The equilibrium distribution is a widely accepted tool in the context of analysis of ageing phenomena. The equilibrium distribution associated with X is defined by the density function,

$$g(x) = \mu^{-1}\int_x^\infty \bar{F}(t)dt. \tag{5.3.38}$$

Suppose Z is the random variable associated with the density (5.3.38). Then from Nair et al. [105], we have,

$$H_Z(u) = \frac{1}{M_X(u)}. \tag{5.3.39}$$

where $H_Z(u)$ and $M_X(u)$ are the hazard quantile function of Z and the mean residual quantile function of X respectively. From (5.3.39), it follows that the hazard quantile function of the equilibrium random variable is the reciprocal of the mean residual quantile function of the baseline distribution.

Theorem 5.3.5. The distribution of X belongs to the class of distributions with quadratic mean residual quantile function if and only if the equilibrium distribution of X is also a member of the same class of distributions.

Proof. Suppose X belongs to the class of distributions with quadratic mean residual quantile function, $M(u) = A + Bu + Cu^2$. Now from (5.3.39), we obtain

$$H_Z(u) = \frac{1}{A + Bu + Cu^2}. \tag{5.3.40}$$

Since $H_Z(u)$ is a reciprocal quadratic function in u, from Theorem 5.3.1, we have the equilibrium random variable Z belongs to the class of distributions with quadratic mean residual quantile function. The proof of the converse part is direct from the identity (5.3.39). ■

Another measure, which has received wide applications in economics is the Ginis mean difference, defined as

$$G(t) = 2\int_t^\infty F_t(x)\bar{F}_t(x)dx.$$

Nair and Vineshkumar [100] presented the quantile-based Ginis mean difference as

$$\Delta(u) = G(Q(u)) = 2\int_u^1 \frac{(1-p)(p-u)}{(1-u)^2}dp = 2\alpha_2(u). \tag{5.3.41}$$

From (5.3.41), we have $\alpha_2(0)$ is half the mean difference of X, which is a widely accepted measure of dispersion in the analysis of income and poverty in economic studies.

Theorem 5.3.6. Let X be a random variable with quantile function $Q(u)$. Then X has quadratic mean residual life if and only if the second L-moment of residual life is a quadratic function in u.

Proof. Suppose $M(u) = A + Bu + Cu^2$. We have,

$$\alpha_2(u) = \frac{1}{(1-u)^2}\int_u^1 (1-p)M(p)dp$$

$$(1-u)^2\alpha_2(u) = \int_u^1 (1-p)M(p)dp. \tag{5.3.42}$$

Differentiating (5.3.42) with respect to u and simplifying, we get

$$\alpha_2(u) = (B - 2A) + u(2C - 3B) - 4Cu^2, \tag{5.3.43}$$

which is quadratic in u. Conversely, for the class of distributions (5.2.2), from (5.3.23), we get $\alpha_2(u)$ as a quadratic function in u. This completes the proof. ■

Corrolary 5.3.6.1. Let X be a random variable with quantile function $Q(u)$. Then X has quadratic mean residual life if and only if the quantile-based Gini's mean difference $\Delta(u)$ of X is a quadratic function in u.

Proof. The proof follows from Theorem 5.3.6 once we note that $\Delta(u) = 2\alpha_2(u)$. ■

5.4 Estimation of parameters

In the literature, there are different methods for the estimation of parameters of the quantile functions (Gilchrist [42]). The L-moment estimates are likely to have less bias compared to other estimates. To estimate the parameters of the proposed model (5.2.2), we use the method of L-moments. We equate sample L-moments to corresponding population L-moments. The first three sample L-moments are given by

$$\begin{aligned}
l_1 &= \left(\frac{1}{n}\right) \sum_{i=1}^{n} X_{(i)}, \\
l_2 &= \left(\frac{1}{2}\right) \binom{n}{2}^{-1} \sum_{i=1}^{n} \left(\binom{i-1}{1} - \binom{n-i}{1} \right) X_{(i)}, \\
l_3 &= \left(\frac{1}{3}\right) \binom{n}{3}^{-1} \sum_{i=1}^{n} \left(\binom{i-1}{2} - 2\binom{i-1}{1}\binom{n-i}{1} + \binom{n-i}{2} \right) X_{(i)},
\end{aligned}$$

where $X_{(i)}$ is the ith order statistic in a sample of size n. For estimating the parameters α, β and γ, we equate the above three sample L-moments to the corresponding population L-moments given in (5.3.6) through (5.3.8). The parameters are obtained by solving the equations,

$$l_r = L_r; \quad r = 1, 2, 3. \tag{5.4.1}$$

For the class of distributions (5.2.2), using Theorem 1.1.1, we have,

$$\sqrt{n}\begin{pmatrix} l_1 - L_1 \\ l_2 - L_2 \\ l_3 - L_3 \end{pmatrix} \sim \quad N(0, \Lambda),$$

where the variance-covariance matrix Λ has the form,

$$\Lambda = \begin{pmatrix} \Lambda_{1,1} & \Lambda_{1,2} & \Lambda_{1,3} \\ \Lambda_{2,1} & \Lambda_{2,2} & \Lambda_{2,3} \\ \Lambda_{3,1} & \Lambda_{3,2} & \Lambda_{3,3} \end{pmatrix},$$

with,

$$\begin{aligned}
\Lambda_{1,1} &= \frac{1}{180}\left(105\alpha^2 + 5\alpha(5\beta + 7\gamma) + 4\beta^2 + 7\beta\gamma + 4\gamma^2\right), \\
\Lambda_{1,2} &= \frac{1}{720}\left(350\alpha^2 + 59\alpha\beta + 101\alpha\gamma + 6\beta^2 + 14\beta\gamma + 10\gamma^2\right), \\
\Lambda_{1,3} &= \frac{1}{50400}\left(17115\alpha^2 + 784\alpha\beta + 3311\alpha\gamma - 260\beta^2 - 110\beta\gamma + 160\gamma^2\right), \\
\Lambda_{2,1} &= \frac{1}{720}\left(350\alpha^2 + 59\alpha\beta + 101\alpha\gamma + 6\beta^2 + 14\beta\gamma + 10\gamma^2\right), \\
\Lambda_{2,2} &= \frac{1}{5040}\left(2072\alpha^2 + 238\alpha\beta + 518\alpha\gamma + 17\beta^2 + 50\beta\gamma + 45\gamma^2\right), \\
\Lambda_{2,3} &= \frac{1}{100800}\left(29890\alpha^2 + 718\alpha\beta + 5092\alpha\gamma - 165\beta^2 - 80\beta\gamma + 245\gamma^2\right), \\
\Lambda_{3,1} &= \frac{1}{50400}\left(17115\alpha^2 + 784\alpha\beta + 3311\alpha\gamma - 260\beta^2 - 110\beta\gamma + 160\gamma\right),
\end{aligned}$$

$$\Lambda_{3,2} = \frac{1}{100800}\left(29890\alpha^2 + 718\alpha\beta + 5092\alpha\gamma - 165\beta^2 - 80\beta\gamma + 245\gamma^2\right),$$
$$\text{and}\quad \Lambda_{3,3} = \frac{1}{100800}\left(23025\alpha^2 - 199\alpha\beta + 2869\alpha\gamma + 158\beta^2 + 59\beta\gamma + 128\gamma^2\right).$$

From (5.4.1), we have $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ are linear functions of sample L-moments given by,

$$\hat{\alpha} = 3l_1 - 9l_2 + 15l_3, \quad \hat{\beta} = 9l_1 - 21l_2 + 15l_3, \quad \text{and} \quad \hat{\gamma} = -21l_1 + 69l_2 - 75l_3. \tag{5.4.2}$$

Now it is straight forward from the variance covariance matrix and (5.4.2) that,

$$\sqrt{n}(\hat{\alpha} - \alpha) \sim N(0, \sigma_\alpha^2), \tag{5.4.3}$$
$$\sqrt{n}(\hat{\beta} - \beta) \sim N(0, \sigma_\beta^2), \tag{5.4.4}$$
$$\text{and} \qquad \sqrt{n}(\hat{\gamma} - \gamma) \sim N(0, \sigma_\gamma^2), \tag{5.4.5}$$

where

$$\sigma_\alpha^2 = \frac{9\left(3533\alpha^2 - 79\alpha\beta + 293\alpha\gamma + 88\beta^2 + 63\beta\gamma + 22\gamma^2\right)}{2240}, \tag{5.4.6}$$
$$\sigma_\beta^2 = \frac{2397\alpha^2 + 825\alpha\beta + 821\alpha\gamma + 288\beta^2 + 487\beta\gamma + 222\gamma^2}{2240}, \tag{5.4.7}$$
$$\text{and} \qquad \sigma_\gamma^2 = \frac{204517\alpha^2 + 28609\alpha\beta + 28557\alpha\gamma + 25128\beta^2 + 28367\beta\gamma + 9062\gamma^2}{2240}. \tag{5.4.8}$$

The asymptotic $100(1-\eta)\%$ confidence intervals for α, β and γ are given by

$$C.I(\alpha) = \hat{\alpha} \pm z_{\eta/2}\sqrt{\frac{\hat{\sigma}_\alpha^2}{n}},$$
$$C.I(\beta) = \hat{\beta} \pm z_{\eta/2}\sqrt{\frac{\hat{\sigma}_\beta^2}{n}},$$
$$\text{and} \qquad C.I(\gamma) = \hat{\gamma} \pm z_{\eta/2}\sqrt{\frac{\hat{\sigma}_\gamma^2}{n}},$$

where $z_{\eta/2}$ is the $100(1-\eta/2)$th percentile of the standard normal distribution. Further, $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ are consistent estimators of α, β and γ respectively, which follows from the fact that sample L-moments are consistent estimators of population L-moments (see Hosking [60]).

5.4.1 Akaike information criterion (AIC)

Let X be a random variable with density function $f(X, \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ be a vector of parameters. Suppose $\hat{\theta}_i$ is the estimate of the parameter $\theta_i, i = 1, 2, ...k$ based on a random sample $X_1, X_2, ...X_n$. Then the AIC of the model is given by

$$\mathrm{AIC} = 2k - 2\log(\hat{L}), \tag{5.4.9}$$

where $\hat{L} = \prod_1^n f(X_i, \theta)$. Suppose $Q(u)$ be the quantile function corresponding to the random variable X. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the ordered sample. Then the quantile version of (5.4.9) is obtained as

$$AIC = 2k - 2\log\left(\frac{1}{\prod_1^n q(u_i, \theta)}\right), \tag{5.4.10}$$

where the values of $u_i, i = 1, 2, 3, ...n$ are calculated using the following algorithm (Gilchrist [42], Nair et al. [105]).

(i) Take $u_i^0 = \frac{i}{n+1}, i = 1, 2, ..., n$ as the initial estimate of u_i corresponding to $X_{(i)}$.

(ii) Compute $D_i = Q(u_i^0) - X_{(i)}$ for $i = 1, 2, ..., n$, and update each u_i^0 with $u_i^1 = u_i^0 + \frac{X_{(i)} - Q(u_i^0)}{q(u_i^0)}$, when $|D_i| > \epsilon$, a small positive quantity, say $\epsilon = 10^{-7}$.

(iii) Iterate step-(ii) until $|D_i| \leq \epsilon$ for all i.

The performance of the quantile function models with no closed form expression for their distribution function can be assessed using the AIC measure (5.4.10).

5.5 An application

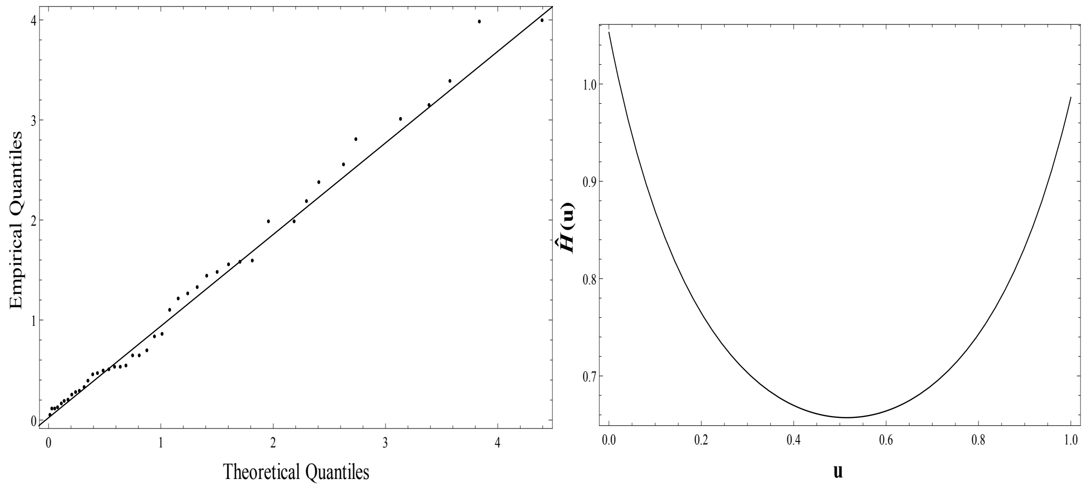
We give an application of the proposed family using a well-known data set to demonstrate the flexibility and applicability of the proposed model over other lifetime parametric models. The method of L-moments is used for estimating the parameters. We consider the data set reported in Bekker et al. [15], which corresponds to the survival times (in years) of a group of 45 patients given chemotherapy treatment alone. The sample L-moments are obtained as

$$l_1 = 1.341 \qquad l_2 = 0.676 \quad \text{and} \quad l_3 = 0.205. \tag{5.5.1}$$

Then the estimates of the parameters are

$$\hat{\alpha} = 1.014 \qquad \hat{\beta} = 0.949 \quad \text{and} \quad \hat{\gamma} = 3.11. \tag{5.5.2}$$

The standard error of the estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are obtained as 0.099, 0.046 and 0.33 respectively. To examine the goodness of fit of the model, we use a Q-Q plot, presented in Figure 5.6(a), which shows that our model fit's well to the data. We also carry out the chi square goodness of fit test. The chi-square value is 0.73 with $p-$value 0.69, which does not reject the model. AIC value obtained is 122.106. Recently Handique and Chakraborty [54] fitted this data with Kumaraswamy Weibull (K-W), beta generalized Weibull (B-W) and beta generalized Kumaraswamy Weibull (BKw-W) distributions. The AIC values of these models are 124.14, 123.44, and 122.92 respectively. It is evident for this data that our model gives a better fit than the other three with respect to the AIC values. Plot of the estimated hazard quantile function is given in Figure 5.6(b). Note that $\hat{H}(u)$ has bathtub shape, which supports our claim in Table 5.1.



(a) Q-Q plot (b) Estimate of H(u)

Figure 5.6: Q-Q plot and estimated hazard quantile function for the chemotherapy data set.

5.6 Conclusion

In this chapter, we proposed a new class of distributions with quadratic mean residual quantile function. The proposed class of distributions generalizes the class of distributions with linear mean residual quantile function, proposed and studied by Midhu et al. [89]. The distributional properties and various reliability characteristics of the proposed model are tractable. The analysis of the hazard quantile function over the whole parameter space can be done without using numerical methods. Various characterizations of the proposed lifetime model were developed in terms of the quantile-based reliability measures. We discussed the L-moment method of estimation and derived the asymptotic variance-covariance matrix of the estimates. The model was applied to a real life data. The proposed model can be used in different real life situations due to the flexibility of hazard quantile function.

Chapter 6

Proportional Odds Model- A Quantile Approach

6.1 Introduction

The concept of proportional hazards model (PHM) introduced by Cox [31] is extensively used for the analysis of survival data. The assumption of constant hazard rate ratio in the PHM is unreasonable in many practical situations as pointed by Bennett [17], Kirmani and Gupta [71] and Rossini and Tsiatis [126]. As an alternative to this, Bennett [17] developed the proportional odds model (POM), which studies the effectiveness of cure when the mortality rate of a group having some disease approaches that of a (disease-free) control group as time progresses using the concept of POM. Followed by the work of Bennett [17], several applications of the POM were developed by various researchers such as Collett [28], Pettitt [119] and Rossini and Tsiatis [126]. Various ageing properties of the POM were studied by Kirmani and Gupta [71].

Let X be a random variable with distribution function $F(x)$ and density function $f(x)$.

Results in this chapter have been published as entitled "Proportional odds model-a quantile approach" in the "Journal of Applied Statistics" (See Dileep et al. [36]).

Then the odds function is defined by

$$\phi_X(t) = \frac{P(X > t)}{P(X \leq t)} = \frac{\bar{F}_X(t)}{F_X(t)} = \frac{\lambda_X(t)}{h_X(t)},$$

where $h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}$ is the hazard rate and $\lambda_X(t) = \frac{d}{dt}\log(F_X(t))$ is the reversed hazard rate of X. Note that the odds function $\phi_X(t)$ is a decreasing function of t.
Consider two non-negative random variables X and Y with survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(x)$ respectively. Then Y is the POM of X with proportionality constant α if

$$\phi_Y(t) = \alpha\,\phi_X(t), \tag{6.1.1}$$

or equivalently

$$\frac{\bar{F}_Y(t)}{F_Y(t)} = \alpha\frac{\bar{F}_X(t)}{F_X(t)}. \tag{6.1.2}$$

Marshall and Olkin [87] studied the model (6.1.1) as a method for the construction of more flexible new families of distributions by introducing an additional parameter α in the baseline distribution. For more details on the Marshall-Olkin family of distributions, one can refer to Caroni [22], Ghitany et al. [41], Cordeiro et al. [30], Sankaran and Jayakumar [129] and Cordeiro et al. [29].

In literature, the properties and applications of the POM are studied in terms of the distribution functions. However, as pointed out earlier, any probability distribution can also be specified in terms of its quantile function. Recently, quantile regression models are employed for the analysis of censored lifetime data, which does not restrict the variation of the coefficients for different quantiles, in contrast to the well-known PHM or accelerated failure time models. The analysis of Stanford heart transplant data was done in Gorfine et al. [44] using quantile regression models. Nair et al. [106] presented various properties and applications of PHM in terms of the quantile functions. Motivated by these facts, in the present chapter, we study the properties and applications of POM in quantile set up. The proposed quantile-based approach has several advantages. It provides an alternative methodology for the analysis of lifetime data. Further, the proposed method develops a new class of lifetime models that do not have tractable distribution function but have simple and closed form quantile function. It gives new results in reliability analysis which are useful for the study of ageing phenomena as well as for the comparison of lifetime of systems.

The rest of the chapter is organized as follows. Section 6.2 presents the quantile-based definition of POM and it's basic properties. In Section 6.3, we establish various ageing concepts in the context of POM. The role of POM in constructing new flexible quantile functions is illustrated in Section 6.4. Finally, Section 6.5 summarizes major conclusions of the study.

6.2 Quantile based proportional odds model

Suppose X and Y be two non-negative random variables as described in Section 6.1. The quantile functions of X and Y are denoted by $Q_X(u)$ and $Q_Y(u)$. Let $q_X(u)$ and $q_Y(u)$ respectively denote the quantile density functions of X and Y. Then from (6.1.2), we get

$$\frac{\bar{F}_Y(Q_X(u))}{F_Y(Q_X(u))} = \alpha\left(\frac{1-u}{u}\right) \tag{6.2.1}$$

or equivalently

$$F_Y(Q_X(u)) = \left(\frac{u}{\alpha(1-u)+u}\right). \tag{6.2.2}$$

Note that $F_Y(Q_X(u)) \neq u$.

Then (6.2.1) can be written in terms of quantile function as

$$Q_X(u) = Q_Y\left(\frac{u}{\alpha(1-u)+u}\right), \tag{6.2.3}$$

or

$$Q_Y(u) = Q_X\left(\frac{\alpha\, u}{1+(\alpha-1)\, u}\right). \tag{6.2.4}$$

From (6.2.4), we note that $Q_Y(0) = Q_X(0)$ and $Q_Y(1) = Q_X(1)$. Thus the support of the random variable Y is the same as that of X. Cordeiro et al. [30] have shown that (6.2.4) is the quantile function of the family of distributions discussed in Marshall and Olkin [87] and they derived several characteristics of (6.2.1) in the quantile formulation.

6.3 Ageing properties

The preservation of reliability ageing classes under different reliability operations is a relevant topic in reliability theory. In this section, we study different ageing properties of Y in relation to those of X. Let $H_X(u)$, $M_X(u)$ and $\Lambda_X(u)$ denote the hazard quantile function, mean residual quantile function, and reversed hazard quantile function of the random variable X. Then the POM model given in (6.2.1) reduces to

$$\frac{\bar{F}_Y(Q_X(u))}{F_Y(Q_X(u))} = \alpha\left(\frac{1-u}{u}\right) = \alpha\left(\frac{\Lambda_X(u)}{H_X(u)}\right). \tag{6.3.1}$$

From (6.2.4), we have

$$H_Y(u) = \left(\frac{(1+(\alpha-1)\,u}{\alpha}\right) H_X\left(\frac{\alpha\,u}{1+(\alpha-1)\,u}\right), \;\; 0<u<1. \tag{6.3.2}$$

We now state a few structural properties of the variables X and Y involved in POM in terms of the hazard quantile function.

(i) $H_Y(u)$ lies in $\left(\frac{H_X(u')}{\alpha},\, H_X(u')\right)$ for $\alpha \geq 1$ and lies in $\left(H_X(u'),\, \frac{H_X(u')}{\alpha}\right)$ for $0 < \alpha \leq 1$, where $u' = \frac{\alpha u}{1+(\alpha-1)u}$. Note that $u' = 0$ when $u = 0$ and $u' = 1$ when $u = 1$, $0 < u' < 1$ and further u' and $\frac{\alpha u}{1+(\alpha-1)u}$ are in one-to-one correspondence. This is an interesting property of POM that geometrically $H_Y(u)$ curve within a band of width $\left(\frac{\alpha-1}{\alpha}\right) H_X(u')$ below the $H_X(u')$ curve for $\alpha \geq 1$ and in a band of width $\left(\frac{1-\alpha}{\alpha}\right) H_X(u')$ whenever $0 < \alpha \leq 1$.

(ii) $\lim_{u\to 0} H_Y(u) = \frac{1}{\alpha} \lim_{u\to 0} H_X(u)$ and $\lim_{u\to 1} H_Y(u) = \lim_{u\to 1} H_X(u)$. Thus at the lower endpoint $H_X(u)$ is below or above $H_Y(u)$ depending on whether $\alpha \geq 1$ or $0 < \alpha \leq 1$ and the two coincide at the extremity of the unit square.

(iii) From property (i), for $\alpha \geq 1$,

$$\begin{aligned} H_Y(u) \geq \frac{H_X(u')}{\alpha} &\Leftrightarrow (1-u)\,q_Y(u) \;\leq\; \alpha(1-u')\,q_X(u') \\ &\Leftrightarrow u\,q_Y(u) \;\leq\; \frac{\alpha\,u}{1+(\alpha-1)u}\,q_X(u') \\ &\Leftrightarrow u\,q_Y(u) \;\leq\; u'\,q_X(u') \qquad (6.3.3) \\ &\Leftrightarrow \Lambda_Y(u) \;\geq\; \alpha^{-1}\,\Lambda_X(u'). \end{aligned}$$

Similar arguments show that the reversed hazard quantile function of Y lies within $(\Lambda_X(u'),\, \alpha^{-1}\Lambda_X(u'))$ for $\alpha \geq 1$ and within $(\alpha^{-1}\Lambda_X(u'),\, \Lambda_X(u'))$ when $0 < \alpha \leq 1$.

Note that, $\Lambda_Y(u)$ also lies within a band above or below $\Lambda_X(u')$, which is similar to $H_Y(u')$.

