THERMODYNAMICS AND GEOMETROTHERMODYNAMICS OF BLACK HOLES IN MODIFIED THEORIES OF GRAVITY

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Thermodynamics and Geometrothermodynamics of Black holes in Modified Theories of Gravity
PhD thesis in the field of Black Hole Physics

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"In this house, we obey the laws of thermodynamics!"

Dan Castellaneta.

Department of Physics,
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CERTIFICATE

Certified that the work presented in this thesis is a bonafide research work done by Mr. Jishnu Suresh, under our guidance in the Department of Physics, Cochin University of Science and Technology, Kochi- 682022, India, and has not been included in any other thesis submitted previously for the award of any degree. All the relevant corrections and modifications suggested by the audience during the pre-synopsis seminar and recommendations by doctoral committee of the candidate have been incorporated in the thesis.

Kochi-22
June, 2016

Prof. V. C. Kuriakose
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Prof. Ramesh Babu T.
(Joint Guide)

DECLARATION

I hereby declare that the work presented in this thesis is based on the original research work done by me under the guidance of Prof. V. C. Kuriakose (Rtd.) and Prof. Ramesh Babu T., Department of Physics, Cochin University of Science and Technology, Kochi-682022, India, and has not been included in any other thesis submitted previously for the award of any degree.

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Contents

Preface

Thermodynamics was born in the nineteenth century along with the industrial revolution and it deals with macroscopic description of energy conversion involving heat and other forms of energy. The great generality of thermodynamics is due to the fact that its axioms, called the laws, are based on well established empirical results and are applicable to most of the systems in nature. In spite of its generality, some aspects of thermodynamics remain to be uncertain and are not evident in the classical description, as for example the non-extensive systems in thermodynamics. In this regard, many attempts have been made to describe the thermodynamic behaviour of the systems using the language of geometry, from the pioneering works of Gibbs and Caratheodory. In all these descriptions, contact geometry of thermodynamic phase space and Riemannian geometry of the equilibrium space have been explored. Understanding the geometric description of thermodynamics received great attention because of many reasons. Among them, the important one is that, geometry provides a language that can be used to describe the fundamental theories of physics and can be used to infer the new physical ideas that exhibits by the system. The discovery of the analogue between the laws of thermodynamics and those of black hole mechanics by Bekenstein and Hawking has received the attention of many researchers. These observations led to the idea of regarding the existence of a strong connection between geometrical aspects of gravitation and their thermodynamic equivalence, like, the area of the event horizon and the entropy of a black hole, commonly referred to as Bekenstein-Hawking area theorem. Hence it is believed that a deeper understanding of the geometrical description of the thermodynamics of black holes could shed some light on the relationship among thermodynamics, gravity and quantum mechanics. On the other hand, the shortcomings of Einstein's general relativity made the theory to undergoing many explorations and accordingly modifications are introduced into the formalism. Modified theories of gravity are among these efforts which shed light in to such a development. Thermodynamic aspects of black holes in such modified theories will definitely help the scientific community to perfectly model a theory of gravity that withstand all the observational tests as well as they may give information regarding the microscopic origin of entropy and quantum gravity.

The subject of this thesis work is an attempt to provide a general and consistent way of studying the geometric description of black holes in modified theories of gravity and their thermodynamic properties and interactions through a new geometric approach, known as Geometrothermodynamics (GTD).

In **chapter 1** of the thesis, we will present a brief review of Einstein's General theory of relativity and modified theories of gravity with a special emphasis on Hořava-Lifshitz gravity and Massive gravity. The black hole solutions in both these theories are discussed. Moreover, a detailed introduction to black hole thermodynamics, entropy spectrum of the black holes as well as the Geometrothermodynamics of the system are also given.

The thermodynamics of black holes in modified theories of gravity is discussed in **chapter 2**. Black hole solutions in Hořava-Lifshitz gravity, like Kehagias-Sfetsos, Lü-Mei-Pope and Park solutions are studied in this chapter. We have also studied different black hole solutions in Massive gravity that include the dRGT black hole and (2+1) BTZ black hole in New Massive Gravity. Their thermodynamic properties have been analyzed in detail with a special emphasis on phase transitions shown by these black holes. We have obtained the variation of temperature, mass and heat capacity of these black hole systems. In KS black hole and LMP black hole cases, it is evident from the existence of an infinite discontinuity in the heat capacity diagram that the systems exhibits phase transitions. The transition is from a positive heat capacity phase to a negative heat capacity phase, change in sign of heat capacity in turn indicates the transition from a thermodynamically stable phase to an unstable phase. For the Park black hole case also there exists a phase transition and this solution exhibits different anomalous behaviours in the form of negative temperature and existence of mass bound. In dRGT massive gravity black hole solution, we obtained Reissner-Nordstorm black hole solution in de Sitter and anti de Sitter space-time by varying the coupling parameters of the theory. One expects that thermodynamics of these black holes would be the same as in general relativity, taking into account that massive gravity differs from general relativity by a non-derivative coupling to a fiducial metric. But the present studies show that, even though the results agree with general relativity when massive parameter tends to zero, there are significant changes in the phase transition structure. Whereas in the BTZ black hole case in new massive gravity, black hole does not possess an infinite discontinuity transition, but shows a continuous transition from a thermodynamically stable to an unstable phase.

In **chapter 3**, we have analyzed the entropy spectrum of these black hole solution in modified theories of gravity. The method suggested by Majhi, Vagenas, Jiang and Han, in which they have incorporated the ideas form the adiabatic invariance, tunneling mechanism, Bohr-Sommerfeld quantization rule and near horizon approximations, is used to calculate the entropy spectrum of these black holes. For the black hole case in Hořava-Lifshitz gravity, the entropy spectrum is equally spaced and are independent of black hole parameters. On the other hand for black hole in Massive gravity, particularly in the charged BTZ solution, the entropy spectrum depends on the black hole parameters, even though the spectrum is equispaced. This may shed some light in to the microscopic origin of entropy.

Chapter 4 is devoted for the discussion of the new geometric approach, Geometrothermodynamics, proposed by Hernando Quevedo as applied for the case of black holes mentioned earlier. Just as in general relativity the physical reality cannot depend on a particular choice of coordinates, thermodynamics is independent of the potential one uses to describe a given system. Hence, Legendre invariance should be an essential ingredient of any geometric construction of thermodynamics. GTD formalism preserves this Legendre invariance. Usual black hole thermodynamic studies discussed in Chapter 2 show the presence of phase transition in black hole systems of modified theories of gravity. But these calculations were unable to confirm the order of phase transition. Even though one can eliminate the possibility of first order phase transition from temperature variation, but divergence in heat capacity can be of second or higher orders in nature. Here in GTD formalism, thermodynamic interaction can be reflected from the curvature of the metric defined on equilibrium spaces. If thermodynamic curvature is free of singularities, then GTD interprets it as non-existence of singular points at the level of the heat capacity and no (second order) phase transitions occur in the system. Interestingly the curvature of the GTD metric reflects all abnormalities shown by the system in terms of negative temperature, mass bound, etc. The phase transition structure of black hole solutions in Hořava-Lifshitz and Massive gravity is studied in this chapter. Whenever the system shows a phase transition, the scalar curvature exactly reproduces the behaviour and as a result, the order of phase transition can also be verified.

Finally, in **chapter 5** a summary of the new results are presented and a possible applications and future developments are also discussed. It is expected that this work will be a concrete step towards a clearer description of thermodynamics of black holes in modified theories of gravity and hopefully a deeper understanding of the relation between thermodynamic geometry and the thermodynamics of black hole systems.

List of Publications

1. “Area spectrum and thermodynamics of KS black holes in Hořava gravity”, Jishnu Suresh and V. C. Kuriakose, Gen. Relativ. Gravit (2013) 45: 1877.

2. “Thermodynamics and Quasinormal modes of Park black hole in Hořava gravity”, Jishnu Suresh and V. C. Kuriakose, Eur. Phys. J. C (2013) 73: 2613.

3. “The thermodynamics and thermodynamic geometry of the Park black hole”, Jishnu Suresh, R. Tharanath, Nijo Varghese and V C Kuriakose, Eur. Phys. J. C (2014) 74: 2819.

4. “A unified thermodynamic picture of Hořava-Lifshitz black hole in arbitrary space time”, Jishnu Suresh, Tharanath R and V C Kuriakose, JHEP 01 (2015) 019.

5. “Thermodynamics and Geometrothermodynamics of Charged black holes in Massive Gravity”, Jishnu Suresh , C P Masroor, Geethu Prabhakar and V. C. Kuriakose, arXiv:1603.00981 (Communicated for publication).

6. “Entropy spectrum of BTZ black hole in massive gravity”, Jishnu Suresh and V. C. Kuriakose, arXiv:1605.00142 (Communicated for publication).

7. "Geometrothermodynamics of BTZ black hole in new massive gravity", Jishnu Suresh and V. C. Kuriakose, arXiv:1606.06098 (Communicated for publication).

8. "Thermodynamic Geometry of Reissener-Nordstörm de Sitter black hole and its extremal case", R. Tharanath, Jishnu Suresh, Nijo Varghese and V C Kuriakose, Gen. Relativ. Gravit (2014) 46: 1743.

9. "Phase transitions and geometrothermodynamics of regular black holes", R. Tharanath, Jishnu Suresh and V C Kuriakose, Gen. Relativ. Gravit (2014) 47 (4), 1-20.

10. "Thermodynamics of Charged Lovelock-AdS Black Holes", C. B. Prasobh, Jishnu Suresh and V. C. Kuriakose, Eur. Phys. J. C (2016) 76:207

11. "Entropy spectrum of (1+1) dimensional stringy black holes", Jishnu Suresh and V C Kuriakose, Eur. Phys. J. C (2015) 75:214.

This thesis is based on the publications numbered [1,2,3,4,5,6,7].

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1
Introduction

1.1 General theory of relativity and Black holes

The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time.

-S. Chandrashekar.

In 1687 Newton published the world famous Philosophi Naturalis Principia Mathematica [1]. He showed a mathematical treatment of gravity, which resulted in the well known formula for gravitational forces:

$$F = G\frac{m_1 m_2}{r^2} \ . \tag{1.1}$$

However this way of understanding gravity turned out to be only a part of the whole story of gravitation. In this part of the story, Newton's law of universal gravitation seemed to account for the motion of all the planets as well as their satellites. But, when the orbit of Uranus was found, it became evident that there exists some irregularities that could not be understood or explained using Newton's law of gravitation. This discrepancy led to the discovery of a new planet named Neptune. A similar discrepancy between observation and Newton's law arose in the case of planet Mercury. The perihelion of the trajectory precesses more slowly than the result expected from Newton's Law of gravitation. In order to explain this scenario, it was suggested that a planet named Vulcan caused this discrepancies. The majority of astronomers believed this, but when convincing evidence could not be found, astronomers started to rethink their opinion and doubted the idea of an unobserved planet. This failure of Newton's law remained unexplained until Einstein proposed a new law for gravity: the theory of general relativity, about 200 years later in 1915 [2]. One of the major challenges for general relativity was to explain this anomaly - the precession of the perihelion of the trajectory of Mercury. Einstein calculated the orbit using his theory of general relativity, and found that it could predict the observed precession of the perihelion of the trajectory of Mercury without introducing any new objects in the solar system. This theory of general relativity is the other part, major part of the story. Einstein's theory explains that gravitation isn't a force at all, but is the effect of distortion of the four dimensional space-time under consideration by the mass presented in it. Comparing these two independent theories, Newton's theory allowed information to travel at infinite speeds, whereas Einstein showed that nothing can exceed the speed of light, not even gravity. As a result of these predictions, general relativity got transformed as the cornerstone of our theoretical knowledge of the gravitational interaction, and its predictions are in excellent agreement with all weak-field experiments. This geometric theory of gravity unified the description of gravity as a property of space-time. In general relativity, curvature of space-time is directly related to the stress-energy tensor through the Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \ . \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor,

$T_{\mu\nu}$ is the energy-momentum tensor of matter present in the universe and the constant $\kappa = \frac{8\pi G}{c^4}$, where G is the gravitational constant and c is the speed of light. Among the predictions made by this theory, the most striking one among them, is the existence of black holes. Black holes are the most mysterious exotic entities encountered in physics of the present time. They are solutions to the classical theories of gravity coupled to matter, which are characterized by the existence of an event horizon. The verbal definition of the black hole: *it is a region of space-time surrounded by a boundary known as the event horizon inside which the force of gravity is so strong that not even light can escape, hence it is invisible.* Physically, a black hole is defined as a region where gravity is so strong that nothing can escape. When a sufficiently large quantity of matter is compactified into a small region, a space-time singularity occurs; black holes describe the endpoints of gravitational collapse.

