# ASPECTS OF DETERMINISTIC ELECTRON THEORY

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**DOCTOR OF PHILOSOPHY**

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By

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# ASPECTS OF DETERMINISTIC ELECTRON THEORY

***Ph.D. thesis in the field of Theories of matter, Spinors & Geometric Algebra***

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# CERTIFICATE

This is to certify that the work reported in this thesis entitled "**ASPECTS OF DETERMINISTIC ELECTRON THEORY**" is a bonafide record of the research work carried out by Mr. Didimos K. V. under my supervision in the Department of Mathematics, Cochin University of Science & Technology. The results embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate have been incorporated in the thesis entitled "**ASPECTS OF DETERMINISTIC ELECTRON THEORY**".

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# DECLARATION

I, DIDIMOS K V, hereby declare that the work presented in this thesis entitled "**ASPECTS OF DETERMINISTIC ELECTRON THEORY**" is based on the original research work carried out by me under the supervision and guidance of Dr.R. S. CHAKRAVARTI, Reader(Rtd), Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted previously for the award of any degree.

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*Didimos K.V.*

*To*
*Issa, Dave and Andrea*

# Contents

# Chapter 1

# Introduction

This thesis is a study of some questions arising from the work of Toyoki Koga (1912-2010) on the foundations of quantum physics. We begin with a few words about Koga's work.

## 1.1 Background

Around the turn of the 20th century, investigators like Lorentz, Poincare, Abraham and Mie speculated that the electron's structure and properties were of electromagnetic origin. This line of thought was abandoned by physicists in the wake of the successes of quantum mechanics from the 1920s onwards.

During the 1950s and 1960s, Toyoki Koga studied the foundations of quantum mechanics with a view to removing ambiguities and contradictions. He was not satisfied with either the Copenhagen interpretation or the work of de Broglie and Bohm. This finally led him to a deterministic theory of the electron including its own internal gravitational field ([9], Chapter VI of [10]). He then applied this theory to quantum electrodynamics and nuclear physics. (chapters VII-X of [10]). His approach was influenced by, among others, Einstein and the investigators mentioned above.

Before he included gravitation in his theory, Koga gave a treatment of the Schrödinger and Dirac equations for the electron ([13], [14], [12], [7], [8] and Chapters IV and V of [10]). He interpreted his solutions to these equations as localised fields centred around the centre of mass of the electron. He showed that a de Broglie wave for a free electron could be obtained by averaging over an ensemble of solutions to the Schrödinger equation as given by him. This would suggest that the Schrödinger and Dirac theories of Koga are the deterministic theories underlying quantum mechanics which Einstein believed to exist.

Koga studied the Schrödinger equation and the Dirac equation (Chapter V of [10]) but not the Pauli equation. This may be because he found it of no use in developing his general relativistic theory of the electron, including its internal gravitational field (Chapter VI of [10]). The latter was motivated by his solution to the Dirac equation. But the deterministic theory of the Pauli equation is a good illustration of his ideas.

Koga also showed ([8], Chapter V of [10])that the solution to a system of equations obtained from the Dirac equation could be interpreted in such a way that the Maxwell equations could be derived from them as an approximation under certain conditions.

Koga worked with specific matrix entries and obtained a system of partial differential equations. Without solving the system, he then gave names to certain quantities obtained from the solutions to the PDEs and then interpreted these as the electric field, the magnetic field and so on which appear in Maxwell's equations. He did not explain how he arrived at these definitions. It would seem to be difficult or even impossible to get an insight into his derivation.

As a consequence of his approximation procedure Koga obtained the correct value of the magnetic moment of the electron, namely, the Bohr magneton. This showed that his derivation was not an empty mathematical exercise.

Later, Koga ([11], Chapter V) wrote out an explicit solution to the Dirac equation and suggested that the solution represented a spinning field. In [18], a solution to the Dirac equation closely related to Koga's was given using the Geometric Algebra of David Hestenes [4]. The solution is the sum of three terms: a Klein-Gordon field, a spinning field and another field symmetric about the spin axis.

After Schrödinger discovered the equation named after him, it was found that this equation did not completely describe the electron. It is necessary to ascribe to the electron an intrinsic angular momentum and magnetic moment. This phenomenon is called *electron spin* since it appears that the electron is spinning.

A non-relativistic theory of the electron, including its magnetic moment, was developed in 1927 by Pauli [21] (see [6] for a modern outline and references to textbooks) who showed how to extend the Schrödinger theory. In the following year, Dirac gave a theory of the electron incorporating special relativity.

It should be noted that by 1970, Dirac had come to believe that a deterministic theory of matter ought to hold ([17], lecture by Dirac).

## 1.2 Outline of this thesis

In this thesis, we focus on a relatively small but crucial part of Koga's work: his study of the Schrödinger and Dirac equations, especially their solutions for free electrons.

We first give a brief description of Koga's solution to the Schrödinger equation (which he called a *wavelet* in his early papers and an *elementary field* in his books). Then we discuss and elaborate on his claim that the de Broglie wave associated to a free electron can be obtained by averaging over an ensemble of elementary fields. His treatment of this topic is rather cursory and inadequate, although it is a key part of his work. We give a more detailed explanation.

In the next chapter, we develop the non-relativistic theory of electron spin, as Pauli did, by extending Koga's solution of the Schrödinger equation. We find that the electron has a definite spin axis at any point of time. An external magnetic field exerts a torque which rotates the spin axis. The Pauli equation holds.

We also discuss the relation of the Hopf map ([5], [20], [15], [19]) to the Pauli spin theory. It has been mentioned by several authors that the Hopf map gives the spin direction of a spin 1/2 particle such as an electron. We show the consistency of this assertion with the Pauli spin theory, which seems to have been taken for granted in the literature so far.

After this, we consider Koga's solution to the Dirac equation. We show that four one-dimensional solutions to the Klein-Gordon equation each lead to a solution to the Dirac equation containing a term representing a rotating field. Two of these solutions are significant. They represent opposing spins. An electron field with arbitrary spin axis can be represented as a linear combination of the two. The Hopf map is used in proving this.

Then we continue the study of Koga's work on the Dirac equation by applying Geometric Algebra to Koga's approximate derivation of Maxwell's equations. The notation and methods of Geometric Algebra make the relation between the electromagnetic and Dirac fields easy to see, in fact almost obvious.

We finally summarise the thesis with some concluding remarks.

An appendix dealing with questions posed by an examiner has been added. Errors that he pointed out have been corrected there.

# Chapter 2

# The Schrödinger Equation

## 2.1 Introduction

We work in an inertial frame, i.e., Newton's first and third laws are assumed to hold. The second law needs to be modified for small objects like electrons. This section is a very brief outline of some of this research done in the early 20th century.

Space and time are taken to be independent; this is Galilean space-time.

The wave nature of electromagnetic radiation such as light was accepted by the end of the 19th century. But in 1900, Max Planck, in his study of radiation from a cavity, introduced the quantum hypothesis: matter emits and absorbs radiation of frequency $\nu$ in units (quanta) of $h\nu$. Here $h$ is called Planck's constant.

In 1905 Albert Einstein extended this further to explain the photoelectric effect. Einstein proposed that radiation also travels as quanta, each quantum being a particle of energy $h\nu$ and momentum $p = h\nu/c$ where $c$ is the velocity of light.

In 1923 Debye and Compton explained the change of wavelength of X-rays during scattering using Einstein's ideas.

The following year, Louis de Broglie, guided by the analogy between Fermat's principle in optics (photons) and the least-action principle in mechanics extended the concept of wave-particle duality to material particles: the wavelength $\lambda$ of a particle with velocity $v$ is $h/mv$.

