Stochastic Modelling: Analysis and Applications

Analysis of Queueing-Inventory Systems - with Several Modes of Service; Reservation, Cancellation and Common Life Time; of the GI/M/1 Type (Two Commodity) and an Inventory Problem Associated with Crowdsourcing.

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Analysis of Queueing-Inventory Systems- with Several Modes of Service; Reservation, Cancellation and Common Life Time; of the GI/M/1 Type (Two Commodity) and an Inventory Problem Associated with Crowdsourcing.

Ph.D. thesis in the field of Stochastic Modelling: Analysis & Applications

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June 2018

To
My Parents
and
Teachers

June 2018

Certificate

Certified that the work presented in this thesis entitled "**Analysis of Queueing-Inventory Systems- with Several Modes of Service; Reservation, Cancellation and Common Life Time; of the GI/M/1 Type (Two Commodity) and an Inventory Problem Associated with Crowdsourcing**" is based on the authentic record of research carried out by Ms.Binitha Benny under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Kochi-682 022 and has not been included in any other thesis submitted for the award of any degree. Also certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis and the work done is adequate and complete for the award of Ph. D. Degree.

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Declaration

I, Binitha Benny, hereby declare that the work presented in this thesis entitled "**Analysis of Queueing-Inventory Systems- with Several Modes of Service; Reservation, Cancellation and Common Life Time; of the GI/M/1 Type (Two Commodity) and an Inventory Problem Associated with Crowdsourcing**" is based on the original research work carried out by me under the supervision and guidance of Dr. A. Krishnamoorthy, formerly Professor, Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted previously for the award of any degree.

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Praise the Lord. Give thanks to the Lord,

for he is good;
his love endures forever.

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Analysis of Queueing-Inventory Systems-with Several Modes of Service; Reservation, Cancellation and Common Life Time; of the GI/M/1 Type (Two Commodity) and an Inventory Problem Associated with Crowdsourcing

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Notations and Abbreviations

$\mathbf{e}$	:	Column vector consisting of 1's of appropriate dimension
I	:	Identity matrix of appropriate dimension
PH	:	Phase type
$CTMC$	:	Continuous Time Markov Chain
QBD	:	Quasi-birth-and-death
$LIQBD$	:	Level Independent Quasi-Birth-and-Death process
CLT	:	Common Life Time

Chapter 1

Introduction

Stochastic Modelling is the application of probability theory to the description and analysis of real world phenomena. These are usually so complex that deterministic laws cannot be formulated, a circumstance that leads to pervasive use of stochastic concepts. Stochastic modelling is a science with close interactions between theory and practical applications. It combines the possibility of theoretical beauty with a real world meaning of its key concepts. Application fields as telecommunication or insurance bring methods and results of stochastic modelling to the attention of applied sciences such as engineering, economics.

One of the most important domains in stochastic modelling is the field of queueing theory. Many real systems can be reduced to components which can be modelled by the concept of queue. The basic idea of this concept has been borrowed from the every-day experience of queues at the checkout counters in a supermarket. A queue consists of a system into which there comes a stream of users who demand some capacity of the system over a certain time interval before they leave the system.

Users are served in the system by one or many servers. Thus a queueing system can be described by a stochastic specification of the arrival stream and of the system demand for every user as well as a definition of the service mechanism. The former describe the input into a queue, while the latter represents the function of the inner mechanisms of a queueing system.

Computer networks(the most prominent example is the internet) have increasingly become the object of applications of queueing theory. Queues find further applications in airport traffic and computer science. More complicated queueing models have been developed for the design of traffic lights at crossroads.

One of the important tasks in a business world is to manage inventory. Any resource that is stored to satisfy the current as well as future needs is called an inventory. Examples of inventory are spare parts, raw materials, work-in-process etc. Inventory models are widely used in hospitals, educational institutions, agriculture, industries, banks etc. Two questions faced while dealing with inventory models are: how much to order and when to order. First is the order quantity and second is reorder level. The number of items ordered when an order is placed to minimize total running cost is called the optimum order quantity. Reorder level is determined based on the inventory models. In inventory management we try to find a balance between two conflicting goals- one is to make available the required item at a time of need and second is to minimize related costs.

For inventory transaction several control policies are considered. Some of the control policies are:

- (s, S) **policy-** In (s, S) policy s is the reorder level and S is the maximum inventory level. At the replenishment epoch the order quantity is that many units required to bring the level back to S.

- (s, Q) **policy-** In (s, Q) policy s is the reorder level and Q is the fixed order quantity. Here the number of items to be replenished is fixed and is equal to $Q = S - s$.

- $(S - 1, S)$**- policy**, an order is placed for exactly one unit at each epoch of occurrence of a demand. This is used for controlling the stock levels of expensive and slow moving items.

- **Random order policy-** Replenishment order is placed whenever inventory level is at some point in the set $\{0, 1, 2, ..., s\}$. Once an order is placed, the next order goes only after the replenishment against the first order is realized.

For solving an inventory problem, an appropriate cost function is needed. A typical cost function consists of following type of costs.

- Variable Procurement Cost- Cost of buying items. This cost is the actual price per unit paid for the procurement of items.

- Holding Cost-Cost incurred for carrying or holding inventory items in the warehouse.

- Fixed Ordering(set-up) Cost-Cost incurred each time an order is placed for procuring items from the vendors.

- Stock-out(Shortage Cost)- Shortage occurs when items cannot be supplied due to non availability.

In classical queues the availability of item to be served need not be considered whereas in classical queueing-inventory models at least one customer and at least an item in inventory is needed to provide service. A queue is formed when time taken to serve the items is positive. If service time is negligible a queue is formed only during stock-out period and when unsatisfied customers are allowed to wait. For inventory models with positive service time a queue is formed even when items are available. This is because new customers join while a service is going on. Also a queue is formed when time between placement of an order and its receipt (**lead time**) is positive.

1.1 Stochastic Processes

The theory of stochastic processes is concerned with the investigation of the structure of families of random variables X_t, where t is a parameter running over a suitable index set T. The index set t may correspond to discrete units of time $T = \{0, 1, 2, 3, ...\}$ or $T = [0, \infty]$. If $T = \{0, 1, 2, 3, ...\}$ then $\{X_t\}$ is a discrete time stochastic process. If $T = [0, \infty]$, then $\{X_t\}$ is called a continuous time process. State space is the space in which the possible values of each X_t lie.

Markov Processes

A Markov process is a process with the property that, given the value of X_t, the values of X_s, $s > t$, do not depend on the value of $X_u, u < t$; that is, the probability of any particular future behaviour of the process, when its present state is known exactly, is not altered by additional knowledge concerning its past behaviour. In formal terms a process is said to be

Markov if

$$Pr\{a < X_t \leq b | X_{t_1} = x_1, X_{t_2} = x_2....X_{t_n} = x_n\}$$

$$= Pr\{a < X_t \leq b | X_{t_n} = x_n\}$$

whenever $t_1 < t_2 < .... < t_n < t$.

Markov Chain

A Markov process having a finite or denumerable state space is called a Markov chain.

Continuous time Markov Chain

A continuous time stochastic process $\{X(t), t \geq 0\}$ with discrete state space I is said to be a continuous time Markov chain if

$$Pr\{X(t_n) = i_n | X(t_0) = i_0, ..., X(t_{n-1}) = i_{n-1}\}$$

$$= Pr\{X(t_n) = i_n | X(t_{n-1}) = i_{n-1}\}$$

for all $0 \leq t_0 < ... < t_n$ and $i_0, ...i_{n-1}, i_n \epsilon I$

1.2 The Exponential Distribution

A nonnegative random variable X has an exponential distribution if its probability distribution function is given by

$$F(t) = Pr\{X \leq t\} = 1 - exp(-\lambda t), t \geq 0$$

where λ is a positive real number. We call X an exponential random variable with parameter λ. The exponential distribution is widely used in queueing models because of the memoryless property,

$$Pr\{X > t + s | X > s\} = Pr\{X > t\}$$

holds for $t \geq 0$ and $s \geq 0$ of this distribution.

The Poisson Process

The counting process $\{N(t), t \geq 0\}$ where $N(t)$ is the number of events occurring in $[0, t]$, is called a Poisson Process having rate λ, $\lambda > 0$, if

1. $N(0)$=0.

2. The process has stationary and independent increments.

3. $P\{N(h) = 1\} = \lambda h + o(h)$.

4. $P\{N(h) \geq 2\} = o(h)$.

A counting process is said to possess independent increments if the number of events that occur in disjoint time intervals are independent. A counting process is said to possess stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval.

1.3 Phase Type distribution (Continuous time)

Phase type distributions provide a versatile set of tractable models for applied probability. They are based on the method of stages, a technique introduced by A.K.Erlang and generalized to its full potential by M.F.Neuts. The key idea is to model random time intervals as being made up of a possibly random number of exponentially distributed segments and to exploit the resulting Markovian structure to simplify the analysis. It is possible to approximate any distribution on the nonnegative real numbers by a phase type distribution, and the resulting queueing models can be analyzed almost as if we have dealt with the exponential distribution.

Let $\mathcal{X} = \{X(t) : t \geq 0\}$ be a homogeneous Markov chain with finite state space $\{1, ..., m, m+1\}$ and generator

$$\mathcal{Q} = \begin{pmatrix} \mathcal{T}_{m\times m} & \mathcal{T}^0 \\ \mathbf{0} & 0 \end{pmatrix}$$

where the elements of the matrices $\mathcal{T}$ and $\mathcal{T}^0$ satisfy $\mathcal{T}_{ii} < 0$ for $1 \leq i \leq m$, $\mathcal{T}_{ij} \geq 0$ for $i \neq j$; $\mathcal{T}_i^0 \geq 0$ and $\mathcal{T}_i^0 > 0$ for at least one i, $1 \leq i \leq m$ and $\mathcal{T}\boldsymbol{e} + \mathcal{T}^0 = \mathbf{0}$.

Let the initial distribution of $\mathcal{X}$ be the row vector $(\boldsymbol{\alpha}, \alpha_{m+1})$, $\boldsymbol{\alpha}$ being a row vector of dimension m with the property that $\boldsymbol{\alpha}\boldsymbol{e} + \alpha_{m+1} = 1$. The states $1, 2, \ldots, m$ shall be transient, while the state $m+1$ is absorbing.

Let $\mathcal{Z} = \inf\{t \geq 0 : X(t) = m+1\}$ be the random variable representing the time until absorption in state $m+1$. Then the distribution of $\mathcal{Z}$ is Phase type distribution (or shortly PH distribution) with representation

$(\boldsymbol{\alpha}, \mathcal{T})$. The dimension m of $\mathcal{T}$ is called the order of the distribution. The states $1, 2, ..., m$ are also called phases.

- The distribution function of $\mathcal{Z}$ is given by

  $$F(t) = P\left(X \leq t\right) = 1 - \boldsymbol{\alpha} \exp(Tt)\boldsymbol{e} \equiv 1 - \boldsymbol{\alpha} \left(\sum_{r=0}^{\infty} \frac{t^r T^r}{r!} \right) \boldsymbol{e}, \quad t \geq 0$$

  where,

  $\boldsymbol{\alpha}$ is row vector of non-negative elements of order $m(> 0)$ satisfying $\boldsymbol{\alpha} e \leq 1$. and T is an $m \times m$ matrix such that
  i) all off-diagonal elements are nonnegative
  ii) all main diagonal elements are negative
  iii) all row sums are non-positive and
  iv) T is invertible.
  The 2- tuple $(\boldsymbol{\alpha}, T)$ is called a phase-type representation of order m for the PH distribution and T is called a generator of the PH distribution..

- The density function is

  $$f(t) = \boldsymbol{\alpha}\ exp(\mathcal{T}.t)\ \mathcal{T}^0 \ \text{ for every } t > 0$$

- $E[X^n] = (-1)^n n! \boldsymbol{\alpha} \mathcal{T}^{-n} \mathbf{e},\ n \geq 1.$

- The Laplace-Stieltjes transform of $F(.)$ is

  $$\phi(s) = \alpha_{m+1} + \boldsymbol{\alpha}(sI - \mathcal{T})^{-1}\ \mathcal{T}^0 \ \text{ for } Re(s) \geq 0.$$

Theorem 1.3.1 (see, *Latouche and Ramaswami* [40]). *Consider a*

PH distribution $(\boldsymbol{\alpha}, \mathcal{T})$. *Absorption into state* $m+1$ *occurs with probability 1 from any phase* i *in* $\{1, 2, \ldots, m\}$ *if and only if the matrix* $\mathcal{T}$ *is nonsingular.*

Moreover, $(-\mathcal{T}^{-1})_{i,j}$ *is the expected total time spent in phase* j *during the time until absorption, given that the initial phase is* i.

For further information about the PH distribution, see, *Neuts* [48], *Breuer and Baum* [15], *Latouche and Ramaswami* [41] and *Qi-Ming He*[50]. Usefulness of PH distribution as service time distribution in telecommunication networks is elaborated, e.g., in *Pattavina and Parini* [49] and *Riska, Diev and Smirni* [53].

1.4 Quasi-birth-death processes

Quasi-birth-death processes (QBDs) are matrix generalizations of simple birth-and-death processes on the nonnegative integers. A birth increases the size by one and a death decreases its size by one. Consider a Markov Chain $\{X_t,\ t \in \mathbf{R}^+\}$ on the two dimensional state space $\Omega = \bigcup_{n \geq 0} \{(n, j) : 1 \leq j \leq m\}$. The first coordinate n is called the level, and the second coordinate j is called a phase of the n^{th} level. The number of phases in each level may be either finite or infinite. The Markov chain is called a QBD process if one-step transitions from a state are restricted to phases in the same level or to the two adjacent levels. In other words,

$$(n-1, j') \rightleftharpoons (n, j) \rightleftharpoons (n+1, j'') \ \text{ for } n \geq 1.$$

If the transition rates are level independent, the resulting QBD process is called level independent quasi-birth-death process ($LIQBD$); else it is called level dependent quasi-birth-death process ($LDQBD$). Arranging the elements of Ω in lexicographic order, the infinitesimal generator of a $LIQBD$ process is block tridiagonal and has the following form:

$$\boldsymbol{Q} = \begin{pmatrix} B_1 & A_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

where the sub matrices A_0, A_1, A_2 are square and have the same dimension; matrix B_1 is also square and need not have the same size as A_1. Also, the matrices B_2, A_2 and A_0 are nonnegative and the matrices B_1 and A_1 have nonnegative off-diagonal elements and strictly negative diagonals. The row sums of Q are equal to zero, so that we have $B_1\boldsymbol{e}+A_0\boldsymbol{e} = B_2\boldsymbol{e}+A_1\boldsymbol{e}+A_0\boldsymbol{e} = (A_0+A_1+A_2)\boldsymbol{e} = \mathbf{0}$. Among the several tools that we employed in this thesis Matrix geometric method plays a key role. A brief description of this is given below.

1.5 Matrix Geometric Method

Matrix Geometric Method, introduced by M. F. Neuts is popular as modelling tools because they give one the ability to construct and analyze, in a unified way and in algorithmically tractable manner, a wide class of stochastic models. The methods are applied in several areas, of which the performance analysis of telecommunication systems is one of the most notable at the present time. In matrix geometric methods the distribution of a random variable is defined through a matrix; its density function, moments etc., are expressed with this matrix.

Theorem 1.5.1 (see Theorem 3.1.1. of *Neuts* [48]). *The process* $\boldsymbol{Q}$ *in (1.1) is positive recurrent if and only if the minimal non-negative solution* R *to the matrix-quadratic equation*

$$R^2 A_2 + R A_1 + A_0 = O \tag{1.2}$$

has all its eigenvalues inside the unit disk and the finite system of equations

$$\begin{aligned} \boldsymbol{x}_0 \left(B_1 + R B_2\right) &= \boldsymbol{0} \\ \boldsymbol{x}_0 (I - R)^{-1} \boldsymbol{e} &= 1 \end{aligned} \tag{1.3}$$

has a unique positive solution $\boldsymbol{x}_0$.

If the matrix $A = A_0 + A_1 + A_2$ *is irreducible, then* $sp(R) < 1$ *if and only if*

$$\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e} \tag{1.4}$$

where $\boldsymbol{\pi}$ *is the stationary probability vector of* A.

The stationary probability vector $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ *of* $\boldsymbol{Q}$ *is given by*

$$\boldsymbol{x}_i = \boldsymbol{x}_0 R^i \quad \text{for} \;\; i \geq 1. \tag{1.5}$$

Once R, the rate matrix, is obtained, the vector $\boldsymbol{x}$ can be computed. We can use an iterative procedure or logarithmic reduction algorithm (see *Latouche and Ramaswami* [40]) or the cyclic reduction algorithm (see *Bini and Meini* [11]) for computing R.

1.6 G/M/1 type model

Consider a Markov chain with bivariate state space

$$\{(i,j), i \geq 0, 1 \leq j \leq k\},$$

where i represent the level and j the phase of the chain. Its generator $\boldsymbol{Q}$ has the form:

$$\boldsymbol{Q} = \begin{pmatrix} B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & \\ B_2 & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \qquad (1.6)$$

where the off-diagonal elements of $\boldsymbol{Q}$ are non-negative and diagonal elements are negative such that

$$\sum_{r=0}^{n} A_r \mathbf{e} + B_n \mathbf{e} = 0, n = 0, 1, ...$$

Such a model is called G/M/1 type model.

Theorem 1.6.1. *The irreducible Markov process* $\boldsymbol{Q}$ *is positive recurrent if and only if the minimal non negative solution* R *of the equation*

$$\sum_{k=0}^{\infty} R^k A_k = 0$$

has $sp(R) < 1$ *and if there exists a positive vector* x_0 *such that*

$$x_0 \ \boldsymbol{B[R]} = 0.$$

The matrix $\boldsymbol{B[R]} = \sum_{k=0}^{\infty} R^k B_k = 0$ *is a generator. The stationary probability vector* x*, satisfying* $x\boldsymbol{Q} = 0$*,* $x\mathbf{e} = 1$*, is then given by*

$$x_k = x_0 R^k,\ for\ k \geq 0,$$

and x_0 *is normalized by*

$$x_0(I - R)^{-1}\mathbf{e} = 1.$$

The matrix R *has a positive maximal eigen value* η*. If the generator* A *is irreducible, the left eigen vector* u^* *of* R *corresponding to* η*, is determined up to a multiplicative constant and may be chosen to be positive. The matrix* R *then satisfies* $sp(R) < 1$*, if and only if*

$$\pi A_0 \mathbf{e} < \sum_{k=2}^{\infty} (k-1)\pi A_k \mathbf{e},$$

where π *is given by* $\pi A = 0$*,* $\pi\mathbf{e} = 1$*.*

1.7 Computation of R matrix

There are several algorithms for computing rate matrix R.

Iterative algorithm

From (1.2), we can evaluate R in a recursive procedure as follows.

Step 0: $R(0) = \mathrm{O}$.

Step 1:

$$R(n+1) = A_0(-A_1)^{-1} + R^2(n)A_2(-A_1)^{-1}, \quad n = 0, 1, \ldots$$

Continue **Step 1** until $R(n+1)$ is close to $R(n)$.

That is, $||R(n+1) - R(n)||_\infty < \epsilon$.

For $G/M/1$ type models, we use

$$R(0) = \mathbf{0},$$

and

$$R(n+1) = -A_0A_1^{-1} - R^2(n)A_2A_1^{-1} - R^3(n)A_3A_1^{-1} - \ldots, n \geq 0.$$

Uniformization Technique

Uniformization or Randomization technique is a powerful method which allows one to interpret a continuous time Markov process as a discrete time Markov chain for which one merely replaces the constant unit of time between any two transitions by independent exponential random variables with the same parameter. On the other hand it allows us to evaluate the transition matrix $P(t)$ without recourse to any differential equations.

Consider a Markov process $\{X(t); t \geq 0\}$ with generator Q such that $|q_{ii}| \leq c < \infty$ for all i for some constant c. Then, the matrix $K = \frac{1}{c}Q + I$ is stochastic. Define the stochastic process $\{Y(t) : t \geq 0\}$ as follows. Take a Poisson process with rate c and denote by $0 = t_0, t_1, t_2, ...$ the epochs of events in that process. Take a discrete time Markov chain $\{Z_n : n \geq 0\}$ with transition matrix K independent of the Poisson process. Define the process $\{Y(t) : t \geq 0\}$ such that $Y(t) = Z_n$ for $t_n \leq t < t_{n+1}$, for $n \geq 0$. $\{Y(t) : t \geq 0\}$ happens to be a Markov chain with generator Q. If we define the transition matrix $P(t)$ where $P_{ij}(t) = P[Y(t) = j|Y(0) = i]$, by a simple conditioning argument on the number of Poisson events in $(0, t]$ that

$$\begin{aligned} P(t) &= \sum_{n\geq 0} e^{-ct}\frac{(ct)^n}{n!}K^n \\ &= exp(Qt) \end{aligned}$$

which is the transition matrix of the process $X(t); t \geq 0$.

Computation of density and distribution function of $PH(\tau, T)$ random variable

Uniformize the associated Markov process with generator

$$\mathcal{Q} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{t} & T \end{bmatrix}$$

by choosing $c = max(-T_{ii} : 1 \leq i \leq n)$ and $K = \frac{1}{c}Q + I$=

$$= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{p} & P \end{bmatrix}$$

where $P = \frac{1}{c}T + I$, and $\mathbf{p} = \frac{1}{c}\mathbf{t}$. Then we get

$$exp(Tx) = \sum_{k\geq 0} e^{-cx}\frac{(cx)^k}{k!}P^k$$

so that

$$F(x) = 1 - \sum_{k\geq 0} e^{-cx}\frac{(cx)^k}{k!}\tau P^k \mathbf{1}$$

and

$$f(x) = c\sum_{k\geq 0} e^{-cx}\frac{(cx)^k}{k!}\tau P^k \mathbf{p}$$

1.8 Review of related work

Inventory with positive service time was introduced independently by Melikov and Molchanov[45] and Sigman and Levi ([59]). In [59] the authors introduced the concept of positive service time into inventory models with arbitrarily distributed service duration, exponentially distributed lead time with customer arrival constituting a Poisson process. A light traffic heuristic approximation procedure was used to find performance measures of the system.

Among queueing-inventory problems, of particular interest are those which yield **product form solution**. Product form refers to the observation that the steady state distribution of the models with a vector valued state process is the product of the marginal steady state distributions. In queueing-inventory models this means that the asymptotic and stationary distribution of the joint(queue length and inventory size) process factorizes into the stationary queue length and inventory size distributions. In the long run and in equilibrium the queue length process and the inventory process behave as if they are independent. This is a rather strange phenomenon because the processes strongly interact, whether being in equilibrium or otherwise.

The literature providing product form solution for system state distribution is quite scarce. The reason for this could be attributed to the fact that we have to impose severe restrictions on the structure of the system under consideration. The product form solution is of great significance since it provides asymptotic independence of the components of the state space which are highly correlated. The high degree of correlation comes through the fact that the number of customers joining during lead time and the length of lead time are strongly dependent.

Schwarz et al. [57] derived stationary distributions of joint queue length and inventory processes in explicit product form for various $M/M/1$-systems with inventory under continuous review and different inventory management policies, and with lost sales. It is assumed that demand is Poisson, service times and lead times are exponentially distributed. Schwarz et al. [58] investigated a new class of stochastic networks that exhibit a product form steady state distribution. The stochastic models developed here are integrated models for networks of service stations and inventories. Here they integrate a server with attached inventory under (r, Q) or (r, S)-policy into Jackson or Gordon-Newell networks. Replenishment lead times are non-zero and random and depend on the load of the system. While the inventory is depleted the server with attached inventory does not accept new customers but they assume that lost sales are not lost to the system. Three different approaches are used to handle routing with respect to this node during the time the inventory is empty. The stationary distributions of joint queue lengths and inventory process is derived in explicit product form.

Saffari et al.[56] considers an $M/M/1/\infty$ queueing system with inventory under continuous review (s,Q) policy with lead times mixed exponentially distributed. During stock out, arriving demands are lost They derive stationary distribution of product form of joint queue length and on-hand inventory. Saffari et al. [55] consider an $M/M/1$ queueing system with inventory under the (r,Q) policy with lost sales. Demands occur according to a Poisson process and service times are exponentially distributed. Customers arriving during stock-out period are lost. They derive the stationary distributions of the joint queue length and on- hand inventory when lead time is random.

Krishnamoorthy and Viswanath [39] consider an (s,S) production inventory system where demand process is Poisson, duration of each service and time required to add an item to the inventory when the production is on, are independent non-identically distributed exponential random variables. An explicit product form solution for the steady state probability vector is obtained under the assumption that no customer joins the queue when inventory level is zero. We refer to the survey paper by Krishnamoorthy et al. [34] for details on queueing-inventory models with positive service time. Quite recently, Krishnamoorthy et al. ([30], [35], [36]) have analyzed a single server queueing-inventory system with positive service time. In all these cases explicit product form solution for the system state is obtained.

Krenzler and Daduna ([29], [28]) have analyzed a single server system with positive service time in a random environment. The service system and the environment interact in both directions. Whenever the environment enters a specific subset of its state space, the service process is completely blocked and new arrivals are lost. They obtain a necessary and sufficient condition for a product form steady state distribution of the joint queueing-environment process. This blocking set cannot be enlarged to a partial blocking set to obtain product form solution.

Discrete time (s, S) inventory model in which the stored items have a random common life time with a discrete phase type distribution where demands arrive in batches following a discrete phase type renewal process is considered by Lian et al. [42]

Inventory systems dealing with several/distinct commodities are very common, (see for example [46][1]). Such systems are more complex than single commodity system which could be attributed to the reordering procedures. Whether the ordering policies of joint, individual or some mixed type are superior will depend on the particular problem at hand.

Balintfy [7] evaluates and compares multi-item inventory problems where joint order of several items may save a part of the set up cost. The comparisons call for the necessity of a new policy for reorder point-triggered random output multi-item systems. This policy, the "random joint order policy" operates through the determination of a reorder range within which several items can be ordered. The existence of an optimum reorder range is proved, and a computational technique is demonstrated with the help of a machine-interference type queueing model.

Federgruen et al.[19] considered a continuous review multi-item inventory system with compound Poisson demand processes; excess demands are backlogged and each replenishment requires a lead time. There is a major setup cost associated with any replenishment of the family of items, and a minor (item dependent) setup cost when including a particular item in this replenishment. Moreover, there are holding and penalty costs. An algorithm which searches for a simple coordinated control rule which minimizes the long run average cost per unit time subject to a service level constraint per item on the fraction of demand satisfied directly from on hand inventory is presented. This algorithm is based on a heuristic decomposition procedure and a specialized policy -iteration method to solve the single-item subproblems generated by the decomposition procedure.

Two commodity continuous review inventory system without lead time is considered by Krishnamoorthy et al.[33] where each demand is for one unit of the first commodity or one unit of the second commodity or one unit each of both commodities with a prefixed probability. Krishnamoorthy and Varghese[37] considered two commodity inventory problem without lead time and with Markovian shift in demand for first commodity, second commodity and both commodities. Using results from Markov renewal theory Sivasamy and Pandiyan[61] derived various results by the application of filtering techniques for the same problem.

A two commodity continuous review inventory system with independent Poisson demands is considered by Anbazhagan and Arivarignan [2]. Here the maximum inventory level for $i-th$ commodity is fixed as $S_i, i = 1, 2$ and net inventory level at time t for the $i-th$ commodity is denoted by $I_i(t), i = 1, 2$. If the total net inventory level $I(t) = I_1(t) + I_2(t)$ drops to a prefixed level, s [$\leq \frac{S_1-2}{2}$ or $\frac{S_2-2}{2}$] an order is placed for $(S_i - s)$ units of $i-th$ commodity $(i = 1, 2)$. Here the probability distribution for inventory level and mean reorders and shortage rates in the steady state are computed. Two commodity continuous review inventory system with renewal demands and ordering policy as a combination of individual and joint ordering policies is considered by Sivakumar et al.[60]. Two commodity stochastic inventory system with lost sales, Poisson arrivals with joint and individual ordering policies is considered by Yadavalli et al.[62]

Two commodity continuous review inventory system with substitutable items and Markovian demands is considered by Anbazhagan et al.[3]. Here reordering for supply is initiated as soon as the sum of the on-hand inventory levels of the two commodities reaches a certain level s.

The last chapter considers a queueing - inventory model under the context of crowdsourcing. The concept of crowdsourcing is used by many industries such as food, consumer products, hotels, electronics and other large retailers. A number of examples of crowdsourcing can be found in [51].

According to Howe[23], "Crowdsourcing represents the act of a company or institution taking a function once performed by employees and outsourcing it to a large network of people in the form of an open call. This can take the form of peer production(when the job is performed collaboratively), but is also often undertaken by sole individuals. The crucial prerequisite is the use of the open call format and the large network of potential labourers".

In the paper by Chakravarthy and Dudin[16], they use crowdsourcing in the context of service sectors getting possible help from one group of customers who first receive service from them and then opt to execute similar service to another group of customers. They consider a multi-server queueing system with two type of customers, Type-I and Type-II. Type-I customers visit the store to procure items while Type-II customers orders over some medium such as internet and phone and expects them to be delivered. The store management use the customers visiting them as couriers to serve the other type of customers. Since not all in-store customers may be willing to act as servers, a probability is introduced for in-store customers to opt for serving the other type. They assumed that Type-I have non-preemptive priority over Type-II. This is the first reported work on crowdsourcing modelled in the queueing theory context. A multi-server priority queue with preemption in crowdsourcing is considered in Krishnamoorthy et al [32]. Here they assume that arrival of a Type-I customer interrupts the ongoing service of any one of Type-II customers if any in service, and hence this preempted customer joins back as the head of the Type-II queue.

This thesis analyzes models providing explicit solution for system state distribution and also those that need algorithmic analysis. The matrix-geometric structure of the steady-state distributions introduced by Neuts[48] is used in the models for obtaining solutions.

1.9 Summary of the thesis

This thesis includes analysis of some queueing inventory models which we face in many real life situations. They are studied by means of continuous time Markov chains.In all the models we have assumed that arrival process is a Poisson process and service times are exponential.

This thesis is divided into 6 chapters. including the introductory chapter. Chapter 2 deals with queueing inventory models with several modes of service and chapters 3 and 4 deal with queueing inventory models with reservation, cancellation and common life time. Chapter 5 is on queueing inventory model with two commodites and the last chapter is on queueing inventory model under the context of crowdsourcing.

In chapter 2 we study an M/M/1 queue with an attached inventory system. Customers arrive to the system according to a Poisson process, and are served by a single server. The stock is replenished by (s, Q)-policy and (s, S)-policy which has an exponentially distributed lead time. The service time is exponentially distributed with parameter μ_2 whenever the inventory level is above s and $\alpha\mu_2$ $(0 < \alpha \leq 1)$ whenever the inventory level is below $s + 1$. This is to reduce customer loss on account of the inventory level droping to zero – we assume that customers do not join when the inventory level is zero, thereby leading to product form solution. Using the joint distribution, we introduce long-run performance measures and a cost function. We also provide several numerical examples.

In chapter 3 we consider a single server queueing - inventory system having capacity to store S items at a time which have a common-life time (CLT), exponentially distributed with parameter γ. On realization of CLT a replenishment order is placed so as to bring the inventory level back to S, the lead time of which follows exponential distribution with parameter β. Items remaining are discarded on realization of CLT. Customers waiting in the system stay back on realization of common life time. Reservation of items and cancellation of sold items before its expiry time is permitted. Cancellation takes place according to an exponentially distributed inter-occurrence time with parameter $i\theta$ when there are $(S-i)$ items in the inventory. We assume that the time required to cancel the reservation is negligible. Customers arrive according to a Poisson process of rate λ and service time follows exponential distribution with parameter μ. The main assumption that no customer joins the system when inventory level is zero, leads to a product form solution of the system state distribution. Several system performance measures are obtained.

In chapter 4 we study an M/M/1 queue with a storage system having capacity S which have a common life time (CLT), exponentially distributed. On realization of common life time or the first time inventory level drops to zero in a cycle whichever occurs first, a replenishment order is placed so as to bring the inventory level back to S (zero lead time).

Customers arrive to the system according to a Poisson process and their service time is exponentially distributed. Reservation of items and cancellation of sold items is permitted before the realization of common life time. Cancellation takes place according to an exponential distribution. In this chapter we assume that the time required to cancel the reservation is negligible. When the inventory level becomes zero through service completion or CLT realization, a replenishment order is placed which is realized instantly. We first derive the stationary joint distribution of the queue length and the on-hand inventory in product form. Using the joint distribution, long-run performance measures and a revenue function. The case of positive lead time is also investigated. Numerical illustrations are provided.

A two commodity inventory system with a single server is considered in chapter 5. We assume that the buffer sizes(to store the two types of commodities) are finite. Customers (or demands) arrive according to a Poisson Process and the requirement for either type or both type of commodities are assigned certain probabilities. Customers are lost when their demands are not met due to shortage at the time of offering of service as opposed to getting lost when the inventory level is zero at the time of arrival. This is to allow the possibility of inventory being replenished during the time of existing service. A customer's demand for both items will be met with only one item if their is a situation in which only one type of inventory is readily available and the other is zero at the time of initiating a service. The processing time for meeting the demands are random and modelled using exponential distribution with parameters depending on the type of demands being processed. We adopt (s,S)- type replenishment policy which depends on the type of commodity. Assuming the lead time to be exponentially distributed with parameters depending on the type of commodity, we employ matrix-analytic methods to study the queueing inventory system and report interesting results including an optimization problem dealing with various costs.