When one revisits the history of black hole, it can be found that the existence of black hole was predicted back in 1784 by John Michell, who discussed classical bodies which have escape velocity greater than the speed of light. Later in 1875, Pierre Laplace obtained the gravitational radius in the scenario of Newtonian gravity. But exact theoretical solution was obtained in 1916, when Karl Schwarzschild [3] solved the Einstein field equation (1.2) in vacuum for a spherically symmetric uncharged distribution of matter. For that he considered Minkowski space-time whose line element in general is given by,

$$ds^2 = A\ dt^2 - (B\ dr^2 + C\ r^2 d\theta^2 + D\ r^2 \sin^2\theta d\phi^2)\ . \tag{1.3}$$

Solving the equation by considering the gravitational effects and weak field approximations he arrived at the most frequently written-down metric in black hole physics, known as the Schwarzschild metric, given by,

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(\frac{1}{1 - \frac{2GM}{c^2 r}}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\ . \tag{1.4}$$

The above metric which is the solution Einstein field equation describes nothing but a spherical massive object, a black hole. In (1.4) r, θ and ϕ are spherical polar coordinates; M is the mass of the corresponding black hole known as the Schwarzschild black hole. For each black hole there associates an event horizon, a boundary in space-time, beyond which events inside this boundary cannot affect an outside observer. For the Schwarzschild black hole system, the event horizon can be obtained by solving the corresponding metric (1.4) and is given by, $r = 2M$.

In 1918, Reissner and Nordström [4, 5] independently considered an electro-vacuum solution of the Einstein field equation (1.2) and obtained a black hole solution, which is spherically symmetric and electrically charged. It is usually referred to as the Reissner-Nordström black hole (RN). This RN black hole solution can be obtained by considering the Einstein field equation coupled to Maxwell's equations. Hence the energy momentum tensor $T_{\mu\nu}$ of (1.2) is given by,

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda} \ , \tag{1.5}$$

where $\nabla_{\mu}F^{\mu\nu} = 0$. Solving these equations, one arrives at the Reissner-Nordström black hole metric as,

$$ds^2 = -f(r)dt^2 - \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ , \tag{1.6}$$

where,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \ , \tag{1.7}$$

and M and Q are mass and charge of the RN black hole respectively. Now the event horizon of the RN black hole is obtained by solving for $f(r) = 0$, and is given by,

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}, \tag{1.8}$$

where r_+ is the outer horizon and r_- denotes the Cauchy horizon.

Not much studies have been done in the direction of black hole solution until Roy Kerr found a solution to Einstein field equation in vacuum in 1963, about four decades later. He found an uncharged rotating black hole solution named as Kerr black hole solution [6]. Kerr black hole metric with a nonzero angular momentum can be written as,

$$\begin{aligned} ds^2 = & -\frac{\Delta - a^2 sin^2\theta}{\rho^2} dt^2 - 2a\frac{2Mr sin^2\theta}{\rho^2} dt d\phi \\ & + \frac{(r^2+a^2)^2 - a^2\Delta sin^2\theta}{\rho^2} sin^2\theta d\phi^2 \\ & + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \ , \end{aligned} \tag{1.9}$$

where M and a are mass of the black hole and angular momentum respectively and,

$$\Delta = r^2 - 2Mr + a^2 \ , \tag{1.10}$$

$$\rho^2 = r^2 + a^2 cos^2\theta \ . \tag{1.11}$$

Kerr black hole can be considered as the generalization of Schwarzschild black hole or in other words Kerr solution reduces to Schwarzschild solution when the angular momentum reduces to zero. From the metric, its event horizons are at,

$$r_\pm = M \pm \sqrt{M^2 - a^2} \ . \tag{1.12}$$

An electrically charged rotating black hole solution was obtained by solving electro-vacuum field equations in 1965 by Newman [7]. By adding electric charge to Kerr black hole, one can obtain a new solution known as Kerr-Newman black hole solution. The Kerr-Newman black hole space-time is similar to the Kerr black hole space-time except in the definition of Δ, given by,

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \ . \tag{1.13}$$

Like the Kerr black hole, one can obtain the event horizons at,

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2 - a^2}\ . \tag{1.14}$$

It is interesting to note that the Kerr-Newman black holes are the largest family of black holes in four dimensional space-time.

1.2 Modified theories of gravity and Black holes

There is no wonder that Einstein's general relativity, which gives an accurate model for describing macroscopic gravitational interactions and the framework of relativistic quantum field theory (QFT), upon which the standard model of particle physics is built, are among the most successful achievements in the history of science. Decade long efforts to incorporate these two into a unified theory led us to the conclusion that for such a theory one has to change the view of the universe at the fundamental level. When one views this problem on the theoretical side, it can be inferred that in the weak field regime, the characteristic energies are low and quantum effects do not come in to play. But, when the energy increases, at some point quantum mechanics will come in to play and as a result quantum effects starts to dominate. As we know that general relativity is a classical theory of gravity that does not incorporate the effects of quantum mechanics, hence it is impossible to explain what was going on at energies as high as the Planck energy or at distances as small as the Planck length. On the other hand, on the observational side, there are many regions of the sky where the mass content expected from general theory relativity exceeds the mass estimated from astronomical observations. It is believed that such discrepancies can be accounted by considering the presence of an unknown form of mass, known as dark matter. Its presence is evident from many observations, including motions of galaxies and gravitational lensing. So the general relativity prediction fails and hence the gravity laws should be modified. Another discrepancy in the case of general relativity is regarding the universe at large scales. Observations of distance versus redshift for type Ia supernovae indicate that the universe is now expanding in an accelerated fashion. It is confirmed by the measurements done on cosmic microwave background radiation (CMBR). This accelerated expansion of the universe demands a positive cosmological constant to exist. The measured value of cosmological constant is many orders of magnitude ($\sim 10^{-120}$) smaller than the estimated value in quantum field theoretic calculations. This difference can not be incorporated in Einstein's general relativity. These issues related to general relativity point towards the incompleteness in our understanding of either matter or gravity, or both. So there might be some unknown entity of matter that is remaining hidden, that makes the theory of general relativity incomplete. Perhaps we have to change our understanding of both matter and gravity. These thoughts act as the main motivation for many investigations in theoretical high-energy physics going on now. Modified theories of gravities are among these efforts which shed light in to such a development. In the history of modified theories of gravity, a number of models were proposed, which includes Lovelock gravity, Einstein-Yang-Mills theory, $f(R)$ theories, Massive gravity, Hořava gravity, etc [8, 9]. Our investigations will involve Hořava Gravity and Massive gravity. This thesis is based on the studies of the black hole solutions and the resulting physics in these two models.

1.2.1 Hořava-Lifshitz gravity

Hořava gravity attempts to create a renormalizable theory of gravity by giving up Lorentz invariance. The model, proposed in 2009, rapidly gained a large amount of interest. Here, we review the theory and the associated literature regarding Hořava gravity. Hořava gravity is a non-relativistic modification to general theory of relativity, proposed by Petr Hořava in 2009 [10–12] in which the construction of the theory is based on the observations on a class of models known in the condensed matter physics referred to as the Lifshitz scalar field theory. The first ingredient in the discussions regrading Hořava gravity is its anisotropic scaling relation,

$$x \to bx, \qquad t \to b^z t; \tag{1.15}$$

where b is a constant and the degree of anisotropy characterized by z, the dynamical critical exponent. It measures the degree of anisotropy between space and time coordinates and it takes the value $z = 1$ in relativistic theories. Hořava's theory is based on the basic assumption that the Lorentz symmetry is fundamentally broken at high scales of energy and restores only in the infrared (IR) limit. Since Hořava gravity theory gained the basic ideas from Lifshitz scalar field theory, one can easily arrive at different properties of the theory. Among them, one is when $z > 1$, this model does not respect Lorentz symmetry which requires space and time to have the same dimension. Second, when $z = d$, where d is the dimension of space-time, the scalar field itself is dimensionless. This indicates that the theory might be UV complete. It is important to note that these properties are not based on any symmetry of the action, but the solutions of the theory respect this scaling. This relation clearly singles out time coordinate, which not only breaks diffeomorphism invariance (which can be seen as a statement of coordinate independence) but forces a further distinction between space and time absent in relativistic physics. From the experimental tests conducted on Lorentz invariance it becomes evident that, it is not necessary to consider Lorenz invariance as a fundamental principle of nature, and hence it is not a problem to effectively consider the violation of Lorentz invariance at scales of our interest. For the time being we leave z as unspecified, but later we will choose it appropriately such that gravitational coupling is dimensionless. Hořava defined the new theory of gravity, Hořava-Lifshitz gravity, by a path integral,

$$\int \mathcal{D}g_{ij}\,\mathcal{D}N_i\,\mathcal{D}N\ \exp\{iS\}\ . \tag{1.16}$$

Here $\mathcal{D}g_{ij}\,\mathcal{D}N_i\,\mathcal{D}N$ denotes the path integral measure and S is the most general action compatible with the requirements of gauge symmetry. He has used the ADM formalism in which the four dimensional metric of general relativity is parametrized as,

$$ds_4^2 = -N^2dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\ , \tag{1.17}$$

where N and N^i denote the lapse and shift functions, respectively. In the UV region, the action of Hořava-Lifshitz theory can be written [10] as,

$$S = \frac{1}{16\pi G}\int dt\ d^3\mathrm{x}\ \sqrt{g}N\{(K_{ij}K^{ij} - \lambda K^2) - \frac{1}{k_W^4}C_{ij}C^{ij}\}\ . \tag{1.18}$$

Here the first two terms represent the kinetic term and the last term is the potential term. As well as G is Newton's gravitational constant, R is the curvature scalar and K_{ij} is the extrinsic curvature that takes the form,

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \tag{1.19}$$

here the dot denotes differentiation with respect to the time coordinate t. Considering the potential term in (1.18), C_{ij} represents the Cotton tensor and it has the form,

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R^j_l - \frac{1}{4}R\ \delta^j_l\right) = \epsilon^{ikl}\nabla_k R^j_l - \frac{1}{4}\epsilon^{ikj}\partial_k R, \tag{1.20}$$

where $\kappa^2,\ \mu,\ \omega,\ \lambda$ and Λ are constants. Varying the action with respect to lapse and shift functions will yield equation of motion of the system, and solving that will add further information to the theory.

The previous discussion of Hořava-Lifshitz is based on UV action of the theory. Now we will consider the full action containing the description of theory in UV and IR regions. In this scenario, the detailed balance principle restricts the action to take the form [11],

$$\begin{aligned} S &= \int dtd^3\mathrm{x}\sqrt{g}N\{(K_{ij}K^{ij}-\lambda K^2)-\frac{1}{k_W^4}C_{ij}C^{ij} \\ &+\frac{\mu}{k_W^2}\epsilon^{ijk}R_{il}\nabla_j R^l_k-\frac{\mu^2}{4}R_{ij}R^{ij} \\ &+\frac{\mu^2}{4(1-3\lambda)}(\frac{1-4\lambda}{4}R^2+\Lambda_W R-3\Lambda_W^2)\}, \end{aligned} \tag{1.21}$$

with the emergent speed of light and cosmological constant term can be extracted as,

$$c=\frac{\mu}{2}\sqrt{\frac{\Lambda_W}{1-3\lambda}}, \qquad \Lambda=\frac{3}{2}\Lambda_W. \tag{1.22}$$

In this thesis, we will explore some of the black hole solutions existing in Hořava-Lifshitz gravity and will study their thermodynamics, spectroscopy as well as their thermodynamic geometry aspects.

1.2.2 Massive gravity

We know that general relativity describes nonlinear self interactions of a massless spin 2 excitations. A most natural way of modifications of general theory of relativity would be by adding a mass term for the spin 2 field. As a result, the modified theory would explain the nonlinear interactions of a massive spin 2 field, and this theory is known as Massive gravity. In 1939, Fierz and Pauli [13] considered the modification by adding mass to a linearized theory of gravity. The proposed theory was unique in such a way that there were only a single way to add mass term so that the theory becomes physically significant. Later in two independent articles, van Dam and Veltman as well as Zakharov claimed that Fierz-Pauli theory does not converge to Einstein's theory in the zero mass limit [14, 15]. Later in 1972, Vainshtein introduced a new mechanism to overturn the above mentioned vDVZ discontinuity [16]. In this mechanism he considered the full non linear formulation of massive gravity and as a result in the zero mass limit Einstein's original equations are retrieved. In the same year, Boulware and Deser proved that any non linear massive gravity theory which uses the Vainshtein's mechanism contain a 'ghost' [17, 18], known as the BD ghost. The presence of BD ghost remained as an unsolved problem until 2010, when de Rham, Gabadadze and Tolley (dRGT) proposed the first non linear completion of the FP theory free of BD ghost instability [19, 20]. They showed that the potential to be ghost free up to the quartic order in perturbation and to all orders in decoupling limit, and as a result of this, many extensions of this theory have been made [21–28]. Besides these extensions, alternative theories with massive graviton have also been under rigorous investigations. These theories include the DGP model [29, 30], Kaluza-Klein models [31, 32], New massive gravity [33] and Topological Massive gravity [34, 35].

dRGT massive gravity

The dRGT massive gravity model can be described using the action [19, 20],

$$S = \int d^D x \left[\frac{M_{pl}^2}{2} \sqrt{-g} \left(R + m^2 \, U(g, H) \right) \right], \tag{1.23}$$

where the first term is the usual Einstein-Hilbert action and the second term is arising from the contributions of mass of the graviton m, and from the nonlinear higher derivative term U corresponding to the massive graviton. It is given by

$$U = U_2 + \alpha_3 U_3 + \alpha_4 U_4, \tag{1.24}$$

where,

$$\begin{aligned} U_2 &= [K]^2 - [K^2] \\ U_3 &= [K]^3 - [K][K^2] + 2[K^3] \\ U_4 &= [K]^4 - 6[K]^2[K^2] + 8[K^3][K] - 6[K^4]. \end{aligned} \quad (1.25)$$

In the above set of equations, the tensor K^μ_ν is defined as,

$$K^\mu_\nu = \delta^\mu_\nu - \sqrt{\partial^\mu \phi^\alpha \partial_\nu \phi^\beta f_{\alpha\beta}}, \quad (1.26)$$

where ϕ^α and ϕ^β are the corresponding Stuc̆kelberg field and $f_{\alpha\beta}$ is a fixed symmetric tensor usually called as the reference metric.