(iv) Note that $u \leq (\geq) \frac{\alpha u}{1+(\alpha-1)u}$ when $\alpha \geq (\leq) 1$. Then we have

$$Q_X(u) \geq Q_X\left(\frac{\alpha u}{1+u(\alpha-1)}\right) = Q_Y(u), \quad \text{when } 0 < \alpha \leq 1,$$

and

$$Q_X(u) \leq Q_X\left(\frac{\alpha u}{1+u(\alpha-1)}\right) = Q_Y(u), \quad \text{when } \alpha > 1.$$

Further,

$$Q_X(u) \; \leq \; Q_Y(u) \; \leq \; \alpha\, Q_X(u'), \; \alpha \geq 1 \tag{6.3.4}$$

and

$$\alpha\, Q_X(u') \; \leq \; Q_Y(u) \; \leq \; Q_X(u), \; 0 < \alpha \leq 1. \tag{6.3.5}$$

The last two inequalities give bounds for the distribution of Y in the event of Y being POM of X. Further, they can be used for comparing lifetimes of two systems under POM set up.

Remark 6.3.1. The result (iii) is difficult to obtain in the distribution function approach to POM.

Example 6.3.1. Consider the generalized lambda distribution,

$$Q_X(u) = \lambda_1 + \frac{1}{\lambda_2}\left[\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right], \, 0 \leq u \leq 1. \tag{6.3.6}$$

with condition $\lambda_1 - \frac{1}{\lambda_2\,\lambda_3} \geq 0$ to make it the quantile function of a lifetime X. The distribution has support $\left(\lambda_1 - \frac{1}{\lambda_2\,\lambda_3}, \, \lambda_1 + \frac{1}{\lambda_2\,\lambda_4}\right)$, $\lambda_3, \, \lambda_4 > 0$ and $\left(\lambda_1 - \frac{1}{\lambda_2\,\lambda_3}, \, \infty\right)$, $\lambda_3 > 0, \, \lambda_4 \leq 0$ and it becomes the exponential as $\lambda_3 \longrightarrow \infty$, $\lambda_4 \longrightarrow 0$. It is a highly flexible model capable of representing a wide range of data sets. In this case, $Q_Y(u)$ lies above

$Q_X(u)$ when $\alpha > 1$ and below $Q_X(u)$ when $0 < \alpha \leq 1$, the other bounds for $Q_Y(u)$ are obtained directly from (6.3.4) and (6.3.5) respectively. Similar bounds with distribution functions are difficult to get as $Q_X(u) = x$ has no closed form solution in u.

Remark 6.3.2. Equation (6.3.4) or (6.3.5) provides a graphical diagnostic procedure for POM. Using the baseline quantile function $Q_X(u)$ and uniform random numbers in $[0, 1]$, the band $(Q_X(u),\, \alpha Q_X(u'))$ can be constructed for a given α. If the plot of sample values falls within the band, then $Q_Y(u)$ in (6.2.4) will be a reasonable choice as POM of X.

We have

$$H_Y(u) = \left(\frac{(1 + (\alpha - 1)u}{\alpha}\right) H_X\left(\frac{\alpha\, u}{1 + (\alpha - 1)u}\right). \tag{6.3.7}$$

Kirmani and Gupta [71] have proved that if X is IHR (DHR) and $\alpha > 1\,(0 < \alpha \leq 1)$ then Y is IHR (DHR). This result does not consider the other two cases such as X is IHR with $0 < \alpha \leq 1$ and X is DHR with $\alpha > 1$. Moreover, its application requires the monotonicity of the hazard rate of X. Note that the monotonicity properties of the hazard rate and the hazard quantile function are equivalent (see Nair et al. [105]). We now give a result that ascertains the behaviour of $H_Y(u)$ without any assumptions on the hazard quantile function of X. The identity (6.3.7) gives

$$\frac{d\,H_Y(u)}{du} = \left(\frac{(1 + (\alpha - 1)u}{\alpha}\right)\left(\frac{d}{du}H_X\left(\frac{\alpha u}{1 + (\alpha - 1)u}\right)\right) + \left(\frac{\alpha - 1}{\alpha}\right) H_X\left(\frac{\alpha u}{1 + (\alpha - 1)u}\right).$$

Hence Y is IHR if $\frac{d}{du}H_X\left(\frac{\alpha u}{1+(\alpha-1)u}\right) \geq -\frac{d}{du}\log(1 + (\alpha - 1)u)$. Integrating from 0 to u,

$$H_X\left(\frac{\alpha u}{1 + (\alpha - 1)u}\right) \geq \frac{H_X(0)}{1 + (\alpha - 1)u}. \tag{6.3.8}$$

If we set $u' = \frac{\alpha u}{1+(\alpha-1)u}$, the above inequality is equivalent to

$$H_X(u') \geq (\alpha - (\alpha-1)u')\, H_X(0), \text{ for all } 0 \leq u' \leq 1. \tag{6.3.9}$$

Similarly, Y is DHR if the above inequality is reversed. Further, when

$$H_X(u_0') = (\alpha - (\alpha-1)u_0')H_X(0), \tag{6.3.10}$$

for some $0 \leq u_0' \leq 1$, then Y has either bathtub-shaped or upside down bathtub-shaped hazard rate.

Note that $H_Y(u)$ can be more flexible compared to $H_X(u)$, since it can accommodate non-monotonic shapes even if X is monotonic. This suggests the use of POM (6.2.4) for constructing more flexible families of distributions. We illustrate this in Section 6.4.

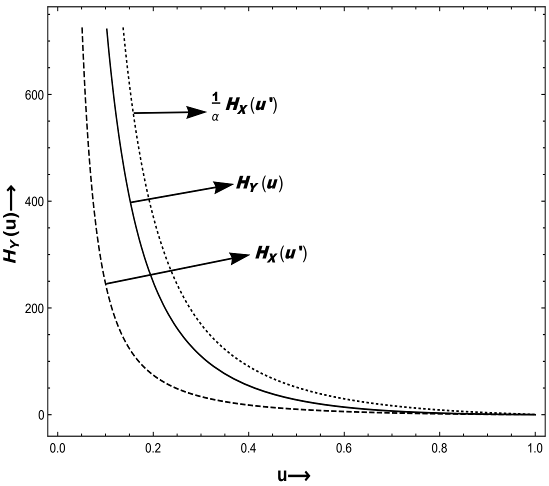
Example 6.3.2. Let X follows Weibull distribution with quantile function

$$Q_X(u) = \sigma(-\log(1-u))^{\frac{1}{\lambda}}, \ u \in (0,1). \tag{6.3.11}$$

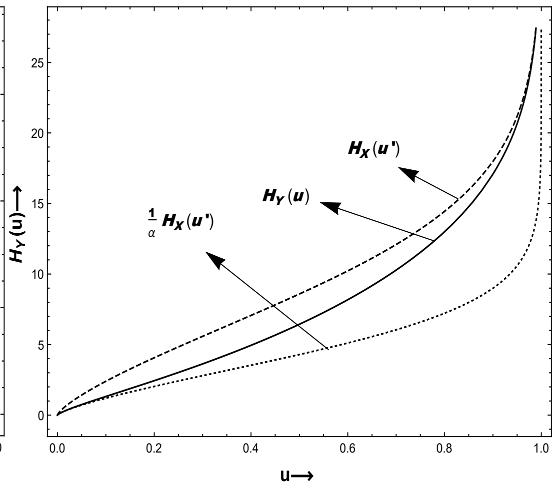
Then

$$Q_Y(u) = \sigma\left(-\log\left(\frac{1-u}{1+(\alpha-1)\,u}\right)\right)^{\frac{1}{\lambda}}. \tag{6.3.12}$$

Plots of $H_X(u)$ and $H_Y(u)$ for various parameter combinations are presented in Figures 6.1(a) and 6.1(b). From Figure 6.1(a), since $\lambda = 0.4$, we see that, $H_X(u) \leq H_Y(u)$ for $\alpha = 2\,(> 1)$ and $H_X(u) \geq H_Y(u)$ for $\alpha = 0.5\,(< 1)$. In Figure 6.1(b), we consider the case where X is IHR ($\lambda = 2$). Here we observe that the plots of $H_X(u)$ and $H_Y(u)$ intersects. The Weibull distribution given in (6.3.11) is DHR when $0 < \lambda < 1$ and IHR when $\lambda > 1$. Figure 6.2 represents non-monotonic shapes of $H_Y(u)$ for various parameter



(a) $\alpha = 0.2,\ \lambda = 0.4,\ \sigma = 0.5$



(b) $\alpha = 2,\ \lambda = 4,\ \sigma = 0.5$

Figure 6.1: Bounds of $H_Y(u)$.

combinations. Thus the model (6.3.12) will be a reasonable choice in modelling different types of lifetime datasets.

Definition 6.3.1. The random variable X is new better (worse) than used in hazard rate (NBUHR (NWUHR)) if and only if $H_X(u) \leq (\geq) H_X(0)$ for all $u \in (0,1)$ ((Nair et al. [105])).

Theorem 6.3.1. Suppose Y is the POM of X. Then the following results hold;

(i) If X is NBUHR and $\alpha > 1$, then Y is NBUHR

(ii) If X is NWUHR and $0 < \alpha \leq 1$, then Y is NWUHR.

Proof. Assume X is NBUHR and $\alpha > 1$. Then,

$$\begin{aligned} H_Y(u) &= \left(\frac{(1+(\alpha-1)u}{\alpha}\right) H_X\left(\frac{\alpha\, u}{1+(\alpha-1)u}\right) \\ &\geq H_X(0)\left(\frac{(1+(\alpha-1)u}{\alpha}\right) \quad \text{(since } X \text{ is NBUHR)} \end{aligned}$$

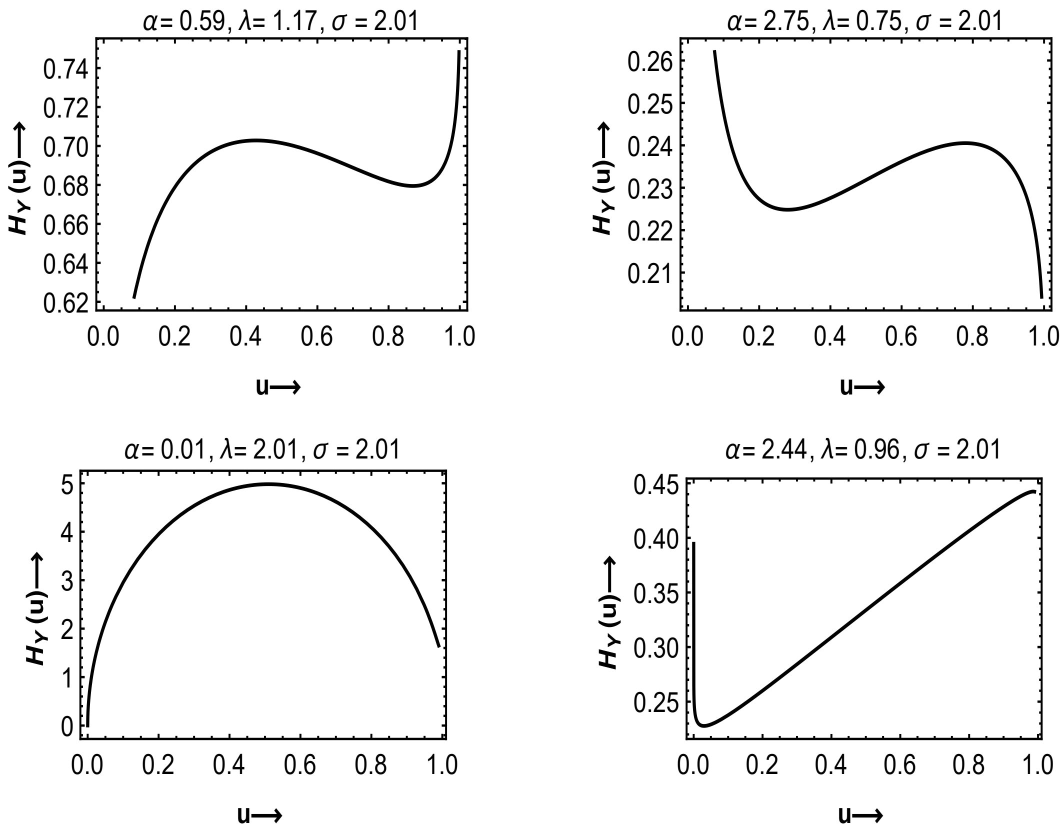


Figure 6.2: Non-monotonic shapes of $H_Y(u)$.

$$\geq H_X(0)\left(\frac{1}{\alpha}\right) \qquad (\text{since}(1+(\alpha-1)u) \geq 1 \text{ when } \alpha \geq 1)$$

$$= H_Y(0), \quad \text{which implies } Y \text{ is NBUHR.}$$

This completes the proof of (i). Proof for (ii) is similar. ■

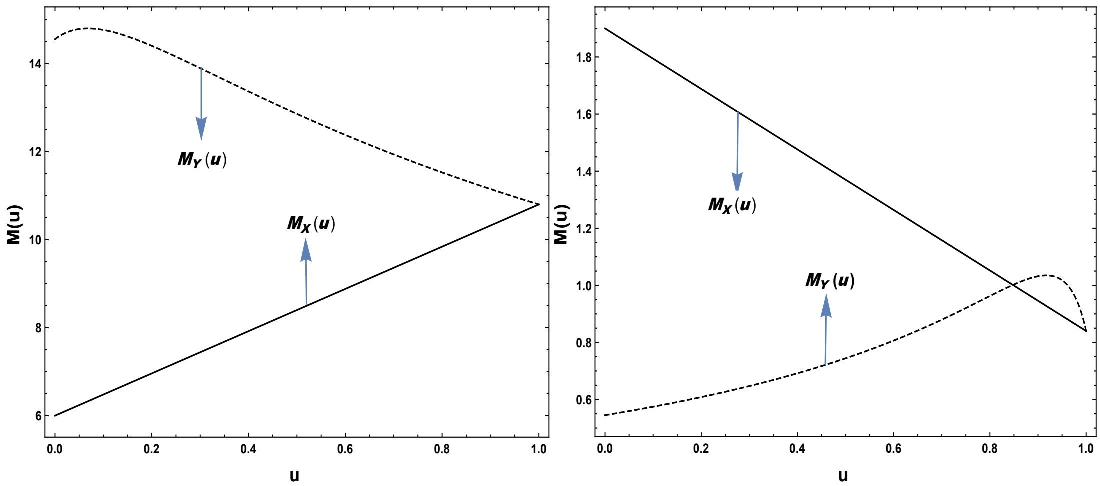
We say that X is decreasing (increasing) mean residual life if $M_X(u)$ is decreasing (increasing) in u. In general, the DMRL (IMRL) property is not preserved under POM. To illustrate this, consider a random variable X with quantile function

$$Q_X(u) = -(c+\mu)\log(1-u) - 2\,c\,u,\ \ \mu \geq 0,\ \ -\mu \leq c \leq \mu \ (\text{ Midhu et al. [89]}).$$

Then

$$M_Y(u) = \frac{\alpha\left(((\alpha-1)\mu+(\alpha+1)c)\log\left(\frac{\alpha}{(\alpha-1)u+1}\right)+\frac{2(\alpha-1)c(u-1)}{(\alpha-1)u+1}\right)}{(\alpha-1)^2(1-u)}.$$

Plots of $M_X(u)$ and $M_Y(u)$ for selected values of parameters are given in Figure 6.3. From this, it is clear that in general monotonicity properties of $M_Y(u)$ are different from $M_X(u)$. We establish a necessary and sufficient condition for Y to have monotone mean residual life in the next Theorem.



(a) $\mu = 6,\ c = 4.8,\ \alpha = 5$ (b) $\mu = 1.9,\ c = -1.06,\ \alpha = 0.09$

Figure 6.3: $M_X(u)$ and $M_Y(u)$ for selected values of parameters.

Theorem 6.3.2. The random variable Y is DMRL (IMRL) if and only if

$$-\frac{d}{du}\log\left[\int_u^1\left(\left(\frac{1+(\alpha-1)p}{\alpha}\right)\ H_X\left(\frac{\alpha p}{1+(\alpha-1)p}\right)\right)^{-1}\right]dp \geq (\leq)\,\frac{1}{1-u}, \tag{6.3.13}$$

for all $u \in (0,1)$.

Proof. Suppose Y is DMRL (IMRL). From (1.2.11) and (6.2.4),

$$(1-u)M_Y(u) = \int_u^1 (1-p)\,q_Y(p)dp$$

$$= \int_u^1 \left(\frac{\alpha\,(1-p)}{(1+(\alpha-1)p)^2} \right) q_X \left(\frac{\alpha p}{1+(\alpha-1)p} \right) dp. \tag{6.3.14}$$

Differentiating (6.3.14) with respect to u, since Y is DMRL (IMRL),

$$M_Y(u) \;\leq (\geq) \left(\frac{\alpha\,(1-u)}{(1+(\alpha-1)u)^2} \right) q_X \left(\frac{\alpha u}{1+(\alpha-1)u} \right),$$

equivalently,

$$\begin{aligned} \frac{1}{1-u} \int_u^1 \left(\frac{\alpha\,(1-p)}{(1+(\alpha-1)p)^2} \right) q_X \left(\frac{\alpha p}{1+(\alpha-1)p} \right) dp & \\ \leq (\geq) \left(\frac{\alpha\,(1-u)}{(1+(\alpha-1)u)^2} \right) q_X \left(\frac{\alpha u}{1+(\alpha-1)u} \right). & \end{aligned}$$

This implies

$$\frac{\left(\frac{\alpha\,(1-u)}{(1+(\alpha-1)u)^2} \right) \; q_X \left(\frac{\alpha u}{1+(\alpha-1)u} \right)}{\int_u^1 \left(\frac{\alpha\,(1-p)}{(1+(\alpha-1)p)^2} \right) \; q_X \left(\frac{\alpha p}{1+(\alpha-1)p} \right) dp} \geq (\leq) \frac{1}{1-u},$$

which is equivalent to (6.3.13).

Conversely, we have (6.3.13) is true, which gives

$$-\frac{d}{du} \log \left[\int_u^1 \left(\frac{\alpha\,(1-p)}{(1+(\alpha-1)p)^2} \right) \; q_X \left(\frac{\alpha p}{1+(\alpha-1)p} \right) \right] dp + \frac{d}{du} \log(1-u) \;\geq (\leq)\, 0.$$

This gives

$$-\frac{d}{du} \log \left[\frac{1}{1-u} \int_u^1 \left(\frac{\alpha\,(1-p)}{(1+(\alpha-1)p)^2} \right) \; q_X \left(\frac{\alpha p}{1+(\alpha-1)p} \right) \right] dp \;\geq (\leq)\, 0,$$

which implies

$$-\frac{d}{du} \log\, M_Y(u) \;\geq (\leq)\, 0,$$

or $M_Y(u)$ is decreasing (increasing). ■

To illustrate this, consider the case when X is exponential with quantile function,

$$Q_X(u) = -\lambda \log(1-u). \tag{6.3.15}$$

Note that $M_X(u) = \lambda$, a constant. Now to analyse the monotonicity property of $M_Y(u)$, the left hand side of (6.3.13) is obtained as

$$-\frac{d}{du}\log\left[\int_u^1 \left(\left(\frac{1+(\alpha-1)p}{\alpha}\right) \quad H_X\left(\frac{\alpha p}{1+(\alpha-1)p}\right)\right)^{-1}\right] dp$$
$$= \frac{\alpha-1}{((\alpha-1)u+1)\log\left(\frac{\alpha}{(\alpha-1)u+1}\right)} \tag{6.3.16}$$

We observe that

$$\frac{\alpha-1}{((\alpha-1)u+1)\log\left(\frac{\alpha}{(\alpha-1)u+1}\right)} \leq (\geq) \frac{1}{1-u} \text{ for } \alpha \leq (\geq) 1.$$

Thus from Theorem 6.3.2, we have $M_Y(u)$ is IMRL (DMRL) for $\alpha \leq (\geq) 1$.

Two other models used in similar contexts that have received considerable attention are the proportional hazards (PHM) and proportional reversed hazards (PRHM) models. In the quantile framework, Y is the PHM of X if

$$h_Y(Q_X(u)) = \alpha H_X(u),$$

and Y is the PRHM of X if

$$\lambda_Y(Q_X(u)) = \alpha \Lambda_X(u).$$

See Nair et al. [106] for a discussion of PHM. We now show that POM does not imply PHM or PHRM for the same pair of random variables X and Y. As an illustration, let X

be a Pareto random variable with survival function,

$$\bar{F}_X(x) = x^{-\beta},\ x > 1,\ \beta > 0 \text{ and } Q_X(u) = (1-u)^{-\frac{1}{\beta}}.$$

Then $H_X(u) = \beta\,(1-u)^{\frac{1}{\beta}}$. The POM has quantile function,

$$Q_Y(u) = \left(\frac{1-u}{1+(\alpha-1)u}\right)^{-\frac{1}{\beta}},$$

and survival function,

$$\bar{F}_Y(x) = \frac{\alpha\, x^{-\beta}}{1+(\alpha-1)x^{-\beta}}.$$

Accordingly,

$$h_Y(x) = \big(x(1+(\alpha-1)\,x^{-\beta})\big)^{-1},$$

or

$$h_Y(Q_X(u)) = \frac{(1-u)^{-\frac{1}{\beta}}}{1+(\alpha-1)(1-u)} \neq \alpha H_X(u).$$

Hence Y is not PHM of X.

In similar lines, suppose X follows power distribution with $Q_X(u) = u^{\frac{1}{\beta}}$, $\beta > 0$, so that $\Lambda_X(u) = \beta\, u^{-\frac{1}{\beta}}$. Under the assumption of POM,

$$Q_Y(u) = \left(\frac{\alpha u^{\frac{1}{\beta}}}{1+(\alpha-1)u^{\frac{1}{\beta}}}\right)$$

and

$$F_Y(x) = \left(\frac{x}{\alpha-(\alpha-1)x}\right)^{\beta}, \text{ giving, } \lambda_Y(x) = \frac{\alpha\,\beta}{x(\alpha-(\alpha-1)x)}.$$

Thus

$$\lambda_Y(Q_X(u)) = \frac{\alpha\beta}{u^{\frac{1}{\beta}}\left(\alpha-(\alpha-1)u^{\frac{1}{\beta}}\right)} \neq \alpha\Lambda_X(u),$$

showing that Y is not PRHM of X.

Although POM does not imply both PHM and PRHM in the same variables, our next Theorem reveals that in the presence of POM there is indeed PHM and PRHM in two other random variables.

Theorem 6.3.3. Suppose that Y is POM of X. Then there exist random variables $X_1(X_2)$ and $Y_1(Y_2)$ such that $Y_1(Y_2)$ is the PHM (PRHM) of $X_1(X_2)$.

Proof. Under the POM assumption, it follows from (6.2.1) that,

$$\frac{\bar{F}_Y(Q_X(u))}{F_Y(Q_X(u))} = \alpha\left(\frac{1-u}{u}\right),$$

and hence

$$\begin{aligned}\exp\left[-\left(\frac{u}{1-u}\right)\right] &= \exp\left[-\alpha\left(\frac{F_Y(Q_X(u))}{\bar{F}_Y(Q_X(u))}\right)\right] \\ &= \left(\exp\left[-\left(\frac{F_Y(Q_X(u))}{\bar{F}_Y(Q_X(u))}\right)\right]\right)^{\alpha}. \end{aligned} \tag{6.3.17}$$

It is not difficult to recognize the left side of (6.3.17) as a survival function of a non-negative random variable Y_1 in $[0,\, 1]$ and by the same way $\exp\left[-\left(\frac{F_Y(Q_X(u))}{\bar{F}_Y(Q_X(u))}\right)\right]$ as the survival function of some random variable X_1. Thus,

$$\bar{F}_{Y_1}(u) = \left(\bar{F}_{X_1}(u)\right)^{\alpha},$$

and hence Y_1 is the PHM of X_1.

