STUDIES ON THE DOMINATION GAME IN GRAPHS

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By

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To

My Parents and Wife

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Certificate

Certified that the work presented in this thesis entitled **"Studies on the Domination Game in Graphs"** is based on the authentic record of research carried out by Mr. Tijo James under my guidance in the Department of Mathematics, Cochin University of Science and Technology and has not been included in any other thesis submitted for the award of any degree.

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Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the doctoral committee of the candidate has been incorporated in the thesis entitled **"Studies on the Domination Game in Graphs"**.

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Declaration

I, Tijo James, hereby declare that this thesis entitled **"Studies on the Domination Game in Graphs"** is based on the original work carried out by me under the guidance of Dr. A. Vijayakumar, Emeritus Professor, Department of Mathematics, Cochin University of Science and Technology and contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or Institution. To the best of my knowledge and belief, this thesis contains no material previously published by any person except where due references are made in the text of the thesis.

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STUDIES ON THE DOMINATION GAME
IN GRAPHS

Contents

List of Symbols

$V(G)$	:	the vertex set of G
$E(G)$	:	the edge set of G, a collection of 2-element subsets of $V(G)$
$\|V(G)\|$	:	the order of G
$\|E(G)\|$	:	the size of G
$d_G(v)$ or $d(v)$	:	the degree of v in G
$\delta(G)$	:	the minimum degree of G
$\Delta(G)$	:	the maximum degree of G
$r(G)$	:	the radius of G
$diam(G)$	:	the diameter of G
$N_G(v)$ or $N(v)$	:	the open neighbourhood of v in G
$N_G[v]$ or $N[v]$	:	the closed neighbourhood of v in G
$N_G(S)$ or $N(S)$	:	the open neighbourhood of a subset S of $V(G)$
$N_G[S]$ or $N[S]$	:	the closed neighbourhood of a subset S of $V(G)$
$N_S(x)$	:	$N_G(x) \cap S$

$deg_S(x)$	:	$\vert N_G(x) \cap S\vert$
$[m]$	:	$\{1, 2, \ldots m\}$
$d_G(x, y)$	:	the standard shortest-path distance between vertices x and y of G
$ecc_G(u)$	:	the eccentricity of u in G
$G \cup H$	:	union of G and H
P_n	:	the path on n vertices
C_n	:	the cycle on n vertices
K_n	:	the complete graph on n vertices
$K_{1,n}$	:	the star of size n
$K_{m,n}$	:	the complete bipartite graph where m and n are the cardinalities of the partitions
$G - v$	:	the subgraph of G obtained by deleting the vertex v
$G - e$	:	the subgraph of G obtained by deleting the edge e
$G - A$	:	the subgraph of G obtained by the deletion of the vertices in A
$G - B$	:	the subgraph of G obtained by the deletion of the edges in B
$G\vert S$	:	the partially dominated graph in which vertices from S are dominated
$\gamma(G)$	:	domination number of G
$\gamma_g(G)$	:	game domination number of G
$\gamma'_g(G)$	:	staller start game domination number

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Chapter 1

Introduction

Graph theory is a branch of mathematics originated in the 18^{th} century when Leonhard Euler solved the Konigsberg bridge problem and it provides mathematical models for complex real world situations. The term 'graph' refers to a set of vertices (points or nodes) and of edges (links or lines) that connect the vertices. A graph without loops and with at most one edge between any two vertices is called a simple graph. Unless stated otherwise the term graph in this thesis refers to a simple graph with finite number of vertices. The origin of graph theory is well written in [33].

In 1862, C. F. De Jaenisch [14] studied the problem of determining the minimum number of queens required to cover an $n \times n$ chess board and W. W. Rouse Ball [51] reported the following three basic types of problems of a chess player.

- Determine the minimum number of a given type of chess pieces that are necessary to attack every square of an $n \times n$ chess board.
- Determine the smallest number of mutually non-attacking chess pieces of a particular type that are necessary to attack every square of an $n \times n$ board.
- Determine the maximum number of a particular type of chess pieces that can be placed on an $n \times n$ chess board such that no two pieces attack each other.

The above problems are the motivation for domination related problems in graph theory. Oystein Ore [35] introduced the terms "dominating set" and "domination number" of a graph G. The concept of dominating set is applied in various fields such as communication networks and facility location problems. There are many variations of domination [48] and generalizations including the total domination [30], the connected domination [17] and the power domination [45].

In this thesis we study a competitive optimization variant of domination, introduced by Brešar, Klavžar and Rall [2]. One of the main problems in graph theory is to find some special structures that are optimal with respect to some condition. For example, the problem in graph matching is to find the largest possible matching; however, when we view a maximal matching as an edge-dominating set, we suddenly want to find the smallest maximal matching. This motivates the study of "competitive optimization" parameters on graphs. A competitive optimization parameter can be viewed as a game in which two players, Min and Max, collaboratively build some desired structure. In the process, Min aims to minimize the cardinality of the structure produced, while Max aims to maximize it.

The domination game played on a finite, undirected graph G consists of two players, Dominator (D) and Staller (S), who alternately taking turns choosing a vertex from G such that whenever a vertex is chosen by either player, at least one additional vertex is dominated. Dominator uses a strategy to dominate the graph in as few steps as possible, and Staller uses a strategy to delay the process as much as possible. **D game** and **S game** are two variants of the domination game in which Dominator and Staller has respectively the first move. The **game domination number**, denoted by $\gamma_g(G)$, the number of vertices chosen in a D game when both players play optimally. The **Staller-start game domination number**, denoted by $\gamma_g'(G)$, is the number of vertices chosen in an S game when both players play optimally. (This differs from the parameter called 'game domination number' by Alon, Balogh, Bollobas, and Szabo [32].)

1.1 Definitions

The basic notations, terminology and definitions are from [38].

Definition 1.1.1.

- A **graph** is an ordered pair $G = (V, E)$ where V is a non-empty set and E is an unordered pair of elements of V.

- A graph $G = (V, E)$ is **trivial or empty** if its vertex set is a singleton set and it contains no edges.

- A graph G is **non-trivial** or non-empty if it has at least one edge.

- A vertex of degree zero is an **isolated vertex** and of degree one is a **pendant vertex**. An edge incident to a pendant vertex is a **pendant edge**.

- If G is a graph of order n, then a vertex of degree $n - 1$ is called a **universal vertex**.

- Let u and v be any two vertices of a graph $G = (V, E)$, then u **dominates** v if $u = v$ or u is adjacent to v.

- The **open neighborhood** $N_G(u) = \{v \in V : uv \in E(G)\}$ and the **closed neighborhood** $N_G[u] = N_G(u) \cup \{u\}$ will be abbreviated to $N(u)$ and $N[u]$ when G will be clear from the context.

- If $u \in V(G)$ and $S \subseteq V(G)$, then let $N_S(u) = N_G(u) \cap S$.

- The eccentricity $ecc_G(x)$ of x is $max\{d(x,y) : y \in V(G)\}$ where $d_G(x,y)$ be the standard shortest-path distance between vertices x and y of G.

- A subgraph H of G is isometric if $d_H(u,v) = d_G(u,v)$ holds for every pair of vertices u and v of H and that H is 2-isometric if for $d_G(u,v) = 2$, $u,v \in V(H)$, it follows that $d_H(u,v) = 2$.

Definition 1.1.2. A dominating set of a graph $G = (V,E)$ is a set $S \subseteq V$ such that $V = N[S] = \bigcup_{v \in S} N[v]$. **Domination number** of G is the minimum number of vertices in a dominating set of G, denoted by $\gamma(G)$.

Definition 1.1.3. A **partially dominated graph** is a graph together with the declaration that some vertices $S \subseteq V$ are already dominated.
If $S \subseteq V$ then let $G|S$ denote the partially dominated graph in which vertices from S are already dominated.

Definition 1.1.4. A vertex u of a partially dominated graph $G|S$ is **saturated** if each vertex in $N[u]$ is dominated.

Definition 1.1.5.

- Two vertices u and v in G are **true twins** if $N[u] = N[v]$.

- Two vertices u and v in G are **false twins** if $N(u) = N(v)$.

- A vertex u of a graph G is a **support vertex** if the vertex u is adjacent to at least one pendant vertex in G.
- A set of vertices $S \subseteq V(G)$ is an **independent set** if no two vertices in S are adjacent in G.
- A set of vertices $S \subseteq V(G)$ is a **clique** if every two vertices in S are adjacent in G.
- An independent set (resp. a clique) is **maximal** if no other independent set (resp. a clique) contains it

Definition 1.1.6. [41] A graph $G = (V, E)$ is a **split graph**, if $V(G)$ can be partitioned into (possibly empty) sets K and I, where K is a clique and I is an independent set. The pair (K, I) is called a **split partition** of G.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [23] in 1955 developed an interesting graph transformation called Mycielskian of a graph.

Definition 1.1.7. For a graph $G = (V, E)$, the **Mycielskian of G** is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{w\}$, where $V' = \{u' : u \in V\}$ and edge set $E \cup \{uv' : uv \in E\} \cup \{v'w : v' \in V'\}$. The vertex v' is called the twin of the vertex v and vice versa. The vertex w is called the root of $\mu(G)$.

Definition 1.1.8. A **subdivision of an edge** $e = uv$ of a graph G is obtained by replacing the edge e by the path uwv

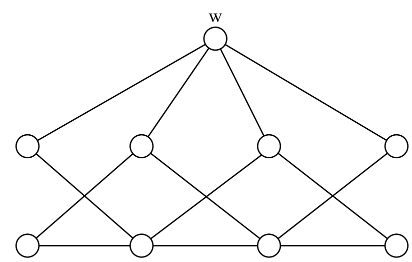

Figure 1.1: Mycielskian of P_4

of length two for some vertex w not in $V(G)$. It is denoted by $G \odot e$.

Definition 1.1.9. The graph obtained by **contraction of an edge** $e = uv$, denoted by $G.e$, is obtained from $G-e$ by replacing u and v by a new vertex w (contracted vertex) which is adjacent to all vertices in $N_{G-e}(u) \cup N_{G-e}(v)$.

Definition 1.1.10. A **chordal graph** is one in which every cycle of length at least four has a chord, that is, an edge that connects two non-consecutive vertices of the cycle.

Definition 1.1.11. [1]

- A vertex $u \in N[v]$ is a **maximum neighbour** of v if for all $w \in N[v]$, $N[w] \subseteq N[u]$. Let $G_i = G(\{v_i, v_{i+1} \dots v_n\})$ be the subgraph induced by $\{v_i, v_{i+1} \dots v_n\}$ and $N_i[v]$ be the closed neighbourhood of v in G_i.

- A vertex ordering $(v_1, v_2, \dots, v_n)$ of G is a **maximum neigh-**

bourhood ordering of G if for all $i \in 1, \ldots, n$, the vertex v_i has a maximum neighbour $u_i \in G_i$, that is for all $w \in N_i[v_i]$, $N_i[w] \subseteq N_i[u_i]$.

- G is **dually chordal** if G has a maximum neighbourhood ordering.

Definition 1.1.12. [37] A graph G is a **tri-split graph** if $V(G)$ can be partitioned into three disjoint sets $A \neq \emptyset$, B, and C with the following properties. The set A induces a clique, B induces an independent set, and C an arbitrary graph. Each vertex of A is adjacent to each vertex of C and no vertex of B is adjacent to a vertex in C.

1.2 A survey of previous results

The domination game was introduced by B. Brešar, S. Klavžar, and D. F. Rall in [2]. One of the main tools for proving most of the results in this area is the Continuation Principle.

The Continuation Principle [50]: Let G be a graph and fix $A, B \subseteq V(G)$. Let $G|A$ and $G|B$ be the partially-dominated graphs arising from G with A and B dominated, respectively. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma_g'(G|A) \leq \gamma_g'(G|B)$.

It is known [2, 50] that the difference between $\gamma_g(G)$ and $\gamma_g'(G)$ is at most one. Exact game domination number of a few classes of graphs are known and some of them are proved in

[21, 22] by G. Košmrlj.

- $\gamma_g(C_n) = \gamma_g(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil - 1 & n \equiv 3(\mathrm{mod} 4) \\ \lceil \frac{n}{2} \rceil & otherwise \end{cases}$
- $\gamma_g^{'}(C_n) = \begin{cases} \lceil \frac{n-1}{2} \rceil - 1 & n \equiv 2(\mathrm{mod} 4) \\ \lceil \frac{n-1}{2} \rceil & otherwise \end{cases}$
- $\gamma_g^{'}(P_n) = \lceil \frac{n}{2} \rceil$.

  The **Comb** P_k' is the graph obtained from a path of length k by adding a vertex of degree 1 adjacent to each vertex of the path.

- $\gamma_g(P_k') = k + \lceil \frac{k-7}{10} \rceil$.
- $\gamma_g^{'}(P_k') = k + \lceil \frac{k-2}{10} \rceil$

A new class of graphs related to game domination number is defined by Dorbec et al. in [37] known as **no-minus graphs**. A graph G is a no-minus graph if for any set $S \subseteq V(G)$, $\gamma_g(G|S) \leq \gamma_g^{'}(G|S)$. Trees and split graphs are some examples of no-minus graphs. A graph G is an **equal graph** if $\gamma_g(G) = \gamma_g'(G)$. Dorbec et al. also studied the possible values of the domination game parameters $\gamma_g(G)$ and $\gamma_g'(G)$ of the disjoint union of two graphs according to the values of these parameters in the initial graphs.

The concept of **guarded subgraph** of a graph was introduced in [5]. Every convex subgraph is a guarded subgraph and every guarded subgraph is a 2-isometric subgraphs. A guarded subgraph is not comparable to isometric subgraphs. A subgraph H of a graph G is guarded in G if for any vertex u in G there exists a vertex $v \in V(H)$ such that $N[u] \cap V(H) \subseteq N[v] \cap V(H)$. Such a vertex v will be called a guard of u in H. It is proved [5] that if H is guarded in G, then $\gamma_g(H) \leq \gamma_g(G)$ and $\gamma_g'(H) \leq \gamma_g'(G)$.

B. Brešar et al. answered the following question in [6]: How much $\gamma_g(G)$ and $\gamma_g'(G)$ can change if an edge is removed. It is proved [2] that $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ holds for any graph G.

Critical graph with respect to the domination game is introduced in [12]. A graph G is **domination game critical** or shortly γ_g-critical if $\gamma_g(G) > \gamma_g(G-v)$ holds for every $v \in V(G)$. Also G is $k\gamma_g$ - critical if $\gamma_g(G) = k$.

Kinnersley, West, and Zamani in [50] posed a celebrated 3/5-***conjecture*** asserting that if G is an isolate-free forest of order n or an isolate-free graph of order n, then $\gamma_g(G) \leq 3n/5$. Henning and Kinnersley proved this conjecture for all graphs with minimum degree at least 2 in [25]. **Now this conjecture is open for graphs with minimum degree 1.**

The concept of **bluff graph** is introduced in [8]. A graph is a bluff graph if every vertex is an optimal start vertex for Dominator in D-game and every vertex is also an optimal start vertex for Staller in S-game. It is proved [8] that every minus graph is a bluff graph. Moreover, if G is a connected graph with $\gamma_g(G) \geq 2$ and $\delta(G) = 1$, then G is a bluff graph if and only if G is a minus graph.

Two new techniques cutting lemma and union lemma are introduced in [36]. The **cutting lemma** bounds the game domination number of a partially dominated graph with the game domination number of suitably modified partially dominated graph. The **union lemma** bounds the S-game domination number of a disjoint union of paths using appropriate weighting functions.

It is proved [42] that the game domination number of a connected graph can be bounded above in terms of the size of minimal edge cuts. In particular, if C a minimum edge cut of a connected graph G, then $\gamma_g(G) \leq \gamma_g(G \setminus C) + 2\kappa'(G)$.

Motivated by all these results which had profound impact in the theory of game domination, we shall in the next section sum up our research work.

1.3 Summary of the thesis

This thesis entitled **Studies on the Domination game in Graphs** is divided into five chapters. The first chapter is on the basic definitions and terminology and contains the literature on the domination game in graphs.

The second chapter deals with the effect of $\gamma_g(G)$ as well as $\gamma_g'(G)$ when an edge or vertex is removed. Here we give a partial answer to the following problem posed in [6] by B. Brešar, P. Dorbec, S. Klavžar and G. Košmrlj .

- **Problem: Which of the subsets of** $\{-2,-1,0,1,2\}$ **can be realized as** $\{\gamma_g(G)-\gamma_g(G-e) : e \in E(G)\}$ **within the family of all (respectively connected) graphs** G**?.**

Considering the above problem in the class of no-minus graphs we have the following result.

- If G is a no-minus graph and $e \in E(G)$, then $\{\gamma_g(G) - \gamma_g(G-e)\} \subseteq \{-1,0,1\}$ and $\{\gamma_g'(G)-\gamma_g'(G-e)\} \subseteq \{-1,0,1\}$.

Trees are no-minus graphs. We have analyzed all possibilities of γ_g and γ_g' in trees when an edge is removed. Also we have studied the effect of γ_g and γ_g' when a vertex is removed in the class of no-minus graphs .

- If v is a pendant vertex of a no-minus graph G, then

$$\begin{aligned} \gamma_g(G)-1 &\leq \gamma_g(G-v) \leq \gamma_g(G), \\ \gamma_g'(G)-1 &\leq \gamma_g'(G-v) \leq \gamma_g'(G). \end{aligned}$$

We have also studied the effect of γ_g and γ_g' in trees when a vertex is removed.

In the third chapter, we discuss the following.

- **Problem: Find a graph operation that involves a monotone behaviour of γ_g and $\gamma_g^{'}$ of the graphs in the sense that these parameters either increase or decrease but not both.**

This motivated us to study the contraction of an edge and the subdivision of an edge in domination game. The main results in this chapter are listed below. In the following, $G.e$ denotes the graph obtained from G by contracting the edge e and $G \odot e$ denotes the graph obtained from G by subdividing the edge e.

- Let G be a graph and $e \in E(G)$ then
  $\gamma_g(G) - 2 \leq \gamma_g(G.e) \leq \gamma_g(G)$
  $\gamma_g^{'}(G) - 2 \leq \gamma_g^{'}(G.e) \leq \gamma_g^{'}(G)$.

- Let G be a graph and $e \in E(G)$ then
  $\gamma_g(G) \leq \gamma_g(G \odot e) \leq \gamma_g(G) + 2$
  $\gamma_g^{'}(G) \leq \gamma_g^{'}(G \odot e) \leq \gamma_g^{'}(G) + 2$.

- Let G be a no minus graph and $e \in E(G)$ then
  $\gamma_g(G) \leq \gamma_g(G \odot e) \leq \gamma_g(G) + 1$
  $\gamma_g^{'}(G) \leq \gamma_g^{'}(G \odot e) \leq \gamma_g^{'}(G) + 1$.

Some examples are also provided here.

In chapter four, we discuss domination game in split graphs. In the class of split graphs we have,

- If G is a connected split graph, then $\{\gamma_g(G) - \gamma_g(G - e) : e \in$

$E(G)\} \subseteq \{-1, 0\}$ and $\{\gamma_g'(G) - \gamma_g'(G - e) : e \in E(G)\} \subseteq \{-1, 0\}$.

- For any connected split graph G , $\gamma_g(G) - 1 \leq \gamma_g(G - v)$ and $\gamma_g'(G) - 1 \leq \gamma_g'(G - v)$ for any $v \in V(G)$.

Here we also discuss 3/5-**conjecture**. Kinnersley, West, and Zamani in [50] posed a celebrated 3/5-**conjecture**.

- **Conjecture: If G is an isolate-free forest of order n or an isolate-free graph of order n, then $\gamma_g(G) \leq 3n/5$.**

Motivated by the above conjecture this chapter also deals with the bounds of game domination number as well as staller start game domination number in the class of split graphs. The main results are listed below.

- If G is a connected split graph with $n(G) \geq 2$, then $\gamma_g(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$.

- If G is a connected split graph with $n(G) \geq 2$, then $\gamma_g'(G) \leq \left\lfloor \frac{n(G)+1}{2} \right\rfloor$.

- A split graph G is a 1/2 split graph if $\gamma_g(G) = \frac{1}{2}n$.

- A connected split graph of even order is a 1/2-split graph if and only if every vertex in K is adjacent to at least one leaf in I and $deg_I(x_i) \in [2]$ for $i \in [k]$.