In 1926 Schrödinger introduced a wave function $\psi$ to represent the wave character of a particle and derived a wave equation for the electron in a hydrogen atom. This is known as the time-independent Schrödinger equation. Solving it, he obtained the exact Bohr energy levels of the hydrogen atom.

A few months later he discovered a more general equation for the time evolution of the wave function, now called the time-dependent Schrödinger equation. This is also called the wave equation.

Louis de Broglie in 1927 and David Bohm in 1952 suggested that a particle accompanies the wave.

In 1972, Koga published a paper giving a new solution to this equation representing a localised field around a centre, which he called a *wavelet* in his papers. Later, in his books, he used the term *elementary field*. Using this solution, he explained various phenomena involving a free electron. Koga actually took up and developed a 1927 idea of de Broglie. A sequel in 1974 considered the effect of external electromagnetic fields which are present in the real world in atoms, etc..

## 2.2 The Schrödinger equation and Koga's solution

According to Schrödinger, an electron is described by a complex-valued function $\psi$ of time $t$ and position vector $\mathbf{r} = (x, y, z)$. Let $m$ be the mass of the electron and $\hbar = h/2\pi$ where $h$ is Planck's constant. Suppose $U$ stands for potential energy. Then Schrödinger's equation is

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - U\psi = 0. \tag{2.1}$$

We substitute $\psi = a\exp(iS/\hbar)$ in the above equation and separate real and imaginary parts, getting

$$\frac{\partial S}{\partial t} + \frac{(\mathrm{grad}S)^2}{2m} + U - \frac{\hbar^2\nabla^2 a}{2ma} = 0 \tag{2.2}$$

and

$$\frac{\partial a^2}{\partial t} + div\left\{\frac{a^2\,(\mathrm{grad}S)}{m}\right\} = 0. \tag{2.3}$$

Suppose we have a stationary free electron centred at the origin, i.e., the momentum is $\mathbf{p} = 0$. Let us assume that $\mathrm{grad}U = 0$. There is a solution to the Schrödinger equation given by

$$a = A\exp(-\kappa r)/r \tag{2.4}$$

with $\kappa > 0$, a constant, and

$$S = -Et + \mathbf{p}\cdot\mathbf{r} \tag{2.5}$$

where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $E = -(\hbar^2\kappa^2/2m)$ is the total energy.

The elementary field contains a singularity because $\exp(-\kappa r)/r$ becomes infinite as $r$ approaches 0. Therefore it does not represent the real electron accurately at such points. Koga's explanation is that this is a consequence of using a linear equation to describe nonlinear reality.

Also, the value of the constant $\kappa$ is not given by the Schrödinger equation and must be chosen to reflect reality. Koga uses ideas of Yukawa to estimate the value of $\kappa$.

Koga mentions that in the Schrödinger theory, we are not considering the relativistic rest energy, which is why we can get negative energy as above.

More generally, suppose the electron is at position $\mathbf{R}$ at $t = 0$ and is moving with constant velocity $\mathbf{v}$. Then the momentum $\mathbf{p}$ is $m\mathbf{v}$. Now we have to replace $r$ with $|\mathbf{r} - (\mathbf{v}t + \mathbf{R})|$ in the expression for $a$ and take

$$E = (mv^2/2 - \hbar^2\kappa^2/2m) \tag{2.6}$$

where $v = |\mathbf{v}|$.

This is Koga's solution to the Schrödinger equation for a free electron. It satisfies

$$\frac{\nabla^2 a}{a} = \kappa^2. \tag{2.7}$$

## 2.3 The de Broglie wave and its relation to elementary fields

The following question arises: suppose we have a free electron, i.e., an electron which is described by an elementary field, but whose location is unknown, all points of $\mathbb{R}^3$ being equally likely.

It can be considered an infinite ensemble of (possible) electrons, one at every point of $\mathbb{R}^3$. These electrons don't interact, since there is actually only one.

Is there a solution of the Schrödinger equation that describes this situation? And if so, how is it related to elementary fields?

Koga's answer is that this is nothing but a de Broglie wave, and it is obtained by applying a certain averaging process to a collection (the technical term is ensemble) of elementary fields. The rest of this section is an attempt to make explicit what Koga seems to suggest in [13], subsection 6.1 (page 66) and [10], subsection 4.3.b (page 50).

Suppose we take $U = 0$ in the Schrödinger equation. There is a solution $S = \mathbf{p} \cdot \mathbf{r} - Et$ with $E = p^2/2m =$ constant, $a =$ constant. This can be expressed as

$$\psi = a \exp[i(\mathbf{p} \cdot \mathbf{r} - Et)/\hbar] \tag{2.8}$$

which is a plane wave with direction $\mathbf{p}$. This describes a de Broglie wave; we now consider how it can be obtained from elementary fields.

Since the Schrödinger equation is linear and homogeneous, a finite sum of elementary fields is also a solution to the equation. We can approximate the infinite ensemble mentioned above with a finite collection of elementary fields, uniformly distributed in space with their centres forming a finite set $S_n$. We then consider the limit as the number of electrons increases without bound and the distance between adjacent electrons approaches zero, their distribution remaining uniform. We will assume that $S_n \subseteq S_{n+1}$ and $|S_n|$ increases with $n$. We also require that every point of $\mathbb{R}^3$ is a limit point of the union of the sets $S_n$.

Here is an example of how all this can be accomplished. Let $n$ be a nonnegative integer. Consider the set $I_n$ of rational numbers, of size $2^{2n+1} + 1$, defined by

$$I_n = \left\{ -2^n, -2^n + \frac{1}{2^n}, -2^n + \frac{2}{2^n}, ..., 2^n \right\} \tag{2.9}$$

and let

$$S_n = I_n^3, \tag{2.10}$$

a subset of $\mathbb{R}^3$ with $(2^{2n+1}+1)^3$ elements. Each point of $S_n$ is at a distance $1/2^n$ from 6 nearby points. These sets $S_n$ form an ascending chain.

If we enclose each point $\mathbf{P} = (x, y, z)$ of $S_n$ in a small cube $K_{\mathbf{P}}$ centred at $\mathbf{P}$ with edge length $1/2^n$ and edges parallel to the coordinate axes, their union is a cube $C_n$ of edge length $2(2^n + 1/2^{n+1})$ centred at the origin,

$$C_n = \left[-2^n - \frac{1}{2^{n+1}}, 2^n + \frac{1}{2^{n+1}}\right]^3 \tag{2.11}$$

where the brackets denote a closed interval in $\mathbb{R}$. For each face of $C_n$, there are $(2^{2n+1}+1)^2$ points of $S_n$ at distance $1/2^{n+1}$ from that face.

Let

$$D_n = [-n, n]^3, \tag{2.12}$$

a cube of edge length $2n$ contained in $C_n$, also centred at the origin. As $n$ increases, both cubes get larger but $C_n$ grows far more rapidly than $D_n$; the ratio of their edge lengths, $n2^{n+1}/(2^{2n+1}+1)$, approaches 0.

Suppose we have a collection of possible (i.e., non-interacting) free electrons, one centred at $\mathbf{R}$ at $t = 0$, for each point $\mathbf{R}$ of $S_n$. Suppose all of them are moving at velocity $\mathbf{v}$. The elementary field of one of them with centre $\mathbf{R}$ (at $t = 0$) is given by

$$\psi_{\mathbf{R}} = A\left(\frac{\exp(-\kappa r_{\mathbf{R},t})}{r_{\mathbf{R},t}}\right)\exp(iS/\hbar) \tag{2.13}$$

where $A$ is a constant, $r_{\mathbf{R},t} = |\mathbf{r} - (\mathbf{R} + \mathbf{v}t)|$ and $S$ is the same for all the electrons and is as given in the last section.