To prove the second part, we write (6.2.1) as

$$\exp\left[-\left(\frac{\bar{F}_Y(Q_X(u))}{F_Y(Q_X(u))}\right)\right] = \exp\left[-\alpha\left(\frac{1-u}{u}\right)\right],$$

and observe that $\exp\left[-\left(\frac{1-u}{u}\right)\right]$ is a distribution function of a random variable X_2 and the left side is a distribution function of Y_2 and they satisfy

$$F_{Y_2}(u) = (F_{X_2}(u))^{\alpha},$$

showing that Y_2 is the PRHM of X_2. ■

As an illustration, Suppose X follows unit exponential distribution with quantile function $Q_X(u) = -\log(1-u)$. Then the distribution of Y is

$$F_Y(x) = \frac{1-e^{-x}}{1+(\alpha-1)e^{-x}}, \ x > 0.$$

and

$$Q_Y(u) = \frac{-\log(1-u)}{1+(\alpha-1)u}.$$

Also the distribution function of X_1 is

$$\begin{aligned} F_{X_1}(u) &= \left(\exp\left[-\left(\frac{F_Y(Q_X(u))}{\bar{F}_Y(Q_X(u))}\right)\right]\right)^{\alpha} \\ &= \exp\left[-\alpha\left(\frac{u}{1-u}\right)\right]. \end{aligned}$$

Always $F_{Y_1}(u) = \exp\left[-\left(\frac{u}{1-u}\right)\right]$, so that the condition for PHM is satisfied.

The analysis of data using POM can be done by transforming random variables in such a way that the resultant random variables satisfy PHM (PRHM) property. The well-known estimation methods such as partial likelihood could be employed for the inference procedures.

The survival function of the equilibrium random variable Z associated with X has the form,

$$\bar{F}_Z(x) = \frac{1}{\mu_X} \int_x^{\infty} \bar{F}_X(t)dt,$$

where $\mu_X = E(X)$. Setting $x = Q_X(u)$, from Nair et al. [105], we obtain

$$F_Z(Q_X(u)) = \frac{1}{\mu_X} \int_0^u (1-p)q_X(p)dp, \tag{6.3.18}$$

where the integral

$$\zeta_X(u) = \frac{1}{\mu_X} \int_0^u (1-p)q_X(p)dp, \tag{6.3.19}$$

is called the scaled total time on test transform of the random variable X. For various properties and applications of $\zeta_X(u)$, one could refer to Nair et al. [105].

From (6.3.18), we have,

$$Q_X(u) = Q_Z\left(\zeta_X(u)\right) \text{ or, } Q_Z(u) = Q_X(\zeta_X^{-1}(u)), \tag{6.3.20}$$

where $Q_Z(.)$ is the quantile function corresponding to the equilibrium distribution of X.

Theorem 6.3.4. Suppose Y is the POM of X and Z denotes the equilibrium random variable of X. The random variables Y and Z are identically distributed if and only if

$$q_X(u) = \mu\,\alpha\left(\left(1-u\right)\left(\alpha + \left(1-\alpha\right)u\right)^2\right)^{-1}.$$

Proof. Assume that the distributions of Y and Z are identical. Thus we have $Q_Z(u) = Q_Y(u)$, which implies,

$$Q_X(\zeta_X^{-1}(u)) = Q_X\left(\frac{\alpha\,u}{1+(\alpha-1)u}\right).$$

Thus,

$$\zeta_X^{-1}(u) = \frac{\alpha\, u}{1 + (\alpha - 1)u}, \text{ or equivalently } \zeta_X(u) = \frac{u}{\alpha + (1 - \alpha)u}.$$

Now from the identity, $q_X(u) = \frac{\zeta'_X(u)\, \mu}{1-u}$ (Nair et al. [105]), we get

$$q_X(u) = \mu\, \alpha \left((1 - u)\, (\alpha + (1 - \alpha)\, u)^2\right)^{-1}.$$

Conversely, we have $q_X(u) = \mu\, \alpha\, ((1 - u)\, (\alpha + (1 - \alpha)\, u)^2)^{-1}$, which gives $\zeta_X(u) = \frac{u}{\alpha+(1-\alpha)u}$ and $\zeta_X^{-1}(u) = \frac{\alpha\, u}{1+(\alpha-1)u}$. Now from (6.2.4) and (6.3.20), we obtain

$$Q_Z(u) = Q_Y(u) = Q_X\left(\frac{\alpha\, u}{1 + (\alpha - 1)u}\right).$$

Thus Y and Z are identically distributed. which completes the proof. ■

6.4 Generalized homographic hazard quantile function model and its applications

The POM in (6.2.4) with $0 < \alpha < \infty$ gives a method of introducing a new parameter α to an existing quantile function for obtaining a more flexible new family of distributions based on quantile function. To illustrate this, we consider the class of distributions with homographic hazard quantile function (HGHQ) introduced by Sankaran et al. [135], with quantile function,

$$Q_X(u) = \frac{(A - B)\log(Au + 1) - A(B + 1)\log(1 - u)}{A(A + 1)K}, \; K > 0, B \geq -1 \text{ and } A > 0. \tag{6.4.1}$$

We now introduce a generalization of the class of distributions with homographic hazard quantile function (GHGHQ) by taking $Q_X(u)$ as the baseline distribution in the POM given in (6.2.4). The resulting quantile function is denoted by $Q_Y(u)$ and has the form

$$Q_Y(u) = \frac{(A-B)\log\left(\frac{\alpha A u}{(\alpha-1)u+1}+1\right) - A(B+1)\log\left(1-\frac{\alpha u}{(\alpha-1)u+1}\right)}{A(A+1)K}, \tag{6.4.2}$$

$$\alpha > 0,\ K > 0, B \geq -1, \text{and } A > 0.$$

Note that the class of distributions (6.4.2) does not have a tractable form for its distribution function and density function. In such situations, conventional methods of analysis using distribution functions are not appropriate. The support of the proposed class of distributions (6.4.2) is $(0, \infty)$. The quantile density function is obtained as

$$q_Y(u) = \frac{\alpha(u(\alpha+\alpha B-1)+1)}{K(1-u)((\alpha-1)u+1)(u(\alpha+\alpha A-1)+1)}. \tag{6.4.3}$$

Using the relation $f_Y(Q_Y(u)) = \frac{1}{q_Y(u)}$, we present the plots of the density function for different combinations of parameters in Figure 6.4. We observe that the proposed class of distributions accommodates increasing, decreasing and bell-shaped density curves. The proposed family also includes symmetric, positively and negatively skewed distributions as its special cases. The hazard quantile function is obtained as

$$H_Y(u) = \frac{K((\alpha-1)u+1)((\alpha+\alpha A-1)u+1)}{\alpha((\alpha+\alpha B-1)u+1)}, \tag{6.4.4}$$

which accommodates increasing, decreasing, linear, bathtub and upside-down bathtub shapes for different choices of parameters. Plots of hazard quantile function for different values of parameters is given in Figure 6.5. Note that the baseline distribution has only increasing or decreasing hazard quantile function.

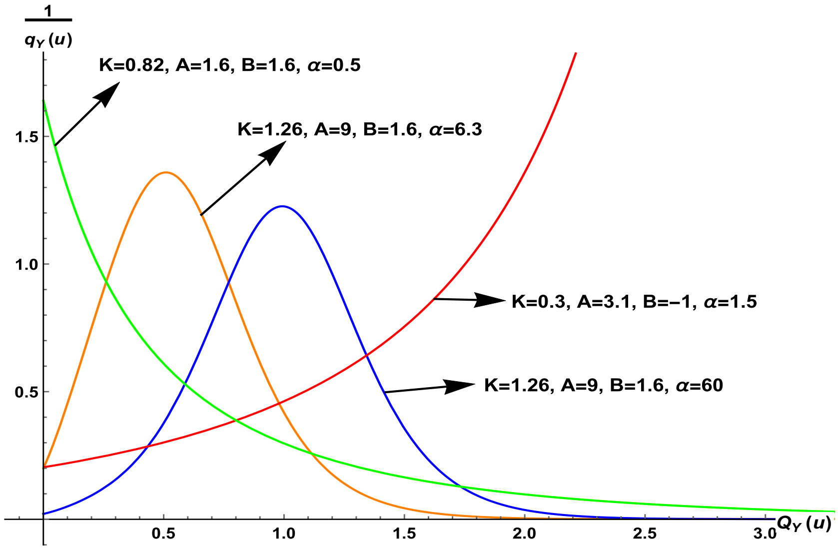


Figure 6.4: Plots of the density function for different values of parameters.

As mentioned in Chapter 1, the $L-$moments are often found to be more desirable than the conventional moments in describing the characteristics of the distributions as well as for inference. For the model (6.4.2), first four $L-$moments are obtained as follows;

$$L_1 = \frac{\alpha((\alpha-1)(A-B)\log(A+1) + A\log(\alpha)(\alpha+\alpha B-1))}{(\alpha-1)AK(\alpha+\alpha A-1)},$$

$$L_2 = \frac{1}{(\alpha-1)^2AK(\alpha+\alpha A-1)^2}\left[(\alpha-1)A(\alpha+\alpha A-1)(\alpha+\alpha B-1)\right.$$

$$\left.(\alpha-1)^2(-(A-B))\log(A+1) - A\log(\alpha)(\alpha(\alpha+B(\alpha(A+2)-2)-2)+1)\right],$$

$$L_3 = \left(\frac{\alpha}{(\alpha-1)^3AK(\alpha+\alpha A-1)^3}\right)\alpha(-2(\alpha-1)A(\alpha+\alpha A-1)\left(\alpha\left(\alpha\right.\right.,$$

$$+B(\alpha(A+2)-2)-2)+1)(\alpha-1)^3(\alpha+\alpha A+1)(A-B)\log(\alpha(A+1))$$

$$+(\alpha+1)B\log(\alpha)(\alpha+\alpha A-1)^3\Big),$$

and,

$$L_4 = \left(\frac{\alpha}{6(\alpha-1)^4AK(\alpha+\alpha A-1)^4}\right)\left[(\alpha-1) - 6\left(\alpha^2+3\alpha+1\right)B\log(\alpha)(\alpha+\alpha A-1)^4\right.$$

$$\left(A\alpha+\alpha A-1+\left(\alpha^2\left(-\left(A^2+A(47B-37)+94B+10\right)\right)+\alpha(-13A+61B+35)\right.\right.$$

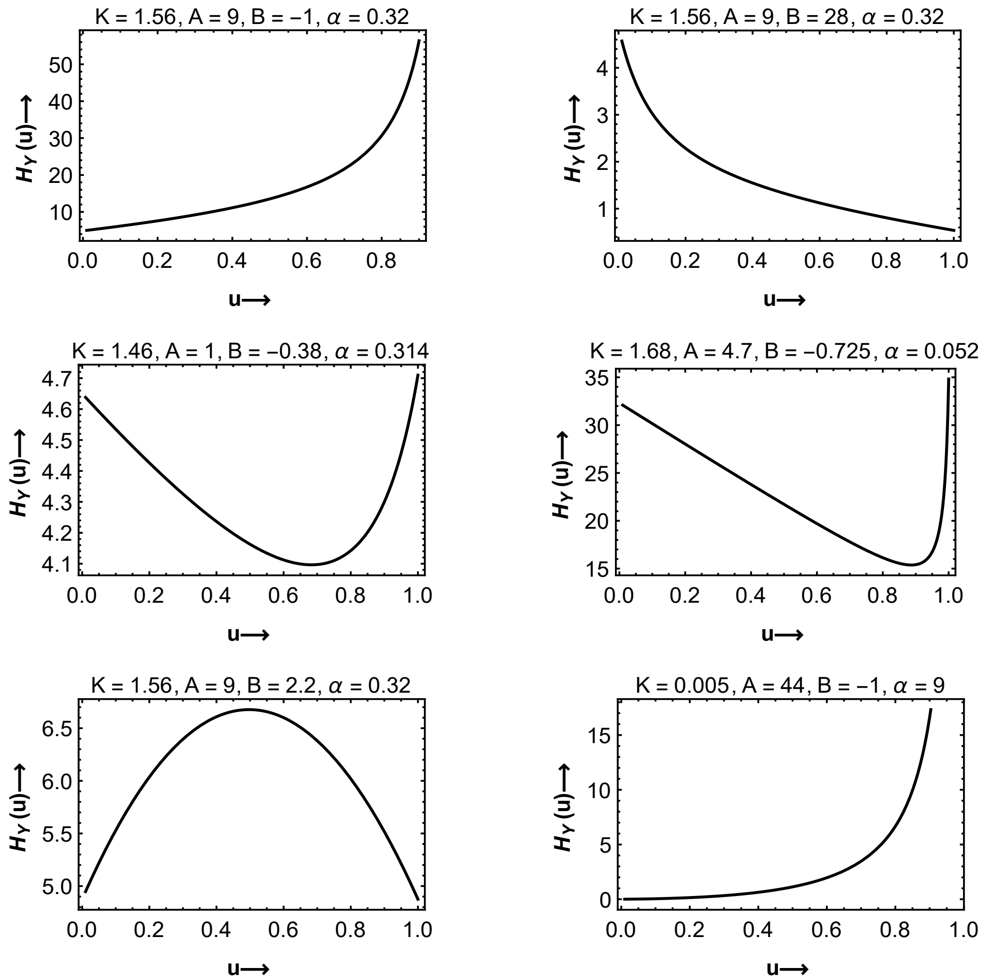


Figure 6.5: Plots of hazard quantile function for different values of parameters.

$$-16A^2(16B+3)+A(6B-33)+6B-20\big)\,\alpha^3+\alpha^4(A+1)(A(13B-3)+26B+10)$$
$$+\alpha^5(A+1)^2(B+1)\big)\ -6(\alpha-1)^3\left(\alpha^2(A+1)^2+3\alpha(A+1)+1\right)$$
$$(A-B)\log(\alpha(A+1)))). \tag{6.4.5}$$

For the estimation of parameters in (6.4.2), we employ the method of $L-$moments. We equate sample $L-$moments to corresponding population $L-$moments. For estimating the parameters A, B, α and K, we equate first four sample $L-$moments to corresponding population $L-$moments. The parameters are obtained by solving the equations,

$$l_r = L_r; \quad r = 1, 2, 3, 4. \tag{6.4.6}$$

Hosking [60] showed that sample $L-$moments are consistent estimators of population $L-$moments. Asymptotic normality of the $L-$moment estimators is given in Theorem 1.1.1. Since the set of equations (6.4.6) are non-linear in $A,\ B,\ \alpha$ and K, we need to adopt a proper numerical method to find out the estimates.

6.5 Applications

To illustrate the application of the proposed class of distributions, we consider two real data sets. First, we consider the data reported in Birnbaum and Saunders [18]. The data consist of measurements of fatigue life (thousands of cycles until rupture) of rectangular strips of 101 aluminium sheeting. A special testing machine designed and constructed by members of the instrument development unit of the physical research staff, Boeing airplane company, was employed for recording the data. The method of L-moments is employed for the estimation of parameters. We equate the population $L-$moments $L_1,\ L_2,\ L_3$ and L_4 given in (6.4.5) to corresponding sample $L-$moments. Since the equations are non-linear functions of parameters, we use Newton-Raphson method for finding the estimates. Least square method of estimation is employed for choosing the initial estimates. The initial estimates thus obtained are $A_0 = 6.101,\ B_0 = -0.424,\ \alpha_0 = 18.102$ and $K_0 = 0.008$. Then the estimates of the parameters are obtained as

$$\hat{A} = 6.781, \quad \hat{B} = -0.364, \quad \hat{\alpha} = 22.442 \ \text{ and } \quad \hat{K} = 0.003.$$

Recently Sankaran et al. [136] employed a four parameter quantile function model, which

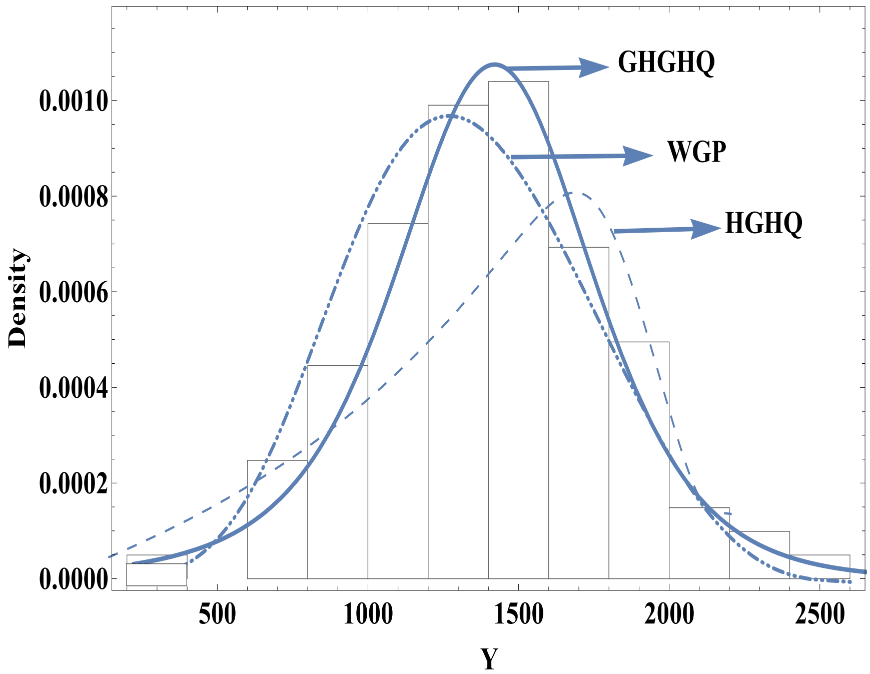


Figure 6.6: Fitted densities and histogram of aluminium strip data.

was formed by taking the sum of quantile functions of Weibull and generalized Pareto distributions to model the above data. We denote this model by Weibull generalized Pareto model (WGP). The estimated density with the histogram of the observed data for GHGHQ, WGP and HGHQ models are presented in Figure 6.6. This shows that the GHGHQ distribution gives a better fit than the other two models for the data set. To examine the adequacy of the fitted model, we carried out the chi-square goodness of fit test. The chi-square value of 2.83 with P-value 0.94 does not reject the model (6.4.2) for the given set of data. Chi-square values obtained for the models WGP and HGHQ are 3.386 (p-value=0.90) and 4.15 (p-value=0.84) respectively. On the basis of chi-square values, our model gives a better fit. Figure 6.7 presents the Q-Q plot which also shows the adequacy of the model.

Next, we consider another data set reported in Nichols and Padgett [114]. The data consist of the breaking stress of a sample of 100 carbon fibres. Carbon fibres of 50 mm in length

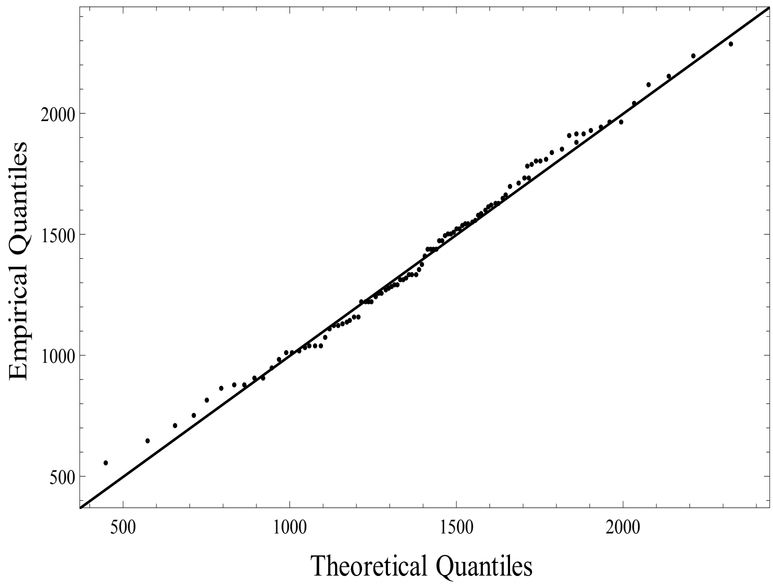


Figure 6.7: Q-Q plot of the aluminium strips data.

were sampled, tested and their tensile strengths were observed. The initial estimates obtained are $A_0 = 40.225,\ B_0 = 5.213,\ \alpha_0 = 37.121$ and $K_0 = 0.075$. Then the estimates of the parameters using method of $L-$moments are obtained as

$$\hat{A} = 44.974, \quad \hat{B} = 5.198, \quad \hat{\alpha} = 37.422 \text{ and } \quad \hat{K} = 0.219.$$

The estimated density with the histogram of the observed data for GHGHQ, WGP and HGHQ models are presented in Figure 6.8, which shows that the GHGHQ distribution gives a better fit than the other two models for the data set. To examine the adequacy of the fitted model, we carried out the chi-square goodness of fit test. The chi-square value of 1.016 with p-value 0.60 does not reject the model (6.4.2) for the given set of data. Chi-square values obtained for the models WGP and HGHQ are 2.587 (p-value=0.274) and 2.512 (p-value=0.473) respectively. These values indicate that the proposed model performs better as compared to the competing alternatives. Figure 6.9 presents the Q-Q plot of the proposed model to the data, which ensures the appropriateness of our model.

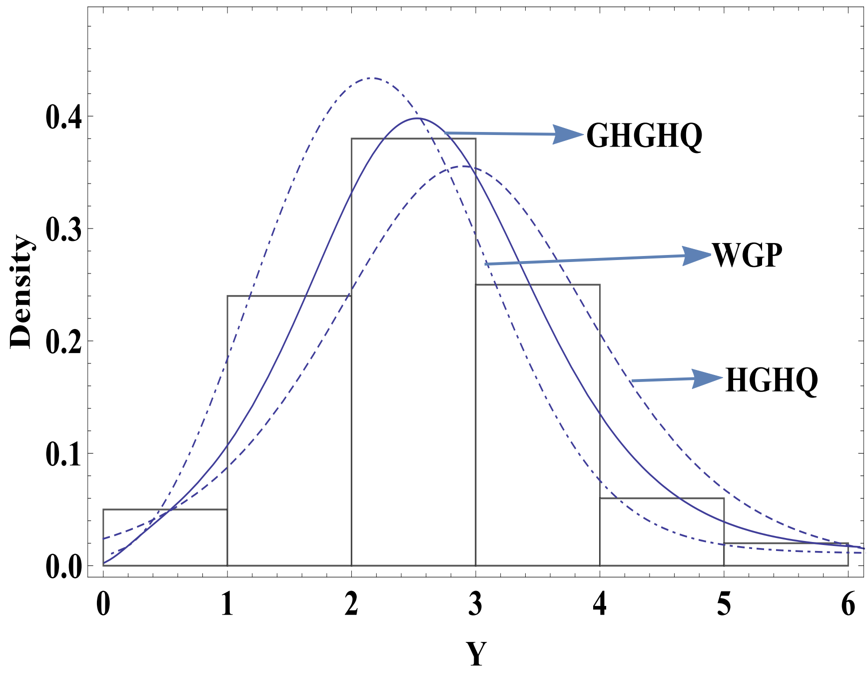


Figure 6.8: Fitted densities and histogram of the carbon fibres data.

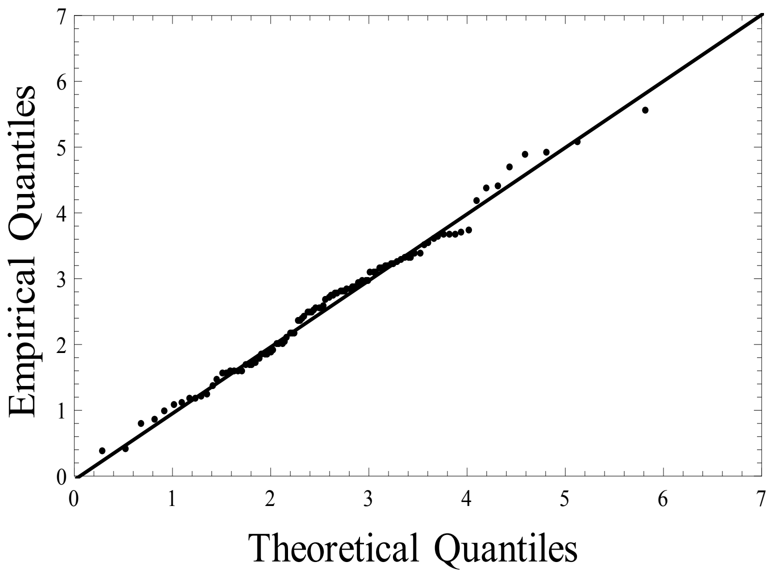


Figure 6.9: Q-Qplot of the carbon fibres data.