In chapter five, the domination game in the Mycielskian of a graph is studied. The main results are,

- For any graph G, $\gamma_g(\mu(G)) = 2$ if and only if $G \cong K_1$.
- For any graph G, $\gamma_g'(\mu(G)) = 2$ if and only if $G \cong K_n$.
- For any disconnected graph G, $\gamma_g(\mu(G)) = 3$ if and only if $G \cong 2K_1$.
- Let G be a connected graph with at least two vertices, then $\gamma_g(\mu(G)) = 3$ if and only if every vertex of G lies in a connected dominating set of order 2.
- For any connected graph G, $\gamma_g'(\mu(G)) = 3$ if and only if $G \ncong K_n$ and $\Delta(G) = n - 1$.

In the concluding remarks, we have listed some open problems.

Chapter 2

Domination Game: Effect of Edge and Vertex Removal in Graphs

In this chapter, we describe the behaviour of the game domination number by the removal of an edge or a vertex in the class of no-minus graphs. In general, the game domination number can either increase or decrease by at most 2 when an edge is removed and it can either increase arbitrary large or decrease by at most 2 when a vertex is removed.

[0]Some results of this chapter are included in the following paper.
Tijo James, Paul Dorbec, A. Vijayakumar, Further progress on the heredity of the game domination number, *Lecture Notes in Comput. Sci.*(Springer) 18 (3) (2016), 435 – 445.

2.1 Edge Removal

Here, we prove that removing an edge from a no-minus graph can either increase or decrease its game domination number by at most 1.

Theorem 2.1.1. *If G is a no-minus graph and $e \in E(G)$, then* $\left|\gamma_g(G) - \gamma_g(G-e)\right| \le 1$ *and* $\left|\gamma_g'(G) - \gamma_g'(G-e)\right| \le 1$.

Proof. First we prove that $\gamma_g(G) \le \gamma_g(G-e)+1$. It is enough to show that Dominator has a strategy on G such that at most $\gamma_g(G-e)+1$ moves will be played for any move of Staller. Consider a D game on G and at the same time Dominator imagines another D game on $G-e$ with at most $\gamma_g(G-e)$ steps. Dominator's strategy on G is as follows. He will copy every move of Staller in the real game on G to the imagined game if it is a legal move in the imagined game and responds optimally in the imagined game on $G-e$. Each response by Dominator in the imagined game is then copied back to the real game on G if it is a legal move in G. Let $e = uv$ and if every move of Dominator in the imagined game is a legal move in the real game and every move of Staller in the real game is also a legal move in the imagined game, then both the games end at the same time. Since the imagined game on $G-e$ has at most $\gamma_g(G-e)$ steps, Staller plays optimally in the real game and possibily not Dominator, so $\gamma_g(G) \le \gamma_g(G-e)$.

Suppose at the k^{th} move, Dominator chooses a vertex in the imagined game that is not a legal move in the real game. This is possible only if Dominator chooses a vertex that dominates either u or v itself and all other neighbours of that vertex are already dominated. Suppose that the k^{th} move dominates only the vertex v which is already dominated in G. After this move in the imagined game, the set of vertices dominated in both the games are same. At this stage the number of moves in the real game is $k-1$ and the next turn is that of Dominator. Since Staller plays optimally in the real game and Dominator may not, $\gamma_g(G) \leq k - 1 + \gamma_g(G|D)$, where D denotes the set of vertices already dominated in G (it is to be noted that both u and v are already dominated in the real game on G). If possible suppose that u is not dominated in the real game on G, then the vertex v is a legal move in G and by the same argument as above we get $\gamma_g(G) \leq \gamma_g(G-e)$. But in the imagined game, the next turn is that of Staller and the number of moves at this stage is k. Since Dominator plays optimally in $G - e$ and Staller may not, $k + \gamma_g'(G - e|D) \leq \gamma_g(G - e)$.

So,

$$\begin{aligned}
\gamma_g(G) &\leq k - 1 + \gamma_g(G|D) \\
&= k - 1 + \gamma_g(G - e|D) \quad \text{(both ends of } e \text{ are already dominated)} \\
&\leq k + \gamma_g(G - e|D) \\
&\leq k + \gamma_g^{'}(G - e|D) \quad (G \text{ is a no-minus graph}) \\
&\leq \gamma_g(G - e).
\end{aligned}$$

Hence, in this case the real game ends in at most $\gamma_g(G - e)$ steps.

Suppose at the k^{th} move, Staller chooses a vertex in the real game which is not a legal move in the imagined game. This is possible only if Staller chooses one of the end vertices of e and the other end vertex is the only vertex which is newly dominated. Let v denotes the newly dominated vertex and D denotes the set of vertices dominated in the real game on G after the k^{th} move. At this stage, the set of vertices dominated in the imagined game on $G - e$ is $D - v$. In the real game, k vertices are already selected by both the players and the next is Dominator's turn. Since Staller plays optimally in the real game, we have $\gamma_g(G) \leq k + \gamma_g(G|D)$. But in the imagined game both the players have selected $k - 1$ vertices and the next turn is that of Staller. Since

Dominator plays optimally in the imagined game, we have

$$k - 1 + \gamma_g^{'}(G - e|D - v) \leq \gamma_g(G - e).$$

So,

$$\begin{aligned}
\gamma_g(G) &\leq k + \gamma_g(G|D) \\
&\leq k + \gamma_g(G - e|D) \quad \text{(both ends of } e \text{ are already dominated)} \\
&\leq k + \gamma_g(G - e|D - v) \quad \text{(by the Continuation Principle)} \\
&\leq k + \gamma_g^{'}(G - e|D - v) \quad (G \text{ is a no-minus graph}) \\
&= k - 1 + \gamma_g^{'}(G - e|D - v) + 1 \\
&\leq \gamma_g(G - e) + 1.
\end{aligned}$$

Thus $\gamma_g(G) \leq \gamma_g(G - e) + 1$. Note that in the above proof of this inequality, it does not matter whether D game or S Game is played. Hence analogous arguments also give us $\gamma_g'(G) \leq \gamma_g'(G - e) + 1$.

Now we prove that $\gamma_g(G - e) - 1 \leq \gamma_g(G)$. This proof is analogous to the proof of $\gamma_g(G - e) \leq \gamma_g(G) + 2$ in [6] but we substitute the condition $\gamma_g(G|D) \leq \gamma_g^{'}(G|D)$ instead of $\gamma_g(G|D) \leq 1 + \gamma_g^{'}(G|D)$. For the sake of completeness we shall give the proof mentioned in [6].

To prove the bound $\gamma_g(G - e) \leq \gamma_g(G) + 1$, it suffices to show that Dominator has a strategy on $G - e$ such that at most

$\gamma_(G) + 1$ moves will be played. His strategy is to play the game on $G - e$ as follows. In parallel to the real game, he is playing an imagined game on G by copying every move of Staller to this game and responding optimally in G. Each response in the imagined game is then copied back to the real game in $G - e$. Let $e = uv$ and consider the following possibilities.

Suppose that neither Staller nor Dominator plays on either of u and v in the course of the real game. Then all the moves in both games are legal and so the imagined game on G lasts no more than $\gamma_g(G)$ moves. (Recall that Dominator plays optimally on G but Staller might not play optimally.) Since the game on $G - e$ uses the same number of moves, we conclude that in this case the number of moves played in the real game is at most $\gamma_g(G)$.

Assume now that at some point of the game, the strategy of Dominator on G is to play a vertex incident with e, say u, but this move is not legal in the real game. This can happen only in the case when v is the only vertex in $N_G[u]$, which is not yet dominated. In this case Dominator plays v in the real game and by the Continuation Principle it ensures that the game is finished in no more than $\gamma_g(G)$ moves.

Assume next that in the course of the game one of the players played a vertex incident with e, say u, which is a legal move. This means that, after this move is copied into the imagined game on G, the vertex v is dominated in this game but may not yet be dominated in the real game. If all the moves are legal in the real game (played on $G-e$), then after at most $\gamma_g(G)$ moves all vertices except may be v are dominated. Hence the real game finishes in no more than $\gamma_g(G)+1$ moves. In the other case, Staller played a move in $G-e$ with only v was newly dominated, and this is not a legal move in G. Let this move of Staller in $G-e$ be the k^{th} move of the game. Note that after this move of Staller, the sets of dominated vertices are the same in both games and is denoted by D. Since after the $(k-1)^{th}$ move it is Staller's turn in the imagined game, we derive that

$$(k-1)+\gamma_g'(G|D) \leq \gamma_g(G).$$

(This inequality holds because Staller need not necessarily play optimally in the imagined game.) Now, Dominator does not copy the move of Staller into the imagined game but simply optimally plays the next moves. Therefore, since the number of moves left to end each of the games is $\gamma_g((G-e)|D)$, we have:

$$\begin{aligned}
\gamma_g(G-e) &\leq k+\gamma_g((G-e)|D) \\
&= k+\gamma_g(G|D) \\
&\leq k+\gamma_g'(G|D) \quad (G \text{ is a no-minus graph}) \\
&\leq \gamma_g(G)+1.
\end{aligned}$$

We have thus proved that $\gamma_g(G - e) - 1 \leq \gamma_g(G)$. Since in the above proof of this inequality it does not matter whether D Game or S Game is played on $G-e$, it follows that $\gamma_g'(G-e)-1 \leq \gamma_g'(G)$. $\square$

2.2 Edge Removal in Trees

Trees are no-minus graphs [50] and for any tree T we have $|\gamma_g(T) - \gamma_g(T-e)| \leq 1$ and $|\gamma_g'(T) - \gamma_g'(T-e)| \leq 1$, by Theorem 2.1.1.

Lemma 2.2.1. *If T is the graph obtained by subdividing each edge of the star $K_{1,n}$, $(n \geq 2)$, then $\gamma_g(T) = \gamma_g(T|u) = \gamma_g'(T) = \gamma_g'(T|u) = n+1$, where u is the centre of T.*

Proof. First we show that $\gamma_g(T) = \gamma_g(T|u) = n+1$. For that we prove $\gamma_g(T) \leq n+1$ and $\gamma_g(T|u) \geq n+1$. Therefore by the Continuation Principle $\gamma_g(T|u) \leq \gamma_g(T) \leq n+1$ and $\gamma_g(T) \geq \gamma_g(T|u) \geq n+1$. Thus we get $\gamma_g(T) = \gamma_g(T|u) = n+1$.

First we prove that $\gamma_g(T) \leq n+1$. It is enough to show that Dominator has a strategy for any move of Staller on T which ensures that the game has at most $n+1$ moves. Dominator first chooses the vertex u and the residual graph after this move is the disjoint union of n copies of K_2 with one of its end vertices is dominated. So this game has $n+1$ moves. It is noted that the first move of Dominator may not be optimal in this game and hence we conclude that $\gamma_g(T) \leq n+1$.

Now we prove $\gamma_g(T|u) \geq n+1$ by showing that for any move of Dominator in $T|u$, there is a strategy for Staller which ensures that the game on $T|u$ has at least $n+1$ moves. Suppose that Dominator chooses the vertex u as his first optimal move. In this case the residual graph after this move is the disjoint union of n copies of K_2 with one of its end vertices is dominated and hence a total of $n+1$ moves in this game. On the other hand suppose that Dominator chooses a vertex other than u as his first optimal move. By the Continuation Principle, Dominator prefers to select a vertex other than a pendant vertex. So Dominator chooses a vertex adjacent to u and after that move Staller selects u as next move. This is a legal move since u is adjacent to at least two vertices in $T|u$ (note that only one vertex adjacent to u is dominated). Now the residual graph at this stage is the disjoint union of $n-1$ copies of K_2 with one of its end vertices is dominated . Therefore this game has $2+n-1 = n+1$ moves. Note that the vertex u may not be an optimal move for Staller in this game on $T|u$ and hence $\gamma_g(T|u) \geq n+1$.

Now we show that $\gamma'_g(T) = \gamma'_g(T|u) = n+1$. For that we prove $\gamma'_g(T) \leq n+1$ and $\gamma'_g(T|u) \geq n+1$. Therefore by the Continuation Principle $\gamma'_g(T|u) \leq \gamma'_g(T) \leq n+1$ and $\gamma'_g(T) \geq \gamma'_g(T|u) \geq n+1$. Thus we get $\gamma'_g(T) = \gamma'_g(T|u) = n+1$.

First we prove that $\gamma'_g(T|u) \geq n+1$. It is enough to show that for any move of Dominator there is a strategy for Staller which ensures that an S game on $T|u$ has at least $n+1$ moves. Staller selects her first move as u and the residual graph after this move becomes n copies of K_2 with one of its end vertices is dominated. So this game has $n+1$ moves. It is noted that u may not be an optimal move for Staller in this game on $T|u$ and hence $\gamma_g'(T|u) \geq n+1$.

Now we prove that $\gamma_g'(T) \leq n+1$. Suppose that if Staller chooses u as her first optimal move, then the residual graph is the disjoint union of n copies of K_2 with one of its end vertices is dominated. So this game has $n+1$ moves. On the other hand if Staller chooses a vertex other than u as her first optimal move, then Dominator selects u as his next move and clearly u is a legal move. So the residual graph after these two moves is the disjoint union of $n-1$ copies of K_2 with one of its end vertices is dominated. In this case this game has $2+n-1 = n+1$ moves. So in this game Staller plays optimally and Dominator may not. Hence $\gamma_g'(T) \leq n+1$.

Thus we have $\gamma_g'(T) = \gamma_g'(T|u) = n+1$. $\square$

Lemma 2.2.2. *If T is the graph obtained by subdividing $n-1$ edges of the star $K_{1,n}$, $(n \geq 3)$, then $\gamma_g(T) = \gamma_g(T|v) = n$ and $\gamma_g'(T) = \gamma_g'(T|v) = n+1$, where v is the pendant vertex adjacent to the centre of T.*

Proof. Let T be the graph obtained by subdividing each edge except one, say $e = uv$ of the star $K_{1,n}$ with centre u.

There are n support vertices in T and this ensures that $\gamma_g(T) \geq \gamma(T) \geq n$. Now we show that $\gamma_g(T) \leq n$ by a strategy of Dominator which ensures that at most n moves in a D game on T for any move of Staller. Dominator first chooses the vertex u and then the residual graph is the disjoint union of $n-1$ copies of K_2 with one of its end vertices is dominated. It is to be noted that u may not be an optimal move of Dominator in T. Thus $\gamma_g(T) \leq n - 1 + 1 = n$.

Now we prove that $\gamma_g(T|v) \geq n$. It is enough to show that for any move of Dominator there exists a strategy for Staller on $T|v$ has at least n moves. It is known by the Continuation Principle that Dominator prefers to select non-pendant vertices in $T|v$. Suppose that Dominator chooses the vertex u as his first optimal move. So the residual graph after this move is the disjoint union of $n-1$ copies of K_2 with one of the end vertices is dominated and this game has n moves. Again suppose that Dominator chooses a non-pendant vertex other than u as his first optimal move. In this case Staller selects u as her next move. Since $n \geq 3$, u is adjacent to atleast 3 vertices in $T|v$ and hence u is a legal move. The residual graph after these two moves is the disjoint union of $n-2$ copies of K_2 with one of the end vertices is dominated. So there are $2 + n - 2 = n$

moves in this game. Dominator plays optimally in this game and Staller may not. Therefore $\gamma_g(T|v) \geq n$. It is known that $\gamma_g(T|v) \leq \gamma_g(T) = n$ and hence $\gamma_g(T|v) = n$.

Now we show that $\gamma'_g(T|v) = n+1$. It is known [50] that $\gamma'_g(T|v) \leq 1 + \gamma_g(T|v) = 1+n$. So it is enough to show that $\gamma'_g(T|v) \geq n+1$. Staller first chooses the vertex v and this may not be an optimal move. It is to be noted that v is a legal move because v dominates u in $T|v$. Therefore $\gamma'_g(T|v) \geq 1 + \gamma_g(T')$, where T' is the residual graph of $T|v$ after the first move. It is clear that the residual graph T' is a tree in Lemma 2.2.1 with $n-1 \geq 2$ support vertices. Therefore $\gamma_g(T') = n-1+1 = n$. So $\gamma'_g(T|v) \geq n+1$ and hence $\gamma'_g(T|v) = n+1$.

Also $n+1 = \gamma'_g(T|v) \leq \gamma'_g(T) \leq 1 + \gamma_g(T) = 1+n$. Thus $\gamma'_g(T) = n+1$. $\square$

Now we discuss all the possibilities of edge removal in trees.

Proposition 2.2.3. *For any $k \geq 3$ there is a tree T with an edge e such that $\gamma_g(T) = k$ and $\gamma_g(T-e) = k-1$.*

Proof. For $k = 3$, we have $T = P_5$ as the desired tree and e is an edge of T which is not a pendant edge. It is known that $\gamma_g(P_5) = 3$ and $\gamma_g(P_5 - e) = \gamma_g(P_3 \cup P_2)$. Clearly $\gamma_g(P_2) = \gamma'_g(P_2) = 1$. Therefore it follows from [37] that $\gamma_g(P_3 \cup P_2) = \gamma_g(P_3) + \gamma_g(P_2) = 1+1 = 2$.

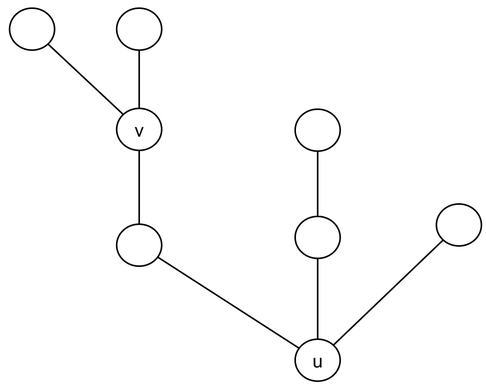

Figure 2.1: T_4

For $k = 4$, we construct a tree T_4 in the following way. Let $K_{1,3}$ be a star with u as its centre. T_4 is obtained from $K_{1,3}$ by subdividing each edge except one edge and attaches two pendant vertices to a vertex v, which is not adjacent to the vertex u. Let e be the edge incident to u and a vertex in $N(v)$. See the Figure 2.1. It can be easily verified that $\gamma_g(T_4) = 4$ and $\gamma_g(T_4 - e) = \gamma_g(K_{1,3} \cup P_4) = \gamma_g(K_{1,3}) + \gamma_g(P_4) = 1 + 2 = 3$ (note that $\gamma_g(P_4) = \gamma'_g(P_4) = 2$).

For $k \geq 5$, we have a general construction T_k in the following way. Subdividing each edge of the star $K_{1,k-2}$ with centre u. The graph T_k is obtained from the subdivided star by attaching two vertices to one of the pendant vertices say v. Let e be the edge incident to u and a vertex in $N(v)$. See the Figure 2.2. We

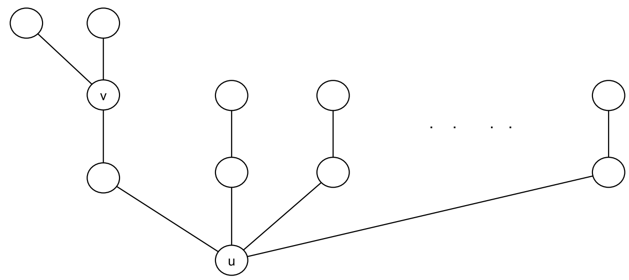

Figure 2.2: T_k

have to show that $\gamma_g(T_k) = k$. For that, first we show $\gamma_g(T_k) \geq k$ and it is enough to show that for any move of Dominator there is a strategy for Staller which ensures that the game has at least k moves. Suppose that Dominator chooses the vertex v as his first optimal move. In this case Staller selects the neighbour vertex of v which is adjacent to u. This is a legal move since u is additionally dominated by this move. The residual graph after these two moves is the graph $T_k|u$ in Lemma 2.2.1 with $k-3$ support vertices. So in this case the game has at least $2+k-3+1 = k$ moves. Now suppose that Dominator chooses the vertex u as his first optimal move . So Staller selects a pendant vertex attached to v as her next move. The residual graph after these two moves is the disjoint union of $k-2$ copies of K_2 with one of its end vertices is dominated. So in this case the game has at least $2+k-2 = k$ moves. Again suppose that Dominator chooses a vertex adjacent to u and v as his first optimal move. In this case Staller chooses u as her next move. Clearly this is a legal move since u is adjacent to at least 3 vertices. So the residual graph after these two moves is the disjoint union of $k-3$ copies of K_2 with one of its end vertices is dominated and a $K_{1,2}$ with the centre vertex is dominated. So in this case the game has at least $2+1+k-3=k$ moves. By the Continuation Principle, Dominator prefers to select a vertex of degree at least two to a pendant vertex of an edge. Now again suppose that Dominator chooses a vertex adjacent to u and a pendant vertex as his first optimal move. In this case Staller chooses the vertex adjacent to u and v as her next move and the residual graph after theses two moves is the disjoint union of $T_k|u$ in Lemma 2.2.1 with $k-4$ support vertices and a $K_{1,2}$ with the centre is dominated. So in this case the game has at least $2+1+k-3=k$ moves. It is noted that Dominator plays optimally and Staller may not. Thus we conclude that $\gamma_g(T_k) \geq k$.