It was mentioned earlier that the sigularity in the elementary field is unrealistic. We therefore modify the definition of $\psi$ as follows. We choose a "small" $r_0$ and let

$$a = \text{constant} = \exp(-\kappa r_0)/r_0 \tag{2.14}$$

for $r \le r_0$. This removes the singularity. We shall use this modified definition in the rest of this section without any change in notation.

What follows is a plausibility argument rather than a rigorous proof. Let $V_n$ be the volume of any one of the cubes $K_{\mathbf{P}}$. For large $n$, the sum of these elementary fields, multiplied by the volume of a small cube,

$$V_n \sum_{\mathbf{R}\in S_n} \psi_{\mathbf{R}}, \tag{2.15}$$

is a good approximation to

$$\iiint_{C_n} \psi_{\mathbf{R}}\, dR_x\, dR_y\, dR_z \tag{2.16}$$

where $\mathbf{R} = (R_x, R_y, R_z)$. If the point $\mathbf{r} - \mathbf{v}t$ is sufficiently close to the origin, the integral over $C_n$ approximates the integral over all of $\mathbb{R}^3$ because the contributions of distant points are negligible.

For sufficiently large $n$, points in $D_n$ can be considered close to the origin for this purpose.

The last integral, over all of $\mathbb{R}^3$, is independent of $\mathbf{r}$ and $t$. It can be concluded that the same property holds approximately for $V_n \sum_{\mathbf{R}\in S_n} \psi_{\mathbf{R}}$, at least for points in $D_n$.

Since $V_n = (1/2^n)^3 = 1/(2^{3n})$, we have shown that the amplitude $a$ of $\psi$ is constant and obtained by averaging.

We now turn our attention to the factor $\exp(iS/\hbar)$. Since we now have a solution $\psi = a\exp(iS/\hbar)$ of the Schrödinger equation with $a =$ constant, and

$$\mathbf{p} = \nabla S \tag{2.17}$$

$$E = -\frac{\partial S}{\partial t}, \tag{2.18}$$

the real part of the equation reduces to the Hamilton-Jacobi equation. The result is

$$E = p^2/(2m), \mathbf{p} = m\mathbf{v} \tag{2.19}$$

and finally

$$\psi = a\exp[i(-p^2t/2m + m\mathbf{v}\cdot\mathbf{r})/\hbar]. \tag{2.20}$$

This is the de Broglie wave.

# Chapter 3

# Deterministic Electron Spin

## 3.1 Introduction

In 1927 Pauli [21] showed how to extend the Schrödinger theory of the electron to account for magnetic effects (which others suggested was due to spin) without taking relativity into account. He introduced an electron field that took values in $\mathbb{C}^2$ rather than $\mathbb{C}$. His modification of Schrödinger's equation is now called the Pauli equation. This contains an additional term due to the torque exerted by the external magnetic field on an electron. We proceed as Pauli did but extend the Schrödinger theory of Koga described in the last chapter. This means our theory is deterministic.

We also study the Hopf map

$$f : \mathbb{S}^3 \to \mathbb{S}^2$$

in the context of electron theory. We address its consistency with the Pauli theory. This has been mentioned in the literature but has apparently not been explicitly explained.

It should be noted that in the Pauli theory, the magnetic nature of the electron is postulated, not explained. The conventional term “spin” was not used by Pauli. But we find it expedient to use it.

## 3.2 A free electron

By a free electron we mean one with no external forces acting on it. We take $U = 0$ for a free electron. For convenience, we assume that we are studying an electron which is at rest in our frame. In order to study electron spin, we postulate that $\psi_1$ and $\psi_2$ are related solutions to the Schrödinger equation for a free electron:

$$\psi_j = a_j \exp(iS/\hbar) \tag{3.1}$$

where $a_j = |\psi_j|$ are complex valued functions of position and time.

We also assume that the ratio $\psi_2/\psi_1$ is constant (independent of time as well as position) for a free electron. If this ratio gives the spin direction, then the assumption above is equivalent to the assertion that in the absence of a torque, an electron does not precess.

This account is motivated by Koga’s work. See [13] for Koga’s treatment of the Schrödinger equation. When an electron is in an external electric field, Koga studies what happens in [14]. These matters are also explained in Chapter IV of his book [10] where he changes his terminology for the wavefunction from “wavelet” to “elementary field”.

We can write

$$\psi = \psi_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.2}$$

where $\psi_0$ is a solution to the Schrödinger equation and $\alpha$ and $\beta$ are constants (for a free electron) with $|\alpha|^2+|\beta|^2= 1$. This expression is unique up to multiplication by a complex number of absolute value 1. We can make it unique by (for example) assuming that $\alpha$ is real and positive.

If we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ as in [20], we have $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in S^3$, the unit sphere in $\mathbb{R}^4$. This brings up the question: what is the direction of $\psi$, if any? In other words, what is the spin axis? There is a possible answer to this. In 1931, Hopf [5] defined a map which we denote

$$f : S^3 \to S^2 \tag{3.3}$$

as an example of a continuous map between spheres that is not null-homotopic. Here $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$.

We define the Hopf map $f$ here and justify its use later. The following definition and several equivalent ones are given in [20]; this one is the most convenient for us.

A general point $P \in S^3$ can be described as

$$P = e^{i\xi} \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} \tag{3.4}$$

where $0 \leq \theta \leq \pi$. Here $\theta$ is unique and $\phi$ can also be made unique by putting suitable bounds on it; $\xi$ is arbitrary. Let

$$f(P) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in S^2. \tag{3.5}$$

We see that

(i) $f$ is continuous, independent of $\xi$ and depends only on the ratio of the components of $P$, $e^{i\phi}\tan(\theta/2)$,

(ii) every point of $S^2$ is $f(P)$ for some $P \in S^3$,

(iii) $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0,0,1)$ (the north pole of $S^2$),

(iv) $f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0,0,-1)$ (the south pole of $S^2$), and

(v) $f(P)$ uniquely determines $P$ (except for the value of $\xi$).

As a consequence of (i), we can extend the domain of $f$ to all the points of $\mathbb{C}^2$ except the origin. There is also an $S^2$-valued map $f(\psi)$ where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. It should be noted that $f(\psi)$ depends on $t$ alone; for a free electron, $f(\psi)$ is constant. Thus, $f$ is a candidate for the direction map. But is $f$ compatible with the Pauli spin theory? In other words, for a free electron, do $f(\psi)$ and the spin angular momentum vector $\mathbf{s}$ have the same direction in $\mathbb{R}^3$?

By properties (iii) and (iv), $f(\psi)$ gives the spin direction of a spin-up or spin-down electron.

## 3.3 An electron in an electromagnetic field

We accept the Pauli equation as given in the literature. This is the Schrödinger equation with one more term due to an external magnetic field. If there is no magnetic field, it reduces to a pair of identical Schrödinger equations, one for each component of the field. One difference in our approach is that we always consider angular momentum as a vector in $\mathbb{R}^3$.

In the Schrödinger (or Pauli) equation, the potential energy $U$ is a sum of terms due to various forces acting on the electron. The vector potential of the electromagnetic field also modifies the kinetic energy term. These are all scalar operators; they simply multiply $\psi$. None of them take into account the fact that the electron has an intrinsic magnetic moment.

Suppose an electron is placed in a magnetic field $\mathbf{B}$. Then it experiences a torque $\boldsymbol{\mu} \times \mathbf{B}$ where $\boldsymbol{\mu}$ is its magnetic moment. The potential energy term due to the magnetic field is

$$V = -\mathbf{B} \cdot \boldsymbol{\mu} \tag{3.6}$$

which leads to

$$V = \frac{e}{m}(B_1 s_1 + B_2 s_2 + B_3 s_3) \tag{3.7}$$

where $B_1$, $B_2$, $B_3$ are the components of $\mathbf{B}$.