6.6 Conclusion

The quantile version of the POM has been introduced and studied important reliability properties and applications of the POM based on quantile functions. The quantile version of POM has a different structure as compared to the POM using traditional distribution function approach. We have investigated certain characterizations and ageing concepts and illustrated using examples. A new class of distributions defined in terms of the quantile function was introduced. The applications of the proposed class of distributions were studied with two real life data sets. The performance of the proposed model was compared with existing quantile function models.

Chapter 7

The Cause Specific Hazard Quantile Function

7.1 Introduction

In survival studies, it is common that the failure of subjects may be attributed to more than one cause. Competing risks models are usually employed to analyse such type of data. In the competing risks set up, for each subject under study we observe a random vector (X, J) where X represents lifetime (possibly censored) and $J = \{1, 2, \ldots, k\}$ is a set of possible causes of failure. Assume that the causes of failure are mutually exclusive. Two frameworks are often employed to deal with standard competing risks data such as cumulative incidence function formulations and cause specific hazard formulations.

The cumulative incidence function $F_j(x)$ is the probability of failure before time x due to cause j given by

$$F_j(x) = P[X \leq x, J = j], \quad j = 1, 2, \ldots, k. \tag{7.1.1}$$

Note that $F(x) = \sum_{j=1}^{k} F_j(x)$ is the distribution function of X.

Results in this chapter have been published as entitled "The cause specific hazard quantile function" in the "Austrian Journal of Statistics" (see Sankaran et al. [137]).

The cause specific hazard function $h_j(x)$ of X is defined as

$$h_j(x) = \lim_{\Delta t \to 0} \frac{P[x < X + \Delta t, J = j | X \geq X]}{\Delta t}, \quad j = 1, 2, \ldots, k. \tag{7.1.2}$$

The $h_j(x)$ is the instantaneous rate of failure due to the cause j at time x given the subject has survived up to time x. Let $f_j(x) = \frac{d}{dt} F_j(x)$ be the cause specific density of X. If the density $f_j(x)$ exists, (7.1.2) can be written as

$$h_j(x) = \frac{f_j(x)}{S(x)}, \tag{7.1.3}$$

where $S(x) = 1 - F(x)$ is the survival function of X.

Another important function of interest used for the analysis of competing risks data is the sub-survival function $S_j(x)$, defined by

$$S_j(x) = \exp\left(- \int_0^x h_j(t) dt \right). \tag{7.1.4}$$

The function (7.1.4) does not represent a proper survival function of an observable random variable (Lawless [83]). Further $S_j(x) \neq 1 - F_j(x)$.

When the causes of failure are mutually exclusive and exhaustive, then the hazard rate of X, $h(x)$ can be written as

$$h(x) = \sum_{j=1}^{k} h_j(x).$$

Thus, $S(x)$ is uniquely determined by the following identity

$$S(x) = \exp\left(- \sum_{j=1}^{k} \int_0^x h_j(u) du \right) = \prod_{j=1}^{k} S_j(x).$$

From (7.1.3), we get the cumulative incidence function $F_j(x)$ as

$$F_j(x) = \int_0^x S(u)h_j(u)du.$$

For properties and applications of (7.1.1), (7.1.2) and (7.1.4), see Carriere and Kochar [23], Lawless [83] and Crowder [32].

As mentioned earlier an alternative approach for modelling and analysis of statistical data is to use the quantile function. Recently, Peng and Fine [118] developed non-parametric inference procedures for competing risks data using the quantile function. Sankaran et al. [134] derived a test procedure for comparing various risks using sub-quantile functions and Soni et al. [147] proposed tests for successive comparison of quantiles using the quantile functions. Soni et al. [146] developed a non-parametric estimator of the quantile density function.

The objective of the present chapter is to supplement the work of Peng and Fine [118] by introducing quantile-based concepts in the competing risks set up. We define the cause specific hazard quantile function which is the quantile version of (7.1.2). The proposed study has several advantages. In many practical situations, the well-known parametric models are not appropriate for the analysis of lifetime data. The quantile approach provides new quantile function models, as shown in Section 7.3, which are useful for the modelling and analysis of lifetime data. In survival studies, censoring is common. In such contexts, quantile-based analysis is more appropriate as quantiles are more robust (Nair et al. [105]). Finally, the quantile approach gives an alternative methodology for the statistical analysis of competing risks data.

The rest of the chapter is organized as follows. In Section 7.2, we present definitions of quantile-based reliability concepts useful in competing risks theory. Using the proposed methodology, we provide some new quantile function models in Section 7.3. Section 7.4 discusses non-parametric estimation of the cause specific hazard quantile functions and study asymptotic properties of the estimators. A simulation study to assess finite sample properties of the estimators and two real life applications are presented in Section 7.5. Finally, Section 7.6 provides major conclusions of the chapter.

7.2 Cause specific hazard quantile functions

Let X be a non-negative continuous random variable representing the lifetime of a subject with distribution function $F(x)$ and density function $f(x)$. Assume that $F(x)$ is strictly increasing. Denote $Q(u) = \inf\{t : F(x) \geq u\}$ as the quantile function of X. Since $F(x)$ is strictly increasing, we have $Q(u) = F^{-1}(u)$. Let $Q_j(u)$ be the sub-quantile function defined by

$$Q_j(u) = \inf\{t : F_j(t) \geq u\}. \tag{7.2.1}$$

Since $F_j(\infty) < 1, Q_j(1) = v_j < \infty$. Let $q(u) = \frac{d}{du}Q(u)$ and $q_j(u) = \frac{d}{du}Q_j(u)$ be the quantile density and the sub-quantile density functions, respectively (see Peng and Fine [118]). We now define the cause specific hazard quantile function as

$$H_j(u) = h_j(Q(u)) = \frac{f_j(Q(u))}{1 - F(Q(u))} = \frac{f_j(Q(u))}{(1-u)}. \tag{7.2.2}$$

The quantity $H_j(u)$ is interpreted as the conditional probability of failure of the subject in the next small interval of time due to cause j given the survival of the subject at $100(1-u)\%$ point of the distribution.

Note that,

$$\frac{d}{du}F_j(Q(u)) = q(u)f_j(Q(u)) = \frac{f_j(Q(u))}{f(Q(u))}. \tag{7.2.3}$$

It is easy to see from (1.2.4) and (7.2.2), that $\sum_{j=1}^{k} H_j(u) = H(u)$.

Thus the hazard quantile function is the sum of cause specific hazard quantile functions. Further, note that,

$$\frac{d}{du}\left(F_j(Q(u))\right) = \frac{H_j(u)}{H(u)}.$$

Therefore,

$$F_j(Q(u)) = \int_0^u \frac{H_j(p)}{H(p)}dp,$$

or

$$Q(u) = Q_j\left(\int_0^u \frac{H_j(p)}{H(p)}dp\right). \tag{7.2.4}$$

The identity (7.2.4) enables us to determine $Q(u)$ or $Q_j(u)$ from $H_j(u)$.

7.3 Competing risks models

In this section, we discuss competing risks models that arise using different functional forms of the cause specific hazard quantile function $H_j(u)$.

(1) Constant cause specific hazard quantile function

Assume that the cause specific hazard quantile function corresponding to jth risk is constant. That is, $H_j(u) = a_j, \;\; a_j > 0 \quad \text{for} \;\; 0 < u < 1$, we get,

$$H(u) = \sum_{j=1}^{k} a_j, \;\; q(u) = \left((1-u)\sum_{j=1}^{k} a_j\right)^{-1} \quad \text{and} \;\; Q(u) = \frac{-log(1-u)}{\sum_{j=1}^{k} a_j}.$$

Thus the constant cause specific hazard quantile function leads to the fact that lifetime X has an exponential distribution with parameter $\sum_{j=1}^{k} a_j$. From (7.2.4), we have

$$Q_j\left(\frac{a_j u}{\sum_{j=1}^{k} a_j}\right) = Q(u) = -\frac{log(1-u)}{\sum_{j=1}^{k} a_j},$$

so that

$$Q_j(u) = -\frac{log(1 - \frac{\sum_{j=1}^{k} a_j}{a_j} u)}{\sum_{j=1}^{k} a_j}.$$

Thus,

$$F_j(t) = \frac{a_j}{\sum_{j=1}^{k} a_j}(1 - e^{-\sum_{j=1}^{k} a_j t}).$$

The form of such sub-distributions have been discussed by Crowder [32].

(2) Linear cause specific hazard quantile function (Midhu et al. [90])

Suppose that the cause specific hazard quantile function for the cause j is given by the function, $H_j(u) = a_j + b_j u,\ a_j > 0,\ a_j + b_j > 0,\ 0 < u < 1$. Then we obtain

$$\begin{aligned} H(u) &= A + Bu, \text{ where } A = \sum_{j=1}^{k} a_j \text{ and } B = \sum_{j=1}^{k} b_j, \\ q(u) &= \frac{1}{(1-u)(A+Bu)}, \\ \text{and} \quad Q(u) &= \log\left(\frac{A+Bu}{A(1-u)}\right)^{\frac{1}{A+B}}. \end{aligned}$$

We also have

$$\frac{d}{du}F_j(Q(u)) = \frac{a_j + b_j u}{A + Bu},$$

which leads to

$$Q(u) = Q_j\left(\frac{b_j}{B}u + \frac{Ba_j - Ab_j}{B^2}\log\left(\frac{A+Bu}{A}\right)\right).$$

When $b_j = 0$, the model reduces to the exponential model.

If $a_j = 0$, then $H_j(u) = b_j u$ and $H(u) = u\sum_{j=1}^{k} b_j$. Thus, $Q(u) = \left(\log\left(\frac{u}{1-u}\right)\right)^{\frac{1}{\sum_{j=1}^{k} b_j}}$.

In this case,

$$Q_j\left(\frac{b_j}{\sum_{j=1}^{k} b_j}\right) = Q(u) = \left(\log\left(\frac{u}{1-u}\right)\right)^{\frac{1}{\sum_{j=1}^{k} b_j}}.$$

Thus,

$$Q_j(u) = \left(\log\left(\frac{\frac{\sum_{j=1}^{k} b_j}{b_j}u}{1-\frac{\sum_{j=1}^{k} b_j}{b_j}u}\right)\right)^{\frac{1}{\sum_{j=1}^{k} b_j}},$$

so that

$$q_j(u) = \frac{b_j}{\sum_{j=1}^{k} b_j u(b_j - \sum_{j=1}^{k} b_j u)}.$$

Now $F_j(t)$ can be written as

$$F_j(t) = \frac{b_j}{\sum_{j=1}^{k} b_j}\left(\frac{e^{t(\sum_{j=1}^{k} b_j)}}{1+e^{t^{\sum_{j=1}^{k} b_j}}}\right).$$

(3) Weibull cause specific hazard model.

Suppose the cause specific hazard function is given by

$$h_j(t) = \phi\xi_j^{-\phi}t^{\phi-1},\ \phi > 0,\ \ \xi_j > 0.$$

Thus the k risks have the same shape parameter but different scale parameters. Then the hazard rate of X is given by

$$h(t) = \phi\beta t^{\phi-1}, \quad (\beta = \sum_{j=1}^{k}\xi_j^{-\phi}).$$

The cause specific hazard quantile function for the risk j is

$$H_j(u) = \phi \xi_j^{-\phi} \left(-\frac{1}{\beta} \log(1-u) \right)^{1-\frac{1}{\phi}}.$$

The hazard quantile function and quantile function are

$$H(u) = \phi\beta \left(-\frac{1}{\beta} \log(1-u) \right)^{1-\frac{1}{\phi}}$$

$$Q(u) = \left(-\frac{1}{\beta} \log(1-u) \right)^{\frac{1}{\phi}},$$

and

$$Q_j(u) = \left[-\frac{1}{\beta} \log \left(1 - \frac{\beta u}{\xi_j^{-\phi}} \right) \right]^{\frac{1}{\phi}}$$

respectively. It follows that $H_j(u) = \frac{\xi_j^{-\phi}}{\beta} H(u)$ and thus cause specific hazard quantile functions are proportional.

(4) Exponential mixture distribution.

For this model, we have

$$F_j(t) = \pi_j(1 - e^{-a_j t}), \;\; a_j, \pi_j > 0, \tag{7.3.1}$$

then, we obtain,

$$Q_j(u) = -\frac{1}{a_j} \log \left(1 - \frac{u}{\pi_j} \right). \tag{7.3.2}$$

We do not have explicit forms for $H_j(u)$ and $H(u)$ in this case.

(5) Proportional hazards model.

When

$$h_j(t) = \pi_j h(t), \;\; \pi_j > 0,$$

we obtain,

$$H_j(u) = \pi_j H(u),$$

and $Q(u) = Q_j(\pi_j u)$. Thus cause specific hazard functions are proportional.

(6) Consider the model specified by **Dewan and Kulathinal [34].** The cumulative incidence functions for two risks are given by,

$$F_1(t) = \phi F^a(t) \text{ and } F_2(t) = F(t) - \phi F^a(t), \text{ where } 1 \leq a \leq 2 \text{ and } 0 \leq \phi \leq 0.5.$$

Then,

$$f_1(t) = \phi a(F(t))^{a-1} f(t) \text{ and } f_2(t) = f(t) - \phi a(F(t))^{a-1} f(t).$$

We then obtain,

$$h_1(t) = \frac{f_1(t)}{\bar{F}(t)} = \frac{\phi a(F(t))^{a-1} f(t)}{\bar{F}(t)} \text{ and } h_2(t) = \frac{f_2(t)}{\bar{F}(t)} = \frac{f(t) - \phi a(F(t))^{a-1} f(t)}{\bar{F}(t)}.$$

Finally note that, $Q_1(u) = Q\big((\frac{u}{\phi})^{\frac{1}{a}}\big)$.

7.4 Non-parametric estimation of cause specific hazard quantile function

We develop a non-parametric estimator of $H_j(u)$ under right censoring using the kernel density estimation approach. Suppose that the lifetime X is randomly right censored by a variable Z. Then, we observe a random vector $(Y, \delta, \delta J)$ where $Y = \min(X, Z)$ and $\delta = I(X \leq Z)$. Note that δJ is 0 for a censored observation, otherwise it is the cause of failure. Denote $G(x)$ and $H(x)$ as the distribution functions of Z and Y, respectively. When Z and X are independent, we have

$$1 - H(x) = (1 - F(x))(1 - G(x)).$$

The tuples $(Y_i, \delta_i, \delta_i J_i)$ are assumed to be realizations of random variables $(Y, \delta, \delta J)$, for subjects $1, 2, \cdots, n$. If censoring is assumed, the Kaplan-Meier estimator of $S(x)$ for the ordered failure times $Y_{(1)} < Y_{(2)} < ... < Y_{(n)}$, corresponding to Y_i, $i = 1, 2, 3, ...n$ is given by

$$\hat{S}(x) = \prod_{k:Y_{(k)}<t} \left(1 - \frac{d_k}{n_k}\right), \tag{7.4.1}$$

where d_k is the number of failures at $Y_{(k)}$ and n_k is the number of subjects at risk in $Y_{(k)}$, $k = 1, 2, \ldots, n$. Then the non-parametric estimator of $F(x)$ is $\hat{F}(x) = 1 - \hat{S}(x)$. Let $Y_{j(1)} < Y_{j(2)} < ... < Y_{j(n_j)}$ be ordered failure times due to risk j. The Kaplan-Meier estimator of $S_j(x)$ is obtained as

$$\hat{S}_j(x) = \prod_{k:Y_{j(k)}<t} \left(1 - \frac{d_{jk}}{n_{jk}}\right), \tag{7.4.2}$$

where d_{jk} is the number of failures at $Y_{j(k)}$ and n_{jk} is the number of subjects at risk in $Y_{j(k)}$.

Let

$$\hat{S}_{ji} = \begin{cases} \hat{S}_j(Y_{j(i-1)}) - \hat{S}_j(Y_{j(i)}) & i = 2, \ldots, n_j - 1 \\ \hat{S}_j(Y_{j(n_j)}) & i = n_j \end{cases}$$

and

$$S^*(i) = \begin{cases} 0 & \text{if } i = 0 \\ \hat{F}(Y_{(i)}) & \text{if } i = 1, 2, ..., n-1 \\ 1 & \text{if } i = n \end{cases}$$

A simple non-parametric estimator of $H_j(u)$ is given by

$$\hat{H}_j(u) = \frac{\hat{f}_j(\hat{Q}(u))}{1-u}, \tag{7.4.3}$$

where $\hat{Q}(u) = \inf\{x : \hat{F}(x) > u\}$ is the non-parametric estimator of $Q(u)$ and

$$\hat{f}_j(\hat{Q}(u)) = \frac{1}{h(n)} \int_0^1 K\left(\frac{p-u}{h(n)}\right) d(\hat{F}_j(\hat{Q}(p))). \tag{7.4.4}$$

Function $K(x)$ is a kernel function satisfying following conditions:

(a) $K(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} K(x)dx = 1$;

(b) $K(x)$ is symmetric about zero;

(c) $K(x)$ has finite support and

(d) $K(x)$ satisfies the Lipschitz condition.

Denote $\delta_{(i)}$ as the indicator function corresponding to $Y_{(i)}$. Then a non-parametric estimator of $\hat{f}_j(\hat{Q}(u))$ given in (7.4.4) becomes

$$\hat{f}_j(\hat{Q}(u)) = \frac{1}{h(n)} \sum_{i=1}^{n} \hat{S}_{ji} K\left(\frac{S^*(i)-u}{h(n)}\right) I(\delta_{(i)} = 1,\ J = j), \quad j = 1, 2, 3, \ldots, k. \tag{7.4.5}$$

Substituting (7.4.5) in (7.4.3), we get an estimator of $H_j(u)$.

We now establish asymptotic properties of $H_j(u)$. We first prove strong consistency of $H_j(u), j = 1, 2, \ldots, k$.

Theorem 7.4.1. Suppose that $K(x)$ satisfies conditions (a) to (d). Assume that both $F(x)$ and $K(x)$ are differentiable. Then $\sup_u |\hat{H}_j(u) - H_j(u)| \to 0$ as $n \to \infty$ for $j = 1, \ldots, k$.

Proof. From equation (7.2.3), we have $dF_j(Q(u)) = f_j(Q(u))q(u)du$. Then,

$$\hat{H}_j(u) - H_j(u) = \frac{1}{(1-u)h(n)} \int_0^1 K\left(\frac{p-u}{h(n)}\right) d\hat{F}_j(\hat{Q}(p)) - \frac{f_j(Q(u))}{1-u}. \tag{7.4.6}$$

We can write (7.4.6) as

$$\begin{aligned}\hat{H}_j(u) - H_j(u) =& \frac{1}{(1-u)h(n)} \int_0^1 K\left(\frac{p-u}{h(n)}\right) d\left[\hat{F}_j(\hat{Q}(p)) - \hat{F}_j(Q(p)) + \hat{F}_j(Q(p))\right. \\ &\left. - F_j(Q(p))\right] + \frac{1}{(1-u)h(n)} \int_0^1 K\left(\frac{p-u}{h(n)}\right) dF_j(Q(p)) - \frac{f_j(Q(u))}{1-u}.\end{aligned} \tag{7.4.7}$$

Since, $\sup_u |\hat{Q}(u) - Q(u)| \to 0$ as $n \to \infty$ (Andersen et al. [5]) and $\sup_x |\hat{F}_j(x) - F_j(x)| \to 0$ as $n \to \infty$ (Lawless [83]), the first term on the right side of (7.4.7) tends to zero when n is large. Now consider

$$\begin{aligned}\frac{1}{h(n)} \int_0^1 K\left(\frac{p-u}{h(n)}\right) d(F_j(Q(p))) =& \frac{1}{h(n)} \int_{-\infty}^{\infty} K\left(\frac{t-x}{h(n)}\right) d(F_j(t)) \\ =& \frac{1}{h(n)} \int_{-\infty}^{\infty} K\left(\frac{t-x}{h(n)}\right) f_j(t)dt.\end{aligned} \tag{7.4.8}$$

Let $\frac{t-x}{h(n)} = z$. Then (7.4.8) becomes

$$\int_{-\infty}^{\infty} K(z)\, f_j(x + zh(n))dz. \tag{7.4.9}$$

By Taylor series expansion of $f_j(x + zh(n))$, we obtain (7.4.9) as

$$\int_{-\infty}^{\infty} K(z) \left[f_j(x) + zh(n)f_j'(x) + ... \right] dz, \tag{7.4.10}$$

where prime denote derivative with respect to x.

As $n \to \infty$, $h(n) \to 0$ and hence (7.4.10) tends to

$$\int_{-\infty}^{\infty} K(z) f_j(x) dz = f_j(x). \tag{7.4.11}$$

Using (7.4.8) and substituting $x = Q(u)$ in (7.4.11), the equation (7.4.7) becomes $\sup_u |\hat{H}_j(u) - H_j(u)| \to 0$ as $n \to \infty$. ■

In the following theorem, we prove the limiting distribution of $\sqrt{n}(\hat{H}_j(u) - H_j(u))$.

Theorem 7.4.2. As $n \to \infty$, for fixed u $(0 < u < 1)$, $\sqrt{n\, h^2(n)}(\hat{H}_j(u) - H_j(u)),\ j = 1, \cdots, k$ follows a normal distribution with mean 0 and variance $\sigma_j^2(u)$, where, $\sigma_j^2(u) = \mathbb{E}\Big[\frac{1}{(1-u)} \int_0^1 K^*(u,p) Z(p) dp\Big]^2$, with $Z(p) = \sqrt{n}[\hat{F}_j(\hat{Q}(p)) - F_j(Q(p))]$.

Proof. From Theorem 7.4.1 as $n \to \infty$ and $h(n) \to 0$, the expression (7.4.7) asymptotically reduces to

$$\begin{aligned}\sqrt{n}(\hat{H}_j(u) - H_j(u)) = \frac{\sqrt{n}}{h(n)\,(1-u)} \int_0^1 K\left(\frac{p-u}{h(n)}\right) d\Big[\hat{F}_j(\hat{Q}(p)) - F_j(\hat{Q}(p)) \\ + F_j(\hat{Q}(p)) - F_j(Q(p))\Big].\end{aligned} \tag{7.4.12}$$

Using integration by parts, (7.4.12) becomes

$$\sqrt{n}(\hat{H}_j(u) - H_j(u)) = \frac{\sqrt{n}}{h(n)\,(1-u)} \left(\int_0^1 K^*(u,p)[\hat{F}_j(\hat{Q}(p)) - F_j(\hat{Q}(p))]dp \right.$$
$$\left. + \int_0^1 K^*(u,p)[F_j(\hat{Q}(p)) - F_j(Q(p))]dp \right), \quad (7.4.13)$$

where $K^*(u,p) = \frac{dK}{dp}\left(\frac{p-u}{h(n)}\right)$.

From Andersen et al. [5], for $0 < u < 1$, we have $\sqrt{n}\left(\hat{Q}(u) - Q(u)\right)$ is asymptotically normal with mean zero and variance $\sigma_1^{2*}(u) = (S(u))^2 \int_0^u \frac{(-dS(t))}{S(t)S^*(t)}$, where $S^*(t)$ is the probability that a unit is alive and uncensored at time t.