In chapter 6, we consider a multi- server queueing inventory system with two type of customers: Type I and Type II. Type II customers are virtual ones. Arrival of both Type I and Type II customers follow two independent Poisson processes. Type I are to be served by one of the servers and service time is assumed to be exponential. Type II customer may be served by a Type I customer having already been served and ready to act as a server or by one of the servers with exponentially distributed service time. Type I customer has non preemptive priority over Type II. Type II is served by a Type I only if inventory is available after attaching inventory to the existing Type-I customers available in the system. Type II is served by a Type I with probability p and with complementary probability $q = 1 - p$ served Type I leaves the system. Arrival of both type of customers is permitted only when *excess inventory*, which is defined as the difference between on hand inventory and number of busy servers, is positive. There is a limited system capacity for Type I, where as Type II has unlimited waiting area. When inventory level drops to $c + s$, an order for replenishment is placed to bring the inventory level to $c + S$. The ordered items are received after a random amount of time which is exponentially distributed. An optimization problem is numerically analyzed.

Finally a section "concluding remarks and suggestions for future study" is included.

Chapter 2

Queueing-Inventory System with Several Modes of Service

In this chapter a queueing-inventory model under (s, Q) and (s, S) policies with several modes of service is analyzed. We introduce distinct rates of service based on whether inventory level is above s or less than or equal to s and proved that under certain assumptions stochastic decomposition of the vector process is possible for the (s, Q) and (s, S) policies. The purpose of introducing different service rates is to minimize 'customer loss' which is a consequence of the assumption that no customer joins the system when inventory level is zero. It is this assumption that enables us to derive stochastic decomposition of the system state and consequent product form solution. The minimization of customer loss is achieved by

Some results in this chapter are included in the paper. *Dhanya Shajin, Binitha Benny, Rostislav V. Razumchik and A. Krishnamoorthy* : **Queueing Inventory system with two modes of service**, Journal of Automation and Control of Russian Academy Of Sciences. (To appear in October 2018 issue; the English translation will appear subsequently in the same journal) switching over to a reduced service rate during lead time. However, it is done at the expense of an increase in the waiting cost of customers. We try to have a trade off between the two. To this end we construct a cost function with the objective of minimizing "total expected cost". It is seen that (s, Q) policy outperforms the (s, S) policy.

A continuous review (s, S) inventory system at a service facility with two types of services and finite waiting hall was considered by Anbazhagan et al. [4]. Demands arrive according to a Poisson Process and the server provides two types of services, type 1 with probability p_1 and type 2 with probability p_2 with the service time following distinct exponential distributions. They derived the joint probability distribution of both the inventory level and the number of customers in the steady state case where the lead times are negative exponential and demands during stock -out periods are lost.

2.1 Mathematical formulation

Consider a single server queueing-inventory system where service rule is FIFO. Arrival process is assumed to be Poisson with rate λ. Service time follows exponential distribution with parameter μ_1 if the inventory level lies between 1 and s both inclusive, else it is μ_2 with $\mu_1 = \alpha\mu_2$ $(0 \leq \alpha \leq 1)$. The maximum capacity of the inventory level is fixed as S, when the inventoried items reach the level $s \geq 0$, an order for replenishment by fixed quantity Q, where $Q = S - s$, is placed. The lead time is exponentially distributed with parameter β which is independent of the service and arrival processes. No customer is allowed to join the queue when the inventory level is zero. Further in the absence of inventory, service cannot take place even when customers are present. Let

$N(t)$: Number of customers in the system at time t
$I(t)$: Number of items in the inventory at time t

Then $\Omega = \{(N(t), I(t)), t \geq 0\}$ forms a $CTMC$ with state space

$$\{(n, i); n \geq 0, 0 \leq i \leq S\}.$$

We now describe the infinitesimal generator matrix $\mathcal{Q}$ of this $CTMC$. Note that by the assumptions made above the $CTMC$ Ω is a $LIQBD$. We have

$$\mathcal{Q} = \begin{bmatrix} A_{00} & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.1}$$

Each matrix A_{00}, A_0, A_1, A_2 is a square matrix of order $(S+1)$ where

$$(A_{00})_{ij} = \begin{cases} \beta & j = i + Q, 1 \leq i \leq s+1 \\ -\beta & j = i, i = 1 \\ -(\lambda + \beta) & j = i, 2 \leq i \leq s+1 \\ -\lambda & j = i, s+2 \leq i \leq S+1 \\ 0 & \text{otherwise} \end{cases}$$

$$(A_0)_{ij} = \begin{cases} \lambda & j = i, 2 \leq i \leq S+1 \\ 0 & \text{otherwise} \end{cases}$$

$$(A_1)_{ij} = \begin{cases} \beta & j = i + Q, 1 \leq i \leq s + 1 \\ -\beta & j = i, i = 1 \\ -(\lambda + \beta + \mu_1) & j = i, 2 \leq i \leq s + 1 \\ -(\lambda + \mu_2) & j = i, s + 2 \leq i \leq S + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(A_2)_{ij} = \begin{cases} \mu_1 & j = i - 1, 2 \leq i \leq s + 1 \\ \mu_2 & j = i - 1, s + 2 \leq i \leq S + 1 \\ 0 & \text{otherwise} \end{cases}$$

2.1.1 Stability condition

Next we examine the system stability. Define $A = A_0 + A_1 + A_2$. Then

$$A = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ Q \\ Q+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & 1 & \dots & s & s+1 & \dots & Q & Q+1 & \dots & S \end{array} \\ \left(\begin{array}{cccccccccc} -\beta & & & & & & \beta & & & \\ \mu_1 & -(\beta+\mu_1) & & & & & & \beta & & \\ & \ddots & \ddots & & & & & & \ddots & \\ & & \mu_1 & -(\beta+\mu_1) & & & & & & \beta \\ & & & \mu_2 & -\mu_2 & & & & & \\ & & & & \ddots & \ddots & & & & \\ & & & & & \mu_2 & -\mu_2 & & & \\ & & & & & & \mu_2 & -\mu_2 & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & \mu_2 & -\mu_2 \end{array} \right) \end{array}.$$

This is the infinitesimal generator of the finite state $CTMC$ $\Omega' = \{I(t), t \geq 0\}$ corresponding to the inventory level in the system $\{0, 1, ..., S\}$. Let $\boldsymbol{\pi} = (\pi_0, \pi_1, ..., \pi_S)$ be the steady state probability vector of A. Then

$$\boldsymbol{\pi} A = 0, \quad \boldsymbol{\pi}\mathbf{e} = 1. \tag{2.2}$$

From (2.2) we have

$$\pi_i = \begin{cases} \frac{\beta}{\mu_1}\left(\frac{\beta+\mu_1}{\mu_1}\right)^{i-1}\pi_0 & 1 \le i \le s \\ \frac{\beta}{\mu_1}\left(\frac{\beta+\mu_1}{\mu_1}\right)^{s-1}\left(\frac{\beta+\mu_1}{\mu_2}\right)\pi_0 & s+1 \le i \le Q \\ \frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^{i-(Q+1)}\left[\left(\frac{\beta+\mu_1}{\mu_1}\right)^{S-(i-1)} - 1\right]\pi_0 & Q+1 \le i \le S \end{cases}$$

where π_0 is obtained from the normalizing condition as

$$\pi_0 = \left[1 + \left(\frac{\mu_2-\mu_1}{\mu_2}\right)\left(\left(\frac{\beta+\mu_1}{\mu_1}\right)^s - 1\right) + Q\frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^s\right]^{-1}.$$

The following lemma establishes the stability condition of the queueing-inventory system under study.

Lemma 2.1.1. The system under study is stable if and only if

$$\lambda < \frac{Q\beta\left(\frac{\beta+\mu_1}{\mu_1}\right)^s}{\left(\frac{\mu_2-\mu_1}{\mu_2}\right)\left(\left(\frac{\beta+\mu_1}{\mu_1}\right)^s - 1\right) + Q\frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^s}. \tag{2.3}$$

Proof. The queueing-inventory system under study with the QBD type generator given in (2.1) is stable if and only if the left drift rate exceeds the right drift rate. In the present case these drift rates are respectively $\boldsymbol{\pi} A_2\mathbf{e}$ and $\boldsymbol{\pi} A_0\mathbf{e}$(see Neuts [48]) Thus the above condition reduces to,

$$\boldsymbol{\pi} A_0\mathbf{e} < \boldsymbol{\pi} A_2\mathbf{e} \tag{2.4}$$

From the matrices A_0, A_2 we have $\boldsymbol{\pi} A_0\mathbf{e} = \lambda\sum_{i=1}^{S}\pi_i = \lambda(1-\pi_0)$ and

$$\boldsymbol{\pi} A_2\mathbf{e} = \mu_1 \sum_{i=1}^{s} \pi_i + \mu_2 \sum_{i=s+1}^{S} \pi_i = Q\beta \left(\frac{\beta+\mu_1}{\mu_1}\right)^s \pi_0.$$

Using relation (2.4) we obtain the stability condition as

$$\begin{array}{rrl}
 & \lambda(1-\pi_0) & < Q\beta\left(\frac{\beta+\mu_1}{\mu_1}\right)^s \pi_0 \\
\Rightarrow & \lambda & < \left[Q\beta\left(\frac{\beta+\mu_1}{\mu_1}\right)^s + \lambda\right]\pi_0 \\
\Rightarrow & \lambda & < \left[Q\beta\left(\frac{\beta+\mu_1}{\mu_1}\right)^s + \lambda\right]\left[1+\left(\frac{\mu_2-\mu_1}{\mu_2}\right)\left(\left(\frac{\beta+\mu_1}{\mu_1}\right)^s - 1\right) + Q\frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^s\right]^{-1}
\end{array}$$

$$\begin{array}{rl}
\Rightarrow \lambda\left[1+\left(\frac{\mu_2-\mu_1}{\mu_2}\right)\left(\left(\frac{\beta+\mu_1}{\mu_1}\right)^s - 1\right) + Q\frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^s\right] & < \left[Q\beta\left(\frac{\beta+\mu_1}{\mu_1}\right)^s + \lambda\right] \\
\Rightarrow \lambda + \lambda\left[\left(\frac{\mu_2-\mu_1}{\mu_2}\right)\left(\left(\frac{\beta+\mu_1}{\mu_1}\right)^s - 1\right) + Q\frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^s\right] & < Q\beta\left(\frac{\beta+\mu_1}{\mu_1}\right)^s + \lambda \\
\Rightarrow \lambda\left[\left(\frac{\mu_2-\mu_1}{\mu_2}\right)\left(\left(\frac{\beta+\mu_1}{\mu_1}\right)^s - 1\right) + Q\frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^s\right] & < Q\beta\left(\frac{\beta+\mu_1}{\mu_1}\right)^s.
\end{array}$$

From the above inequality we get the stated result (2.3). □

2.2 Steady state analysis

For finding the steady state probability vector of the $CTMC$ Ω, we first consider the system where the serving of the inventory is instantaneous. Thus the infinitesimal generator is given by

$$\tilde{A} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ Q \\ Q+1 \\ \vdots \\ S \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & \dots & s & s+1 & \dots & Q & Q+1 & \dots & S \end{array} \\
\left(\begin{array}{cccccccccc}
-\beta & & & & & & \beta & & & \\
\lambda & -(\beta+\lambda) & & & & & & \beta & & \\
 & \ddots & \ddots & & & & & & \ddots & \\
 & & \lambda & -(\beta+\lambda) & & & & & & \beta \\
 & & & \lambda & -\lambda & & & & & \\
 & & & & \ddots & \ddots & & & & \\
 & & & & & \lambda & -\lambda & & & \\
 & & & & & & \lambda & -\lambda & & \\
 & & & & & & & \ddots & \ddots & \\
 & & & & & & & & \lambda & -\lambda
\end{array}\right)
\end{array}.$$

Let $\boldsymbol{\xi} = (\xi_0, \xi_1, ..., \xi_S)$ be the steady state vector of $\tilde{A}$. Then $\boldsymbol{\xi}$ satisfies the equations

$$\boldsymbol{\xi}\tilde{A} = 0, \qquad \boldsymbol{\xi}\mathbf{e} = 1. \tag{2.5}$$

From $\boldsymbol{\xi}\tilde{A} = 0$ we have

$$\begin{aligned}
-\beta\xi_0 + \lambda\xi_1 &= 0, \\
-(\beta+\lambda)\xi_i + \lambda\xi_{i+1} &= 0, 1 \le i \le s \\
-\lambda\xi_i + \lambda\xi_{i+1} &= 0, s+1 \le i \le Q-1 \\
\beta\xi_{i-Q} - \lambda\xi_i + \lambda\xi_{i+1} &= 0, Q \le i \le S-1 \\
\beta\xi_s - \lambda\xi_S &= 0
\end{aligned}$$

and ξ_i can be obtained as

$$\xi_i = \begin{cases} \frac{\beta}{\lambda}\left(\frac{\beta+\lambda}{\lambda}\right)^{i-1}\xi_0 & 1 \le i \le s \\ \frac{\beta}{\lambda}\left(\frac{\beta+\lambda}{\lambda}\right)^{s}\xi_0 & s+1 \le i \le Q \\ \frac{\beta}{\lambda}\left[\left(\frac{\beta+\lambda}{\lambda}\right)^{s} - \left(\frac{\beta+\lambda}{\lambda}\right)^{i-(Q+1)}\right]\xi_0 & Q+1 \le i \le S \end{cases}$$

The unknown probability ξ_0 can be found from the normalizing condition

$$\xi_0 = \left[1 + Q\frac{\beta}{\lambda}\left(\frac{\beta+\lambda}{\lambda}\right)^{s}\right]^{-1}.$$

Now using the vector $\boldsymbol{\xi}$, we can find the steady state vector of the given system. Let $\mathbf{x}$ be the steady state vector of the generator $\mathcal{Q}$. Then $\mathbf{x}$ must satisfy the set of equations

$$\mathbf{x}\mathcal{Q} = 0, \qquad \mathbf{x}\mathbf{e} = 1. \tag{2.6}$$

Partition $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$. Then the above system of equations reduces to:

$$\mathbf{x}_0 A_{00} + \mathbf{x}_1 A_2 = 0, \tag{2.7}$$

$$\mathbf{x}_{i-1} A_0 + \mathbf{x}_i A_1 + \mathbf{x}_{i+1} A_2 = 0, \quad i \geq 1. \tag{2.8}$$

Then by using the relations (2.7) and (2.8), we get

$$\begin{array}{ll}
-\beta x_i(0) + \mu_1 x_{i+1}(1) & = 0, i \geq 0 \\
-(\beta+\lambda) x_0(j) + \mu_1 x_1(j+1) & = 0, 1 \leq j \leq s-1 \\
\lambda x_{i-1}(j) - (\beta+\lambda+\mu_1) x_i(j) + \mu_1 x_{i+1}(j+1) & = 0, i \geq 1, 1 \leq j \leq s-1 \\
-(\beta+\lambda) x_0(s) + \mu_2 x_1(s+1) & = 0, \\
\lambda x_{i-1}(s) - (\beta+\lambda+\mu_1) x_i(s) + \mu_1 x_{i+1}(s+1) & = 0, i \geq 1 \\
-\lambda x_0(j) + \mu_2 x_1(j+1) & = 0, s+1 \leq j \leq Q-1 \\
\lambda x_{i-1}(j) - (\lambda+\mu_2) x_i(j) + \mu_2 x_{i+1}(j+1) & = 0, i \geq 1, s+1 \leq j \leq Q-1 \\
\beta x_0(j-Q) - \lambda x_0(j) + \mu_2 x_1(j+1) & = 0, Q \leq j \leq S-1 \\
\lambda x_{i-1}(j) + \beta x_i(j-Q) - (\lambda+\mu_2) x_i(j) + \mu_2 x_{i+1}(j+1) & = 0, i \geq 1, Q \leq j \leq S-1 \\
\beta x_0(s) - \lambda x_0(S) & = 0, \\
\lambda x_{i-1}(S) + \beta x_i(s) - (\lambda+\mu_2) x_i(S) & = 0, i \geq 1
\end{array}$$

Solving the above system of linear relations we get

$$x_i(j) = \begin{cases} \vartheta^{-1} \mathcal{C}_i(j) \left(\frac{\lambda}{\mu_1}\right)^i \xi_j & \text{for } i \geq 0, 0 \leq j \leq s \\ \vartheta^{-1} \mathcal{C}_i(j) \left(\frac{\lambda}{\mu_2}\right)^i \xi_j & \text{for } i \geq 0, s+1 \leq j \leq S \end{cases} \tag{2.9}$$

where $\mathcal{C}_i(j)$ are constants to be determined.

The constants $\mathcal{C}_i(j)$ are given by

$$\mathcal{C}_i(j) = \begin{cases} \mathcal{C}_0(0) & \text{for } i = 0, 1 \le j \le S \\ \mathcal{C}_0(0) & \text{for } i = 1, 0 \le j \le S \\ \mathcal{C}_0(0) & \text{for } i \ge 2, 0 \le j \le s \\ w_i(j)\mathcal{C}_0(0) & \text{for } i \ge 2, s+1 \le j \le S \end{cases} \tag{2.10}$$

where

$$w_i(j) = \begin{cases} \left(\frac{\mu_2}{\mu_1}\right)^{i-1} & i \ge 2, j = s+1 \\ 1 & i = 2, s+2 \le j \le Q \\ \mathcal{U}\left[h^s - \frac{\mu_2}{\mu_1}\right] & i = 2, j = Q+1 \\ \mathcal{V}\left[h^s - h^{j-(Q+2)}\left(\frac{\beta\mu_2+\lambda\mu_1}{\lambda\mu_1}\right)\right] & i = 2, Q+2 \le j \le S \\ \frac{\mu_2}{\lambda}\left[\frac{\lambda+\mu_2}{\mu_2} w_2(j-1) - 1\right] & i = 3, s+2 \le j \le Q \\ \frac{\mu_2}{\lambda}\left[\frac{\lambda+\mu_2}{\mu_2} w_{i-1}(j-1) - w_{i-2}(j-2)\right] & i \ge 4, s+2 \le j \le Q \\ \mu_2^2\mathcal{U}\left[h^s\left(\frac{\lambda+\mu_2}{\lambda\mu_2^2} w_2(Q) - \frac{1}{\lambda\mu_2}\right) - \frac{1}{\mu_1^2}\right] & i = 3, j = Q+1 \\ \mu_2^{i-1}\mathcal{U}\left[h^s\left(\frac{\lambda+\mu_2}{\lambda\mu_2^{i-1}} w_{i-1}(Q) - \frac{w_{i-2}(Q)}{\lambda\mu_2^{i-2}}\right) - \frac{1}{\mu_1^{i-1}}\right] & i \ge 4, j = Q+1 \\ \frac{\mu_2^2}{\lambda}\mathcal{V}\left[h^s \frac{\lambda+\mu_2}{\mu_2^2} w_2(j-1) - h^{j-(Q+2)}\left(\frac{1}{\mu_2} - \frac{\beta}{\mu_1^2}\right)\right] & i = 3, Q+2 \le j \le S \\ \frac{\mu_2^{i-1}}{\lambda}\mathcal{V}\left[h^s \frac{\lambda+\mu_2}{\mu_2^{i-1}} w_2(j-1) - h^{j-(Q+2)}\left(\frac{w_{i-2}(j-1)}{\mu_2^{i-2}} - \frac{\beta}{\mu_1^{i-1}}\right)\right] & i \ge 4, Q+2 \le j \le S \end{cases}$$

with $h = \left(\frac{\beta+\lambda}{\lambda}\right), \mathcal{U} = [h^s - 1]^{-1}, \mathcal{V} = \left[h^s - h^{j-(Q+1)}\right]^{-1}$.

Thus we have

$$\sum_{i=0}^{\infty}\left[\sum_{j=0}^{s}\mathcal{C}_i(j)\left(\frac{\lambda}{\mu_1}\right)^i \xi_j + \sum_{j=s+1}^{S}\mathcal{C}_i(j)\left(\frac{\lambda}{\mu_2}\right)^i \xi_j\right]$$

$$= \left\{ h^s \sum_{i=0}^{\infty}\left(\frac{\lambda}{\mu_1}\right)^i + \frac{\lambda+\mu_2}{\mu_2}\left[h^s\left(Q\frac{\beta}{\lambda} - 1\right) + 1\right] + \frac{\beta}{\lambda}\sum_{i=2}^{\infty}\left(\frac{\lambda}{\mu_2}\right)^i \right.$$

$$\left[h^s \sum_{j=s+1}^{S} w_i(j) - \sum_{j=Q+1}^{S} h^{j-(Q+1)} w_i(j)\right]\Bigg\} \left[1 + Q\frac{\beta}{\lambda}h^s\right]^{-1} \mathcal{C}_0(0).$$

If we note $\mathbf{xe} = 1$ and (2.9) we have

$$\vartheta^{-1} \sum_{i=0}^{\infty} \left[\sum_{j=0}^{s} \mathcal{C}_i(j) \left(\frac{\lambda}{\mu_1}\right)^i \xi_j + \sum_{j=s+1}^{S} \mathcal{C}_i(j) \left(\frac{\lambda}{\mu_2}\right)^i \xi_j\right] = 1.$$

Write $\vartheta = \Bigg\{ h^s \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu_1}\right)^i + \frac{\lambda+\mu_2}{\mu_2}\left[h^s\left(Q\frac{\beta}{\lambda} - 1\right) + 1\right] + \frac{\beta}{\lambda} \sum_{i=2}^{\infty} \left(\frac{\lambda}{\mu_2}\right)^i \Bigg[h^s \sum_{j=s+1}^{S} w_i(j) - \sum_{j=Q+1}^{S} h^{j-(Q+1)} w_i(j)\Bigg]\Bigg\} \left[1 + Q\frac{\beta}{\lambda}h^s\right]^{-1} \mathcal{C}_0(0).$

Hence we have the theorem:

Theorem 2.2.1. *If the stability condition (2.3) holds, then the components of the steady-state probability vector are*

$$x_i(j) = \begin{cases} \vartheta^{-1}\mathcal{C}_i(j)\left(\frac{\lambda}{\mu_1}\right)^i \xi_j & \text{for } i \geq 0, 0 \leq j \leq s \\ \vartheta^{-1}\mathcal{C}_i(j)\left(\frac{\lambda}{\mu_2}\right)^i \xi_j & \text{for } i \geq 0, s+1 \leq j \leq S \end{cases}$$

the probabilities $\xi_j, 0 \leq j \leq S$ corresponds to the distribution of number of items in the inventory in the system.

2.3 Performance Measures

- Mean number of customers in the system, $E_N = \sum_{i=1}^{\infty} i\mathbf{x}_i\mathbf{e}$
- Mean number of customers in the system whenever the inventory

level is less than $s+1$, $N_1 = \sum_{i=1}^{\infty}\sum_{j=0}^{s} i x_i(j)$

- Mean number of customers in the system whenever the inventory level is above s, $N_2 = \sum_{i=1}^{\infty}\sum_{j=s+1}^{S} i x_i(j)$

- Expected number of items in the inventory, $E_I = \sum_{i=0}^{\infty}\sum_{j=1}^{S} j x_i(j)$

- Expected reorder rate, $E_R = \mu_2 \sum_{i=1}^{\infty} x_i(s+1)$

- Expected loss rate of customers, $E_L = \lambda \sum_{i=0}^{\infty} x_i(0)$

- Expected number of customers arriving per unit time, $E_A = \lambda \sum_{i=0}^{\infty}\sum_{j=1}^{S} x_i(j)$

- Expected waiting time of the customers in the system, $E_W = \dfrac{E_N}{E_A}$

- Mean number of customers waiting in the system when inventory is available, $E_{N_1} = \sum_{i=1}^{\infty}\sum_{j=1}^{S} i x_i(j)$

- Mean number of customers waiting in the system during the stock out period, $E_{N_2} = \sum_{i=1}^{\infty} i x_i(0)$

- Mean number of replenishment per unit time, $E_{NR} = \beta \sum_{i=0}^{\infty}\sum_{j=0}^{s} x_i(j)$

Next we proceed to determine α so as to have at least a desired probability $1-\epsilon$ of replenishment preceding the sale of s items from the epoch at which order for the former is placed. In other words we wish to attain a high probability for no customer loss for want of inventory. We consider different cases and obtain the following results.

2.3.1 Max. prob(lead time process < time required to serve s demands)

Before going further let us introduce some notations. Let ξ denote the lead time and η denote the time to serve s customers. Assume that at instant τ the replenishment order is placed. Introduce the following probabilities

$$a_{ij} = \mathbf{P}\{\xi < \eta | I(\tau) = i, N(\tau) = j\},\ 1 \leq i \leq s,\ j \geq 0.$$

According to the considered replenishment rule, the replenishment order is placed if and only if the inventory level drops down to s. Thus we are interested in the probabilities

$$a_{s,j},\ j \geq 0.$$

These probabilities are of interest because, as it was mentioned in the description of the system, when the inventory level is zero, no customers are allowed to enter the system. Thus once the inventory level reaches 0, there is a chance for potential customer losses. Indeed, the probability that after the replenishment order has been made, at least one customer will be lost is equal to

$$(1 - a_{s,j})\frac{\lambda}{\lambda + \beta}.$$

When speaking about the inventory system one is usually interested in choosing such parameter values which lead to the optimal value of a certain value function. We will discuss the value function and its optimization in section 2.5.1. But we notice that such function may include additional costs for customer losses. The only way to influence the customer loss probability is to adjust the service rate $\alpha\mu_2$ i.e. to manipulate the value of α.

Now we will show how to calculate $a_{s,j}$, $j \geq 0$.

Firstly notice that in order to calculate $a_{s,j}$, $j \geq 0$, one has to be able to calculate other probabilities $a_{i,j}$, $1 \leq i \leq s$, $j \geq 0$.

Secondly notice that we have to distinguish 2 cases:

1. $j \geq s$;
2. $0 \leq j < s$.

Case 1: Number of customers in the system at the epoch of placing an order for replenishment is $\geq s$

Let us calculate the $\mathbf{P}$(lead time process $<$ time required to serve s demands). If the number of customers in the system at the epoch of placing an order for replenishment $\geq s$, using the notation introduced above we have

$$\mathbf{P}(\text{lead time process } < \text{ time required to serve } s \text{ demands}) = a_{s,j},\ j \geq s.$$

Due to the fact that for each j the probabilities $a_{s,j}$ are the same (i.e. $a_{s,s} = a_{s,s+1} = a_{s,s+2} = \dots$) all we need is to calculate one of them. Let us calculate $a_{s,s}$.

In this case we need not consider future arrivals for the computation of the required probability because s customers are already present in the system. For illustration purposes in this case we will use the matrix notation. Consider the inventory level process $\{I(t), t \geq 0\}$ whose state space $\{1 \leq i \leq s\} \bigcup \{\Delta_s\} \bigcup \{\Delta_r\}$ where $\{\Delta_s\}$ is the absorbing state meaning service of s demands has occurred before replenishment and $\{\Delta_r\}$ is the absorbing state meaning the replenishment occurred before service of s demands. Thus its infinitesimal generator is of the form

$$\mathcal{W} = \begin{bmatrix} -(\alpha\mu_2+\beta) & \alpha\mu_2 & & & 0 & \beta \\ & -(\alpha\mu_2+\beta) & \alpha\mu_2 & & 0 & \beta \\ & & \ddots & \ddots & & \vdots \\ & & & -(\alpha\mu_2+\beta) & \alpha\mu_2 & \beta \\ 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} T & \vec{t^s} & \vec{t^r} \\ \mathbf{0} & \vec{0} & \vec{0} \end{bmatrix}$$

with the matrix T of size $s \times s$.

If the initial probability vector is $\vec{\gamma} = (1, 0, ..., 0)$ of order s, then the probability $a_{s,s}$, that the replenishment occurs before the service of s demands if the replenishment order was placed when the total number of customers in the system was $\geq s$, is equal to

$$a_{s,s} = -\gamma T^{-1}\vec{t^r}.$$

Using the explicit form of the inverse of T (which is an upper triangular matrix), after some simple computations one obtains

$$a_{s,s} = \frac{\beta}{\alpha\mu_2 + \beta} \sum_{n=0}^{s-1} \left(\frac{\alpha\mu_2}{\alpha\mu_2 + \beta} \right)^n .$$

This expression has a clear probabilistic interpretation.

In general it is easy to see that the probabilities $a_{i,i}$, $1 \leq i \leq s$, are equal to

$$a_{i,i} = \frac{\beta}{\alpha\mu_2 + \beta} \sum_{n=0}^{i-1} \left(\frac{\alpha\mu_2}{\alpha\mu_2 + \beta} \right)^n , \ 1 \leq i \leq s.$$

The complementary probability, the replenishment occurs *later* than the service of s demands if the replenishment order was placed when the total number of customers in the system was $\geq s$, is equal to

$$\mathbf{P}(\text{ time required to serve } s \text{ demands} < \text{lead time process }) = 1 - a_{s,s}$$

$$= -\vec{\gamma} T^{-1} \vec{t}^s = \left(\frac{\alpha\mu_2}{\alpha\mu_2 + \beta} \right)^s .$$

Finally, notice that the probability $\pi_{\geq s}$ that at least one customer will be lost, if at the epoch of placing an order for replenishment the number of customers in the system $\geq s$, is equal to

$$\pi_{\geq s} = \left(\frac{\alpha\mu_2}{\alpha\mu_2 + \beta} \right)^s \frac{\lambda}{\lambda + \beta}. \tag{2.11}$$

Case 2: Number of customers in the system at the epoch of placing an order for replenishment is $< s$

In this case we have to consider future arrivals for the computation of the required probability because there are less than s customers in the system at the epoch of placing the order for replenishment.

Assume that at the epoch of placing the order for replenishment there are j, $0 \le j < s$ customers in the system. Thus the probability we have to find is $a_{s,j}$.

We will not use the matrix notation as in the previous section, instead will use the first step analysis.

Let us start with $j = s - 1$ i.e. there are $(s - 1)$ customers in the system at the epoch of placing the order for replenishment. Using the first step analysis we can write out the following (finite) system of algebraic equations for finding $a_{s,s-1}$:

$$
\begin{aligned}
a_{s,s-1} &= \frac{\beta}{\lambda + \mu_2\alpha + \beta} + \frac{\lambda}{\lambda + \mu_2\alpha + \beta} a_{s,s} + \frac{\mu_2\alpha}{\lambda + \mu_2\alpha + \beta} a_{s-1,s-2}, \\
a_{s-1,s-2} &= \frac{\beta}{\lambda + \mu_2\alpha + \beta} + \frac{\lambda}{\lambda + \mu_2\alpha + \beta} a_{s-1,s-1} + \frac{\mu_2\alpha}{\lambda + \mu_2\alpha + \beta} a_{s-2,s-3}, \\
&\cdots \\
a_{2,1} &= \frac{\beta}{\lambda + \mu_2\alpha + \beta} + \frac{\lambda}{\lambda + \mu_2\alpha + \beta} a_{2,2} + \frac{\mu_2\alpha}{\lambda + \mu_2\alpha + \beta} a_{1,0}, \\
a_{1,0} &= \frac{\beta}{\lambda + \beta} + \frac{\lambda}{\lambda + \beta} a_{1,1}.
\end{aligned}
$$

Notice that the values of $a_{i,i}$, $1 \le i \le s$, have already been found in the previous section.

The above system of equations can be solved recursively, starting from the last equation. Denoting $d = \frac{\mu_2\alpha}{\lambda+\mu_2\alpha+\beta}$, the solution can be written out in the following form:

$$
a_{k+1,k} = \frac{1}{\lambda + \mu_2\alpha + \beta} \sum_{i=2}^{k+1} [\lambda a_{i,i} + \beta] d^{k+1-i} + \frac{\lambda a_{1,1} + \beta}{\lambda + \beta} d^k, \ 0 \le k \le s - 1.
$$

The expression for the required probability $a_{s,s-1}$ is found by putting $k = s-1$ in the previous relation:

$$a_{s,s-1} = \frac{1}{\lambda+\mu_2\alpha+\beta}\sum_{i=2}^{s}[\lambda a_{i,i}+\beta]d^{s-i} + \frac{\lambda a_{1,1}+\beta}{\lambda+\beta}d^{s-1}.$$

In order to find other probabilities $a_{s,s-2},\ a_{s,s-3},\ \ldots,\ a_{s,0}$ we can proceed in the same way i.e. we can use the first step analysis, then write out the system of equations and solve it. By doing so we can arrive at the following expression for the computation of any probability $a_{i,j}$, $0 \le k \le s-i$, $1 \le i \le s$:

$$a_{k+i,k} = \frac{1}{\lambda+\mu_2\alpha+\beta}\sum_{n=2}^{k+1}[\lambda a_{n-1+i,n}+\beta]d^{k+1-n} + \frac{\lambda a_{i,1}+\beta}{\lambda+\beta}d^{k},\ 0 \le k \le s-i, 1 \le i \le s. \tag{2.12}$$

But the computation has to be performed sequentially. At first one fixes $i = 1$ and computes $a_{k+1,k}$ for $0 \le k \le s-1$. Then one fixes $i = 2$ and computes $a_{k+2,k}$ for $0 \le k \le s-2$ and so on until $i = s$.

Now we can calculate the probability π_j, $0 \le j \le s-1$, that at least one customer will be lost, if at the epoch of placing an order for replenishment the number of customers in the system is j:

$$\pi_j = (1-a_{s,j})\frac{\lambda}{\lambda+\beta}. \tag{2.13}$$

Thus,we have the lemma,

Lemma 2.3.1. If at the epoch of placing an order for replenishment,

(i) P {at least one customer will be lost, where the number of customers in the system is j,$0 \leq j \leq s-1\}$= $(1-a_{s,j})\frac{\lambda}{\lambda+\beta}$.

(ii) P {at least one customer will be lost, where the number of customers in the system is j,$j \geq s\}$=$\left(\frac{\alpha\mu_2}{\alpha\mu_2+\beta}\right)^s \frac{\lambda}{\lambda+\beta}$.