Now we prove that $\gamma_g(T_k) \leq k$. Dominator chooses his first move as u and the residual graph T'_k after this move is the disjoint union of $k-3$ copies of K_2 with one of its end vertices is dominated and a $K_{1,3}$ with one of its pendant vertices is dominated. So $\gamma'_g(T'_k) = k-3+2 = k-1$ moves. The first move of Dominator may not be an optimal move and hence $\gamma_g(T_k) \leq 1+k-1 = k$.

The graph $T_k - e$ is the disjoint union of a $K_{1,3}$ and a graph T in Lemma 2.2.1 with $k-3$ support vertices. By Lemma 2.2.1 the graph T is an equal graph, and $K_{1,3}$ and T are no-minus graphs. So $\gamma_g(T_k - e) = \gamma_g(K_{1,3} \cup T) = \gamma_g(K_{1,3}) + \gamma_g(T) = 1 + k - 3 + 1 = k - 1$. □

Note 2.2.4. For $k = 1, 2$, there is no tree T with $\gamma_g(T) = k$ and $\gamma_g(T - e) = k - 1$.

Proposition 2.2.5. *For any $k \geq 1$, there is a tree T with an edge e such that $\gamma_g(T) = k$ and $\gamma_g(T - e) = k + 1$.*

Proof. For $k = 1$, $K_{1,2}$ is the desired graph. It is known that $\gamma_g(K_{1,2}) = 1$ and $\gamma_g(K_{1,2} - e) = 2$ for any edge e of $K_{1,2}$.

For $k = 2$, let T_2 be the graph obtained from $K_{1,2}$ by attaching two vertices to a pendant vertex of $K_{1,2}$. Clearly $\gamma_g(T_2) = 2$ and $\gamma_g(T_2 - e) = 3$ for any edge e incident to a newly attached vertex of T_2 .

For $k \geq 3$, T_k is the graph obtained from the star $K_{1,k}$ with centre u by subdividing each edge except one, say e. It is known by Lemma 2.2.2 that $\gamma_g(T_k) = k$.

It is clear that $T_k - e$ is the disjoint union of an isolated vertex and a tree T in Lemma 2.2.1 with $k-1$ support vertices. Thus $\gamma_g(T_k - e) = \gamma_g(K_1) + \gamma_g(T) = 1 + k - 1 + 1 = k + 1$. □

Proposition 2.2.6. *For any $k \geq 2$, there is a tree with an edge* *e such that* $\gamma_g(T) = \gamma_g(T-e) = k$.

Proof. For $k = 2$, P_4 is the desired graph. It is clear that $\gamma_g(P_4) = 2$ and $\gamma_g(P_4 - e) = 2$, where e is the middle edge of P_4.

For $k \geq 3$, let T_k be the graph obtained by subdividing each edge of $K_{1,k-1}$ with centre u . It is known by Lemma 2.2.1 that $\gamma_g(T_k) = k$.

Let e be any edge of T_k adjacent to u. It is clear that $T_k - e$ is the disjoint union of a K_2 and a tree T in Lemma 2.2.1 with $k-2$ support vertices. Thus $\gamma_g(T_k - e) = \gamma_g(K_2 \cup T) = \gamma_g(K_2) + \gamma_g(T) = 1 + k - 2 + 1 = k$. $\square$

Note 2.2.7. There is no tree T with an edge e such that $\gamma_g(T) = 1$ and $\gamma_g(T - e) = 1$. Clearly $T - e$ is disconnected for any edge e of a tree T and hence $\gamma_g(T - e) \geq 2$.

Proposition 2.2.8. *For any* $k \geq 4$*, there is a tree* T *with an edge* e *such that* $\gamma_g^{'}(T) = k$ *and* $\gamma_g'(T - e) = k - 1$

Proof. For $k = 4$, T_4 is the graph obtained by subdividing an edge of a $K_{1,2}$ with centre u and attaching three pendant vertices to the pendant vertex which is adjacent to u . Let e be the edge of T_4 which is adjacent to u and a degree 2 vertex. Clearly $\gamma_g^{'}(T_4) = 4$ and $\gamma_g^{'}(T_4 - e) = 3$.

For $k = 5$, T_5 is the graph obtained by subdividing two edges of a star $K_{1,3}$ with centre u and attaching two vertices to a pendant vertex which is not adjacent to u. Let e be the edge with one end vertex is u and the other end vertex is a degree two vertex which is neighbour of a degree 3 vertex. Clearly $\gamma_g^{'}(T_5) = 5$ and $\gamma_g^{'}(T_5 - e) = 4$.

For $k \geq 6$, T_k is the graph obtained by subdividing all edges of the star $K_{1,k-3}$ with centre v and attaching three vertices to pendant vertex u and let e be the edge incident to v and a vertex $w \in N(v)$.

Now we show that $\gamma_g'(T_k) = k$. For that, first we show that $\gamma_g'(T_k) \geq k$ and it is enough to show that for any move of Dominator there is a strategy for Staller which ensures that the game has at least k moves. Suppose that Staller first chooses a pendant vertex adjacent to u as her move. Suppose that Dominator chooses the vertex u as his first optimal move. In this case Staller selects the neighbour vertex of u which is adjacent to v. This is a legal move since v is additionally dominated by this move. The residual graph after these three moves is the graph $T_k|v$ in Lemma 2.2.1 with $k-4$ support vertices. So in this case the game has at least $3+k-4+1 = k$ moves. Now suppose that Dominator chooses the vertex v as his first optimal move . So Staller selects a pendant vertex attached to u as her next move. The residual graph after these three moves is the disjoint union of $k-3$ copies of K_2 with one of its end vertices is dominated. So in this case the game has at least $3+k-3 = k$ moves. Again suppose that Dominator chooses a vertex adjacent to u and v as his first optimal move. In this case Staller chooses v as her next move. Clearly this is a legal move since v is adjacent to at least 3 vertices. So the residual graph after these three moves is the disjoint union of $k-4$ copies of K_2 with one of its end vertices is dominated and a $K_{1,2}$ with the centre vertex is dominated. So in this case the game has at least $3+1+k-4 = k$ moves. By the Continuation Principle, Dominator prefers to select a vertex of degree at least two to a pendant vertex of an edge. Now again suppose that Dominator chooses a vertex adjacent to v and a pendant vertex as his first optimal move. In this case Staller chooses the vertex adjacent to u and v as her next move and the residual graph after theses three moves is the disjoint union of $T_k|v$ in Lemma 2.2.1 with $k-5$ support vertices and a $K_{1,2}$ with the centre is dominated. So in this case the game has at least $3+1+k-5+1 = k$ moves. It is noted that Dominator plays optimally and Staller may not. Thus we conclude that $\gamma_g'(T_k) \geq k$.

Now we prove that $\gamma_g'(T_k) \leq k$. Suppose that an optimal first move of Staller is a pendant vertex adjacent to u. Now Dominator chooses his first move as v and the residual graph T_k' after these two moves is the disjoint union of $k-4$ copies of K_2 with one of its end vertices is dominated and a $K_{1,2}$ with centre is dominated. So $\gamma_g'(T_k') = k-4+2 = k-2$ moves. So in this case this game has at most $2 + k - 2 = k$ moves.

Now Suppose that an optimal first move of Staller is a pendant vertex which is not adjacent to u. Now Dominator chooses his first move as v and the residual graph T'_k after these two moves is the disjoint union of $k - 5$ copies of K_2 with one of its end vertices is dominated and a $K_{1,4}$ with one pendant vertex is dominated. So $\gamma'_g(T'_k) = k - 5 + 2 = k - 3$ moves. So in this case this game has at most $2 + k - 3 = k - 1$ moves.

Suppose that an optimal first move of Staller is the vertex adjacent to u and v. Now Dominator chooses his first move as v and the residual graph T'_k after these two moves is the disjoint union of $k - 4$ copies of K_2 with one of its end vertices is dominated and a $K_{1,3}$ with centre is dominated. So $\gamma'_g(T'_k) = k - 4 + 2 = k - 2$ moves. So in this case this game has at most $2 + k - 2 = k$ moves.

Suppose that an optimal first move of Staller is v and the residual graph T'_k after this move is the disjoint union of $k - 4$ copies of K_2 with one of its end vertices is dominated and a $K_{1,4}$ one pendant vertex is dominated.. So $\gamma_g(T'_k) = k - 4 + 1 = k - 3$ moves. So in this case this game has at most $1 + k - 3 = k - 2$ moves. By the Continuation Principle, it is clear that no other vertex of T_k is an optimal first move for Staller (it is to be noted that Staller prefers pendant vertex to a support vertex). In all the cases Dominator may not play optimally and hence

$\gamma_g'(T_k) \leq k$. Thus $\gamma_g^{'}(T_k) = k$.

It is clear that $T_k - e$ is the disjoint union of a $K_{1,4}$ and a tree T in Lemma 2.2.1 with $k-4$ support vertices. Therefore $\gamma_g'(T) = k - 4 + 1 = k - 3$. It is known [37] that $\gamma_g'(T_k - e) = \gamma_g'(K_{1,4} \cup T) = \gamma_g'(K_{1,4}) + \gamma_g'(T) = 2 + k - 3 = k - 1$. $\square$

Proposition 2.2.9. *For any $k \geq 1$, there is a tree T with an edge e such that $\gamma_g^{'}(T) = k$ and $\gamma_g'(T - e) = k + 1$.*

Proof. For $k = 1$, K_2 is the desired graph. Clearly $\gamma_g'(K_2) = 1$ and $\gamma_g^{'}(K_2 - e) = 2$.

For $k = 2$, P_4 is the desired graph. Clearly $\gamma_g^{'}(P_4) = 2$ and $\gamma_g^{'}(P_4 - e) = 3$ for any pendant edge e of P_4.

For $k = 3$, let T_3 be the graph obtained from P_3 by attaching three vertices at one of the end points of P_3. Clearly $\gamma_g^{'}(T_3) = 3$. It is clear that $\gamma_g^{'}(T_3 - e) = 4$ for any pendant edge e is incident to the highest degree vertex of T_3.

For $k \geq 4$, let T_k be the graph obtained by subdividing $k-2$ edges of $K_{1,k}$ with centre u. Let e be any pendant edge incident to u.

It can be proved that $\gamma_g^{'}(T_k) = k$ by analogous arguments of Lemma 2.2.2.

It is clear that $T_k - e$ is the disjoint union of K_1 and a tree T in Lemma 2.2.2 with the centre is adjacent to k vertices. Therefore $\gamma_g'(T) = k - 1 + 1 = k$ and $\gamma_g'(T_k - e) = \gamma_g'(K_1 \cup T) = \gamma_g'(K_1) +$

$\gamma_g'(T) = k + 1$. $\square$

Proposition 2.2.10. *For any $k \geq 2$, there is a tree T with an edge e such that $\gamma_g^{'}(T) = k$ and $\gamma_g'(T - e) = k$.*

Proof. For $k \geq 2$, let T_k be the graph obtained by subdividing each edge of $K_{1,k-1}$. It is known by Lemma 2.2.1 that $\gamma_g^{'}(T_k) = k$. If e is any pendant edge, then it is clear that $T_k - e$ is the disjoint union of K_1 and a tree T in Lemma 2.2.2 with $k-1$ vertices adjacent to the centre. Thereore $\gamma_g'(T) = k - 1 + 1 = k$ and it is known [37] that $\gamma_g^{'}(T_k - e) = \gamma_g'(K_1 \cup T) = \gamma_g'(K_1) + \gamma_g'(T) = k + 1$. $\square$

Note 2.2.11. For $k = 1$, there is no tree with $\gamma_g^{'}(T) = 1$ and $\gamma_g'(T - e) = 1$.

2.3 Vertex Removal

If a vertex from a graph G is removed, its game domination number either increases arbitrary large or decreases by at most two [6]. However, if G is a no-minus graph having a pendant vertex v, we have the following lemma.

Lemma 2.3.1. *Let G be a no-minus graph and if v is a pendant vertex, then*

$$\begin{array}{rcccl} \gamma_g(G) - 1 & \leq & \gamma_g(G - v) & \leq & \gamma_g(G) \\ \gamma_g^{'}(G) - 1 & \leq & \gamma_g^{'}(G - v) & \leq & \gamma_g^{'}(G). \end{array}$$

Proof. First we prove that $\gamma_g(G - v) \leq \gamma_g(G|v)$. For that we need to show that Dominator has a strategy on $G - v$ such that at most $\gamma_g(G|v)$ moves will be played. The strategy is as follows. Dominator and Staller play an ordinary D game on $G - v$ and at the same time Dominator imagines another D game on $G|v$. He copies every move of Staller in the real game to the imagined game and responds optimally in the imagined game. He then copies back every optimal response in the imagined game to the real game. Every move of Staller in the real game is a legal move in the imagined game. By the Continuation Principle, Dominator prefers to select a vertex other than v in the imagined game, so every move of Dominator in the imagined game is a legal move in the real game. Hence, the real game ends by at most $\gamma_g(G|v)$ steps. Note that Staller plays optimally in $G - v$ and Dominator may not. Thus $\gamma_g(G - v) \leq \gamma_g(G|v)$. By the Continuation Principle $\gamma_g(G|v) \leq \gamma_g(G)$. Hence $\gamma_g(G - v) \leq \gamma_g(G)$.

Now, we prove that $\gamma_g(G) \leq \gamma_g(G - v) + 1$. It is enough to show that Dominator has a strategy on G such that at most $\gamma_g(G - v) + 1$ moves will be played. Dominator imagines D game on $G - v$ simultaneously with the D game on G. He is copying every move of Staller in the real game to the imaginary game and responding optimally in it. Every optimal response in the imagined game is then copied back to the real game.

If all the moves are legal, then $\gamma_g(G) \leq \gamma_g(G - v)$. Suppose at the k^{th} move, Staller chooses a vertex that is not a legal move in $G - v$ and this is possible only if Staller chooses a vertex whose neighbours are already dominated except v. Let D denote the set of vertices dominated in the real game after the k^{th} move. But in the imagined game both the players have played $k - 1$ moves and the next move is that of Staller. It is noted that Dominator plays optimally in $G - v$ and Staller may not. Therefore $k - 1 + \gamma_g'(G - v|D - v) \leq \gamma_g(G - v)$ and hence

$$\begin{aligned}
\gamma_g(G) &\leq k + \gamma_g(G|D) \\
&= k + \gamma_g(G - v|D - v) \\
&\leq k + \gamma_g'(G - v|D - v) \\
&= k - 1 + \gamma_g'(G - v|D - v) + 1 \\
&\leq \gamma_g(G - v) + 1.
\end{aligned}$$

Hence, $\gamma_g(G) - 1 \leq \gamma_g(G - v) \leq \gamma_g(G)$. The proof is independent of who plays first. So $\gamma_g'(G) - 1 \leq \gamma_g'(G - v) \leq \gamma_g'(G)$. □

2.4 Vertex Removal in Trees

Here, we consider the effect of vertex removal in trees. It may be noted that, there are trees T whose game domination number becomes arbitrarily large after removing a vertex from T. It is proved [6] that there is no graph G with $\gamma_g(G) = k$ and

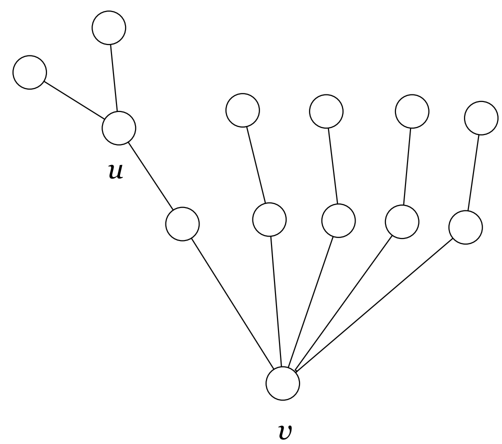

Figure 2.3: A tree T with $\gamma_g(T) = 7$ and $\gamma_g(T - v) = 5$

$\gamma_g(G - v) = k - 2$ for $k \leq 4$. We give examples of trees with $\gamma_g(T) = k$ and $\gamma_g(T - v) = k - t$ for any $t \in \{0, 1, 2\}$ and any integer $k \geq 5$.

Proposition 2.4.1. *For any $k \geq 5$ there exists a tree T with a vertex v such that $\gamma_g(T) = k$ and $\gamma_g(T - v) = k - 2$.*

Proof. Let T_k be the tree obtained by subdividing each edge of $K_{1,k-2}$ with centre v and attaching two vertices to an end vertex u of a subdivided edge as in Figure 2.3.

First we show that $\gamma_g(T_k) = k$. Dominator first chooses the vertex v in a D game on T_k and this may not be an optimal move. So $\gamma_g(T_k) \leq 1 + \gamma_g'(T_k - v|N(v))$. It is clear that $T_k - v|N(v)$ is the disjoint union of $k - 3$ copies of K_2 with one end vertex is dominated and a $K_{1,3}$ with one of its pendant vertices is dominated. So $\gamma_g'(T_k - v|N(v)) = 2+k-3$ and hence $\gamma_g(T_k) \leq k$.

Now we show that $\gamma_g(T_k) \geq k$. It is known by the Continuation Principle that Dominator prefers to select non-pendant vertices in T_k. Suppose that Dominator first chooses the vertex u as an optimal move. The residual graph after this move is the tree T_k' by removing the vertex u and two pendant vertices adjacent to u together with the vertex adjacent to u and v is considered as dominated. It is known by Lemma 2.2.2 that $\gamma_g'(T_k') = 1+k-2 = k-1$ and hence there are at least $1+k-1 = k$ moves in T_k.

Suppose that Dominator first chooses a non-pendant vertex other than u and v in T_k as an optimal move. In this case, Staller chooses a pendant vertex adjacent to u. If second move of Dominator is u, then the game ends with at least k moves. If second move of Dominator is a vertex other than u, then Staller chooses the other pendant vertex adjacent to u. In this case, the game ends with at least k moves.

Suppose that an optimal first move of Dominator in T_k is v. The the residual graph after this move is the disjoint union of $k-3$ copies of K_2 and a $K_{1,3}$. So in this case there are at least k moves. Thus we conclude that for any move of Dominator there is a strategy for Staller in T_k which ensures that there are at least k moves and hence $\gamma_g(T_k) \geq k$.

Dominator first chooses the vertex u in $T_k - v$ and the residual graph after the first move is the disjoint union of $k-3$ copies of K_2. Therefore this game has 1+k-3=k-2 moves. The first move of Dominator may not be an optimal move and hence $\gamma_g(T_k - v) \leq k-2$. It is known [6] that the game domination number decreases at most 2 when a vertex is removed. Thus $\gamma_g(T_k - v) = k-2$. □

Proposition 2.4.2. *For any $k \geq 1$ there exists a tree T with $\gamma_g(T) = k$ and $\gamma_g(T-v) = k-1$ for some vertex $v \in V(T)$.*

Choose $T = P_n$, $n \geq 1$. This satisfies the above proposition, as mentioned in [6].

Proposition 2.4.3. *For any $k \geq 1$ there exists a tree T with $\gamma_g(T) = k$ and $\gamma_g(T-v) = k$ for some vertex $v \in V(T)$.*

Proof. Let k be a positive integer and let T' be an arbitrary tree with $\gamma_g(T') = k$. Let x be an optimal first move of Dominator in T'. Let T be the tree obtained from T' by attaching a vertex u to x as mentioned in [6]. In that case, T and $T-u$ have the same game domination number. □

Proposition 2.4.4. *For any $k \geq 1$ there exists a tree T with $\gamma_g^{'}(T) = k$ and $\gamma_g^{'}(T-v) = k$ for some vertex $v \in V(T)$.*

Proof. For $k = 1$, K_2 is the desired tree.

For $k \geq 2$, let T_k be the tree obtained by subdividing each edge of the star $K_{1,k-1}$. It is known by Lemma 2.2.1 that

$\gamma_g'(T_k) = k$

It is clear that $T_k - v$ is a tree in Lemma 2.2.2 for any pendant vertex v of T_k. Therefore $\gamma_g'(T_k - v) = k$. $\square$

Proposition 2.4.5. *For any $k \geq 1$ there exists a tree T with $\gamma_g'(T) = k$ and $\gamma_g'(T - v) = k - 1$*

Proof. For $k = 1$, K_1 is the desired tree.

For $k \geq 2$, let T_k be the tree obtained by subdividing each edge of the star $K_{1,k-1}$ with centre v. It is known by Lemma 2.2.1 that $\gamma_g'(T_k) = k$.