Since $\psi$ has two components, the three real components $s_1$, $s_2$, $s_3$ of the spin angular momentum $\mathbf{s}$ must be represented in the Pauli equation by $2 \times 2$ matrices, say $S_1$, $S_2$, $S_3$. These satisfy well known commutativity relations and consequently we can make the choice

$$S_j = \frac{\hbar}{2}\sigma_j \tag{3.8}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

These are the so called Pauli spin matrices.

Hestenes [4] and Doran and Lasenby [2] explain that these matrices merely represent three orthogonal unit vectors in $\mathbb{R}^3$ and have nothing to do with spin.

Thus, in the Pauli equation, corresponding to $V$ is the operator

$$\frac{\hbar e}{2m}(B_1\sigma_1 + B_2\sigma_2 + B_3\sigma_3). \tag{3.9}$$

For us, the spin vector $\mathbf{s}$ is a real vector in $\mathbb{R}^3$, not a triple of matrices as stated in most textbooks. A spin measurement, such as passing an electron through a Stern-Gerlach apparatus, is actually a rotation of the spin axis due to the torque exerted by the magnetic field. This is compatible with the view of Doran and Lasenby in their book [2] that spin measurement is really spin polarisation.

## 3.4 Pauli and Hopf

We wish to study the Hopf map in the context of a free electron. But the effect of spin is not seen unless there is an external magnetic field. So we assume its existence and then consider the limit of its effect as it approaches 0.

Let $\mathbf{B} = |\mathbf{B}|\mathbf{n}$ where $\mathbf{n}$ is a unit vector with components $n_1$, $n_2$, $n_3$. This means that $B_j = |\mathbf{B}|n_j$ for $j = 1, 2, 3$ and $\mathbf{n} \in S^2$ is the direction of $\mathbf{B}$.

Now there is a unique $\theta$ such that $0 \leq \theta \leq \pi$ and $\cos\theta = n_3$. Then, since ${n_1}^2 + {n_2}^2 + {n_3}^2 = 1$, there is $\phi$ such that $\sin\theta\cos\phi = n_1$ and

$\sin\theta\sin\phi = n_2$. Note that for $P \in S^3$ the Hopf map satisfies

$$f(P) = \mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3.10}$$

if and only if $P$ is of the form

$$P = e^{i\xi}\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{3.11}$$

for some $\xi$.

Similarly, we find $Q \in S^3$ such that $f(Q) = -\mathbf{n}$. Since $-\mathbf{n}$ is obtained from $\mathbf{n}$ by replacing $\theta$ with $\pi - \theta$ and $\phi$ with $\phi + \pi$, we get

$$Q = e^{i\xi}\begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}cos(\theta/2) \end{pmatrix}. \tag{3.12}$$

Without loss of generality, we will take $\xi = 0$ or any other convenient value. With the usual inner product, $\{P, Q\}$ forms an orthonormal basis for $\mathbb{C}^2$. The two complex vectors $P$ and $Q$ are eigenvectors of the matrix $n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ corresponding to the eigenvalues 1, $-1$. See the paper [20] for details.

We are now concerned with the question of the relation, if any, between $\mathbf{n}$ and the directions of $\psi$ and $\mathbf{s}$.

First consider a special case. Suppose $\mathbf{B}$ is parallel to the $+z$-axis, i.e., $n_1 = n_2 = 0$, $n_3 = 1$. We can take $P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In this case, using $\psi = \psi_1 P + \psi_2 Q$, $\sigma_3 P = P$ and $\sigma_3 Q = -Q$, we see that the Pauli equation degenerates into a pair of independent scalar equations, one for each $\psi_j$. If $\psi_2 = 0$, we have a spin-up electron; by definition, its spin axis has the same direction, $(0, 0, 1)$, as $\mathbf{B}$.

In the general case, there are complex-valued maps $\psi_P$ and $\psi_Q$, uniquely determined by $\psi$, such that $\psi = \psi_P P + \psi_Q Q$. Again, the Pauli equation degenerates into a pair of independent scalar equations, one for $\psi_P$ and the other for $\psi_Q$. If $\psi_Q = 0$ then $f(\psi) = f(P) = \mathbf{n}$ while if $\psi_P = 0$ then $f(\psi) = f(Q) = -\mathbf{n}$.

Conversely, if $f(\psi) = \mathbf{n}$ then $\psi_Q = 0$; if $f(\psi) = -\mathbf{n}$ then $\psi_P = 0$.

Suppose we rotate the $z$-axis in $\mathbb{R}^3$, making $\mathbf{n}$ the north pole of $S^2$ (and $-\mathbf{n}$ the south pole), and correspondingly change bases in $\mathbb{C}^2$ from $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ to $\{P, Q\}$. The vectors $\mathbf{s}$ and $\mathbf{B}$ are unchanged, but now $\mathbf{B}$ is parallel to the new $+z$-axis. With the new basis of $\mathbb{C}^2$, $P$ is represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Q$ by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Similarly, $\mathbf{n}$ is now represented by $(0, 0, 1)$. Although the Hopf map changes, its values at $P$ and $Q$ remain the same: $\mathbf{n}$ and $-\mathbf{n}$. Thus, as in the special case, the direction of $\psi_P P$ is $\mathbf{n}$, which is the direction of $\mathbf{B}$. As in the special case, $\psi_P P$ stands for a spin-up electron. So its direction coincides with those of $\mathbf{n}$ and $\mathbf{s}$.

Considering the limit as $\mathbf{B} \to 0$, we see that for a free electron $\psi$ and $\mathbf{s}$ have the same direction. In other words, the Hopf map gives the spin axis.

# Chapter 4

# Deterministic Dirac Theory

## 4.1 Introduction

In 1928 Dirac published an equation for the electron field which was apparently compatible with special relativity. Dirac found that his electron field had to be 4-dimensional.

The Dirac equation is closely related to another equation, compatible with relativity, whose solution is a complex scalar electron field (or an n-tuple of such fields). This equation was discovered and rejected by Schrödinger and was rediscovered by Klein and Gordon, all independently. It is named after the latter two.

Koga solved these two equations in a manner similar to his slightly earlier work on Schrödinger theory. In this chapter we study some properties of Koga's solution to the Dirac equation. He conjectured that the electron was a rotating field but could not prove it.

We describe all this in detail and show that there is more to the solution than what he suspected.

We also show that the Hopf map can be used to obtain a Dirac electron field with arbtrarily chosen spin axis in $\mathbb{R}^3$.

## 4.2 The Klein-Gordon and Dirac Equations

Suppose $\Phi(x, y, z, t)$ is a solution to the Klein-Gordon equation for a free electron (this is the only case treated here). This is the following equation:

$$\left(\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + m^2 c^4\right) \Phi = 0.$$

Here $\Phi$ can be a complex-valued scalar map or an $n$-tuple of such maps, for any $n$.

We take $n = 4$ in order to use $\Phi$ to get a solution $\Psi$ to the Dirac equation.

Then the Klein-Gordon equation can be written as

$$D_0 D_1 \Phi = 0 \text{ or } D_1 D_0 \Phi = 0$$

where the operators $D_0$ and $D_1$ commute:

$$D_0 = i\hbar\beta \frac{\partial}{\partial t} + i\hbar c\beta \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z}\right) - mc^2,$$

$$D_1 = i\hbar\beta \frac{\partial}{\partial t} + i\hbar c\beta \left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z}\right) + mc^2.$$

Here $\beta,\ \alpha_1,\ \alpha_2$ and $\alpha_3$ are $4 \times 4$ matrices satisfying certain commutativity conditions; following Koga, we choose them to be the matrices which were first given by Dirac. Later, we adopt the modern usage of the term "Dirac equation" of which Koga's is a special case.