It follows from Lawless [83] that for $0 < x < \infty$, $\sqrt{n}(\hat{F}_j(x) - F_j(x))$ is asymptotically normal with mean zero and variance $\sigma_2^{2*}(x)$, which can be estimated as given in Section 9.2 of Lawless [83]. Using the functional delta method and Slutsky's theorem (Serfling [140]), we get that for $0 < u < 1$, $\sqrt{n\,h^2(n)}(\hat{H}_j(u) - H_j(u))$ follows normal distribution with mean as zero and variance $\sigma_j^2(u)$, where $\sigma_j^2(u) = \mathbb{E}\left[\frac{1}{(1-u)}\int_0^1 K^*(u,p)Z(p)dp\right]^2$, with $Z(p) = \sqrt{n}\left(\hat{F}_j(\hat{Q}(p)) \; - F_j(Q(p))\right)$. This completes the proof. ■

Remark 7.4.1. Since the analytical expressions of $\sigma_j^2(u)$ is complex, we have to use the bootstrap procedure for estimating the variance of $\hat{H}_j(u), j = 1, 2, ..., k$. The bootstrap method is based on the resampling method from the original data. We take B samples of size n from the original data using random sampling with replacement. The bootstrap samples are $(Y_i^{(k)}, \delta_i^{(k)}, J_i^{(k)}), k = 1, 2, ..., B; i = 1, 2, ..., n$. We then compute $\hat{H}_j(u)$, using original data set and the estimate of $H_j(u)$ using the bootstrap sample k is $\hat{\Lambda}_j^{(k)}(u), k = 1, 2, ..., B$. We then compute the bias by taking differences $\hat{\Lambda}_j^{(k)}(u) - \hat{H}_j(u), j = 1, 2; k = 1, 2, ..., B$. Then using these differences, the average bias and MSE are calculated.

7.5 Simulations

We now carry out extensive simulation studies to find out mean square error (MSE) and bias of the estimator $\hat{H}_j(u)$ for the uncensored as well as the censored case. We consider two causes of failure. We take different samples of size 50, 100 and 200. We generated 5000 data sets in each scenario. The order of sub-quantiles considered are $u = 0.2,\ 0.4,\ 0.6$ and 0.8. Simulations are carried out for uncensored and censored cases to find the average bias and MSE of the estimators. We have employed the triangular, uniform and Epanechnikov kernel functions in simulation studies. However, results are being reported for the Epanechnikov kernel as this provides the smallest MSE. The Epanechnikov kernel is defined by

$$K(u) = 0.75(1 - u^2)I(|u| \leq 1).$$

To generate random numbers, we consider the linear and Weibull cause specific hazard quantile function models given in Section 7.3. Since the proposed estimator of the cause specific hazard quantile function is based on the kernel function, the choice of bandwidth is an important issue. For the construction of kernel type estimator of a quantile function, Padgett [116] has considered separate bandwidths for different regions of $u \in (0, 1)$ in such a way that the mean squared error (MSE) is minimum. In our study, we calculate the optimum bandwidths corresponding to different values of u such as 0.2, 0.4, 0.6 and 0.8. The average of the optimal bandwidths obtained for different values of u is employed for the construction of the proposed estimators.

To perform the simulation study, we use the same parameter combinations for the linear cause specific hazard quantile function model in both censored as well as uncensored cases. The same procedure is adopted for the Weibull cause specific hazard quantile function model. The parameter values chosen for the linear cause specific hazard quantile function model are $a_1 = \frac{1}{2}$, $b_1 = 3$, $a_2 = \frac{1}{3}$ and $b_2 = 2$. For the Weibull model, we take $\phi = 3$, $\xi_1 = 1$ and $\xi_2 = 2$.

7.5.1 Results for the uncensored case

We first consider the linear cause specific hazard quantile function for different sample sizes $n = 50, 100$ and 200. The estimators $\hat{H}_j(u), j = 1, 2$ are calculated for all values of u ($0 < u < 1$), which provides the smooth curves. Then the average bias and MSE of the estimators are computed. The bandwidths for $\hat{H}_1(u)$ and $\hat{H}_2(u)$ are obtained as 0.52 and 0.64 respectively. Table 7.1 presents the average bias and MSE of $H_j(u), j = 1, 2$, for n= 50, 100 and 200. Both average bias and MSE decrease as sample size increases.

Table 7.1: Average bias and MSE of $\hat{H}_1(u)$ and $\hat{H}_2(u)$ for the linear cause specific hazard model (uncensored) for the optimal bandwidths.

n			u			
			0.2	0.4	0.6	0.8
50	$\hat{H}_1(u)$	MSE	0.0562	0.2758	0.3318	0.1729
		BIAS	-0.1445	-0.4912	-0.4314	0.5814
	$\hat{H}_2(u)$	MSE	0.0384	0.1477	0.1686	0.3029
		BIAS	-0.1225	-0.3077	-0.2773	0.2433
100	$\hat{H}_1(u)$	MSE	0.0453	0.2400	0.2766	0.1525
		BIAS	-0.1312	-0.4732	-0.4040	0.3315
	$\hat{H}_2(u)$	MSE	0.0328	0.1303	0.1417	0.2478
		BIAS	-0.1221	-0.3008	-0.2727	0.6292
200	$\hat{H}_1(u)$	MSE	0.0406	0.2238	0.2523	0.1380
		BIAS	-0.1239	-0.4638	-0.3921	0.3307
	$\hat{H}_2(u)$	MSE	0.0303	0.1221	0.1293	0.1734
		BIAS	-0.1203	-0.2961	-0.2566	0.1922

We then consider the Weibull cause specific hazard model (7.3.1). The estimators $\hat{H}_j(u), j = 1, 2$ are calculated. The bandwidths which give minimum MSE for $\hat{H}_1(u)$ and $\hat{H}_2(u)$ are

0.72 and 0.44 respectively. Table 7.2 gives average bias and MSE of the estimators of the cause specific hazard quantile functions. Note that both average bias and MSE decrease as sample size increases.

Table 7.2: Average bias and MSE for $\hat{H}_1(u)$ and $\hat{H}_2(u)$ for the Weibull cause specific hazard model (uncensored) for the optimal bandwidths.

n			u			
			0.2	0.4	0.6	0.8
50	$\hat{H}_1(u)$	MSE	0.0395	0.1511	0.1546	0.1776
		BIAS	-0.1670	-0.3421	-0.2833	0.4186
	$\hat{H}_2(u)$	MSE	0.0160	0.0253	0.0301	0.0296
		BIAS	-0.0560	-0.1248	-0.1296	0.1410
100	$\hat{H}_1(u)$	MSE	0.0354	0.1387	0.1368	0.1594
		BIAS	-0.1555	-0.3319	-0.2822	0.3956
	$\hat{H}_2(u)$	MSE	0.0142	0.0207	0.0231	0.0232
		BIAS	-0.0554	-0.1009	-0.0932	0.1125
200	$\hat{H}_1(u)$	MSE	0.0335	0.1328	0.1285	0.1262
		BIAS	-0.1533	-0.3231	-0.1541	0.3492
	$\hat{H}_2(u)$	MSE	0.0136	0.0189	0.0207	0.0165
		BIAS	-0.0447	-0.0804	-0.0748	0.0168

7.5.2 Results for the censored case

The censored observations are generated using uniform distribution $U(0, C)$, where C is chosen such that 20% observations are censored. We first consider the linear cause specific hazard quantile function model. We compute the average bias and MSE of the estimators $\hat{H}_j(u),\ j = 1, 2$. The bandwidths which give minimum MSE for $\hat{H}_1(u)$ and $\hat{H}_2(u)$ are 0.67 and 0.31 respectively. Table 7.3 presents the average bias and MSE under censoring. Both average bias and MSE decrease as sample size increases.

We generate observations from the Weibull cause specific hazard model with the censoring scheme given above. The $\hat{H}_j(u), j = 1, 2$ are calculated and the average bias and MSE of

Table 7.3: Average bias and MSE for $\hat{H}_1(u)$ and $\hat{H}_2(u)$ for the linear cause specific hazard model (censored) for the optimal bandwidths.

<table>
<tr><td rowspan="2">n</td><td rowspan="2"></td><td></td><td colspan="4">u</td></tr>
<tr><td></td><td>0.2</td><td>0.4</td><td>0.6</td><td>0.8</td></tr>
<tr><td rowspan="4">50</td><td rowspan="2">$\hat{H}_1(u)$</td><td>MSE</td><td>0.0680</td><td>0.2764</td><td>0.3901</td><td>0.7679</td></tr>
<tr><td>BIAS</td><td>-0.2179</td><td>-0.5230</td><td>-0.5746</td><td>0.6104</td></tr>
<tr><td rowspan="2">$\hat{H}_2(u)$</td><td>MSE</td><td>0.0548</td><td>0.1622</td><td>0.1862</td><td>0.4400</td></tr>
<tr><td>BIAS</td><td>-0.1691</td><td>-0.3778</td><td>-0.4054</td><td>0.6832</td></tr>
<tr><td rowspan="4">100</td><td rowspan="2">$\hat{H}_1(u)$</td><td>MSE</td><td>0.0618</td><td>0.2401</td><td>0.1875</td><td>0.7560</td></tr>
<tr><td>BIAS</td><td>-0.1898</td><td>-0.4870</td><td>-0.5239</td><td>0.6020</td></tr>
<tr><td rowspan="2">$\hat{H}_2(u)$</td><td>MSE</td><td>0.0337</td><td>0.1991</td><td>0.1856</td><td>0.4110</td></tr>
<tr><td>BIAS</td><td>-0.1506</td><td>-0.3533</td><td>-0.3699</td><td>0.2796</td></tr>
<tr><td rowspan="4">200</td><td rowspan="2">$\hat{H}_1(u)$</td><td>MSE</td><td>0.0587</td><td>0.2284</td><td>0.1582</td><td>0.7032</td></tr>
<tr><td>BIAS</td><td>-0.1760</td><td>-0.4697</td><td>-0.4997</td><td>0.5468</td></tr>
<tr><td rowspan="2">$\hat{H}_2(u)$</td><td>MSE</td><td>0.0323</td><td>0.1978</td><td>0.1816</td><td>0.3591</td></tr>
<tr><td>BIAS</td><td>-0.1415</td><td>-0.3412</td><td>-0.3523</td><td>0.2642</td></tr>
</table>

the estimators are computed. The bandwidths which give minimum MSE for $\hat{H}_1(u)$ and $\hat{H}_2(u)$ are 0.59 and 0.38 respectively. Table 7.4 presents average bias and MSE of the estimators of $H_j(u), j = 1, 2$. It follows that the average bias and MSE of $\hat{H}_j(u), j = 1, 2$ are small and both decrease as sample size increases.

Hoel data (Hoel [57]) The data were obtained from a laboratory experiment on two groups of RFM strain male mice which had received a radiation dose of 300r at an age of 5-6 weeks. The first group of mice lived in a conventional laboratory environment while the second group was in a germ-free environment. There are three major causes for death such as thymic lymphoma, reticulum cell sarcoma and other cause. All mice died at the end of the study so that there is no censoring. We considered data from first group of 99 mice for analysis. We combine the last two causes since the number of deaths due to reticulum cell sarcoma is small. Thus two causes for the analysis are thymic lymphoma (J_1) and other causes (J_2) which include reticulum cell sarcoma. The interest is to compare the mortality from these two modes of death. The estimates of $H_j(u), j = 1, 2$ are computed as described in Section 7.4. The bandwidth which minimizes the bootstrap MSE has been

Table 7.4: Average bias and MSE of $\hat{H}_1(u)$ and $\hat{H}_2(u)$ for the Weibull cause specific hazard model (censored) for the optimal bandwidths.

n			u			
			0.2	0.4	0.6	0.8
50	$\hat{H}_1(u)$	MSE	0.0406	0.1512	0.1885	0.4045
		BIAS	-0.1750	-0.3827	-0.3864	0.6372
	$\hat{H}_2(u)$	MSE	0.0168	0.0275	0.0339	0.0315
		BIAS	-0.0697	-0.1264	-0.1360	0.1430
100	$\hat{H}_1(u)$	MSE	0.0363	0.1415	0.1879	0.4041
		BIAS	-0.1619	-0.3658	-0.3619	0.6267
	$\hat{H}_2(u)$	MSE	0.0149	0.0276	0.0341	0.0260
		BIAS	-0.0562	-0.1059	-0.1059	0.1344
200	$\hat{H}_1(u)$	MSE	0.0349	0.1501	0.1730	0.3757
		BIAS	-0.1554	-0.3574	-0.3497	0.6232
	$\hat{H}_2(u)$	MSE	0.0157	0.0247	0.0295	0.0157
		BIAS	-0.0468	-0.0962	-0.0917	0.0177

chosen. Bandwidths thus obtained for $\hat{H}_j(u)$, $j = 1, 2$, are 0.71 and 0.29 respectively. Figure 7.1 shows the cause specific hazard quantile functions. From Figure 7.1, it is clear that

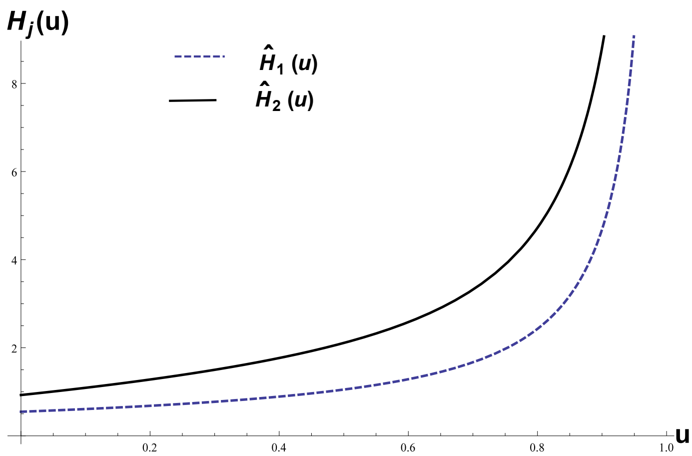


Figure 7.1: Estimates of cause specific hazard quantile functions for Hoel data.

the cause specific hazard quantile function due to thymic lymphoma is uniformly smaller than that due to other causes. We also observe that the two cause specific hazard functions are closer to each other at the tails. The major cause of failure is not thymic lymphoma J_1, but other causes J_2.

Davis and Lawrance data (Davis and Lawrance [33]). They considered the tyre-testing data, which measure the failure times at hourly intervals of 171 tyres with 12% right censoring. The major causes of failures are

(i) an open joint on the inner lines,

(ii) rubber chunking on the shoulder

(iii) loose chunking, low on the shoulder,

(iv) cracking of tread rubber,

(v) cracking on the side wall,

and (vi) all other causes of failures.

Since there are few failures due to certain causes, we grouped the causes into three major categories as, cause 1 (J_1) - for causes (iii) and (v) - 34 failures, cause 2 (J_2) - for cause (iv) - 69 failures and cause 3 (J_3) - for causes (i), (ii), and (vi) - 48 failures.

The optimal bandwidths for $\hat{H}_1(u)$, $\hat{H}_2(u)$ and $\hat{H}_3(u)$ are 0.47, 0.61 and 0.51 respectively. Figure 7.2 shows the estimates of cause specific hazard quantile function due to three different causes. From Figure 7.2, it follows that the cause specific hazard quantile function due to cause 1 is larger than that of cause 2. Further, the cause specific hazard quantile function due to cause 3 lies between the other two causes of failure. The major cause of failure is due to loose chunking, low on side wall and cracking on side wall (J_1).

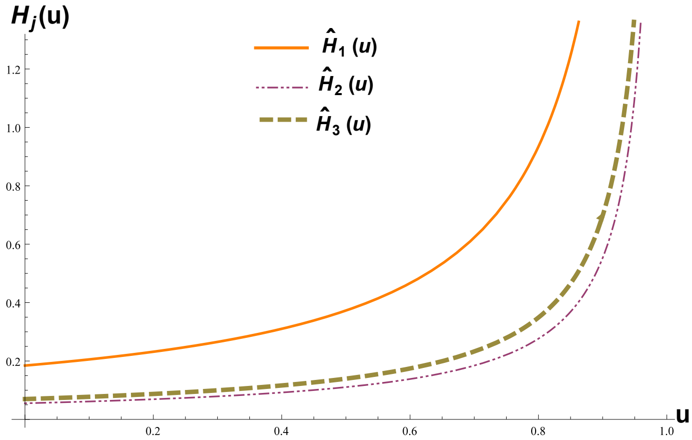


Figure 7.2: Estimates of cause specific hazard quantile functions for Davis and Lawrence data.

7.6 Conclusion

The present chapter introduced the concept of cause specific hazard quantile function, which is the quantile version of the cause specific hazard rate. The proposed methodology provided new lifetime models useful for the analysis of competing risks data. The smooth kernel type estimator of cause specific hazard quantile function has been developed for uncensored as well as censored data. Asymptotic properties of the proposed estimator were studied. The estimator performs well in terms of average bias and MSE for linear cause specific hazard model as well as for Weibull cause specific hazard model. The procedure has been applied to two real life data sets.

The proposed work based on the cause specific hazard quantile functions is an alternative method of modelling and analysis of competing risks data. This technique has the ability to pick up the differences at extreme values of the data. The quantile models presented here will enable the practitioner to differentiate between the effects of various risks.

Chapter 8

Relevation Transforms and their Applications

8.1 Introduction

Relevation transforms, introduced by Krakowski [74] have attracted considerable interest of researchers in survival analysis and reliability theory. Let X and Y be two absolutely continuous non-negative random variables, with survival functions $\bar{F}(\cdot)$ and $\bar{G}(\cdot)$ respectively. Consider a component of a system whose lifetime X has survival function $\bar{F}(x)$; suppose that the component is replaced at the time of its failure at age x, by another component of the same age x, where lifetime Y of the second component has survival function $\bar{G}(x)$. Let $X\#Y$ be the total lifetime of the random variable Y given that it exceeds a random time X. Then the survival function $\bar{T}_{X\#Y}(x)$ given by

$$\bar{T}_{X\#Y}(x) = \bar{F}\#\bar{G}(x) = \bar{F}(x) - \bar{G}(x)\int_0^x \frac{1}{\bar{G}(t)} d\bar{F}(t), \tag{8.1.1}$$

Results in this chapter have been published in the Journals "Metrika" as entitled "Reliability properties of proportional hazards relevation transform" (See Sankaran and Kumar [130]) and in "Statistica" as entitled "Quantile based relevation transform and its properties" (See Dileep et al. [35]).

The probability density function of the relevation random variable is obtained as

$$t_{X\#Y}(x) = T'_{X\#Y}(x) = g(x)\int_0^x \frac{f(t)}{\bar{G}(t)}dt. \tag{8.1.2}$$

Grosswald et al. [47] provided two characterizations of the exponential distribution based on relevation transform. The concept of dependent relevation transform and its importance in reliability analysis were given in Johnson and Kotz [62]. Applications of the multiple relevation transforms, denoted by $X = Y_1\#...\#Y_n,\ \ n = 2, 3, ...$, where Y_n is a sequence of independent and identical random variables were considered from the reliability point of view by Baxter [13]. Shanthikumar and Baxter [142] provided closure properties of ageing concepts in the context of relevation transforms. Improved versions of the results in Grosswald et al. [47] were given in Lau and Rao [80]. Chukova et al. [27] derived characterizations of the class of distributions with almost lack of memory property based on the relevation transform. Belzunce et al. [16] established the relationship between the relevation transform and the distribution of record values. Further, they have shown that the distribution of epoch times of a non-homogeneous pure birth process (NHPBP) is identical to the relevation transform. Some results for the comparison of the failure times and inter failure times of two systems based on a replacement policy were given in Sordo and Psarrakos [148]. Psarrakos and Di Crescenzo [120] introduced an inaccuracy measure concerning the relevation transform of two non-negative continuous random variables.

The present chapter is arranged into two main sections. In Section 8.2, we study the basic reliability properties of relevation transform under proportional hazards assumption and its applications. In this connection, we discuss the ageing properties and stochastic orders. We also introduce a new lifetime model by considering Weibull distribution as the baseline of the proposed model. The proposed lifetime model is applied to two real-life data sets.

Then in Section 8.3, we introduce the quantile version of the relevation transform and study its basic properties. The quantile-based relevation transform in the context of proportional hazards model and equilibrium distribution are discussed in sequel. Finally, Section 8.4 provides major conclusions of the study.

8.2 Proportional hazards relevation transform

In reliability theory, the proportional hazards model (PHM) plays a vital role in the comparison of lifetime of two components. The random variables X and Y satisfy PHM if,

$$h_Y(x) = \theta h_X(x), \quad \theta > 0, \tag{8.2.1}$$

where $h_Y(x)$ and $h_X(x)$ are the hazard rate functions of X and Y. An equivalent representation of (8.2.1) is

$$\bar{G}(x) = (\bar{F}(x))^{\theta}, \quad \theta > 0. \tag{8.2.2}$$

For more details on PHM, one could refer to Kalbfleisch and Prentice [67] and Lawless [84]. When Y is the PHM of X with survival functions related as in (8.2.2), we call the transformation given in (8.1.1) as the proportional hazards relevation transform (PHRT).

Let X and Y be two non-negative random variables with absolutely continuous distribution functions $F(x)$ and $G(x)$ respectively. When Y is the PHM of X, from (8.1.1), the relevation random variable $X\#Y$ has the survival function given by

$$\bar{T}_{PH}(x) = \bar{F}(x) - (\bar{F}(x))^{\theta} \int_0^x \frac{1}{(\bar{F}(t))^{\theta}} d\bar{F}(t). \tag{8.2.3}$$

We interpret $\bar{T}_{PH}(x)$ in the same manner but with the difference that the replacement of the item is done by another, whose hazard rate is proportional to that of the original item. Further, (8.2.3) provides a family of life distributions in its own right, which is quite flexible with respect to the reliability properties. Krakowski [74] has shown that PHRT posses commutative property, which means that the random variables $Y\#X$ and $X\#Y$ are identically distributed.

Note that, for $\theta \to 1$,

$$\underset{\theta\to 1}{\text{Lim}}\ \bar{T}_{PH}(x) = \bar{F}(x)(1-\log(\bar{F}(x))) = \bar{F}\#\bar{F}(x). \tag{8.2.4}$$

The survival function (8.2.4) is known as the auto relevation of $\bar{F}(x)$. Kapodistria and Psarrakos [68] studied properties and applications of a sequence of random variables with weighted tail distribution functions based on the auto relevation transform.

In the present chapter, we consider the case of $\theta \neq 1$, which means X and Y are not identically distributed. The survival function $\bar{T}_{PH}(x)$ of the relevation random variable $X\#Y$ under PHRT has a closed form expression in terms of the baseline survival function $\bar{F}(x)$, as seen from the following theorem.