It is clear that $T_k - v$ is the disjoint union $k - 1$ copies of K_2 and hence $\gamma_g'(T_k - v) = k - 1$. $\square$

Note 2.4.6. It is proved [6] that there is no graph G with $\gamma_g'(G) = k$ and $\gamma_g'(G - v) = k - 2$ for $k < 4$ and there exist graphs G with $\gamma_g'(G) = k$ and $\gamma_g'(G - v) = k - 2$ for $k \geq 4$.

Proposition 2.4.7. *For any $k \geq 6$ there exists a tree T with a vertex v such that $\gamma_g'(T) = k$ and $\gamma_g'(T - v) = k - 2$.*

Proof. Let T_k be the tree obtained by subdividing each edge of $K_{1,k-3}$ with centre v and attaching three vertices to one of the end points say u of a subdivided edge as in Figure 2.4. It is known by Proposition 2.2.8 that $\gamma_g'(T_k) = k$.

It is clear that $T_k - v$ is the disjoint union of a $K_{1,4}$ and $k - 4$ copies of K_2. Therefore $\gamma_g'(T_k - v) = 2 + k - 4 = k - 2$. $\square$

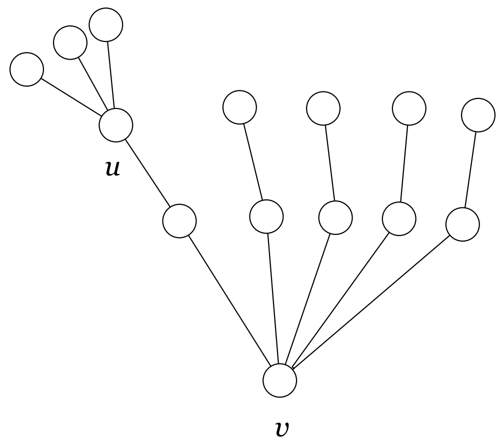

Figure 2.4: A tree T with $\gamma_g^{'}(T) = 8$ and $\gamma_g^{'}(T - v) = 6$

Proposition 2.4.8. *There is no tree T with $\gamma_g^{'}(T) = 4$ and $\gamma_g^{'}(T - v) = 2$ for any vertex $v \in T$.*

Proof. Assume the contradiction. Let T be a tree with a vertex v such that $\gamma_g^{'}(T) = 4$ and $\gamma_g^{'}(T - v) = 2$. First, consider the case that if v is a pendant vertex of T then it is known by Lemma 2.3.1 that $\gamma_g^{'}(T) - 1 \leq \gamma_g^{'}(T - v) \leq \gamma_g^{'}(T)$. Therefore $\gamma_g^{'}(T)$ is at most 3 and this contradicts $\gamma_g^{'}(T) = 4$.

Now, consider the case that if v is a cut vertex. In this case $T - v$ is disconnected with exactly two components. If possible suppose that $T - v$ has more than two components then $\gamma_g^{'}(T - v)$ is at least 3. This is not possible. So clearly $T - v$ has exactly two components T_1 and T_2. Each component is either K_1 or K_2, otherwise it contradicts that $\gamma_g^{'}(T - v)$ is 2. Since T is a tree,

v is adjacent to exactly one vertex in each component. In this case $\gamma_g^{'}(T)$ is at most 3. This contradicts $\gamma_g^{'}(T) = 4$. $\square$

Proposition 2.4.9. *There is no tree with $\gamma_g^{'}(T) = 5$ and $\gamma_g^{'}(T - v) = 3$ for any vertex v in T.*

Proof. Assume the contradiction. Let T be a tree with a vertex v such that $\gamma_g^{'}(T) = 5$ and $\gamma_g^{'}(T - v) = 3$. It is known by Lemma 2.3.1 that removal of a pendant vertex from a tree decreases its game domination number by at most 1. So clearly v is a cut vertex and $T - v$ has at most 3 components. It is to be noted that if $T - v$ has at least 4 components, then $\gamma_g'(T - v) \geq 4$.

Now, we prove that the vertex v is not an optimal first move of Staller in T. If possible let v be an optimal first move of Staller in T. Then

$$\begin{aligned}
\gamma_g^{'}(T) &= 1 + \gamma_g(T|N[v]) \\
&\leq 1 + \gamma_g^{'}(T|N[v]) \\
&= 1 + \gamma_g^{'}(T - v|N(v)) \\
&\leq 1 + \gamma_g^{'}(T - v).
\end{aligned}$$

Hence, $\gamma_g'(T - v)$ is decreased by at most 1.

First, we consider the case that $T - v$ has 3 components. In this case each component of $T - v$ is either K_1 or K_2 otherwise a contradiction to the assumption that $\gamma_g'(T - v) = 3$. For an S

game on T, Staller first chooses a vertex of T other than v. Now Dominator can finish this game by next 3 moves by selecting the vertex v. Here Staller plays optimally and Dominator may not. Therefore $\gamma_g'(T) \leq 4$ and is contradiction to the assumption that $\gamma_g'(T) = 5$.

Now consider the case that $T - v$ has exactly two components say T_1 and T_2. In this case one component say T_1 has $\gamma_g'(T_1) = 1$ and the other component T_2 has $\gamma_g'(T_2) = 2$. So T_1 is either K_1 or K_2 and it is known [43] that every vertex of T_2 is in a dominating set of order 2 in T_2. Consider an S game on T and an optimal first move of Staller is either from T_1 or from T_2. If an optimal first move of Staller in T is a vertex from T_1 then Dominator chooses v as next move. Now Staller chooses any vertex from T_2 and then Dominator can finish this game by next move. It is to be noted that every vertex of T_2 is a member of Dominating set of order 2 in T_2. So this game on T is finished in at most 4 steps. In this game on T Staller plays optimally and Dominator may not. Therefore $\gamma_g'(T) \leq 4$ and is a contradiction that $\gamma_g'(T) = 5$. If an optimal first move of Staller is a vertex from T_2, then Dominator chooses a vertex from T_2 in which all vertices of T_2 are dominated by these two moves. This is possible because every vertex is a member of a dominating set of order 2 in T_2. Now it is clear that this game is finished by at most two moves. Here also Staller plays optimally and Dominator may not. Therefore $\gamma_g'(T) \leq 4$ and is a contradiction that $\gamma_g'(T) = 5$. So there is no tree with $\gamma_g^{'}(T) = 5$ and $\gamma_g^{'}(T - v) = 3$. $\square$

Chapter 3

Domination Game: Effect of Edge Contraction and Edge Subdivision

In this chapter we discuss two graph operations, the edge contraction and the edge subdivision. These operations have a monotone behaviour on γ_g and γ_g' of the graphs, in the sense that these parameters either increase or decrease but not both.

3.1 Edge Contraction and Edge Subdivision

We first prove the following bounds for the game domination number of the graph obtained by edge contraction.

Theorem 3.1.1. *Let G be a graph and $e \in E(G)$. If $G.e$ is the* *graph obtained from G by contracting the edge e then*

$$\begin{aligned}\gamma_g(G) - 2 &\leq \gamma_g(G.e) \leq \gamma_g(G)\\ \gamma_g'(G) - 2 &\leq \gamma_g'(G.e) \leq \gamma_g'(G).\end{aligned}$$

Proof. Let G be a graph and $e = uv$ be an edge in G. In $G.e$, we denote by w the new vertex obtained by the identification of u and v.

We first prove the upper bounds by describing a strategy for Dominator. We use the imagination strategy, as used in [2]. During the course of the game on $G.e$, Dominator imagines another game played on G. Every time in his turn to play, he plays an optimal move in the imagined game and copies this move to the real game. Then he copies Staller's answer in the real game to his imagined game. In addition, in the imagined game, Dominator may consider some extra vertices dominated during the course of the game, and adapt his strategy. Note that by the Continuation Principle, considering more vertices dominated in the imagined game cannot make the imagined game last longer. If Staller plays the vertex w, then Dominator will consider that she played the vertex u in G and also adds the neighbourhood of v to the set of dominated vertices. Similarly, if Dominator is supposed to copy to the real game a move in the imagined game on the vertex u or v, then Dominator plays to w and adds both the neighbourhoods of u and v to the set of dominated vertices in the imagined game. Finally, if any player moves on a neighbour of w in $G.e$, Dominator will assume that both u and v get dominated in the imagined game. We know that Dominator and possibly not Staller, is playing optimally in the imagined game. This guarantees that the imagined game in G should last no longer than $\gamma_g(G)$ for the D game or than $\gamma'_g(G)$ for the S game. Staller and possibly not Dominator, is playing optimally in the real game on $G.e$. This implies that the total number of moves made in the real game is at least $\gamma_g(G.e)$ for the D game and at least $\gamma'_g(G.e)$ for the S game. Moreover, at each stage of the game, if S is the set of dominated vertices in the real game, either S does not contain w or the set of dominated vertices in the imagined game is precisely $(S \setminus \{w\}) \cup \{u, v\}$. Eventually, when the imagined game is over, the real game is also finished and Dominator ensured that the number of moves in the real game was no more than the number of moves in the imagined game. Thus $\gamma_g(G.e) \leq \gamma_g(G)$ which proves the upper bound.

Now we prove that $\gamma_g(G) - 2 \leq \gamma_g(G.e)$. Consider a real D game played on G and at the same time Dominator imagines another D game played on $G.e$. Again, Dominator copies every move of Staller in the real game except u and v to the imagined game and copies back his optimal response in the imagined game except w to the real game on G. Every move of Dominator in the imagined game except w is a legal move in the real game.

Suppose at some stage Dominator chooses w in the imagined game, then he chooses either u or v in the real game instead of copying w. Clearly one of u or v is a legal move in G. If Staller plays either u or v in the real game then Dominator plays w for Staller in the imagined game when it is a legal move. Assume first that every move of Staller in the real game is also a legal move in the imagined game. There may be vertices remaining undominated in the real game when the imagined game is finished. If neither Dominator nor Staller played u or v, then the only undominated vertices must be one among $\{u, v\}$. Otherwise, all the undominated vertices must be included either in $N(v)$ or in $N(u)$. In both cases the real game can be finished by playing either u or v depending on the game. Thus the game finishes in at most two more moves.

Assume now that the k^{th} move of Staller is not a legal move in the imagined game. Again, the only vertices that may be dominated in the imagined game but not in the real game are vertices from $N[u] \cup N[v]$. More precisely, if any of u or v was played in the real game then these vertices are contained in $N(v)$ or in $N(u)$ respectively, otherwise only u or v may be such a vertex. In any case Dominator plays any legal move x in the real game. Let S be the set of vertices dominated in the real game on G after the $(k+2)^{th}$ move and let S' be the set of vertices dominated in the imagined game after the k^{th}

move. The residual graph after the $(k+2)^{th}$ move on G is $G|S$ and the residual graph after the k^{th} move is $G.e|S'$. Defining $S'' = S' \cup N[x]$ by adding the newly dominated vertices in $N[x]$ to the set S' of dominated vertices in the imagined game after the k^{th} move. By the Continuation Principle, we get that $\gamma_g'(G.e|S'') \leq \gamma_g'(G.e|S')$. It is clear that $G.e|S''$ and $G|S$ are isomorphic. Therefore $\gamma_g(G.e|s'') = \gamma_g(G|S)$. Staller and possibly not Dominator, is playing optimally in the real game on G. We then have

$$\begin{aligned}\gamma_g(G) &\leq k + 2 + \gamma_g'(G|S)\\ &= k + 2 + \gamma_g'(G.e|S'')\\ &\leq k + 2 + \gamma_g'(G.e|S').\end{aligned}$$

Also Dominator and possibly not Staller, is playing optimally in the imagined game on $G.e$. Thus we get

$$k + \gamma_g'(G.e|S') \leq \gamma_g(G.e).$$

Therefore $\gamma_g(G) \leq \gamma_g(G.e) + 2$.

The same arguments also hold for the staller start game domination number and hence the bounds proposed for the S game can be proved similarly. $\square$

Now we consider the case of edge subdivision. Since in $G \odot e$, for any edge e' incident to the added vertex of degree 2, $(G \odot e).e'$ is the initial graph G, we get as a corollary of Theorem 3.1.1.

Corollary 3.1.2. *Let G be a graph and $e \in E(G)$. The graph $G \odot e$ obtained from G by subdividing the edge e satisfies*

$$\begin{aligned} \gamma_g(G) &\leq \gamma_g(G \odot e) \leq \gamma_g(G) + 2 \\ \gamma'_g(G) &\leq \gamma'_g(G \odot e) \leq \gamma'_g(G) + 2. \end{aligned}$$

3.2 Edge Contraction in No-minus Graphs

In a no-minus graph it is of no advantage for either player to pass a move. It is known already that forests [50], tri-split graphs and dually chordal graphs [37] are no-minus graphs. We have just proved that $0 \leq \gamma_g(G) - \gamma_g(G.e) \leq 2$ and $0 \leq \gamma'_g(G) - \gamma'_g(G.e) \leq 2$. We shall now describe no-minus graphs, especially trees, which attain all possible values for these differences.

Proposition 3.2.1. *For any $l \geq 3$ there exists a no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G.e) = l - 2$.*

Proof. For $l \geq 3$, we construct the following family of no-minus graphs denoted by $G_l, l \geq 0$. Let G_0 be the graph constructed in the following way. Take two copies of $K_{1,2}$ and label their centre vertices as u and v. Join u and v by the edge e. For $l \geq 1$, the graph G_l is obtained from G_0 by identifying the end vertices of l copies of P_3 with x. See Figure 3.1. We claim that $\gamma_g(G_l) = l+3$

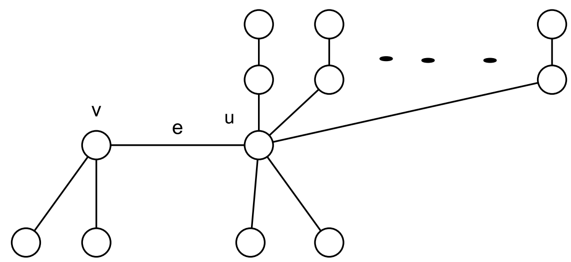

Figure 3.1: The graph G_l

and $\gamma_g(G_l.e) = l+1$. Note that if Dominator plays his first move on x, then only $l+2$ vertices remain undominated which yields $\gamma_g(G_l) \leq l+3$.

Now we present a strategy for Staller which ensures that at least $l+3$ moves are needed to finish the game on G_l. If Dominator starts by playing on u, then Staller selects a leaf adjacent to v. The resulting residual graph at this stage is a partially dominated graph consisting of $l+1$ copies of K_2 and hence at least $l+1$ more moves are needed to finish the game. Therefore a total of $l+3$ moves will be played. Otherwise, if Dominator does not start by playing on u, then Staller responds by playing on a leaf adjacent to u. Note that all vertices of G_l are either leaves or support vertices and the number of support vertices in G_l is $l+2$. By the Continuation Principle, Dominator prefers to select support vertices to leaves. The number of support vertices of the residual graph G'_l after the first two moves of G_l is $l+1$. Clearly $\gamma_g(G_l) \geq 2 + \gamma_g(G'_l)$ and $\gamma_g(G'_l) \geq \gamma(G'_l) \geq l+1$

(using the fact that the domination number of a graph is at least as that of the number of support vertices of that graph). Thus $\gamma_g(G_l) \geq l+3$ and get that $\gamma_g(G_l) = l+3$.

Let $G_l.e$ be the graph obtained from G by contracting the edge e in G_l. Here u and v are identified by a new vertex say w. If Dominator selects the vertex w then only l vertices remain undominated which yield $\gamma_g(G_l.e) \leq l+1$. There are $l+1$ support vertices in $G_l.e$ and hence $\gamma_g(G_l.e) \geq \gamma(G_l.e) \geq l+1$. Thus we get that $\gamma_g(G_l.e) = l+1$. □

Proposition 3.2.2. *For any $l \geq 2$ there exists a no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G.e) = l-1$.*

Proof. For $l \geq 2$, we construct the graph H_l from a star $K_{1,l}$ by subdividing each edge except one. We claim that $\gamma_g(H_l) = l$. Note that if Dominator plays his first move on the centre vertex, then only $l-1$ vertices remain undominated which yields $\gamma_g(H_l) \leq l$. On the other hand the number of support vertices in H_l is l and by the Continuation Principle Dominator prefers to select a support vertex to a leaf. After the first move of Dominator there remain $l-1$ support vertices and hence $\gamma_g(H_l) \geq 1+l-1 = l$. Consider the graph $H_l.e$ where e is an edge not incident to the centre of H_l. We claim that $\gamma_g(H_l.e) \leq l-1$. If Dominator plays his first move on the centre vertex of $H_l.e$, then only $l-2$ vertices remain undominated which yields $\gamma_g(H_l.e) \leq l-1$.

It is clear that $H_l.e$ has $l-1$ support vertices. Hence $\gamma_g(H_l.e) \geq \gamma(H_l.e) \geq l-1$. Thus $\gamma_g(H_l.e) = l-1$.

□

Proposition 3.2.3. *For any $l \geq 1$ there exists a no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G.e) = l$.*

Proof. For $l \geq 1$, construct the graph F_l from a star $K_{1,l+1}$ by subdividing each edge except two. Clearly F_l has $(l+1)-2+1 = l$ support vertices. Hence $\gamma_g(F_l) \geq \gamma(F_l) \geq l$. On the other hand if Dominator plays his first move on the centre vertex then only $l+1-2$ vertices remain undominated in F_l. So $\gamma_g(F_l) \leq 1+l+1-2 = l$ and thus $\gamma_g(F_l) = l$.

The graph $F_l.e$ is obtained from F_l by contracting an edge e incident with the centre and a leaf. By a similar argument we can prove that $\gamma_g(F_l.e) = l$. □

Proposition 3.2.4. *There is no graph G with an edge e such that $\gamma'_g(G) = 3$ and $\gamma'_g(G.e) = 1$.*

Proof. We know that $\gamma'_g(G) = 1$ if and only if G is complete. Assume that G is a graph with $\gamma'_g(G) = 3$ and hence G has at least two non adjacent vertices say u and v. Clearly u and v are non adjacent in $G.e$ for any edge e of G. Therefore $\gamma'_g(G.e) \geq 2$ and conclude that there is no graph G with an edge e such that

$\gamma_g'(G) = 3$ and $\gamma_g'(G.e) = 1$. $\square$

Proposition 3.2.5. *For any $l \geq 4$ there exists a no-minus graph G with an edge e such that $\gamma_g'(G) = l$ and $\gamma_g'(G.e) = l - 2$.*

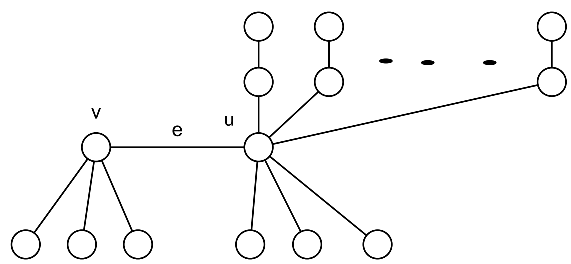

Figure 3.2: The graph S_k

Proof. For $l \geq 4$, we construct the following family of no-minus graphs denoted by S_k , $k = l - 4 \geq 0$. Let S_0 be the graph constructed in the following way. Take two copies of $K_{1,3}$ and label their centre vertices as u and v. Join u and v by the edge e. For $k \geq 1$ the graph S_k is obtained from S_0 by identifying the end vertices by k copies of P_3 with u. See figure 3.2. We claim that $\gamma_g'(S_k) = k + 4$ and $\gamma_g'(S_k.e) = k + 2$. By Theorem 3.1.1 it suffices to show that $\gamma_g'(S_k) \geq k + 4$ and $\gamma_g'(S_k.e) \leq k + 2$. First we show that $\gamma_g'(S_k) \geq k + 4$ by presenting a strategy for Staller which ensures that the game ends with at least $k + 4$ moves. Staller first plays a leaf adjacent to u and we know that all vertices of S_k are either support vertices or leaves. Dominator prefers to select a support vertex to a leaf. If Dominator plays a support vertex other than u, then Staller chooses another leaf adjacent to u otherwise Staller chooses a leaf adjacent to v. Let S'_k be the residual graph after these three moves. We know that the number of support vertices of S_k is $k+2$ and the number of support vertices of S'_k is $k+1$. Therefore $\gamma_g S'_k \gamma(S'_k) \geq k+1$. Thus $\gamma'_g(S_k) = 3 + \gamma_g(S'_k) \geq 3 + k + 1 = k + 4$.