The Dirac equation is nothing but

$$D_0\Psi = 0$$

and hence, as Koga and others mention, a solution to it is given by $\Psi = D_1\Phi$.

Another equivalent formulation of the Dirac equation uses the so-called gamma matrices. We are not concerned with this here.

## 4.3 Solutions to the Dirac equation

Suppose we consider a free electron at rest in our inertial frame with its centre at the origin of our coordinate system. This case suffices for our purposes; there is no loss of generality. There exists a (complex) scalar solution to the Klein-Gordon equation, given by

$$\varphi = a\exp{(iS/\hbar)}$$

where

$$\begin{aligned} a &= \exp{(-\kappa r)}/r, \\ S &= -Ect \\ \text{with} & \\ r &= |\mathbf{r}|, \\ E^2 &= m^2c^2 - \hbar^2\kappa^2 \end{aligned}$$

where $\kappa$ is a positive constant, $\mathbf{r}$ = position vector, $cE$ = energy, (the value of $\kappa$ is not given by the theory and is to be chosen to make the result conform to reality).

If the electron is moving with velocity $\mathbf{u}$, then the expressions above need to be modified because the Lorentz transformation of special relativity applies. For our purposes it is not necessary to consider all this.

This solution was given by Koga ([11], Chapter V); it is very similar to his solution to the Schrödinger equation. He then took, as a 4-dimensional solution to the Klein-Gordon equation, $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$ with

$$\varphi_j = a \exp{(iS/\hbar)} A_j$$

where $A_j$ are arbitrary complex constants.

Koga wrote down a 4-dimensional solution to the Dirac equation, $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, by evaluating $D_1\Phi$ with $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T$. He then tried to demonstrate that this solution, with arbitrary $A_1, A_2, A_3, A_4$, represents a rotating field, similar to a spinning top. He was not very successful, probably because he did not assign specific, suitable values to the constants $A_j$ as we shall do in this paper, but kept them arbitrary.

Pandey and Chakravarti ([18]) translated the complex scalar solution of the Klein-Gordon equation $\psi = a \exp{(iS/\hbar)}$ into Geometric Algebra and got a solution to the Dirac equation, but did not interpret the terms properly. One purpose of this paper is to correct the error in their paper. It turns out that although Geometric Algebra was initially very helpful in finding a solution, it did not clearly display some other solutions and the relation between them; the conventional approach makes things clear.

We shall assume, with no loss of generality, that the electron is at rest in our inertial frame:

$$\mathbf{u} = 0.$$

Koga's solution to the Dirac equation is as follows. He defined

$$\mathbf{R} = (\mathbf{r} - \mathbf{u}t)\left(\frac{1}{|\mathbf{r} - \mathbf{u}t|^2} + \frac{\kappa}{|\mathbf{r} - \mathbf{u}t|(1 - u^2/c^2)^{1/2}}\right).$$

For us, of course, $\mathbf{u} = 0$ in the above equation. Now the solution $\Psi$, with arbitrary complex constants $A_j$, has components

$$\begin{aligned}
\psi_1 &= a \exp(iS/\hbar)\left[A_1(Ec + mc^2) - A_4\, i\hbar c\,(R_x - iR_y) - A_3\ i\hbar c\ R_z\right],\\
\psi_2 &= a \exp(iS/\hbar)\left[A_2(Ec + mc^2) - A_3\, i\hbar c\,(R_x + iR_y) + A_4\ i\hbar c\ R_z\right],\\
\psi_3 &= a \exp(iS/\hbar)\left[A_3(Ec - mc^2) + A_2\, i\hbar c\,(R_x - iR_y) + A_1\ i\hbar c\ R_z\right],\\
\psi_4 &= a \exp(iS/\hbar)\left[A_4(Ec - mc^2) + A_1\, i\hbar c\,(R_x + iR_y) - A_2\ i\hbar c\ R_z\right].
\end{aligned}$$

These expressions can be obtained directly or by putting $\mathbf{u} = 0$ in the solution given by Koga.

## 4.4 The choice of the coefficients

Not all such solutions can be expected to be physically realistic; Koga mentioned that the arbitrary coefficients $A_j$ need to be appropriately chosen. But he did not make any choice and tried to prove, using arbitrary $A_j$, that the solution represents a spinning field.

We propose four possible solutions, each obtained by taking one $A_j$ to be 1 and the others to be 0.

Taking $A_1 = 1,\ A_2 = A_3 = A_4 = 0$ gives the solution corresponding to that obtained by Pandey and Chakravarti [18] using Geometric

Algebra. We call this solution $\Psi_f$ ($f$ stands for first).

$$\Psi_f = a \exp{(iS/\hbar)} \begin{pmatrix} Ec + mc^2 \\ 0 \\ i\hbar c R_z \\ i\hbar c (R_x + iR_y) \end{pmatrix}.$$

This can be written as the sum of four column vectors, the second term being the zero vector:

$$\begin{aligned} \Psi &= a \exp{(iS/\hbar)} \begin{pmatrix} Ec + mc^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &+ a \exp{(iS/\hbar)} \begin{pmatrix} 0 \\ 0 \\ i\hbar c R_z \\ 0 \end{pmatrix} \\ &+ a \exp{(iS/\hbar)} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i\hbar c (R_x + iR_y) \end{pmatrix}. \end{aligned}$$

The three nonzero vectors can be interpreted as follows: since $Ec + mc^2$ is constant, the first vector is a solution to the Klein-Gordon equation which represents a field without spin. The second vector is a field which has rotational symmetry about the z-axis. Finally, the last vector represents a spinning field with angular velocity

$$\omega = Ec/\hbar.$$

This can be verified by observing that if we take a rotating frame with the same origin, $z$-axis and angular velocity $\omega$, the field is a constant field in this frame. We do this in the next section.

At this point it can be mentioned that Pandey and Chakravarti [18] made the error of mixing the last two components. In the present approach, it is impossible to do this.

Similarly, if we take $A_2 = 1$ and $A_1 = A_3 = A_4 = 0$ we get

$$\begin{aligned}
\psi_1 &= 0, \\
\psi_2 &= a\exp(iS/\hbar)(Ec + mc^2), \\
\psi_3 &= a\exp(iS/\hbar)i\hbar c(R_x - iR_y), \\
\psi_4 &= a\exp(iS/\hbar)(-i\hbar cR_z).
\end{aligned}$$

Again there are three nonzero column vectors. They can be interpreted exactly as in the previous case (but in a different order) except that the direction of rotation is reversed because $R_x - iR_y$ is the complex conjugate of $R_x + iR_y$ (their arguments are the negatives of each other). If the first solution $\Psi_f$ is defined to be spin up, then the second, which we call $\Psi_s$, is spin down.

Two similar choices can be made, $A_3 = 1$ and $A_4 = 1$. The solutions are similar to the ones described above. But the Klein-Gordon term contains $Ec - mc^2$ instead of $Ec + mc^2$.

These four solutions have some common features. Each has three nonzero components: a constant component, a component independent of $x$ and $y$ with the same magnitude in all four, and a component giving a rotating field with angular velocity $|Ec/\hbar|$. The first two solutions have a constant component of the same magnitude, $|Ec+mc^2|$. Similarly for the last two. It was mentioned earlier that $E$ satisfies $E^2 = m^2c^2 - \hbar^2\kappa^2$. If we assume the positive root for $E$ in the first two solutions and the negative root in the last two, the constant component has the same value in all four. All this suggests that these solutions really represent the electron. The last two can be considered negative energy solutions. We will not say any more about them.