Table 2.1 shows the probability of loss of customers in the cases discussed in the above lemma for varying values of α when we fix $(S, s, \lambda, \mu_2, \beta) = (15, 7, 3, 10, 4)$.

α	$n \geq s$	$n = 0$	$0 < n < s$ ($n = 4$)
1	0.0474	0.0007	0.0149
0.9	0.0381	0.0006	0.0128
0.8	0.0293	0.0005	0.0105
0.75	0.0251	0.0005	0.0094
0.7	0.0211	0.0004	0.0082
0.6	0.0140	0.0003	0.0059
0.5	0.0082	0.0002	0.0038
0.4	0.0039	0.0001	0.0020
0.3	0.0013	0.0001	0.0008
0.25	0.0006	0	0.0004
0.2	0.0002	0	0.0001
0.1	0	0	0

Table 2.1: α verses loss probability

The numerical output shown in Table 2.1 are on expected lines. We notice that with number of customers (n) at order placement epoch (inventory level = s) is at least equal to s, the inventory level depletes faster to go down to zero, resulting in high loss probability of customers. The loss probability is least for the case $n = 0$. This is no surprise since it required s new arrivals and their services completed before replenishment, in order that customers are lost.

2.4 System under (s, S) policy

Now we turn to a brief description of the system under the (s, S) policy. The generator matrix is similar to the system under (s, Q) policy (see (2.1)) but with

$$(A_{00})_{ij} = \begin{cases} \beta & j = S+1, 1 \le i \le s+1 \\ -\beta & j = i, i = 1 \\ -(\lambda+\beta) & j = i, 2 \le i \le s+1 \\ -\lambda & j = i, s+2 \le i \le S+1 \\ 0 & \text{otherwise.} \end{cases}$$

$$(A_{1})_{ij} = \begin{cases} \beta & j = S+1, 1 \le i \le s+1 \\ -\beta & j = i, i = 1 \\ -(\lambda+\beta+\mu_1) & j = i, 2 \le i \le s+1 \\ -(\lambda+\mu_2) & j = i, s+2 \le i \le S+1 \\ 0 & \text{otherwise.} \end{cases}$$

2.4.1 Stability Condition

To establish the stability condition, define $A = (A_0 + A_1 + A_2)$. This is the infinitesimal generator of the finite state $CTMC$ $\{I(t):t \ge 0\}$, where $I(t)$ is as defined earlier whose state space is given by$\{0, 1, 2, \cdots S\}$. Let $\boldsymbol{\phi} = (\phi_0, \phi_1, ..., \phi_S)$ be the steady-state probability vector of A. Then $\boldsymbol{\phi}$ satisfies the equation

$$\boldsymbol{\phi} A = 0, \quad \boldsymbol{\phi}\mathbf{e} = 1. \tag{2.14}$$

$$
A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S-1 \\ S \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 \cdots & s & s+1\cdots & S-1 & S \end{array} \\
\begin{pmatrix}
-\beta & & & & & & & \beta \\
\mu_1 & -(\mu_1+\beta & & & & & & \beta \\
 & \mu_1 & -(\mu_1+\beta) & & & & & \beta \\
 & & & & & & & \\
 & & & & & & & \beta \\
 & & & & -\mu_2 & & & 0 \\
 & & & & & & & 0 \\
 & & & & & \mu_2 & -\mu_2 & 0 \\
 & & & & & & \mu_2 & -\mu_2
\end{pmatrix}
\end{array}
$$

Then the components of $\boldsymbol{\phi}$ can be obtained as

$$
\phi_i = \begin{cases} \frac{\beta}{\mu_1}\boldsymbol{\phi}_0 & i = 1 \\ \frac{\beta}{{\mu_1}^i}(\beta+\mu_1)^{i-1}\boldsymbol{\phi}_0 & 2 \leq i \leq s \\ \frac{\beta}{{\mu_1}^s\mu_2}(\beta+\mu_1)^s\boldsymbol{\phi}_0, & i = s+1 \leq i \leq S \end{cases}
$$

The unknown probability $\boldsymbol{\phi}_0$ can be found from the normalizing condition $\boldsymbol{\phi}\mathbf{e} = 1$ as

$$
\boldsymbol{\phi}_0 = \left(\left(\frac{\mu_1}{\mu_1+\beta}\right)^s\left(\frac{\mu_2+(S-s)\beta}{\mu_2}\right)\right)^{-1} \tag{2.15}
$$

The $LIQBD$ description of the model indicates that the queueing system is stable (see Neuts [48]) if and only if

$$
\boldsymbol{\phi}A_0\mathbf{e} < \boldsymbol{\phi}A_2\mathbf{e} \tag{2.16}
$$

which on simplification gives the stability condition as

$$(\lambda - \mu_1)\left[\left(\frac{\beta+\mu_1}{\mu_1}\right)^s - 1\right] < (\mu_2 - \lambda)(S-s)\frac{\beta}{\mu_2}\left(\frac{\beta+\mu_1}{\mu_1}\right)^s. \quad (2.17)$$

2.4.2 Steady-State probability Vector

Assuming that stability condition is satisfied, we compute the steady state probability of the original system. Let $\tilde{\mathbf{x}}$ be the steady-state probability vector of the generator $\mathcal{Q}$. Then

$$\tilde{\mathbf{x}}\mathcal{Q} = 0 \text{ and } \tilde{\mathbf{x}}\mathbf{e} = 1. \quad (2.18)$$

Partitioning $\tilde{\mathbf{x}}$ as $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_0, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, ...)$ where $\tilde{\mathbf{x}}_i = (\tilde{x}_i(0), \tilde{x}_i(1), ...\tilde{x}_i(S)$ for $i \geq 0$. Then by the relation (2.18) we get

$$\begin{aligned} \tilde{\mathbf{x}}_0 A_{00} + \tilde{\mathbf{x}}_1 A_2 &= 0, \\ \tilde{\mathbf{x}}_{i-1} A_0 + \tilde{\mathbf{x}}_i A_1 + \tilde{\mathbf{x}}_{i+1} A_2 &= 0, \quad i \geq 1. \end{aligned}$$

From the above relations, we have

$$\begin{aligned} -\beta\tilde{x}_i(0) + \mu_1\tilde{x}_{i+1}(1) &= 0, \quad i \geq 0 \\ -(\lambda+\beta)\tilde{x}_0(j) + \mu_1\tilde{x}_1(j+1) &= 0, \quad 1 \leq j \leq s-1 \\ -(\lambda+\beta)\tilde{x}_0(s) + \mu_2\tilde{x}_1(s+1) &= 0, \\ -\lambda\tilde{x}_0(j) + \mu_2\tilde{x}_1(j+1) &= 0, \quad s+1 \leq j \leq S-1 \\ \beta\sum_{j=0}^{j=s}\tilde{x}_0(j) - \lambda\tilde{x}_0(S) &= 0, \\ \lambda\tilde{x}_{i-1}(j) - (\lambda+\beta+\mu_1)\tilde{x}_i(j) + \mu_1\tilde{x}_{i+1}(j+1) &= 0, \quad i \geq 1, 1 \leq j \leq s-1 \\ \lambda\tilde{x}_{i-1}(s) - (\lambda+\beta+\mu_1)\tilde{x}_i(s) + \mu_2\tilde{x}_{i+1}(s+1) &= 0, \\ \lambda\tilde{x}_{i-1}(j) - (\lambda+\mu_2)\tilde{x}_i(j) + \mu_2\tilde{x}_{i+1}(j+1) &= 0, \quad i \geq 1, s+1 \leq j \leq S-1 \\ \lambda\tilde{x}_{i-1}(S) + \beta\sum_{j=0}^{s}\tilde{x}_i(j) - (\lambda+\mu_2)\tilde{x}_i(S) &= 0. \end{aligned}$$

We seek the solution in the form

$$\tilde{x}_i(j) = \begin{cases} \mathcal{D}_i(j)\left(\dfrac{\lambda}{\mu_1}\right)^i \phi_j, & 0 \le j \le s \\ \mathcal{D}_i(j)\left(\dfrac{\lambda}{\mu_2}\right)^i \phi_j, & s+1 \le j \le S \end{cases} \tag{2.19}$$

where $\mathcal{D}_i(j)$ are constants to be determined and

$$\phi_j = \begin{cases} \dfrac{\beta}{\lambda}\left(\dfrac{\beta+\lambda}{\lambda}\right)^{j-1}\phi_0, & 1 \le j \le s \\ \dfrac{\beta}{\lambda}\left(\dfrac{\beta+\lambda}{\lambda}\right)^{s}\phi_0, & s+1 \le j \le S \\ \left(\dfrac{\lambda}{\beta+\lambda}\right)^{s}\left[1+(S-s)\dfrac{\beta}{\lambda}\right]^{-1}, & j=0 \end{cases} \tag{2.20}$$

which represent the inventory level probabilities.

The constants $\mathcal{D}_i(j)$ are given by

$$\mathcal{D}_i(j) = \begin{cases} \mathcal{D}_0(0), & i=0, 1 \le j \le S \\ \mathcal{D}_0(0), & i=1, 0 \le j \le S \\ \mathcal{D}_0(0), & i \ge 2, 0 \le j \le s \\ \psi_i(j)\mathcal{D}_0(0), & i \ge 2, s+1 \le j \le S \end{cases} \tag{2.21}$$

where

$$\psi_i(j) = \begin{cases} \left(\frac{\mu_2}{\mu_1}\right)^{i-1} & i \ge 2 j = s+1 \\ 1 & i \ge 2, s+i \le j \le S \\ \left(\frac{\mu_2}{\mu_1}\right)^{i-3}\frac{\mu_2}{\lambda}\left[\left(\frac{\lambda+\mu_2}{\mu_2}\right)\frac{\mu_2}{\mu_1}-1\right] & i \ge 3, j = s+2 \\ \frac{\mu_2}{\lambda}\left[\frac{\lambda+\mu_2}{\mu_2}\psi_{i-1}(j-1)-\psi_{i-2}(j-1)\right] & i \ge 4, s+3 \le j \le S. \end{cases} \tag{2.22}$$

From the normalizing condition we get

$$\mathcal{D}_0(0) = \left[1 + (S-s)\frac{\beta}{\lambda}\right]\left[\sum_{i=0}^{\infty}\left(\frac{\lambda}{\mu_1}\right)^i + \frac{\beta}{\lambda}\left((S-s)\left(1+\frac{\lambda}{\mu_2}\right) + \sum_{i=2}^{\infty}\sum_{j=s+1}^{S}\left(\frac{\lambda}{\mu_2}\right)^i \psi_i(j)\right)\right]^{-1}.$$

Performance Measures

In order to compare the performance with that of model under (s, Q) policy, we consider the following basic measures.

- Expected number of customers in the system, $E_N = \sum_{i=1}^{\infty} i\tilde{\mathbf{x}}_i\mathbf{e}$
- Expected number of item in the inventory, $E_I = \sum_{i=0}^{\infty}\sum_{j=1}^{S} j\tilde{x}_i(j)$
- Expected loss rate of customers, $E_L = \lambda \sum_{i=0}^{\infty} \tilde{x}_i(0)$

2.5 Numerical illustration

In this section we provide numerical illustrations to compare the relative performance of the two queueing-inventory models.

The increase in the expected number of customers increase drastically for (s, Q) policy in comparison with that for (s, S) policy for increasing value of λ. However, the expected inventory level decreases with increase in value of λ. These are on expected lines (see Table 2.2).

Table 2.3 provides a comparison of E_N, E_I and E_L values for (s, Q) and (s, S) policies, with variation in μ_2. The expected loss rate is seen

λ	(s,Q) policy			(s,S) policy		
	E_N	E_I	E_L	E_N	E_I	E_L
4	0.6604	10.3905	9.6477×10^{-5}	0.6116	9.9907	2.6392×10^{-4}
5	1.0709	10.1786	2.3277×10^{-4}	0.9620	9.7931	5.8555×10^{-4}
6	1.7307	9.9831	5.2460×10^{-4}	1.4938	9.6162	0.0011
7	2.8822	9.8002	0.0013	2.3567	9.4570	0.0020
8	5.0723	9.6336	0.0028	3.8746	9.3161	0.0038
9	9.5550	9.5006	0.0050	6.7654	9.1996	0.0064

Table 2.2: Effect of λ for $(\alpha, \beta, \mu_2, s, Q, S) = (0.1, 3, 15, 7, 8, 15)$

to be minimum for the (s,Q) policy. The same observation is applicable when we consider the effect of replenishment rate (see Table 2.4).

μ_2	(s,Q) policy			(s,S) policy		
	E_N	E_I	E_L	E_N	E_I	E_L
11	3.3399	10.0785	5.1419×10^{-4}	5.5761	9.5803	0.0017
12	2.6808	10.0497	4.4167×10^{-4}	4.1438	9.5392	**0.0016**
13	2.2516	10.0248	$\mathbf{4.2652}\times10^{-4}$	3.2934	9.5066	0.0017
14	1.9517	10.0028	4.5693×10^{-4}	2.7410	9.4797	0.0018
15	1.7307	9.9831	5.2460×10^{-4}	2.3567	9.4570	0.0020
16	1.5611	9.9654	6.2479×10^{-4}	2.0751	9.4376	0.0024

Table 2.3: Effect of μ_2 for $(\alpha, \beta, \lambda, s, Q, S) = (0.1, 3, 6, 7, 8, 15)$

Table 2.5 provides a comparison between (s,Q) and (s,S) policies based on the measures E_N, E_I and E_L. Here the behaviour of the first two measures are on expected lines. However, E_L, the expected loss rate shows higher values for (s,S) for $\alpha = 0.2$ and 0.3 and for still higher values of α, the (s,Q) policy has larger values.

β	(s, Q) policy			(s, S) policy		
	E_N	E_I	E_L	E_N	E_I	E_L
2	3.7928	9.7141	0.0013	3.0021	9.3840	0.0019
2.5	2.6808	10.0497	3.6806×10^{-4}	2.2896	9.6739	5.4854×10^{-4}
3	2.1565	10.2754	1.2508×10^{-4}	1.9226	9.8722	1.9090×10^{-4}
3.5	1.8633	10.4379	4.9690×10^{-5}	1.7056	10.0170	7.5970×10^{-5}
4	1.6802	10.5608	2.2620×10^{-5}	1.5649	10.1277	3.3840×10^{-5}
4.5	1.5565	10.6573	1.1574×10^{-5}	1.4675	10.2153	1.6615×10^{-5}

Table 2.4: Effect of β for $(\alpha, \lambda, \mu_2, s, Q, S) = (0.1, 5, 10, 7, 8, 15)$

α	(s, Q) policy			(s, S) policy		
	E_N	E_I	E_L	E_N	E_I	E_L
0.2	11.3777	8.4804	0.0443	6.2383	8.5352	0.0553
0.3	7.2463	8.1600	0.1113	4.1143	8.4738	0.1162
0.4	4.7245	7.9475	0.1817	2.9964	8.4597	0.1710
0.5	3.3075	7.8120	0.2410	2.3688	8.4675	0.2146
0.6	2.5004	7.7229	0.2863	1.9853	8.4838	0.2482
0.7	2.0139	7.6605	0.3201	1.7332	8.5024	0.2741

Table 2.5: Effect of α for $(\lambda, \beta, \mu_2, s, Q, S) = (4, 1, 7, 7, 8, 15)$

2.5.1 Optimization Problem

We consider a cost minimization problem associated with the (s, Q) policy.

For computing the minimal costs of the given queueing-inventory model we introduce the cost function $\mathcal{F}(\alpha)$ as

$$\mathcal{F}(\alpha) = C\left(\frac{N_1}{\alpha} + N_2\right) + (C_0 + QC_1)E_R + C_2E_I + C_3E_L$$

where

C : unit holding cost of customer for one unit of time
C_0 : fixed cost for placing an order
C_1 : variable procurement cost per item
C_2 : unit holding cost of inventory for one unit of time
C_3 : cost incurred due to loss per customer

N_1, N_2, E_R, E_I and E_L are defined in section on the performance measures.

Effect of α

α	$\mathcal{F}(\alpha)$
0.1	154.4116
0.2	151.2455
0.3	150.1005
0.4	149.4784
0.5	149.0754
0.6	148.7881
0.7	148.5706
0.8	**148.3992**
0.9	148.5803
1	148.6452

Table 2.6: Effect of α on $\mathcal{F}(\alpha)$

For $(\lambda, \beta, \mu_2, s, Q, C, C_0, C_1, C_2, C_3) = (2, 3, 5, 5, 15, \$10, \$500, \$25, \$2, \$20)$ and α from 0.1 to 1, Table 2.6 provides the effect of α on the expected system cost per unit time. $\mathcal{F}(\alpha)$ decreases first with increasing value of α and after a certain stage in starts increasing. There is a global minimum. Of course this has heavy dependence on the input parameter values. Nevertheless, the existence of a global minimum seems to be guaranteed, though quite hard to prove mathematically.

Chapter 3

Queueing Inventory System with Reservation, Cancellation and Common Life Time

Whereas in chapter 2 we considered varying service rates, with items sold never to be returned, nor items perish separately or together. In the present chapter we are concerned with features other than the first described above. This problem is again based on real life situation. It is common to purchase an item in the inventory and later cancel (return) it. We shall refer purchase of an item from inventory as reservation (for

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example, reservation of a seat in bus/train/flight for a future journey). Sometimes a few of the purchased items are returned. We call this as cancellation (for example, canceling a reserved seat). Each item on hand may have an expiry date which in some cases is common to all. Several examples can be cited for this: batch of medicines that were manufactured together have a common expiry date; once a bus/train/flight departs, the vacant seats have no use, those could not board the transport system before departure, miss it. In this chapter we study a queueing inventory process consisting of S items which have an expiry time, called common life time where reservation of items and cancellation of sold items within the expiry time is allowed. The common life time (CLT) of items is exponentially distributed with parameter γ, on realization of which the remaining items are discarded, but the waiting customers stay back in the system. Cancellation and reservation are permitted as long as common life time is not realized. Inter-cancellation time follows exponential distribution with parameter depending on the number of items in the reservation list at that moment. Time required to cancel a reservation is assumed to be negligible. Demands for the item form a Poisson process of rate λ; one unit of item is supplied to a customer at the end of his service. The service time follows exponential distribution with parameter μ.

Queueing inventory with reservation, cancellation, common life time and retrial is introduced by Krishnamoorthy et al. [31]. They assumed that a customer on arrival to an idle server with at least one item in inventory is immediately taken for service or else he joins the buffer of varying size depending on the number of items in the inventory. If there is no item in the inventory the arriving customer first queue up in a finite waiting space of capacity K. When it overflows an arrival goes to an orbit of infinite capacity with probability p or is lost forever with probability $1 - p$. From the orbit he retries for service. However the authors could not produce a "product form solution", namely joint system state distribution equal to product of the marginal distributions.

For the model discussed here we do away with the buffer, waiting space and orbit; instead a single queue is considered. This is at the expense of loss of some crucial information - the finite waiting list is gone and is replaced by the number in the waiting room at any time. Nevertheless, under the crucial assumption that no customer joins the system when inventory level is zero, we establish the stochastic decomposition property of the system state.

3.1 Mathematical formulation

We consider a single server queueing-inventory system consisting of a homogeneous items having a CLT. The time duration from the epoch at which we start with maximum inventory level S at a replenishment epoch, to the moment when the CLT is realized is called a *cycle*. The CLT of items is exponentially distributed with parameter γ. On realization of CLT customers waiting in the system stay back in the system. When CLT is reached a replenishment order is placed, which is realized on completion of a positive lead time, exponentially distributed with parameter β. Reservation of items and cancellation of sold items before the CLT realization is permitted in each cycle. Cancellation takes place according to an exponentially distributed inter-occurrence time with parameter $i\theta$, when $(S - i)$ items are present in the inventory. Through cancellation of purchased item, inventory gets added to the existing one;

nevertheless inventory level will not go above S (the sum of items in sold list and those in store equal to S). The customers arrive according to a Poisson process of rate λ. Each customer requires exactly one item from the inventory, which is served to him at the end of a random duration of service which follows exponential distribution with parameter μ. No customer joins the system when inventory level is zero.

The above system is modelled as a continuous time Markov Chain Γ=$\{(N(t), I(t)), t \geq 0\}$ with state space

$$\{(n, 0^*), n \geq 0\} \bigcup \{(n, i), n \geq 0, 0 \leq i \leq S\},$$

where 0^* is inventory level on common life time realization but before the replenishment and

$N(t)$: Number of customers in the system at time t
$I(t)$: Number of items in the inventory at time t.

The transitions in the Markov Chain are

- Transitions due to arrival:

  $(n, i) \rightarrow (n+1, i)$ at the rate λ for $n \geq 0$, $1 \leq i \leq S$.

- Transitions due to service completions:

  $(n, i) \rightarrow (n-1, i-1)$ at the rate μ for $n \geq 1$, $1 \leq i \leq S$.

- Transitions due to common life time realization:

  $(n, i) \rightarrow (n, 0^*)$ at the rate γ for $n \geq 0$, $0 \leq i \leq S$.

- First transition that is counted after CLT is realized (which is due to replenishment):

$(n, 0^*) \to (n, S)$ at the rate β for $n \geq 0$.

- Transition due to cancellation:

  $(n, i) \to (n, i+1)$ at the rate $(S-i)\theta$ for $n \geq 0$, $0 \leq i \leq S-1$.

The infinitesimal generator of Γ with entries governed as described above is

$$\mathcal{Q} = \begin{bmatrix} B & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where B contains transitions within level 0; A_0 represents transitions from n to $n+1$ for $n \geq 0$, A_1 represents transitions within n for $n \geq 1$ and A_2 represents transitions from n to $n-1$ for $n \geq 1$. All these are square matrices of order $S+2$.

$$B = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \end{array} \\ \begin{pmatrix} b_S & S\theta & & & & & \gamma \\ & b_{S-1} & (S-1)\theta & & & & \gamma \\ & & b_{S-2} & (S-2)\theta & & & \gamma \\ & & & \ddots & \ddots & & \vdots \\ & & & & b_1 & \theta & \gamma \\ & & & & & b_0 & \gamma \\ & & & & & \beta & -\beta \end{pmatrix} \end{array},$$

$$
A_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \end{array} \\
\left(\begin{array}{ccccccc}
b_S & S\theta & & & & & \gamma \\
 & a_{S-1} & (S-1)\theta & & & & \gamma \\
 & & a_{S-2} & (S-2)\theta & & & \gamma \\
 & & & \ddots & \ddots & & \vdots \\
 & & & & a_1 & \theta & \gamma \\
 & & & & & a_0 & \gamma \\
 & & & & & \beta & -\beta
\end{array}\right)
\end{array},
$$

$$
A_0 = \begin{pmatrix} 0 & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & & & & \\ \mu & 0 & & & \\ & \ddots & \ddots & & \\ & & \mu & 0 & \\ & & & 0 & 0 \end{pmatrix}.
$$

with $b_S = -(\gamma + S\theta), b_i = -(\lambda + i\theta + \gamma), a_i = -(\lambda + \mu + i\theta + \gamma)$ for $0 \leq i \leq S-1$.

3.1.1 Stability Condition

To establish the stability condition, we consider the Markov chain $\{I(t){:}t \geq 0\}$, where $I(t)$ is as defined earlier with state space given by $\{0, 1, 2, \cdots S, 0^*\}$. Let $\boldsymbol{\phi} = (\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, ..., \boldsymbol{\phi}_S, \boldsymbol{\phi}_0^*)$ be the steady-state probability vector of this Markov chain. Then $\boldsymbol{\phi}$ satisfies the equations

$$\boldsymbol{\phi} A = 0, \qquad \boldsymbol{\phi}\mathbf{e} = 1. \tag{3.1}$$

where $A = (A_0 + A_1 + A_2) =$ is the infinitesimal generator of this Markov chain.

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \end{array} \\ \begin{pmatrix} b_S & S\theta & & & & & \gamma \\ \mu & b'_{S-1} & (S-1)\theta & & & & \gamma \\ & \mu & b'_{S-2} & (S-2)\theta & & & \gamma \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & \mu & b'_1 & \theta & \gamma \\ & & & & \mu & b'_0 & \gamma \\ & & & & & \beta & -\beta \end{pmatrix} \end{array}$$

with $b_S = -(\gamma + S\theta), b'_i = -(\mu + i\theta + \gamma)$ for $0 \leq i \leq S-1$.

The components of $\boldsymbol{\phi}$ are obtained as

$$\phi_i = \begin{cases} V_i \phi_0 & 1 \leq i \leq S \\ V_0^* \phi_0 & i = 0^* \end{cases}$$

where

$$V_i = \begin{cases} 1 & i = 0 \\ \dfrac{\gamma + S\theta}{\mu} & i = 1 \\ \dfrac{(\gamma + \mu + (S-(i-1))\theta)\, V_{i-1} - (S-(i-2))\theta)\, V_{i-2}}{\mu} & 2 \leq i \leq S \\ \dfrac{\gamma}{\beta} \displaystyle\sum_{i=0}^{S} V_i & i = 0^* \end{cases}$$

The unknown probability $\boldsymbol{\phi}_0$ can be found from the normalizing condition

$\boldsymbol{\phi}\mathbf{e} = 1$ as

$$\boldsymbol{\phi}_0 = \left(\sum_{i=0}^{S} V_i + V_0^*\right)^{-1}. \tag{3.2}$$

The $LIQBD$ description of the model indicates that the queueing-inventory system is stable (see Neuts [48]) if and only if the left drift exceeds that of right drift. That is,

$$\boldsymbol{\phi}A_0\mathbf{e} < \boldsymbol{\phi}A_2\mathbf{e} \tag{3.3}$$

which on simplification gives the stability condition as

$$\lambda < \mu. \tag{3.4}$$

This leads to

Lemma 3.1.1. The process $\Gamma = \{(N(t), I(t)), t \geq 0\}$ is stable if and only if $\lambda < \mu$.

3.2 Steady-state Analysis

For finding the steady state vector of the process Γ, we first consider an inventory system with negligible service time and no backlog of demands. The corresponding Markov Chain may be defined as $\tilde{\Gamma}=\{I(t), t \geq 0\}$ where $I(t)$ has the same definition as described in Section 3.1. Its infinitesimal generator is given by

$$\mathcal{H} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ S-1 \\ S \\ 0^* \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & S-1 & S & 0^* \end{array} \\ \begin{pmatrix} b_S & S\theta & & & & & \gamma \\ \lambda & b_{S-1} & (S-1)\theta & & & & \gamma \\ & \lambda & b_{S-2} & (S-2)\theta & & & \gamma \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & \lambda & b_1 & \theta & \gamma \\ & & & & \lambda & b_0 & \gamma \\ & & & & & \beta & -\beta \end{pmatrix} \end{array}$$

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \cdots, \pi_S, \pi_0^*)$ be the steady state vector of the process $\tilde{\Gamma}$. Then $\boldsymbol{\pi}$ satisfies the equations

$$\boldsymbol{\pi}\mathcal{H} = 0, \qquad \boldsymbol{\pi}\mathbf{e} = 1. \tag{3.5}$$

Then the components of $\boldsymbol{\pi}$ can be obtained as

$$\pi_i = \begin{cases} U_i\pi_0 & 1 \le i \le S \\ U_0^*\pi_0 & i = 0^* \end{cases}$$

where

$$U_i = \begin{cases} 1 & i = 0 \\ \dfrac{\gamma + S\theta}{\lambda} & i = 1 \\ \dfrac{(\gamma + \lambda + (S-(i-1))\theta)\, U_{i-1} - (S-(i-2))\theta)\, U_{i-2}}{\lambda} & 2 \le i \le S \\ \dfrac{\gamma}{\beta}\displaystyle\sum_{i=0}^{S} U_i & i = 0^* \end{cases}$$

The unknown probability π_0 can be found from the normalizing condition $\pi\mathbf{e} = 1$ as

$$\pi_0 = \left(\sum_{i=0}^{S} U_i + U_0^* \right)^{-1}. \tag{3.6}$$

Assuming that (3.4) is satisfied, we compute the steady state probability of the original system. Let $\mathbf{x}$ denote the steady-state probability vector of this system. Then

$$\mathbf{x}\mathcal{Q} = 0, \qquad \mathbf{x}\mathbf{e} = 1. \tag{3.7}$$

Partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ where

$$\mathbf{x}_i = (x_i(0), x_i(1), ...x_i(S), x_i(0^*)), for i \geq 0$$

.Then by (3.7) we get

$$\mathbf{x}_0 B + \mathbf{x}_1 A_2 = 0, \tag{3.8}$$

$$\mathbf{x}_i A_0 + \mathbf{x}_{i+1} A_1 + \mathbf{x}_{i+2} A_2 = 0; i \geq 0. \tag{3.9}$$

We produce a solution of the form

$$\mathbf{x}_i = K \left(\frac{\lambda}{\mu} \right)^i \boldsymbol{\pi}; i \geq 0 \tag{3.10}$$

where K is a constant to be determined. With these $\mathbf{x}_i$ substituted in $\mathbf{x}\mathcal{Q} = 0$ we get

$$\mathbf{x}_0 B + \mathbf{x}_1 A_2 = K\boldsymbol{\pi} \left(B + \frac{\lambda}{\mu} A_2 \right) = K\boldsymbol{\pi}\mathcal{H} = 0,$$

$$\mathbf{x}_i A_0 + \mathbf{x}_{i+1} A_1 + \mathbf{x}_{i+2} A_2 = K \left(\frac{\lambda}{\mu}\right)^{i+1} \boldsymbol{\pi} \left(B + \frac{\lambda}{\mu} A_2\right) = K \left(\frac{\lambda}{\mu}\right)^{i+1} \boldsymbol{\pi}\mathcal{H} = 0.$$

Thus we can see that (3.10) satisfy the equations (3.8) and (3.9). Now applying the normalizing condition $\mathbf{xe} = 1$ we get

$$K\left(1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + ...\right) = 1.$$

Hence under the condition $\lambda < \mu$, we have $K = 1 - \frac{\lambda}{\mu}$.

Thus under the condition that $\lambda < \mu$, the steady state probability vector of the process Γ with generator matrix $\mathcal{Q}$ is given by $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$, where

$$\mathbf{x}_i = K\left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi}; i \geq 0 \tag{3.11}$$

where

$$K = 1 - \frac{\lambda}{\mu}. \tag{3.12}$$

Thus, the system state distribution under the stability condition is the product of marginal distributions of the number of customers in an $M/M/1$ system and the number of items in the inventory.

Now we look at a few of the system characteristics that throw light on the performance of the system.

3.2.1 Performance Measures

We have the following entities providing information on the system.

1. Expected number of customers in the system, $E_C = \dfrac{\lambda}{\mu - \lambda}$.

2. Expected number of item in the inventory, $E_I = \sum_{i=1}^{S} i\pi_i$.

3. Expected cancellation rate, $E_{CR} = \sum_{i=0}^{S}(S-i)\theta\pi_i$.

4. Expected number of cancellation, $E_{CN} = \frac{\sum_{i=0}^{S}(S-i)\theta\pi_i}{\gamma}$.

5. Expected inventory purchase rate by customers, $E_{PR} = \lambda\sum_{i=1}^{S}\pi_i$.

6. Expected number of inventory purchased by customers in a cycle, $E_{PN} = \frac{\lambda\sum_{i=1}^{S}\pi_i}{\gamma}$.

7. Expected loss rate of customers, $E_L = \lambda\pi_0$.

8. Probability that all items are in sold list before CLT realization, $Pvacant = \pi_0$.

9. Probability that all items are in the system before CLT realization, $Pfull = \pi_S$.

3.2.2 Expected sojourn time in zero inventory level in a cycle before realization of CLT

This is the expected time during which the system stays with no inventory. We derive this for a finite capacity system. For that consider the Markov Chain $\{(N(t), I(t)) : t \geq 0\}$. The state space is $\{(n,0) : 0 \leq n \leq K\}\bigcup\{\Delta\}$ where $\{\Delta\}$ denotes the absorbing state of the Markov chain which is realization of CLT or cancellation and K

is the maximum number of customers accommodated in the system. Its infinitesimal generator is of the form

$$\mathcal{H}_1 = \begin{bmatrix} T & T^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$T = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ K-1 \\ K \\ \Delta \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & K-1 & K & \Delta \end{array} \\ \begin{pmatrix} -(S\theta+\gamma) & & & & & & (S\theta+\gamma) \\ & -(S\theta+\gamma) & & & & & (S\theta+\gamma) \\ & & -(S\theta+\gamma) & & & & (S\theta+\gamma) \\ & & & & & & \vdots \\ & & & & -(S\theta+\gamma) & & (S\theta+\gamma) \\ & & & & & -(S\theta+\gamma) & (S\theta+\gamma) \\ 0 & & & & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Thus expected sojourn time in zero inventory level, $E_T^0 = -\alpha_K T^{-1}\mathbf{e}$ where $\alpha_K = (x_0(0), x_1(0), \cdots, x_K(0))$. Expected number of visits $= \frac{\mu\rho}{\gamma}\pi_1$. Thus the expected sojourn time in zero inventory level in a cycle $= \frac{\mu\rho}{\gamma}\pi_1(-\alpha_K T^{-1}\mathbf{e})$.