In the unitary gauge, defined as $\phi^a = x^a$, the term $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the gravitational analogue of the Proca field of massive electrodynamics [36]. By introducing the Stückelberg field ϕ^a, which can be considered as background field plus a pion contribution, $\phi^a = x^a + \pi^a$ [37], and replacing the Minkowski metric by,

$$g_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} + H_{\mu\nu},$$

where $H_{\mu\nu}$ is the covariantized metric perturbation, one can restore the diffeomorphism invariance. As given in [36, 37], two new coefficients α and β are introduced which relate the coefficients α_3 and α_4 in (1.24) by,

$$\alpha_3 = -\frac{(-\alpha + 1)}{3}, \quad (1.27)$$

and

$$\alpha_4 = \frac{-\beta}{2} + \frac{(-\alpha + 1)}{12}. \quad (1.28)$$

In empty space, the equation of motion is given as,

$$G_{\mu\nu} + m^2 X_{\mu\nu} = 0, \quad (1.29)$$

where $X_{\mu\nu}$ is the effective energy-momentum tensor contributed by the graviton mass m, which is given by,

$$\begin{aligned} X_{\mu\nu} &= -\frac{1}{2}\Bigg[Kg_{\mu\nu} - K_{\mu\nu} + \alpha\left(K^2_{\mu\nu} - KK_{\mu\nu} + \frac{1}{2}g_{\mu\nu}([K]^2 - [K^2])\right) \\ &\quad +6\beta\left(K^3_{\mu\nu} - KK^2_{\mu\nu} + \frac{1}{2}K_{\mu\nu}([K]^2 - [K^2])\right. \\ &\quad \left.\left. -\frac{1}{6}g_{\mu\nu}([K]^3 - 3[K][K^2] + 2[K^3])\right)\right] . \end{aligned} \tag{1.30}$$

Now applying the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$ in (1.29), we arrive at the constraint equation,

$$m^2\nabla^\mu X_{\mu\nu} = 0. \tag{1.31}$$

As given in [36–38], we concentrate on a particular family of the ghost-free theory, for that we will assume,

$$\beta = -\frac{\alpha^2}{6}. \tag{1.32}$$

For this particular choice, (1.31) is automatically satisfied for a certain and time-independent metrics in spherical polar coordinates. Using (1.29), (1.30) and (1.31), a spherically symmetric and time independent metric in de Sitter space can be obtained, by choosing,

$$m^2 X_{\mu\nu} = \lambda g_{\mu\nu}, \tag{1.33}$$

where λ is a constant. Here, in this thesis we will explore the solutions of this dRGT model of massive gravity.

New massive gravity

Independently of the previously discussed massive gravity models in four dimensions, there has been interest in developing a three dimensional theory of massive gravity. Such a theory developed is referred to as New

Massive Gravity (NMG). This three dimensional higher derivative gravity model was proposed by Bergshoeff, Hohm and Townsend in 2009 [33]. Their action can be written as a higher curvature term in addition to the usual Einstein-Hilbert action,

$$S_{\rm NMG} = S_{\rm EH} + S_{\rm FOT} \ , \tag{1.34}$$

$$S_{\rm EH} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\ (R-2\lambda)\ , \tag{1.35}$$

$$S_{\rm FOT} = -\frac{1}{16\pi G m^2}\int d^3x\sqrt{-g}\ \left(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2\right)\ , \tag{1.36}$$

where m^2 is a mass parameter with the dimension of mass and G is a three dimensional Newton constant. The equation of motion is given by,

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = 0\ , \tag{1.37}$$

where $G_{\mu\nu}$ is the Einstein tensor given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\ ,$$

and

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{\Box R}{2}g_{\mu\nu} + 4R_{\mu\rho\nu\sigma}R^{\rho\sigma} - \frac{3R}{2}R_{\mu\nu} - R^2_{\rho\sigma}g_{\mu\nu} + \frac{3R^2}{8}g_{\mu\nu}. \tag{1.38}$$

From the analysis of the above discussed action and equations of motion, the non existence of BD ghost was proved. As a result NMG is a completely consistent ghost free theory of a fully interacting massive graviton in three dimensions. These peculiarities of the theory gained reputation to NMG from the scientific community as having interesting features [39]. We will explore the thermodynamics, spectroscopy and geometrothermodynamics of black hole solutions in this massive gravity model by keeping in mind the fact that the lower dimensional theory and the black hole solutions usually provide an interesting playground to obtain a simplified insight into the thermodynamic properties of black holes.

1.3 Black Hole Thermodynamics

From the discovery of Hawking radiation [40], black holes are believed to be thermodynamic systems which have a characteristic temperature which is directly related to the event horizon of the black hole. Classically black holes can be thought of as an end stage of a massive star with zero temperature. But these thoughts got reverted after Bekenstein's suggestion [41] that black holes possesses entropy and this entropy is proportional to area of its event horizon and Hawking's path breaking discovery that the black hole radiates thermally [41–43]. Initially the idea that the black hole is associated with an entropy got noticed when the event horizon surface area exhibiting a remarkable tendency to increase when undergoing any transformation as noticed by Floyd and Penrose [44] and it was supported by Christodoulou's [45] observations. Here we will discuss the thermodynamic properties of black holes, beginning from the discovery of black hole mechanics.

In 1973 Bardeen, Carter and Hawking introduced the four laws of black hole mechanics [46] which are analogues to the ordinary laws in classical thermodynamics.

- ***Z****eroth Law: The event horizon is described by a surface gravity κ which is constant over the horizon of a stationary black hole.*

  This is analogous to the zeroth law in ordinary thermodynamics, which states that the temperature is constant throughout a body in thermal equilibrium. Hence it can be noted here that the surface gravity is analogous to temperature. The surface gravity(κ) is the gravitational acceleration experienced at its surface and is related to the physical temperature of the black hole by,

$$T_H = \frac{\kappa}{2\pi} \ . \tag{1.39}$$

As an example let us consider the Schwarzschild black hole, where $\kappa = 1/4GM$, then the Hawking temperature becomes,

$$T_H = \frac{\hbar}{8\pi G k_B M} \approx 6.2 \times 10^{-8} \ \frac{M_\odot}{M} K \ . \tag{1.40}$$

From this it is obvious that, the Hawking temperature associated with Schwarzschild black hole where mass of the black hole is sufficiently high, is negligibly small. So one can say that the Hawking radiation does not play any significant role in the case of large-sized black holes, whereas it is prominent in the case of mini black holes which might have been formed in the primordial stages of the universe.

- ***F****irst Law: For perturbations of stationary black holes, the change in energy (dE) is related to changes in area (A), angular momentum (J), and electric charge (Q):*

$$dE = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \ . \tag{1.41}$$

It is quite evident that the above equation is analogous to the first law of thermodynamics in classical thermodynamics, which is given as,

$$dE = TdS + \text{"work terms"} \ . \tag{1.42}$$

Comparing the above two relations, the entropy of the black hole can be represented as a quarter of the area of the event horizon of the black hole under consideration, hence,

$$S_{\rm BH} = \frac{A}{4} \ . \tag{1.43}$$

From thermodynamic relationship among mass, temperature and entropy, Hawking was able to confirm Bekenstein's conjecture: the black hole entropy is proportional to the area of its event horizon divided by the Planck area, and established the constant of proportionality as 1/4, i.e, $S_{BH} = \frac{kA}{4\ell_{\mathrm{P}}^2}$, where A is the area of the event horizon, k is Boltzmann's constant, and $\ell_{\mathrm{P}} = \sqrt{G\hbar/c^3}$ is the Planck length. Usually the entropy S_{BH} is referred to as Bekenstein-Hawking entropy.

- ***S****econd Law: The area of the event horizon is a non-decreasing function of time,*

$$dA \geq 0. \tag{1.44}$$

The second law of black hole thermodynamics is the statement of Hawking's area theorem where it states that the change in entropy of an isolated system will be greater than or equal to zero for a spontaneous process, suggesting a link between entropy and the area of a black hole horizon. However, this theorem violates the second law of thermodynamics when a black hole emits radiation there must be an increase in entropy of the surroundings. According to No-Hair theorem, regardless of the specific details of the structure and properties of a collapsing body, the resulting stationary black hole is described by a geometry specified by the externally observable parameters, namely mass, angular momentum and charge. Hence violation of second law is obvious when some one puts a box of entropy in to the black hole. In order to make the second law of black hole thermodynamics consistent with second law of classical thermodynamics, Bekenstein [41, 42] introduced the idea of generalized entropy. The generalized entropy is defined as

$$S' = S_{\mathrm{BH}} + S_{\text{surrounding matter}}\ . \tag{1.45}$$

Then one can immediately note down the Generalized Second Law

(GSL) as,

$$dS' \geq 0 \ . \tag{1.46}$$

Through this GSL, the universal behaviour of entropy is validated in the case of black hole too, where the total entropy of the universe increases when matter is fallen in to the black hole. As an example, let us obtain the entropy of Kerr-Newman black hole from the area law (1.43). The event horizon of KN black hole is given by,

$$r_+ = M + M\sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}} \ . \tag{1.47}$$

The area of the event horizon can be written as,

$$A = 4\pi r_+^2 \ . \tag{1.48}$$

Now using the Bekenstein-Hawking area law, the entropy of the KN black hole can be written in terms of its mass, charge and angular momentum as,

$$S = 2M^2 - Q^2 + 2M\sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}} \ . \tag{1.49}$$

- ***T****hird Law: It is not possible to form a black hole with vanishing surface gravity, i.e., $\kappa = 0$ can not be achieved.* Stating that κ cannot go to zero is analogous to the third law of thermodynamics where, the entropy of a system at absolute zero is a well defined constant, or absolute zero of temperature is unattainable. This is often referred to as Nernst's Heat theorem.

When we look in to this scenario in a pure classical way, black holes in general relativity obey certain laws which bear a remarkable mathematical as well as structural resemblance to the ordinary laws of thermodynamics. This analogy has taken the idea of black hole thermodynamics to the present level and the same thing drives the present intense activities in the current field. Thermodynamic properties of black holes have been studied during all these years. During this period, it became well known that the black hole space-time can possess phase structures along with the standard thermodynamic variables like temperature, entropy, etc. Hence it causes to believe in the existence of a complete analogy between black hole systems and non-gravitational thermodynamic systems. Various black hole thermodynamic variables and their properties have been extensively studied. Thermodynamic stability of a black hole space-time can be studied by investigating the behaviour of its specific heat or heat capacity in the corresponding equilibrium phase space. It is well known that the thermodynamic stability of the system is completely related to the sign of heat capacity. If the heat capacity is positive, then the black hole is stable and it is unstable when the heat capacity is negative in sign. As an example, it is found that, for Schwarzschild black hole the heat capacity is negative and hence thermodynamically unstable. Phase transition is an important phenomenon in thermodynamics, so it is natural to look for the same in black hole thermodynamics. In 1983, Hawking and Page [47] discovered the phase transition phenomena in the Schwarzschild AdS background. This became a turning point in the study of black hole phase transition. As a result of this many studies are done in the same direction [48–56]. The phase transition is always identified with the sign change of heat capacity or having infinite discontinuities at the critical points in the heat capacity variation. Davies [57] argued that the point at which the specific heat changes from positive to negative values through an infinite discontinuity marks a phase transition, commonly referred to as Davies phase transition. No laboratory is yet equipped with the tools necessary to probe the formation of a black hole in a phase transition of thermal space-time [58], although such situations might have existed in the very early universe. Here, we want to see how far our understanding of more accessible physical systems can be applied to similar processes in space-times.

1.4 Area spectrum and Entropy spectrum of black holes

After the discovery of Hawking effect, it was widely believed that the studies on black hole physics may shed light to formulating a quantum theory of gravity. Bekesntein [59] was the first to begin the investigation in this direction. He started his investigations based on the fact that, classically, the horizon area of a non-extremal black hole behaves as an adiabatic invariant quantity. From Ehrenfest's principle, any classical adiabatic invariant quantity corresponds to a quantum variable with discrete spectrum. As a result of these observations, Bekenstein concluded that the horizon area of a non-extremal black hole must have a discrete spectrum. In order to construct the discrete area levels, Bekenstein found [41] a lower bound for the increase in the black hole event horizon area as,

$$(\Delta A)_{min} = 8\pi \ell_{\rm P}^2 \ , \tag{1.50}$$

where $\ell_{\rm P} = (\frac{G\hbar}{c^3})^{1/2}$ is the Planck length. It is interesting enough to note that the lower bound of the change in area does not depend on black hole parameters. Hence it was considered as an evidence for equispaced area spectrum devoid of black hole parameters of a quantum black hole.