Let $S_k.e$ is the graph obtained from S_k by contracting the edge $e = uv$ and let w be the new vertex due to the contraction. We show that $\gamma'_g(S_k.e) \leq k+2$ by presenting a strategy for Dominator which ensures that at most $k+2$ moves are needed to finish the game. Staller's first move is a leaf on $S_k.e$. Now Dominator plays w as his next move and the number of vertices remain undominated in $S_k.e$ is at most k. Hence $\gamma'_g(S_k.e) \leq 2+k$ and we conclude that $\gamma'_g(S_k) = k+4$ and $\gamma'_g(S_k.e) = k+2$. $\square$

Proposition 3.2.6. *For any $l \geq 2$ there exists a no-minus graph G with an edge e such that $\gamma'_g(G) = l$ and $\gamma'_g(G.e) = l-1$.*

Proof. For the general case $l \geq 2$, consider the graph P_{2l-1}. It is known that $\gamma'_g(P_{2l-1}) = l$ and $\gamma'_g(P_{2l-1}.e) = \gamma'_g(P_{2l-2}) = l-1$ for any edge e. $\square$

Proposition 3.2.7. *For any $l \geq 1$ there exists a no-minus graph G with an edge e such that $\gamma'_g(G) = l$ and $\gamma'_g(G.e) = l$.*

Proof. For the general case $l \geq 1$, consider the graph P_{2l}. It is known that $\gamma'_g(P_{2l}) = l$ and $\gamma'_g(P_{2l}.e) = \gamma'_g(P_{2l-1}) = l$ for any edge e. $\square$

3.3 Edge Subdivision in No-minus Graphs

By Corollary [3.1.2] we have $\gamma_g(G \odot e) - \gamma_g(G) \leq 2$

We here prove that the result can be strengthened in the case of no-minus graphs, as follows.

Theorem 3.3.1. *Let G be a no minus graph and $e \in E(G)$. The graph $G \odot e$ satisfies*

$$\begin{array}{ccccc} \gamma_g(G) & \leq & \gamma_g(G \odot e) & \leq & \gamma_g(G) + 1 \\ \gamma'_g(G) & \leq & \gamma'_g(G \odot e) & \leq & \gamma'_g(G) + 1. \end{array}$$

Proof. We know that for any graph G, $\gamma_g(G) \leq \gamma_g(G \odot e)$ and $\gamma'_g(G) \leq \gamma'_g(G \odot e)$ by Corollary 3.1.2. So this is true in the case of no-minus graphs.

Now we prove that $\gamma_g(G \odot e) \leq \gamma_g(G) + 1$. Let uv be the subdivided edge and w be the vertex added in the subdivision. Consider a real dominator start game played on $G \odot e$. At the same time Dominator imagines another dominator start Staller-pass game played on G. Dominator copies every move of Staller in the real game except w to the imagined game and copies back his optimal response. Every move of Dominator in the imagined game is legal in the real game. If every move of Staller in the real game is also legal in the imagined game then only one vertex may remain undominated in $G \odot e$ at the end of the game, and it is either u or v or w. Thus the real game is finished within at most one move more than in the imagined game. Suppose at the k^{th} stage Staller chooses a vertex in the real game that is not a legal move in the imagined game. This is possible only if that move additionally dominates either w itself in the real game or u or v itself and u and v are already dominated in the imagined game. Let S be the set of vertices dominated in the real game after the k^{th} move and S' be the set of vertices dominated in the imagined game after the $(k-1)^{th}$ move. Clearly $S' = S - w$ and the residual graph after the k^{th} move in the real game is isomorphic with the residual graph after the $(k-1)^{th}$ move in the imagined game. Staller but possibly not Dominator, is playing optimally in the real game. This implies that

$$\begin{aligned}\gamma_g(G \odot e) &\leq k + \gamma_g(G \odot e|S)\\ &= k + \gamma_g(G|S').\end{aligned}$$

Dominator, but possibly not Staller, is playing optimally in the imagined game and the k^{th} move is of Staller's. Staller skips that move in the imagined game and for a no-minus graph G we have in [37], $\gamma_g^{sp}(G) = \gamma_g(G)$. Thus

$$\begin{aligned} k - 1 + \gamma_g(G|S') &\leq \gamma_g^{sp}(G) \\ &= \gamma_g(G). \end{aligned}$$

Therefore

$$\begin{aligned} \gamma_g(G \odot e) &\leq k + \gamma_g(G|S') \\ &= k - 1 + \gamma_g(G|S') + 1 \\ &\leq \gamma_g(G) + 1. \end{aligned}$$

This concludes the proof.

The same argument also holds for staller start game domination number. □

Proposition 3.3.2. *For any $l \geq 1$, there is a no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G \odot e) = l$.*

Proof. For the case $l = 1$, consider the graph $G = K_2$. It is clear that $\gamma_g(G \odot e) = 1$ and $\gamma_g(G) = 1$.
For the case $l = 2$, consider the graph $G = K_2 \cup K_2$. It is clear that $\gamma_g(G) = 2$ and $\gamma_g(G \odot e) = 2$ for any edge e of G.
For the general case when $l \geq 3$, Consider the graph G_{l-3} in Proposition 3.2.1 and it is known that $\gamma_g(G_{l-3}) = l,\ \ l \geq 3$. The edge e as the edge joining the vertices x and y in the graph G_{i-3}. Consider the graph $G_{l-3} \odot e$ and claim that $\gamma_g(G_{l-3} \odot e) = l$. We

present a strategy for Dominator which yields $\gamma_g(G_{l-3} \odot e) \leq l$. Dominator selects his first move as u and by the Continuation Principle, v is not an optimal first move of Staller because leaves are adjacent to v. So Dominator selects v after the first move of Staller and there are at most $l-2$ vertices remain undominated. Therefore $\gamma_g(G_{l-3} \odot e) \leq 2 + l - 2 = l$. By Theorem 3.3.1 we have $l = \gamma_g(G_{l-3}) \leq \gamma_g(G_{l-3} \odot e)$. Thus $\gamma_g(G_{l-3} \odot e) = l$. □

Proposition 3.3.3. *For any $l \geq 1$, there is a no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G \odot e) = l + 1$.*

Proof. For $l \geq 1$, construct the graph G from a star $K_{1,l}$ by subdividing each edge except one. Clearly $\gamma_g(G) = l$ and let $G \odot e$ be the graph obtained from G by subdividing the remaining edge and we get $\gamma_g(G \odot e) = l + 1$. □

Note: It is obvious that $\gamma_g'(G) = 1$ if and only if G is a complete graph . So there is no graph G with $\gamma_g'(G) = \gamma_g'(G \odot e) = 1$.

Proposition 3.3.4. *For any $l \geq 2$, there is a no-minus graph G with an edge e such that $\gamma_g^{'}(G) = \gamma_g^{'}(G \odot e) = l$.*

Proof. For the general case $l \geq 2$, consider the graph $G = P_{2l-1}$. It is known that $\gamma_g'(P_{2l-1}) = l$. We know that $P_{2l-1} \odot e = P_{2l}$ for any edge e of P_{2l-1} and hence $\gamma_g'(P_{2l}) = \gamma_g'(P_{2l-1} \odot e) = l$. □

Proposition 3.3.5. *For $l \geq 1$, there is a no-minus graph G with an edge e such that $\gamma_g'(G) = l$ and $\gamma_g'(G \odot e) = l + 1$.*

Proof. For the general case $l \geq 2$, consider the graph $G = P_{2l}$. It is known that $\gamma_g'(P_{2l}) = l$. We know that $P_{2l} \odot e = P_{2l+1}$ for any edge e of P_{2l} and hence $\gamma_g'(P_{2l+1}) = \gamma_g'(P_{2l} \odot e) = l + 1$. $\square$

3.4 Edge Subdivision in General

For all no-minus graphs we have $0 \leq \gamma_g(G \odot e) - \gamma_g(G) \leq 1$ and $0 \leq \gamma_g'(G \odot e) - \gamma_g'(G) \leq 1$ by Theorem 3.3.1. But in general $0 \leq \gamma_g(G \odot e) - \gamma_g(G) \leq 2$ and $0 \leq \gamma_g'(G \odot e) - \gamma_g'(G) \leq 2$. Here we discuss all possibilities of realizing $\gamma_g(G \odot e) = \gamma_g(G) + 2$ and $\gamma_g'(G \odot e) = \gamma_g'(G) + 2$. Note that all graphs with $\gamma_g(G) \leq 2$ are no-minus graphs. Hence in the following we consider only $l \geq 3$.

Proposition 3.4.1. *For any $l \geq 3$ there is a graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G \odot e) = l + 2$.*

Proof. We present two families of graphs U_k and V_k that realize odd and even values of l respectively. Construct U_0 in the following way. Take the disjoint union of C_6 and $K_{1,2}$ having u as its centre. We get U_0 by connecting u with one of the vertices of C_6 say v. The graph U_k, $k \geq 1$ is obtained from U_0 by identifying one end vertex of $2k$ copies of P_3 with x. We set e to be the edge between u and v. See Figure 3.3.

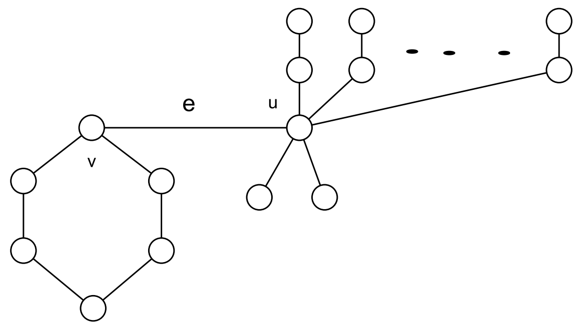

Figure 3.3: The graph U_k

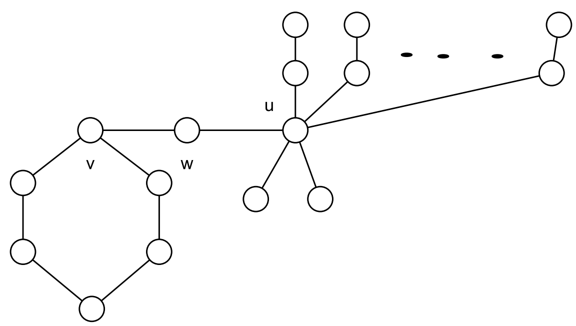

Figure 3.4: The graph $U_K \odot e$

We claim that $\gamma_g(U_k) = 2k+3$ and $\gamma_g(U_k \odot e) = 2k+5 \;\; \forall k \geq 0$. By corollary 3.1.2, it suffices to show that $\gamma_g(U_k) \leq 2k+3$ and $\gamma_g(U_k \odot e) \geq 2k+5$. First we prove $\gamma_g(U_k) \leq 2k+3$ by presenting a strategy for Dominator which ensures that the game ends with at most $2k+3$ moves. Dominator starts the game by playing the vertex x. Any move of Staller on one of the $2k$ attached paths is followed by a move of Dominator on some other path in the same 2k attached paths, so that all vertices of this $2k$ paths are dominated. Therefore Staller is forced to be the first to play in the subgraph C_6 and it is known that $\gamma_g'(C_6|y) = 2$. Hence Dominator can ensure that at most 1 + 2k + 2 = 2k + 3 moves are needed to finish the game. Thus we get $\gamma_g(U_k) \leq 2k + 3$.

Now we show that $\gamma_g(U_k \odot e) \geq 2k+5$ by presenting a strategy for Staller which ensures that at least $2k+5$ moves are needed to finish the game. We set w as the new vertex obtained due to the subdivision of the edge e which is adjacent to u and v. Whenever Dominator plays on one of the $2k$ attached paths then Staller follows a move on some other path in the $2k$ attached paths. If Dominator plays on u, then Staller responds by playing on w. On the other hand if Dominator plays on w then Staller selects a leaf adjacent to u [This is a legal move because u is not selected and the leaves which are adjacent to u are not dominated yet]. By this strategy Staller forces Dominator to be the first to play in the subgraph C_6 and it is known that $\gamma_g(C_6|y) = 3$. On the other hand if Dominator starts to play a vertex in C_6 then Staller selects a vertex adjacent to the vertex selected by Dominator in C_6 and two more vertices remain undominated in C_6. Now there are two possibilities either Dominator selects a vertex in C_6 that dominates the remaining undominated vertices in C_6 or selects a vertex from $\{u, w\}$. In the first case Staller responds by playing on w and the game is finished with $2k + 1$ moves (2k moves in the attached paths and one for x). In the other case Staller selects a vertex in C_6 which dominates only one new vertex. So one more move is needed to dominate all vertices in C_6 and $2k$ moves are needed in the $2k$ attached paths. Hence in any case at least $2k+5$ moves are needed to finish the game and thus conclude the proof.

The family V_k realizes the case when l is even. Construct V_0 in the following way. Take disjoint union of the graph F'

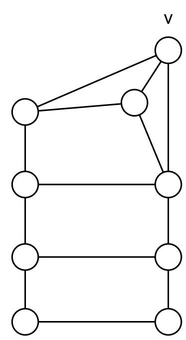

Figure 3.5: The graph F'

in Figure 3.5 and $K_{1,2}$ having u as its centre. We get V_0 by connecting one of the vertices of F' having degree two say v with u. The graph V_k, $k \geq 1$ is obtained from V_0 by identifying one end vertex of $2k$ copies of P_3 with u. We set e to be the edge between u and v. By using a similar argument in the previous case we get $\gamma_g(V_k) \leq 2k+4$ and $\gamma_g(V_k \odot e) \geq 2k+6$. $\square$

Proposition 3.4.2. *For any $l \geq 2$ there is a graph G with an edge e such that $\gamma_g'(G) = l$ and $\gamma_g'(G \odot e) = l+2$.*

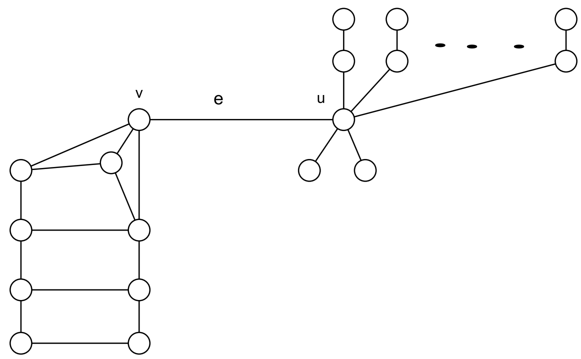

Figure 3.6: The graph V_k

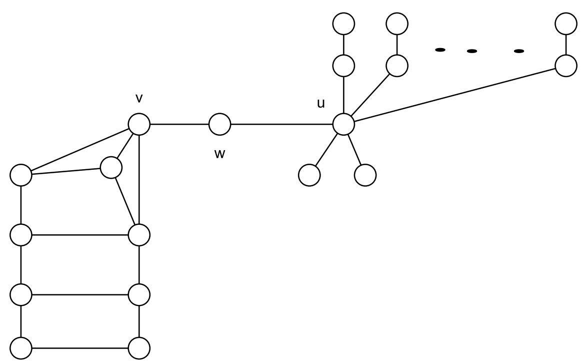

Figure 3.7: The graph $V_k \odot e$

Proof. For the case when $l = 2$, consider the Domino graph D and set the edge e as the chord. It is known that $\gamma'_g(D) = 2$ while after subdividing the edge e we get $\gamma'_g(D \odot e) = 4$.

For the case $l = 3$, Construct the graph F'' from F by attaching two vertices v' and v'' as true twins of v. It is known that $\gamma_g(F) = 4$ and $\gamma'_g(F) = 3$ and the game domination number remains the same after attaching true twins. Therefore $\gamma'_g(F) =$

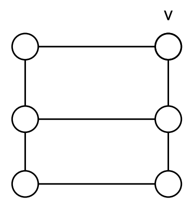

Figure 3.8: The Domino

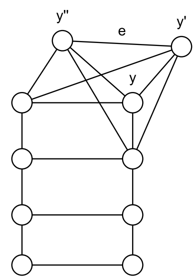

Figure 3.9: The graph F''

$\gamma_g'(F'') = 3$ and we set e as the edge between v' and v''. After subdividing the edge e we get $\gamma_g'(F'' \odot e) = 5$

For the general case $l \geq 5$, we present two different infinite families U_k and V_k realizing even and odd l respectively. By using analogous arguments as in the previous case we conclude that $\gamma_g'(U_k) = 2k+4$ and $\gamma_g'(U_k \odot e) = 2k+6$ and $\gamma_g'(V_k) = 2k+5$ and $\gamma_g'(V_k \odot e) = 2k+7$, where e is the edge joining x and y. $\square$

Chapter 4

Domination Game on Split Graphs

This chapter deals with the effect of edge removal and vertex removal on the game domination number as well as staller start game domination number of split graphs. Here we also establish the bounds for game domination number as well as staller start game domination number and prove that $\gamma_g(G) \leq \frac{n}{2}$ for an isolate free n-vertex split graph G. We also characterise split graphs of even order with $\gamma_g(G) = \frac{n}{2}$.

Split graphs can be characterized in several different ways, in particular as the graphs that contain no induced subgraphs

[0]Some results of this chapter are included in the following paper.
Tijo James, Sandi Klavzar, A. Vijayakumar, Domination game on split graphs, *Bull. Aus. Math. Soc.99(2019),327-337*

isomorphic to a graph in $\{2K_2, C_4, C_5\}$, [41]. If G is a split graph with a split partition (K, I), then a maximal clique of G is either K or it is induced with the closed neighborhood of a vertex from I. Throughout this chapter we may assume that if $(K,\ I)$ is a split partition of a split graph G, then $|K| = \omega(G)$, that is, K is the largest clique of G. We will also set $k = |K|$ and $i = |I|$ and label $K = \{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots y_i\}$. Here $deg_I(x) = |N_I(x)|$, that is, the number of vertices I which are adjacent to x in G.

4.1 Edge Removal in Split Graphs

In this section we consider the effect of edge removal on the game domination number of split graphs. In general it is known [6] that if e is an edge of a graph G, then $\gamma_g(G) - \gamma_g(G - e) \subseteq \{-2, -1, 0, 1, 2\}$ as well as $\gamma_g'(G) - \gamma_g'(G - e) \subseteq \{-2, -1, 0, 1, 2\}$. It is known by Theorem 2.1.1 that for a no-minus graph G, $\{\gamma_g(G) - \gamma_g(G - e) : e \in E(G)\} \subseteq \{-1, 0, 1\}$ and $\{\gamma_g'(G) - \gamma_g'(G - e) : e \in E(G)\} \subseteq \{-1, 0, 1\}$. Here we strengthen these bounds for split graphs as follows.

Lemma 4.1.1. *Let G be a connected split graph with $V = K \cup I$. If $e \in E(K)$, then $\gamma_g(G) \leq \gamma_g(G - e)$ and $\gamma_g'(G) \leq \gamma_g'(G - e)$.*

Proof. To prove $\gamma_g(G) \leq \gamma_g(G - e)$, it suffices to show that Dominator has a strategy on G which ensures that at most $\gamma_g(G - e)$

moves will be played in a D game on G. Consider a D game on G and at the same time Dominator imagines another D game on $G - e$. The strategy of Dominator is as follows: he copies every move of Staller in the real game to the imagined game if it is a legal move in the imagined game and copies back his optimal response in the imagined game to the real game if it is a legal move in the real game. By the Continuation Principle, Dominator prefers vertices from K. All vertices in K must be dominated after the first move of Dominator in G and all vertices in K except at most one vertex must be dominated after the first move of Dominator in $G-e$. It is clear that all moves of Staller in the real game on G is a legal move to the imagined game on $G - e$. If all moves of Dominator in the imagined game is also a legal move to the real game then the real game and the imagined game are finished at the same time. Suppose that at the k^{th} stage, Dominator's move in the imagined game is not a legal move in the real game. This is possible when K has a vertex which is not dominated after the first move of Dominator in $G - e$ and the only vertex which is newly dominated by this move of Dominator is the undominated vertex in K. Now the set of vertices dominated in G and $G - e$ are equal and is denoted by S. The residual graph after the k^{th} move of Dominator in the imagined game on $G-e$ is $G-e|S$ and the residual graph after the $(k-1)^{th}$ move of Staller in the real game on G is $G|S$. It is clear that both $G|S$ and $G-e|S$ are isomorphic. Staller plays optimally in G and possibly not by Dominator, so $\gamma_g(G) \le K - 1 + \gamma_g(G|s)$. Dominator plays optimally in the imagined game on $G - e$ and possibly not by Staller, so $k + \gamma'_g(G - e|S) \le \gamma_g(G - e)$. Thus $\gamma_g(G) \le k - 1 + \gamma_(G|S) \le k + \gamma'_g(G|S) \le \gamma_g(G - e)$.