3.2.3 Expected sojourn time in maximum inventory level S in a cycle before realization of CLT

This is the expected time system stays with maximum inventory. Here also derivation is done in case of finite number of customers. For that consider the Markov Chain $\{(N(t), I(t)), t \geq 0\}$. The state space is $\{(n, S) : 0 \leq n \leq K\}\bigcup\{\Delta\}$, where $\{\Delta\}$ denotes the absorbing state of the Markov chain which represents realization of CLT or service completion. Its infinitesimal generator is of the form

$$\mathcal{H}_2 = \begin{bmatrix} T_1 & T_1^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$T_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ K-1 \\ K \\ \Delta \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & \cdots & K-1 & K & \Delta \end{array} \\ \begin{pmatrix} -(\lambda+\mu+\gamma) & \lambda & & & & & (\mu+\gamma) \\ & -(\lambda+\mu+\gamma) & \lambda & & & & (\mu+\gamma) \\ & & -(\lambda+\mu+\gamma) & \lambda & & & (\mu+\gamma) \\ & & & & & & \vdots \\ & & & & -(\lambda+\mu+\gamma) & & (\mu+\gamma) \\ & & & & & -(\lambda+\mu+\gamma) & (\mu+\gamma) \\ 0 & & & & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Thus the expected sojourn time in maximum inventory level, $E_{T_1}^S = -\alpha_K T_1^{-1}\mathbf{e}$ where $\alpha_K = (x_0(S), x_1(S), \cdots, x_K(S))$ and expected number of visits to $S = \frac{\theta}{\gamma}\pi_{S-1}$. Thus, expected sojourn time in maximum inventory level in a cycle $= (-\alpha_K T_1^{-1}\mathbf{e})\frac{\theta}{\gamma}\pi_{S-1}$.

3.3 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

Effect of λ on various performance measures

Table 3.1 indicates that increase in λ value makes increase in expected number of customers in the system, expected loss rate, expected purchase rate, expected cancellation rate. As λ increases there is a decrease in the expected number of items in the inventory. Also, as λ increases probability that all items are in the sold list prior to realization of CLT

λ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
9	1.4992	16.2817	1.3246×10^{-6}	8.2979	8.5697	1.4717×10^{-7}	0.0602
10	1.9946	15.9726	6.1139×10^{-6}	9.2258	9.5154	6.1139×10^{-7}	0.0459
11	2.7185	15.6629	2.1047×10^{-5}	10.1563	10.4442	1.9133×10^{-6}	0.0355
12	3.8342	15.3587	5.5471×10^{-5}	11.0717	11.3313	4.6226×10^{-6}	0.0280
13	5.6433	15.0756	1.1340×10^{-4}	11.9242	12.1367	8.7234×10^{-6}	0.0229

Table 3.1: Effect of λ:Fix $S = 20, \theta = 3, \mu = 15, \gamma = 0.1, \beta = 2$

increases and probability that all items are in the system just prior to realization of CLT decreases. These are all natural consequences as arrival rate increases.

Effect of the service rate μ

μ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
12	7.7781	15.7020	8.3553×10^{-5}	10.0471	10.1282	7.5957×10^{-6}	0.0380
13	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
14	3.5499	15.6618	3.7134×10^{-5}	10.1616	10.4034	3.3759×10^{-6}	0.0357
15	2.7185	15.6629	2.1047×10^{-5}	10.1563	10.4442	1.9133×10^{-6}	0.0355
16	2.1905	15.6665	1.1363×10^{-5}	10.1490	10.4622	1.0330×10^{-6}	0.0354
17	1.8302	15.6665	5.9801×10^{-6}	10.1440	10.4700	5.4364×10^{-7}	0.0353

Table 3.2: Effect of μ: $S = 20, \theta = 3, \lambda = 11, \gamma = 0.1, \beta = 2$

Table 3.2 indicates that increase in μ values leads to decrease in the expected number of customers and expected loss rate of customers in the system. As service rate increases, it is natural that loss rate of customers and expected number of customers in the system decreases. As μ increases expected number of items in the inventory shows a decreasing tendency first and then it increases. This could be attributed to the increase in cancellation of purchased items. Expected purchase rate increases, which is on expected lines. However, expected cancellation rate increases first and then decreases as μ value increases. The initial increase in cancellation rate is due to large number of purchases taking place consequent to increasing value of μ; however with further increase in value of μ, the traffic intensity decreases and so the number of actual purchase decreases, which in turn results in the decrease of the rate of cancellations. Probability for all items in sold list prior to CLT realization decreases, so also that for all items in system.

Effect of common life time parameter γ

In Table 3.3, there are few surprises. These are in the behaviour of E_I, E_{CR} and E_{PR} with increase in value of γ. Increase in γ means the CLT realization is faster. We observe that as γ increases there is a decrease in expected number of items in the inventory, expected loss rate of customers. Shorter the CLT, lesser will be the purchase rate, so cancellation rate also decreases. Also, we observe that as CLT realization decreases probability that all items are in sold list just prior to CLT realization decreases and probability that all items are in system prior to CLT realization increases.

γ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
0.1	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
0.2	4.9865	15.0569	5.6983×10^{-5}	9.3877	9.8460	5.1803×10^{-6}	0.0449
0.3	4.9871	14.4928	5.4177×10^{-5}	8.7134	9.4188	4.9251×10^{-6}	0.0524
0.4	4.9875	13.9706	5.1615×10^{-5}	8.1101	9.0272	4.6923×10^{-6}	0.0590
0.5	4.9880	13.4857	4.9268×10^{-5}	7.5680	8.6669	4.4789×10^{-6}	0.0648
0.6	4.9884	13.0342	4.7109×10^{-5}	7.0791	8.3342	4.2827×10^{-6}	0.0699

Table 3.3: Effect of γ: $S = 20, \theta = 3, \lambda = 11, \mu = 13, \beta = 2$

Effect of cancellation rate θ

Table 3.4, shows that as cancellation rate increases expected number of customers in the system initially show a slight increase and then it remains constant. Expected number of items in the inventory and expected cancellation rate show an upward trend, which is a consequence of increasing value of θ. Expected purchase rate increases first and then remains constant and expected loss rate of customers decrease with respect to increase in θ. Also, we observe that as cancellation rate increases probability that all items are in sold list just prior to CLT realization decreases and probability that all items are in system just prior to CLT realization increases.This tendency is a consequence of higher cancellation rate for the same CLT parameter value.

θ	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
1	4.9854	9.7793	0.0278	9.2706	10.2863	0.0025	0.0099
2	4.9859	14.0978	1.5007×10^{-4}	9.9043	10.3136	1.3643×10^{-5}	0.0153
3	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
4	4.9860	16.4797	3.4360×10^{-5}	10.2809	10.3137	3.1236×10^{-6}	0.0726
5	4.9860	16.9753	2.3040×10^{-5}	10.3736	10.3137	2.0946×10^{-6}	0.1165
6	4.9860	17.3093	1.7012×10^{-5}	10.4440	10.3137	1.5466×10^{-6}	0.1624

Table 3.4: Effect of θ: $S = 20, \gamma = 0.1, \lambda = 11, \mu = 13, \beta = 2$

Effect of replenishment rate β

From Table 3.5, we observe that as replenishment rate increases expected number of customers in the system show a slight decreasing tendency and expected loss rate of customers increase. There is an increase in expected number of items in the inventory, expected cancellation rate, expected purchase rate. Also, we observe that as replenishment rate increases probability that all items are in sold list just prior to CLT realization and probability that all items are in system just prior to CLT realization increases.

β	E_C	E_I	E_L	E_{CR}	E_{PR}	Pvacant	Pfull
1	4.9865	14.9579	5.720×10^{-5}	9.6846	9.8460	5.2001×10^{-6}	0.0347
2	4.9860	15.6684	6.0072×10^{-5}	10.1447	10.3136	5.4611×10^{-6}	0.0364
3	4.9858	15.9205	6.1094×10^{-5}	10.3079	10.4796	5.5540×10^{-6}	0.0370
4	4.9857	16.0496	6.1618×10^{-5}	10.3915	10.5646	5.6017×10^{-6}	0.0373
5	4.9856	16.1281	6.1937×10^{-5}	10.4423	10.6162	5.6306×10^{-6}	0.0374
6	4.9856	16.1808	6.2151×10^{-5}	10.4764	10.6509	5.6501×10^{-6}	0.0376

Table 3.5: Effect of β: $S = 20, \theta = 3, \lambda = 11, \mu = 13, \gamma = 0.1$

3.3.1 Optimization Problem

Based on the above performance measures we construct a cost function to check the maximality of profit function.

We define a revenue function as $\mathcal{RF}$ as

$$\mathcal{RF} = C_1 E_{PR} + C_2 E_{CR} - h_I E_I - h_C E_C$$

$$= \pi_0\{C_1\lambda \sum_{1=1}^{S} U_i + C_2 \sum_{i=0}^{S-1} (S-i)\theta U_i - h_I \sum_{i=1}^{S} iU_i\} - h_C \frac{\lambda}{\mu - \lambda}$$

where

- C_1 = revenue to the system due to per unit purchase of item in the inventory
- C_2 = revenue to the system due to per unit cancellation of inventory purchased
- h_I = holding cost per unit time per item in the inventory
- h_C = holding cost of customer per unit per unit time

In order to study the variation in different parameters on profit function we first fix the costs $C_1 = \$150, C_2 = \$50, h_I = \$20, h_C = \$5$.

Effect of variation in S, γ and θ on $\mathcal{RF}$

Table 3.6 shows that the change in revenue function with respect to S and θ. The revenue function increases first with θ and then keep going down. It may be noted that cancellation to some extent prior to common life realization results in higher profit to the system since there is a cancellation penalty imposed on the customer. As common life time realization decreases profit becomes less. This is due to lower cancellation rate. Table 3.7 shows that the change in revenue function with respect to S and γ keeping rate of cancellation a constant. Table 3.8 shows the change in revenue function with respect to γ and θ.

$S \downarrow \theta \rightarrow$	1	1.5	2	2.5	3	3.5	4
10	1511.2	1789.9	1882.4	1904.6	1906.8	1904.3	1901.2
11	1602	1835.4	1890.2	1895.8	1892.1	1887.6	1883.7
12	1677	1859.5	1885.2	1881.5	1875.2	1869.9	1865.8
13	1735.7	1867.3	1873.1	1864.7	1857.4	1851.9	1847.8
14	1778.6	1863.9	1857.5	1847.1	1839.4	1833.9	1829.8
15	1806.7	1853.5	1840.4	1829.1	1821.4	1815.8	1811.7
16	1882	1839.1	1822.6	1811.1	1803.3	1797.8	1793.7

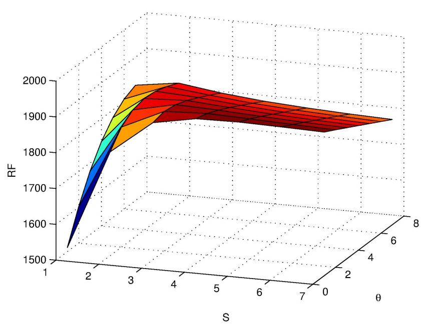
Table 3.6: Effect of S and θ. Fix $\lambda = 11, \mu = 13, \gamma = 0.1, \beta = 2$

$S \downarrow \gamma \rightarrow$	0.1	0.15	0.2	0.25	0.3	0.35	0.4
10	1882.4	1828.1	1776.7	1728	1681.6	1637.6	1595.6
11	1890.2	1834.5	1781.7	1731.7	1684.2	1639.2	1596.4
12	1885.2	1828.5	1775.5	1725	1677.1	1631.7	1588.6
13	1873.1	1816.6	1763.2	1712.7	1664.8	1619.4	1576.3
14	1857.5	1801.2	1748	1697.7	1650	1604.8	1561.8
15	1840.4	1784.4	1731.5	1681.4	1634.1	1589.1	1546.5
16	1822.6	1767	1714.4	1664.7	1617.7	1573.1	1530.7

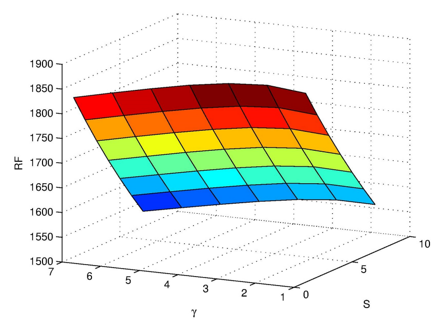
Table 3.7: Effect of γ and S.Fix $\lambda = 11, \mu = 13, \beta = 2, \theta = 2$

$\gamma \downarrow \theta \rightarrow$	1	1.5	2	2.5	3
0.1	1806.7	1747.2	1691.3	1638.8	1589.4
0.15	1853.5	1793.3	1736.7	1683.5	1633.3
0.2	1840.4	1784.4	1731.5	1681.4	1634.1
0.25	1829.1	1776	1725.7	1677.9	1632.6
0.3	1821.4	1770.2	1721.6	1675.4	1631.4
0.35	1815.8	1766.0	1718.7	1673.6	1630.7
0.4	1811.7	1763	1716.6	1672.4	1630.3

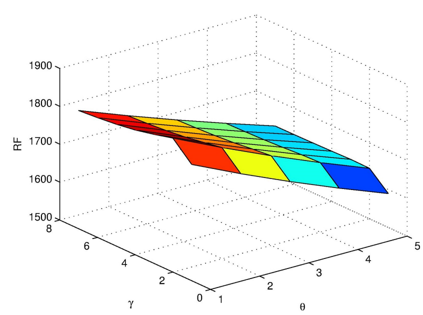
Table 3.8: Effect of θ and γ. Fix $S = 15$ $\lambda = 11, \mu = 13, \beta = 2$



(a) Effect of S and θ



(b) Effect of S and γ



(c) Effect of γ and θ

Chapter 4

Queueing Inventory System with Reservation, Cancellation, Common Life Time- Case of Zero and Positive Lead Time

4.1 Introduction

In this chapter we continue our investigation of queueing-inventory with reservation,cancellation and CLT. Two different scenarios are discussed:(i)the case of zero lead time,in which on realization of CLT or the first time

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Some results in this chapter appeared in Communications in Applied Analysis : *Dhanya Shajin, Binitha Benny, Deepak T.G and A. Krishnamoorthy* : **A relook to queueing-inventory system with reservation, cancellation and common life time**,Communications in Applied Analysis.20(2016),545-574 inventory level hits zero for the first time in the cycle, whichever occurs first, the inventory reaches its maximum level S through an instantaneous replenishment for the next cycle. (ii)the case of positive lead time which is exponentially distributed. When the inventory level is zero, new arrivals and cancellation of purchased items are not permitted. In both cases we produce product form solutions. Assumption concerning arrival process, service time distribution, distribution of CLT are as in Chapter-3.

To start with we need to define a few terms.

Definition 4.1.1. A ***cycle*** *is the time starting from the maximum inventory S in stock at an epoch, until the next epoch of replenishment, that is, duration between two consecutive S to S transitions.*

The end of a cycle and hence the beginning of the next cycle can be either due to CLT realization or by a service completion when there was just one item left in the inventory (the customer completing service, walks away with this item), whichever occurs first.

We define two types of events that causes the beginning of a new cycle. We call these two events A and B, respectively.

Definition 4.1.2. ***A Event***
A event *is the one, occurrence of which causes the end of a cycle in the following way: suppose a service is going on with just one item of inventory left. Assume that neither CLT realization nor a cancellation takes place before this service is completed. Thus at the end of the present service the customer walks away with the single item left in the inventory.*

If this happens for the first time starting from the moment the inventory is replenished most recently, we refer to it as **A event**. This means that we don't allow cancellation once the inventory level goes down to zero.

Definition 4.1.3. ***B Event***
*When a cycle ends (and so the new cycle begins) with occurrence of CLT we say that a **B event** has occurred resulting in the cycle completion.*

The significance of the model rests in the fact that a replenishment is triggered by either realization of CLT or by a demand when there is only one item left in the inventory, whichever occurs first. This means that the cycle time (length of a cycle) is given by min(exp(α), time until inventory level drops to zero from S (starting from the epoch of replenishment in that cycle)). The distribution of this, which is phase type, will be derived at a later stage in this chapter.

4.2 Mathematical formulation

We have a single server queueing-inventory system with a storage space for a maximum of S items of the inventory at the beginning of a cycle. Customers arrive according to Poisson process of rate λ, each demanding exactly one unit of the item. To deliver one unit of the item to a customer, it requires an exponentially distributed amount of time with parameter μ for service. The inventoried items have a common life time (CLT) which means that they all perish together on realization of this time. We assume that this common life time is exponentially distributed with parameter α. On realization of CLT or the first time inventory level hits zero for the first time in the cycle, whichever occurs first, the inventory reaches its maximum level S (denoted by S^* for identification purpose) through an instantaneous replenishment for the next cycle. In addition the possibility of cancellation of purchased item (return of the item with a penalty), is introduced here. Inter cancellation time follows exponential distribution with parameter $i\beta$ when there are $(S-i)$ items present in the inventory.

4.3 Steady state analysis

In this section we analyze the queueing-inventory model described in 4.2 in steady state. Let

$N(t)$: Number of customers in the system at time t
$I(t)$: Number of items in the inventory at time t

The process $\Omega = \{(N(t), I(t)), t \geq 0\}$ is a continuous time Markov chain with state space given by

$$\{(n,i), n \geq 0, 1 \leq i \leq S\} \bigcup \{(n, S^*), n \geq 0\}.$$

where S^* denotes inventory level on realization of common life time (consequent to the replenishment. This is same as S; however, just to distinguish the beginning of the next cycle we use it as a purely temporary notation). The transition rates are:

(a) Transitions due to arrival:

$(n,i) \to (n+1,i)$: rate λ for $n \geq 0, \quad 1 \leq i \leq S$
$(n,S^*) \to (n+1,S^*)$: rate λ for $n \geq 0$.

(b) Transitions due to service completions:

$$\begin{array}{lll}(n,i)\to(n-1,i-1): & \text{rate } \mu & \text{for } n\geq 1,\quad 2\leq i\leq S\\ (n,1)\to(n-1,S^*): & \text{rate } \mu & \text{for } n\geq 1,\\ (n,S^*)\to(n-1,S-1): & \text{rate } \mu & \text{for } n\geq 1.\end{array}$$

(c) Transitions due to CLT realization:

$$(n,i)\to(n,S^*): \quad \text{rate } \alpha \quad \text{for } n\geq 0, 1\leq i\leq S.$$

(d) Transition due to cancellation:

$$(n,i)\to(n,i+1): \quad \text{rate } (S-i)\beta \quad \text{for } n\geq 0, 1\leq i\leq S-1.$$

Other transitions have rate zero.

Thus the infinitesimal generator of Ω is of the form

$$\mathcal{Q}=\begin{pmatrix} A_{00} & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{4.1}$$

Each matrix A_{00}, A_0, A_1, A_2 is a square matrix of order $S+1$.

Entries of A_0 are given in (a); that of A_2 are given in (b) and those in $A_{0,0}$ and A_1 correspond to transition rates given by (c) and (d). In addition diagonal entries in A_{00} and A_1 are non-positive, having numerical value equal to but with negative sign the sum of other elements of the same row found in A_{00}, A_0, A_1 and A_2. All other transitions have rate zero.

4.3.1 Stability condition

Let $\boldsymbol{\pi}$ be the steady state probability vector of $A = A_0 + A_1 + A_2$, where $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_S, \pi_{S^*})$. That is, $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi} A = \mathbf{0},\ \boldsymbol{\pi}\mathbf{e} = 1. \tag{4.2}$$

where

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-2 \\ S-1 \\ S \\ S^* \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & \cdots & \cdots & S-1 & S & S^* \end{array} \\ \begin{pmatrix} b_{S-1} & a_{S-1} & & & & & & \alpha+\mu \\ \mu & b_{S-2} & a_{S-2} & & & & & \alpha \\ & \mu & b_{S-3} & a_{S-3} & & & & \alpha \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & \mu & b_2 & a_2 & & \alpha \\ & & & & \mu & b_1 & a_1 & \alpha \\ & & & & & \mu & b_0 & \alpha \\ & & & & & \mu & & -\mu \end{pmatrix} \end{array}$$

with $b_i = -(\mu+\alpha+i\beta), 0 \le i \le S-1$ and $a_j = j\beta, 1 \le j \le S-1$. Then π can be obtained as

$$\pi_i = \mathcal{U}_i \pi_1,\ 1 \le i \le S, S^*$$

where

$$\mathcal{U}_i = \begin{cases} 1 & i = 1, \\ \dfrac{\mu + \alpha + (S-1)\beta}{\mu} & i = 2, \\ \dfrac{\mu + \alpha + (S-i+1)\beta}{\mu}\mathcal{U}_{i-1} - \dfrac{(S-i+2)\beta}{\mu}\mathcal{U}_{i-2} & 3 \leq i \leq S-1, \\ \dfrac{\beta}{\mu+\alpha}\mathcal{U}_{S-1} & i = S \\ \dfrac{\mu+\alpha+\beta}{\mu}\mathcal{U}_{S-1} - \dfrac{2\beta}{\mu}\mathcal{U}_{S-2} - \mathcal{U}_S & i = S^*. \end{cases}$$

The unknown probability π_1 can be found from the normalizing condition

$$\pi_1 = \left[\sum_{i=1}^{S}\mathcal{U}_i + \mathcal{U}_{S^*}\right]^{-1}.$$

The following theorem establishes the stability condition of the queueing-inventory system under study.

Theorem 4.3.1. *The queueing-inventory system under study is stable if and only if $\lambda < \mu$.*

Proof. The queueing-inventory system under study with the $LIQBD$ type generator given in (4.1) is stable if and only if (see Neuts [48])

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}. \tag{4.3}$$

Note that from the transition rates (a) (which give the elements of A_0), and (b) (which give the form of A_2), we get

$$\boldsymbol{\pi} A_0 \mathbf{e} = \lambda(\pi_1 + ... + \pi_S + \pi_{S^*}) \text{ and } \boldsymbol{\pi} A_2 \mathbf{e} = \mu(\pi_1 + ... + \pi_S + \pi_{S^*}). \tag{4.4}$$

From the normalizing condition we have $\pi_1 + ... + \pi_S + \pi_{S^*} = 1$. Substituting this expression into (4.4) and using (4.3) we get the stated result. □

4.3.2 Steady state probability vector

Let $\mathbf{x}$ be the steady state probability vector of $\mathcal{Q}$. Then $\mathbf{x}$ must satisfy the set of equations

$$\mathbf{x}\mathcal{Q} = \mathbf{0}, \quad \mathbf{xe} = 1. \tag{4.5}$$

Note that the vector $\mathbf{x}$ partitioned as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$, is such that the i^{th} component of $\mathbf{x}_n$ gives the steady state probability that there are n customers in the system and i items in the inventory. Then the above set of equations reduce to:

$$\mathbf{x}_0 A_{00} + \mathbf{x}_1 A_2 = 0, \tag{4.6}$$

$$\mathbf{x}_{n-1} A_0 + \mathbf{x}_n A_1 + \mathbf{x}_{n+1} A_2 = 0, \quad n \geq 1. \tag{4.7}$$

For computing the steady state probability vector of the $CTMC$ Ω, we first consider the system with negligible service time. Thus the infinitesimal generator is given by

$$\tilde{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-2 \\ S-1 \\ S \\ S^* \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & \cdots & \cdots & S-1 & S & S^* \end{array} \\ \begin{pmatrix} d_{S-1} & a_{S-1} & & & & & & \alpha+\lambda \\ \lambda & d_{S-2} & a_{S-2} & & & & & \alpha \\ & \lambda & d_{S-3} & a_{S-3} & & & & \alpha \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & \lambda & d_2 & a_2 & & \alpha \\ & & & & \lambda & d_1 & a_1 & \alpha \\ & & & & & \lambda & d_0 & \alpha \\ & & & & & \lambda & & -\lambda \end{pmatrix} \end{array}$$

with $d_i = -(\lambda + \alpha + i\beta), 0 \leq i \leq S-1$ and $a_j = j\beta, 1 \leq j \leq S-1$.

Let $\boldsymbol{\xi} = (\xi_1, ..., \xi_S, \xi_{S^*})$ be the steady state vector of $\tilde{A}$. Then $\boldsymbol{\xi}$ satisfies the equations

$$\boldsymbol{\xi}\tilde{A} = \mathbf{0}, \qquad \boldsymbol{\xi}\mathbf{e} = 1. \tag{4.8}$$

From $\boldsymbol{\xi}\tilde{A} = 0$ we have

$$\begin{aligned} -(\lambda + \alpha + (S-1)\beta)\xi_1 + \lambda\xi_2 &= 0, \\ -(\lambda + \alpha + (S-i+1)\beta)\xi_{i-1} + (S-i)\beta\xi_i + \lambda\xi_{i+1} &= 0, \quad 2 \leq i \leq S-2 \\ 2\beta\xi_{S-2} - (\lambda + \alpha + \beta)\xi_{S-1} + \lambda\xi_S + \lambda\xi_{S^*} &= 0, \\ \beta\xi_{S-1} - (\lambda + \alpha)\xi_S &= 0, \\ \alpha(\xi_1 + ... + \xi_S) + \lambda\xi_1 - \lambda\xi_{S^*} &= 0 \end{aligned}$$

and ξ_i can be obtained as $\xi_i = \mathcal{V}_i \xi_1,\ 1 \leq i \leq S, S^*$ where

$$\mathcal{V}_i = \begin{cases} 1 & i = 1, \\ \dfrac{\lambda + \alpha + (S-1)\beta}{\lambda} & i = 2, \\ \dfrac{\lambda + \alpha + (S-i+1)\beta}{\lambda}\mathcal{V}_{i-1} - \dfrac{(S-i+2)\beta}{\lambda}\mathcal{V}_{i-2} & 3 \leq i \leq S-1, \\ \dfrac{\beta}{\lambda+\alpha}\mathcal{V}_{S-1} & i = S \\ \dfrac{\lambda+\alpha+\beta}{\lambda}\mathcal{V}_{S-1} - \dfrac{2\beta}{\lambda}\mathcal{V}_{S-2} - \mathcal{V}_S & i = S^*. \end{cases}$$

The unknown probability ξ_1 can be found from the normalizing condition

$$\xi_1 = \left[\sum_{i=1}^{S} \mathcal{V}_i + \mathcal{V}_{S^*}\right]^{-1}.$$

Now using the vector $\boldsymbol{\xi}$ we proceed to compute the steady state probability vector of the original system. It is seen that

$$\mathbf{x}_n = \mathcal{K}\left(\frac{\lambda}{\mu}\right)^n \boldsymbol{\xi} \text{ for } n \geq 0 \tag{4.9}$$

where $\mathcal{K}$ is a constant to be determined, is the unique solution to (4.5). From (4.6), we have

$$\mathbf{x}_0 A_{00} + \mathbf{x}_1 A_2 = \mathcal{K}\boldsymbol{\xi}\left(A_{00} + \frac{\lambda}{\mu}A_2\right) = \mathcal{K}\boldsymbol{\xi}\tilde{A} = 0 \tag{4.10}$$

and from relation (4.7), we have

$$\begin{aligned}\mathbf{x}_{n-1}A_0 + \mathbf{x}_n A_1 + \mathbf{x}_{n+1}A_2 &= \mathcal{K}\left(\tfrac{\lambda}{\mu}\right)^n \boldsymbol{\xi}\left(\tfrac{\mu}{\lambda}A_0 + A_1 + \tfrac{\lambda}{\mu}A_2\right)\\ &= \mathcal{K}\left(\tfrac{\lambda}{\mu}\right)^n \boldsymbol{\xi}\left(\tfrac{\mu}{\lambda}A_0 + A_{00} - \tfrac{\mu}{\lambda}A_0 + \tfrac{\lambda}{\mu}A_2\right)\\ &= \mathcal{K}\left(\tfrac{\lambda}{\mu}\right)^n \boldsymbol{\xi}\tilde{A} = 0.\end{aligned} \tag{4.11}$$

Thus (4.9) satisfies (4.6) and (4.7). Now applying the normalizing condition $\mathbf{xe} = 1$, we get

$$\mathcal{K}\boldsymbol{\xi}\left[1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + ...\right]\mathbf{e} = 1.$$

Hence under the condition that $\lambda < \mu$, we have

$$\mathcal{K} = 1 - \frac{\lambda}{\mu}. \tag{4.12}$$

Thus we arrive at our main result:

Theorem 4.3.2. *Under the necessary and sufficient condition $\lambda < \mu$ for stability, the components of the steady state probability vector of the CTMC Ω, with generator $\mathcal{Q}$, is given by (4.9) and (4.12). That is,*

$$\mathbf{x}_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \boldsymbol{\xi} \text{ for } n \geq 0. \tag{4.13}$$

4.3.3 Probability that the next cycle starts with service completion / realization of common life time in the previous cycle

In this section we analyze the probability that a cycle starts with service completion / realization of common life time in the previous cycle. First choose K such that

$$\sum_{n=0}^{K}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n} > 1-\epsilon \text{ for any preassigned } \epsilon.$$

Consider the Markov chain $\{(I(t), N(t)), t \geq 0\}$ whose state space

$$\{(i,n), 1 \leq i \leq S, 0 \leq n \leq K\} \bigcup \{\Delta_{\mu}\} \bigcup \{\Delta_{CLT}\}$$

where $\{\Delta_{\mu}\}$ is the absorbing state consequent to the replenishment order placed on realization of event A and $\{\Delta_{CLT}\}$ represents the realization of common life time. Thus its infinitesimal generator is of the form

$$\mathcal{P} = \begin{pmatrix} \mathcal{T} & \mathcal{T}_{\mu}^{0} & \mathcal{T}_{CLT}^{0} \\ \mathbf{0} & 0 & 0. \end{pmatrix}$$

where

$$\mathcal{T} = \begin{array}{c} S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \end{array} \overset{\begin{array}{ccccccc} S & S-1 & S-2 & \cdots & 3 & 2 & 1 \end{array}}{\begin{pmatrix} B_1 & B_2 & & & & & \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2 \\ & & & & & B_0^{(S-1)} & B_1^{(S-1)} \end{pmatrix}},$$

$$\mathcal{T}_\mu^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B_2' \end{pmatrix}, \mathcal{T}_{CLT}^0 = \begin{pmatrix} \alpha\mathbf{e} \\ \vdots \\ \alpha\mathbf{e} \\ \alpha\mathbf{e} \end{pmatrix}$$

with

$$B_2' = \begin{pmatrix} 0 \\ \mu \\ \vdots \\ \mu \end{pmatrix}, B_2 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0 \end{pmatrix}, B_0^{(i)} = \begin{pmatrix} i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta \end{pmatrix},$$

$$B_1 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, B_1^{(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix}$$

with $a_0 = -(\lambda + \alpha), a = -(\lambda + \mu + \alpha), a_K = -(\mu + \alpha), b_0 = -(\lambda + i\beta + \alpha), b = -(\lambda + \mu + i\beta + \alpha), b_K = -(\mu + i\beta + \alpha), 1 \leq i \leq S - 1.$

Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_S, 0, ..., 0)$ be the initial probability vector of order $S(K+1)$ where $\boldsymbol{\gamma}_S = \dfrac{1}{(1-\rho^{K+1})}((1-\rho), (1-\rho)\rho, ..., (1-\rho)\rho^K)$ with $\rho = \dfrac{\lambda}{\mu}$. Thus we arrive at

Theorem 4.3.3. **(a)** *Probability that the inventory level drops to zero before realization of common life time,* $p_\mu = -\boldsymbol{\gamma}\mathcal{T}^{-1}\mathcal{T}_\mu^0$.

(b) *Probability that the common life time realizes before inventory level becomes zero,* $p_{CLT} = -\boldsymbol{\gamma}\mathcal{T}^{-1}\mathcal{T}_{CLT}^0 = -\boldsymbol{\gamma}\mathcal{T}^{-1}\alpha\mathbf{e}$.

(c) *Mean duration of the time until either the inventory level becomes zero or realization of common life time whichever occurs first,* $\mu_T = -\gamma\mathcal{T}^{-1}\boldsymbol{e}$.

4.3.4 System performance measures

In this section we consider system performance measures.

- Expected number of customers in the system

$$E_N = \sum_{n=1}^{\infty}\sum_{i=1}^{S} n x_n(i) = \frac{\lambda}{\mu-\lambda}\sum_{i=1}^{S}\xi_i.$$

- Expected number of items in the inventory

$$E_I = \sum_{n=0}^{\infty}\sum_{i=1}^{S} i x_n(i) = \sum_{i=1}^{S} i\xi_i.$$

- Expected rate of purchase

$$E_{PR} = \mu\sum_{n=1}^{\infty}\sum_{i=1}^{S} x_n(i) = \lambda\sum_{i=1}^{S}\xi_i.$$

- Expected cancellation rate

$$E_{CR} = \sum_{n=0}^{\infty}\sum_{i=1}^{S}(S-i)\beta x_n(i) = \sum_{i=1}^{S}(S-i)\beta\xi_i.$$

- Expected number of reservations for inventory made in a cycle

$$E_{PN} = \frac{E_{PR}}{\mu_T}.$$

- Expected number of cancellations in a cycle

$$E_{CN} = \frac{E_{CR}}{\mu_T}.$$

4.4 Case of positive lead time

In this section we consider the system with positive lead time. Thus on realization of CLT or when the inventory level reaches zero through a service completion, an order for replenishment is placed. The lead time is exponentially distributed with parameter θ. Subsequently the inventory level reaches its maximum S (denoted by S^* for convenience in identification). When the inventory level is zero, new arrivals and cancellation of purchased items are not permitted. The above condition is imposed since inventory level can drop to zero through a demand or through realization of CLT. The significance of this assumption is that a passenger bus leaves the station with all seats full and so cancellation thereafter has no meaning. Remaining assumptions are as in Section 4.2. We have the $CTMC$ $\{(N(t), I(t)), t \geq 0\}$ with state space

$$\{(n, i), n \geq 0, 0 \leq i \leq S\} \bigcup \{(n, S^*), n \geq 0\}.$$

Thus the infinitesimal generator is the same as that given in (4.1). But with entries of A_0 as given in (i); that of A_2 as given in (ii) and that in $A_{0,0}$ and A_1 correspond to transition rates given by $(iii), (iv)$ and

(v) below. In addition diagonal entries in A_{00} and A_1 are non-positive, having numerical value equal to the sum of other elements of the same row found in A_{00}, A_0, A_1 and A_2. All other transitions have rate zero.