These ideas put forwarded by Bekenstein became the corner stone of the investigation in the direction of the calculations and derivations of the area and hence the entropy spectrum of the black hole. Quasinormal mode

(QNM) frequencies are known as the characteristic sound of the black hole. Identifying the adiabatic invariant quantity from QNMs, Hod [60] derived the area as well as entropy spectrum of black hole from QNMs. Hod showed that in the case of Schwarzschild black hole, if one considers the real part of the QNM frequency, it acts as an adiabatic invariant quantity and it relates to the area spectrum of the black hole event horizon. Using Bohr-Sommerfield quantization rule, given as,

$$I_{\text{adiabatic}} = n\hbar \ , \tag{1.51}$$

he found that the area spectrum of Schwarzschild black hole is equispaced. From this area spectrum one can arrive at the entropy spectrum of the black hole by using the well known Bekenstein-Hawking area law as,

$$\Delta S_{\text{bh}} = \ln 3 \ . \tag{1.52}$$

Later Kunstatter [61] calculated the area spectrum of d-dimensional spherically symmetric black holes by considering the explicit form of the adiabatic invariant quantity as,

$$I_{\text{adiabatic}} = \int \frac{dE}{\Delta\omega(E)} \ , \tag{1.53}$$

$$\Delta\omega = \omega_{n+1} - \omega_n \ , \tag{1.54}$$

where E and ω are respectively energy and frequency of the QNM. He obtained the area spectrum of the black hole by considering (1.51), and that yields the same spacing obtained by Hod. In this work, Hod and Kunstatter considered the real part of the QNM frequency to calculate the area spectrum. Maggiore refined the idea proposed by Hod by providing a new interpretation [62] where, black hole is found to behave like a damped harmonic oscillator for which the physical frequency of QNM is determined by its real and imaginary parts as,

$$\omega = \sqrt{\omega_R^2 + \omega_I^2} \ . \tag{1.55}$$

From this he derived the area spectrum which is consistent with that of the relation obtained by Bekenstein earlier. These calculations directly affect the studies on spectroscopic aspects of black holes. As a result, entropy spectrum of many black holes have been calculated which include the calculations for a most general black hole [63–65]. Hence it is found that the quantization of entropy is more fundamental than the quantization of area.

Recently Majhi and Vagenas [66] proposed a new method to quantize the entropy. By relating an adiabatic invariant quantity to the Hamiltonian of the black hole, obtained an equally spaced entropy spectrum with its quantum to be equal to the one obtained by Bekenstein. It is noteworthy that in this proposal one need not relay on QNMs to calculate the black hole area spectrum. Classically, general relativity gives the picture of the black hole from which nothing, even light, can escape. This way of picturing the black hole has been changed when Hawking discovered the radiation from the black hole as a quantum effect. As a result of intense research in this field, quantum mechanical tunneling picture was treated as the source for Hawking radiation, where this picture resembles that of electron-positron pair creation in a constant electric field. In this tunneling picture, the black hole horizon can be assumed to oscillate periodically when the particle tunnels in or out. Interestingly, this approach follows Maggiore's method in which the perturbed black hole behaves as damped harmonic oscillator. In this tunneling picture, we begin with the adiabatic invariant quantity of the form, which is basically the action of the oscillating horizon,

$$I = \int p_i dq_i \ , \tag{1.56}$$

where p_i is the corresponding conjugate momentum of the coordinate q_i and $i = 0, 1$ for which $q_0 = \tau$ and $q_1 = r_h$. Here, τ represents the Euclidean time and r_h is the horizon radius. By implementing the Hamilton's equation $\dot{q}_i = \dfrac{\partial H}{\partial p_i}$, where H denotes the Hamiltonian of the system, which acts as total energy for the black hole case, into (1.56), one can rewrite the action as,

$$I = \int p_i dq_i = \int \int_0^H dH' d\tau + \int \int_0^H \frac{dH'}{\dot{r}_h} dr_h = 2 \int \int_0^H \frac{dH'}{\dot{r}_h} dr_h \ . \tag{1.57}$$

where $\dot{r}_h = \frac{dr_h}{d\tau}$. In order to calculate the above adiabatic invariant quantity, let us consider a static, spherically symmetric black hole solution, in general given by,

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ . \tag{1.58}$$

One can obtain r_h, namely the black hole horizon radius from the relation $N(r_h) = f(r_h) = 0$. To evaluate the integral (1.57), one must find the oscillating velocity of the black hole horizon. In the tunneling picture, when a particle tunnels in or out, the black hole horizon will expand or shrink due to gain and loss of the black hole mass. Since the tunneling and oscillation happen simultaneously, the tunneling velocity of particle is equal and opposite to the oscillating velocity of the black hole horizon,

$$\dot{r}_h = -\dot{r} \ . \tag{1.59}$$

In (1.57), τ denotes the Euclidean time, hence one has to Euclideanize the metric given by (1.58), by introducing the transformation $t \to -i\tau$. Then,

$$ds^2 = N(r)^2 d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ . \tag{1.60}$$

Now, when a photon travels across the black hole horizon, the radial null path, often called as radial null geodesic ($ds^2 = 0$ and $d\Omega^2 = 0$)is given by,

$$\dot{r} = \frac{dr}{d\tau} = \pm i \sqrt{N(r)^2 f(r)} \ , \tag{1.61}$$

where the positive sign denotes the outgoing radial null paths and negative sign represents the incoming radial null paths. Among these we will consider the outgoing paths for area spectrum calculations, since these paths are more related to the quantum behaviours under consideration. Hence the shrinking velocity of the black hole horizon is,

$$\dot{r}_h = -\dot{r} = -i\sqrt{N\left(r\right)^2 f\left(r\right)}\ . \tag{1.62}$$

Then, (1.57) is now read off,

$$\int p_i dq_i = -2i \int\int_0^H \frac{dH'}{\sqrt{N\left(r\right)^2 f\left(r\right)}} dr\ . \tag{1.63}$$

To find this adiabatic invariant quantity, one has to execute this integration by determining $N\left(r\right)^2$ and $f\left(r\right)$. To perform the τ integration, one has to consider the periodicity of τ given as $\frac{2\pi}{\kappa}$, where κ is the surface gravity which in turn is given by,

$$\kappa = \frac{1}{2}\sqrt{N'\left(r\right)^2 f\left(r\right)}\ . \tag{1.64}$$

Since we rely on outgoing paths, the integration limit for τ will be, $0 \leq \tau \leq \frac{\pi}{\kappa}$. Now from Hawking's discovery on temperature of the black holes, we know that temperature of the black hole is proportional to the surface gravity as,

$$T_H = \frac{\hbar\kappa}{2\pi}\ . \tag{1.65}$$

Along with these findings, one can directly apply Bohr-Sommerfiled quantization rule to this scenario and that will lead to the area spectrum and eventually the entropy spectrum of the black hole. Many black hole systems are analyzed using these tunneling picture which is free from QNM calculations and the results obtained were in good agreement with the Bekenstein's original proposal.

The studies related to this proposal proved that, the proposed adiabatic invariant quantity $I_{\text{adiabatic}} = \int p_i dq_i$ apparently depends on the choice of coordinates. As a result, the area spectrum as well as the entropy spectrum spacing changes with respect to the change in coordinate transformations. To account for this discrepancy, Akhmedova [67–71], Jiang and Han [72] proposed and argued that the closed contour integral $\oint p_i dq_i$ is invariant under coordinate transformations and hence the adiabatic invariant quantity must be of the covariant form,

$$I = \oint p_i dq_i \tag{1.66}$$

Here the closed contour integral can be considered as a closed path that goes from q_i^{out} (outside the event horizon) to q_i^{in}(inside the event horizon). That is,

$$I = \oint p_i dq_i = \int_{q_i^{in}}^{q_i^{out}} p_i^{out} dq_i + \int_{q_i^{out}}^{q_i^{in}} p_i^{in} dq_i. \tag{1.67}$$

Here p_i^{in} or p_i^{out} is the conjugate momentum corresponding to the coordinate q_i^{in} or q_i^{out}, respectively, and $i = 0, 1, 2...$ It is also to be considered that $q_1^{in} = r_h^{in}(q_1^{out} = r_h^{out})$ and $q_0^{in}\left(q_0^{out}\right) = \tau$ where r_h is the horizon radius and τ is the Euclidean time with a periodicity $\frac{2\pi}{\kappa}$ in which κ is the surface gravity. Proceeding the tunneling method using this covariant action, one can arrive at the area spectrum of the black hole. Using this, the black hole spectroscopy is quantized independently of the choice of coordinate transformations.

In this thesis, we adopted this method to calculate the area as well as the entropy spectrum of the black holes in modified theories of gravity.

1.5 Thermodynamics Geometry and Geometrothermodynamics

Thermodynamics describes how systems respond to the thermal changes occurring in their surroundings. The ideas gained by the people about thermodynamics are codified in terms of four laws. One can apply these laws to a wide variety of systems in physics which include black holes too! In spite of the generality of these laws, some aspects of thermodynamics remain to be uncertain and are not evident in the classical description, as for example the non-extensive systems in thermodynamics. Answering these anomalies was a challenging problem for the scientific community. For this purpose, the ideas of differential geometry is incorporated with ordinary thermodynamics. Gibbs [73] and Caratheodory [74], followed by Hermann [75] and later by Mrugala [76, 77], proposed a differential geometric approach to study the ordinary thermodynamics of different systems based on the contact manifold of the thermodynamic phase space $\mathcal{T}$. This space is a $(2n+1)$ dimensional phase space and they are coordinatized by the thermodynamic potential Φ along with n extensive variables E^a and their corresponding n dual intensive variables I^a.

The first attempt to describe the thermodynamic systems in terms of differential geometry was done by Weinhold [78]. Later in 1979 Ruppeiner [79] proposed a new way of exploring the thermodynamics of a system using the Riemannian geometry ideas. In this geometric theory, he included the theory of fluctuations in to the propositions of equilibrium thermodynamics. Using this language the thermodynamic equilibrium states that can be represented as points in a two dimensional manifold and the distance between these states or between these points, are related to the thermodynamic fluctuations existing in the systems. These ideas are directly related to the probability, if the thermodynamic fluctuation between different states is less probable, then the distance between these points in the equilibrium manifold are far apart. So, the line element or the distance between two equilibrium states can be written as,

$$ds^2 = g_{ij} dx^i dx^j \ , \tag{1.68}$$

where g_{ij} is the symmetric metric tensor. The Ruppeiner metric defined on the thermodynamic phase space is written as,

$$g^R_{ij} = -\partial_i \partial_j S(M, N^a) \ , \tag{1.69}$$

where S is the entropy, M represents the energy or mass of the system and N^a are other extensive variables which characterizes the thermodynamic system. Extensive variables in a system should satisfy the definition of extensivity [145, 146]: *Given a system of n particles and a joint physical observable* $X(x_1, ..., x_n)$ *of the individual states of the particles, we say that* X *is (asymptotically) extensive if,*

$$\lim_{n \to \infty} \frac{X(n)}{n} < \infty \ .$$

The thermodynamic geometry defined through the Ruppeiner metric physically describes the thermodynamic fluctuation theory in equilibrium manifold. Then it follows that, the thermodynamic interaction of the system can be studied using the curvature of the thermodynamic geometry. If the Ruppeiner geometry is flat, then one can arrive at a conclusion that the underlying system undergoes no thermodynamic interactions at all. Hence any curvature in the geometry in turn indicates the interaction of the system. He proved in the case of ideal gas that, if the thermodynamic curvature vanishes then it corresponds to the absence of thermodynamic interactions. In Weinhold's method, he introduced a metric on the space of equilibrium states whose components are given as the Hessian of the internal energy of the thermodynamic system under consideration as,

$$g^W_{ij} = -\partial_i \partial_j M(S, N^a) \ , \tag{1.70}$$

similar to the Ruppeiner geometry, here M represents the energy or mass of the system, S is the entropy and N^a are other extensive variables which constitutes the thermodynamic system. Interestingly, one can conclude that the Weinhold geometry is conformally related to the Ruppeiner geometry as,

$$ds_R^2 = \frac{1}{T} ds_W^2 \ , \tag{1.71}$$

A number of investigations have been done to analyze the thermodynamic geometry of various thermodynamic systems. The Weinhold and Ruppeiner geometries have been used to analyze several black hole systems to explain the thermodynamic interactions as well as abnormalities [80–90].

According to Davies: *"The phenomenon which occurs at the critical values of α and β may be then classified as a second order phase transition. Such a transition is characterized by continuity of G and its first derivatives, but a discontinuity in the second derivatives, e.g. heat capacity"*. This argument resembles Ehrenfest's classification scheme (discussion is included in chapter 4), but it is not treated as the standard definition in the modern treatments of thermodynamics [91, 92]. On the other hand, the points where heat capacities diverge or change sign from positive to negative through a zero are the points where the thermodynamic potential changes the concavity. At these points the local conditions of equilibrium fail. Now one can't expect that by means of the tools of thermodynamic geometry that apply to ordinary systems would recover the correct results for black hole thermodynamics (as for any thermodynamic systems with long-range interactions). This can perhaps explain the ambiguous results obtained from Weinhold and Ruppeiner metrics [93–96] for black hole phase transitions. Geometrothermodynamics (GTD) [97–99] is the latest attempt in this direction. This new method describes the phase transitions of a thermodynamic system by incorporating the ideas of differential geometry and Legendre invariance. Then it is proposed that [100] from the the GTD program, one which has been shown in the literature to apply to the case of black holes [98, 101–103], could be the exact one to account for the transitions that emerge in the case of "non-extensive" systems (thermodynamics is ensemble dependent). In GTD the main requirement for defining a Riemannian metric on the thermodynamic phase space is that it must satisfy the condition of Legendre invariance. So we have many possibilities of constructing such a metric with these salient features. In general the Legendre invariant metrics found so far can be classified in to three classes each can be used to describe thermodynamic systems with particular phase transitions [104]. Here we are considering systems with second order phase transitions. In GTD method one can consider the curvature singularities as the phase transition points and hence the system interactions can be explained well. The order of phase transition can be determined using the GTD program since they are found in agreement with Davies' phase transitions structure [105].