Theorem 8.2.1. Let X and Y be two independent non-negative random variables with distribution functions $F(x)$ and $G(x)$ respectively. Then Y is the PHM of X if and only if the relevation survival function $\bar{T}_{PH}(x)$ satisfies,

$$\bar{T}_{PH}(x) = \frac{1}{\theta-1}\left(\theta\bar{F}(x) - (\bar{F}(x))^{\theta}\right). \tag{8.2.5}$$

Proof. Assume that Y is the PHM of X. Now from (8.2.3), we have

$$\begin{aligned}\bar{T}_{PH}(x) &= \bar{F}(x) - (\bar{F}(x))^{\theta}\int_{t=0}^{x}\frac{1}{(\bar{F}(t))^{\theta}}d\bar{F}(t)\\ &= \bar{F}(x) - \frac{1}{1-\theta}\left(\bar{F}(x) - (\bar{F}(x))^{\theta}\right)\end{aligned}$$

$$= \left(\frac{\theta}{\theta-1}\right)\bar{F}(x) - \left(\frac{1}{\theta-1}\right)(\bar{F}(x))^{\theta}.$$

Conversely assume that the relation (8.2.5) holds. Then from (8.1.1) and (8.2.5), we get

$$\bar{F}(x) - \bar{G}(x)\int_{t=0}^{x}\frac{1}{\bar{G}(t)}d\bar{F}(t) = \left(\frac{\theta}{\theta-1}\right)\bar{F}(x) - \left(\frac{1}{\theta-1}\right)(\bar{F}(x))^{\theta},$$

which can be written as

$$\int_{t=0}^{x}\frac{1}{\bar{G}(t)}d\bar{F}(t) = \frac{1}{1-\theta}\left(\frac{(\bar{F}(x))^{\theta}-\bar{F}(x)}{\bar{G}(x)}\right). \tag{8.2.6}$$

Differentiating both sides of (8.2.6) with respect to x,

$$\frac{(\bar{F}(x))'}{\bar{G}(x)} = \frac{1}{\theta-1}\left(\frac{\theta(\bar{F}(x))^{\theta-1}(\bar{F}(x))'-(\bar{F}(x))'}{\bar{G}(x)} - \frac{((\bar{F}(x))^{\theta}-(\bar{F}(x)))((\bar{G}(x))')}{(\bar{G}(x))^2}\right), \tag{8.2.7}$$

where prime denote the derivative. Simplifying (8.2.7), we get

$$\frac{(\bar{G}(x))'}{\bar{G}(x)} = \theta\frac{(\bar{F}(x))'}{\bar{F}(x)} \text{ or } h_Y(x) = \theta h_X(x),$$

as required. ■

Remark 8.2.1. $\bar{T}_{PH}(x)$ in (8.2.5) has a mixture representation given by

$$\bar{T}_{PH}(x) = \phi\bar{F}(x) + (1-\phi)(\bar{F}(x))^{\theta},$$

where $\phi = \frac{\theta}{1-\theta}$. Note that one of the weights is negative (generalized mixture) depending on $\theta > 1$ or $0 < \theta < 1$.

In the context of coherent systems with 'n' identical components, Navarro et al. [111]

established that the component survival function $\bar{F}_c(x)$ and the system survival function $\bar{F}_S(x)$ are connected through the relation

$$\bar{F}_S(x) = w(\bar{F}_c(x)), \tag{8.2.8}$$

where $w(u)$ is a distortion function, which is a concave non-decreasing function from $[0, 1]$ to $[0, 1]$, such that $w(0) = 0$ and $w(1) = 1$.

From (8.2.5), the survival function $\bar{T}_{PH}(x)$ satisfies,

$$\bar{T}_{PH}(x) = w(\bar{F}(x)), \quad \text{where } w(u) = \frac{1}{\theta - 1}(\theta u - u^{\theta}),\ u \in [0, 1]. \tag{8.2.9}$$

The function $w(u)$ is a concave distortion function. From this we can infer that, $X\#Y$ is the distorted random variable obtained from X by the distortion $w(u)$. Distorted random variables have many applications in reliability theory. Navarro et al. [112] and Navarro et al. [111] developed various stochastic orders and preservation properties of ageing classes for the general distorted distributions in the context of coherent systems. For more details on this topic, one could refer to Wang [155], Sordo and Suárez-Llorens [149], Sordo et al. [150] and Navarro et al. [113].

From (8.2.5), we obtain the mean of $X\#Y$ as

$$\mu_{X\#Y} = \int_0^{\infty} \bar{T}_{PH}(x)dx = \frac{\theta\mu_X - \mu_Y}{\theta - 1},$$

where μ_X, μ_Y and $\mu_{X\#Y}$ denotes the mean of the random variables X, Y and $X\#Y$ respectively. Let $t_{PH}(x)$ be the relevation density function under PHM. Then,

$$t_{PH}(x) = \frac{\theta}{\theta - 1} f(x) \left(1 - (\bar{F}(x))^{\theta - 1}\right). \tag{8.2.10}$$

From (8.2.3) and (8.2.10), the hazard rate of the relevation random variable $X\#Y$ under PHM is given by

$$\begin{aligned} h_{X\#Y}(x) &= \frac{t_{PH}(x)}{\bar{T}_{PH}(x)} \\ \Leftrightarrow h_{X\#Y}(x) &= \theta\ h_X(x) \left(\frac{1 - (\bar{F}(x))^{\theta-1}}{\theta - (\bar{F}(x))^{\theta-1}} \right). \end{aligned} \tag{8.2.11}$$

From (8.2.5), we have

$$\int_x^{\infty} \bar{T}_{PH}(t)dt = \frac{\theta}{\theta - 1} \int_x^{\infty} \bar{F}(t)dt - \frac{1}{\theta - 1} \int_x^{\infty} (\bar{F}(t))^{\theta} dt.$$

Dividing with $\bar{T}_{PH}(x)$ and noting that $\frac{1}{(\bar{F}(x))^{\theta}} \int_x^{\infty} (\bar{F}(t))^{\theta} dt = m_Y(x)$, we have the relation,

$$m_{X\#Y}(x) = \left(\frac{\theta}{\theta - (\bar{F}(x))^{\theta-1}} \right) m_X(x) + \left(\frac{1}{1 - \theta(\bar{F}(x))^{1-\theta}} \right) m_Y(x), \tag{8.2.12}$$

where $m_X(x)$ and $m_Y(x)$ are the mean residual life functions of X and Y respectively.

Expressions (8.2.5), (8.2.11) and (8.2.12) enable us to derive several results about the reliability aspects of $X\#Y$ in terms of the corresponding results of X. These results are helpful in assessing the properties of $X\#Y$ directly from X without having to find and use the expressions of $\bar{T}_{PH}$, $h_{X\#Y}$ and $m_{X\#Y}$. The survival function $\bar{T}_{PH}(x)$ represents a family of life distributions depending on the baseline survival function $\bar{F}(x)$. It is advantageous to know some criteria by which certain members of the family can be distinguished. We now present few characterizations to meet this objective. When X follows the generalized Pareto distribution with

$$\bar{F}(x) = \left(1 + \frac{ax}{b}\right)^{-\left(\frac{a+1}{a}\right)}, \quad x > 0; b > 0, a > -1, \tag{8.2.13}$$

we obtain

$$\bar{T}_{PH}(x) = \frac{1}{\theta - 1}\left[\theta\left(1 + \frac{ax}{b}\right)^{-\left(\frac{a+1}{a}\right)} - \left(1 + \frac{ax}{b}\right)^{-\left(\frac{\theta(a+1)}{a}\right)}\right]. \tag{8.2.14}$$

The distribution (8.2.13) reduces to the exponential distribution with mean b as $a \to 0$, the Pareto-II as $a > 0$ and the scaled beta as $-1 < a < 0$ and uniform as its special cases. Even though the hazard rate and mean residual life of (8.2.14) are generally not of simple forms, there is a simple relationship between the two functions that characterize the model (8.2.14).

Theorem 8.2.2. The random variable $X\#Y$ has distribution (8.2.14) if and only if

$$m_{X\#Y} = \frac{\theta(a+1)+1}{\theta(a+1)-a}(ax+b) + \frac{(ax+b)^2}{\theta(a+1)-a}h_{X\#Y}(x), \quad \theta > \text{maximum}\left\{0, \frac{a}{a+1}\right\}. \tag{8.2.15}$$

Proof. By direct calculation from (8.2.14), we obtain

$$h_{X\#Y} = \frac{(a+1)\theta}{b}\frac{\left[\left(1+\frac{ax}{b}\right)^{-\frac{a+1}{a}-1} - \left(1+\frac{ax}{b}\right)^{-\frac{\theta(a+1)}{a}-1}\right]}{\left[\theta\left(1+\frac{ax}{b}\right)^{-\frac{a+1}{a}} - \left(1+\frac{ax}{b}\right)^{-\frac{\theta(a+1)}{a}}\right]} \tag{8.2.16}$$

and

$$m_{X\#Y} = b\frac{\left[\theta\left(1+\frac{ax}{b}\right)^{-\frac{a+1}{a}+1} - (\theta(a+1)-a)^{-1}\left(1+\frac{ax}{b}\right)^{-\frac{\theta(a+1)}{a}+1}\right]}{\left[\theta\left(1+\frac{ax}{b}\right)^{-\frac{a+1}{a}} - \left(1+\frac{ax}{b}\right)^{-\frac{\theta(a+1)}{a}}\right]}. \tag{8.2.17}$$

From the last two expressions, we get the relation (8.2.15). To prove the ‘only if’ part, we note that (8.2.15) is equivalent to,

$$\int_x^{\infty} \bar{T}_{PH}(t)dt = (ax+b)\frac{\theta(a+1)+1}{\theta(a+1)-a}\bar{T}_{PH}(x) + \frac{(ax+b)^2}{\theta(a+1)-a}t_{PH}(x). \qquad (8.2.18)$$

Setting $y = \int_x^{\infty} \bar{T}_{PH}(t)dt$, (8.2.18) reduces to the Euler-type differential equation,

$$y = -(ax+b)\left(\frac{\theta(a+1)+1}{\theta(a+1)-a}\right)\frac{dy}{dx} + \left(\frac{(ax+b)^2}{\theta(a+1)-a}\right)\frac{d^2y}{dx^2}. \qquad (8.2.19)$$

Taking $e^z = ax+b, \quad \frac{dy}{dx} = a\frac{dy}{dz}e^{-z}; \quad \frac{d^2y}{dx^2} = a^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)e^{-2z}$, and substituting this in (8.2.19), we have the second order homogeneous differential equation,

$$a^2\frac{d^2y}{dz^2} + (a^2 + a(\theta a + \theta + 1))\frac{dy}{dz} + (\theta(a+1)-a)y = 0,$$

which has the auxiliary equation,

$$m^2a^2 + m(a^2 + a(\theta a + \theta + 1)) + (\theta(a+1)-a) = 0,$$

with solutions $m = -\frac{1}{a}, -\left(\frac{\theta(a+1)}{a} - 1\right)$. Thus,

$$y = C_1(ax+b)^{-\frac{1}{a}} + C_2(ax+b)^{-\frac{\theta(a+1)}{a}+1},$$

giving

$$\bar{T}_{PH}(x) = K_1(ax+b)^{-\frac{a+1}{a}} + K_2(ax+b)^{-\frac{\theta(a+1)}{a}},$$

for arbitrary constants K_1 and K_2. The constants are chosen in such a way that $t_{PH}(x)$ is a density function, which leads to (8.2.18) and this completes the proof. ■

Remark 8.2.2. When X has exponential distribution with mean b, the characteristic property reduces to

$$m_{X\#Y}(x) = \frac{(1+\theta)b}{\theta} + \frac{b^2}{\theta} h_{X\#Y}(x).$$

In this case, the mean residual life is a linear function of the hazard rate and therefore the points representing the empirical values of $m_{X\#Y}$ and $h_{X\#Y}$ lies along a straight line, which is easy to verify.

Another special case of interest in (8.1.1) arises when $F(\cdot)$ is the equilibrium distribution of $G(\cdot)$, so that $f(x) = \frac{\bar{G}(x)}{\mu_Y}$, where μ_Y is the mean of Y. From (8.1.2),

$$t_{X\#Y}(x) = \frac{xg(x)}{\mu_Y},$$

the length-biased model corresponding to $G(\cdot)$ and hence,

$$\bar{T}_{X\#Y}(x) = \frac{1}{\mu_Y}\left[\int_x^\infty \bar{G}(t)dt - x\bar{G}(x)\right].$$

We can write the identity connecting hazard rates of $X\#Y$ and Y as

$$h_{X\#Y}(x) = \frac{x}{m_Y(x) - x} h_Y(x),$$

The random variable Y can be written as the product of $Y\#X$ with an independent uniform random variable U over $[0,1]$. Thus Y is stochastically smaller than $Y\#X$ and also $E[X\#Y] = 2E[Y]$. For a detailed discussion of the role of length-biased models in reliability, see Gupta and Kirmani [49]

8.2.1 Ageing properties

We describe ageing properties of the proportional hazards relevation random variable $X\#Y$ in connection with the ageing behaviour of the baseline random variable X. For a non-negative random variable X with survival function $\bar{F}(x)$, hazard rate $h(x)$ and reversed hazard rate $\lambda(x)$, we shall consider the following ageing classes,

(i) X has increasing or decreasing hazard rate, denoted by IHR (DHR), if the hazard rate $h(x)$ is increasing (decreasing) in x.

(ii) X is new better (worse) than used, denoted by NBU (NWU) if $\bar{F}(x+t) \leq (\geq)\, \bar{F}(x)\, \bar{F}(t)$ for all $x, t > 0$.

(iii) X has increasing (decreasing) hazard rate average, denoted by IHRA (DHRA) if $\frac{1}{x}\int_0^x h(t)dt$ is increasing (decreasing).

(iv) X is new better (worse) than used in hazard rate denoted by NBUHR (NWUHR) if $h(0) \leq (\geq)\, h(x)$ for all $x > 0$ (Loh [86]).

The basic properties and applications of these ageing classes can be seen in Barlow and Proschan [11], Shaked and Shanthikumar [141] and Nair et al. [105]. From (8.2.11), we have the identity,

$$h_{X\#Y}(x) = \theta\ h_X(x) \left(\frac{1-(\bar{F}(x))^{\theta-1}}{\theta-(\bar{F}(x))^{\theta-1}} \right).$$

Since,

$$\frac{d}{dx}\left(\frac{1-(\bar{F}(x))^{\theta-1}}{\theta-(\bar{F}(x))^{\theta-1}} \right) = \frac{\theta(\theta-1)^2(\bar{F}(x))^{\theta-1}}{(\theta-(\bar{F}(x))^{\theta-1})^2} > 0,$$

$\left(\frac{1-(\bar{F}(x))^{\theta-1}}{\theta-(\bar{F}(x))^{\theta-1}}\right)$ is an increasing function in x for all $\theta > 0$. Thus, when X is IHR then $X\#Y$ is also IHR. The IHR property is preserved for $\bar{F} \neq \bar{G}\,(=\bar{F}^{\theta})$. However, the case when X is DHR gives different options as will be seen subsequently.

Now, we recall the following results from Navarro et al. [112] for the general distorted distributions (8.2.8) discussed in the context of a coherent system having identical components. Let X and S denote the lifetimes of the component and system respectively. Then,

(a) if X is NBU (NWU) and $w(u)$ is submultiplicative (supermultiplicative) on [0,1], that is $w(u\,v) \leq (\geq)\, w(u)\, w(v)$ holds for all $0 \leq u,\ v \leq 1$, then S is NBU (NWU),

(b) if X is IHRA (DHRA) and $w(u^a) \geq (\leq)\, (w(u))^a$ holds for all $0 \leq u,\ v \leq 1$ and $0 < a < 1$, then S is IHRA (DHRA).

Using these results, we establish the following theorem for the model (8.2.5).

Theorem 8.2.3. Let X and Y be two non-negative random variables with distribution functions $F(x)$ and $G(x)$ respectively and $X\#Y$ be the relevation of X and Y. Suppose Y is the PHM of X. Then,

(i) if X is NBU then $X\#Y$ is NBU,

and,

(ii) if X is IHRA then $X\#Y$ is IHRA.

Proof. For the model (8.2.5), we have $X\#Y$ is the distorted random variable of X, with distortion function,

$$w(u) = \frac{1}{\theta-1}(\theta u - u^{\theta}),\ \ u \in [0,1]. \tag{8.2.20}$$

We can easily verify that $w(u)$ is submultiplicative and satisfies the condition $w(u^a) \geq (\leq)(w(u))^a$ for all $0 \leq u, v \leq 1$ and $0 < a < 1$. Now from (a) and (b), proof for (i) and (ii) follows. ■

Remark 8.2.3. We can present an alternative proof for the above theorem by adopting the results given in Shanthikumar and Baxter [142]. Let $CH_X(\cdot)$ and $CH_Y(\cdot)$ be the cumulative hazard functions of X and Y respectively. When Y is the PHM of X, we have $h_Y(x) = \theta h_X(x)$. Moreover, we get

$$\frac{CH_X(x)}{CH_Y(x)} = \frac{\int_0^x h_X(t)dt}{\int_0^x h_Y(t)dt} = \frac{1}{\theta}. \tag{8.2.21}$$

Note that if G is NBU and $\frac{CH_X(x)}{CH_Y(x)}$ is non decreasing then $T_{X\#Y}(x)$ is also NBU (Shanthikumar and Baxter [142]). Since Y is the PHM of X, from (8.2.21), we have $\frac{CH_X(x)}{CH_Y(x)}$ is a constant. Now it is straightforward that if G is NBU then $T_{PH}(x)$ is also NBU. Since $h_Y(x) = \theta h_X(x)$, it is clear that X is IHR if and only if Y is IHR. This implies X is NBU if and only if Y is NBU, from which the result (i) follows. Shanthikumar and Baxter [142] proved that if G is IHRA and $\frac{CH_X(x)}{CH_Y(x)}$ is decreasing, then $T_{X\#Y}(x)$ is IHRA. Under PHM, similar to (i) we have G is IHRA if and only if F is IHRA.

Remark 8.2.4. Lai and Xie [78] established that NBU (NWU) implies NBUHR (NWUHR). Thus the NBUHR (NWUHR) property is preserved under PHRT by Theorem 8.2.3.

8.2.2 Stochastic orders

There are many situations in practice where we need to compare the characteristics of two distributions. Stochastic orders are used for the comparison of lifetime distributions. In this section, we provide some important stochastic orders between the random variables X and $X\#Y$. We shall consider the following stochastic orders. Their basic properties and interrelations can be seen in Shaked and Shanthikumar [141] and Barlow and Proschan [11].

Suppose X and Y are two lifetime random variables with absolutely continuous distribution functions $F(x)$ and $G(x)$ respectively. Let $f(x)$ and $g(x)$ are the corresponding probability density functions. Then we have the following;

(i) X is smaller than Y in the usual stochastic order denoted by $X \leq_{st} Y$ if and only if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

(ii) X is smaller than Y in hazard rate order, denoted by $X \leq_{hr} Y$, if and only if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing in x.

(iii) X is smaller than Y in the likelihood ratio order, denoted by $X \leq_{lr} Y$, if and only if $\frac{g(x)}{f(x)}$ is increasing in x in the union of their supports.

(iv) X is smaller than Y in the increasing convex order, denoted by $X \leq_{icx} Y$, if and only if $\int_x^\infty \bar{F}(t)\,dt \leq \int_x^\infty \bar{G}(t)\,dt$ for all x.

(v) X is less than Y in convex order, $X \underset{c}{\leq} Y$, if $G^{-1}(F(x))$ is a convex function.

Let $\bar{F}_1(x) = w_1(\bar{F}(x))$ and $\bar{F}_2(x) = w_2(\bar{F}(x))$, where $\bar{F}_1(x)$ and $\bar{F}_2(x)$ are the survival functions obtained by the distortion of $\bar{F}(x)$ using the distortion functions $w_1(\cdot)$ and $w_2(\cdot)$ respectively. Suppose S_1 and S_2 be the random variables corresponding to $\bar{F}_1(x)$ and $\bar{F}_2(x)$ respectively. We now recall the following result from Navarro et al. [112] (Theorem 2.5).

$$S_1 \leq_{lr} (\geq_{lr})\, S_2 \text{ if and only if } \frac{w_1'(u)}{w_2'(u)} \text{ is increasing (decreasing) in } u \in (0,1), \quad (8.2.22)$$

where $w_i'(u)$ is the derivative of $w_i(u)$, $i = 1, 2$. To establish different stochastic order relations between X and $X\#Y$, we take $S_1 = X\#Y$ and $S_2 = X$, with distortion functions $w_1(u) = \frac{(\theta u - u^\theta)}{\theta - 1}$ and $w_2(u) = u$ respectively.

Theorem 8.2.4. Suppose Y is the PHM of X. Then $X \leq_{lr} X\#Y$.

Proof. Note that,

$$\frac{d}{du}\left(\frac{w_1'(u)}{w_2'(u)}\right) = \frac{d}{du}\left(\frac{\left(\theta - \theta u^{\theta-1}\right)}{\theta - 1}\right) = -\theta\, u^{\theta-2} \leq 0.$$

Thus $\frac{w_1'(u)}{w_2'(u)}$ is decreasing in $u \in (0,1)$. Now from (8.2.22), we get $X \leq_{lr} X\#Y$. ■

Remark 8.2.5. The proof for the above theorem can also be obtained from (8.2.5) by noting that,

$$\frac{t_{PH}(x)}{f(x)} = \frac{\theta}{\theta - 1}\left(1 - (\bar{F}(x))^{\theta-1}\right).$$

Now, for $\theta > 0$ it holds that,

$$\frac{d}{du}\left(\frac{t_{PH}(x)}{f(x)}\right) = \theta(\bar{F}(x))^{\theta-2} f(x) \geq 0.$$

This implies $X \leq_{lr} X\#Y$. Moreover, from Shaked and Shanthikumar [141], we have the following implications,

$$X \leq_{lr} X\#Y \implies X \leq_{hr} X\#Y \implies X \leq_{st} X\#Y.$$

Kochar and Wiens [73] have defined an IHR order by saying that X is more IHR than Y if $X \underset{c}{\leq} Y$. Further, X is more IHRA (NBU) than Y if $G^{-1}(F(x))$ is star-shaped denoted by $X \leq_* Y$ (super-additive denoted by $X \leq_{su} Y$). Also, from Nair et al. [105] we have the following,

(a) $X \leq_{DMRL} Y$ if $\frac{m_X(x)}{m_Y(x)}$ is non-decreasing in x,

(b) $X \leq_{NBUE} Y$ if $\frac{m_X(x)}{m_Y(x)} \leq \frac{E(X)}{E(Y)}$ for all x,

(c) $X \leq_{NBUHR} Y$ if $\frac{h_X(x)}{h_Y(x)} \geq \frac{h_X(0)}{h_Y(0)}$ for all x,

(d) $X \leq_{NBUHRA} Y$ if $F_Y^{-1}(F_X(x)) \geq x\left(F_Y^{-1}(F_X(x)\right)'_{x=0}$.

Further, it follows that $X \leq_c Y \implies X \leq_{DMRL} Y \implies X \leq_{NBUE} Y$ and $X \leq_{NBU} Y \implies X \leq_{NBUHRA} Y$. Later Sengupta and Deshpande [139] proved that $X \underset{c}{\leq} Y$ if and only if $\frac{h_X(x)}{h_Y(x)}$ is non-decreasing in x, provided $h_Y(x) \neq 0$. The following theorem establishes various interrelationships among these orderings, in the context of PHRT.

Theorem 8.2.5. Let X and Y be two non-negative random variables and $X\#Y$ be the random variable corresponding to the relevation of X and Y with survival function (8.2.5). If Y is the PHM of X, then $X\#Y \underset{c}{\leq} X$.

Proof. From (8.2.11), we have

$$\frac{h_{X\#Y}(x)}{h_X(x)} = \theta\left(\frac{1-\bar{F}^{\theta-1}(x)}{\theta-\bar{F}^{\theta-1}(x)}\right).$$

Differentiating with respect to x, we obtain

$$\frac{d}{du}\left(\frac{h_{X\#Y}(x)}{h_X(x)}\right) = \theta(\theta-1)f(x)\bar{F}^{\theta-2}(x)\left[\frac{\left(1-\bar{F}^{\theta-1}(x)\right)+\left(\theta-\bar{F}^{\theta-1}(x)\right)}{\left(\theta-\bar{F}^{\theta-1}(x)\right)^2}\right].$$

Since $\left(1-\bar{F}^{\theta-1}(x)\right)+\left(\theta-\bar{F}^{\theta-1}(x)\right) \geq (\leq)\, 0$ for $\theta \geq (\leq)\, 1$, we observe that

$$\frac{d}{du}\left(\frac{h_{X\#Y}(x)}{h_X(x)}\right) \geq 0 \text{ for all } \theta > 0.$$

Thus, $\frac{h_{X\#Y}(x)}{h_X(x)}$ is non-decreasing in x and hence $X\#Y$ is more IHR than X. ∎

The implications, consequence of Theorem 8.2.5, are exhibited in the following diagram;

$$\begin{array}{ccccccc}
X\#Y \leq_c X & \Longrightarrow & X\#Y \leq_* X & \Longrightarrow & X\#Y \leq_{su} X & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
X\#Y \leq_{DMRL} X & \Longrightarrow & X\#Y \leq_{NBUE} X & \Longrightarrow & X\#Y \leq_{NBUHR} X & \Longrightarrow & X\#Y \leq_{NBUHRA} X.
\end{array}$$

Theorem 8.2.6. Let X_1 and X_2 be two random variables with survival functions $\bar{F}_1(x)$ and $\bar{F}_2(x)$. Suppose Y_1 and Y_2 are the proportional hazards random variables associated with X_1 and X_2 respectively. Then the following properties hold;

(i) If $X_1 \leq_{st} X_2$ then $X_1\#Y_1 \leq_{st} X_2\#Y_2$.