(i) *Transitions due to arrival:*

$$\begin{aligned}(n,i) &\to (n+1,i): &&\text{rate } \lambda \quad \text{for } n \geq 0, \quad 1 \leq i \leq S\\ (n,S^*) &\to (n+1,S^*): &&\text{rate } \lambda \quad \text{for } n \geq 0.\end{aligned}$$

(ii) *Transitions due to service completions:*

$$\begin{aligned}(n,i) &\to (n-1,i-1): &&\text{rate } \mu \quad \text{for } n \geq 1, \quad 1 \leq i \leq S\\ (n,S^*) &\to (n-1,S-1): &&\text{rate } \mu \quad \text{for } n \geq 1.\end{aligned}$$

(iii) *Transitions due to common life time realization:*

$$(n,i) \to (n,0): \quad \text{rate } \alpha \quad \text{for } n \geq 0, 1 \leq i \leq S.$$

(iv) *Transition due to cancellation:*

$$(n,i) \to (n,i+1): \quad \text{rate } (S-i)\beta \quad \text{for } n \geq 0, 1 \leq i \leq S-1.$$

(v) *Transition due to lead time:*

$$(n,0) \to (n,S^*): \quad \text{rate } \theta \quad \text{for } n \geq 0.$$

Each matrix A_{00}, A_0, A_1, A_2 is a square matrix of order $S+2$.

Stability condition

Let $\boldsymbol{\phi} = (\phi_0, \phi_1, ..., \phi_S, \phi_{S^*})$ be the steady state probability vector of $\mathcal{A} = A_0 + A_1 + A_2$. Then

$$\boldsymbol{\phi}\mathcal{A} = \mathbf{0},\ \ \boldsymbol{\phi}\mathbf{e} = 1.$$

The Markov chain is stable if and only if (see Neuts [48]) the left drift rate exceeds the right drift rate. That is,

$$\boldsymbol{\phi} A_0 \mathbf{e} < \boldsymbol{\phi} A_2 \mathbf{e}.$$

Using this relation we have the following

Theorem 4.4.1. *The system under study is stable if and only if* $\lambda < \mu$.

4.4.1 Stochastic decomposition of system states

Let $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, ...)$ be the steady-state probability vector of $\mathcal{Q}$ where each component
$\mathbf{y}_n = (y_n(0), y_n(1), ..., y_n(S), y_n(S^*)), n \geq 0$. Then

$$\mathbf{y}\mathcal{Q} = \mathbf{0},\ \ \mathbf{y}\mathbf{e} = 1.$$

$$y_n(i) = \lim_{t \to \infty} Prob.(N(t) = n, I(t) = i), n \geq 0, 0 \leq i \leq S \text{ and } i = S^*.$$

Assume

$$\mathbf{y}_n = \mathcal{K}\rho^n \boldsymbol{\psi}, n \geq 0$$

where $\boldsymbol{\psi}$ is steady-state probability vector when the service time is negligible, $\mathcal{K}$ is a constant and $\rho = \dfrac{\lambda}{\mu}$.

Now we first consider the system with instantaneous service time. The infinitesimal generator is given by

$$\tilde{\mathcal{A}} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-2 \\ S-1 \\ S \\ S^* \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & \cdots & \cdots & S-1 & S & S^* \end{array} \\ \begin{pmatrix} -\theta & & & & & & & & \theta \\ \alpha+\lambda & f_{S-1} & h_{S-1} & & & & & & \\ \alpha & \lambda & f_{S-2} & h_{S-2} & & & & & \\ \alpha & & \lambda & f_{S-3} & h_{S-3} & & & & \\ \vdots & & & \ddots & \ddots & \ddots & & & \\ \alpha & & & & \lambda & f_2 & h_2 & & \\ \alpha & & & & & \lambda & f_1 & h_1 & \\ \alpha & & & & & & \lambda & f_0 & \\ & & & & & & \lambda & & -\lambda \end{pmatrix} \end{array}$$

with $f_i = -(\lambda + \alpha + i\beta), 0 \le i \le S-1$ and $h_j = j\beta, 1 \le j \le S-1$.

Let $\boldsymbol{\psi} = (\psi_0, \psi_1, ..., \psi_S, \psi_{S^*})$ be the steady state vector of $\tilde{\mathcal{A}}$. Then $\boldsymbol{\psi}$ satisfies the equations

$$\boldsymbol{\psi}\tilde{\mathcal{A}} = \mathbf{0}, \ \ \boldsymbol{\psi}\mathbf{e} = 1.$$

Each ψ_i can be obtained as

$$\psi_i = \begin{cases} \mathcal{U}_i\psi_1 & 0 \le i \le S, \\ \mathcal{U}_{S^*}\psi_1 & i = S^*, \\ \left[\displaystyle\sum_{i=0}^{S} \mathcal{U}_i + \mathcal{U}_{S^*}\right]^{-1} & i = 1, \end{cases} \tag{4.14}$$

where

$$\mathcal{U}_i = \begin{cases} \dfrac{\lambda}{\theta}\mathcal{U}_{S^*} & i = 0, \\ 1 & i = 1, \\ \dfrac{\alpha + \lambda + (S-1)\beta}{\lambda}\mathcal{U}_1 & i = 2, \\ \dfrac{\alpha + \lambda + (S-i+1)\beta}{\lambda}\mathcal{U}_{i-1} - \dfrac{(S-i+2)\beta}{\lambda}\mathcal{U}_{i-2} & 3 \leq i \leq S-1, \\ \dfrac{\beta}{\alpha + \lambda}\mathcal{U}_{S-1} & i = S, \\ \dfrac{\alpha + \lambda + \beta}{\lambda}\mathcal{U}_{S-1} - \dfrac{2\beta}{\lambda}\mathcal{U}_{S-2} - \mathcal{U}_S & i = S^*. \end{cases} \tag{4.15}$$

Now from $\mathbf{y}\mathcal{Q} = 0$ and $\mathbf{y}_n = \mathcal{K}\rho^n\boldsymbol{\psi}, n \geq 0$, we have

$$\mathbf{y}_0 A_{00} + \mathbf{y}_1 A_2 = \mathcal{K}\boldsymbol{\psi}\tilde{\mathcal{A}} = 0,$$

and

$$\mathbf{y}_{n-1}A_0 + \mathbf{y}_n A_1 + \mathbf{y}_{n+1}A_2 = \mathcal{K}\rho^n\boldsymbol{\psi}\tilde{\mathcal{A}} = 0.$$

Using $\mathbf{ye} = 1$ we get $\mathcal{K} = 1 - \rho$.

Theorem 4.4.2. *The steady-state probability vector $\mathbf{y}$ of $\mathcal{Q}$ is obtained as*

$$y_n(i) = (1-\rho)\rho^n\psi_i, n \geq 0, 0 \leq i \leq S \text{ and } i = S^* \text{ at the beginning of the new cycle} \tag{4.16}$$

where $\rho = \dfrac{\lambda}{\mu}$ and ψ_i represents the inventory level probabilities when service time is negligible and are given in (4.14).

4.4.2 Probability that the next cycle starts with service completion or realization of common life time

Unlike in section 4.3.3, we cannot compute the probabilities of a new cycle starting with a service completion/ realization of CLT, with the help of the same infinitesimal generator since the lead time is positive. In this section we analyze the probabilities of the next cycle starting with service completion and realization of common life time. Choose K sufficiently large that

$$\sum_{n=0}^{K}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n} > 1-\epsilon, \text{ for arbitrary small } \epsilon > 0.$$

Except for heavy traffic (that is, $\frac{\lambda}{\mu}$ close to 1) the above approximation is very much near to exact value.

First we compute the probability that the inventory level becomes zero before realization of CLT. Consider the Markov chain

$$\{(I(t), N(t)), t \geq 0\}$$

whose state space $\{(i,n), 0 \leq i \leq S, 0 \leq n \leq K\} \bigcup \{\Delta_\mu\}$ where $\{\Delta_\mu\}$ is the absorbing state which means the replenishment order is placed after realization of event A. Thus its infinitesimal generator is of the form

$$\mathcal{P}_1 = \begin{pmatrix} \mathcal{T}_1 & \tilde{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix} \text{ where}$$

$$
\mathcal{T}_1 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \\ 0 \end{array}
\begin{array}{c} \begin{array}{cccccccc} S & S-1 & S-2 & \cdots & 3 & 2 & 1 & 0 \end{array} \\
\begin{pmatrix}
B_1^0 & B_2 & & & & & & \\
B_0^{(1)} & B_1^{0(1)} & B_2 & & & & & \\
 & B_0^{(2)} & B_1^{0(2)} & B_2 & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & B_0^{(S-3)} & B_1^{0(S-3)} & B_2 & & \\
 & & & & B_0^{(S-2)} & B_1^{0(S-2)} & B_2 & \\
 & & & & & B_0^{(S-1)} & B_1^{0(S-1)} & B_2 \\
 & & & & & & & -\theta I
\end{pmatrix} \end{array},
$$

$$
\tilde{\mathcal{T}}^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta \mathbf{e} \end{pmatrix} \text{ where } B_2 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0 \end{pmatrix}, B_0^{(i)} = \begin{pmatrix} i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta \end{pmatrix},
$$

$$
B_1^0 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, B_1^{0(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix}
$$

with $a_0 = -\lambda, a = -(\lambda + \mu), a_K = -\mu, b_0 = -(\lambda + i\beta), b = -(\lambda + \mu + i\beta), b_K = -(\mu + i\beta), 1 \leq i \leq S - 1$.

Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_S, 0, ..., 0)$ be initial probability vector of order $(S+1)(K+1)$ where $\boldsymbol{\eta}_S = \dfrac{1}{(1-\rho^{K+1})}((1-\rho), (1-\rho)\rho, ..., (1-\rho)\rho^K)$.
Thus we have

Theorem 4.4.3. *Probability that the inventory level becomes zero before realization of common life time,* $p_\mu = -\boldsymbol{\eta}\,(\mathcal{T}_1)^{-1}\,\tilde{\mathcal{T}}^0$.

Next we compute the probability that CLT realized before inventory level becomes zero. Consider the Markov chain $\{(I(t), N(t)), t \geq 0\}$ whose state space $\{(i, n), 0 \leq i \leq S, 0 \leq n \leq K\} \bigcup \{\Delta_{CLT}\}$ where $\{\Delta_{CLT}\}$ is the absorbing state which means the replenishment order is placed after realization of event B. Thus its infinitesimal generator is of the form

$$\mathcal{P}_2 = \begin{pmatrix} \mathcal{T}_2 & \tilde{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix}$$

where

$$\mathcal{T}_2 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccccccc} S & S-1 & S-2 & \cdots & 3 & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} B_1 & B_2 & & & & & & \alpha I \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & & \alpha I \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & & \alpha I \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & & \alpha I \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2 & \alpha I \\ & & & & & B_0^{(S-1)} & B_1^{'(S-1)} & \alpha I \\ & & & & & & & -\theta I \end{pmatrix} \end{array},$$

$$\tilde{\mathcal{T}}^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta \mathbf{e} \end{pmatrix} \text{ where } B_0^{(i)} = \begin{pmatrix} i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta \end{pmatrix},$$

$$B_1^{'(S-1)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b' & \lambda & & \\ & & \ddots & \ddots & \\ & & & b' & \lambda \\ & & & & b'_K \end{pmatrix}, B_2 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, B_1^{(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix}$$

with $a_0 = -(\alpha + \lambda), a = -(\lambda + \alpha + \mu), a_K = -(\alpha + \mu), b_0 = -(\lambda + \alpha + i\beta), b = -(\lambda + \alpha + \mu + i\beta), b_K = -(\mu + \alpha + i\beta), 1 \leq i \leq S - 2, b' = -(\lambda + \alpha + i\beta), b'_K = -(\alpha + i\beta)$.

Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_S, 0, ..., 0)$ be the initial probability vector of order $(S + 1)(K + 1)$ where $\boldsymbol{\eta}_S = \frac{1}{(1 - \rho^{K+1})}((1 - \rho), (1 - \rho)\rho, ..., (1 - \rho)\rho^K)$. The above discussion leads us to

Theorem 4.4.4. *Probability that common life time realizes before the inventory level becomes zero,* $p_{CLT} = -\boldsymbol{\eta}\,(\mathcal{T}_2)^{-1}\,\tilde{\mathcal{T}}^0$.

4.4.3 Mean duration of the time between two successive replenishment

Consider the Markov chain $\{(I(t), N(t)), t \geq 0\}$ whose state space

$$\{(i, n), 0 \leq i \leq S, 0 \leq n \leq K\} \bigcup \{\Delta\}$$

where $\{\Delta\}$ is the absorbing state which means the replenishment order is placed after realization of event A or event B. Thus its infinitesimal generator is of the form

$$\mathcal{P}_3 = \begin{pmatrix} \mathcal{T}_3 & \tilde{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix}$$

where

$$\mathcal{T}_3 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccccccc} S & S-1 & S-2 & \cdots & 3 & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} B_1 & B_2 & & & & & & \alpha I \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & & \alpha I \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & & \alpha I \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & & \alpha I \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2 & \alpha I \\ & & & & & B_0^{(S-1)} & B_1^{(S-1)} & \alpha I + B_2 \\ & & & & & & & -\theta I \end{pmatrix} \end{array},$$

$$\tilde{\mathcal{T}}^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \theta \mathbf{e} \end{pmatrix} \text{ where } B_2 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \ddots & \ddots & \\ & & \mu & 0 \end{pmatrix}, B_0^{(i)} = \begin{pmatrix} i\beta & & & \\ & i\beta & & \\ & & \ddots & \\ & & & i\beta \end{pmatrix},$$

$$B_1 = \begin{pmatrix} a_0 & \lambda & & & \\ & a & \lambda & & \\ & & \ddots & \ddots & \\ & & & a & \lambda \\ & & & & a_K \end{pmatrix}, B_1^{(i)} = \begin{pmatrix} b_0 & \lambda & & & \\ & b & \lambda & & \\ & & \ddots & \ddots & \\ & & & b & \lambda \\ & & & & b_K \end{pmatrix}$$

with $a_0 = -(\alpha+\lambda), a = -(\lambda+\alpha+\mu), a_K = -(\alpha+\mu), b_0 = -(\lambda+\alpha+i\beta), b = -(\lambda+\alpha+\mu+i\beta), b_K = -(\mu+\alpha+i\beta), 1 \le i \le S-1$.

Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_S, 0, ..., 0)$ be the initial probability vector of order $(S+1)(K+1)$ where $\boldsymbol{\eta}_S = \dfrac{1}{(1-\rho^{K+1})}((1-\rho),(1-\rho)\rho,...,(1-\rho)\rho^K)$. The above discussions lead us to

Lemma 4.4.1. Mean duration of the time between two successive replenishment, $\mu_T = -\boldsymbol{\eta}\,(\mathcal{T}_3)^{-1}\,\mathbf{e}$.

Mean duration of the time for the inventory level to reach zero through realization of event A or event B

Consider the Markov chain $\{(I(t), N(t)), t \geq 0\}$ whose state space $\{(i, n), 1 \leq i \leq S, 0 \leq n \leq K\} \bigcup \{\Delta'\}$ where $\{\Delta'\}$ is the absorbing state which means the inventory level becomes zero after realization of event A or event B. Thus its infinitesimal generator is of the form

$$\mathcal{P}_4 = \begin{pmatrix} \mathcal{T}_4 & \tilde{\mathcal{T}}'^0 \\ \mathbf{0} & 0 \end{pmatrix} \text{ where}$$

$$\mathcal{T}_4 = \begin{array}{c} \\ S \\ S-1 \\ S-2 \\ \vdots \\ 3 \\ 2 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccccc} S & S-1 & S-2 & \cdots & 3 & 2 & 1 \end{array} \\ \begin{pmatrix} B_1 & B_2 & & & & & \\ B_0^{(1)} & B_1^{(1)} & B_2 & & & & \\ & B_0^{(2)} & B_1^{(2)} & B_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & B_0^{(S-3)} & B_1^{(S-3)} & B_2 & \\ & & & & B_0^{(S-2)} & B_1^{(S-2)} & B_2 \\ & & & & & B_0^{(S-1)} & B_1^{(S-1)} \end{pmatrix} \end{array},$$

$$\tilde{\mathcal{T}}'^0 = \begin{pmatrix} \alpha\mathbf{e} \\ \alpha\mathbf{e} \\ \alpha\mathbf{e} \\ \vdots \\ \alpha\mathbf{e} \\ \alpha\mathbf{e} \\ B_2' \end{pmatrix} \text{ with } B_2' = \begin{pmatrix} \alpha \\ \alpha+\mu \\ \vdots \\ \alpha+\mu \end{pmatrix}.$$

Let $\boldsymbol{\eta}' = (\boldsymbol{\eta}_S, 0, ..., 0)$ be the initial probability vector of $\mathcal{T}_4$ (see Sec-

tion 4.3) is of order $S(K+1)$.
The above discussion lead us to

Lemma 4.4.2. The mean time until the inventory level becomes zero, $\mu'_T = -\boldsymbol{\eta}' \left(\mathcal{T}_4\right)^{-1} \mathbf{e}$.

4.4.4 Waiting time distribution of a tagged customer

To derive the waiting time distribution of a tagged customer who joins the queue as the r^{th} customer, $r > 0$, we consider the Markov process $W(t) = \{(N'(t), I(t)), t \geq 0\}$ where $N'(t)$ is the rank of the customer and $I(t)$ is the size of the inventory at time t. The rank $N'(t)$ of the customer is assumed to be i if he is the i^{th} customer in the queue at time t. His rank decreases to 1 as the customers ahead of him leave the system after completing their service. Since the customers who arrive after the tagged customer can not change that rank, level changing transitions in $W(t)$ is only to one side of the diagonal. We arrange the state space of $W(t)$ as $\{r, r-1, \ldots, 2, 1\} \times \{0, 1, 2, \ldots, S-1, S, S^*\} \cup \{\Delta\}$, where $\{\Delta\}$ is the absorbing state denoting that the tagged customer is selected for service. Thus the infinitesimal generator $\mathbf{W}$ of the process $W(t)$ assumes the form

$$\mathbf{W} = \begin{pmatrix} \tilde{\mathbf{T}} & \tilde{\mathbf{T}}^0 \\ \mathbf{0} & 0 \end{pmatrix} \text{ where}$$

$$\tilde{\mathbf{T}} = \begin{pmatrix} A_1 & A_2 & & & \\ & A_1 & A_2 & & \\ & & \ddots & \ddots & \\ & & & A_1 & A_2 \\ & & & & A_1 \end{pmatrix} \text{ and } \tilde{\mathbf{T}}^0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ A'_2 \end{pmatrix} \text{ with } A'_2 = \begin{pmatrix} 0 \\ \mu \\ \vdots \\ \mu \end{pmatrix}.$$

Note that A_1 and A_2 are the same matrices as defined at the beginning of 4.3.

Now, the waiting time W of a customer, who joins the queue as the r^{th} customer is the time until absorption of the Markov chain $W(t)$. Thus the waiting time of this particular customer is a PH-variate with representation $PH(\boldsymbol{\phi}, \tilde{\mathbf{T}})$, where $\boldsymbol{\phi} = (\boldsymbol{\psi}, \mathbf{0}, ..., \mathbf{0})$ with $\boldsymbol{\psi} = (0, \psi_1', \psi_2', \ldots, \psi_S', \psi_{S^*}')$ and $\psi_i' = \frac{\psi_i}{1-\psi_0}$ for $i \in \{1, 2, ..., S, S^*\}$ (see Section 4.1). Thus we have arrived at

Theorem 4.4.5. *The waiting time distribution function and the expected waiting time of a tagged customer are given by*

$$F(t) = 1 - \boldsymbol{\phi} \exp\{\tilde{\boldsymbol{T}}t\}\boldsymbol{e}$$

and

$$E_W^T = -\boldsymbol{\phi}(\tilde{\boldsymbol{T}})^{-1}\boldsymbol{e}$$

respectively.

For the computation of $F(t)$ in the above theorem we employ the uniformization procedure (see Latouche and Ramaswami [41]).

Essentially, the uniformization approach associates the infinitesimal generator $\mathbf{W}$ of the Markov chain with another matrix $\mathbf{K}$ which can be viewed as the transition matrix for a discrete time Markov chain. The two matrices are related via $\mathbf{K} = (1/c)\mathbf{W} + I = \begin{pmatrix} \tilde{P} & \tilde{\mathbf{p}} \\ \mathbf{0} & 0 \end{pmatrix}$ where c is at least as big as the maximum of the absolute value of the diagonal elements of $\mathbf{W}$; ordinarily it equals this maximum. Now we have

$$F(t) = 1 - \sum_{k=0}^{\infty} e^{-ct} \frac{(ct)^k}{k!} \boldsymbol{\phi} \tilde{P} \mathbf{e}.$$

Algorithm to compute the distribution function of a continuous $PH(\boldsymbol{\phi}, \tilde{\mathbf{T}})$ random variable

$$
\begin{aligned}
M &:= \boldsymbol{\phi}(I - \tilde{P})^{-1}\mathbf{e}; \\
a_0 &:= \boldsymbol{\phi}\mathbf{e}; \\
k &:= 0; \\
\nu &:= \tilde{P}\mathbf{e}; \\
& \textbf{repeat} \\
k &:= k + 1; \\
a_k &:= \boldsymbol{\phi}\nu; \\
\nu &:= \tilde{P}\nu; \\
& \textbf{until} \left| \sum_{i=0}^{k} a_i - M \right| < \epsilon; \\
K_1 &:= k; \\
& \textbf{for} \text{ any } t \text{ of interest } \textbf{do} \\
p &:= \exp(-ct); \\
F_1 &:= pa_0;
\end{aligned}
$$

$$
\begin{aligned}
& \textbf{for } k := 1 \textbf{ to } K_1 \textbf{ do} \\
& p := ctp/k; \\
& F_1 := F_1 + pa_k \\
& \textbf{end} \\
& \textbf{end}
\end{aligned}
$$

$$F := 1 - F_1$$

p	t	$F(t)$
0.1	0.1460	0.7555
0.2	0.1020	0.6268
0.3	0.0763	0.5219
0.4	0.0581	0.4299
0.5	0.0439	0.3465
0.6	0.0324	0.2692
0.7	0.0226	0.1967
0.8	0.0141	0.1281
0.9	0.0067	0.0627
1	0	0

Table 4.1: Values of $F(t)$: Fix $(S, \lambda, \mu, \beta, \alpha, \theta) = (8, 2, 3, 0.25, 0.1, 0.2)$

4.4.5 System performance measures

In this section we obtain system performance measures as under.

- Expected number of customers in the system

$$E'_N = \sum_{n=1}^{\infty} \sum_{i=0}^{S} n y_n(i) = \frac{\lambda}{\mu - \lambda} \sum_{i=0}^{S} \psi_i.$$

- Expected number of items in the inventory

$$E'_I = \sum_{n=0}^{\infty} \sum_{i=1}^{S} i y_n(i) = \sum_{i=1}^{S} i \psi_i.$$

- Expected rate of purchase

$$E'_{PR} = \mu \sum_{n=1}^{\infty} \sum_{i=1}^{S} y_n(i) = \lambda \sum_{i=1}^{S} \psi_i.$$

- Expected cancellation rate

$$E'_{CR} = \sum_{n=0}^{\infty} \sum_{i=1}^{S} (S-i)\beta y_n(i) = \sum_{i=1}^{S} (S-i)\beta \psi_i.$$

- Expected loss rate of customers during the lead time

$$E'_{LR} = \lambda \sum_{n=0}^{\infty} y_n(0).$$

- Expected number of purchases up to order placement in a cycle

$$E'_{PN} = \frac{E'_{PR}}{\mu'_T}.$$

- Expected number of cancellations up to order placement in a cycle

$$E'_{CN} = \frac{E'_{CR}}{\mu'_T}.$$

- Expected number of customers lost during lead time

$$E'_{LN} = \frac{E'_{LR}}{\theta}.$$

4.5 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

Effect of arrival rate λ

Changes in arrival rate has no significant impact on the measures irrespective of the lead time (see Table 4.2 (a) and (b)). This is so since no customer joins when inventory is zero.

λ	E_N	E_I	E_{PR}	E_{CR}
4	0.5714	17.6074	3.9024	3.8095
5	0.8333	17.2267	4.9018	4.7623
6	1.1993	16.8134	5.8999	5.7170
7	1.7420	16.3790	6.8895	6.6762
8	2.6027	15.9334	7.8473	7.6310
9	4.0594	15.5035	8.7253	8.5332

(a) Zero lead time

λ	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
4	0.5714	17.1882	3.8095	3.7188	0.0238
5	0.8333	16.8146	4.7845	4.6483	0.0239
6	1.1993	16.4099	5.7583	5.5798	0.0240
7	1.7420	15.9850	6.7238	6.5157	0.0241
8	2.6027	15.5495	7.6582	7.4472	0.0241
9	4.0594	15.1292	8.5146	8.3273	0.0241

(b) Positive lead time for $\theta = 4$

Table 4.2: Effect of λ: Fix $(S, \mu, \beta, \alpha) = (20, 11, 2, 0.1)$

Effect of service rate μ

Tables 4.3 (a)and (b) tell us that there is significant impact of lead time on expected inventory held and moderate impact on expected purchase and cancellation rates with respect to service time parameter.

μ	E_N	E_I	E_{PR}	E_{CR}
9	3.3118	20.2354	6.8135	6.6075
10	2.2982	20.2516	6.8692	6.5899
11	1.7420	20.2631	6.8895	6.5754
12	1.3979	20.2687	6.8969	6.5680
13	1.1661	20.2712	6.8997	6.5648
14	0.9998	20.2722	6.9007	6.5634

(a) Zero lead time

μ	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
9	3.3125	16.9061	5.6925	5.5204	0.1645
10	2.2984	16.9174	5.7382	5.5050	0.1646
11	1.7420	16.9262	5.7550	5.4925	0.1647
12	1.3979	16.9305	5.7610	5.4863	0.1647
13	1.1661	16.9325	5.7633	5.4835	0.1647
14	0.9998	16.9333	5.7641	5.4824	0.1647

(b) Positive lead time for $\theta = 0.5$

Table 4.3: Effect of μ: Fix $(S, \lambda, \beta, \alpha) = (25, 7, 1.5, 0.1)$

Effect of cancellation rate β

Impact of lead time with respect to cancellation rate β, is significant on measures such as expected inventory, expected purchase, loss and cancellation rates (see Table 4.4 (a) and (b)).

β	E_N	E_I	E_{PR}	E_{CR}
1	4.0594	21.5214	8.7258	8.1372
1.5	4.0594	23.9795	8.7260	8.5197
2	4.0594	25.2908	8.7260	8.7378
2.5	4.0594	26.1062	8.7261	8.8842
3	4.0594	26.6623	8.7261	8.9930
3.5	4.0594	27.0659	8.7261	9.0795

(a) Zero lead time

β	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
1	4.0592	14.3995	5.8383	5.4448	0.3309
1.5	4.0611	16.0525	5.8414	5.7036	0.3306
2	4.0619	16.9342	5.8428	5.8509	0.3304
2.5	4.0624	17.4825	5.8436	5.9497	0.3303
3	4.0628	17.8564	5.8441	6.0230	0.3303
3.5	4.0630	18.1279	5.8445	6.0813	0.3302

(b) Positive lead time for $\theta = 0.2$

Table 4.4: Effect of β for $(S, \lambda, \mu, \alpha) = (30, 9, 11, 0.1)$

Effect of common life time parameter α

A look at Tables 4.5(a) and (b) tell the sharp difference between zero lead time and positive lead time. Since during the lead time inventory level stays at zero, the sharp decrease seen in Table 4.5(b) is justified in contrast to quite moderate decrease rate indicated in Table 4.5(a).

α	E_N	E_I	E_{PR}	E_{CR}
0.1	3.3118	15.3454	6.8132	6.5482
0.2	3.3118	15.3216	6.7177	6.1714
0.3	3.3118	15.2779	6.6248	5.8359
0.4	3.3118	15.2182	6.5344	5.5352
0.5	3.3118	15.1457	6.4465	5.2642
0.6	3.3118	15.0629	6.3609	5.0187

(a) Zero lead time

α	E'_N	E'_I	E'_{PR}	E'_{CR}	E'_{LR}
0.1	3.3120	10.2792	4.5639	4.3865	0.3301
0.2	3.3142	7.7755	3.4091	3.1320	0.4925
0.3	3.3154	6.2744	2.7207	2.3968	0.5893
0.4	3.3161	5.2717	2.2636	1.9175	0.6536
0.5	3.3165	4.5531	1.9380	1.5826	0.6994
0.6	3.3168	4.0120	1.6942	1.3367	0.7336

(b) Positive lead time for $\theta = 0.2$

Table 4.5: Effect of α for $(S, \lambda, \mu, \beta) = (20, 7, 9, 1.5)$

The common life time parameter α plays a significant role on measures such as p_μ, p_{CLT}, μ_T. We see from Table 4.6 that for the zero lead time case the measures p_μ and μ_T decrease sharply with respect to increasing α, whereas p_{CLT} shows a fast increasing trend with increasing value of α. The latter tendency is on account of faster CLT realization.

α	p_μ	p_{CLT}	μ_T
0.1	0.8212	0.1788	1.7878
0.2	0.6768	0.3232	1.6158
0.3	0.5597	0.4403	1.4675
0.4	0.4644	0.5356	1.3389
0.5	0.3866	0.6134	1.2269
0.6	0.3227	0.6773	1.1288
0.7	0.2702	0.7298	1.0426
0.8	0.2268	0.7732	0.9665
0.9	0.1909	0.8091	0.8989
1	0.1611	0.8388	0.8388

Table 4.6: Effect of α on p_μ, p_{CLT}, μ_T

4.5.1 Cost analysis

Based on the above performance measures we define the following two revenue (profit) functions as:
For zero lead time,

$$F(\alpha, \beta, S) = C_1 E_{PR} + C_2 E_{CR} - C_3 E_I - C_4 E_N$$

For positive lead time,

$$F_{PL}(\alpha, \beta, S) = C_1 E'_{PR} + C_2 E'_{CR} - C_3 E'_I - C_4 E'_N - C_5 E'_{LR}$$

where

- C_1 = revenue to the system due to per unit purchase (by a customer at the end of his service)
- C_2 = revenue to the system due to per unit cancellation
- C_3 = holding cost per inventoried item per unit time
- C_4 = holding cost per customer per unit time
- C_5 = cost due to customer lost per unit time (applicably only to positive lead time case)

In order to study the variation in different parameters on profit function we first take the values $(C_1, C_2, C_3, C_4, C_5) = (\$100, \$30, \$10, \$2, \$10)$.

Zero lead time

Table 4.7 is indicative of the fact that as cancellation rate increases optimal S value decreases. This could be explained as follows: with cancellation rate increasing, the trend for accumulation of lesser quantity of inventory increases at the time of realization of CLT, the items left in the inventory also tend to be longer which brings down the profit (see figure 4.1(a)).

In Table 4.8 (see figure 4.2(a)) we notice that optimal S value stays at 11 as rate of realization of CLT moves from 0.1 to 0.35. This could be attributed to the fact that the expected number of cancellations is brought down thereby the left over items at CLT realization becomes smaller and smaller.

Table 4.9 (figure 4.3(a)) shows a decreasing trend for profit for fixed cancellation rate(s) as CLT is varied from 0.1 to 0.35. This is so since

$S \downarrow \beta \rightarrow$	0.6	0.7	0.8	0.9	1	1.1
12	695.7005	715.6908	733.6266	748.914	761.226	770.5668
13	705.4946	726.1356	743.6778	757.4794	767.4486	**774.0164**
14	714.3682	734.999	751.1854	762.5007	**769.4356**	773.0453
15	722.2044	741.9401	755.7128	**763.7802**	767.5236	768.5599
16	728.7723	746.5474	**756.9911**	761.5639	762.5276	761.7047
17	733.7456	**748.473**	755.1026	756.5191	755.4446	753.4817
18	736.7633	747.5921	750.5101	749.4988	747.1415	744.5712
19	**737.5331**	744.0955	743.905	741.2767	738.2117	735.3537
20	735.9427	738.447	735.9863	732.4085	728.994	726.0119

Table 4.7: Effect of S and β on $F(\alpha, \beta, S)$ for $(\lambda, \mu, \alpha) = (7, 9, 0.1)$

$S \downarrow \alpha \rightarrow$	0.1	0.15	0.2	0.25	0.3	0.35
10	783.0723	774.2038	765.587	757.2099	749.0611	741.1301
11	**787.4177**	**777.6512**	**768.2145**	**759.088**	**750.2537**	**741.6951**
12	786.0369	775.6812	765.7166	756.1166	746.8573	737.917
13	780.696	770.0332	759.7997	749.9644	740.4992	731.379
14	773.0865	762.3077	751.9776	742.0626	732.5323	723.3596
15	764.3822	753.5897	743.2543	733.3410	723.8184	714.6585

Table 4.8: Effect of S and α on revenue $F(\alpha, \beta, S)$ for $(\lambda, \mu, \beta) = (7, 9, 1.5)$

the number of cancellations decrease thereby decreasing the revenue from canceled items.