## 4.5 Verification of the rotating field

In this section, the axis of rotation is always the $z$-axis. We first consider how a rotating field can be described in general. Let $F = F(x, y, z, t)$ be a field. The argument is independent of where $F$ takes its values.

Suppose a general point with coordinates $(x, y, z, t)$ in our inertial frame has coordinates $(x', y', z, t)$ in a rotating frame with the same z-axis and angular velocity $\omega$ relative to the inertial frame.

A point is stationary in the rotating frame if and only if it is moving along a circle centred at a point on the $z$-axis, in a plane parallel to the $xy$-plane, with angular velocity $\omega$.

The field $F$ can be said to be rotating with angular velocity $\omega$ if it is constant (time-independent) when observed by an observer fixed in the rotating field.

Let $x + iy = \rho e^{i\theta}$ where $\rho = |x + iy|$. Then $x' + iy' = \rho e^{i(\theta - \omega t)}$.

It follows from the above that the condition

$$F(x, y, z, t) = F(x', y', z, 0) \text{ for all points}$$

is equivalent to the statement that the field $F$ is rotating with angular velocity $\omega$ relative to the inertial frame.

We now apply the above results to Koga's solutions. It suffices to consider $\Psi = \Psi_f$; the other solutions behave similarly.

Suppose $F = \psi_4$ and the angular velocity is $\omega = Ec/\hbar$. Then, using $S = -Ect$ and $x' + iy' = \exp(-i\omega t)(x + iy)$ we get

$$\begin{aligned}
\psi_4(x,y,z,t) &= a\exp\left(iS/\hbar\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ i\hbar c(R_x + iR_y) \end{pmatrix} \\
&= a\exp\left(-i\omega t\right)(i\hbar c)\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ x + iy \end{pmatrix}
\end{aligned}$$

and so

$$\begin{aligned}
\psi_4(x',y',z,0) &= a(i\hbar c)\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ x' + iy' \end{pmatrix} \\
&= \psi_4(x,y,z,t).
\end{aligned}$$

This shows that $\psi_4$ is a rotating field as desired.

## 4.6 An arbitrary spin axis

Suppose $\mathbf{n} \in \mathbb{S}^2$. This means $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ with $n_1^2 + n_2^2 + n_3^2 = 1$. We can describe the components of $\mathbf{n}$ using spherical coordinates:

$$\begin{aligned} n_1 &= \sin\theta\cos\phi, \\ n_2 &= \sin\theta\sin\phi, \\ n_3 &= \cos\theta \end{aligned}$$

where $\theta$ is unique if we assume $0 \le \theta \le \pi$, and $\phi$ is unique modulo $2\pi$. Let $f$ denote the Hopf map. Then we have seen in the last chapter that

$$\mathbf{n} = f\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

and

$$-\mathbf{n} = f\begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}.$$

Let

$$\Psi_{\mathbf{n}} = \cos(\theta/2)\Psi_f + e^{i\phi}\sin(\theta/2)\Psi_s.$$

In this section we shall explain how $\Psi_{\mathbf{n}}$ can be transformed into a Dirac field with spin-up axis $\mathbf{n}$.

The idea is to consider what happens when we rotate the $\mathbf{n}$-axis to make it coincide with the $z$-axis. Corresponding to this, there is a unitary linear operator $T$ on $\mathbb{C}^2$, i.e., a $2 \times 2$ unitary matrix, which takes

$$\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \text{ to } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix} \text{ to } \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

We apply $T$ to the first two components of $\Psi_{\mathbf{n}}$ and, separately, to the last two components. This amounts to multiplying the column vector $\Psi_{\mathbf{n}}$ by the $4 \times 4$ matrix

$$M = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

. We will show that after rewriting appropriately, the result is an expression identical to $\Psi_f$. The rewriting consists of replacing the components of $\mathbf{R}$ by expressions in the new components. This can be done as soon as the new coordinate axes have been chosen.

In the operator $D_0$ which defines the Dirac equation, we have to change the matrices $\alpha_j$ and $\beta$ in such a way as to preserve the commutativity relations. So we replace $\alpha_j$ with $M\alpha_j M^{-1}$ and $\beta$ with $M\beta M^{-1}$. This yields a new Dirac equation of which $M\Psi_{\mathbf{n}}$ is a solution.

Now we analyse the effect of applying $T$. The first two components of $\Psi_{\mathbf{n}}$ form a constant multiple of $\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$. It therefore suffices to consider the effect on the last two components of $\Psi_{\mathbf{n}}$.

There is a useful simplification. The desired change of axis can be obtained by composing two computationally far simpler rotations as follows.

The vector $\mathbf{n}$ can first be taken into the $xz$-plane by a rotation through the angle $\phi$ about the $z$-axis. This amounts to assuming that $\theta = 0$. In this case, $R_z$ is unchanged and the polar forms for $R_x + iR_y$ and $R_x - iR_y$ help in finding the new components of $\mathbf{R}$.

Assume that this has been done. Then, by a rotation about the $y$-axis through the angle $\theta$, we can take $\mathbf{n}$ (which is now in the $xz$-plane) to the $z$-axis. In this case we have $\phi = 0$ and $R_y$ doesn't change. It helps to use the polar forms for $R_x + iR_z$ and $R_x - iR_z$.

# Chapter 5

# The Electromagnetic Field of an Electron

## 5.1 Introduction

This chapter is a description of a proof by Koga that for a free electron, the Dirac field that Koga describes is roughly equivalent to the Maxwell field, i.e., the electromagnetic field described by Maxwell's equations. This is only a rough equivalence but it has two goals for Koga: firstly a derivation of the magnetic moment of the electron, secondly a suggestion that one should try to look for equations that imply both the Dirac equation and Maxwell equations, as limiting cases for two different limits. Koga accomplished this in [9]; see also Chapter VI of [10], taking into account the internal gravitational field of the electron. This was the culmination of his theoretical work. He later applied this theory to various physical problems in the last four chapters of [10].

## 5.2 Background

We introduce the geometric or Clifford algebra of 4 dimensional spacetime, the study of which was begun by Dirac in the 1920s. This is a brief outline, sufficient for our purposes. It should not be considered an introduction to the subject (see [2] or [4]).

The Einstein summation convention applies. Greek indices range from 0 to 3 and Latin indices from 1 to 3.

### 5.2.1 Minkowski space

For us, Minkowski space (the spacetime of special relativity) is a 4 dimensional real space with coordinates $x^0 = ct$, $x^1$, $x^2$ and $x^3$. We assume that an origin has been chosen and a corresponding basis of unit vectors is given: $\gamma_0$, $\gamma_1$, $\gamma_2$ and $\gamma_3$. There is also a reciprocal basis: $\gamma^0 = \gamma_0$, $\gamma^i = -\gamma_i$ for $i = 1, 2, 3$. The metric is

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

### 5.2.2 The Geometric Algebra of Minkowski space

This algebra, also called the spacetime algebra, is the associative algebra generated by $\gamma_\mu, \mu = 0, 1, 2, 3$ (as above), with relations $(\gamma_0)^2 = 1$, $(\gamma_i)^2 = -1$ for $i = 1, 2, 3$ and $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ when $\mu \neq \nu$.
It is a 16 dimensional real vector space; we call its elements multivectors. A basis is

$$\{1\} \cup \{\gamma_\mu\} \cup \{\gamma_\mu\gamma_\nu \mid \mu < \nu\} \cup \{\gamma_\mu\gamma_\nu\gamma_\tau \mid \mu < \nu < \tau\} \cup \{\gamma_0\gamma_1\gamma_2\gamma_3\}.$$

This algebra contains Minkowski space as a subspace.

We call $I = \gamma_0\gamma_1\gamma_2\gamma_3$ the unit pseudoscalar. It satisfies $I^2 = -1$.