The same arguments also hold for the staller start game domination number and hence $\gamma'_g(G) \le \gamma'_g(G - e)$. □

Lemma 4.1.2. *Let G be a connected split graph with $V = K \cup I$. If $e \in E(G)$ with one end in K and the other end in I, then $\gamma_g(G) \le \gamma_g(G - e)$ and $\gamma'_g(G) \le \gamma'_g(G - e)$.*

Proof. To prove $\gamma_g(G) \le \gamma_g(G-e)$, it suffices to show that Dominator has a strategy on G which ensures that at most $\gamma_g(G-e)$ moves will be played in a D game on G. Consider a D game on G and at the same time Dominator imagines another D game on $G - e$. The strategy of Dominator is as follows: he copies every move of Staller in the real game on G to the imagined game on $G-e$ if it is a legal move in the imagined game and copies back his optimal response in the imagined game to the real game if it is a legal move in the real game. If every move of Staller in the real game is a legal move in the imagined game and every move of Dominator in the imagined game is a legal move in the real game then the imagined game has at most one vertex (note that the end vertex of e in I may not be dominated in $G - e$) that remains to be dominated when the real game is finished. Suppose that at some stage of the real game on G, Staller chooses a vertex which is not a legal move in the imagined game on $G - e$. This is possible when Staller chooses a vertex in K and that vertex additionally dominates only the end vertex of e in I. In this case, Dominator selects the end vertex of e in I for the corresponding move of Staller in the imagined game instead of copying the move in the real game and continues the game. In the above cases the real game has no more moves than the imagined game. It is noted that Staller plays optimally in the real game and possibly not Dominator, so $\gamma_g(G) \leq \gamma_g(G - e)$.

Again suppose that at the k^{th} move of Dominator in the imagined game is not a legal move in the real game. This is possible when Dominator chooses the end vertex of e in I which is already dominated in the real game on G. In this case the set of vertices Dominated in the real game after the $(k-1)^{th}$ move and the set of vertices dominated in the imagined game after the k^{th} move of Dominator in the imagined game are same. Let S be the set of vertices dominated after the $(k)^{th}$ move of Dominator in the imagined game on $G - e$. Thus the residual graph $G|S$ after the $(k-1)^{th}$ move of Staller in the real game and the residual graph $G - e|S$ after the $(k)^{th}$ move of Dominator in the imagined game are isomorphic. It is noted that Staller plays optimally in the real game and possibly not

Dominator, so $\gamma_g(G) \leq k - 1 + \gamma_g(G|S)$. Also Dominator plays optimally in the imagined game and possibly not Staller, so $\gamma_g(G-e) \geq k+\gamma'_g(G-e|S)$. Therefore $\gamma_g(G) \leq k-1+\gamma_g(G|S) \leq k + \gamma'_g(G|S) = k + \gamma'_g(G - e|S) \leq \gamma_g(G - e)$. $\square$

Theorem 4.1.3. *If G is a connected split graph, then $\{\gamma_g(G) - \gamma_g(G - e) : e \in E(G)\} \subseteq \{-1, 0\}$ and $\{\gamma'_g(G) - \gamma'_g(G - e) : e \in E(G)\} \subseteq \{-1, 0\}$.*

Proof. It is known by Theorem 2.1.1 that for a no-minus graph G, $\{\gamma_g(G) - \gamma_g(G - e) : e \in E(G)\} \subseteq \{-1, 0, 1\}$ and $\{\gamma'_g(G) - \gamma'_g(G - e) : e \in E(G)\} \subseteq \{-1, 0, 1\}$. Each edge of a split graph G has either both end vertices are in K or one end vertex in K and the other in I. Hence the proof follows from Lemma 4.1.1 and Lemma 4.1.2. $\square$

4.2 Vertex Removal in Split Graphs

In this section we consider the effect of vertex removal on the game domination number of split graphs. In general it is known [6] that if v is a vertex of a graph G, then $\gamma_g(G) - \gamma_g(G - v) \leq 2$ as well as $\gamma'_g(G) - \gamma'_g(G - v) \leq 2$. Here we strengthen these bounds for split graphs as follows.

Lemma 4.2.1. *If G is a connected split graph and $y \in I$, then $\gamma_g(G) - 1 \leq \gamma_g(G - y) \leq \gamma_g(G)$ and $\gamma'_g(G) - 1 \leq \gamma'_g(G - y) \leq \gamma'_g(G)$.*

Proof. To prove the bound $\gamma_g(G-y) \leq \gamma_g(G)$, it suffices to show that Dominator has a strategy on $G-y$ such that at most $\gamma_g(G)$ moves will be played in a D game on $G-y$. His strategy is to play the game on $G-y$ as follows. In parallel to the real game, he is playing an imagined game on G by copying every move of Staller to this game and responds optimally in G. Each response in the imagined game is then copied back to the real game played on $G-y$.

Note that every move of Staller in the real game on $G-y$ is a legal move in the imagined game on G. Let a and b be the number of moves played on $G-y$ and on G, respectively. Suppose first that all the moves of Dominator in the imagined game (played on G) are legal moves in the real game (played on $G-y$). Then the real game will end in no more moves as the imagined game, that is, $a \leq b$. Since Staller is playing optimally in the real game (Dominator may not), $\gamma_g(G-y) \leq a$. On the other hand, since Dominator is playing optimally in the imagined game (Staller may not), $b \leq \gamma_g(G)$. Hence $\gamma_g(G-y) \leq a \leq b \leq \gamma_g(G)$.

Suppose now that at some stage of the game Dominator's move on G is not a legal move on $G-y$, let this be the r^{th} move of the game in G. This can happen only in the case when y is the only newly dominated vertex in that move of Dominator in the imagined game. Let X denote the set of vertices of G which are already dominated after the r^{th} move in the imagined game. So $r + \gamma_g'(G|X) \leq \gamma_g(G)$. (This inequality holds because Staller does not necessarily play optimally in the imagined game.) Now Dominator is not able to copy his optimal move from the imagined game on G to the real game on $G - y$. The number of moves in the real game to this move is $r - 1$ and it is now the chance of Dominator to play. The set of vertices dominated at this stage in the real game on $G - y$ is $X - y$. This gives the first equality in the following estimation:

$$\begin{aligned}
\gamma_g(G - y) &\leq r - 1 + \gamma_g(G - y|X - y) \\
&= r - 1 + \gamma_g(G|X) \\
&\leq r - 1 + (\gamma_g'(G|X) + 1) \\
&= r + \gamma_g'(G|X) \\
&\leq \gamma_g(G)
\end{aligned}$$

Hence in any case, $\gamma_g(G - y) \leq \gamma_g(G)$.

To prove the bound $\gamma_g(G) - 1 \leq \gamma_g(G - y)$ it suffices to show that Dominator has a strategy on G such that at most $\gamma_g(G - y) + 1$ moves will be played. His strategy is to play the game on G as follows. In parallel to the real game, he is playing an imagined game on $G - y$ by copying every move of Staller to this game and responds optimally in $G - y$. Each response in the imagined game is then copied back to the real game in G.

Every move of Dominator in the imagined game on $G - y$ is a legal move in the real game on G. If in addition all the moves of Staller in the real game on G are legal moves in the imagined game on $G - y$, then since Dominator plays optimally on $G - y$ but Staller might not, and since the number of moves played in the real game is the same as in the imagined game, we infer (analogously as above) that $\gamma_g(G) \leq \gamma_g(G - y)$.

Suppose now that the r^{th} move is played in the real game, which is a Staller's move, and this is not a legal move on $G - y$. This can happen only if y is the only newly dominated vertex in the r^{th} move in G. Let X denote the set of vertices of G which are already dominated after the r^{th} move in G, so that $\gamma_g(G) \leq r + \gamma_g(G|X) = r + \gamma_g(G - y|X - y)$. (Again we have used the fact that Dominator may not be playing optimally in the real game.) Now Dominator does not copy the optimal move of Staller from the real game to $G - y$ since it is not legal. In the imagined game $r - 1$ moves were played so far and it is Staller's turn. Clearly, the set of vertices dominated at this stage in the imagined game on $G - y$ is $X - y$. Therefore

$$
\begin{aligned}
\gamma_g(G - y) &\geq r - 1 + \gamma_g^{'}(G - y|X - y) \\
&\geq r - 1 + \gamma_g(G - y|X - y) \\
&\geq (r - 1) + (\gamma_g(G) - r) \\
&= \gamma_g(G) - 1\,.
\end{aligned}
$$

Hence $\gamma_g(G) - 1 \le \gamma_g(G - y)$.

The above arguments are independent of who moves the first. Therefore a similar result holds also for the S-game. $\square$

Proposition 4.2.1 cannot be extended to vertices from the clique K. For instance, consider the (split) graph $K_{1,n}$. Then $\gamma_g(K_{1,n}) = 1$ but $\gamma_g(K_{1,n} - x) = n$, where x is the degree n vertex. A more interesting (connected) example is the following. Let G'_k be the split graph obtained from G_k (the latter being defined at the end of Section 4.3) by adding the edges $x_1y_2, \ldots, x_1y_k$. Then $\gamma_g(G'_k) = 1$ because x_1 is a universal vertex. On the other hand, $G'_k - x_1 = G_{k-1}$ and thus $\gamma(G'_k - x_1) = k - 1$. But we can still state the following result. $\square$

Lemma 4.2.2. *If G is a connected split graph with at least two vertices and x is a vertex of K with $deg_I(x) = 0$, then $\gamma_g(G - x) = \gamma_g(G)$.*

Proof. By the Continuation Principle, the first move of Dominator in G or in $G - x$ is a vertex from $K - x$. After such a move the residual graph is the same in both cases, hence the assertion. $\square$

Lemma 4.2.3. *If G is a connected split graph with at least two vertices and $x \in K$, then $\gamma_g(G) \le \gamma_g(G - x)$ and $\gamma_g'(G) \le \gamma_g'(G - x)$.*

Proof. To prove the bound $\gamma_g(G) \leq \gamma_g(G-x)$ it suffices to show that Dominator has a strategy on G such that at most $\gamma_g(G-x)$ moves will be played. His strategy is to play the game on G as follows. In parallel to the real game, he is playing an imagined game on $G - x$ by copying every move of Staller to this game and respond optimally in $G - x$. Each response in the imagined game is then copied back to the real game played on G.

By the Continuation Principle, Dominator prefers to select vertices from K. This implies that every move of Dominator in the imagined game on $G-x$ is a legal move to the real game on G. Every move of Staller in the real game on G except the vertex x is a legal move to the imagined game on $G-x$. It is to be noted that $deg_I(x) > 0$ otherwise by Lemma4.2.2 that $\gamma_g(G - x) = \gamma_g(G)$. So by the Continuation Principle Staller prefers to select a vertex adjacent to y in I to x. Thus every move of Staller in the real game is legal to the imagined game and every move of Dominator in the imagined game is legal to the real game. So the real game on G has no more moves than the imagined game on $G - x$. Let the real game has a moves and imagined game has b moves and hence $a \leq b$. Staller plays optimally in the real game and Dominator possibly not, therefore $\gamma_g(G) \leq a$. Also Dominator plays optimally in the imagined game on $G - x$ and possibly not Staller, therefore $b \leq \gamma_g(G - x)$. Thus $\gamma_g(G) \leq a \leq b \leq \gamma_g(G-x)$. The above arguments also valid for S game. Therefore

$\gamma_g'(G) \leq \gamma_g'(G - x)$. $\square$

Theorem 4.2.4. *If v is a vertex of a connected split graph G, then $\gamma_g(G) - \gamma_g(G - v) \leq 1$ and $\gamma_g'(G) - \gamma_g'(G - v) \leq 1$.*

Proof. A vertex of a split graph G is either in K or in I. Therefore by Lemma 4.2.1 and Lemma 4.2.3 we get $\gamma_g(G) - \gamma_g(G-v) \leq 1$ and $\gamma_g'(G) - \gamma_g'(G - v) \leq 1$. $\square$

4.3 The $1/2$ Upper Bound

In general the game domination number is bounded in terms of its order. It is proved [50] that the game domination number of an isolate free graph on n vertices is at most $\left\lceil \frac{7n}{10} \right\rceil$. W. B. Kinnersley, D. B. West, and R. Zamani posed a conjecture in [50] that the game domination number of an isolate free graph is at most $\frac{3n}{5}$. In this section we first prove that the game domination number of an isolate free split graph is $1/2$ of its order and then the bound for the S-game. At the end the sharpness of both the bounds is demonstrated.

Theorem 4.3.1. *If G is a connected split graph with $n(G) \geq 2$, then $\gamma_g(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$.*

Proof. The proof is by induction on $n(G)$. We first check the cases when $2 \leq n(G) \leq 5$. If $n(G) = 2$, then $G = K_2$, and if $n(G) = 3$, then $G \in \{K_3, P_3\}$. For all these three (split) graphs the assertion clearly holds. From [50, Proposition 5.3] we recall that if G is a (partially dominated, isolate-free) chordal graph, then $\gamma_g(G) \leq 2n(G)/3$. As split graphs are chordal, the same conclusion holds for split graphs. Hence, if $n(G) = 4$, then $\gamma_g(G) \leq 2n(G)/3 = 8/3$, that is, $\gamma_g(G) \leq 2$. Suppose finally that $n(G) = 5$. If $k = 2$, then since G is connected, at least one of the vertices, say x_1, of K has at least two neigbors in I. Then the move $d_1 = x_1$ yields $\gamma_g(G) \leq 2$. If $k = 3$, then Dominator starts the game with $d_1 = x_1$ where x_1 is a vertex of K having at least one neighbor in I. If the game is not finished yet, then Staller must finish the game in her first move by dominating the only undominated vertex in I. Hence again $\gamma_g(G) \leq 2$. Finally, if $k \in \{4, 5\}$, then $\gamma_g(G) = 1$. This proves the basis of the induction.

Assume now that the result is true for all split graphs up to and including $n - 1$ vertices, where $n \geq 6$. We distinguish two cases. **Case 1**: $deg_I(x_r) \leq 1$, $r \in [k]$.

In this case we clearly have $|I| \leq |K|$. If $i = 0$, then $G = K_k$ and the assertion is clear. Otherwise, let Dominator start the game by playing a vertex of K with a neighbor in I. Then, in every subsequent move (either by Staller or by Dominator), exactly one new vertex (in I) will be dominated. It follows that

$\gamma_g(G) = |I|$. Consequently,

$$\gamma_g(G) = |I| = \frac{|I| + |I|}{2} \leq \frac{|K| + |I|}{2} = \frac{n(G)}{2}\,.$$

Case 2: $deg_I(x_r) \geq 2$, for some $r \in [k]$.

We may without loss of generality assume that $x_1y_1, x_1y_2 \in E(G)$. The initial strategy of Dominator is to play $d_1 = x_1$. After that Staller selects a vertex optimally which means that she plays y_s, where $s \notin [2]$, unless, of course, the game is over after the move $d_1 = x_1$. (We note that because of the Continuation Principle if $N[x] \subseteq N[w]$ and both x and w are legal moves, we may assume Staller will play x over w.) Set $Z = \{x_1, y_1, y_2, y_s\}$. Then, since Staller has played optimally (Dominator may not), after the first two moves we have,

$$\gamma_g(G) \leq 2 + \gamma_g\left(G\middle| \cup_{z\in Z} N[z]\right)\,.$$

Set $G' = G \setminus \{x_1, y_1, y_2, y_s\}$. After x_1 and y_s have been played, the vertices x_1, y_1, y_2, and y_s are saturated. Therefore, by the Continuation Principle,

$$\gamma_g\left(G\middle| \cup_{z\in Z} N[z]\right) \leq \gamma_g(G')\,.$$

Since $n(G') = n(G) - 4$, we can combine the above two inequalities with the induction hypothesis into

$$\begin{aligned}
\gamma_g(G) &\le 2 + \gamma_g\left(G\middle| \cup_{z\in Z} N[z]\right) \\
&\le 2 + \gamma_g(G') \\
&\le 2 + \left\lfloor \frac{n(G)-4}{2} \right\rfloor \\
&= \left\lfloor \frac{n(G)}{2} \right\rfloor
\end{aligned}$$

and we are done. □

The assumption in Theorem 4.3.1 that G is connected is essential. For instance, for the complement $\overline{K}_n$ of K_n (both of these graphs being split graphs) we have $\gamma_g(\overline{K}_n) = n$.

Combining Theorem 4.3.1 with $|\gamma_g(G) - \gamma_g'(G)| \le 1$ we get that if G is a connected split graph with $n(G) \ge 2$, then

$$\gamma_g{}'(G) \le \gamma_g(G) + 1 \le \left\lfloor \frac{n(G)}{2} \right\rfloor + 1 = \left\lfloor \frac{n(G)+2}{2} \right\rfloor . \quad (4.3.1.1)$$

To slightly improve this bound, we first show the following:

Lemma 4.3.2. *Let G be a connected split graph. If there exists a vertex $x_r \in K$ with $deg_I(x_r) = 0$, then x_r is an optimal first move of Staller in S-game.*

Proof. Suppose that $s_1' = x_r$. Then Dominator has an optimal reply in K, say $d_1' = x_s$, $s \ne r$. Indeed, the Continuation Principle implies that if $d_1' = y_t \in I$, then any neighbor of

y_t is at least as good for Dominator as y_t. After the moves $s'_1 = x_r$ and $d'_1 = x_s$ are played, the set of vertices dominated is $X = K \cup N_G(x_s)$. Hence if Staller had played some other vertex, Dominator can still play x_s, unless Staller played x_s. In any case, if Y is the set of vertices dominated after such two moves, then $X \subseteq Y$. By the Continuation Principle it follows that $s'_1 = x_r$ is an optimal move. $\square$

Now we can improve (4.3.1.1) as follows:

Theorem 4.3.3. *If G is a connected split graph with $n(G) \geq 2$, then $\gamma'_g(G) \leq \left\lfloor \frac{n(G)+1}{2} \right\rfloor$.*

Proof. The assertion is clearly true for K_2, hence we may assume in the rest that $n(G) \geq 3$. By Lemma 4.3.2 and the Continuation Principle, Staller's first move s'_1 is either a vertex of I, or a vertex from K with no neighbour in I. Let $G' = G \setminus s'_1$. Clearly, G' is a connected split graph with $n(G') = n(G) - 1 \geq 2$, hence from Theorem 4.3.1 we get $\gamma_g(G') \leq \lfloor (n(G)-1)/2 \rfloor$. Therefore, applying the Continuation Principle again, we have

$$\begin{aligned}\gamma'_g(G) &= 1 + \gamma_g(G|N[s'_1]) \\ &\leq 1 + \gamma_g(G') \\ &\leq 1 + \left\lfloor \frac{n(G)-1}{2} \right\rfloor \\ &= \left\lfloor \frac{n(G)+1}{2} \right\rfloor\end{aligned}$$

as claimed. $\square$

In view of Theorem 4.3.1 we say that G is a 1/2-*split graph* if $\gamma_g(G) = \lfloor n(G)/2 \rfloor$. To conclude the section we present two families of 1/2-split graphs.

Let G_k, $k \geq 2$, be the split graph with the split partition (K, I), where $K = \{x_1, \ldots, x_k\}$ and $I = \{y_1, \ldots, y_k\}$ (that is, $i = k$), and where $x_r y_r$, $r \in [k]$, are the only edges between K and I. Then it is straightforward to see that $\gamma_g(G_k) = {\gamma_g}'(G_k) = k$, that is, G_k is a 1/2-split graph and the bounds of Theorems 4.3.1 and 4.3.3 cannot be improved in general.

The above graphs G_k are of even order, hence the bounds of Theorems 4.3.1 and 4.3.3 are the same. Let next H_k, $k \geq 2$, be a split graph obtained from G_k by adding one more vertex y_{k+1} to I and the edge $x_k y_{k+1}$. Then $deg_I(x_k) = 2$. From Dominator's first move $d_1 = x_k$ in D-game and Staller's first move $s_1' = y_{k+1}$ in S-game we respectively infer that $\gamma_g(H_k) = k$ and ${\gamma_g}'(H_k) = k + 1$. These values again achieve the upper bounds in the respective theorems.

4.4 1/2-Split Graphs of Even Order

We now characterize the 1/2-split graphs that have even order. In the following two lemmas we first exclude split graphs that are not such.

Lemma 4.4.1. *Let G be a connected split graph of even order and suppose that at least one of the following conditions is fulfilled:*

(i) $i < k$;

(ii) $i > 2k$;

(iii) there exists a vertex $x_r \in K$ *with* $deg_I(x_r) = 0$;

(iv) there exists a vertex $x_r \in K$ *with* $deg_I(x_r) \geq 3$;

(v) there exist $x_r, x_s \in K$ *with* $deg_I(x_s) = 2$ *and* $N_I(x_r) \subseteq N_I(x_s)$.

Then G is not a $1/2$*-split graph.*

Proof. In view of Theorem 4.3.3 we need to show that if one of the conditions (i)-(v) holds, then $\gamma_g(G) < \left\lfloor \frac{n(G)}{2} \right\rfloor$.