$\alpha \downarrow \beta \rightarrow$	1	1.5	2	2.5	3	3.5
0.1	767.5236	764.3822	758.8378	755.3327	753.0308	751.4426
0.15	754.3746	753.5897	749.7732	747.3268	745.732	744.6488
0.2	741.9535	743.2543	740.999	739.529	738.5952	737.9888
0.25	730.191	733.341	732.4984	731.9296	731.6143	731.4584
0.3	719.0269	723.8184	724.2561	724.52	724.7837	725.0534
0.35	708.408	714.6585	716.2579	717.2917	718.0978	718.7699

Table 4.9: Effect of α and β on profit $F(\alpha, \beta, S)$ for $(\lambda, \mu, S) = (7, 9, 15)$

Positive lead time

Tabulations in Tables 4.10 to 4.12 (see figure 4.1(b) to 4.3(b))pertain to positive lead time. A comparison between Tables 4.7 and 4.10 reveal that revenue is less for positive lead time case. This could be attributed to loss of customers during time in the latter. Within Table 4.10 we notice that there is a decreasing trend in the optimal value of S with increase in cancellation rate for which the same explanation as given for Table **??** is valid.

$S \downarrow \beta \rightarrow$	0.6	0.7	0.8	0.9	1	1.1
14	308.8123	352.6232	395.7066	433.407	462.5295	482.5812
15	339.5486	387.1692	429.6957	462.0075	483.024	495.012
16	369.6888	418.3271	456.1647	480.1111	492.7432	**498.406**
17	398.0574	443.9862	473.7589	488.8044	**494.8443**	496.5527
18	423.1877	462.6233	482.9697	**490.6662**	492.4892	492.1152
19	443.6235	473.9276	**485.6684**	488.3703	487.8934	486.5574
20	458.3777	478.8463	484.0916	483.8823	482.2969	480.558
21	467.2944	**479.0377**	480.0756	478.3637	476.2921	474.3943
22	**471.0636**	476.1888	474.8192	472.4018	470.1294	468.169
23	470.8868	471.6039	468.9873	466.2601	463.9054	461.9177

Table 4.10: Effect of S and β on profit $F_{PL}(\alpha, \beta, S)$ for $(\lambda, \mu, \alpha, \theta) = (7, 9, 0.1, 0.2)$

The results in Table 4.11 could be explained on the same lines as that for Table 4.8.

$S \downarrow \alpha \rightarrow$	0.1	0.15	0.2	0.25	0.3	0.35
10	418.2949	367.6324	327.1204	294.0112	266.465	243.2032
11	466.8609	403.2336	353.9116	314.5891	282.5283	255.9052
12	495.1468	422.7614	367.7864	324.6489	289.9221	261.3835
13	506.7439	429.9542	**372.2739**	**327.3945**	**291.5059**	**262.1714**
14	**508.2789**	**429.9933**	371.484	326.1333	289.9766	260.4944
15	505.0019	426.6814	368.2638	323.0535	287.0522	257.7253

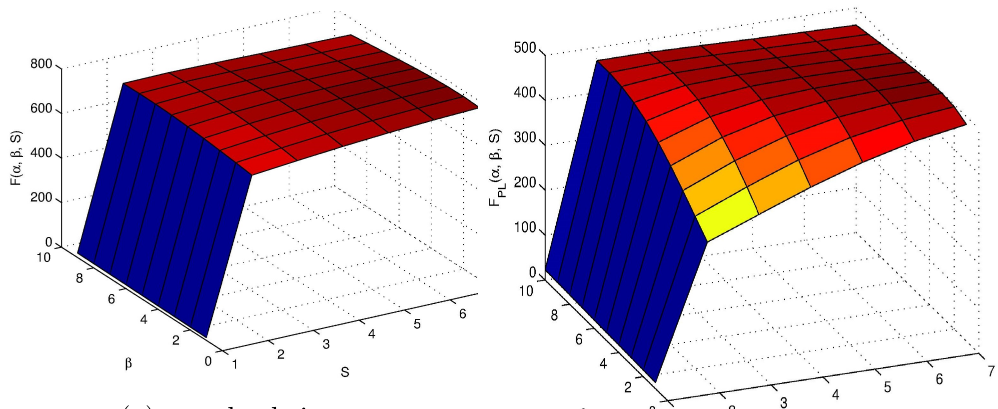
Table 4.11: Effect of S and α on revenue $F_{PL}(\alpha, \beta, S)$ for $(\lambda, \mu, \beta, \theta) = (7, 9, 1.5, 0.2)$

Finally, coming to Table 4.12, we notice that for fixed cancellation rates, the revenue decreases with increase in the rate of realization of CLT, which is on expected lines. On the other hand for fixed rates of realization of CLT, profit is seen to reach a maximum and then starts decreasing with increasing cancellation rates, until α grows up to 0.25. This should be due to a trend of holding cost and revenue from cancellations. However, for higher rates of CLT realization, the profit due to increase in cancellation dominate the loss due to higher rate of realization of CLT. A comparison between values in Tables (for example 4.7 and

$\alpha \downarrow \beta \rightarrow$	1	1.5	2	2.5	3	3.5
0.1	483.024	505.0019	502.6008	500.4557	499.0076	498.0052
0.15	411.3161	426.6814	425.3642	424.096	423.2434	422.6629
0.2	356.7699	368.2638	367.7428	367.1023	366.6791	366.4049
0.25	313.9579	323.0535	323.1262	322.9488	322.845	322.8004
0.3	279.5138	287.0522	287.5732	287.7448	287.8844	288.016
0.35	251.2406	257.7253	258.588	259.0269	259.3551	259.6245

Table 4.12: Effect of α and β on revenue $F_{PL}(\alpha, \beta, S)$ for $(\lambda, \mu, S, \theta) = (7, 9, 15, 0.2)$

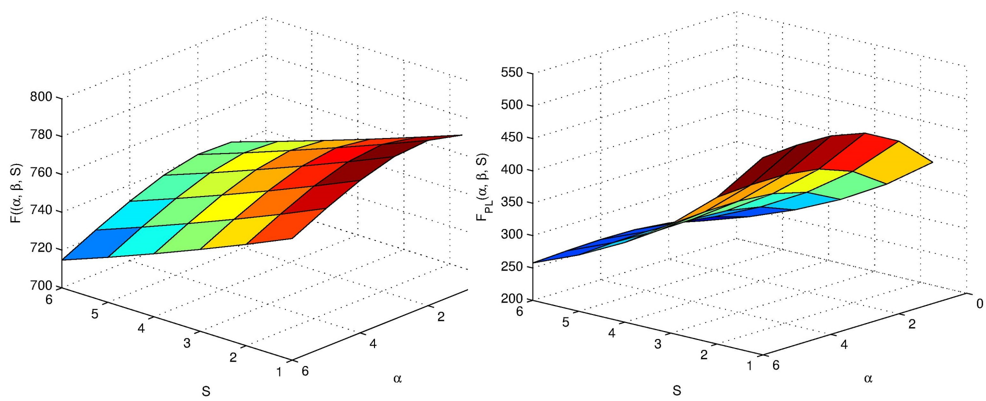
4.10) indicate that lead time plays a crucial role in the revenue generation of the system. For zero lead time revenue is much larger than that corresponding to positive lead time. This is due to customer loss during lead time. Thus if we additionally introduce a cost for reduction in lead time, we will be able to have a trade off between duration of lead time and customer loss.



(a) zero lead time

(b) positive lead time

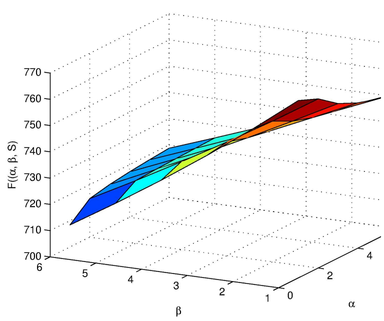
Figure 4.1: Effect of S and β on revenue



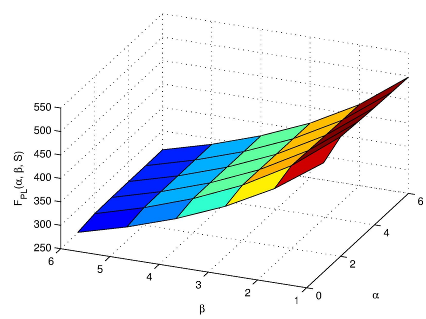
(a) zero lead time

(b) positive lead time

Figure 4.2: Effect of S and α on profit



(a) zero lead time



(b) positive lead time

Figure 4.3: Effect of α and β on profit function

Chapter 5

Queueing-Inventory Model with Two Commodities

So far we were concentrating on single commodity problems. In Chapters 2-4 we were specifically looking for product form solution. This was of interest on its own right. A suitable choice of blocking set helped in arriving at product form solution. However, when we move from single to two or more commodity problems, the identification of blocking set seems to be complex.

In this chapter we analyse a two- commodity queueing inventory problem. The QBD structure of the CTMC's in earlier chapters was a consequence of the fact that in the absence of inventory the server stays idle. However in the two commodity problem that we have at hand we relax the above assumption. So when an item is demanded by the customer at the head of the queue and that turned out to be out-of-stock, that customer leaves the system. This results in the QBD structure of the CTMC at hand being lost. In fact what we get is a $GI/M/1$ type infinitesimal generator. Thus our analysis of the problem needs more sophistication.

Inventory systems dealing with several/distinct commodities are very common, (see for example [46][1]). Such systems are more complex than single commodity system which could be attributed to the reordering procedures. Whether the ordering policies of joint, individual or some mixed type are superior will depend on the particular problem at hand.

For commodities which are of clearly distinct types and are subject to different supply systems, the individual strategies would be the first choice. The individual order policy consists of the calculation of optimum order quantities and/or time periods from item to item, disregarding any economic interaction between them. This policy has considerable flexibility in selecting the individually best inventory models for each single item, as well as in the possibility of modifying independently any constant entering the calculations.

The joint policies may have advantages in situations where a procurement is made from the same suppliers/ items produced on the same machine/ items have to be supplied by the same transport facility, so that joint ordering policy might be superior with regard to cost efficiency. The modelling of multi-item inventory systems are getting more attention now a days. In this chapter we will use 'item' and 'commodity' interchangeably. The replenishment policy is to place order for an item when its level drops to the reorder level.

5.1 Model Description

Consider a two commodity inventory system with a single server. The maximum storage capacity for the $i-th$ commodity is S_i units for $i = 1, 2$.

Demands arrive according to a Poisson Process of rate λ and demand for each commodity is of unit size. Customers are not allowed to join the system when inventory levels of both commodities are zero. However, customers join the system even when the server is busy with no excess inventory available at hand. This is with the hope that during the current service the replenishment of the items would take place, so that at the epoch when taken for service, the item demanded by the customer can be provided. Also Customers are lost when no item of the commodity demanded by them is available at the time of offering service. At the time when taken for service the customer demands item C_i with probability p_i, for $i = 1, 2$ or both C_1 and C_2 with probability p_3 such that $p_1 + p_2 + p_3 = 1$. The demanded item is delivered to the customer after a random duration of service. The service times for processing orders for C_1, C_2 or both C_1 and C_2 are exponentially distributed with parameters μ_1, μ_2 and μ_3 respectively. We adopt (s_i, S_i) replenishment policy for commodity C_i, $i = 1, 2$. That is, whenever the inventory level of commodity C_i falls to s_i an order is placed for that alone to bring the inventory level back to S_i, $i = 1, 2$ at the time of replenishment. The time till replenishment from the epoch at which order is placed(lead time) is exponentially distributed with parameters β_i for $C_i, i = 1, 2$.
The above problem can be modelled as a continuous time Markov chain of the GI/M/1 type

$$\{(N(t), I_1(t), I_2(t), J(t)), t \geq 0\}$$

where

$N(t)$: Number of customers in the queue at time t
$I_i(t)$: Excess inventory level of commodity C_i , $i = 1, 2$ at time t
$J(t)$: State of the server at time$t$

and

$$J(t) = \begin{cases} 0 & \text{if server is idle;} \\ 1 & \text{if server is busy processing } C_1; \\ 2 & \text{if server is busy processing } C_2; \\ 3 & \text{if server is busy processing } C_1 \text{ and } C_2. \end{cases}$$

The state space of the above process is $\mathbf{\Omega} = \bigcup_{n=0}^{\infty} \boldsymbol{\ell(n)}$ where $\boldsymbol{\ell(n)}$ denotes level n,

$$\boldsymbol{\ell(0)} = \{(0, j_1, j_2, r) : 0 \leq j_1 \leq S_1, 0 \leq j_2 \leq S_2, 0 \leq r \leq 3\}$$

and

$$\boldsymbol{\ell(n)} = \{(n, j_1, j_2, r) : 0 \leq j_1 \leq S_1, 0 \leq j_2 \leq S_2, 1 \leq r \leq 3\}, n \geq 1$$

Thus, the infinitesimal generator matrix of the Markov chain has the form

$$\mathcal{Q} = \begin{bmatrix} B_1 & B_0 & & & & \\ B_2 & A_1 & A_0 & & & \\ B_3 & A_2 & A_1 & A_0 & & \\ B_4 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.1}$$

where
$B_{n+1}, n \geq 0$ contains transitions from $\boldsymbol{\ell(n)}$ to $\boldsymbol{\ell(0)}$,

B_0 contains the transition from $\boldsymbol{\ell(0)}$ to $\boldsymbol{\ell(1)}$,

A_1 contains the transitions within $\boldsymbol{\ell(n)}$ $n \geq 1$,

A_0 contains transitions from $\boldsymbol{\ell(n)}$ to $\boldsymbol{\ell(n+1)}$ $i \geq 1$

and A_{k+1} contains transitions from $\boldsymbol{\ell(n)}$ to $\boldsymbol{\ell(n-k)}$, $1 \leq k \leq n-1, n \geq 2$.

Then A_0, A_1, A_2, $\cdots$ are square matrices of dimension a, where $a = 3(S_1+1)(S_2+1)$. B_1 is a square matrix of dimension b, $b = 4(S_1+1)(S_2+1)$. B_0, B_i, $i \geq 2$, are of dimensions $b \times a$, $a \times b$, respectively.

Transitions in the Markov chain and the corresponding rates are described below: The matrix B_1 governs,

1. $(0, j_1, j_2, r) \to (0, j_1, j_2, 0)$ with rate μ_r for $1 \leq r \leq 3$, $0 \leq j_1 \leq S_1$, $0 \leq j_2 \leq S_2$

2. $(0, j_1, j_2, r) \to (0, S_1, j_2, r)$ with rate β_1 for $0 \leq r \leq 3$ $0 \leq j_1 \leq s_1$, $0 \leq j_2 \leq S_2$

3. $(0, j_1, j_2, r) \to (0, j_1, S_2, r)$ with rate β_2 for $0 \leq r \leq 3$ $0 \leq j_1 \leq S_1$, $0 \leq j_2 \leq s_2$

4. $(0, 0, 0, r) \to (1, 0, 0, r)$ with rate λ for $1 \leq r \leq 3$

5. $(0, 0, j_2, 0) \to (0, 0, j_2 - 1, 2)$ with rate $\lambda(p_2 + p_3)$ for $1 \leq j_2 \leq S_2$

6. $(0, j_1, 0, 0) \to (0, j_1 - 1, 0, 1)$ with rate $\lambda(p_1 + p_3)$ for $1 \le j_1 \le S_1$
7. $(0, j_1, j_2, 0) \to (0, j_1 - 1, j_2, 1)$ with rate λp_1 for $1 \le j_1 \le S_1, 1 \le j_2 \le S_2$
8. $(0, j_1, j_2, 0) \to (0, j_1, j_2 - 1, 2)$ with rate λp_2 for $1 \le j_1 \le S_1, 1 \le j_2 \le S_2$
9. $(0, j_1, j_2, 0) \to (0, j_1 - 1, j_2 - 1, 3)$ with rate λp_3 for $1 \le j_1 \le S_1, 1 \le j_2 \le S_2$

The matrix, $B_{n+1}, n \ge 1$, governs

1. $(n, 0, 0, r) \to (0, 0, 0, 0)$ with rate μ_r for $1 \le r \le 3$
2. $(n, 0, j_2, r) \to (0, 0, j_2, 0)$ with rate $\mu_r p_1^n$ for $1 \le j_2 \le S_2$, $1 \le r \le 3$
3. $(n, 0, j_2, r) \to (0, 0, , j_2 - 1, 2)$ with rate $\mu_r p_1^{n-1}(p_2 + p_3)$ for $1 \le j_2 \le S_2$, $1 \le r \le 3$
4. $(n, j_1, 0, r) \to (n, j_1, 0, 0)$ with rate $\mu_r p_2^n$ for $1 \le j_1 \le S_1$,$1 \le r \le 3$
5. $(n, j_1, 0, r) \to (0, j_1 - 1, 0, 1)$ with rate $\mu_r p_2^{n-1}(p_1 + p_3)$ for $1 \le j_1 \le S_1$, $1 \le r \le 3$
6. $(1, j_1, j_2, r) \to (0, j_1 - 1, j_2, 1)$ with rate $\mu_r p_1$ for $1 \le j_1 \le S_1$,$1 \le j_2 \le S_2$,$1 \le r \le 3$
7. $(1, j_1, j_2, r) \to (0, j_1, j_2 - 1, 2)$ with rate $\mu_r p_2$ for $1 \le j_1 \le S_1$,$1 \le j_2 \le S_2$, $1 \le r \le 3$
8. $(1, j_1, j_2, r) \to (0, j_1 - 1, j_2 - 1, 3)$ with rate $\mu_r p_3$ for $1 \le j_1 \le S_1$,$1 \le j_2 \le S_2$, $1 \le r \le 3$

The matrix, $A_{k+1}, 1 \leq k \leq n-1, n \geq 3$, governs

1. $(n, 0, j_2, r) \rightarrow (n-k, 0, j_2-1, 2)$ with rate $\mu_r p_1^{k-1}(p_2+p_3)$ for $1 \leq j_2 \leq S_2$, $1 \leq r \leq 3$

2. $(n, j_1, 0, r) \rightarrow (n-k, j_1-1, 0, 1)$ with rate $\mu_r p_2^{k-1}(p_1+p_3)$ for $1 \leq j_1 \leq S_1$, $1 \leq r \leq 3$

3. $(n, j_1, j_2, r) \rightarrow (n-1, j_1-1, j_2, 1)$ with rate $\mu_r p_1$ for $1 \leq j_1 \leq S_1$, $1 \leq j_2 \leq S_2$, $1 \leq r \leq 3$

4. $(n, j_1, j_2, r) \rightarrow (n-1, j_1, j_2-1, 2)$ with rate $\mu_r p_2$ for $1 \leq j_1 \leq S_1$, $1 \leq j_2 \leq S_2$, $1 \leq r \leq 3$

5. $(n, j_1, j_2, r) \rightarrow (n-1, j_1-1, j_2-1, 3)$ with rate $\mu_r p_3$ for $1 \leq j_1 \leq S_1$, $1 \leq j_2 \leq S_2$, $1 \leq r \leq 3$

The matrix, A_1, governs:

1. $(n, j_1, j_2, r) \rightarrow (n, S_1, j_2, r)$ with rate β_1 for $0 \leq j_1 \leq s_1, 0 \leq j_2 \leq S_2, 1 \leq r \leq 3$

2. $(n, j_1, j_2, r) \rightarrow (n, j_1, S_2, r)$ with rate β_2 for $0 \leq j_1 \leq S_1, 0 \leq j_2 \leq s_2, 1 \leq r \leq 3$

Thus, the elements of the matrices can be described as

$$B_0(n, i_1, i_2, r; m, j_1, j_2, l) = \begin{cases} \lambda, & m = n+1, 0 \leq i_1 \leq S_1, 0 \leq i_2 \leq S_2, r = l = 1, 2, 3; \\ & j_1 = i_1, j_2 = i_2, l = r; \\ 0, & \text{otherwise.} \end{cases}$$

$$A_0(n, i_1, i_2, r; m, j_1, j_2, l) = \begin{cases} \lambda, & m = n+1, 0 \leq i_1 \leq S_1, 0 \leq i_2 \leq S_2, r = 1, 2, 3, \\ & j_1 = i_1, j_2 = i_2, l = r; \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leq k \leq i-1$, $i \geq 3$,

$$A_{k+1}(n,i_1,i_2,r;m,j_1,j_2,l) = \begin{cases} \mu_r p_1^{k-1}(p_2+p_3), & m=i-k, 1\leq i_2 \leq S_2, \\ & i_1=j_1=0, r=1,2,3, l=2; \\ \mu_r p_2^{k-1}(p_1+p_3), & m=i-k, 1\leq i_1 \leq S_1, \\ & i_2=j_2=0, r=1,2,3, l=1; \\ 0, & \text{otherwise.} \end{cases}$$

$$B_2(n,i_1,i_2,r;m,j_1,j_2,l) = \begin{cases} \mu_r, & m=n-1, i_1=i_2=j_1=j_2=0, \\ & r=1,2,3, l=0; \\ \mu_r(p_2+p_3), & m=n-1, i_1=j_1=0, 1\leq i_2\leq S_2, \\ & j_2=i_2-1, r=1,2,3, l=2\ ; \\ \mu_r p_1, & m=n-1, i_1=j_1=0, 1\leq i_2\leq S_2, \\ & j_2=i_2, r=1,2,3, l=0\ ; \\ \mu_r(p_1+p_3), & m=n-1, 1\leq i_1\leq S_1, i_2=j_2=0, \\ & j_1=i_1-1, r=1,2,3, l=1\ ; \\ \mu_r p_2, & m=n-1, 1\leq i_1\leq S_1, i_2=j_2=0, \\ & j_1=i_1, r=1,2,3, l=0\ ; \\ \mu_r p_1, & m=n-1, 1\leq i_1\leq S_1, \\ & 1\leq i_2\leq S_2, j_1=i_1-1, r=1,2,3, l=1; \\ \mu_r p_2, & m=n-1, 1\leq i_1\leq S_1, \\ & 1\leq i_2\leq S_2, j_2=i_2-1, r=1,2,3, l=2; \\ \mu_r p_3, & m=n-1, 1\leq i_1\leq S_1, \\ & 1\leq i_2\leq S_2, j_1=i_1-1, j_2=i_2-1, \\ & r=1,2,3, l=3; \\ 0, & \text{otherwise.} \end{cases}$$

For $i \geq 2$

$$B_{i+1}(n,i_1,i_2,r;m,j_1,j_2,l) = \begin{cases} \mu_r, & m=0, i_1=i_2=j_1=j_2=0, \\ & r=1,2,3, l=0; \\ \mu_r p_1^i, & m=0, i_1=j_1=0, \\ & 1\leq i_2, j_2\leq S_2, r=1,2,3, l=0\ ; \\ \mu_r p_1^{i-1}(p_2+p_3), & m=0, i_1=j_1=0, \\ & 1\leq i_2\leq S_2, j_2=i_2-1, r=1,2,3, l=2\ ; \\ \mu_r p_2^{i-1}(p_1+p_3), & m=0, 1\leq i_1\leq S_1, \\ & j_1=i_1-1, i_2=j_2=0, r=1,2,3, l=1; \\ \mu_r p_2^i & m=0, 1\leq i_1\leq S_1, i_2=j_2=0, \\ & r=1,2,3, l=0; \\ 0, & \text{otherwise.} \end{cases}$$

$$A_2(n,i_1,i_2,r;m,j_1,j_2,l)=\begin{cases}\mu_r(p_2+p_3), & m=n-1, i_1=0, 1\leq i_2\leq S_2,\\ & j_2=i_2-1, r=1,2,3, l=2;\\ \mu_r(p_1+p_3), & m=n-1, 1\leq i_1\leq S_1,\\ & i_2=0, j_1=i_1-1, r=1,2,3, l=1;\\ \mu_r p_1, & m=n-1, 1\leq i_1\leq S_1,\\ & 1\leq i_2\leq S_2, j_1=i_1-1, r=1,2,3, l=1;\\ \mu_r p_2, & m=n-1, 1\leq i_1\leq S_1,\\ & 1\leq i_2\leq S_2, j_2=i_2-1, r=1,2,3, l=2;\\ \mu_r p_3, & m=n-1, 1\leq i_1\leq S_1,\\ & 1\leq i_2\leq S_2, j_1=i_1-1, j_2=i_2-1,\\ & r=1,2,3, l=3;\\ 0, & \text{otherwise.}\end{cases}$$

$$A_1(n,i_1,i_2,r;m,j_1,j_2,l)=\begin{cases}\beta_2, & n=m, 0\leq i_1,j_1\leq S_1,\\ & 0\leq i_2\leq s_2, j_2=S_2, r=l=1,2,3;\\ \beta_1, & n=m, 0\leq i_1\leq s_1, j_1=S_1,\\ & 0\leq i_2,j_2\leq 1,2,..S_2, r=l=0,1,2,3;\\ -(\lambda+\mu_r+\beta_1+\beta_2) & n=m, 0\leq i_1,j_1\leq s_1,\\ & 0\leq i_2,j_2\leq s_2, r=l=1,2,3;\\ -(\lambda+\mu_r+\beta_1) & n=m, 0\leq i_1,j_1\leq s_1,\\ & s_2+1\leq i_2,j_2\leq S_2, r=l=1,2,3;\\ -(\lambda+\mu_r+\beta_2) & n=m, s_1+1\leq i_1,j_1\leq S_1,\\ & 0\leq i_2,j_2\leq s_2, r=l=1,2,3;\\ -(\lambda+\mu_r) & n=m, s_1+1\leq i_1,j_1\leq S_1,\\ & s_2+1\leq i_2,j_2\leq S_2, r=1,2,3;\\ 0, & \text{otherwise.}\end{cases}$$

$$B_1(n,i_1,i_2,r;m,j_1,j_2,l)=\begin{cases}-(\beta_1+\beta_2), & n=m=0, i_1=i_2=j_1=j_2=0,\\ & k=l=0;\\ \mu_r, & n=m=0, 0\leq i_1,j_1\leq S_1,\\ & 0\leq i_2,j_2\leq S_2, r=1,2,3, l=0;\\ \beta_2, & n=m=0, 0\leq i_1,j_1\leq S_1,\\ & 0\leq i_2\leq s_2, j_2=S_2, r=l=0,1,2,3;\\ \beta_1, & n=m=0, 0\leq i_1\leq s_1, j_1=S_1,\\ & 0\leq i_2,j_1\leq 1,2,..S_2, r=l=0,1,2,3;\\ \lambda(p_2+p_3) & n=m=0, i_1=j_1=0, 1\leq i_2\leq S_2,\\ & j_2=i_2-1, r=0, l=2;\\ \lambda(p_1+p_3) & n=m=0, 1\leq i_1\leq S_1, j_1=i_1-1,\\ & i_2=j_2=0, r=0, l=1;\end{cases}$$

$$B_1(n,i_1,i_2,r;m,j_1,j_2,l)=\begin{cases}\lambda p_1 & n=m=0, 1\le i_1\le S_1, j_1=i_1-1,\\ & 1\le i_2,j_2\le S_2, r=0, l=1;\\ \lambda p_2 & n=m=0, 1\le i_1,j_1\le S_1,\\ & 1\le i_2\le S_2, j_2=i_2-1, r=0, l=2;\\ \lambda p_3 & n=m=0, 1\le i_1\le S_1, 1\le i_2\le S_2,\\ & j_1=i_1-1, j_2=i_2-1, r=0, l=3;\\ -(\lambda+\mu_r+\beta_1+\beta_2) & n=m=0, 0\le i_1,j_1\le s_1,\\ & 0\le i_2,j_2\le s_2, r=l=1,2,3;\\ -(\lambda(p_2+p_3)+\beta_1+\beta_2) & n=m=0, i_1=j_1=0,\\ & 0\le i_2,j_2\le s_2, r=l=0;\\ -(\lambda(p_1+p_3)+\beta_1+\beta_2) & n=m=0, i_2=j_2=0,\\ & 0\le i_1,j_1\le s_2, r=l=0;\\ -(\lambda+\beta_1+\beta_2) & n=m=0, 1\le i_1,j_1\le s_1,\\ & 1\le i_2,j_2\le s_2, r=l=0;\\ -(\lambda+\mu_r+\beta_1) & n=m=0, 1\le i_1\le s_1,\\ & s_2+1\le i_2\le S_2, r=l=1,2,3;\\ -(\lambda+\mu_r+\beta_2) & n=m=0, s_1+1\le i_1\le S_1,\\ & 1\le i_2\le s_2, r=l=1,2,3;\\ -(\lambda+\beta_2) & n=m=0, s_1+1\le i_1\le S_1,\\ & 1\le i_2\le s_2, r=l=0;\\ -(\lambda+\beta_1) & n=m=0, 1\le i_1\le s_1,\\ & s_2+1\le i_2\le S_2, r=0;\\ -\lambda & n=m=0, s_1+1\le i_1\le S_1, j_1=i_1,\\ & s_2+1\le i_2\le S_2, j_2=i_2, r=0=l;\\ 0, & \text{otherwise.}\end{cases}$$

5.2 Steady- State Analysis

A necessary condition for $\mathcal{Q}$ to be irreducible is B_1 and A_1 are nonsingular. Consider the matrix $A=\sum_{k=0}^{\infty}A_k$. Let the unique stationary distribution of A be $\boldsymbol{\pi}$. Under the condition,

$$\boldsymbol{\pi}A_0\mathbf{e}<\sum_{k=2}^{\infty}(k-1)\boldsymbol{\pi}A_k\mathbf{e},$$

an irreducible Markov chain with generator $\mathcal{Q}$ possesss a unique stationary solution vector $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ satisfying

$$\mathbf{x}\mathcal{Q} = 0, \mathbf{x}\mathbf{e} = 1.$$

Partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ where

$$\mathbf{x}_0 = (x_0(j_1, j_2, r) : 0 \leq j_1 \leq S_1, 0 \leq j_2 \leq S_2, 0 \leq r \leq 3),$$

$$\mathbf{x}_i = (x_i(j_1, j_2, r) : 0 \leq j_1 \leq S_1, 0 \leq j_2 \leq S_2, 1 \leq r \leq 3), fori \geq 1,$$

where $\mathbf{x}_0$ is of dimension $1 \times b$ and $\mathbf{x}_i$ for $i \geq 1$, is of dimension $1 \times a$ Then $\mathbf{x}$ is obtained as

$$\mathbf{x}_i = \mathbf{x}_1 R^{i-1}, i \geq 2$$

where R is the minimal non negative solution of the matrix equation $\sum_{k=0}^{\infty} X^k A_k = 0$. The boundary equations are given by

$$\sum_{r=0}^{\infty} \mathbf{x}_r B_{r+1} = 0$$

$$\mathbf{x}_0 B_0 + \sum_{r=1}^{\infty} \mathbf{x}_r A_r = 0$$

The normalizing condition $\mathbf{x}\mathbf{e} = 1$ gives

$$\mathbf{x}_0\mathbf{e} + \mathbf{x}_1[I - R]^{-1}\mathbf{e} = 1$$

R matrix is obtained using the algorithm:

$$R(0) = 0$$

$$R(n+1) = -A_0A_1^{-1} - R^2(n)A_2A_1^{-1} - R^3(n)A_3A_1^{-1} - \ldots, n \geq 0$$

Theorem 5.2.1. *The Markov chain with infinitesimal generator $\mathcal{Q}$ given by 5.1 is stable.*

Proof. Consider a service completion epoch at which stock of both commodities or atleast one commodity drops to zero. Suppose n customers are waiting in the queue at this epoch. If all of them ask for the same commodity which is not available in stock then all these customers leave the system instantly with the result that queue becomes empty. The probability for the above indicated event is $\sum_{n=1}^{\infty}\sum_{r=1}^{3}\mu_r p_i^n > 0$. Hence from any level the queue size may drop to zero with positive probability, however small(as n becomes very large), in a very short time following a service completion. This can be thought of as a catastrophic model in that the catastrophic events occur at epochs when service is to begin and there is no inventory left. □

5.3 System Characteristics

Next we proceed to compute measures that are indications of the system performance.