The main constituent of GTD is a $(2n+1)$ dimensional manifold, usually referred to as thermodynamic phase space ($\mathcal{T}$). Let the thermodynamic phase space $\mathcal{T}$ be coordinatized by the thermodynamic potential Φ, extensive variables E^a, and their dual intensive variables I^a $(a = 1, ..., n)$. Let the fundamental Gibbs 1-form be defined on $\mathcal{T}$ as,

$$\Theta = d\Phi - \delta_{ab} I^a dE^b, \tag{1.72}$$

where $\delta_{ab} = \text{diag}(1, 1, ..., 1)$. The pair $(\mathcal{T}, \Theta)$ is called a contact manifold [75] if the phase space $\mathcal{T}$ is differentiable and Θ satisfying the condition $\Theta \wedge (d\Theta)^n \neq 0$. Now let us consider the Riemannian metric on the thermodynamic phase space $\mathcal{T}$ as,

$$G = (d\Phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)\ (\delta_{cd} dE^c dI^d). \tag{1.73}$$

This metric is non-degenerate and are invariant with respect to the Legendre transformations defined by [106, 107],

$$\{\Phi, E^a, I^a\} \longrightarrow \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\} \ , \tag{1.74}$$

$$\Phi = \tilde{\Phi} - \delta_{kl}\tilde{E}^k\tilde{I}^l \ , \quad E^i = -\tilde{I}^i, \quad E^j = \tilde{E}^j, \quad I^i = \tilde{E}^i, \quad I^j = \tilde{I}^j \ , \tag{1.75}$$

where $i \cup j$ is any disjoint decomposition of the set of indices $\{1, ..., n\}$ and $k, l = 1, ..., i$. In particular, for $i = \{1, ..., n\}$ and $i = \emptyset$ we obtain the total Legendre transformation and the identity, respectively. We say that the set $(\mathcal{T}, \Theta, G)$ defines a Legendre invariant manifold with a contact Riemannian structure.

The space of thermodynamic equilibrium states is an $n-$dimensional Riemannian submanifold, known as equilibrium manifold, $\mathcal{E} \subset \mathcal{T}$ induced by a smooth mapping $\varphi : \ \mathcal{E} \ \longrightarrow \mathcal{T}$, i.e. $\varphi : (E^a) \longrightarrow (\Phi, E^a, I^a)$ with $\Phi = \Phi(E^a)$ such that,

$$\varphi^*(\Theta) = 0 \ , \qquad \varphi^*(G) = g = \Phi \frac{\partial^2\Phi}{\partial E^a \partial E^b} dE^a dE^b \ , \tag{1.76}$$

where φ^* represents the pullback of φ and g is the Riemannian metric induced on $\mathcal{E}$. This implies the relationships,

$$d\Phi = \delta_{ab} I^a dE^b \ , \qquad \frac{\partial \Phi}{\partial E^a} = \delta_{ab} I^b \ , \tag{1.77}$$

which correspond to the first law of thermodynamics and the standard conditions for thermodynamic equilibrium, respectively [91]. The metric g on $\mathcal{E}$ is Legendre invariant because it is induced by a smooth mapping from the Legendre invariant metric G of $\mathcal{T}$. To distinguish between the thermodynamic systems in the space of equilibrium states $\mathcal{E}$, one must specify the fundamental equation, which contained in the embedded mapping, $\{\Phi, E^a, I^a\} \longrightarrow \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\}$ through the relation $\Phi = \Phi(E^a)$. All geometric properties of the equilibrium manifold are determined by this fundamental equation. This construction is complimented with the second law of thermodynamics through the convexity condition [108, 109],

$$\frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0 \ . \tag{1.78}$$

In general, the thermodynamic potential must satisfy the homogeneity condition, $\Phi(\lambda E^a) = \lambda^\beta \Phi(E^a)$ for constant parameters λ and β. Using the first law of thermodynamics, it can easily be shown by differentiating this condition with respect to λ, that this homogeneity is equivalent to the relationships,

$$\beta \ \Phi(E^a) = \delta_{ab} I^b E^a \ , \qquad (1-\beta) \ \delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0 \ , \tag{1.79}$$

which are known as Euler's identity and Gibbs-Duhem relation, respectively. It is interesting to be note that the metric g given in (1.76) is not the unique Legendre invariant choice, instead there exist an infinite number of Legendre invariant metrics on $\mathcal{E}$. Hence the general GTD metric defined on the equilibrium manifold can be written from the pull back φ^* to the metric (1.73) as,

$$g^{\text{GTD}} = \varphi^*(G) = \left(E^c \frac{\partial \Phi}{\partial E^c} \right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d \right) , \tag{1.80}$$

which depends only on the fundamental potential $\Phi = \Phi(E^a)$, where η_{ab}=diag(-1,1,1,..,1) .

One can study the equilibrium space of different black hole systems using the GTD formalism where one can construct a thermodynamic metric from (1.80) using different thermodynamic potential corresponding to the same system. Like Ruppeiner and Weinhold constructions, the thermodynamic interaction can be reflected from the curvature of the above defined metric. So the curvature reproduces the main thermodynamic properties of the system. The free of singularities in thermodynamic curvature described in GTD is interpreted as a consequence of the non-existence of singular points at the level of the heat capacity, indicating that no (second order) phase transitions occur. It turns out that these results coincide with the predictions of ordinary black hole thermodynamics as proposed by Davies [57]. On the other hand, the existence of singular point or the infinite discontinuities in the level of heat capacity represents the possibility of second order phase transition. This information is coded on the curvature of the corresponding GTD metric used to describe the system.

1.6 Outline of the thesis

In this thesis we study the thermodynamics, spectroscopy and geometrothermodynamics of different black holes in modified theories of gravity, particularly in Hořava-Lifshotz gravity and Massive gravity. In **Chapter 2**, we discuss the thermodynamics of different black hole solutions in these theories like Kehagias-Sfetsos, Lü-Mei-Pope, Park solution, dRGT black hole and (2+1) BTZ black hole. Their thermodynamic properties have been analyzed in detail with a special emphasis on phase transitions shown by these black holes.

The entropy spectrum of these black hole solution have been analyzed in **Chapter 3** by adopting the method suggested by Majhi, Vagenas, Jiang and Han by incorporating the ideas form adiabatic invariance, tunneling mechanism, Bohr-Sommerfeld quantization rule and near horizon approximations.

Chapter 4 is devoted for the discussion of the new geometric approach, geometrothermodynamics, as applied for the case of black holes mentioned earlier. Phase transition structure of these black holes and abnormal behaviours exhibited by different thermodynamic potentials are studied here.

Chapter 5, is devoted to the conclusions of the present work.

2

Thermodynamics of black holes in modified theories of gravity

Thermodynamic properties of Black holes have been studied intensely since Bekenstein's work [41] appeared in 1973. In that study, Bekenstein proposed a conjecture between entropy and black hole event horizon area that the black hole entropy is proportional to the area of its event horizon divided by the Planck area. Now the subject has grown very rich. In this chapter we discuss the thermodynamic properties of black hole solutions explicitly in both Hořava-Lifshitz gravity and Massive gravity, with a special emphasis on phase transitions exhibited by black hole solutions in these gravity models. Studying the variation of temperature, mass and heat capacity of these black hole systems will add informations to the understandings of these theories. According to the ideas of Davies' phase transition picture [57], the transition from a positive heat capacity phase to a negative heat capacity phase or vice versa indicates a thermodynamic phase transition. Here we will consider different black hole solutions in these theories and will explore their phase transition behaviour.

2.1 Black holes in Hořava-Lifshitz gravity

Recently, a field theoretic model that can be interpreted as a complete theory of gravity in the ultraviolet (UV) limit was recently proposed by Hořava [10–12]. The model is renormalizable and non-relativistic in the UV regime. Moreover, in the infrared (IR) limit it can be reduced to Einstein's general theory of relativity with a cosmological constant. Due to the intense study in this field, different black hole solutions have been proposed and analyzed. In this chapter, we will discuss three kinds of black hole solutions, namely Kehagias-Sfetsos (KS) [110] black hole, Lü-Mei-Pope (LMP) [111] black hole and Park black hole [112].

Kehagias-Sfetsos black hole

In Hořava-Lifshitz gravity theory, Hořava used the idea of ADM formalism and arrived at the action as,

$$\begin{aligned}S = &\int dt\; d^3x\sqrt{g}N\Big\{\frac{2}{\kappa^2}\left(K_{ij}K^{ij}-\lambda K^2\right)-\frac{\kappa^2}{2w^4}C_{ij}C^{ij}\\ &+\frac{\kappa^2\mu}{2w^2}\epsilon^{ijk}R^{(3)}_{i\ell}\nabla_j R^{(3)\ell}{}_k-\frac{\kappa^2\mu^2}{8}R^{(3)}_{ij}R^{(3)ij}\\ &+\frac{\kappa^2\mu^2}{8(1-3\lambda)}\left(\frac{1-4\lambda}{4}(R^{(3)})^2+\Lambda_W R^{(3)}-3\Lambda_W^2\right)+\mu^4 R^{(3)}\Big\}\;.\end{aligned}$$

We will now consider the limit of this theory as $\Lambda_W \to 0$. In this particular limit, the theory will reduce to,

$$\begin{aligned}S = &\int dt\; d^3x\sqrt{g}N\Big\{\frac{2}{\kappa^2}\left(K_{ij}K^{ij}-\lambda K^2\right)-\frac{\kappa^2}{2w^4}C_{ij}C^{ij}\\ &+\frac{\kappa^2\mu}{2w^2}\epsilon^{ijk}R^{(3)}_{i\ell}\nabla_j R^{(3)\ell}{}_k-\frac{\kappa^2\mu^2}{8}R^{(3)}_{ij}R^{(3)ij}\\ &+\frac{\kappa^2\mu^2}{8(1-3\lambda)}\frac{1-4\lambda}{4}(R^{(3)})^2+\mu^4 R^{(3)}\Big\}\;. \end{aligned}\tag{2.1}$$

For spherically symmetric, static solution of HL gravity, let us consider the line element,

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ . \tag{2.2}$$

Substituting this metric ansatz in (2.1) and after angular integration, the Lagrangian will reduce to,

$$\begin{aligned}\tilde{\mathcal{L}} \ &= \ \frac{\kappa^2\mu^2 N}{8\,(1-3\lambda)\sqrt{f}}\Big\{\frac{\lambda-1}{2}f'^2 + \frac{(2\lambda-1)\,(f-1)^2}{r^2} \\ &\quad -\frac{2\lambda\,(f-1)}{r}f' - 2\omega\left(1-f-rf'\right)\Big\},\end{aligned} \tag{2.3}$$

where,

$$\omega = \frac{8\mu^2\,(3\lambda-1)}{\kappa^2}. \tag{2.4}$$

Now the equations of motions are,

$$\begin{aligned}&(2\lambda-1)\frac{(f-1)^2}{r^2} - 2\lambda\frac{f-1}{r}f' + \frac{\lambda-1}{2}f'^2 - 2\omega(1-f-rf') = 0\ , \\ &\left(\log\frac{N}{\sqrt{f}}\right)'\Big\{(\lambda-1)f' - 2\lambda\frac{f-1}{r} + 2\omega r\Big\} + \\ &(\lambda-1)\left(f'' - \frac{2(f-1)}{r^2}\right) = 0\ .\end{aligned} \tag{2.5}$$

For the $\lambda = 1$ ($\omega = 16\mu^2\kappa^2$) case, by solving the field equations, one can obtain the asymptotically flat, space-time,

$$N^2 = f_{\text{KS}} = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}\ , \tag{2.6}$$

where M is an integration constant. The black hole described by the metric function (2.6) is called the Kehagias-Sfetsos (KS) black hole solution [110].