(i) Suppose $i < k$. Let Dominator start the game by playing a vertex $x_r \in K$ with at least one neighbor in I. After this move the vertices left undominated are $X = I \setminus N_I(x_r)$. Clearly, $|X| \leq i - 1$. Since in the rest of the game at least one new vertex is dominated on each move, $\gamma_g(G) \leq 1 + (i-1) = i < (k+i)/2 = n(G)/2 = \lfloor n(G)/2 \rfloor$.

(ii) Assume $i > 2k$. Then there exists a vertex $x_r \in K$ with $deg_I(x_r) \geq 3$. Let Dominator start a D-game with $d_1 = x_r$, and let Staller reply with an optimal move. After these two moves the graph G' obtained from G by removing all saturated vertices is again a connected partially dominated split graph with at most $n(G)-5$ vertices. Indeed, G' does not contain $d_1 = x_r$, the neighbors of x_r in I (at least three of them), and s_1. Therefore, $\gamma_g(G) \leq 2+\gamma_g(G') \leq 2+(n(G)-5)/2 = (n(G)-1)/2 < n(G)/2$ $= \lfloor n(G)/2 \rfloor$ where the second inequality holds by Theorem 4.3.1.

(iii) Suppose that there exists a vertex $x_r \in K$ with $deg_I(x_r) = 0$. Because of (i) we can assume that $k \leq i$. Therefore, since $deg_I(x_r) = 0$, there exists a vertex $x_s \in K$ with $deg_I(x_s) \geq 2$. Let Dominator start the game by playing $d_1 = x_s$. Then, after the first move of Staller, the graph G' obtained from G by removing all saturated vertices is a connected partially dominated split graph with at most $n(G) - 5$ vertices because it does not contain $d_1 = x_s$, the neighbors of x_s in I (at least two of them), the first move of Staller s_1, and x_r. The conclusion now follows by the same argument as in (ii).

(iv) If there exists a vertex $x_r \in K$ with $deg_I(x_r) \geq 3$, then after Dominator plays x_r and Staller an arbitrary (optimal) move, we again have a connected partially dominated split graph with at most $n(G) - 5$ vertices after removing all saturated vertices.

(v) Let Dominator start the game by playing $d_1 = x_s$. Then x_r, x_s, and the two neighbors of x_s in I have no role in the continuation of the game. So again, after the first move of Staller, removing all saturated vertices from G we have a partially dominated connected split graph of order at most $n(G) - 5$. $\square$

Lemma 4.4.2. *If G is a connected split graph of even order and there exists a vertex in K which is not adjacent to a leaf in I, then $\gamma_g(G) < \lfloor n(G)/2 \rfloor$.*

Proof. Let $x_1 \in K$ be a vertex that is not adjacent to a leaf in I. If $deg_I(x_1) \geq 3$, then we are done by Lemma 4.4.1(iv).

Suppose next that $deg_I(x_1) = 1$. Let y_1 be the vertex of I adjacent to x_1. Since y_1 is not a leaf, we may assume that $x_2 \in K$ is another neighbor of y_1. If $deg_I(x_2) \geq 2$, then we are done by Lemma 4.4.1(iv) and (v). Suppose therefore that $deg_I(x_2) = 1$. Then $N[x_1] = N[x_2]$, hence by [8, Proposition 1.4] we have $\gamma_g(G) = \gamma_g(G|x_1) = \gamma_g(G - x_1)$. Therefore, by Theorem 4.3.1 and the fact that n is even,

$$\gamma_g(G) = \gamma_g(G - x_1) \leq \lfloor (n(G) - 1)/2 \rfloor < \lfloor n(G)/2 \rfloor .$$

The remaining case to be considerd is that $deg_I(x_1) = 2$. Let $y_1, y_2 \in I$ be the neighbors of x_1 in I. Recall that by our assumption y_1 and y_2 are not pendant vertices. If y_1 and y_2 have a common neighbor x_r in K, $r \neq 1$, then in view of Lemma 4.4.1(iv) we may assume that $deg_I(x_r) = 2$, but then $N_I(x_1) \subseteq N_I(x_r)$ and we are done by Lemma 4.4.1(v). It follows that there exist vertices $x_2, x_3 \in K$ such that x_2 is adjacent to y_2 and x_3 is adjacent to y_1. Using Lemma 4.4.1(v) again we see that $deg_I(x_2) = deg_I(x_3) = 2$. Let y_3 and y_4 be the other neighbors in I of x_3 and x_2, respectively. Let $Z = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and let G_1 and G_2 be the subgraphs of G induced by Z and $V(G) \setminus Z$ respectively. Clearly, G_1 is a connected split graph. The same holds for G_2 unless it is the empty graph. It can be easily verified that $\gamma_g(G_1) = \gamma_g'(G_1) = 3$. Hence by [37, Theorem 2.7] they are no-minus graphs with $\gamma_g(G_1) = \gamma_g'(G_1)$ and hence by [37, Theorem 2.11] we have $\gamma_g(G_1 \cup G_2) = \gamma_g(G_1) + \gamma_g(G_2)$. Moreover, by Theorem 4.3.1 and because n is even we have

$$\gamma_g(G_2) \leq \lfloor (n(G) - 7)/2 \rfloor = (n(G) - 8)/2$$

and consequently

$$\gamma_g(G_1 \cup G_2) \leq 3 + (n(G) - 8)/2 = (n(G) - 2)/2 < \lfloor n(G)/2 \rfloor \ .$$

The argument will be complete by proving that $\gamma_g(G) \leq \gamma_g(G_1 \cup G_2)$. For this sake we proceed by the imagination strategy as follows. Consider a real D-game played on G and at the same time Dominator imagines a D-game played on $G_1 \cup G_2$. Dominator plays optimally in the game on $G_1 \cup G_2$ and copies his moves from there to the real game on G. On the other hand, Staller plays optimally in the real game on G (this is the only game being played by Staller), and Dominator copies each move of Staller to the imagined game. Since a D-game is played in both games, Dominator will first play a vertex of K in the real game which is played on G. Hence every move of Staller will be a vertex from I, thus newly dominating only this vertex. It follows that every move of Staller in the real game is a legal move in the imagined game. On the other hand, a legal move of Dominator in the imagined game may not be legal in the real game. If this happens, Dominator cannot copy this move to the real game; instead, he selects an arbitrary legal move in the real game (if there is such a move available, otherwise the game is over). Under this strategy, the set of vertices dominated in the imagined game is always a subset of the set of vertices dominated in the real game. Hence, if s is the number of moves played in the real game and t the number of moves in the imagined game, then $s \leq t$. Moreover, since Dominator may not play optimally on G (but Staller does), we have $\gamma_g(G) \leq s$. Similarly, as Dominator plays optimally on $G_1 \cup G_2$, we infer that $\gamma_g(G_1 \cup G_2) \geq t$. Therefore, $\gamma_g(G) \leq s \leq t \leq \gamma_g(G_1 \cup G_2)$ which completes the argument. □

Theorem 4.4.3. *A connected split graph of even order is a* $1/2$-*split graph if and only if every vertex in* K *is adjacent to at least one leaf in* I *and* $deg_I(x_i) \in [2]$ *for* $i \in [k]$.

Proof. Suppose that $\gamma_g(G) = \lfloor n(G)/2 \rfloor$. Then by Lemma 4.4.2

every vertex of K is adjacent to at least one leaf in I and by Lemma 4.4.1(iii) and (iv), $deg_I(x_i) \in [2]$ for every vertex $x_i \in K$.

Conversely, suppose that G is a connected split graph of even order in which every vertex in K is adjacent to at least one leaf in I and $deg_I(x_i) \in [2]$ for $i \in [k]$. By Theorem 4.3.1 we need only to prove that Staller has a strategy that guarantees that a D-game will last at least $\lfloor n(G)/2 \rfloor$ moves. After each move we consider that the resulting graph is a partially dominated graph without saturated vertices. The corresponding Strategy of Staller is the following.

First, In Phase 1, she selects vertices which are not pendant vertices in I. After this is no longer possible for Staller, Phase 1 is over and Phase 2 begins. At that time the vertices from I that are not yet dominated are pendent vertices. In Phase 2 Staller selects pendent vertices which are neighbors of degree-2 vertices from K as long as this is possible. Phase 3 starts when the only not yet dominated vertices from I are those that are adjacent to vertices of K with exactly one neighbor in I.

Consider the number of saturated vertices during this game. Since $deg_I(x_i) \in [2]$, $i \in [k]$, after each move of Dominator in Phases 1 and 2 the number of newly saturated vertices is at most three. By the strategy of Staller, after each of her moves in these two phases the number of saturated vertices increases by exactly one. Suppose that Phase 2 is finished with the k^{th}

move of Staller. Then the number of saturated vertices is at most $3k + k = 4k$. If there are l vertices in Phase 3 yet to be dominated, then the game is finished by the next l moves. After each such move, no matter whether it was done either by Dominator or Staller, two newly saturated vertices are created and therefore $n(G) \leq 4k + 2l$. The described strategy of Staller may not be optimal, hence
$\gamma_g(G) \geq 2k + l = \frac{2(2k+l)}{2} \geq \frac{n(G)}{2} = \left\lfloor \frac{n(G)}{2} \right\rfloor$.

Suppose next that Phase 2 is finished with the k^{th} move of Dominator. In this case the number of saturated vertices at this stage of the game at most $3k + k - 1 = 4k - 1$. Let again l be the number of vertices yet to be dominated in Phase 3. Then the number of not yet saturated vertices is exactly $2l$. Since G is of even order, the number of vertices already saturated is at most $4k - 2$. Hence $n(G) \leq 4k - 2 + 2l$ and therefore

$$\begin{aligned}
\gamma_g(G) &\geq (2k-1) + l \\
&= \frac{2(2k-1+l)}{2} \\
&= \frac{4k-2+2l}{2} \\
&\geq \frac{n(G)}{2} \\
&= \left\lfloor \frac{n(G)}{2} \right\rfloor
\end{aligned}$$

and we are done. $\square$

The study of odd order split graphs is in progress.

Chapter 5

Domination Game on Mycielskian of a Graph

In this chapter we establish bounds for the game domination number of Mycielskian of a graph in terms of its domination number and the game domination number. We characterise Mycielskian of a graph (see Definition 1.1.7) with small game domination number.

5.1 Bounds for the Game Domination Number of Mycielskian of a Graph

First we prove bounds for the game domination number of Mycielskian of a graph in terms of its domination number.

Theorem 5.1.1. *For any graph G,*

$$1 + \gamma(G) \leq \gamma_g(\mu(G)) \leq 2\gamma(G) + 1.$$

Proof. It is known [2] that for any graph G, we have $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$. So this inequality is true for $\mu(G)$ and hence $\gamma(\mu(G)) \leq \gamma_g(\mu(G)) \leq 2\gamma(\mu(G)) - 1$. Also it is known [16] that $\gamma(\mu(G)) = 1 + \gamma(G)$. Therefore,

$$1 + \gamma(G) \leq \gamma_g(\mu(G)) \leq 2\gamma(G) + 1.$$

□

Consider the graph K_2 and $\mu(K_2) \cong C_5$. It is clear that $\gamma(K_2) = 1$ and $\gamma_g(\mu(K_2)) = \gamma_g(C_5) = 3$. So K_2 is an example of a graph whose Mycielskian attains the above upper bound for the game domination number. It can be easily verified that C_4 is an example of a graph whose Mycielskian attains the above lower bound for the game domination number.

Theorem 5.1.2. *For any graph G,*

$$1 + \gamma(G) \leq \gamma'_g(\mu(G)) \leq 2\gamma(G) + 2.$$

Proof. It is known [2] that for any graph G, we have $\gamma(G) \leq \gamma'_g(G) \leq 2\gamma(G)$. So this inequality is true for $\mu(G)$ and hence $\gamma(\mu(G)) \leq \gamma'_g(\mu(G)) \leq 2\gamma(\mu(G))$. Also it is known [16] that

$\gamma(\mu(G)) = 1 + \gamma(G)$. Therefore,

$$1 + \gamma(G) \leq \gamma'_g(\mu(G)) \leq 2\gamma(G) + 2.$$

□

Consider the graph K_2 and $\mu(K_2) \cong C_5$. It is clear that $\gamma(K_2) = 1$ and $\gamma'_g(\mu(K_2)) = \gamma'_g(C_5) = 2$. So K_2 is an example of a graph whose Mycielskian attains the above lower bound for the staller start game domination number.

It is proved in [8] that the game domination number remains the same after attaching a true twin. Now we describe how the game domination number changes after attaching a false twin. Let G_u be the graph obtained by attaching a false twin u' of u in G. That is, $V(G_u) = V(G) \cup \{u'\}$ and $E(G_u) = E(G) \cup \{u'v | v \in N(u)\}$.

Lemma 5.1.3. *For any graph G,*

$$\gamma_g(G) \leq \gamma_g(G_u|u') \quad \textit{and} \quad \gamma'_g(G) \leq \gamma'_g(G_u|u').$$

Proof. First we prove that $\gamma_g(G) \leq \gamma_g(G_u|u')$. It is enough to show that there is a strategy for Dominator on G which ensures that a D game on G has at most $\gamma_g(G_u|u')$ moves. Dominator imagines a D game on $G_u|u'$ when a real D game is played on G.

The strategy of Dominator is as follows: he copies every move of Staller in the real game on G to the imagined game if it is legal and responds optimally in the imagined game. He copies back his move in the imagined game to the real game on G if it is legal. Every move of Staller in the real game on G is a legal move in the imagined game on $G_u|u'$ since the set of vertices that are newly dominated in G with a move are also newly dominated in $G_u|u'$. By the Continuation Principle, Dominator prefers to select u in $G_u|u'$ instead of u' in $G_u|u'$. So every move of Dominator in the imagined game is also a legal move in the real game. Thus the real game ends when the imagined game is over. It is noted that Staller plays optimally in the real game and Dominator plays optimally in the imagined game. Hence $\gamma_g(G) \leq \gamma_g(G_u|u')$.

The same arguments also hold for the staller start game domination number and hence $\gamma_g'(G) \leq \gamma_g'(G_u|u')$. □

By the Continuation Principle and Lemma 5.1.3 we get the following theorem.

Theorem 5.1.4. *For any graph G,*

$$\gamma_g(G) \leq \gamma_g(G_u) \quad \textit{and} \quad \gamma_g'(G) \leq \gamma_g'(G_u).$$

It is known that removing a vertex from a graph can either increase its game domination number or decrease it by at most two. It is clear by Theorem 5.1.4 that removing a false twin from a graph never increases its game domination number.

Theorem 5.1.5. *For any graph G,*

$$\gamma_g(G) \leq \gamma_g(\mu(G)) \leq 2\gamma_g(G) + 1.$$

Proof. For any graph G, we have

$$\gamma_g(\mu(G)) \leq 2\gamma(\mu(G)) - 1 = 2\gamma(G) + 1 \leq 2\gamma_g(G) + 1.$$

Now we have to prove the other part $\gamma_g(G) \leq \gamma_g(\mu(G))$. The graph $\mu(G) - w$ is obtained from G by attaching a false twin to each vertex of G . So by applying Lemma 5.1.3 recursively we get $\gamma_g(G) \leq \gamma_g(\mu(G) - w|N[w])$. The game domination number does not change by the removal of a vertex w whose $N[w]$ is already dominated. Therefore $\gamma_g(\mu(G) - w|N[w]) = \gamma_g(\mu(G)|N[w])$ and by the Continuation Principle, $\gamma_g(\mu(G)|N[w]) \leq \gamma_g(\mu(G)$. Hence $\gamma_g(G) \leq \gamma_g(\mu(G))$. □

Theorem 5.1.6. *For any graph G,*

$$\gamma_g'(G) \leq \gamma_g'(\mu(G)) \leq 2\gamma_g'(G) + 2.$$

Proof. For any graph G, we have

$$\gamma_g'(\mu(G)) \leq 2\gamma(\mu(G)) = 2\gamma(G) + 2 \leq 2\gamma_g'(G) + 2.$$

Now we have to prove the other part $\gamma_g'(G) \leq \gamma_g'(\mu(G))$. The graph $\mu(G) - w$ is obtained from G by attaching a false twin to each vertex of G. So by applying Lemma 5.1.3 recursively we get $\gamma_g'(G) \leq \gamma_g'(\mu(G) - w|N[w])$. The staller start game domination number does not change by the removal of a vertex w whose $N[w]$ is already dominated. Therefore

$$\gamma_g'(\mu(G) - w|N[w]) = \gamma_g'(\mu(G)|N[w])$$

and by the Continuation Principle $\gamma_g'(\mu(G)|N[w]) \leq \gamma_g'(\mu(G)$. Hence $\gamma_g'(G) \leq \gamma_g'(\mu(G))$. □

5.2 Mycielskian of a Graph with Small Game Domination Number

Theorem 5.2.1. *There is no graph G with $\gamma_g(\mu(G)) = 1$.*

Proof. It is known by Theorem 5.1.1 that $\gamma_g(\mu(G)) \geq 1+\gamma(\mu(G))$. Therefore $\gamma_g(\mu(G))$ is at least 2. □

Theorem 5.2.2. *There is no graph G with $\gamma_g'(\mu(G)) = 1$.*

Proof. It is known by Theorem 5.1.2 that $\gamma_g'(\mu(G)) \geq 1+\gamma(\mu(G))$. Therefore $\gamma_g'(\mu(G))$ is at least 2. □

Lemma 5.2.3. *If G is a disconnected graph, then $\gamma_g(\mu(G)) > 2$.*

Proof. Suppose that G is a graph with $\omega \geq 2$ components, say $G_1, G_2, \ldots, G_\omega$. Now we show that $\gamma_g(\mu(G)) > 2$. For that it is enough to show that for any move of Dominator in $\mu(G)$, there is a strategy for Staller which ensures that a D game on $\mu(G)$ has at least 3 moves. Suppose that Dominator first chooses an optimal move w, the root vertex of $\mu(G)$. Now Staller chooses her move $v \in V$ in G_1. Clearly each vertex $v \in V$ is a legal move since v dominates itself and that vertex is newly dominated. It is clear that this vertex v is not adjacent to any vertex of other components of G. So there are vertices which are not yet dominated in $\mu(G)$ and hence this game has at least 3 moves. Now suppose that Dominator first chooses an optimal move $v' \in V'$. In this case Staller selects v , the twin vertex of v' in $\mu(G)$ as her move. This is a legal move for Staller since v is not dominated by the previous move of Dominator. Now there are vertices which are not yet dominated in $\mu(G)$ especially for vertices in V which does not belong to the component of G containing the vertex v. Thus there are at least 3 moves in this game. Finally suppose that Dominator selects his first optimal move $v \in V$ and then Staller selects her move $u \in V$ which does not belong to the component of G containing v. It is clear that u is a legal move for Staller since u is not adjacent to v. Now the vertex w is not dominated and hence this game has at least 3 moves. So Staller ensures that there are at least 3 moves in $\mu(G)$ for a D game on $\mu(G)$. Thus $\gamma_g(\mu(G)) > 2$. □

Lemma 5.2.4. *If G is a connected graph with at least two ver-*

tices, then $\gamma_g(\mu(G)) > 2$.

Proof. Suppose that G is a connected graph with at least two vertices. We prove $\gamma_g(\mu(G)) > 2$ by showing that there is a strategy for Staller in $\mu(G)$ which ensures that there are at least 3 moves in a D game on $\mu(G)$. Suppose that Dominator first chooses his optimal move w, the root vertex of $\mu(G)$. Staller chooses a vertex $v' \in V'$ as her first move. Clearly this is a legal move since v' dominates all vertices adjacent to v in G and it is to be noted that the twin vertex v of v' is not dominated. Thus in this case there are at least three moves in $\mu(G)$ to finish the game. Now suppose that Dominator chooses a vertex $v' \in V'$ as first optimal move. Since G has at least two vertices and w is adjacent to at least one vertex other than v' in $\mu(G)$, it is clear that the root vertex w is a legal move for Staller . So Staller selects w as next optimal move and the twin vertex v of v' is not yet dominated. Hence there are at least 3 moves in this game . Finally suppose that Dominator first chooses his optimal move $v \in V$. In this case Staller selects a vertex $u' \in V'$ other than v', which is the twin vertex of v in $\mu(G)$, as her optimal move. It is clear that u' is a legal move because w is dominated by this move. The twin vertex v' of the vertex v is not yet dominated and it ensures that there are at least 3 moves in this case. Hence in all cases there are at least 3 moves for a D game on $\mu(G)$. Hence we conclude that $\gamma_g(\mu(G)) > 2$. $\square$

Theorem 5.2.5. *For any graph G, $\gamma_g(\mu(G)) = 2$ if and only if $G \cong K_1$.*

Proof. It is known by Lemma 5.2.3 that $\gamma_g(\mu(G)) > 2$ for every disconnected graph G. Also by Lemma 5.2.4 that $\gamma_g(\mu(G)) > 2$ for every connected graph G having at least two vertices. Thus if $G \ncong K_1$, then $\gamma_g(\mu(G)) > 2$.