- Expected number of customers in the queue,

$$E_N = \sum_{n=1}^{\infty} n \sum_{j_1=0}^{S_1}\sum_{j_2=0}^{S_2}\sum_{r=1}^{3} x_n(j_1, j_2, r).$$

- Expected number of customers demanding C_1 alone,

$$E_{C_1} = p_1 E_N.$$

- Expected number of customers demanding C_2 alone,

$$E_{C_2} = p_2 E_N.$$

- Expected number of customers demanding both C_1 and C_2,

$$E_{C_{12}} = p_3 E_N.$$

- Expected number of item C_1 in the system,

$$E_{I_1} = \sum_{n=0}^{\infty} \sum_{j_1=1}^{S_1} \sum_{j_2=0}^{S_2} \sum_{r=0}^{3} j_1 x_n(j_1, j_2, r).$$

- Expected number of item C_2 in the system,

$$E_{I_2} = \sum_{n=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=1}^{S_2} \sum_{r=0}^{3} j_2 x_n(j_1, j_2, r).$$

- Probability that server is busy processing a demand for C_1 alone,

$$P_{C_1} = \sum_{n=1}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} x_n(j_1, j_2, 1).$$

- Probability that server is busy processing a demand for C_2 alone,

$$P_{C_2} = \sum_{n=1}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} x_n(j_1, j_2, 2).$$

- Probability that server is busy processing a demand for both C_1

and C_2,

$$P_{C_{12}} = \sum_{n=1}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{S_2} x_n(j_1, j_2, 3).$$

- Probability that server is busy,

$$P_{busy} = \sum_{n=1}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=1}^{S_2} \sum_{r=1}^{3} x_n(j_1, j_2, r) + \sum_{n=1}^{\infty} \sum_{j_1=1}^{S_1} \sum_{r=1}^{3} x_n(j_1, 0, r).$$

- Probability that inventory C_1 alone is zero,

$$P_{C_{10}} = \sum_{n=0}^{\infty} \sum_{j_2=0}^{S_2} \sum_{r=0}^{3} x_n(0, j_2, r).$$

- Probability that inventory C_2 alone is zero,

$$P_{C_{20}} = \sum_{n=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{r=0}^{3} x_n(j_1, 0, r).$$

- Probability that both inventory C_1 and C_2 equal to zero,

$$P_{00} = \sum_{n=0}^{\infty} \sum_{r=0}^{3} x_n(0, 0, r).$$

- Probability that customer demanding C_1 alone is lost,

$$P_{C_1 lost} = p_1 \sum_{n=1}^{\infty} \sum_{j_2=0}^{S_2} \sum_{r=1}^{3} \mu_r x_n(0, j_2, r).$$

- Probability that customer demanding C_2 alone is lost,

$$P_{C_2 lost} = p_2 \sum_{n=1}^{\infty} \sum_{j_1=0}^{S_1} \sum_{r=1}^{3} \mu_r x_n(j_1, 0, r).$$

- Probability that customer demanding both C_1 and C_2 is lost,

$$P_{C_{12} lost} = p_3 \sum_{n=1}^{\infty} \sum_{r=1}^{3} \mu_r x_n(0, 0, r).$$

- Probability that customer demanding both C_1 and C_2 is met with C_1,

$$P_{C_{121}} = p_3 \sum_{n=1}^{\infty} \sum_{j_1=1}^{S_1} \sum_{r=1}^{3} \mu_r x_n(j_1, 0, r).$$

- Probability that customer demanding both C_1 and C_2 is met with C_2,

$$P_{C_{122}} = p_3 \sum_{n=1}^{\infty} \sum_{j_2=1}^{S_2} \sum_{r=1}^{3} \mu_r x_n(0, j_2, r).$$

- Expected rate of replenishments for item C_1,

$$E_{R_{C_1}} = \beta_1 \sum_{n=0}^{\infty} \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{S_2} \sum_{r=0}^{3} x_n(j_1, j_2, r).$$

- Expected rate of replenishments for item C_2,

$$E_{R_{C_2}} = \beta_2 \sum_{n=0}^{\infty} \sum_{j_1=0}^{S_1} \sum_{j_2=0}^{s_2} \sum_{r=0}^{3} x_n(j_1, j_2, r).$$

- Expected reorder rate of commodity C_1,

$$E_{R_1} = \mu_1 \sum_{n=0}^{\infty} \sum_{j_2=0}^{S_2} x_n(s_1+1, j_2, 1).$$

- Expected reorder rate of commodity C_2,

$$E_{R_2} = \mu_2 \sum_{n=0}^{\infty} \sum_{j_1=0}^{S_1} x_n(j_1, s_2+1, 2).$$

- Expected reorder rate of commodity C_1 and C_2,

$$E_{R_{12}} = \mu_3 \sum_{n=0}^{\infty} x_n(s_1+1, s_2+1, 3).$$

We now look for additional information needed to optimally design the system.

5.3.1 Expected loss rate of customers in the queue demanding C_1 alone

In order to compute the expected loss rate of customers in the queue demanding C_1 alone, consider the Markov chain

$$\{(N(t), I_1(t), I_2(t), J(t)), t \geq 0\}$$

where $N(t)$, $I_1(t)$, $I_2(t)$,$J(t)$) were as defined in section 5.1. The state space of the above process is $\{(n, 0, j_2, r) : 1 \leq n \leq K, 0 \leq j_2 \leq S_2, 1 \leq r \leq 3\} \bigcup \mathbf{\{\Delta\}}$ where $\mathbf{\{\Delta\}}$ is the absorbing state which represents the state that number of customers in the queue becomes zero and K(the size of the queue). It is the maximum value to which the queue size can grow. Thus we have a finite state space Markov chain. The possible transitions and corresponding rates are:

- $(n, 0, 0, r) \rightarrow (0, 0, 0, 0)$ at the rate μ_r for $r = 1, 2, 3$
- $(n, 0, j_2, r) \rightarrow (0, 0, j_2, 0)$ at the rate $\mu_r p_1^n$ for $r = 1, 2, 3$
- $(n, 0, j_2, r) \rightarrow (n + 1, 0, j_2, r)$ at the rate λ for $r = 1, 2, 3$
- $(n, 0, j_2, r) \rightarrow (n, 0, S_2, r)$ at the rate β_2 for $r = 1, 2, 3$

The infinitesimal generator $\mathcal{G}$ of the above Markov chain is of the form

$$\mathcal{G}_1 = \begin{bmatrix} T_1 & T_1^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector

$$\boldsymbol{\alpha} = (cx_1(0, j_2, r), cx_2(0, j_2, r), ..., cx_K(0, j_2, r) : 0 \leq j_2 \leq S_2, 1 \leq r \leq 3),$$

where

$$c = \left\{ \sum_{n=1}^{K} \sum_{j_2=0}^{S_2} \sum_{r=1}^{3} x_n(0, j_2, r) \right\}^{-1};$$

T_1 is a matrix of order $3K(S_2 + 1)$ and T_1^0 is a column vector of order $3K(S_2 + 1)$ such that $T_1\mathbf{e} + T_1^0 = 0$.
Hence we arrive at

Theorem 5.3.1. *The expected loss rate of customers in the queue* *demanding C_1 alone is,*

$$E_{L_1} = \left\{-\boldsymbol{\alpha} T_1^{-1}\mathbf{e}\right\}^{-1}$$

On similar lines we can compute the expected loss rate of customers in the queue demanding C_2 alone and both C_1 and C_2. The following results are arrived at, the details of which are omitted.

Theorem 5.3.2. *The expected loss rate of customers in the queue demanding C_2 alone is*

$$E_{L_2} = \{-\boldsymbol{\alpha_1} T_2^{-1}\mathbf{e}\}^{-1}$$

where initial probability vector

$$\boldsymbol{\alpha_1} = (cx_1(j_1, 0, r), cx_2(j_1, 0, r), ..., cx_K(j_1, 0, r) : 0 \leq j_1 \leq S_1, 1 \leq r \leq 3),$$

and

$$c = \left\{\sum_{n=1}^{K}\sum_{j_1=0}^{S_1}\sum_{r=1}^{3} x_n(j_1, 0, r)\right\}^{-1};$$

and T_2 is a matrix of order $3K(S_1+1)$ and T_2^0 is a column vector of order $3K(S_1+1)$ such that $T_2\mathbf{e} + T_2^0 = 0$.

Theorem 5.3.3. *The expected loss rate of customers demanding both C_1 and C_2 is,*

$$E_{L_{12}} = \{-\boldsymbol{\alpha_2} T_3^{-1}\mathbf{e}\}^{-1} \times \sum_{n=1}^{K}\sum_{r=1}^{3} x_n(0, 0, r)$$

with initial probability vector

$$\boldsymbol{\alpha_2} = (cx_1(0,0,r), cx_2(0,0,r), ..., cx_K(0,0,r) : 1 \leq r \leq 3),$$

$$c = \left\{ \sum_{n=1}^{K} \sum_{r=1}^{3} x_n(0,0,r) \right\}^{-1};$$

T_3 is a matrix of order $3K$ and T_3^0 is a column vector of order $3K$ such that $T_3\mathbf{e} + T_3^0 = 0$.

5.3.2 Analysis of C_1 cycle time

The cycle time of item C_1 is defined as the time interval between two consecutive instants at which its inventory level hits S_1 due to replenishment. We assume that with at most M demands the first return to S_1 of C_1 takes place. Let us consider a Markov chain $\{(N(t), I_1(t), I_2(t), J(t), D(t)), t \geq 0\}$ where $D(t)$ denotes the type of the demand of the commodity; and rest of the notations are as defined in section 5.1. The state space of the above process is $\{(n, j_1, j_2, r, d) : 0 \leq n \leq K, 0 \leq i_1 \leq S_1, 0 \leq j_2 \leq S_2, 1 \leq r \leq 3, 1 \leq d \leq M\} \bigcup \{\boldsymbol{\Delta}\}$ where $\{\boldsymbol{\Delta}\}$ is the absorbing state which represents the state that level of C_1 returns to S_1 and K, the maximum size the queue can grow up. Thus we have a finite state space Markov chain. The possible transitions and corresponding rates are:

- $(n, S_1, j_2, r, d) \rightarrow (n-1, S_1-1, j_2, 1, d)$ with rate $\mu_r p_1$
- $(n, 0, j_2, r, d) \rightarrow (0, 0, j_2, 0, d)$ with rate $\mu_r p_1^n$ or $(0, 0, , j_2 - 1, 2, d)$ with rate $\mu_r p_1^{n-1}(p_2 + p_3)$

- $(n, j_1, 0, r, d) \rightarrow (0, j_1, 0, 0, d)$ with rate $\mu_r p_2^n$ or $(0, j_1 - 1, 0, 1, d)$ with rate $\mu_r p_2^{n-1}(p_1 + p_3)$

- $(n, j_1, j_2, r, d) \rightarrow (n-1, j_1 - 1, j_2, 1, d)$ with rate $\mu_r p_1$, or to $(n-1, j_1, j_2 - 1, 2, d)$ with rate $\mu_r p_2$ or to $(n-1, j_1 - 1, j_2 - 1, 3, d)$ with rate $\mu_r p_3$

- $(n, 0, j_2, r, d) \rightarrow (n-k, 0, j_2 - 1, 2, d)$ with rate $\mu_r p_1^{k-1}(p_2 + p_3)$

- $(n, j_1, 0, r, d) \rightarrow (n-k, j_1 - 1, 0, 1, d)$ with rate $\mu_r p_2^{k-1}(p_1 + p_3)$

- $(n, j_1, j_2, r, d) \rightarrow (n, S_1, j_2, r, d)$ with rate β_1 for $0 \leq j_1 \leq s_1, 0 \leq j_2 \leq S_2, 1 \leq r \leq 3$

- $(n, j_1, j_2, r, d) \rightarrow (n+1, j_1, j_2, r, d)$ with rate λ for $0 \leq n \leq K - 1$ for $0 \leq j_1 S_1$, $0 \leq j_2 \leq S_2$, $1 \leq r \leq 3$, $1 \leq d \leq N$

The infinitesimal generator $\mathcal{C}$ of the above Markov chain is of the form

$$\mathcal{C} = \begin{bmatrix} D & D^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector

$$\boldsymbol{\gamma} = (cx_0(S_1, j_2, r), cx_1(S_1, j_2, r), ..., cx_K(S_1, j_2, r), 0, 0, ...) : 0 \leq j_2 \leq S_2, 1 \leq r \leq 3),$$

where

$$c = \left\{ \sum_{n=0}^{K} \sum_{j_2=0}^{S_2} \sum_{r=1}^{3} x_n(S_1, j_2, r) \right\}^{-1};$$

D is a matrix of order $3(K+1)(S_1+1)(S_2+1)$ and D^0 is a column vector of order $3(K+1)(S_1+1)(S_2+1)$ such that $D\mathbf{e} + D^0 = 0$. Hence, the expected cycle length is $-\boldsymbol{\gamma} D^{-1}\mathbf{e}$

Similarly the cycle time of item C_2 has expected value $-\boldsymbol{\gamma_1} D_1^{-1}\mathbf{e}$ where

$$\boldsymbol{\gamma_1} = (cx_0(j_1, S_2, r), cx_1(j_1, S_2, r), ..., cx_K(j_1, S_2, r), 0, 0, ...) : 0 \leq j_1 \leq S_1, 1 \leq r \leq 3),$$

where

$$c = \left\{ \sum_{n=0}^{K} \sum_{j_1=0}^{S_1} \sum_{r=1}^{3} x_n(j_1, S_2, r) \right\}^{-1};$$

D_1 is a matrix of order $3(K+1)(S_1+1)(S_2+1)$.

5.4 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

Effect of λ on various performance measures

Table 5.1 indicates that increase in λ values results in increase in expected number of customers in the queue, expected loss rate of customers demanding C_1 alone, C_2 alone, both C_1 and C_2. As λ increases there is a decrease in the expected number of items in the inventory. Also, as

λ increases reorder rates for C_1 alone, C_2 alone, both C_1 and C_2 also increase. These are all natural consequences of increase in arrival rate.

λ	E_N	E_{I_1}	E_{I_2}	E_{L_1}	E_{L_2}	$E_{L_{12}}$	E_{R_1}	E_{R_2}	$E_{R_{12}}$
1	0.1202	6.7153	9.8163	0.0063	0.0064	1.5765×10^{-6}	0.0135	0.0090	0.0095
2	0.7984	6.4346	9.6441	0.0109	0.0111	1.0875×10^{-4}	0.0256	0.0189	0.0173
2.5	1.8468	6.3055	9.5655	0.0133	0.0136	4.0599×10^{-4}	0.0313	0.0247	0.0205

Table 5.1: Effect of λ: Fix $S_1 = 10, S_2 = 15, s_1 = 3, s_2 = 4, \mu_1 = 2, \mu_2 = 3, \mu_3 = 4, \beta_1 = 2, \beta_2 = 3, p_1 = 0.1, p_2 = 0.1, p_3 = 0.8$

Effect of μ_1 on various performance measures

μ_1	E_N	E_{I_1}	E_{I_2}	E_{L_1}	E_{L_2}	$E_{L_{12}}$	E_{R_1}	E_{R_2}	$E_{R_{12}}$
1	0.3669	6.7168	9.8208	0.0057	0.0058	3.6197×10^{-6}	0.0135	0.0090	0.0095
1.5	0.2570	6.7131	9.8145	0.0059	0.0059	2.1050×10^{-6}	0.0135	0.0090	0.0095
2	0.2196	6.7123	9.8119	0.0059	0.0061	1.7035×10^{-6}	0.0135	0.0090	0.0095
2.5	0.2022	6.7123	9.8107	0.0059	0.0061	1.5593×10^{-6}	0.0135	0.0090	0.0095
3	0.1926	6.7126	9.8110	0.0060	0.0062	1.4964×10^{-6}	0.0135	0.0090	0.0095

Table 5.2: Effect of μ_1:Fix $S_1 = 10, S_2 = 15, s_1 = 3, s_2 = 4, \lambda = 1, \mu_2 = 2, \mu_3 = 3, \beta_1 = 2, \beta_2 = 3, p_1 = 0.1, p_2 = 0.1, p_3 = 0.8$

Table 5.2 indicates that increase in service rateμ_1 for processing commodity 1, makes decrease in expected number of customers in the system. As μ_1 increases there is a slight decrease initially in the expected number of C_1, then it shows increasing tendency. There is increase in expected loss rate of customers demanding C_1 alone initially and then it remains constant and then it increases. Reorder rates for C_1 alone, C_2 alone, for both C_1 and C_2 remains constant. Expected number of C_2 decreases first and then increases. Expected loss rate of customers demanding C_2 alone increases and loss rate demanding both C_1 and C_2 decreases.

μ_2	E_N	E_{I_1}	E_{I_2}	E_{L_1}	E_{L_2}	$E_{L_{12}}$	E_{R_1}	E_{R_2}	$E_{R_{12}}$
1	0.3361	6.7228	9.8150	0.0056	0.0059	2.8812×10^{-6}	0.0135	0.0090	0.0095
1.5	0.2289	6.7154	9.8110	0.0058	0.0061	1.7583×10^{-6}	0.0135	0.0090	0.0095
2	0.1926	6.7126	9.8100	0.0060	0.0062	1.4964×10^{-6}	0.0135	0.0090	0.0095
2.5	0.1758	6.7112	9.8098	0.0061	0.0062	1.4330×10^{-6}	0.0135	0.0090	0.0095
3	0.1666	6.7106	9.8100	0.0061	0.0063	1.4275×10^{-6}	0.0135	0.0090	0.0096

Table 5.3: Effect of μ_2:Fix $S_1 = 10, S_2 = 15, s_1 = 3, s_2 = 4, \lambda = 1, \mu_1 = 3, \mu_3 = 3, \beta_1 = 2, \beta_2 = 3, p_1 = 0.1, p_2 = 0.1, p_3 = 0.8$

Effect of μ_2 on various performance measures

Table 5.3 indicates that increase in μ_2 decreases expected number of customers in the queue. Expected number of C_1 decreases and C_2 decreases first and then it increases. Expected loss rate of customers demanding C_1 alone, C_2 alone increases, but loss rate demanding both C_1 and C_2 decreases. Reorder rates for C_1, C_2 remains constant and that for both C_1 and C_2 remains constant first then it shows a slight increase.

Effect of μ_3 on various performance measures

μ_3	E_N	E_{I_1}	E_{I_2}	E_{L_1}	E_{L_2}	$E_{L_{12}}$	E_{R_1}	E_{R_2}	$E_{R_{12}}$
1	5.6693	6.7262	9.8367	0.0053	0.0056	1.2930×10^{-5}	0.0132	0.0089	0.0092
1.5	1.0176	6.7059	9.8116	0.0055	0.0055	3.4897×10^{-6}	0.0135	0.0090	0.0095
2	0.4562	6.7053	9.8091	0.0057	0.0057	2.2389×10^{-6}	0.0135	0.0090	0.0096
2.5	0.2748	6.7075	9.8101	0.0059	0.0059	1.8148×10^{-6}	0.0135	0.0090	0.0096
3	0.1922	6.7103	9.8120	0.0061	0.0061	1.6445×10^{-6}	0.0135	0.0090	0.0095

Table 5.4: Effect of μ_3:Fix $S_1 = 10, S_2 = 15, s_1 = 3, s_2 = 4, \lambda = 1, \mu_1 = 2, \mu_2 = 3, \beta_1 = 2, \beta_2 = 3, p_1 = 0.1, p_2 = 0.1, p_3 = 0.8$

Table 5.4 indicates, as the service rate for processing both commodities increases, expected number of customers in the queue decreases. Expected number of items C_1 and C_2 first decreases and then it increases. Reorder rates for C_1, C_2 first increases and then remains a constant and for both C_1 and C_2 increases, remains constant and then decrease. Expected loss rate of customers demanding C_1 alone, C_2 alone increases, but loss rate demanding both C_1 and C_2 decreases.

Effect of β_1 on various performance measures

β_1	E_N	E_{I_1}	E_{I_2}	E_{L_1}	E_{L_2}	$E_{L_{12}}$	E_{R_1}	E_{R_2}	$E_{R_{12}}$
1	0.1917	6.4335	9.8120	0.0061	0.0064	4.3735×10^{-6}	0.0127	0.0097	0.0090
1.5	0.1921	6.6183	9.8120	0.0061	0.0062	2.5750×10^{-6}	0.0132	0.0091	0.0093
2	0.1922	6.7103	9.8120	0.0061	0.0061	1.6445×10^{-6}	0.0135	0.0090	0.0095
2.5	0.1923	6.7652	9.8120	0.0061	0.0061	1.1078×10^{-6}	0.0137	0.0089	0.0097
3	0.1923	6.8018	9.8120	0.0061	0.0060	7.7633×10^{-7}	0.0138	0.0089	0.0098

Table 5.5: Effect of β_1:Fix $S_1 = 10, S_2 = 15, s_1 = 3, s_2 = 4, \lambda = 1, \mu_1 = 2, \mu_2 = 3, \mu_3 = 3, \beta_2 = 3, p_1 = 0.1, p_2 = 0.1, p_3 = 0.8$

Table 5.5 indicates that as the replenishment rate for the first commodity increases expected number of customers in the queue increases and then remains constant. Expected number of items C_1 increases but that of C_2 remains constant. Expected loss rate of customers demanding C_1 alone is constant, but those for C_2 alone and both C_1 and C_2 decrease. Reorder rates for C_1 alone and both C_1 and C_2 increases and for C_2 alone decreases and then remains constant.

Effect of β_2 on various performance measures

Table 5.6 indicates as the replenishment rate for the first commodity increases expected number of customers in the queue decreases. Expected number of items C_2 increases and C_1 remains constant. Expected loss rate of customers demanding C_1 alone decreases first and the remains constant, C_2 alone increases and for both C_1 and C_2 decreases. Reorder

β_2	E_N	E_{I_1}	E_{I_2}	E_{L_1}	E_{L_2}	$E_{L_{12}}$	E_{R_1}	E_{R_2}	$E_{R_{12}}$
1	0.1929	6.7103	9.4642	0.0062	0.0059	2.0465×10^{-5}	0.0139	0.0084	0.0090
1.5	0.1924	6.7103	9.6399	0.0061	0.0060	9.2289×10^{-6}	0.0136	0.0088	0.0093
2	0.1923	6.7103	9.7263	0.0061	0.0060	4.7830×10^{-6}	0.0135	0.0089	0.0094
2.5	0.1922	6.7103	9.7778	0.0061	0.0061	2.7138×10^{-6}	0.0135	0.0089	0.0095
3	0.1922	6.7103	9.8120	0.0061	0.0061	1.6445×10^{-6}	0.0135	0.0090	0.0095

Table 5.6: Effect of β_2:Fix $S_1 = 10, S_2 = 15, s_1 = 3, s_2 = 4, \lambda = 1, \mu_1 = 2, \mu_2 = 3, \mu_3 = 3, \beta_1 = 2, p_1 = 0.1, p_2 = 0.1, p_3 = 0.8$

rates for C_1 alone decreases first and then it is a constant and reorder rates for both C_1 and C_2, for C_2 alone increases.

5.5 Optimization Problem

We now construct an optimization problem involving costs for holding, procurement and due to loss of demands when the item asked for is not available. Consider the cost function,

$$hE_N + c_1E_{I_1} + c_2E_{I_2} + c_3E_{L_1} + c_4E_{L_2} + c_5E_{L_{12}} + c_6E_{R_1} + c_7E_{R_2} + c_8E_{R_{12}}$$

where

h : holding cost per customer per unit time,

c_i: per unit holding cost of C_i per unit time,for $i = 1, 2$,

c_i, for $i = 3, 4, 5$: cost due to loss of customer demanding C_1 alone, C_2 alone and both C_1 and C_2 respectively,

c_i for $i = 6, 7, 8$: fixed procurement cost for C_1, C_2, and both C_1 and

C_2 respectively.

In the absence of analytical expressions for system state distribution, discussions on global optimum is impossible. However, costs for various (s_i, S_i) for $i = 1, 2$ is given below:

(S_1, S_2)	(9,10)	(10,11)	(11,12)	(12,13)
Cost	108.9064	112.8852	117.7741	123.1719

Table 5.7: Value of cost function for various (S_1, S_2): Fix $s_1 = 4, s_2 = 5, h = 3, c_1 = 5, c_2 = 8, c_3 = 15, c_4 = 20, c_5 = 10, c_6 = 100, c_7 = 150, c_8 = 200$

Chapter 6

Queueing Inventory Model for Crowdsourcing

In this chapter we focus on a topic-crowdsourcing- hitherto not investigated in the queueing-inventory literature. Hence it is not in any way related to themes in earlier chapters except that positive service time is considered for serving items to customers. Crowdsourcing is the process of getting work usually online from a crowd of people. It is a combination of 'crowd' and 'outsourcing'. The idea is to take work and outsource it to a crowd of workers. The principle of crowdsourcing is that more heads are better than one. By canvassing a large crowd of people for ideas, skills or participation, the quality of content and idea generation will be superior.

Wikipedia, the most comprehensive encyclopedia the world has ever seen, is a famous example of crowdsourcing. Instead of creating an ency-

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clopedia on their own, they gave a crowd the responsibility to create the information on their own. The concept of crowdsourcing is used by many industries such as food, consumer products, hotels, electronics and other large retailers. A number of examples of crowdsourcing can be found in [51].

The motivation for this chapter is from Chakravarthy and Dudin[16] and Krishnamoorthy et al.[34]. In these the authors use the crowdsourcing in the context of service sectors getting possible help from one group of customers who first receive service from them and then opt to execute similar services to another group. The resources for service is assumed to be abundantly available. In the present chapter we assume finiteness of availability of item to be served. Thus, when the item is not available service cannot be provided.

6.1 Model Description

Consider a queueing inventory system with c servers. There are two types of customers: Type I and Type II. Type II customers are virtual ones, ordering through phone or internet or through some other means. Arrival of Type I and Type-II customers follow Poisson process with parameter λ_1 and λ_2, respectively. Type I are to be served by one of the c servers with service time assumed to be exponentially distributed with parameter μ_1. Type II customers may be served by a Type I customer having already been served and ready to act as a server, or by one of c servers. Type II when served by one of the c servers, the service time is exponentially distributed with parameter μ_2. Type I customers has non preemptive priority over Type II. Type II is served by a Type I only if inventory is available after attaching the existing items to the priority customers already present. Type II is served by a Type I with probability p, $0 \leq p \leq 1$ and with complementary probability $q = 1 - p$, served Type I will leave the system. If a Type I customer serves a Type II customer, then that Type II customer is removed from the system immediately on completion of the corresponding Type-I customer's service. Arrival of both type of customers is permitted only when *excess inventory*, which is defined as the difference between on hand inventory and number of busy servers, is positive. A finite waiting space L for Type I is assumed whereas Type II has unlimited waiting area. When inventory level drops to $c + s$, an order for replenishment is placed to bring the inventory level to $c + S$. We assume $c < s < L < S = c + 2L$. The replenishment takes place after a random amount of time which is exponentially distributed with parameter β.

Define

$N_1(t)$: Number of Type II customers in queue (waiting for service) at time t
$N_2(t)$: Number of servers busy with Type II customers at time t
$N_3(t)$: Number of Type I customers in system at time t
$I(t)$: Inventory level at time t

Then

$$\{(N_1(t), N_2(t), N_3(t), I(t)) : t \geq 0\}$$

is a continuous time Markov chain. The state space of the above process is

$$\boldsymbol{\Omega} = \bigcup_{i=0}^{\infty} \boldsymbol{\ell(i)}$$

where $\boldsymbol{\ell(i)}$ denotes level i. The elements of $\boldsymbol{\Omega}$ are as described below:

$$\boldsymbol{\ell(0)} = \{(0, j, k, l) : 0 \leq j \leq c, 0 \leq k \leq L, j \leq l \leq c + S\}$$

and

$$\boldsymbol{\ell(i)} = \{(i, j, k, l) : i > 0, 0 \leq j \leq c, 0 \leq k \leq c - j - 1, j \leq l \leq j + k\}$$

$$\bigcup \{(i, j, k, l) : i > 0, 0 \leq j \leq c, c - j \leq k \leq L, j \leq l \leq c + S\}.$$

The level 0, $\boldsymbol{\ell(0)}$, can be further partitioned as

$$\boldsymbol{\ell(0)} = \{(0, 0), (0, 1), (0, 2), ..., (0, c)\}$$

where the set of states $(0, j)$ corresponds to the case when there is no Type II customer waiting in the queue and j Type II customers are in service and each $\{(0, j) : 0 \leq j \leq c\}$ has $(L + 1)(c + S - j + 1)$ elements for $0 \leq j \leq c$. Similarly, $\boldsymbol{\ell(i)}$ can also be further partitioned as

$$\boldsymbol{\ell(i)} = \{(i, 0), (i, 1), (i, 2), ..., (i, c)\},$$

where the set of states (i, j) corresponds to the case when there are i Type II customer waiting in the queue and j Type II customers are in service, each has $(1 + 2 + 3 + ... + c - j) + (L - (c - j - 1))(c + S - j + 1)$ elements for $0 \leq j \leq c$. The transitions in the above Markov chain can be described as follows:

1. Transition due to arrival of customers:

   - Due to the arrival of Type I customer:

  - $(0, j, k, l) \to (0, j, k+1, l)$ with rate λ_1 if $j+k < l, 0 \le j \le c, 0 \le k \le c-j-1, j \le l \le c+S$
  - $(i, j, k, l) \to (i, j, k+1, l)$ for $i \ge 1$ with rate λ_1 if $0 \le j \le c, c-j \le k \le L-1, c+1 \le l \le c+S$

- Due to the arrival of Type II customer:
  - $(0, j, k, l) \to (0, j+1, k, l)$ with rate λ_2 if $j+k < l, 0 \le j \le c-1, 0 \le k \le c-j-1, j \le l \le c+S$
  - $(0, j, k, l) \to (1, j, k, l)$ with rate λ_2 if $j = c, c-j \le k \le L, c+1 \le l \le c+S$
  - $(i, j, k, l) \to (i+1, j, k, l)$ with rate λ_2 for $0 \le j \le c, c-j \le k \le L, c+1 \le l \le c+S$

2. Transitions due to service completions:

   - $(0, j, k, l) \to (0, j-1, k, l-1)$ with rate $j\mu_2$ for $1 \le j \le c, 0 \le k \le L, j \le l \le c+S$
   - $(0, j, k, l) \to (0, j, k-1, l-1)$ with rate $min(c-j, k, l-j)\mu_1$ for $0 \le j \le c, 1 \le k \le L, j \le l \le c+S$
   - $(i, j, k, l) \to (i, j, k-1, l-1)$ with rate $min(c-j, k, l-j)\mu_1$ for $0 \le j \le c, 1 \le k \le c-j, j \le l \le j+k$
   - $(i, j, k, l) \to (i-1, j, k-1, l-2)$ with rate $p(c-j)\mu_1$ for $0 \le j \le c-1, c-j \le k \le L, j+k+1 \le l \le c+S$
   - $(i, j, k, l) \to (i-1, j+1, k-1, l-1)$ with rate $q(c-j)\mu_1$ for $0 \le j \le c-1, k = c-j, c+1 \le l \le c+S$

3. Transition due to replenishment:

- $(0, j, k, l) \rightarrow (0, j, k, c+S)$ with rate β for $0 \leq j \leq c, 0 \leq k \leq L, 0 \leq l \leq c+s$
- $(i, j, k, l) \rightarrow (0, j+i, k, c+S)$ with rate β for $0 \leq j \leq c-i, 0 \leq k \leq c-j-i, 0 \leq l \leq j+k$
- $(i, j, k, l) \rightarrow (i, j, k, c+S)$ with rate β for $0 \leq j \leq c, c-j \leq k \leq L, j \leq l \leq c+s$

The infinitesimal generator of the above process is

$$\mathcal{Q} = \begin{pmatrix}
A_{0,0} & A_{0,1} & & & & & & & & & & \\
A_{1,0} & A_{1,1} & A_{1,2} & & & & & & & & & \\
A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & & & & & & & & \\
A_{3,0} & A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & & & & & & & \\
\vdots & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
A_{c-2,0} & A_{c-2,1} & A_{c-2,2} & \cdots & \cdots & & \cdots & A_{c-2,c-2} & A_{c-2,c-1} & & & \\
A_{c-1,0} & A_{c-1,1} & A_{c-1,2} & \cdots & \cdots & & \cdots & A_{c-1,c-2} & A_{c-1,c-1} & A_{c-1,c} & & \\
A_{c,0} & A_{c,1} & A_{c,2} & \cdots & \cdots & & & A_{c,c-2} & A_{c,c-1} & A_{c,c} & A_{c,c+1} & \\
 & A_{c+1,1} & A_{c+1,2} & \cdots & \cdots & & \cdots & & & A_{c+1,c} & A_{c+1,c+1} & \\
 & & A_{c+2,2} & A_{c+2,3} & \cdots & \cdots & & & \cdots & & & \\
 & & & \ddots & & & & & & & & \\
 & & & & \ddots & & & & & & & \\
 & & & & & & & & & A_{2c,c} & A_{2c,c+1} & \cdots \\
 & & & & & & & & & & A_{2c+1,c+1} & \cdots \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & \ddots
\end{pmatrix}$$

For $c = 2$ the matrices appearing in $\mathcal{Q}$ are

$$B_0(i,j,k,l;i',j',k',l') = \begin{cases} \lambda_2, & i=0, i'=i+1, 0 \le j \le c, \\ & c-j \le k \le L, c+1 \le l \le c+S; \end{cases}$$

$$A_{10}(i,j,k,l;i',j',k',l') = \begin{cases} p(c-j)\mu_1, & i=1, i'=0, 0 \le j \le c-1, \\ & c-j \le k \le L, j+k+1 \le l \le c+S, \\ & j'=j, k'=k-1, l'=l-2; \\ q(c-j)\mu_1, & i=1, i'=0, 0 \le j \le c-1, \\ & k=c-j, c+1 \le l \le c+S; \\ & j'=j+1, l'=l-1; \\ \beta, & i=1, i'=0, 0 \le j \le c-1, \\ & 0 \le k \le c-1, j'=j+1, \\ & 0 \le l \le j+k, k'=k, l'=c+S; \\ c\mu_2, & i=1, i'=0, j=c, k=c-j, \\ & c+1 \le l \le c+S; \end{cases}$$

$$A_{20}(i,j,k,l;i',j',k',l') = \begin{cases} p(c-j)\mu_1, & i=2, i'=0, 0 \le j \le c-1, \\ & k=c-j, c+2 \le l \le c+S; \\ \beta, & i=2, i'=0, j=0, l=0, \\ & k=0, j'=j+2, k'=k, l'=c+S \end{cases}$$

$$A_2(i,j,k,l;i',j',k',l') = \begin{cases} p(c-j)\mu_1, & 0 \le j \le c-1, k=c-j, l=c+1, \\ & k'=k-1, l'=l-2; \\ p(c-j)\mu_1, & 0 \le j \le c-1, c-j+1 \le k \le L, \\ & j+k+1 \le l \le c+S, j'=j, k'=k-1, l'=l-2; \\ q(c-j)\mu_1, & 0 \le j \le c-1, k=c-j, c+1 \le l \le c+S, \\ & j'=j+1, k'=k-1, l'=l-1 \\ \beta, & j=0, k=1, 0 \le l \le j+k, \\ & j'=j+1, k'=k, l'=c+S, \\ \beta, & j=1, k=0, l=1, \\ & j'=j+1, k'=k, l'=c+S, \\ j\mu_2, & 1 \le j \le c, k=c-j, c+1 \le l \le c+S, \\ & j'=j, k'=k, l'=l-1, \end{cases}$$

$$A_3(i,j,k,l;i',j',k',l') = \begin{cases} p(c-j)\mu_1, & 0 \le j \le c-1, k = c-j, \\ & c+2 \le l \le c+S, j' = j+1, k' = k-1, l' = l-2; \\ \beta, & j = 0, k = 0, l = 0, \\ & j' = j+2, k' = k, l' = c+S, \end{cases}$$

$$A_0(i,j,k,l;i',j',k',l') = \begin{cases} \lambda_2, & i \ge 1, i' = i+1, 0 \le j \le c, \\ & c-j \le k \le L, c+1 \le l \le c+S; \\ 0 & \text{otherwise} \end{cases}$$