Now we will investigate the thermodynamic properties of KS black hole. From the condition $f_{\text{KS}}(r_\pm) = 0$, the outer and inner horizons are given by,

$$r_{\pm} = M \pm \sqrt{M^2 - \frac{1}{2\omega}} \ . \tag{2.7}$$

By considering r_+ from (2.7), we can establish a connection between mass of the black hole and its horizon radius as,

$$M = \frac{r_+}{2} + \frac{1}{4\omega r_+} \ . \tag{2.8}$$

By employing the Bekenstein-Hawking area law, we can write entropy S as,

$$S = \frac{A}{4} = \pi r_+^2 \ . \tag{2.9}$$

and hence,

$$r_+ = \sqrt{\frac{S}{\pi}} \ . \tag{2.10}$$

Therefore, we can rewrite the mass-horizon radius (2.8) as,

$$M = \frac{1}{4\omega}\sqrt{\frac{\pi}{S}} + \frac{1}{2}\sqrt{\frac{S}{\pi}}. \tag{2.11}$$

Now from the classical thermodynamic relations, temperature and specific heat are defined respectively as,

$$T = \left(\frac{\partial M}{\partial S}\right), \tag{2.12}$$

$$C = T\left(\frac{\partial S}{\partial T}\right). \tag{2.13}$$

Then, from these equations, we can have the black hole temperature as,

$$T = \frac{1}{4\sqrt{\pi S}} - \frac{\sqrt{\pi}}{8\omega S^{\frac{3}{2}}}. \tag{2.14}$$

and the heat capacity of the black hole as,

$$C = -\left(\frac{4\omega S^2 - 2\pi S}{2\omega S - 3\pi}\right). \tag{2.15}$$

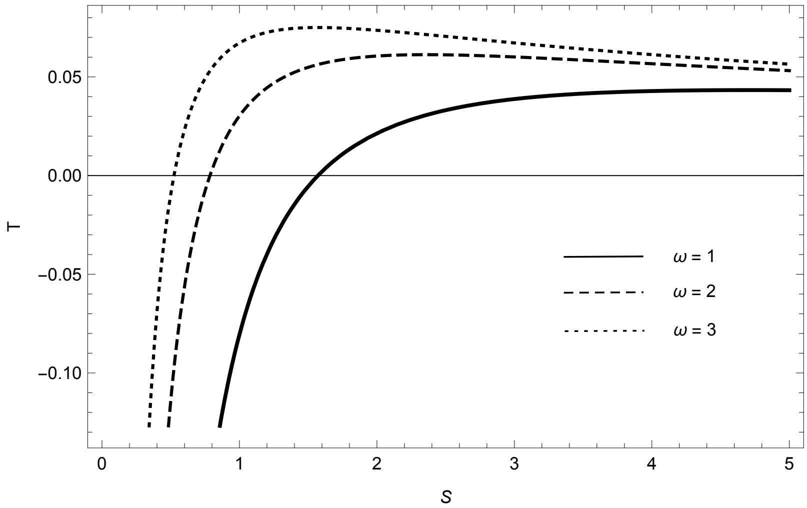

Figure 2.1: Variation of temperature with entropy of KS black hole for different values of ω.

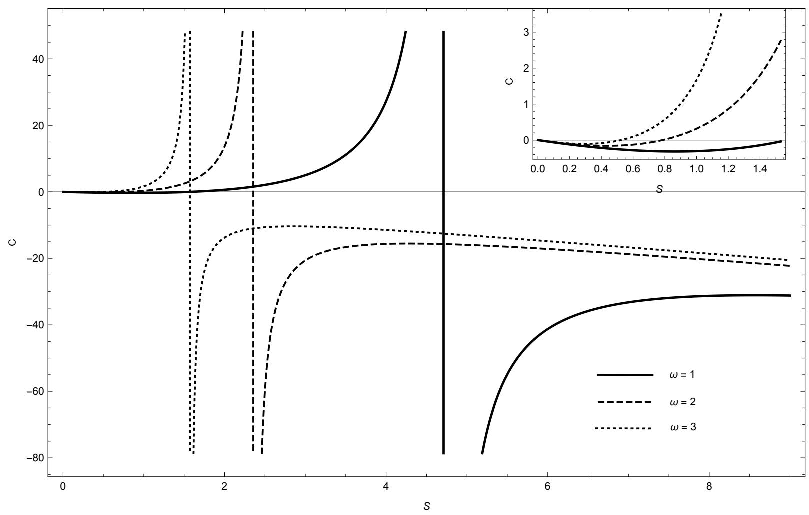

Figure 2.2: Variation of specific heat with entropy of KS black hole for different values of ω. (variation of specific heat for the smaller values of entropy is shown in the box at top right corner.)

In fig.(2.1), variation of temperature with respect to entropy is plotted while in fig.(2.2), the variation of heat capacity with respect to entropy for the different values of ω is plotted. From fig.(2.1), we can note that for each values of the system parameter ω, temperature exhibits anomalous behaviour by entering in to a negative temperature region for small values of entropy. On the other hand temperature become positive for $S > \frac{3\pi}{2\omega}$. Studying the heat capacity of the system will give information about the phase transition structure of the system. In fig.(2.2), there is a discontinuity in the plot, which shows that black hole may undergo a phase transition. Heat capacity is an important thermodynamic quantity because from that we can tell about the stability of the black hole. From (2.15) it is evident that the heat capacity is positive for a range $\frac{\pi}{2\omega} < S < \frac{3\pi}{2\omega}$. Hence, a KS black hole is stable for this range of values of S and for the rest of entropy values, i.e., $S > \frac{3\pi}{2\omega}$, the black hole is unstable.

We have studied the thermodynamic properties of KS black hole in Hořava-Lifshitz gravity and found that they are thermodynamically stable for a certain range of values of the entropy. From the behaviour of heat capacity of the black hole, it is interesting to note that the black hole undergoes an infinite discontinuity transition from thermodynamically stable phase to unstable phase as we probe for different values of entropy. The order of this phase transition can not be confirm using this thermodynamic approach.

Lü-Mei-Pope black hole

The action of the theory given in (2.1) can be rewritten as,

$$S = \int dt d^3\mathbf{x} \left(\mathcal{L}_{\mathbf{0}} + \mathcal{L}_{\mathbf{1}}\right), \tag{2.16}$$

$$\mathcal{L}_0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)} \right\}, \tag{2.17}$$

$$\mathcal{L}_1 = \sqrt{g}N\left\{\frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2}{2w^4}\left(C_{ij} - \frac{\mu w^2}{2}R_{ij}\right)\left(C^{ij} - \frac{\mu w^2}{2}R^{ij}\right)\right\}, \tag{2.18}$$

where $\lambda\,,\kappa\,,\mu\,,w$ and Λ_W are usual constant parameters in the theory, and C_{ij} is the Cotton tensor. We now consider the equations of motion for the action (2.18). Different equation of motion describing the system is obtained by varying this action with respect to N, δN^i and δg^{ij}. As the first case, let us consider the equation following from the variation of N, given as,

$$\begin{aligned}\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2\mu^2(\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)}& \\ -\frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 + \frac{\kappa^2}{2w^4}Z_{ij}Z^{ij} = 0\,,&\end{aligned} \tag{2.19}$$

where,

$$Z_{ij} \equiv C_{ij} - \frac{\mu w^2}{2}R_{ij}\,. \tag{2.20}$$

The variation δN^i implies,

$$\nabla_k(K^{k\ell} - \lambda\, K g^{k\ell}) = 0\,. \tag{2.21}$$

The equations of motion due to the variation of δg^{ij} are more complicated; they are given by,

$$\begin{aligned}\frac{2}{\kappa^2}E_{ij}^{(1)} - \frac{2\lambda}{\kappa^2}E_{ij}^{(2)} + \frac{\kappa^2\mu^2\Lambda_W}{8(1-3\lambda)}E_{ij}^{(3)} + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}E_{ij}^{(4)}& \\ -\frac{\mu\kappa^2}{4w^2}E_{ij}^{(5)} - \frac{\kappa^2}{2w^4}E_{ij}^{(6)} = 0\,,&\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
E_{ij}^{(1)} &= N_i\nabla_k K^k{}_j + N_j\nabla_k K^k{}_i - K^k{}_i\nabla_j N_k - K^k{}_j\nabla_i N_k - N^k\nabla_k K_{ij} \\
&\quad -2NK_{ik}K_j{}^k - \frac{1}{2}NK^{k\ell}K_{k\ell}\,g_{ij} + NKK_{ij} + \dot{K}_{ij}\,, \\
E_{ij}^{(2)} &= \frac{1}{2}NK^2 g_{ij} + N_i\partial_j K + N_j\partial_i K - N^k(\partial_k K)g_{ij} + \dot{K}\,g_{ij}\ , \\
E_{ij}^{(3)} &= N(R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\Lambda_W g_{ij}) - (\nabla_i\nabla_j - g_{ij}\nabla_k\nabla^k)N\,, \\
E_{ij}^{(4)} &= NR(2R_{ij} - \frac{1}{2}Rg_{ij}) - 2\big(\nabla_i\nabla_j - g_{ij}\nabla_k\nabla^k\big)(NR)\ , \\
E_{ij}^{(5)} &= \nabla_k\big[\nabla_j(NZ^k{}_i) + \nabla_i(NZ^k{}_j)\big] - \nabla_k\nabla^k(NZ_{ij}) - \nabla_k\nabla_\ell(NZ^{k\ell})g_{ij}\ , \\
E_{ij}^{(6)} &= -\frac{1}{2}NZ_{k\ell}Z^{k\ell}g_{ij} + 2NZ_{ik}Z_j{}^k - N(Z_{ik}C_j{}^k + Z_{jk}C_i{}^k) + NZ_{k\ell}C^{k\ell}g_{ij} \\
&\quad -\frac{1}{2}\nabla_k\big[N\epsilon^{mk\ell}(Z_{mi}R_{j\ell} + Z_{mj}R_{i\ell})\big] \\
&\quad +\frac{1}{2}R^n{}_\ell\,\nabla_n\big[N\epsilon^{mk\ell}(Z_{mi}g_{kj} + Z_{mj}g_{ki})\big] \\
&\quad -\frac{1}{2}\nabla_n\big[NZ_m^{\ n}\epsilon^{mk\ell}(g_{ki}R_{j\ell} + g_{kj}R_{i\ell})\big] \\
&\quad -\frac{1}{2}\nabla_n\nabla^n\nabla_k\big[N\epsilon^{mk\ell}(Z_{mi}g_{j\ell} + Z_{mj}g_{i\ell})\big] \\
&\quad +\frac{1}{2}\nabla_n\big[\nabla_i\nabla_k(NZ_m^{\ n}\epsilon^{mk\ell})g_{j\ell} + \nabla_j\nabla_k(NZ_m^{\ n}\epsilon^{mk\ell})g_{i\ell}\big] \\
&\quad +\frac{1}{2}\nabla_\ell\big[\nabla_i\nabla_k(NZ_{mj}\epsilon^{mk\ell}) + \nabla_j\nabla_k(NZ_{mi}\epsilon^{mk\ell})\big] \\
&\quad -\nabla_n\nabla_\ell\nabla_k(NZ_m^{\ n}\epsilon^{mk\ell})g_{ij}\,.
\end{aligned} \tag{2.23}$$

Now we will look for a static and spherically symmetric solution with the metric,

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2\ . \tag{2.24}$$

In the present study we are interested in the solution with the choice $\lambda = 1$ in (2.18). This will lead to the Lü-Mei-Pope (LMP) black hole solution [111], given by,

$$f(r) = k - \Lambda_W r^2 - \mathcal{A}\sqrt{\frac{r}{-\Lambda_W}}\ . \tag{2.25}$$

where $\mathcal{A}$ is an integration constant and is related to the black hole mass as $\mathcal{A} = aM$. It is interesting to note that this solution (2.25) is asymptotically AdS_4.

We will dig in to the details of the thermodynamic properties of this system. From the first law of black hole mechanics [46], when a black hole undergoes a change, from a stationary state to another, the change in mass of the black hole is given by,

$$dM = \frac{\kappa}{8\pi}\, dA + \Omega\, dJ + \Phi\, dQ \ . \tag{2.26}$$

Comparing this with the first law of thermodynamics,

$$dM = T\, dS + \Omega\, dJ + \Phi\, dQ, \tag{2.27}$$

one can easily establish the analogy between black hole mechanics and the first law of thermodynamics. We know that Hořava-Lifshitz theory does not possess the full diffeomorphism invariance of general relativity but only a subset in the form of local Galilean invariance. This subset is manifest in the Arnowitt, Deser and Misner (ADM) slicing. Here we have considered the ADM decomposition of the four dimensional metric. Then for a non-rotating uncharged black hole, the entropy can be written as [113],

$$S = \int \frac{dM}{T} = \int \frac{1}{T_H}\frac{\partial H}{\partial r_h} dr_h, \tag{2.28}$$

where H denotes the enthalpy and r_h denotes the horizon radius. The Hawking temperature can be determined from,

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} f'(r)\Big|_{r=r_h}, \tag{2.29}$$

and,

$$T_H = \left(\frac{\partial H}{\partial S}\right)_P, \tag{2.30}$$

where P is the pressure and it is related to the Hawking temperature and entropy as,

$$P = \frac{1}{2} T_H S. \tag{2.31}$$

Hence the volume of the black hole is given by,

$$V = \left(\frac{\partial H}{\partial P} \right)_S . \tag{2.32}$$

The heat capacity at constant pressure and at constant volume can be obtained respectively as,

$$C_P = T \left(\frac{\partial S}{\partial T} \right)_P , \tag{2.33}$$

and,

$$C_V = C_P + V \frac{\partial P}{\partial T}. \tag{2.34}$$

Using these relations we can calculate the thermodynamic quantities of the LMP black holes in arbitrary space curvature. In the cases of spherical ($k = 1$) and flat spaces ($k = 0$), detailed studies are done in [114]. And it is noted that, in both cases the black hole doesn't show any kind of phase transition behaviors. Hence we are interested in the LMP black hole solution in Hyperbolic space ($k = -1$). In this case, (2.25) can be reduced to,