The subspaces of the spacetime algebra have dimension 1, 4, 6, 4 and 1 respectively; their members are called scalars, vectors, bivectors, trivectors and pseudoscalars.

The bivectors generate a subalgebra of dimension 8, which we call the even subalgebra. It is convenient to introduce new notation to deal with this subalgebra. Let $\sigma_i = \gamma_i\gamma_0$ for $i = 1, 2, 3$. Then a basis for the even subalgebra is

$$\{1\} \cup \{\sigma_i\} \cup \{\sigma_i\sigma_j \mid i < j\} \cup \{\sigma_1\sigma_2\sigma_3\}.$$

Note that ${\sigma_i}^2 = 1$, $\sigma_1\sigma_2\sigma_3 = I$ and $I\sigma_1 = \sigma_2\sigma_3 = \gamma_3\gamma_2$ with two more such relations obtained by cyclic permutation of the indices.

Suppose $\psi$ is a function defined on Minkowski space with values in the spacetime algebra. The vector derivative of $\psi$ is defined as

$$\nabla\psi = \gamma^\mu \frac{\partial\psi}{\partial x^\mu}.$$

### 5.2.3 The Dirac Equation in Geometric Algebra

We follow the scheme given in chapter 8 of [2] for mapping the 4 dimensional complex space, in which solutions to the Dirac equation take values, bijectively to the even subalgebra of the spacetime algebra, and replacing the action of the Dirac (gamma) matrices (and the complex number $i$) with multiplication by members of the spacetime algebra on the left and right. A similar scheme is used in [4]. An inertial frame is assumed. The Dirac (or Dirac-Hestenes) equation is

$$\hbar\nabla\psi I\sigma_3 = mc\psi\gamma_0.$$

### 5.2.4 Maxwell's Equation in Geometric Algebra

See chapter 7 of [2] (or [4]). The four Maxwell equations (in differential form) in vacuum reduce to the single equation

$$\nabla F = J$$

where $F = E + IB$ is called the electromagnetic field strength, $E$ is the electric field, $B$ for our purposes is the magnetic field (both are in the space spanned by $\sigma_1$, $\sigma_2$ and $\sigma_3$) and $J = J^\mu \gamma_\mu$ is a vector called the spacetime current density. Here $J^0$ is the charge density, denoted $\rho$ and $(J^i \gamma_i)\gamma_0 = J^i \sigma_i$ is the current density. The quantity $F$ is covariant under Lorentz transformations.

## 5.3 The Dirac Equation implies Maxwell's Equation

Consider the Dirac equation in Geometric algebra,

$$\hbar \nabla \psi I \sigma_3 = mc\psi\gamma_0.$$

We now replace $\psi$ with $\phi \exp\left(\dfrac{-I\sigma_3 mc^2 t}{\hbar}\right)$ in this equation.

This corresponds to a similar transformation made by Koga (with the complex number $i$ instead of the bivector $I\sigma_3$). The same transformation is also used by Gurtler and Hestenes [3] and Doran and Lasenby chapter 8 of [2] in different contexts. In all these cases, the idea is to factor out the fast oscillating or high-energy component of $\psi$ and leave a funtion, $\phi$, which varies relatievly slowly with time. This implies that if we use $I\sigma_3$, the field must spin around an axis parallel to the $\sigma_3$ axis.

We get a "Dirac equation" for $\phi$:

$$\hbar \nabla \phi I \sigma_3 = mc(\phi \gamma_0 - \gamma_0 \phi).$$

Like $\psi$, the function $\phi$ is also even-valued. Hence $\phi$ can be written as

$$\phi = \phi_0 + X + IY + \phi_4 I$$

where $\phi_0$ and $\phi_4$ are scalars and $X$ and $Y$ are bivectors (specifically, linear combinations of $\sigma_1, \sigma_2, \sigma_3$). Suppose

$$X = X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3,\ Y = Y_1\sigma_1 + Y_2\sigma_2 + Y_3\sigma_3$$

In order to compare the Dirac Equation (for $\phi$) and the Maxwell equation, we try to equate $F$ with some bivector obtained from $\phi$. Some possible choices are the bivector part of $\phi I \sigma_3$, $\phi\sigma_2$, $\phi I$ or $\phi$. It turns out that the second of these four gives sensible results.
We have

$$\phi\sigma_2 = \phi_0\sigma_2 + X\sigma_2 + IY\sigma_2 + \phi_4 I\sigma_2$$

with bivector part $[Y_3\sigma_1 + \phi_0\sigma_2 - Y_1\sigma_3] + I[-X_3\sigma_1 + \phi_4\sigma_2 + X_1\sigma_3]$, which we equate to $F = E + IB$. Equating the coefficients of the six bivector basis elements
$(\sigma_1, \sigma_2, \sigma_3, I\sigma_1, I\sigma_2, I\sigma_3)$, we get

$$E = Y_3\sigma_1 + \phi_0\sigma_2 - Y_1\sigma_3,\ B = -X_3\sigma_1 + \phi_4\sigma_2 + X_1\sigma_3.$$

The Dirac equation gives (using $(\sigma_3)^2 = 1$)

$$\nabla\phi\sigma_2 = -\frac{mc}{\hbar}(\phi\gamma_0 - \gamma_0\phi)\sigma_1$$

which can be rewritten as

$$\nabla(X_2 + [Y_3\sigma_1 + \phi_0\sigma_2 - Y_1\sigma_3] + I[-X_3\sigma_1 + \phi_4\sigma_2 + X_1\sigma_3] + Y_2 I) = \left(\frac{2mc}{\hbar}\right)(X + \phi_4 I)\,\gamma_1$$

. From this we subtract the Maxwell equation

$$\nabla F = J.$$

The result is

$$\nabla X_2 + \nabla I Y_2 = \left(\frac{2mc}{\hbar}\right)(X + \phi_4 I)\,\gamma_1 - J.$$

The left hand side of this equation has eight terms: four vector and four trivector terms. The right hand side also has only vector and trivector terms. Equating corresponding terms, we get the following scalar equations. Similar equations were obtained by Koga ([8], Chapter V of [10]).

$$\begin{aligned}
\frac{\partial X_2}{\partial x^0} &= \frac{2mc}{\hbar} X_1 - \rho, \\
\frac{\partial X_2}{\partial x^i} &= J^i, \quad (i = 1, 2, 3), \\
\frac{\partial Y_2}{\partial x^1} &= \left(\frac{2mc}{\hbar}\right) \phi_4, \\
\frac{\partial Y_2}{\partial x^2} &= \left(\frac{2mc}{\hbar}\right) X_3, \\
\frac{\partial Y_2}{\partial x^3} &= -\left(\frac{2mc}{\hbar}\right) X_2, \\
\frac{\partial Y_2}{\partial x^0} &= 0.
\end{aligned}$$

Thus, by taking $F = [Y_3\sigma_1 + \phi_0\sigma_2 - Y_1\sigma_3] + I[-X_3\sigma_1 + \phi_4\sigma_2 + X_1\sigma_3]$, we ensure that all the components of $\phi$ are closely related to the electromagnetic field.

Now assume that the field $\phi$ varies "slowly" with time. In other words, the time derivatives of the components of $\phi$ are small compared to the other terms. By assuming $\dfrac{\partial X_2}{\partial x^0} = 0$, we get

$$\frac{2mc}{\hbar} X_1 = \rho.$$

Assuming that the electron is a field which is nonzero only in a bounded region, we can integrate over all of $\sigma_1\sigma_2\sigma_3$ space to get

$$\frac{2mc}{\hbar} M = e$$

where $M$ is the $\sigma_3$ component of the moment of the magnetic field and $e$ is the electronic charge. Thus we get the correct expression for the Bohr magneton.