Let $a_j = min(c-j,k,l-j)\mu_1$ for $0 \le j \le c$

$$B_1(i,j,k,l;i',j',k',l') = \begin{cases} \beta, & j = 0, k = 0, \\ & j \le l \le c+s, l' = c+S; \\ -\beta, & i = i' = 0, j = 0, \\ & 0 \le k \le L, l = l' = 0; \\ -(\beta+\lambda_1+\lambda_2), & i = i' = 0, j = 0, \\ & k = 0, 1 \le l \le c+s; \\ -(\lambda_1+\lambda_2), & i = i' = 0, j = 0, \\ & k = 0, c+s+1 \le l \le c+S, \\ \lambda_1, & i = i' = 0, 0 \le j, j' \le c-1, \\ & 0 \le k \le c-j-1, j \le l, l' \le c+S, \\ & k' = k+1, j+k < 1; \\ \lambda_1, & i = i' = 0, 0 \le j, j' \le c, \\ & c-j \le k \le L-1, c+1 \le l, l' \le c+S, \\ & k' = k+1, j+k < 1; \\ \lambda_2, & i = i' = 0, 0 \le j \le c-1, \\ & 0 \le k \le c-j-1, j \le l \le c+S, \\ & j' = j+1, j+k < 1; \\ a_j, & i = i' = 0, j = 0, \\ & 1 \le k \le L, k' = k-1, j \le l \le c+S, \\ & l' = l-1; \\ -(\beta+a_j), & j = 0 = j', k = 1 = k', l = 1 = l', \\ -(\beta+a_j+\lambda_1+\lambda_2), & j = 0 = j', k = 1 = k', \\ & 2 \le l \le c+s, \end{cases}$$

$$B_1(i,j,k,l;i',j',k',l') = \begin{cases} -(a_j+\lambda_1+\lambda_2), & j=0=j', k=1=k', \\ & c+s+1\le l\le c+S, \\ -(\beta+a_j), & j=0=j', c\le k\le L, 1\le l\le c, \\ -(\beta+a_j+\lambda_1+\lambda_2), & j=0=j', c\le k\le L-1, \\ & c+1\le l\le c+s, \\ -(a_j+\lambda_1+\lambda_2), & j=0=j', c\le k\le L-1, \\ & c+s+1\le l\le c+S, \\ -(\beta+a_j+\lambda_2), & j=0=j', k=L, \\ & c+1\le l\le c+s, \\ -(a_j+\lambda_2), & j=0=j', k=L, \\ & c+s+1\le l\le c+S, \\ -(\beta+j\mu_2), & i=i'=0, j=1, \\ & 0\le k\le L, l=j; \\ -(\beta+j\mu_2+\lambda_1+\lambda_2), & i=i'=0, j=1, \\ & k=0, 2\le l\le c+s; \\ -(j\mu_2+\lambda_1+\lambda_2), & i=i'=0, j=1, \\ & k=0, c+s+1\le l\le c+S; \\ -(\beta+j\mu_2+a_j), & i=i'=0, j=1, \\ & k=1, l=2; \\ -(\beta+j\mu_2+a_j+\lambda_1+\lambda_2), & i=i'=0, j=1, \\ & 1\le k\le L-1, c+1\le l\le c+s; \\ -(j\mu_2+a_j+\lambda_1+\lambda_2), & i=i'=0, j=1, \\ & 1\le k\le L-1, c+s+1\le l\le c+S; \\ -(\beta+j\mu_2+a_j+\lambda_2), & i=i'=0, j=1, \\ & k=L, c+1\le l\le c+s; \\ -(j\mu_2+a_j+\lambda_2), & i=i'=0, j=1, \\ & k=L, c+s+1\le l\le c+S; \\ j\mu_2, & i=i'=0, 1\le j\le c, \\ & 0\le k\le L, 1\le l\le c+S; \\ -(\beta+j\mu_2), & i=i'=0, j=c, \\ & 0\le k\le L, l=j; \\ -(\beta+j\mu_2+\lambda_1+\lambda_2), & i=i'=0, j=c, \\ & 0\le k\le L-1, c+1\le l\le c+s; \end{cases}$$

$$B_1(i,j,k,l;i',j',k',l') = \begin{cases} -(j\mu_2+\lambda_1+\lambda_2), & i=i'=0, j=c, \\ & 0\le k\le L-1, c+s+1\le l\le c+S; \\ -(\beta+j\mu_2+\lambda_2), & i=i'=0, j=c, \\ & k=L, c+1\le l\le c+s; \\ -(j\mu_2+\lambda_2), & i=i'=0, j=c, \\ & k=L, c+s+1\le l\le c+S; \end{cases}$$

$$A_1(i,j,k,l;i',j',k',l') = \begin{cases} -\beta, & i=i'=1, j=0, \\ & k=0,1, l=0; \\ a_j, & i=i'=1, 0\le j\le 1, \\ & 1\le k\le L, 0\le l\le j+k; \\ q(c-j)\mu_1, & i=i'=1, j=0, \\ & c+1\le k\le L, j+k+1\le l\le c+S; \\ \beta, & i=i'=1, j=0, \\ & c\le k\le L, 0\le l\le c+s, l'=c+S; \\ -(\beta+a_j), & i=i'=1, j=0, \\ & 1\le k\le L, 0\le l\le c; \\ -(\beta+a_j+\lambda_1+\lambda_2), & i=i'=1, j=0, \\ & c\le k\le L-1, c+1\le l\le c+s; \\ -(a_j+\lambda_1+\lambda_2), & i=i'=1, j=0, \\ & c\le k\le L-1, c+s+1\le l\le c+S; \\ -(\beta+a_j+\lambda_2), & i=i'=1, j=0, \\ & k=L, c+1\le l\le c+s; \\ -(a_j+\lambda_2), & i=i'=1, j=0, \\ & k=L, c+s+1\le l\le c+S; \\ j\mu_2, & i=i'=1, j=1, \\ & 0\le k\le L, j\le l\le j+k; \\ j\mu_2, & i=i'=1, j=1, \\ & c\le k\le L, j\le l\le c+S; \\ q(c-j)\mu_1, & i=i'=1, j=1, \\ & c\le k\le L, j+k+1\le l\le c+S; \\ -(\beta+\mu_2), & i=i'=1, j=1, \\ & k=0,1, l=1; \end{cases}$$

$$A_1(i,j,k,l;i',j',k',l') = \begin{cases} -(\beta+\mu_2+\mu_1), & i=i'=1, j=1, \\ & 1\leq k\leq L-1, l=2; \\ -(\beta+\mu_2+\mu_1+\lambda_1+\lambda_2), & i=i'=1, j=1, \\ & 1\leq k\leq L-1, c+1\leq l\leq c+s; \\ -(\mu_2+\mu_1+\lambda_1+\lambda_2), & i=i'=1, j=1, \\ & 1\leq k\leq L-1, c+s+1\leq l\leq c+S; \\ -(\beta+\mu_2+\mu_1+\lambda_2), & i=i'=1, j=1, \\ & k=L, c+1\leq l\leq c+s; \\ -(\mu_2+\mu_1+\lambda_2), & i=i'=1, j=1, \\ & k=L, c+s+1\leq l\leq c+S; \\ -(\beta+j\mu_2), & i=i'=1, j=c, \\ & 0\leq k\leq L, l=j; \\ -(\beta+j\mu_2+\lambda_1+\lambda_2), & i=i'=1, j=c, \\ & 0\leq k\leq L-1, c+1\leq l\leq c+s; \\ -(j\mu_2+\lambda_1+\lambda_2), & i=i'=1, j=c, \\ & 0\leq k\leq L-1, c+s+1\leq l\leq c+S; \\ -(\beta+j\mu_2+\lambda_2), & i=i'=1, j=c, \\ & k=L, c+1\leq l\leq c+s; \\ -(j\mu_2+\lambda_2), & i=i'=1, j=c, \\ & k=L, c+s+1\leq l\leq c+S; \\ j\mu_2, & i=i'=1, j=2, \\ & 0\leq k\leq L, j\leq l\leq c+S; \end{cases}$$

The matrices $A_{i,i-1}$ and $A_{i,i+1}$ represents the transitions from $\boldsymbol{\ell}(i)$ to $\boldsymbol{\ell}(i-1)$ and to $\boldsymbol{\ell}(i+1)$ respectively and $A_{i,i}$ has as elements transition rates within $\boldsymbol{\ell}(i)$. $A_{i,j}$ has as entries transition rates from $\boldsymbol{\ell}(i)$ to $\boldsymbol{\ell}(j)$ for $0 \leq j \leq i-2$ for $i \geq 2$. From the transitions described above we can see that $A_{i,i+1}$ are same for $i \geq 1$ and is denoted by A_0, $A_{i,i}$, for $i \geq 1$, are same and they are denoted by A_1, $A_{i,i-1}$, for $i \geq 1$, are same and they are denoted by A_2. Similarly, $A_{i,i-2}$ for $i \geq 3$, $A_{i,i-3}$ for $i \geq 4$, $A_{i,i-4}$ for $i \geq 5$, …, $A_{i,i-(c-1)}$ for $i \geq c$ and $A_{i,i-c}$ for $i \geq c+1$ are same. They are denoted by $A_3, A_4, A_5, \ldots, A_c, A_{c+1}$ respectively. The model under study can be studied as a QBD process by combining the set of states as follows:

$$L(1) = \{\ell(1), \ell(2), \ell(3), \ldots, \ell(c)\}$$

$$L(2) = \{\ell(c+1), \ell(c+2), \ell(c+3), \ldots, \ell(2c)\}$$

$$L(3) = \{\ell(2c+1), \ell(2c+2), \ell(2c+3), \ldots, \ell(3c)\}$$

and so on. Thus ,the new generator is

$$\mathcal{Q}' = \begin{bmatrix} B_1 & B_0' & & & & \\ A_2' & \tilde{A}_1 & \tilde{A}_0 & & & \\ & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & & \\ & & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where the block entries appearing in $\mathcal{Q}'$ are obtained from those of $\mathcal{Q}$ as follows.

$$B_0' = \begin{bmatrix} B_0 & 0 & \ldots & \end{bmatrix}, \; A_2' = \begin{bmatrix} A_{10} & & \\ A_{20} & & \\ A_{30} & & \\ \vdots & & \\ & & \\ A_{c0} & & \end{bmatrix}$$

$$\tilde{A}_0 = \begin{bmatrix} 0 & & & & \cdots & & & 0 \\ \vdots & & & & & & & \vdots \\ & & & & & & & \\ 0 & & & & & & & \\ A_0 & 0 & 0 & 0 \ldots & & \cdots & \cdots & 0 \end{bmatrix}$$

$$\tilde{A}_2 = \begin{bmatrix} A_{c+1} & A_c & & \cdots & & & & A_2 \\ & A_{c+1} & A_c & & \cdots & & & A_3 \\ & & A_{c+1} & A_c & & & & A_4 \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & A_{c+1} & A_c \\ 0 & & & \cdots & & \cdots & & A_{c+1} \end{bmatrix}$$

$$\tilde{A}_1 = \begin{bmatrix} A_1 & A_0 & & & & & & \\ A_2 & A_1 & A_0 & & & & & \\ A_3 & A_2 & A_1 & A_0 & & & & \\ \vdots & & & & \ddots & & & \\ & & & & & \ddots & & \\ \vdots & & & & & & \ddots & \\ & & & & & & A_1 & A_0 \\ A_c & A_{c-1} & A_{c-2} & \cdots & \cdots & & A_2 & A_1 \end{bmatrix}$$

6.2 Steady State Analysis

We proceed with the steady state analysis of the queueing -inventory system under study. The first step is to look for the condition for stability.

6.2.1 Stability Condition

Define $\tilde{A} = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2$. Then it is the infinitesimal generator of the finite state continuous time Markov chain. Let $\tilde{\pi}$=$(\tilde{\pi}_1, \tilde{\pi}_2, ...\tilde{\pi}_c)$ be the steady state probability vector of this generator $\tilde{A}$. That is $\tilde{\pi}$ satisfies

$$\tilde{\pi}\tilde{A} = 0$$

and

$$\tilde{\pi}\mathbf{e} = 1$$

$\tilde{A}$ is a circulant matrix and so the vector $\tilde{\pi}$ is of the form $\tilde{\pi}$=$(\pi/c, \pi/c, \pi/c, ...\pi/c)$ where π satisfies

$$\pi A = 0$$

and

$$\pi\mathbf{e} = 1$$

with $A = A_0 + A_1 + A_2 + ... + A_{c+1}$, and π=$(\pi_0, \pi_1, \pi_2, ...\pi_c)$. The QBD type generator is stable if and only if

$$\tilde{\pi}\tilde{A}_0\mathbf{e} < \tilde{\pi}\tilde{A}_2\mathbf{e},$$

which on simplification yields

$$\pi/cA_0\mathbf{e} < \pi/c\{cA_{c+1}\mathbf{e}+(c-1)A_c\mathbf{e}+(c-2)A_{c-1}\mathbf{e}+\ldots 2A_3\mathbf{e}+A_2\mathbf{e}\} \quad (6.1)$$

i.e,

$\lambda_2 < Prob$(excess inventory level exceeds number of priority customer waiting)

$$Prob(r\,\text{priority customers are in service})$$

$$r\,\mu_1\,p\,Prob(\text{atleast one low priority waiting})$$

$$+Prob(l\,\text{low priority customers in service})l\,\mu_2$$

6.2.2 Steady state Probability vector

Let $\mathbf{y} = (\mathbf{y_0}, \mathbf{y_1}, \mathbf{y_2}, \ldots)$ denote the steady state probability vector of $\mathcal{Q}'$. Then,

$$\mathbf{y}\mathcal{Q}' = 0, \mathbf{ye} = 1.$$

Note that, $\mathbf{y_0}$=$\mathbf{x_0}$, and $\mathbf{y_1} = (\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \ldots \mathbf{x_c})$, $\mathbf{y_2} = (\mathbf{x_{c+1}}, \mathbf{x_{c+2}}, \mathbf{x_{c+3}}, \ldots \mathbf{x_{2c}})$ and so on where $\mathbf{x} = (\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \ldots)$ being the steady state probability vector of $\mathcal{Q}$. The component vectors are partitioned as

$$\mathbf{x_0} = \{x_0(j,k,l) : 0 \le j \le c, 0 \le k \le L, j \le l \le c + S\}$$

and

$$\mathbf{x_i} = \{x_i(j,k,l) : 0 \le j \le c, 0 \le k \le c - j - 1, j \le l \le j + k\} \bigcup$$

$$\{x_i(j,k,l) : 0 \le j \le c, c - j \le k \le L, j \le l \le c + S\}, \text{for } i \ge 1$$

Under the stability condition (6.1), the steady state probability vector

$$\mathbf{y_i} = \mathbf{y_1} R^{i-1}, i \ge 2$$

where R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2\tilde{A}_2 + R\tilde{A}_1 + \tilde{A}_0 = 0,$$

and the vectors $\mathbf{y_0}$ and $\mathbf{y_1}$ are obtained by solving

$$\mathbf{y_0}B_1 + \mathbf{y_1}A_2' = 0$$

$$\mathbf{y_0}B_0' + \mathbf{y_1}[\tilde{A}_1 + R\tilde{A}_2] = 0$$

subject to the normalizing condition

$$\mathbf{y_0} + \mathbf{y_1}(I - R)^{-1}\mathbf{e} = 1$$

6.3 System Characteristics

1. Expected number of Type-II customers in the queue,

$$E_{TII} = \sum_{i=1}^{\infty} i\mathbf{x_i}\mathbf{e}.$$

2. Expected number of Type-I customers in system,

$$E_{TI} = \sum_{j=0}^{c}\sum_{k=1}^{L} k \sum_{l=j}^{c+S} x_0(j,k,l) + \sum_{i=1}^{\infty}\sum_{j=0}^{c}\sum_{k=1}^{c-j-1} k \sum_{l=j}^{j+k} x_i(j,k,l)$$

$$+ \sum_{i=1}^{\infty}\sum_{j=0}^{c}\sum_{k=c-j}^{L} k \sum_{l=j}^{c+S} x_i(j,k,l).$$

3. Rate at which Type-II customers leave with Type-I customers upon completion of latter's service,

$$R_{TII,TI} = \sum_{i=1}^{\infty}\sum_{j=0}^{c-1} p(c-j)\mu_1 \sum_{k=c-j}^{L} k \sum_{l=j+k+1}^{c+S} x_i(j,k,l).$$

4. Rate at which Type II customers served out by servers,

$$R_{TIIS} = \sum_{j=1}^{c} j\mu_2 \sum_{k=0}^{L} \sum_{l=j}^{c+S} x_0(j,k,l)$$

$$+ \sum_{i=1}^{\infty} \sum_{j=1}^{c} j\mu_2 \sum_{k=0}^{c-j-1} k \sum_{l=j}^{j+k} x_i(j,k,l)$$

$$+ \sum_{i=1}^{\infty} \sum_{j=1}^{c} j\mu_2 \sum_{k=c-j}^{L} \sum_{l=j}^{c+S} x_i(j,k,l).$$

5. Probability that a Type II customer leaves with a Type-I customer,=

$$\frac{1}{\lambda_2} R_{TII,TI}.$$

6. Probability that Type II customer leaves with service from one of c servers=

$$\frac{1}{\lambda_2} R_{TIIS}.$$

7. Probability that Type-I is lost due to no inventory,

$$P_{TInoinv} = \sum_{j=0}^{c} \sum_{k=0}^{c-j} \sum_{l=j}^{j+k} x_0(j,k,l) + \sum_{j=0}^{c} \sum_{k=c-j+1}^{L-1} \sum_{l=j}^{c} x_0(j,k,l)$$

$$+ \sum_{i=1}^{\infty} \sum_{j=0}^{c} \sum_{k=0}^{c-j} \sum_{l=j}^{j+k} x_i(j,k,l)$$

$$+ \sum_{i=1}^{\infty} \sum_{j=0}^{c} \sum_{k=c-j+1}^{L-1} \sum_{l=j}^{c} x_i(j,k,l).$$

8. Expected loss rate of Type-I customer due to no inventory,

$$E_{TIlossrate} = \lambda_1 P_{TInoinv}.$$

9. Expected loss rate of Type-II customer due to no inventory,

$$E_{TIIlossrate} = \lambda_2 P_{TIInoinv}.$$

10. Probability that an arriving Type-I customer is lost due to lack of space in buffer,

$$P_{nospace} = \sum_{i=0}^{\infty} \sum_{j=0}^{c} \sum_{l=j}^{c+S} x_i(j, L, l).$$

11. Probability that Type-II is lost due to no inventory,

$$P_{TIInoinv} = \sum_{j=0}^{c} \sum_{k=0}^{c-j} \sum_{l=j}^{j+k} x_0(j,k,l) + \sum_{j=0}^{c} \sum_{k=c-j+1}^{L} \sum_{l=j}^{c} x_0(j,k,l)$$

$$+ \sum_{i=1}^{\infty} \sum_{j=0}^{c} \sum_{k=0}^{c-j} \sum_{l=j}^{j+k} x_i(j,k,l)$$

$$+ \sum_{i=1}^{\infty} \sum_{j=0}^{c} \sum_{k=c-j+1}^{L} \sum_{l=j}^{c} x_i(j,k,l).$$

12. Probability that all servers are idle,

$$\sum_{l=0}^{c+S} x_0(0,0,l) + \sum_{i=1}^{\infty} \sum_{k=0}^{l} x_i(0,k,0).$$

13. Probability that all servers are busy,

$$\sum_{i=0}^{\infty}\sum_{j=0}^{c}\sum_{k=c}^{L}\sum_{l=c}^{c+S} x_i(j,k,l).$$

14. Probability that all servers are busy with Type-I,

$$\sum_{i=0}^{\infty}\sum_{k=c}^{L}\sum_{l=c}^{c+S} x_i(0,k,l).$$

15. Probability that all servers are busy with Type-II,

$$\sum_{i=0}^{\infty}\sum_{k=c}^{L}\sum_{l=c}^{c+S} x_i(c,k,l).$$

16. Probability that no server is busy with Type-I,

$$\sum_{j=0}^{c}\sum_{l=j}^{c+S} x_0(j,0,l) + \sum_{j=0}^{c}\sum_{k=1}^{L} x_0(j,k,j) + \sum_{k=1}^{L}\sum_{l=c+1}^{c+S} x_0(c,k,l)$$

$$+\sum_{i=0}^{\infty}\sum_{j=0}^{c}\sum_{k=0}^{L} x_i(j,k,j) + \sum_{k=0}^{L}\sum_{l=c+1}^{c+S} x_i(c,k,l).$$

17. Probability that exactly 'm' servers are busy with Type-I, =

$$\sum_{i=0}^{\infty}\sum_{j=0}^{c-m}\sum_{k=m}^{L} x_i(j,k,j+1) + \sum_{j=0}^{c-m}\sum_{l=j+k+1}^{c+S} x_0(j,m,l).$$

18. Probability that no server is busy with Type-II,

$$\sum_{k=0}^{L}\sum_{l=0}^{c+S} x_0(0,k,l) + \sum_{i=1}^{\infty}\sum_{k=0}^{c}\sum_{l=0}^{j+k} x_i(0,k,l)$$

$$+\sum_{k=c+1}^{L}\sum_{l=c+1}^{c+S} x_i(0,k,l).$$

19. Probability that exactly 'm' servers are busy with Type-II,

$$\sum_{k=0}^{L}\sum_{l=m}^{c+S} x_0(m,k,l) + \sum_{i=1}^{\infty}\left\{\sum_{k=0}^{c-m}\sum_{l=m}^{j+k} x_i(m,k,l) + \sum_{k=c}^{L}\sum_{l=m}^{c+S} x_i(m,k,l)\right\}.$$

20. Expected reorder rate,

$$E_R = k\mu_1 \sum_{k=1}^{c} x_0(0,k,c+s+1) + c\mu_1 \sum_{k=c+1}^{L} x_0(0,k,c+s+1)$$

$$+\sum_{j=1}^{c} j\mu_2 \sum_{k=0}^{L} x_0(j,k,c+s+1)$$

$$+\sum_{j=1}^{c}(c-j)\mu_1 \sum_{k=1}^{L} x_0(j,k,c+s+1)$$

$$+\sum_{i=1}^{\infty}\sum_{j=0}^{c-1} q(c-j)mu_1 \sum_{k=c-j}^{L} x_i(j,k,c+s+1)$$

$$+\sum_{i=1}^{\infty} c\mu_2 \sum_{k=0}^{L} x_i(j,k,c+s+1)$$

$$+\sum_{i=1}^{\infty}\sum_{j=0}^{c-1}p(c-j)mu_1\sum_{k=c-j}^{c+s+2-(j+1)}x_i(j,k,c+s+2).$$

21. Expected number of items in the inventory,

$$EI=\sum_{k=0}^{L}\sum_{l=1}^{c+S}lx_0(0,k,l)+\sum_{j=1}^{c}\sum_{k=0}^{L}\sum_{l=j}^{c+S}lx_0(j,k,l)$$

$$+\sum_{i=1}^{\infty}x_i(0,1,1)+\sum_{k=2}^{c-1}\sum_{l=1}^{j+k}lx_i(0,k,l)$$

$$+\sum_{i=1}^{\infty}\sum_{j=1}^{c}\sum_{k=0}^{c-j-1}\sum_{l=j}^{j+k}lx_i(j,k,l)+\sum_{i=1}^{\infty}\sum_{j=0}^{c}\sum_{k=c-j}^{L}\sum_{l=j}^{c+S}lx_i(j,k,l)$$

6.3.1 Optimization Problem

Based on the above performance measures we construct a revenue function. We define this revenue function as $\mathcal{RF}$ as

$$\mathcal{RF}=(C_1-C_2-C_3)R_{TII,TI}+(C_1-C_2)R_{TIIS}-C_4P_{nospace}-C_5P_{noinv}-h_IE_I$$

$$-C_2E_R-h_{C_I}E_{TI}-h_{C_{II}}E_{TII}$$

where

- C_1 = Selling Cost per unit item
- C_2 = Purchase Cost per unit item
- C_3 = Incentive to Type-I for serving Type-II
- C_4=Cost for loss due to lack of space in buffer

- C_5 = Cost for customer loss due to no inventory
- h_I = holding cost per unit time per unit item in the inventory
- h_{C_I} = holding cost per Type-I customer per unit time
- $h_{C_{II}}$ = holding cost per Type-II customer per unit time

In order to study the variation in different parameters on profit function we first fix the costs $C_1 = \$75, C_2 = \$50, C_3 = \$2, C_4 = \$10, C_5 = \$10, h_I = \$5$, $h_{CI} = \$5$, $h_{CI} = \$2$.

Effect of p on $\mathcal{RF}$

The effect of p on the revenue function for c=1,c=2 and c=3 are given below:

p	0	0.25	0.5	0.75	1
$\mathcal{RF}$	-37.4033	-40.8083	-43.0853	-44.4609	-45.2468

Table 6.1: Value of revenue function for various p: Fix $c = 1, L = 8, S = 17, s = 5, \lambda_1 = 0.9, \lambda_2 = 0.8, \beta = 1, \mu_1 = 2, \mu_2 = 3$

p	0	0.25	0.5	0.75	1
$\mathcal{RF}$	-35.2785	-50.1243	-54.6316	-54.6847	-53.7205

Table 6.2: Value of revenue function for various p: Fix $c = 2, L = 8, S = 18, s = 5, \lambda_1 = 0.9, \lambda_2 = 0.8, \beta = 2, \mu_1 = 2, \mu_2 = 3$

As p increases the value of the revenue function decreases for $c = 1, c = 3$. For $c = 2$, revenue function decreases first and then it shows a slight increase. This is because as p increases incentives given increases.

p	0	0.25	0.5	0.75	1
$\mathcal{RF}$	515.4303	404.3992	339.7477	298.8594	267.5835

Table 6.3: Value of revenue function for various p: Fix $c = 3, L = 8, S = 19, s = 5, \lambda_1 = 1, \lambda_2 = 1.1, \beta = 2, \mu_1 = 1.1, \mu_2 = 1.2$

Effect of β on loss rates and number of items in inventory

As β value increases loss rate of Type-I customers, Type-II customers due to no inventory is evaluated and we can see that loss rate of customers decreases, and as β increases expected number of items in the inventory increases.

β	1	1.5	2	2.5	3
$E_{TIlossrate}$	0.0141	0.0049	0.0019	$9,1408 \times 10^{-4}$	4.8460×10^{-4}
$E_{TIIlossrate}$	0.0125	0.0042	0.0017	8.1252×10^{-4}	4.3075×10^{-4}
E_I	10.4024	10.7832	10.9746	11.0889	11.1644

Table 6.4: Value of revenue function for various value of β: Fix $c = 1, L = 8, S = 17, s = 5, \lambda_1 = 0.9, \lambda_2 = 0.8, \mu_1 = 2, \mu_2 = 3, p = 0.5$

β	1	1.5	2	2.5	3
$E_{TIlossrate}$	0.0081	0.0024	9.3577×10^{-4}	4.1559×10^{-4}	1.9997×10^{-4}
$E_{TIIlossrate}$	0.0072	0.0021	8.1711×10^{-4}	3.5942×10^{-4}	1.7045×10^{-4}
E_I	12.8770	13.2197	13.3877	13.4867	13.5519

Table 6.5: Value of revenue function for various values of β: Fix $c = 2, L = 8, S = 18, s = 5, \lambda_1 = 0.9, \lambda_2 = 0.8, \mu_1 = 2, \mu_2 = 3, p = 0.5$

β	1	1.5	2	2.5	3
$E_{TIlossrate}$	0.0749	0.0474	0.0342	0.0270	0.0224
$E_{TIIlossrate}$	0.0837	0.0535	0.0389	0.0307	0.0256
E_I	12.0157	13.1016	13.5791	13.8264	13.9762

Table 6.6: Value of revenue function for various values of β: Fix $c = 3, L = 8, S = 19, s = 5, \lambda_1 = 1, \lambda_2 = 1.1, \mu_1 = 1.1, \mu_2 = 1.2, p = 0.5$

Effect of β on revenue function

As β increases value of revenue function increases for $c = 2$ and $c = 3$, whereas for $c = 1$ it decreases.

β	1	1.5	2	2.5	3
$\mathcal{RF}$	-43.0853	-44.5015	-45.1581	-45.5057	-45.7053

Table 6.7: Value of revenue function for various values of β: Fix $c = 1, L = 8, S = 17, s = 5, \lambda_1 = 0.9, \lambda_2 = 0.8, \mu_1 = 2, \mu_2 = 3, p = 0.5$

β	1	1.5	2	2.5	3
$\mathcal{RF}$	-58.7539	-56.5816	-54.6316	-53.1138	-51.9845

Table 6.8: Value of revenue function for various values of β: Fix $c = 2, L = 8, S = 18, s = 5, \lambda_1 = 0.9, \lambda_2 = 0.8, \mu_1 = 2, \mu_2 = 3, p = 0.5$

β	1	1.5	2	2.5	3
$\mathcal{RF}$	195.6691	278.4610	339.7477	381.2734	410.2780

Table 6.9: Value of revenue function for various values of β: Fix $c = 3, L = 8, S = 19, s = 5, \lambda_1 = 1, \lambda_2 = 1.1, \mu_1 = 1.1, \mu_2 = 1.2, p = 0.5$

Concluding remarks and suggestions for future study:

In this thesis we discussed queueing -inventory models with several modes of service, those with reservation, cancellation and common life time, queueing inventory model with two commodities and inventory problems associated with crowdsourcing. In certain cases explicit product form solution of the system state could be arrived at.

In chapter 2 we investigated a queueing-inventory model under (s, Q) and (s, S) policies. We introduced two distinct rates of service based on whether inventory level is above s or less than or equal to s. The purpose of these distinct service rates is to reduce the customer loss in the absence of inventory. It is seen that (s, Q) policy outperforms the (s, S) policy. In addition to producing product form solution in both cases we investigated the effect of various parameters on different system performance measures.

It is easy to compute distribution of the time between two successive visits to S (or for that matter s). However, it turns out to be extremely hard to compute the distribution of a busy period (starting with a single customer in the system at an arrival epoch, until the system returns to 'no customer' state at a departure epoch). We will take up this in a future investigation.

In chapter 3, we analyzed an inventory system with reservation and CLT for inventory. Purchased items could be returned before expiry of CLT. The CLT of items is exponentially distributed. On realization of CLT customers waiting in the system stay back. When CLT is reached a replenishment order is placed, lead time of which follows exponential distribution. No new arrival joins when inventory level is zero. This leads to a product form solution. Under stability condition we computed the long run system state distribution. These are in turn used for computing several system performance measures. Expected sojourn time in maximum inventory level and zero inventory level in a cycle are derived. An optimization of a revenue function is also done numerically. We propose to examine whether a product form solution exist for a queueing-inventory system with finite capacity.

In chapter 4 we considered a queueing-inventory model with reservation (purchase), cancellation (return of purchased items) when the items in a batch have common life time. The cases of both zero lead time as well as positive lead time were examined. In these two cases we arrived at the stochastic decomposition of the system state and further product form solution in the long run - that asymptotic independence of number of items in the inventory and number of customers in the system. Several performance characteristics of the system were studied. A significant application of the model is indicated in the transport system.

In a future work we propose to analyze the effect of lead time when it is arbitrarily distributed.

In chapter 5 we analyzed a two commodity queueing inventory problem with Poisson arrival of demands. Customers reveal their requirement at the time when taken for service. If item demanded is not available, the customer leaves the system forever. If both items are demanded when taken for service and only one item is available, then the customer is served that item. Service times are exponentially distributed with parameter depending on the type of demand. The lead times for $i - th$ commodity is exponentially distributed with parameter $\beta_i, 1 = 1, 2$. The continuous time Markov chain is seen to be of GI/M/1 type. The system is shown to be stable. Several system performance indices are derived and numerical illustration provided. An optimization problem is set up and its numerical investigation is carried out.

Extension of the model discussed to $n-$ commodity system with MAP and PH type service time with representation depending on the commodity served, is proposed. 'Emergency purchase' made whenever inventory level of an item drops to zero without cancelling replenishment order seems to produce product form solution. This is also proposed to be investigated.

In chapter 6 we considered a queueing inventory system useful in crowdsourcing. We investigated a multi-server queueing inventory model in which one type of customers are encouraged to serve another type of customers which improves the efficiency of the service facility. Here we assumed that resources to be provided to the customer on service completion to be finite. We assumed the arrival process to be Poisson and service times exponentially distributed. A revenue function is constructed and effect of probability p on the revenue function for single server, two server and three server is numerically analyzed. Effect of β on the loss rate of customers and revenue function is numerically analyzed. We propose to extend the above model where arrival is MAP and service time is Phase-type.

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1. *A. Krishnamoorthy, Binitha Benny and Dhanya Shajin* : **A revisit to queueing-inventory system with reservation, cancellation and common life time**, OPSEARCH, Springer, Operational Research Society of India, vol.54(2),pages 336-350,June 2017.

2. *Dhanya Shajin, Binitha Benny, Deepak T.G and A. Krishnamoorthy* : **A relook to queueing-inventory system with reservation, cancellation and common life time**, Communications in Applied Analysis.20(2016),545-574.

3. *Dhanya Shajin, Binitha Benny, Rostislav V. Razumchik and A. Krishnamoorthy* : **Queueing Inventory system with two modes of service**, Journal of Automation and Control of Russian Academy Of Sciences.(To appear in October 2018 issue; the English translation will appear subsequently in the same journal)

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