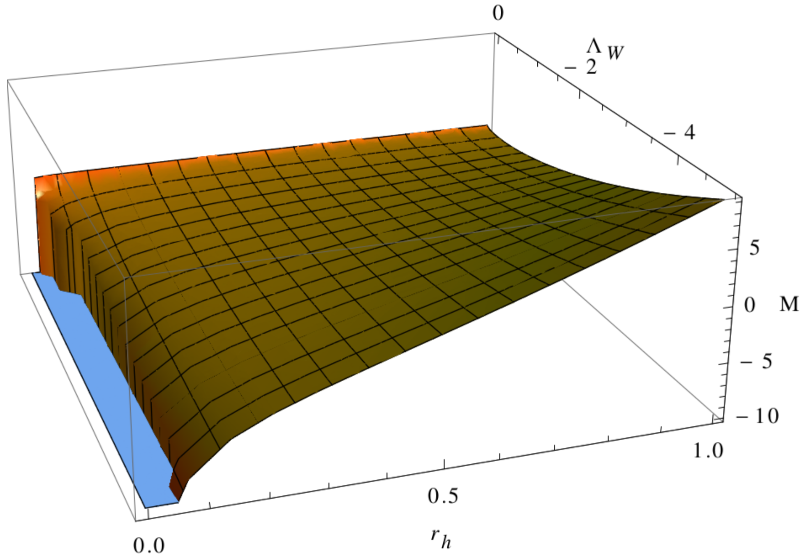

Figure 2.3: 3D Plot of Mass of the LMP black hole with r_h and Λ_W as varying parameters.

$$f(r) = -1 - \Lambda_W r^2 - \mathcal{A}\sqrt{\frac{r}{-\Lambda_W}}. \tag{2.35}$$

The event horizon can be obtained from $f(r_h) = 0$, and from that one can easily arrive at the black hole mass-event horizon radius relation as,

$$M = \frac{1}{a}\sqrt{\frac{-\Lambda_W}{r_h}}\left(-1 - \Lambda_W r_h^2\right). \tag{2.36}$$

In fig.(2.3) we draw the 3D plot of variation of black hole mass with respect to black hole horizon radius for different cosmological constant term (Λ_W). From this figure it can be easily seen that the black hole mass increases with increase in the magnitude of the cosmological constant.

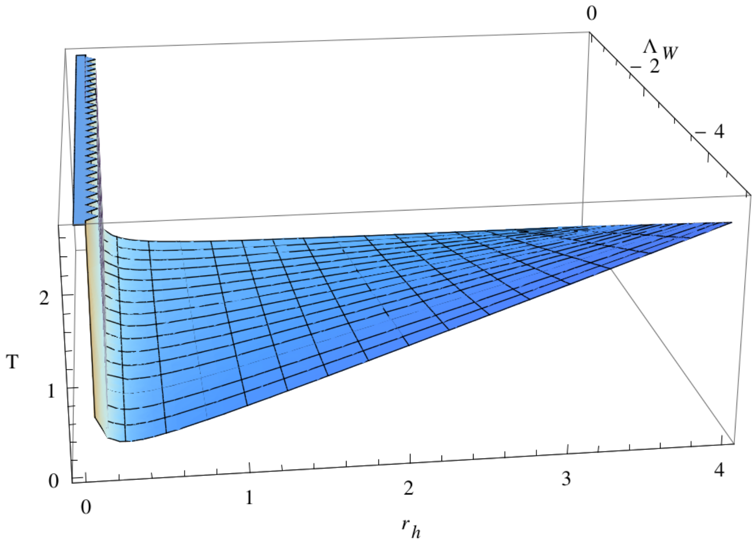

Figure 2.4: 3D Plots of temperature of the LMP black hole with r_h and Λ_W as varying parameters.

Black hole entropy can be obtained from (2.28) as,

$$S = \frac{8\pi\sqrt{-\Lambda_W}}{a}. \tag{2.37}$$

From (2.29) we can derive the Hawking temperature as,

$$T_H = \frac{\left(1 - 3\Lambda_W r_h^2\right)}{8\pi r_h}. \tag{2.38}$$

3D plot of Hawking temperature with respect to black hole horizon radius for varying cosmological constant term is depicted in fig.(2.4). Here also the temperature increases with the magnitude of the cosmological constant. From (2.31), black hole pressure can be found as

$$P = \frac{21 r_h \left(1 - 3\Lambda_W r_h^2\right)}{64}. \tag{2.39}$$

From the above expression, it is obvious that for any negative value of the cosmological constant term Λ_W, the pressure is found to be positive. Using (2.32), black hole volume is given by,

$$V = \frac{32\sqrt{-\Lambda_W}}{21a\, r_h^{3/2}} \frac{\left(1 - 3\Lambda_W r_h^2\right)}{\left(1 - 9\Lambda_W r_h^2\right)}. \tag{2.40}$$

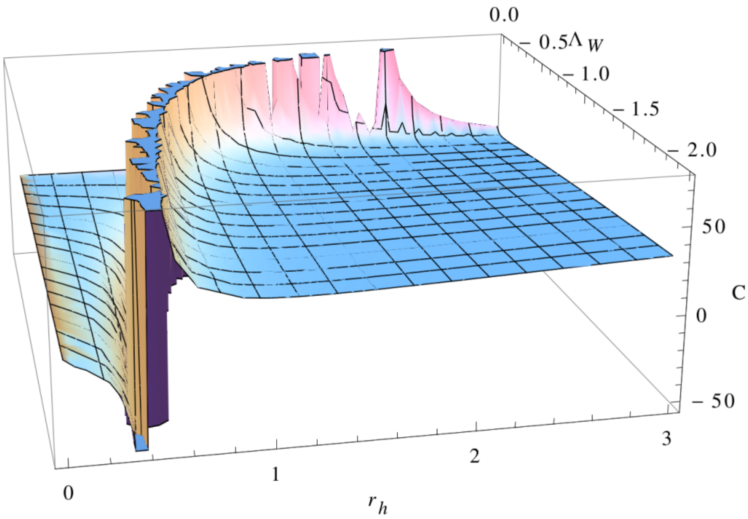

Figure 2.5: 3D Plots of heat capacity of the LMP black hole with r_h and Λ_W as varying parameters.

Using (2.33) and (2.34), the heat capacity at constant pressure and at constant volume are respectively determined as,

$$C_P = \frac{4\pi\sqrt{-\Lambda_W r_h}}{a} \frac{\left(3\Lambda_W r_h^2 - 1\right)}{\left(3\Lambda_W r_h^2 + 1\right)} \tag{2.41}$$

and

$$C_V = \frac{8\pi\sqrt{-\Lambda_W r_h}}{a} \frac{\left(3\Lambda_W r_h^2 - 1\right)}{\left(3\Lambda_W r_h^2 + 1\right)} \tag{2.42}$$

In fig.(2.5) we draw the 3D variations of heat capacity at constant pressure with respect to horizon radius and for varying cosmological constant term. From the figure, it is evident that the black hole has both positive and negative values in certain parametric regions. It is also clear from the figure that, heat capacity has a divergent point. According to Davies [57], phase transitions take place at those points where the heat capacity diverges. So LMP black hole undergoes a phase transition in this case. By investigating the free energy of the black hole we can get further details of the phase transition picture.

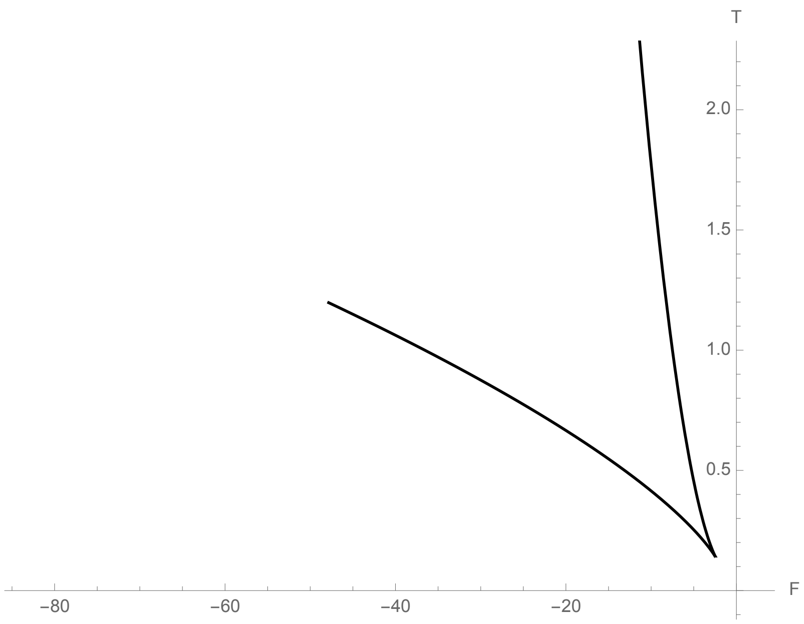

Figure 2.6: Parametric plot of free energy and temperature for $\Lambda_W = -1$, $a = 1$

Free energy of the black hole is given by,

$$F = M - TS. \tag{2.43}$$

Using (2.36), (2.37) and (2.38), the free energy of LMP black hole is obtained as,

$$F = \frac{1}{a}\sqrt{\frac{-\Lambda_W}{r_h}}\left(\frac{2(1-\Lambda_W r_h^2)(1-9\Lambda_W r_h^2)+(3\Lambda_W r_h^2-1)^2}{2(9\Lambda_W r_h^2-1)}\right). \quad (2.44)$$

From above equation one can infer that, free energy is negative for both large and small black holes. Small black holes exhibits negative specific heat and negative free energy, indicating their instability to both perturbative and non-perturbative fluctuations. On the other hand, large black hole have positive specific heat and negative free energy indicating their instability to thermal fluctuations. Now one can plot the parametric variation of free energy and temperature using (2.44) and (2.38). From fig.(2.6), it is evident that there is a cusp like double point (at $T = 0.135$, where $r_h = 0.5742$). This indicates the existence of phase transition. For one of the branches, the free energy decreases and reaches the temperature which corresponds to the minimum free energy. Thereafter, free energy increases with a different slope.

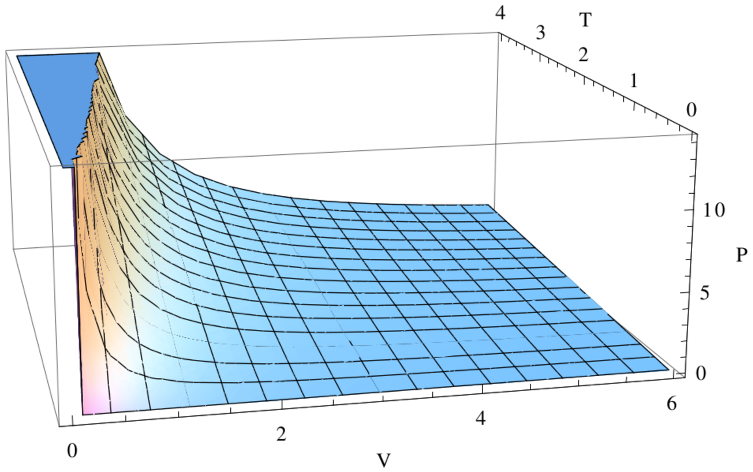

Figure 2.7: Isotherm $P-V$ diagram

For LMP black hole case, since we have defined and calculated pressure and volume, let us connect them through the equation of state and will check whether they have any resemblance with usual thermodynamic systems. An equation of state in general is a thermodynamic equation describing the state of matter under a given set of physical conditions. It is a constitutive equation which provides a mathematical relationship between two or more state functions associated with the matter, such as temperature, pressure, volume, or internal energy and black hole equation of state can be obtained from (2.39), (2.40) and (2.38) as,

$$PV^{\frac{4}{3}} = \frac{4\pi T}{3}\sqrt{\frac{32}{63}}\frac{-\Lambda_W}{a^2}. \tag{2.45}$$

Here, P is the the pressure, V is the thermodynamic volume and T is the black hole temperature. Now, we plot the isotherm $P-V$ diagram in fig.(2.7). From (2.45) and the corresponding fig.(2.7) we can conclude that the behaviour resembles the adiabatic expansion of an ideal gas. Hence no critical point can be found and there would be no $P-V$ criticality. We have systematically analyzed the thermodynamics and phase transition structure of the LMP black hole. From this thermodynamic study, absence of any discontinuity in entropy-temperature relationship eliminates the presence of any first order transition. Whereas the heat capacity of the system is found to be diverging, thereby indicating the presence of a phase transition. But the exact order of phase transition is not revealed. Further studies are needed to get the exact picture. We will address these issues in chapter 4 by analysing the black hole system using geometrothermodynamics formalism.