(ii) If $X_1 \leq_{hr} X_2$ then $X_1\#Y_1 \leq_{hr} X_2\#Y_2$.

(iii) If $X_1 \leq_{icx} X_2$ then $X_1\#Y_1 \leq_{icx} X_2\#Y_2$

Proof. The proof of (i) is direct from (8.2.5). To prove (ii), we have

$$\frac{u\,q'(u)}{w(u)} = \frac{\theta\left(u^{\theta} - u\right)}{u^{\theta} - \theta u}.$$

Since $\frac{d}{du}\left(\frac{u\,q'(u)}{w(u)}\right) = -\frac{(\theta-1)^2\theta u^{\theta}}{\left(u^{\theta}-\theta u\right)^2} \leq 0$ for all $\theta > 0$, from Navarro et al. [112] (Theorem 2.6), we get $X_1\#Y_1 \leq_{hr} X_2\#Y_2$. From Theorem 2.6 of Navarro et al. [112], (iii) follows, since $w(u)$ is concave in (0,1). ■

8.2.3 Proportional hazards relevated Weibull distribution (PHRW) and it's applications

We now present a new lifetime model for illustrating the usefulness of PHRT in the construction of flexible lifetime models. The hazard rate of $X\#Y$ increases at a faster rate than that of X. Hence if the baseline model is DHR, then $X\#Y$ may lead to an upside-down bathtub shaped hazard rate. Thus PHRT models with DHR baseline model provide a means to construct distributions with non-monotone hazard rates.

Let X be a Weibull random variable with survival function $\bar{F}(x) = e^{-\alpha x^{\beta}}$, $\alpha > 0,\ \beta > 0,$ and hazard rate $h_X(x) = \alpha\beta x^{\beta-1}$. From (8.2.5), the survival function of $X\#Y$ is obtained as

$$\bar{T}_{PH}(x) = \frac{\theta e^{-\alpha x^{\beta}} - e^{-\alpha\theta x^{\beta}}}{\theta - 1}. \tag{8.2.23}$$

The distribution (8.2.23) will be referred to as the proportional hazards relevated Weibull distribution (PHRW). The hazard function, $h_{X\#Y}$ has the following expression

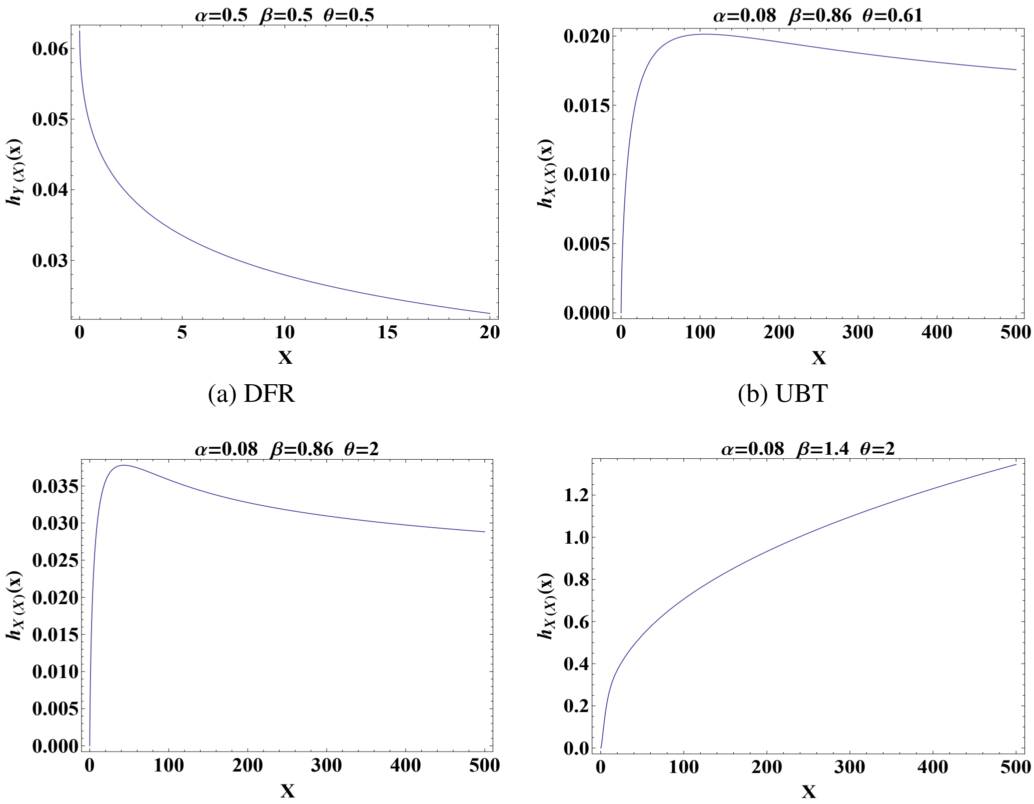
$$h_{X\#Y}(x) = \frac{\alpha\beta\theta x^{\beta-1}\left(e^{-\alpha(\theta-1)x^{\beta}} - 1\right)}{e^{-\alpha(\theta-1)x^{\beta}} - \theta}.$$

Note that $h_X(x)$ is always monotonic, IHR when $\beta \geq 1$ and DHR when $0 < \beta \leq 1$. From Figure 8.1, we can observe that $h_{X\#Y}$ accommodates IHR, DHR and UBT.

We illustrate the utility of the model (8.2.23) with the aid of two real data sets. The first data consists of the time between failures of secondary reactor pumps, which was reported in Salman and Prayoto [127]. The model (8.2.23) is applied to this data. We estimate the parameters using the method of maximum likelihood. The estimates are obtained as

$$\hat{\alpha} = 9.631 \qquad \hat{\beta} = 0.709 \quad \text{and} \quad \hat{\theta} = 0.094.$$

For comparison purposes, we consider some alternative models, such as extended Weibull (Marshall and Olkin [87]), Weibull, flexible Weibull (Bebbington et al. [14]), reduced additive Weibull (Xie and Lai [156], Lai et al. [79]) models. The Kolmogorov-Smirnov (K-S) test statistics with the associated p-values for the PHRW, extended Weibull, Weibull, flex-



(a) DFR (b) UBT

(c) UBT (d) IFR

Figure 8.1: Plots of relevated hazard function ($h_{X\#Y}(x)$) of the Weibull distribution.

ible Weibull and reduced additive Weibull models are presented in Table 8.1. The PHRW

Table 8.1: Kolmogorov-Smirnov statistic and p-values.

Distribution	KS statistic	p-Value
PHRW	0.101895	0.951183
Extended Weibull	0.105796	0.935241
Weibull	0.118395	0.866694
Flexible Weibull	0.138483	0.719057
Reduced Additive Weibull	0.16295	0.522059

distribution provides a better fit than the other models since it has the smallest K-S statistic and largest p-value. To check the goodness of fit, we use Q-Q plot, which is given in Figure 8.2. The proposed model (8.2.23) is also useful for modelling statistical data in other contexts. To illustrate this, we consider another data reported in Kuş [76]. The data represent

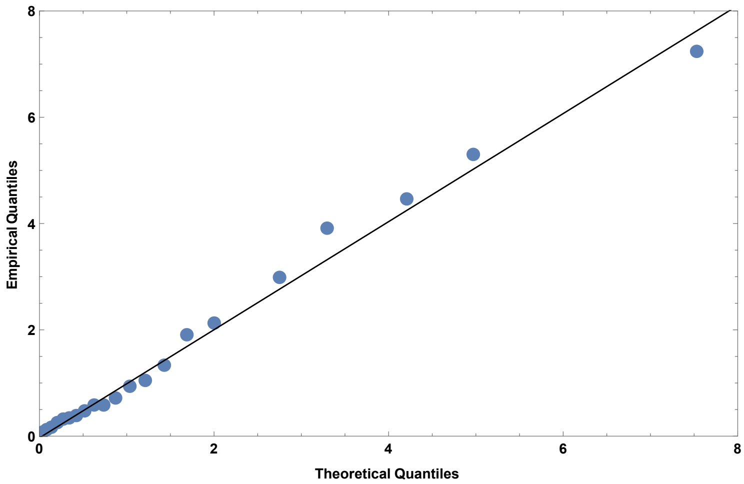


Figure 8.2: Q-Q plot for data set-1.

the period between successive earthquakes in the last century in North Anatolia fault zone. The proposed model is applied and the maximum likelihood estimates are obtained as

$$\hat{\alpha} = 0.018 \qquad \hat{\beta} = 0.609 \quad \text{and} \quad \hat{\theta} = 4.083.$$

Kuş [76] showed that exponential-Poisson distribution (EP) provides a good fit for the data and compared the performance with the Exponential geometric (EG), Weibull and Gamma models. The K-S statistic with the associated p-values for the PHRW, EP, EG, Weibull and Gamma models are presented in Table 8.2. The K-S test statistic takes the smallest value

Table 8.2: Kolmogorov-Smirnov statistic and p-values.

Distribution	KS statistic	p-Value
PHRW	0.0727	0.9985
EP	0.0972	0.9772
Weibull	0.1004	0.9690
Gamma	0.1239	0.8551
EG	0.1839	0.3914

when PHRW model is employed. Thus PHRW distribution provides a better fit than the other four models. Figure 8.3 presents the Q-Q plot, which also shows the adequacy of the model. The ageing properties, stochastic orderings and other reliability characteristics

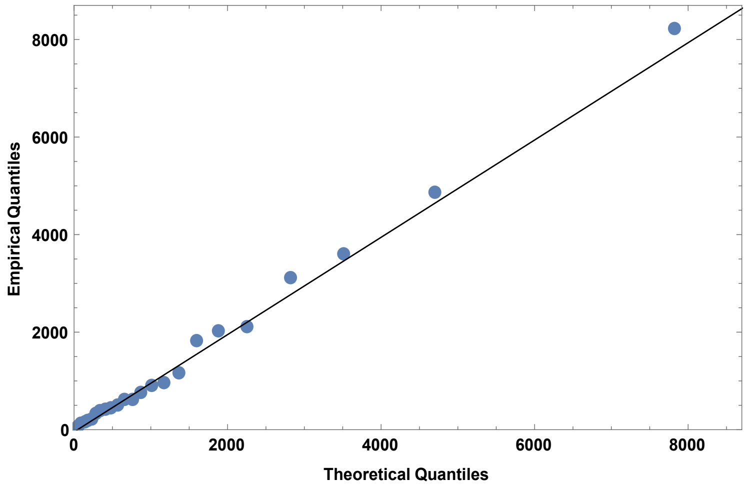


Figure 8.3: Q-Q plot for data set-2.

of PHRW distribution can be explained in terms of the corresponding properties of the Weibull distribution by using the theorems given in Section 8.2.1 and Section 8.2.2.

8.3 Quantile based relevation transform

To introduce the quantile-based relevation transform between X and Y, we denote $Q_X(u)$ and $Q_Y(u)$ as the quantile functions corresponding to the distribution functions $F(\cdot)$ and $G(\cdot)$ respectively. From (8.1.1), by taking $x = Q_X(u)$, we define quantile-based relevation transform as

$$\begin{aligned} T_{X\#Y}(Q_X(u)) = 1 - \bar{T}_{X\#Y}(Q_X(u)) &= u - \bar{G}(Q_X(u)) \int_0^u \frac{1}{\bar{G}(Q_X(p))} dp \\ &= u - (1 - Q_Y^{-1}(Q_X(u))) \int_0^u \frac{1}{(1 - Q_Y^{-1}(Q_X(p)))} dp. \qquad (8.3.1) \end{aligned}$$

Denote $T^*_{X\#Y}(u) = T_{X\#Y}(Q_X(u))$ and $Q_1(u) = Q_Y^{-1}(Q_X(u))$, then (8.3.1) becomes,

$$T^*_{X\#Y}(u) = u - (1 - Q_1(u)) \int_0^u \frac{1}{(1 - Q_1(p))} dp. \tag{8.3.2}$$

From the property 9 of the quantile function given in Section 1.1.1, it follows that if $K(x)$ is a non-decreasing function of x, then $K(Q(u))$ is again a quantile function (Gilchrist [42]). Now, since $T_{X\#Y}(\cdot)$ and $Q_Y^{-1}(\cdot)$ are non-decreasing functions, $T_{X\#Y}(Q_X(u))$ and $Q_Y^{-1}(Q_X(p))$ are the quantile functions of $F(T^{-1}(x))$ and $F(G^{-1}(x))$. We call $T^*_{X\#Y}(u)$ as the relevation quantile function (RQF). Note that, in general, the quantile-based relevation transform is not symmetric, namely $T^*_{X\#Y}(u) \neq T^*_{Y\#X}(u)$. We can interpret $T^*_{X\#Y}(u)$ as the probability that the total lifetime is less than or equal to $100u\%$ point of X, given it exceeds a random time X. From (8.3.1), we have

$$\begin{aligned}
T^*_{X\#Y}(u) &= T_{X\#Y}(Q_X(u)) \\
\Rightarrow Q_{X\#Y}(T^*_{X\#Y}(u)) &= Q_X(u) \\
\Rightarrow \qquad Q_{X\#Y}(u) &= Q_X(T^{*^{-1}}_{X\#Y}(u)).
\end{aligned} \tag{8.3.3}$$

Thus, we can compute the quantile function of the relevation random variable $X\#Y$ from the relevation quantile function $T^*_{X\#Y}(u)$ using the identity (8.3.3).

Theorem 8.3.1. Let X and Y be two random variables with survival functions $\bar{F}(x)$ and $\bar{G}(x)$ with quantile functions $Q_X(u)$ and $Q_Y(u)$ respectively. Then $T^*_{X\#Y}(u) \leq u$ for all $u \in (0, 1)$.

Proof. Denote $T^*_{X\#Y}(u) = u - \xi(u)$, where

$$\xi(u) = (1 - Q_1(u)) \int_0^u \frac{1}{(1 - Q_1(p))} dp \geq 0 \text{ for all } u \in (0, 1). \tag{8.3.4}$$

Since $Q_1(u) = Q_Y^{-1}(Q_X(u)) = F_Y(Q_X(u))$, we have $Q_1(u) \in (0,1)$ for all $u \in (0,1)$. This implies, $\xi(u) \geq 0$ for all $u \in (0,1)$. From this, we get $T^*_{X\#Y}(u) \leq u$ for all $u \in (0,1)$. ■

Remark 8.3.1. From Theorem 8.3.1, we have

$$\begin{aligned} & T^*_{X\#Y}(u) \leq u \text{ for all } u \in (0,1) \\ \Leftrightarrow\ & T_{X\#Y}(Q_X(u)) \leq u \text{ for all } u \in (0,1) \\ \Leftrightarrow\ & Q_X(u) \leq Q_{X\#Y}(u) \text{ for all } u \in (0,1). \end{aligned}$$

Since $Q_X(u) \leq Q_{X\#Y}(u)$ for all $u \in (0,1)$, from Nair et al. [105], we get $X \leq_{st} X\#Y$.

Psarrakos and Di Crescenzo [120] showed that $X \leq_{hr} X\#Y$. From Nair et al. [105], we have X is smaller than Y in hazard rate order, denoted by $X \leq_{hr} Y$, if and only if $\frac{\bar{F}_Y(Q_X(1-u))}{u}$ is decreasing in u. This implies

$$\frac{1 - T_{X\#Y}(Q_X(1-u)}{u} = \frac{1 - T^*_{X\#Y}(1-u)}{u}, \text{ is decreasing in } u.$$

In the next theorem, we establish the relation between hazard quantile functions of the random variable $X\#Y$ and X.

Theorem 8.3.2. Let $H_{X\#Y}(u)$ and $H_X(u)$ be the hazard quantile functions corresponding to the random variables $X\#Y$ and X. Then,

$$H_{X\#Y}(T^*_{X\#Y}(u)) = \frac{1}{q_X(u)} \frac{d}{du} \left(-\log(1 - T^*_{X\#Y}(u)) \right), \tag{8.3.5}$$

or equivalently,

$$\frac{H_{X\#Y}(T^*_{X\#Y}(u))}{H_X(u)} = (1-u) \frac{d}{du} \left(-\log(1 - T^*_{X\#Y}(u)) \right). \tag{8.3.6}$$

Proof. From (8.3.3), we have, $Q_{X\#Y}(T^*_{X\#Y}(u)) = Q_X(u)$. Differentiating both sides with respect to u, we get

$$\begin{aligned}
& q_{X\#Y}(T^*_{X\#Y}(u))\,(T^*_{X\#Y}(u))' = q_X(u). \\
\Rightarrow \quad & \frac{1}{q_{X\#Y}(T^*_{X\#Y}(u))\,(T^*_{X\#Y}(u))'} = \frac{1}{q_X(u)} \\
\Rightarrow \quad & \frac{H_{X\#Y}(T^*_{X\#Y}(u))}{(T^*_{X\#Y}(u))'} = \frac{1}{(1 - T^*_{X\#Y}(u))\,q_X(u)} \\
\Rightarrow \quad & H_{X\#Y}(T^*_{X\#Y}(u))\,q_X(u) = \frac{(T^*_{X\#Y}(u))'}{(1 - T^*_{X\#Y}(u))}.
\end{aligned} \tag{8.3.7}$$

From (8.3.7), we have

$$H_{X\#Y}(T^*_{X\#Y}(u)) = \frac{1}{q_X(u)}\frac{d}{du}\left(-\log(1 - T^*_{X\#Y}(u))\right). \tag{8.3.8}$$

Since $q_X(u) = \frac{1}{(1-u)H_X(u)}$, (8.3.6) follows directly from (8.3.8), which completes the proof. ■

Theorem 8.3.3. Suppose X and Y be two random variables with same support in the non-negative set up and $Q_{Exp}(u)$ be the quantile function of the unit exponential distribution. Then,

$$H_{X\#Y}(T^*_{X\#Y}(u)) = \frac{H_X(u)}{H_Z(u)}, \tag{8.3.9}$$

where $H_Z(u)$ is the hazard quantile function corresponding to the quantile function $Q_Z(u) = Q_{Exp}(T^*_{X\#Y}(u))$.

Proof. Since X and Y have the same support, D, we have, $T^*_{X\#Y}(0) = 0$ and $T^*_{X\#Y}(1) = 1$. From Gilchrist [42], we have, if $Q(u)$ is a quantile function and $K(u)$ is a non-decreasing function of u satisfying the boundary conditions $K(0) = 0$ and $K(1) = 1$, then $Q(K(u))$

is again a quantile function of a random variable with the same support. This gives

$$Q_Z(u) = Q_{Exp}(T^*_{X\#Y}(u)) = -\log(1 - T^*_{X\#Y}(u)), \tag{8.3.10}$$

is a quantile function with support $(0, \infty)$.

From (8.3.10), we have

$$H_Z(u) = \left((1-u)\frac{d}{du}\left(-\log(1 - T^*_{X\#Y}(u))\right)\right)^{-1}, \tag{8.3.11}$$

is the hazard quantile function of $Q_Z(u)$. Now the result (8.3.9) follows from (8.3.6) and (8.3.11), which completes the proof. ∎

Example 8.3.1. Suppose X follows uniform distribution with quantile function $Q_X(u) = \theta u$ and Y follows the exponential distribution with quantile function $Q_Y(u) = -\frac{1}{\lambda}\log(1-u)$. Then $Q_1(u) = Q_Y^{-1}(Q_X(u)) = 1 - \exp(-\lambda\theta u)$, and hence

$$T^*_{X\#Y}(u) = u - \frac{1}{\lambda\theta}\left(1 - \exp(-\lambda\theta u)\right). \tag{8.3.12}$$

The identity (8.3.3) is useful for generating random observations of the relevation random variable $X\#Y$. Since $T^*_{X\#Y}(u)$ given in (8.3.12) is not directly invertible, we generate the random sample of $X\#Y$ by first carry out the numerical inversion of (8.3.12) and then using the relation $Q_{X\#Y}(u) = Q_X(T^{*^{-1}}_{X\#Y}(u))$.

Generally, relevation quantile function is not unique. There exist different distribution pairs with same relevation quantile function. We illustrate this with the following example.

Example 8.3.2. Let X, Y, $W$and Z be four random variables with quantile functions, respectively by

$$Q_X(u) = -\frac{1}{\lambda_1}\log(1-u);\ \lambda_1 > 0, \quad \text{[exponential distribution}(\lambda_1)],$$
$$Q_Y(u) = -\frac{1}{\lambda_2}\log(1-u);\ \lambda_2 > 0, \quad \text{[exponential distribution}(\lambda_2)],$$
$$Q_W(u) = (1-u)^{-\frac{1}{\lambda_1}} - 1;\ \lambda_1 > 0, \quad \text{[Pareto-II distribution}(\lambda_1)],$$

and

$$Q_Z(u) = (1-u)^{-\frac{1}{\lambda_2}} - 1;\ \lambda_2 > 0, \quad \text{[Pareto-II distribution}(\lambda_2)].$$

Now we obtain

$$Q_Y^{-1}(Q_X(u)) = Q_Z^{-1}(Q_W(u)) = 1 - (1-u)^{\frac{\lambda_2}{\lambda_1}}.$$

This gives

$$T^*_{X\#Y}(u) = T^*_{W\#Z}(u) = \frac{\lambda_1\left((1-u)^{\lambda_2/\lambda_1} - 1\right) + u\,\lambda_2}{\lambda_2 - \lambda_1}.$$

Note that $Q_Y^{-1}(Q_X(u))$ is the quantile function of the rescaled beta distribution and $T^*_{X\#Y}(u)$ is the linear combination of the quantile functions of the rescaled beta and the uniform distributions.

Example 8.3.3. Suppose X follows Govindarajalu distribution with quantile function, $Q_X(u) = \sigma((\beta+1)u^\beta - \beta u^{\beta+1})$ and Y is uniform over the interval $(0,1)$. In this case, $Q_1(u) = \beta u^{\beta+1} - (\beta+1)u^\beta + 1$, then

$$T^*_{X\#Y}(u) = \frac{(\beta(u-1)-1)\left(\frac{\beta u}{\beta+1}\right)^\beta B_{\frac{u\beta}{\beta+1}}[1-\beta, 0]}{\beta} + u.$$

8.3.1 Proportional hazards relevation quantile function

When Y is the PHM of X with survival functions as in (8.2.2), then the quantile form of the transformation given in (8.2.3) is obtained as

$$\begin{aligned} T^*_{PH}(u) = T_{X\#Y}((Q_X(u))) &= u - (1-u)^\theta \int_0^u \frac{1}{(1-p)^\theta} dp \\ &= \frac{1-u\theta}{1-\theta} - \frac{(1-u)^\theta}{1-\theta},\ u \in (0,1). \end{aligned} \tag{8.3.13}$$

We call $T^*_{PH}(u)$ as the proportional hazards relevation quantile function (PHRQF). When $\theta = 1$,

$$T^*_{PH}(u) = T_{X\#X}((Q_X(u))) = u + (1-u)\log(1-u),\ u \in (0,1). \tag{8.3.14}$$

Theorem 8.3.4. Let X and Y be two independent random variables. Then Y is the PHM of X if and only if $T^*_{PH}(u)$ satisfies the relation,

$$T^*_{PH}(u) = Q_A(u) - Q_B(u), \tag{8.3.15}$$

where

(i) $Q_A(u)$ and $Q_B(u)$ are the quantile functions of uniform $(0, \frac{\theta}{\theta-1})$ and rescaled beta $(0, \frac{1}{\theta-1})$ respectively, when $\theta > 1$,

and

(ii) $Q_A(u)$ is rescaled beta $(0, \frac{1}{1-\theta})$ and $Q_B(u)$ is uniform $(0, \frac{\theta}{1-\theta})$, when $\theta < 1$.