## 5.4 Conclusions

We have been able to show that from the Dirac equation for a free electron, an approximate derivation of Maxwell's equation can be given. This corresponds to Koga's arguments. The Maxwell field is accepted by the scientific community as something that really exists. The derivation in this paper suggests that Koga's Dirac field that we work with in this paper is equally real.

It may be worthwhile to find the electromagnetic field corresponding to Koga's explicit solution to the Dirac equation ([11], Chapter V and [18]).

We conclude by mentioning that for Koga, the work of his which we study here was a motivation for finding a set of equations which include the electron's gravitational field and imply the Dirac equation and Maxwell's equations as limiting cases under different conditions ([9] and Chapter VI of [10]).

# Chapter 6

# Concluding Remarks

## 6.1 Summary of the thesis

In this thesis we discussed a deterministic theory of the electron based on Toyoki Koga's work with the help of the Hopf map.

The thesis started by analysing a fundamental equation in quantum mechanics: the Schrödinger equation. We showed that the solution to the Schrödinger equation given by Koga yielded a de Broglie wave through an averaging process.

Next we considered the Pauli equation to deal with the spin of a free electron. Using the solution to the Schrödinger equation given by Koga, a solution to the Pauli equation for free electron could be obtained. With the help of the Hopf map, the spin axis was defined for a free electron. Even though this spin axis has been referred to in the literature, nobody gave an explanation for assigning the spin axis. In this thesis, we could explain what is the relation between the spin axis assigned to a free electron and the spin angular momentum vector.

The properties of the solution to the Dirac equation given by Koga were discussed in this thesis. It was shown that the solution to the Dirac equation contains a term which represents a rotating field. We have proved that the Hopf map can be used to obtain a Dirac electron field with arbtrarily chosen spin axis in $\mathbb{R}^3$.

Finally, we discussed Koga's rough comparison between the Maxwell field and the Dirac field of an electron using Geometric Algebra, which greatly clarified his argument.

In his papers and books, Koga devoted considerable space to his difficulties with the foundations of quantum mechanics which were the reasons for his taking a new path. We did not touch upon any of these issues here.

## 6.2 Some Open Questions

- Remove the singularity in $a = \exp(-\kappa r)/r$, i.e., get a better expression for the function $a$ which occurs in all the solutions given by Koga.

- How to take the non-relativistic limit of the solution to the Dirac equation given by Koga?

- Understand the solutions to the Dirac equation containing $Ec-mc^2$.

- Understand the significance of the term containing $R_z$ in the solution of the Dirac equation.

- Understand how the spin theories of Pauli and Dirac, involving spinors, emerge from the theories given here based on Koga's work.

We have examined some mathematical issues arising in Koga's work, especially concerning electron spin. We hope this thesis will help to convince the community of physicists that Koga's ideas are worth studying.

# Appendix

## The questions raised by an examiner are answered here

| S.No | Question | Answer |
|---|---|---|
| 1. | In chapter 2, for a free particle the de Broglie wave is obtained via an ensemble of elementary field solution of the Schrödinger equation in page 9. Does this method extend to the case of the harmonic oscillator, or the hydrogen atom? | Koga discusses the hydrogen atom in subsection 5.2 of his paper [13] and, in more detail, in subsection 4.5.a of his book [10]. He discusses the one-dimensional oscillator in subsection 5.3 of the above paper and, in more detail, section 4.7 of the book mentioned above. He concludes that "it is unavoidable to suspect that the one-dimensional oscillator is a phenomenon fictitiously formulated in order to conclude, as closely as possible, Planck's formula $E = n\hbar\nu$". |

| | | |
|---|---|---|
| 2. | End of page 7,<br>What is $m$ in $mc = h\nu/c$ ? | The last sentence of page 7 has been rewritten as "Einstein proposed that radiation also travels as quanta, each quantum being a particle of energy $h\nu$ and momentum $p = h\nu/c$ where $c$ is the velocity of light". |
| 3. | Koga's elementary field seems to be square integrable and it can be expressed as Fourier integral over plane curves.<br>What does this imply for the proposed framework? | Koga's elementary field suggests that a "particle" such as an electron is a field in a small region of space. The region is spherical for a free particle but gets distorted by external fields and resists the distortion leading to stable atoms. See his paper [14]. More details are in section 4.3 of his book [10]. Square integrability does not seem to be significant. We are interested in individual particles, not ensembles. |

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| 4. | Can each component of the spin has any real value in the range $(-\hbar/2, \hbar/2)$?<br>The Stern Gerlach experiment seems to show only $\pm\hbar/2$ are possible.<br>Do all components of spin have definite values simultenously? Do they not anticommute? | A deterministic interpretation of the Stern-Gerlach experiment follows. Any electron has a unique spin axis at any time. Electrons enter the apparatus with different spin directions. The torque due to the magnetic field rotates each electron and when it leaves, its spin can only have one of two values (directions). The Pauli matrices represent orthogonal unit vectors in $\mathbb{R}^3$. Thus the spin is a vector with numerical components in all directions, simultaneously. See the book by Doran and Lasenby [2], page 273 for a more conventional explanation assuming that the spin measurement apparatus is actually a spin polariser rather than a measurer. |
| 5. | Section 5.2.1. The text says $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are unit vectors in $\mathbb{R}^4$. Then how do we get $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ for $\mu \neq \nu$? | The relation is a definition introduced by Hestenes to encode the invariance of the metric of special relativity in the Spacetime Algebra. See [2], Section 5.1. |

| | | |
|---|---|---|
| 6. | Page 23. First sentence: why 'apparently'? | In his book [10], Koga derives the relation $M_z = (\hbar/2mc)e$ in section 5.4 and concludes that "the relation $M_z = (\hbar/2mc)e$ implies that a field governed by the Dirac equation has a peculiar regularity or order that can be introduced to a field satisfying the Maxwell-Lorentz equations either by a particular boundary condition or by a particular source, i.e., Bohr Magneton. These observations suggest that neither an electromagnetic field nor a field satistying the Dirac equation may fully represent the real field of the electron".<br>That is why we have written 'apparently' in the first sentence. |

| | | |
|---|---|---|
| 7. | Page 38. For the vacuum Maxvwell equation, Should we not have $J = 0$? | We are concerned with points in the interior of the electron. The electron is a field and is not "matter" in the usual sense. So we consider it a vacuum. It is assumed that it is electrically charged. Since the Dirac field has a rotating component, we assume that charges are rotating, producing magnetic fields. So we take $J$ to be nonzero. |
| 8. | There are problems with physical dimensions which should be corrected.<br>Page 24 first equation<br>Page 25 $S/\hbar$ is not dimensionless.<br>Pages 27,28,29. $Ec + mc^2$ is not meaningful. | The first equation on page 24 is the Klein-Gordon equation and has been corrected.<br>In Koga's treatment of the Dirac equation (but not the Schrodinger equation) which is studied in this thesis, the energy of an electron is $cE$ rather than $E$. This should be kept in mind. Therefore there are no problems with physical dimensions on pages 25, 27, 28 and 29. |

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# Publications

## Papers Published and Communicated

- Didimos K. V. and R. S. Chakravarti, Deterministic Electron Spin, *Annales de la Fondation Louis de Broglie* **42**, 299-306 (2017).
- Didimos K.V. and R.S. Chakravarti, The de Broglie Wave as a Representation of an Ensemble (communicated).
- Didimos K.V. and R.S. Chakravarti, Deterministic Dirac Theory (under preparation).

## Papers presented

- *Didimos K.V. and R.S. Chakravarti*: **Deterministic Electron Spin**, International Conference on Emerging Trends in Mathematics and Computer Science, at St. Thomas College, Thrissur, 13-14 December 2016.

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