Now suppose that if $G \cong K_1$, then $\mu(G)$ is the disjoint union of a K_2 and a K_1. Thus $\gamma_g(\mu(K_1)) = 2$. □

Lemma 5.2.6. *If G is a graph with at least two non adjacent vertices, then $\gamma_g'(\mu(G)) > 2$.*

Proof. Suppose that G is a graph having two non adjacent vertices say u and v. Now we show that there is a strategy for Staller which ensures at least 3 moves in an S game on $\mu(G)$. Suppose that Staller selects $v \in V$ as first move in $\mu(G)$. In this case Dominator cannot finish this game with his next move. If Dominator chooses w, the root vertex of $\mu(G)$, then the vertex u is not dominated in $\mu(G)$. Again that if Dominator chooses a vertex in V, then the root vertex w is not dominated. It is clear that the twin vertices u' of u and v' of v are not dominated after the first move of Staller in $\mu(G)$. So if Dominator chooses a vertex in V', then there are vertices in $\mu(G)$ which are not yet dominated. Thus there are at least 3 moves for an S game on $\mu(G)$ and hence $\gamma_g'(\mu(G)) > 2$. □

Theorem 5.2.7. *For any graph G, $\gamma_g'(\mu(G)) = 2$ if and only if $G \cong K_n$.*

Proof. First we show that if $G \cong K_n$, then $\gamma_g'(\mu(K_n)) = 2$. It is known by Theorem 5.2.2 that for any graph G, $\gamma_g'(\mu(G)) \geq 2$. So we need to prove that $\gamma_g'(\mu(K_n)) \leq 2$. It is enough to show that for any move of Staller in $\mu(K_n)$ there is a strategy for Dominator which ensures that there are at most two moves. Suppose that Staller selects her first optimal move as w, the root vertex of $\mu(K_n)$. Now Dominator can finish this game by selecting a vertex $v \in V$. Clearly w dominates the vertex w together with all vertices in V' and the vertex v dominates all vertices in V. So in this case there are at most two moves. Now suppose that Staller selects her first optimal move $v' \in V'$. In this case Dominator can finish this game by selecting the twin vertex v of v'. It is clear that the vertex v dominates all vertices in V together with all vertices in V' except v' and the vertex v' dominates the remaining vertices v' and w. So this game is finished by two moves. Finally suppose that Staller selects her first optimal move $v \in V$. Now Dominator can finish this game by selecting w, the root vertex of $\mu(K_n)$. It is clear that v dominates all vertices in V and w dominates w itself and all vertices in V'. Thus we conclude that $\gamma_g'(\mu(K_n)) \leq 2$ and hence $\gamma_g'(\mu(K_n)) = 2$.

It is known by Lemma 5.2.6 that if G has two non adjacent vertices, then $\gamma'_g(\mu(G)) > 2$. So $\gamma'_g(\mu(G)) = 2$ if and only if $G \cong K_n$. □

Lemma 5.2.8. *If G is a disconnected graph having at least 3 components, then $\gamma_g(\mu(G)) > 3$.*

Proof. Suppose that G is a disconnected graph having at least $\omega \geq 3$ components, say $G_1, \ldots G_\omega$. We prove that $\gamma_g(\mu(G)) > 3$ by showing that for any move of Dominator there is a strategy for Staller which ensures that a D game on $\mu(G)$ has at least 4 moves. First let us suppose that Dominator selects w , the root vertex of $\mu(G)$ as his first optimal move. In this case Staller selects a vertex $v \in V$. It is given that G has at least 3 components and hence there are vertices which are not yet dominated in two different components of G in $\mu(G)$. Thus at least two more moves are needed to finish this game and hence a total of at least 4 moves. Now suppose that Dominator chooses a vertex $v' \in V'$ as his first optimal move. In this case Staller selects the root vertex w as next move. Clearly the twin vertex v of v' is not yet dominated and there are vertices which are not yet dominated in two different components of G in $\mu(G)$. So in this case there are at least 4 moves. Finally we suppose that Dominator selects a vertex $v \in V$ as his first optimal move. In this case Staller selects the twin vertex v' of v and there are vertices in at least two components of G which are not yet dominated in

$\mu(G)$. So there are at least 4 moves needed to finish this game. Thus in any case we conclude that $\gamma_g(\mu(G)) > 3$. $\square$

Lemma 5.2.9. *If G is a disconnected graph having at least one edge, then $\gamma_g(\mu(G)) > 3$.*

Proof. If G is a disconnected graph having at least 3 components then it is known by Lemma 5.2.8 that $\gamma_g(\mu(G)) > 3$. So we assume that G is a disconnected graph having exactly two components say G_1 and G_2 and let $e = uv$ be an edge in G_1. We prove that $\gamma_g(\mu(G)) > 3$ by showing that for any move of Dominator there is a strategy for Staller which ensures that there are at least 4 moves in a D-game on $\mu(G)$. First let us suppose that Dominator selects w, the root vertex of $\mu(G)$ as his first optimal move. In this case Staller selects u' as next move. u' is a legal move for Staller since u is adjacent to v and hence u' dominates v. It is clear that no vertex in V from the component G_2 and the twin vertex $u \in V$ of u' from the component G_1 are not dominated. So two more moves are needed to finish this game. Now suppose that Dominator selects a vertex $x' \in V'$ as his first optimal move. In this case Staller selects the root vertex w as her next move. It is clear that the twin vertex $x \in V$ of x' is not dominated and no vertex of V from the component of G, which does not contain x is dominated. So two more moves are needed to finish this game. Finally suppose that Dominator selects a vertex $x \in V$ of G_1. In this case Staller selects the twin vertex $x' \in V'$ of x. So the vertices and its twin vertices of G_2 are not dominated. Thus at least two more moves are needed to finish this game. Hence $\gamma_g(\mu(G)) > 3$. □

Theorem 5.2.10. *Let G be a disconnected graph, then $\gamma_g(\mu(G)) = 3$ if and only if $G \cong 2K_1$.*

Proof. Suppose that G is a disconnected graph with $\gamma_g(\mu(G)) = 3$. It is known by Lemma 5.2.8 that G has exactly two components and by Lemma 5.2.9 that G has no edge. A graph with exactly two components and no edge is $2K_1$.

Suppose that if $G \cong 2K_1$ then $\mu(2K_1) \cong K_{1,3} \cup 2K_1$. Therefore $\gamma_g(\mu(2K_1)) = \gamma_g(K1, 3) + \gamma_g(2K_1) = 1 + 2 = 3$. □

Lemma 5.2.11. *If $\gamma_g(\mu(G)) = 3$ for a connected graph G, then there exists a vertex $u \in V(G)$ such that $ecc(u) \leq 2$*

Proof. Suppose that G is a connected graph with $ecc(u) \geq 3$ for all $u \in V(G)$. Now we prove that $\gamma_g(\mu(G)) > 3$. It is enough to show that Staller has a strategy on $\mu(G)$ such that there are at least 4 moves in a D game on $\mu(G)$. Suppose that an optimal first move of Dominator is the root vertex w of $\mu(G)$. In this case Staller selects a vertex $u' \in V'$. Now it is clear that u, the twin vertex of u', and vertices in $S_2(u) \cup S_3(u) \in V$ are not dominated. So it is impossible to find a vertex in $\mu(G)$ which dominates u and vertices in $S_3(u)$ and hence there are at least 4 moves.

Suppose that an optimal first move of Dominator is a vertex $u' \in V'$. In this case Staller selects the root vertex w of $\mu(G)$ and it is clear that w is a legal move. Therefore there are at least 4 moves needed by analogous arguments in the above case.

Suppose that an optimal first move of Dominator is a vertex $u \in V$. In this case Staller selects a vertex in $v \in S_1(u)$ and it is clear that v is a legal move. So the root vertex w and vertices in $S_3(u)$ are not dominated. So it is impossible for Dominator to finish this game by next move. So there are at least 4 moves.

In all the cases Dominator plays optimally and possibly not by Staller, therefore $\gamma_g(\mu(G)) \geq 4$ and hence there exists a vertex $u \in V(G)$ such that $ecc(u) \leq 2$. $\square$

Theorem 5.2.12. *Let G be a connected graph with at least two vertices, then $\gamma_g(\mu(G)) = 3$ if and only if every vertex of G lies in a connected dominating set of order 2.*

Proof. Suppose that G is a connected graph with every vertex of G lies in a connected dominating set of order 2. Now we show that $\gamma_g(\mu(G)) = 3$. It is known by Theorem 5.2.1 & 5.2.5 that $\gamma_g(\mu(G)) \geq 3$. Therefore it is enough to show that $\gamma_g(\mu(G)) \leq 3$. Now we show that there exists a strategy for Dominator which ensures at most 3 moves in a D game on $\mu(G)$. An optimal first move of Dominator in $\mu(G)$ is the root vertex w. Suppose that Staller selects a vertex $u' \in V'$. It is known by our assumption that every vertex lies in a connected dominating set of order 2. Therefore there exist a vertex $v \in V$ which is adjacent to u and $\{u, v\}$ forms a dominating set of G. Now Dominator can finish this game on $\mu(G)$ by selecting v as next move. It is clear that w dominates itself together with all vertices in V'. The vertex u' dominates all vertices in $N_G(u)$ and the vertex v dominates u and all the remaining vertices in G. Thus $V(\mu(G)) = \{w\} \cup N_{\mu(G)}[u'] \cup N_{\mu(G)}[v]$ and this game ends by these 3 moves.

Suppose that Staller selects a vertex $u \in V$ after the first move of Dominator. By our assumption there exist a vertex $v \in V$ such that $\{u, v\}$ is a connected dominating set in G. Therefore it is clear that $u, v \& w$ dominates all vertices in $\mu(G)$ and hence Dominator can finish this game by selecting the vertex v as his next move. So we conclude that $\gamma_g(\mu(G)) = 3$.

Conversely suppose that G is a connected graph with a vertex $u \in V(G)$ such that u does not belong to a connected dominating set of order 2. It is clear that G has no universal vertex (if G has a universal vertex v, then $\{u, v\}$ is a connected dominating set of order 2). Now we show that $\gamma_g(\mu(G)) \neq 3$. For that it is enough to show that Staller has a strategy in a D game on $\mu(G)$ such that the game has at least 4 moves.

Suppose that an optimal first move of Dominator is the root vertex w in $\mu(G)$. Now Staller selects the vertex u' and it is clear that u is not dominated in $\mu(G)$. So the game is not finished by the next move of Dominator. Since the vertex u does not belong to a connected dominating set of order 2, it is impossible to find a vertex v which is adjacent to u and $\{u, v\}$ is a dominating set of G.

Suppose that an optimal first move of Dominator is a vertex $v' \in V'$ in $\mu(G)$. If v is adjacent to u, then $\{u, v\}$ is not a dominating set of G. Now Staller chooses u' as next move. It is clear that there exists at least one vertex say u_1 in G such that u_1 is not adjacent to u and v in G. Therefore u_1 and u_1' are not dominated in $\mu(G)$. If Dominator chooses u_1, then u_1' is not dominated in $\mu(G)$ and if Dominator chooses u_1', then u_1 is not dominated in $\mu(G)$. If there exists a vertex u_2 which is adjacent to u, v and u_1, then Dominator selects u_2 and it dominates both u_1 and u_1' but the twin vertex u_2' of u_2 is not dominated in $\mu(G)$.

If v is not adjacent to u in G, then v and u are not dominated in $\mu(G)$ after selecting v' by Dominator. Now Staller chooses u' as her next move. Since u' dominates itself, it is a legal move for Staller. If there exists a vertex $u_1 \in V$ which is adjacent to both u and v, then Dominator plays either u_1 or u_1'. In any case the other twin vertex is not dominated and hence there are at least 4 moves. If there is no vertex which is adjacent to both u and v, then at least two more moves are needed to finish this game. So there are at least 4 moves in this case.

Suppose that an optimal first move of Dominator is a vertex $v \in V$ in $\mu(G)$. If v is not adjacent to u in G, then u, u' and v' are not dominated in $\mu(G)$ after selecting v by Dominator. Now Staller chooses u as her next move. Since u dominates itself, it is a legal move for Staller. If there exists a vertex $u_1 \in V$ which is adjacent to both u and v, then Dominator plays either u_1 or u_1'. In any case the other twin vertex is not dominated and hence there are at least 4 moves. If there is no vertex which is adjacent to both u and v, then at least two more moves are needed to finish this game. So there are at least 4 moves in this case. □

Theorem 5.2.13. *If G is a disconnected graph, then $\gamma_g'(\mu(G)) \geq 4$.*

Proof. Suppose that G is a disconnected graph with at least two components, say G_1 and G_2. Now we show that $\gamma_g'(\mu(G)) \geq 4$. It is enough to show that there exists a strategy for Staller which ensures at least 4 moves in an S game on $\mu(G)$. Let $V_1 = V(G_1)$ and $V_2 = V(G_2)$. Staller chooses her first move as $u' \in V_1'$. It is clear that the twin vertex u of u' in $\mu(G)$ and all vertices in $V_2 \cup V_2'$ are not dominated. Suppose that if Dominator chooses the vertex w as his optimal move, then the vertex u and all vertices in V_1 of $\mu(G)$ are not dominated. So there are at least

4 moves needed to finish this game. If Dominator selects the vertex $v \in V_2'$ as his optimal move, then it is clear that u and v are not dominated. So there are at least 4 moves needed to finish this game. If Dominator chooses a vertex $v' \in V_1$ as his optimal move, then it is clear that all vertices in $V_2 \cup V_2'$ are not dominated. So Staller selects a vertex in V_2' and its twin vertex is not yet dominated. Thus there are at least 4 moves in this game. If Dominator chooses the vertex $v \in V_1$ as his optimal move, then it is clear that all vertices in $V_2 \cup V_2'$ are not dominated. So Staller selects a vertex in V_2' and its twin vertex is not yet dominated. Thus there are at least 4 moves in this game. If Dominator chooses a vertex $v \in V_2$ as his optimal move, then Staller selects u as her next optimal move. It is clear that v', the twin vertex of v in $\mu(G)$, is not dominated. Thus we conclude that$\gamma_g'(\mu(G)) \geq 4$. □

Lemma 5.2.14. *If $\Delta(G) < n-1$ for a connected graph G, then $\gamma_g'(\mu(G)) \geq 4$.*

Proof. Let G be a connected graph with $\Delta(G) < n-1$. It is clear that G has at least 4 vertices. To prove that $\gamma_g'(\mu(G)) \geq 4$, it is enough to show that there is a strategy for Staller which ensures at least 4 moves in an S game on $\mu(G)$. Staller first chooses a vertex $u' \in V'$. Since $\Delta(G) < n-1$, u is not adjacent to at least one vertex say $v \in V(G)$. Therefore it is clear that u

and v are not dominated in $\mu(G)$ after the first move of Staller.

If Dominator chooses the root vertex w as his optimal move in $\mu(G)$, then Staller selects v and it is clear that u is not dominated by thsese 3 moves in $\mu(G)$. So there are at least 4 moves in this case.

If Dominator chooses the vertex u, then it is clear that v and its twin vertex v' are not dominated. Now Staller chooses v and it is clear that v' is not dominated by these 3 moves in $\mu(G)$. So there are at least 4 moves in this case.

If Dominator chooses a vertex $u_1 \in V$ which is adjacent to u, then it is clear that u_1' is not dominated in $\mu(G)$. Since $\Delta(G) < n - 1$, there exists a vertex u_2 in G such that u_2 is not adjacent to u_1. Therefore u_2' is not dominated in $\mu(G)$. Now Staller selects u_1' and it is clear that u_2' is not dominated by thsese 3 moves in $\mu(G)$. So there are at least 4 moves in this case.

If Dominator chooses a vertex $u_1 \in V$ which is not adjacent to u, then it is clear that u_1' is not dominated in $\mu(G)$. Now Staller chooses u_1' and it is clear that u is not dominated by these 3 moves in $\mu(G)$. So there are at least 4 moves in this case.

If Dominator chooses a vertex $u_1 \in V'$, then Staller chooses a vertex in V' other than u' and u_1'. It is known that G has at least 4 vertices and hence in $\mu(G)$ there is at least 1 vertex which is not yet dominated in V'. So there are at least 4 moves in this case. Thus $\gamma'_g(G) \geq 4$. $\square$

Theorem 5.2.15. *For any connected graph G, $\gamma'_g(\mu(G)) = 3$ if and only if $G \ncong K_n$ and $\Delta(G) = n-1$.*

Proof. Let $G \ncong K_n$ be a connected graph with $\Delta(G) = n-1$. It is known by Theorem 5.2.2 and Theorem 5.2.7 that $\gamma'_g(\mu(G)) \geq 3$. So we need to prove that $\gamma'_g(\mu(G)) \leq 3$. It is enough to show that there exist a strategy for Dominator which ensures at most 3 moves in an S game on $\mu(G)$. Suppose that an optimal first move of Staller is a vertex $u \in V$ in $\mu(G)$. Now Dominator chooses a universal vertex $v \in V$ in $\mu(G)$ (this is possible because $\Delta(G) = n-1$). So all vertices in $V \cup V'$ is dominated and the only vertex which is not dominated is w. Hence this game has at most 3 moves.

If an optimal first move of Staller is a vertex $u' \in V'$ in $\mu(G)$. Now Dominator chooses a universal vertex of G say $v \in V$ in $\mu(G)$ (This is possible because $\Delta(G) = n-1$). So all vertices in $\mu(G)$ except v', the twin vertex of v, are dominated in $\mu(G)$. Hence this game has at most 3 moves.

If an optimal first move of Staller is the root vertex w in $\mu(G)$, then Dominator can finish this game by selecting a universal vertex of G say $v \in V$ in $\mu(G)$. So we conclude that $\gamma'_g(\mu(G)) \leq 3$.

Conversely suppose that $\gamma_g'(\mu(G)) = 3$ for a graph G. It is known by Theorem 5.2.7 that $G \ncong K_n$ and by Lemma 5.2.14 that $\Delta(G) \geq n-1$. Therefore if $\gamma_g'(\mu(G)) = 3$, then $G \ncong K_n$ and $\Delta(G) = n-1$. $\square$

Concluding Remarks

In this thesis we have studied the domination game in graphs. We have described the effect of game domination number by the removal of an edge or a vertex in the class of no-minus graphs. We have discussed two graph operations, the edge contraction and the edge sub division. These operations have a monotone behaviour on the game domination number of graphs in the sense that these parameters either increase or decrease but not both. We have also studied the bounds for game domination number in split graphs and characterise Mycielskian of a graph with small game domination number.

Split graphs have several important generalizations. Chordal graphs form one of them. Since trees are chordal graphs and there exist infinite families of the so-called 3/5-trees [4, 28], Theorem 4.3.1 does not extend to chordal graphs. Another important generalization of split graphs are $2K_2$-free graphs [20, 19]. Now, C_5 belongs to this class and $\gamma_g(C_5) = 3$, hence Theorem 4.3.1 also does not extend to $2K_2$-free graphs. Actually we know of one such class (tri-split graphs) but this extension is rather straightforward.

The following problems may be of interest.

1. Whether there is some natural superclass of split graphs to which Theorem 4.3.1 extends?.

2. **Rall's conjecture** [47]: If a graph G contains a hamiltonian path, then $\gamma_g(G) \leq \left\lceil \frac{|V(G)|}{2} \right\rceil$.
   Whether Rall's Conjecture is true?.

3. Find a graph operation other than the edge contraction such that the game domination number never increases.

4. Find a graph operation other than the edge subdivision such that the game domination number never decreases.

5. Find the game domination number in product graphs.
   (We have obtained some results on the lexicographic product of graphs).

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List of Publications

Papers published/ communicated

1. Tijo James, Paul Dorbec, A.Vijayakumar, Further progress on the heredity of the game domination number, *Lecture Notes in Comput. Sci.* (Springer) 10398 (2017), 435–445 (MR3696753).

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Papers presented

1. Domination game on tri-split graphs, 79^{th} Annual Conference of Indian Mathematical Society, Rajagiri School of Engineering and Technology, Kochi, Kerala, 28-31, December 2013.

2. Domination game on trees, International Conference on Algebra and Discrete Mathematics, Government College, Kattappana, Kerala, 04-06, March 2014.

3. Further progress on the heredity of the game domination number, International Conference on Theoretical Computer Science and Discrete Mathematics, Kalasalingam University, Krishnankoil, Tamilnadu, 19-21, December 2016.

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