Proof. From (8.3.13), we have

$$T^*_{PH}(u) = \frac{\theta u}{\theta - 1} - \frac{1}{\theta - 1}\left(1 - (1-u)^{\theta}\right).$$

This can be written as

$$T^*_{PH}(u) = \begin{cases} \frac{\theta u}{\theta-1} - \frac{1}{\theta-1}\left(1 - (1-u)^{\theta}\right) & \text{if } \theta > 1 \\ \frac{1}{1-\theta}\left(1 - (1-u)^{\theta}\right) - \frac{\theta u}{1-\theta} & \text{if } \theta < 1 \end{cases}$$

which completes the proof for the 'if' part of the theorem. Conversely, assume that $T^*_{PH}(u)$ has the form (8.3.15), now for $\theta > 1$, from (8.3.2), we have

$$T^*_{PH}(u) = u - \vartheta(u)\int_0^u \frac{1}{\vartheta(p)}dp = \frac{\theta u}{\theta - 1} - \frac{1}{\theta - 1}\left(1 - (1-u)^{\theta}\right),$$

where $\vartheta(u) = 1 - Q_Y^{-1}(Q_X(p))$.

This implies

$$\vartheta(u)\int_0^u \frac{1}{\vartheta(p)}dp = \frac{((1-u)^{\theta} - (1-u))}{1-\theta}. \tag{8.3.16}$$

Differentiating both sides with respect to u, we get

$$\vartheta'(u)\int_0^u \frac{1}{\vartheta(p)}dp = \frac{\theta}{1-\theta}(1 - (1-u)^{\theta-1}). \tag{8.3.17}$$

Divide (8.3.17) by (8.3.16), we obtain

$$\frac{\vartheta'(u)}{\vartheta(u)} = \frac{\theta(1-(1-u)^{\theta-1})}{(u-1)(1-(1-u)^{\theta-1})} = \frac{-\theta}{1-u},$$

which implies

$$\frac{d}{du}\log(\vartheta(u)) = \frac{-\theta}{1-u}. \tag{8.3.18}$$

On integration, (8.3.18) reduces to

$$\log(\vartheta(u)) = \log(1-u)^{\theta}.$$

This gives $\vartheta(u)) = (1-u)^{\theta}$. Now from (8.3.2), we obtain

$$\vartheta(u) = 1 - Q_Y^{-1}(Q_X(u)) = (1-u)^{\theta},$$

which gives

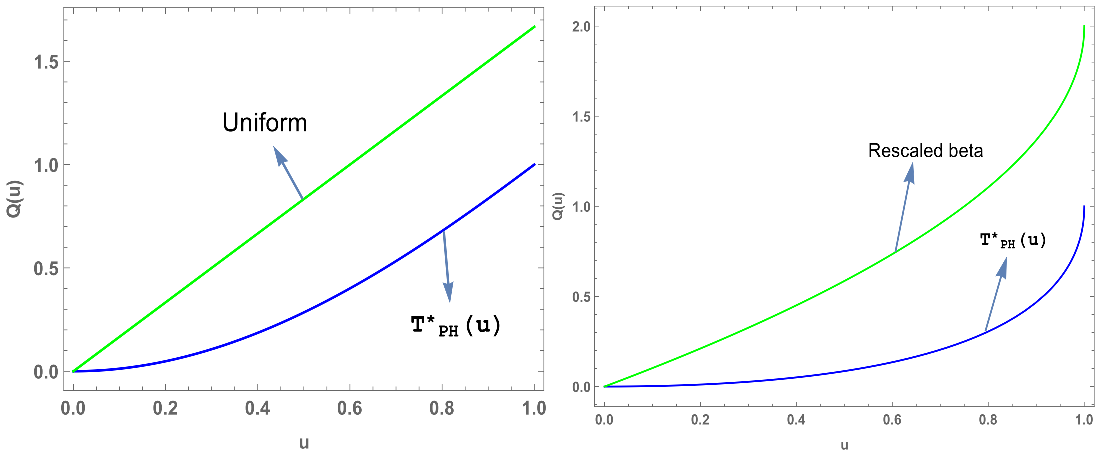
$$Q_X(u) = Q_Y(1-(1-u)^{\theta}), \text{ or equivalently } \bar{G}(x) = (\bar{F}(x))^{\theta}.$$

Thus, Y is the PHM of X. Proof for the case $0 < \theta < 1$ is similar and hence the details are omitted. ■

Remark 8.3.2. From Theorem 8.3.4, we can see that $T^*_{PH}(u)$ lies below uniform $(0, \frac{\theta}{\theta-1})$ quantile function when $\theta > 1$ and it lies below rescaled beta $(0, \frac{1}{1-\theta})$ quantile function when $\theta < 1$. We illustrate this for two particular cases of θ such as 0.5 and 2.5 in Figure 8.4.

Since $T^*_{PH}(u)$ is a unit support quantile function, with an additional scale parameter σ, we can use $T^*_{PH}(u)$ for modelling lifetime data sets. Thus, consider the model,

$$Q^*(u) = \begin{cases} \sigma\left(\frac{1-u\theta}{1-\theta} - \frac{(1-u)^{\theta}}{1-\theta}\right) & \text{if } \theta \neq 1 \\ \sigma(u + (1-u)\log(1-u)) & \text{if } \theta = 1. \end{cases} \tag{8.3.19}$$



(a) $\theta = 2.5$ (b) $\theta = 0.5$

Figure 8.4: (a) Uniform $(0, \frac{\theta}{\theta-1})$ with $T^*_{PH}(u)$, and (b) rescaled beta $(0, \frac{1}{1-\theta})$ with $T^*_{PH}(u)$.

The hazard quantile function has the form

$$H^*(u) = \begin{cases} \frac{1-\theta}{\theta\sigma\left((1-u)^{\theta}+u-1\right)} & \text{if } \theta \neq 1 \\ (\sigma(u-1)\log(1-u))^{-1} & \text{if } \theta = 1. \end{cases} \tag{8.3.20}$$

Note that, when $\theta = 1$, $q^*(u) = \frac{d}{du}Q^*(u) = -\sigma \log(1-u)$, which is the quantile function of an exponential distribution with mean σ. Thus $q^*(u)$ is non-decreasing when $\theta = 1$. $H^*(u)$ is bathtub shaped for all choices of the parameters. Change point of $H^*(u)$ is $u_0 = 1 - \left(\frac{1}{\theta}\right)^{\frac{1}{\theta-1}}$, when $\theta \neq 1$ and for $\theta = 1$, change point $u_0 = 0.63$. Figure 8.5 shows the plots of $H^*(u)$ for some particular values of the parameters.

We illustrate the practical applicability of the above model with the aid of a real data set given in Lawless [84]. The data consists of the number of cycles to failure for a group of 60 electrical appliances in a life test. The parameters of the model are estimated using the

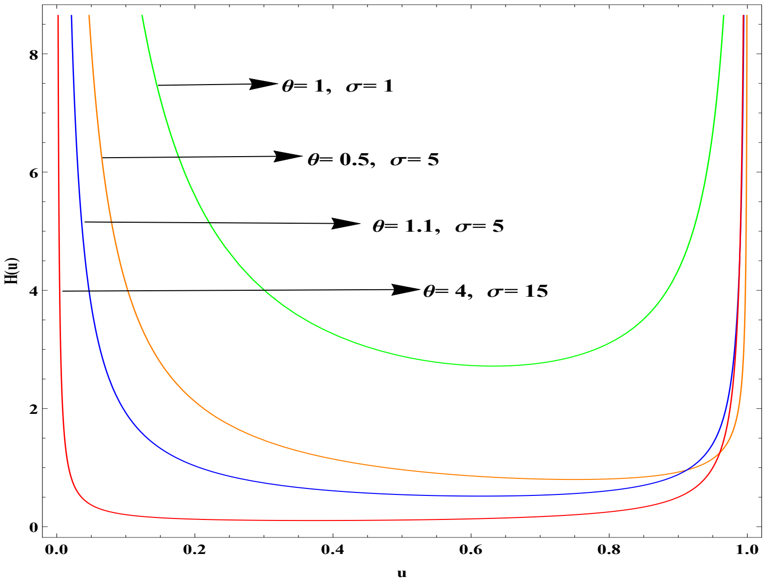


Figure 8.5: Hazard quantile function for different choices of parameters.

method of L-moments. The first and second L-moments are given by

$$L_1 = \frac{\theta\left(\frac{1}{\theta+1} - \frac{1}{2}\right)\sigma}{1-\theta}, \tag{8.3.21}$$

and

$$L_2 = \frac{\theta\left(\frac{1}{\theta^2+3\theta+2} - \frac{1}{6}\right)\sigma}{1-\theta}. \tag{8.3.22}$$

We equate the first two sample $L-$moments to corresponding population L-moments to estimate the parameters. The first two sample $L-$moments are given by

$$l_1 = \left(\frac{1}{n}\right)\sum_{i=1}^{n} X_{(i)}, \tag{8.3.23}$$

$$l_2 = \left(\frac{1}{2}\right)\binom{n}{2}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{1} - \binom{n-i}{1}\right)X_{(i)}, \tag{8.3.24}$$

a where $X_{(i)}$ is the ith order statistic. For estimating the parameters θ and σ, we equate the above two sample $L-$moments to corresponding population L-moments given in (8.3.21) and (8.3.22). The parameters are obtained by solving the equations,

$$l_r = L_r; \quad r = 1, 2. \tag{8.3.25}$$

The estimates of the parameters are obtained as $\hat{\theta} = 2.573$, $\hat{\sigma} = 60.437$. Figure 8.6 presents the Q-Q plot, which shows that the proposed model provides a good fit for the data.

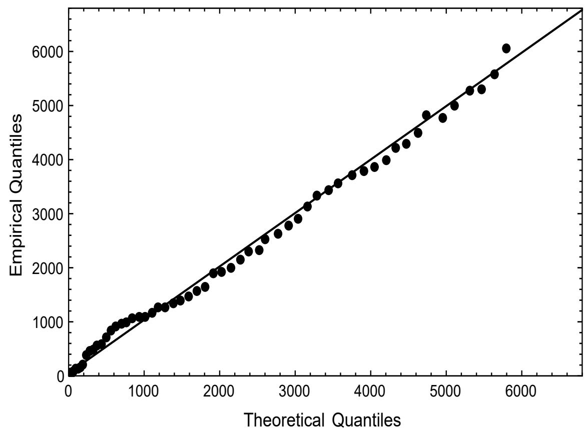


Figure 8.6: Hazard quantile function for different choices of parameters.

8.3.2 Relevation transform with equilibrium distribution

When Y is the equilibrium random variable X, from (6.3.20), we have

$$Q_X(u) = Q_Y(\zeta_X(u)), \tag{8.3.26}$$

where $\zeta_X(u)$ is the scaled total time on test transform of X. Now in the coming theorem, we present the relevation quantile function in the context of equilibrium random variables.

Theorem 8.3.5. Let X and Y be two non-negative random variables. Then Y is the equilibrium random variable of X if and only if

$$T^*_{X\#Y}(u) = u - (1 - \phi_X(u)) \int_0^u \frac{dp}{(1 - \phi_X(p))}. \tag{8.3.27}$$

Proof. Assume Y is the equilibrium random variable of X. From (8.3.1), We have

$$T(Q_X(u)) = u - (1 - Q_Y^{-1}(Q_X(u))) \int_0^u \frac{1}{(1 - Q_Y^{-1}(Q_X(p)))} dp. \tag{8.3.28}$$

Since Y is the equilibrium random variable of X, we have

$$Q_X(u) = Q_Y\left(\phi_X(u)\right). \tag{8.3.29}$$

Now using (8.3.29) in (8.3.28), we get

$$\begin{aligned} T^*_{X\#Y}(u) &= u - \left(1 - Q_Y^{-1}\left(Q_Y\left(\phi_X(u)\right)\right)\right) \int_0^u \frac{1}{\left(1 - Q_Y^{-1}\left(Q_Y\left(\phi_X(p)\right)\right)\right)} dp, \\ &= u - (1 - \phi_X(u)) \int_0^u \frac{dp}{(1 - \phi_X(p))}. \end{aligned} \tag{8.3.30}$$

Conversely, assume (8.3.27) is true. Now from (8.3.2), we have

$$u - (1 - Q_1(u)) \int_0^u \frac{1}{(1 - Q_1(p))} dp = u - (1 - \phi_X(u)) \int_0^u \frac{dp}{(1 - \phi_X(p))}.$$

Taking derivative on both sides with respect u, and simplifying, we get

$$\begin{aligned} & Q_1(u) = \phi_X(u) \\ \Leftrightarrow \quad & Q_Y^{-1}(Q_X(u)) = \phi_X(u) \\ \Leftrightarrow \quad & Q_X(u) = Q_Y\left(\phi_X(u)\right). \end{aligned}$$

Thus Y is the equilibrium random variable of X, which completes the proof. ■

Corollary 8.3.1. Suppose Y is the equilibrium random variable of X. Then $T^*_{X\#Y}(u)$ uniquely determines $\phi_X(u)$ through the identity,

$$\phi_X(u) = 1 - \exp\left(\int_0^u \frac{(T^*_{X\#Y}(p))'}{T^*_{X\#Y}(p) - p} dp\right). \tag{8.3.31}$$

Proof. From (8.3.27), we have

$$(1 - \phi_X(u)) \int_0^u \frac{dp}{(1 - \phi_X(p))} = u - T^*_{X\#Y}(u).$$

Differentiating both sides with respect to u, we get

$$\begin{aligned} 1 - \phi_X'(u) \int_0^u \frac{dp}{(1 - \phi_X(p))} &= 1 - (T^*_{X\#Y}(u))' \\ \Leftrightarrow \qquad \phi_X'(u) \int_0^u \frac{dp}{(1 - \phi_X(p))} &= (T^*_{X\#Y}(u))'. \end{aligned} \tag{8.3.32}$$

From (8.3.32), we have $\int_0^u \frac{dp}{(1-\phi_X(p))} = \frac{u - T^*_{X\#Y}(u)}{(1-\phi_X(u))}$. Inserting this in (8.3.32), we obtain

$$\begin{aligned} \frac{\phi_X'(u)}{1 - \phi_X(u)} &= \frac{(T^*_{X\#Y}(u))'}{u - T^*_{X\#Y}(u)}, \\ \text{or,} \qquad \frac{d}{du}(\log(1 - \phi_X(u))) &= \frac{(T^*_{X\#Y}(u))'}{T^*_{X\#Y}(u) - u}, \end{aligned}$$

which gives

$$\phi_X(u) = 1 - \exp\left(\int_0^u \frac{(T^*_{X\#Y}(p))'}{T^*_{X\#Y}(p) - p} dp\right).$$

This completes the proof. ■

Remark 8.3.3. From Nair et al. [105], it follows that $\phi_X(u)$ uniquely determines the distribution through the relation,

$$Q(u) = \int_0^u \frac{\mu_X \, \phi_X'(p)}{1 - p} dp, \tag{8.3.33}$$

and consequently $T^*_{X\#Y}(u)$ uniquely determines the baseline distribution.

Example 8.3.4. Let X be distributed as generalized Pareto with quantile function,

$$Q_X(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right], \quad b > 0, a > -1. \tag{8.3.34}$$

Since $\mu_X = b$, we get

$$\phi_X(u) = \frac{1}{\mu}\int_0^u (1-p)q_X(p)dp = \left[1 - (1-u)^{\frac{1}{a+1}}\right]. \tag{8.3.35}$$

Hence, the equilibrium random variable Y has its quantile function as, $Q_Y(u) = Q_X(\phi_X^{-1}(u))$. Thus from (8.3.34) and (8.3.35), we obtain

$$Q_Y(u) = \frac{b}{a}\left[(1-u)^{-a} - 1\right]. \tag{8.3.36}$$

Using (8.3.27), we get

$$T^*_{X\#Y}(u) = \frac{1}{a}\left((a+1)\left(1-(1-u)^{\frac{1}{a+1}}\right) - u\right). \tag{8.3.37}$$

From Nair et al. [105], $\phi_X(u)$ and $M_X(u)$ are related through the identity,

$$M_X(u) = \frac{1-\phi_X(u)}{1-u}. \tag{8.3.38}$$

Substituting (8.3.38) in (8.3.27),

$$T^*_{X\#Y}(u) = u - (1-u)M_X(u)\int_0^u \frac{dp}{(1-p)M_X(p)}. \tag{8.3.39}$$

Thus $M_X(u)$ uniquely determines $T^*_{X\#Y}(u)$, when Y corresponds to the equilibrium distribution of X.

Example 8.3.5. Suppose X follows linear mean residual quantile function distribution with $M_X(u) = \mu + c\,u$, then $Q_X(u) = -(c+\mu)\log(1-u) - 2cu,\ \mu > 0,\ -\mu < c < \mu$ (Midhu et al. [89]). In this case,

$$T^*_{X\#Y}(u) = u + \frac{(1-u)(cu+\mu)\log\left(\frac{\mu-\mu u}{cu+\mu}\right)}{c+\mu}.$$

In the next theorem, we provide a characterization for the exponential distribution using $T^*_{X\#Y}(u)$, when Y is the equilibrium random variable of X.

Theorem 8.3.6. Let X be a non-negative random variable and Y be the corresponding equilibrium random variable. Then X has exponential distribution if and only if

$$T^*_{X\#Y}(u) = T^*_{X\#X}(u), \text{ for all } u \in (0,1). \tag{8.3.40}$$

Proof. Assume X follows exponential distribution with quantile function $Q_X(u) = \frac{-1}{\lambda}\log(1-u),\ \lambda > 0$. We get $\mu_X = \frac{1}{\lambda}$ and $\phi_X(u) = u$. Since, $\phi_X(u) = u$, from (8.3.26), we have $Q_X(u) = Q_Y(u)$. This implies, $T^*_{X\#Y}(u) = T^*_{X\#X}(u)$.
Conversely, we have, $T^*_{X\#Y}(u) = T^*_{X\#X}(u)$ for all $u \in (0,1)$. Now from (8.3.2), we have

$$T^*_{X\#Y}(u) = u + (1-u)\log(1-u). \tag{8.3.41}$$

Now using (8.3.31), we have $\phi_X(u) = u$. Thus from (8.3.33), the baseline quantile function of X is obtained as $Q_X(u) = -\mu_X\log(1-u)$, which is exponential. This completes the proof. ∎

8.4 Conclusion

In this chapter, we first presented the proportional hazards relevation transform, which is useful in the context of lifetime studies. Various properties and characterizations in terms of reliability measures of PHRT were presented. Stochastic orders between the relevated random variable $X\#Y$ and the baseline random variable X were developed. We also derived various ageing concepts of $X\#Y$ in connection with the ageing behaviour of X, which will be useful in reliability studies. We introduced the PHRW distribution and compared the performance with existing competing alternatives.

We then provided the concept of relevation quantile function which is the quantile version of the relevation transform. Various properties and applications were discussed. Relevation quantile function in the context of proportional hazards and equilibrium models were stiudied. The PHRQF model was applied to a real life data. We proved that $T^{*}_{X\#Y}(u)$ uniquely determines the distribution of X, when Y is the equilibrium random variable of X.

Chapter 9

Conclusion and Future Study

9.1 Conclusion

The present study discussed the role of quantile functions in modelling and analysis of lifetime data. The basic reliability concepts using quantile functions and their properties were presented in Chapter 1. We have also provided a brief review of literature and the relevance of the present study in Chapter 1.

In Chapters 2, 3, 4 and 5, we have developed some new quantile function models, which are useful for the analysis of lifetime data sets. Various distributional properties and quantile-based reliability measures of the proposed models were studied in detail. Estimation of parameters has been done using the method of L-moments and method of percentiles. The practical applications of these models were established with the help of real life data sets.

Motivated by the special properties of quantile functions, in Chapter 6 we studied the properties and applications of the proportional odds model in quantile set up. The proposed quantile-based approach has several advantages. It provides an alternative methodology for the analysis of lifetime data. Further, the proposed method develops a new class of lifetime distributions that do not have tractable distribution function but have simple and closed form quantile function. It gives new results in reliability analysis which are useful for the study of ageing phenomena as well as for the comparison of lifetime of systems.

We discussed modelling and analysis of competing risks data using quantile functions in Chapter 7. The cause specific hazard quantile function was introduced. We provided certain competing risks models using various functional forms for the cause specific hazard quantile functions. A non-parametric estimator of the cause specific hazard quantile function was derived. Asymptotic properties of the estimators were studied. Simulation studies were carried out to assess the performance of the estimators. Finally, we applied the proposed procedure to two real life data sets.

In Chapter 8, the relevation transform in the context of reliability modelling was discussed. The proportional hazards relevation transform and its reliability properties were studied. We introduced a quantile-based definition for the well-known relevation transform and derived reliability characteristics. Quantile-based relevation transform in the context of proportional hazards and equilibrium models were presented. A new quantile function model which generalizes to various existing models was introduced. The model was applied to a real life data set.

9.2 Future study

The present study developed new distribution models, which are useful for the analysis of various types of lifetime data sets. When the physical process of the system is complex, more flexible quantile functions are needed to model the underlying mechanism. As pointed out in Chapter 1, quantile functions have certain special properties which are not true in the case of distribution functions. There is a scope for generalizing the models proposed in Chapters 2, 3, 4 and 5 using these special properties of quantile function. The work in this direction will be carried out later.

The term competing risks applies in survival studies where a system or an organism is exposed to two or more causes of failure or death, but its eventual failure or death can be attributed to precisely one of the causes. In Chapter 7, we have introduced the concept of cause specific hazard quantile functions and derived non-parametric estimators of this function. In survival studies, it is often interesting to compare various risks. The comparison of various risks can be done by developing non-parametric tests using $\hat{H}_j(u)$, which will be taken up in future research.

Due to the presence of censoring and truncation in lifetime data sets, we cannot employ the parametric inference procedures efficiently. In such contexts, non-parametric estimators are commonly used for inferential problems. To analyse these types of data sets in quantile set up, we need to derive non-parametric estimators for various quantile-based reliability measures. This is an area of research work that remains to be explored.

In this thesis, we deal with only univariate lifetime data. The analysis of high dimensional data sets using quantile functions are yet to be discussed. Quantile-based definitions of various reliability concepts, ageing properties and stochastic orders in multivariate set up are the topics yet to be investigated.

The use of explanatory variables or covariates is an important way to represent heterogeneity in a population. Regression models are usually employed to analyse the relationship between lifetime and covariates. Recently quantile regression models are employed for the analysis of lifetime data. New regression models can be developed using the special properties of quantile functions, which will be of great importance and works in this direction will be carried out in our future work.

Published Papers

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(2) Dileep Kumar, M., Sankaran, P. G.,and Nair, N. U., Quantile-based relevation transform and its properties. *Statistica*, 78.3: 197-214, 2018.

(3) Sankaran, P. G., and Dileep Kumar, M., Reliability properties of proportional hazards relevation transform. *Metrika* 1-16, (2018).

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(1) Dileep Kumar, M., Sankaran, P. G., A new quantile function with applications to reliability analysis.

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