Park black hole

In this section, we consider the black hole solutions in the generalized model with the IR modification term $\mu^4 R^{(3)}$ but with an "arbitrary" cosmological constant in the Hořava gravity. These solutions are analogues to the Schwarzschild-(A)dS solutions in Einstein's general relativity which have been absent in the original Hořava model. By introducing an IR modification term, the modified action can be written as,

$$\begin{aligned} S &= \int dtd^3x\sqrt{g}N[\frac{2}{\kappa}\left(K_{ij}K^{ij}-\lambda K^2\right)-\frac{\kappa^2}{2\nu^4}C_{ij}C^{ij} \\ &+ \frac{\kappa^2\mu}{2\nu^2}\epsilon^{ijk}R^{(3)}_{il}\nabla_j R^{(3)l}_{k}-\frac{\kappa^2\mu^2}{8}R^{(3)}_{ij}R^{(3)ij}+\frac{\kappa^2\mu^2\omega}{8(3\lambda-1)}R^{(3)} \\ &+ \frac{\kappa^2\mu^2}{8(3\lambda-1)}\left(\frac{4\lambda-1}{4}(R^{(3)})^2-\Lambda_W R^{(3)}+3\Lambda_W^2\right), \end{aligned} \tag{2.46}$$

where K_{ij} and C^{ij} are the extrinsic curvature and the Cotton tensor, respectively. In the action, $\kappa, \nu, \mu, \lambda, \Lambda_W, \omega$ are constant parameters. The last term in (2.46) represents a soft violation of the detailed balance condition [10]. For static and spherically symmetric solution, substituting the metric ansatz as,

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.47}$$

in the action (2.46) and after angular integration, we obtain the Lagrangian as,

$$\begin{aligned} \mathcal{L} &= \frac{\kappa^2\mu^2}{8(1-3\lambda)}\frac{N}{\sqrt{f}}[(2\lambda-1)\frac{(f-1)^2}{r^2} - 2\lambda\frac{f-1}{r}f' \\ &+ \frac{\lambda-1}{2}f'^2 - 2(\omega-\Lambda_W)(1-f-rf') - 3\Lambda_W^2 r^2]. \end{aligned} \tag{2.48}$$

Kehagias and Sfetsos [110] obtained only the asymptotically flat solution (with $\Lambda_W = 0$) while Mu-In Park [112] considered an arbitrary Λ_W and obtained a general solution. The equation of motion of this system can be calculated by varying the lapse function, N,

$$\begin{aligned} &(2\lambda-1)\frac{(f-1)^2}{r^2} - 2\lambda\frac{f-1}{r}f' + \frac{\lambda-1}{2}f'^2 \\ &-2(\omega-\Lambda_W)(1-f-rf') - 3\Lambda_W^2 r^2 = 0, \end{aligned} \tag{2.49}$$

and by varying f,

$$\begin{aligned} &\left(\frac{N}{\sqrt{f}}\right)' \left((\lambda-1)f' - 2\lambda\frac{f-1}{r} + 2(\omega-\Lambda_W)r\right) \\ &\qquad + (\lambda-1)\frac{N}{\sqrt{f}}\left(f'' - \frac{2(f-1)}{r^2}\right) = 0. \end{aligned} \tag{2.50}$$

By choosing $\lambda = 1$ and solving these field equations, we arrive at the Park solution [112],

$$N^2 = f_{\text{Park}} = 1 + (\omega-\Lambda_W)r^2 - \sqrt{r[\omega(\omega-2\Lambda_W)r^3 + \beta]}, \tag{2.51}$$

where β is an integration constant which is related to the black hole mass. Now let us consider the asymptotically dS case of Park solution. In this case, the action is given by an analytic continuation [111] of the action given in (2.46) ,

$$\mu \to i\mu,\ \nu^2 \to -i\nu^2,\ \omega \to -\omega. \tag{2.52}$$

From (2.51), for $r \gg [\beta/|\omega(\omega - 2\Lambda_W)|]^{1/3}$, i.e., considering assymptotically de Sitter case with $\Lambda_W > 0$ and $\omega < 0$, we can arrive at,

$$f = 1 - \frac{\Lambda_W}{2}\left|\frac{\Lambda_W}{\omega}\right| r^2 - \frac{2M}{\sqrt{1+2|\Lambda_W/\omega|}}\frac{1}{r} + \mathcal{O}(r^{-4}). \tag{2.53}$$

When we compare (4.3) with the Schwarzschild-dS solution

$$f = 1 - \frac{\Lambda_W}{2}r^2 - \frac{2M}{r}, \tag{2.54}$$

we can see that it agrees up to some numerical factor corrections. In the coming sections we will investigate more about this agreement.

In order to explore the thermodynamics of Park black hole, let us consider (4.3). In general dS solution has two horizons. Larger one, r_{++} corresponding to the cosmological horizon and the smaller one, r_+ for the black hole horizon. By considering this black hole horizon, we can arrive at a relation which connects mass and horizon radius of the black hole,

$$M = \frac{1 + 2(\omega - \Lambda_W)r_+^2 + \Lambda_W^2 r_+^4}{4\omega r_+}. \tag{2.55}$$

Then from the usual definition of temperature in thermodynamics, we can obtain the temperature of the black hole with $\Lambda_W > 0$ and $\omega < 0$ as,

$$T = \frac{3\Lambda_W^2 r_+^4 + 2(\omega - \Lambda_W)r_+^2 - 1}{8\pi r_+(1 + (\omega - \Lambda_W)r_+^2)}. \tag{2.56}$$

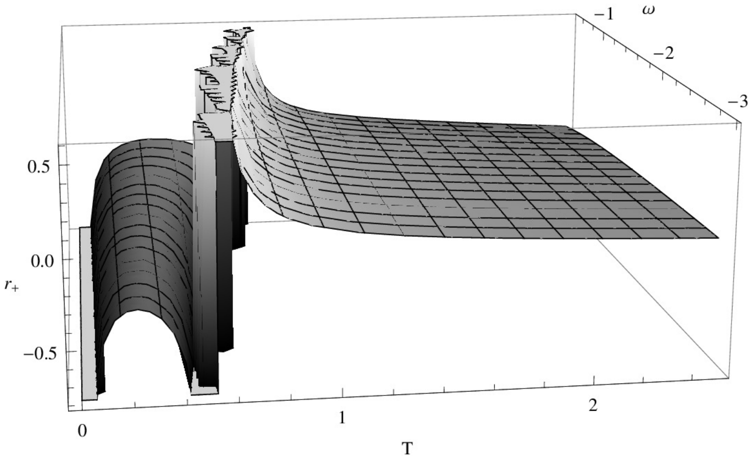

Figure 2.8: Variation of temperature with horizon radius for different values of ω with $\Lambda_W = 0.0001$.

We have plotted the variation of black hole temperature against the horizon radius in fig.(2.8). From this plot it is evident that there is an infinite discontinuity in temperature. It occurs at,

$$\tilde{r}_+ = \frac{1}{\sqrt{\Lambda_W - \omega}}. \tag{2.57}$$

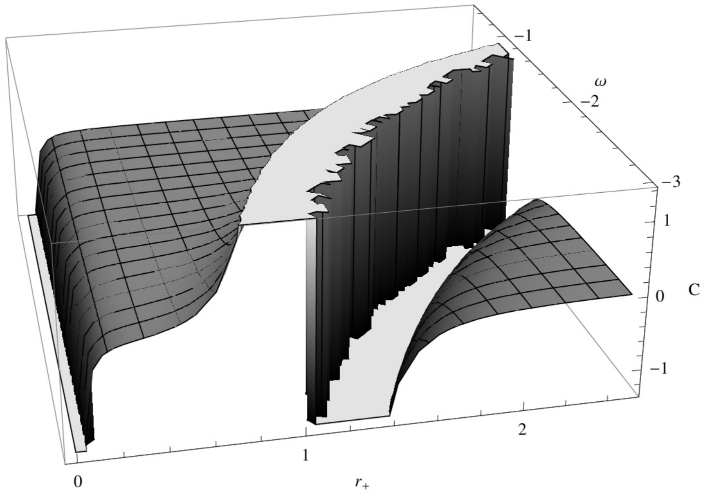

Figure 2.9: Variation of specific heat with horizon radius for different values of ω with $\Lambda_W = 0.0001$.

Interestingly for the region $r_+ < \tilde{r}_+$, the temperature is found to be negative. The heat capacity of the black hole is given by,

$$\begin{aligned} C &= \frac{1}{128\pi^3 r_+^4((\Lambda_W - \omega)r_+^2 - 1)^3}[(\Lambda_W r_+^2 - 1)^3(9\Lambda_W^2 r_+^4 - 1) \\ &+ 3r_+^2(1 + \Lambda_W r_+^2(8 - 6\Lambda_W r_+^2 - 3\Lambda_W^3 r_+^6))\omega \\ &- 12r_+^4(1 + \Lambda_W r_+^2)\omega^2 + 4r_+^6\omega^3]. \end{aligned} \tag{2.58}$$

Variation of heat capacity with respect to the black hole horizon r_+ for different values of coupling parameters are plotted in fig.(2.9). From this we can see that specific heat undergoes a transition from negative values to positive values or in other words black hole changes from a thermodynamically unstable phase to a thermodynamically stable phase. By looking and comparing the two figures, fig.(2.8) and fig.(2.9), we can straightaway say that in the region where temperature shows the anomalous behaviour due to its negative values, the black hole is found to be thermodynamically unstable as its heat capacity is negative. Now for a Schwarzschild-dS black hole, from (2.54) we can write the event horizon, defined by $f(r_+) = 0$, as,

$$r_+ - \frac{\Lambda_W}{2} r_+^3 - 2M = 0. \tag{2.59}$$

From this the mass of the black hole can be written as,

$$M = \frac{r_+}{2} - \Lambda_W r_+^3. \tag{2.60}$$

Employing the Bekenstein-Hawking area law,

$$S = \frac{A}{4} = \pi r_+^2, \tag{2.61}$$

we can rewrite the mass-horizon relation (2.60) as,

$$M = \frac{1}{2}\sqrt{\frac{S}{\pi}} - \Lambda_W \left(\frac{S}{\pi}\right)^{3/2}. \tag{2.62}$$

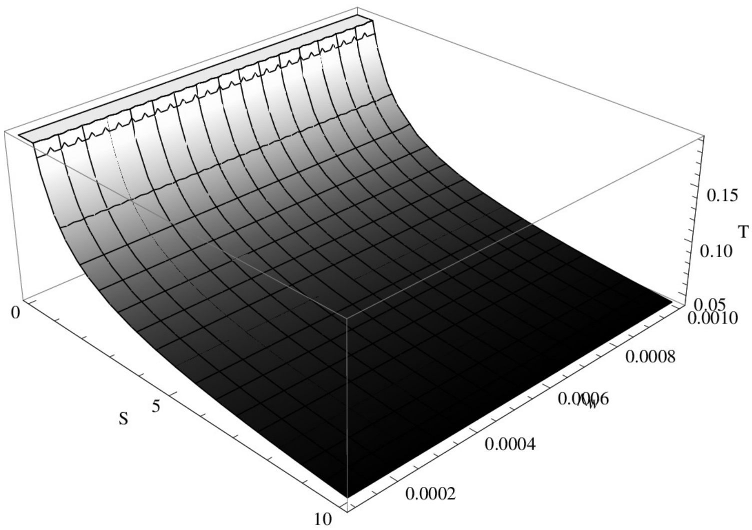

Figure 2.10: Variation of temperature with entropy for SdS black hole.

Using the definition of temperature as $T = \left(\frac{\partial M}{\partial S}\right)$ and that of heat capacity as $C = T\left(\frac{\partial S}{\partial T}\right)$, we can arrive at,

$$T = \frac{1}{4\sqrt{\pi S}} - \frac{3\Lambda_W\sqrt{S}}{2\pi^{3/2}}, \tag{2.63}$$

and,

$$C = \frac{\pi^2}{3\Lambda_W\pi + 18\Lambda_W^2 S} - \frac{3\Lambda_W\sqrt{S}}{2\pi^{3/2}} - \frac{\pi}{3\Lambda_W}. \tag{2.64}$$

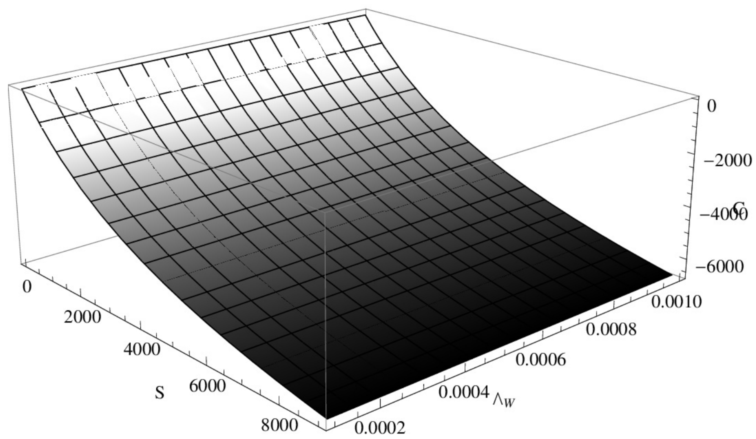

Figure 2.11: Variation of specific heat with entropy for SdS black hole.

Variation of temperature with respect to entropy is plotted in fig.(2.10) while in fig.(2.11) the variation of heat capacity with entropy is plotted. From fig.(2.11) it is evident that Schwarzschild-dS black hole is thermodynamically unstable for all range of entropy values.

Now let us investigate a peculiar behavior of Park black hole. As explained in [112], for a black hole horizon to exist and curvature singularity at $r = 0$ is not naked, one has to consider another constraint on mass parameter that can be obtained from (2.55) and is given by,

$$M \leq \frac{(2\Lambda_W - \omega)}{4} r_+^3. \tag{2.65}$$

Or we can say that, $M < \frac{(2\Lambda_W - \omega)}{4} r_+^3$ for all r_+ except for $r_+ = \tilde{r}_+$. It can verified using fig.4.8 that at $\tilde{r}_+$ mass of the black hole meets the upper bound. Now let us investigate the thermodynamics of the black hole which

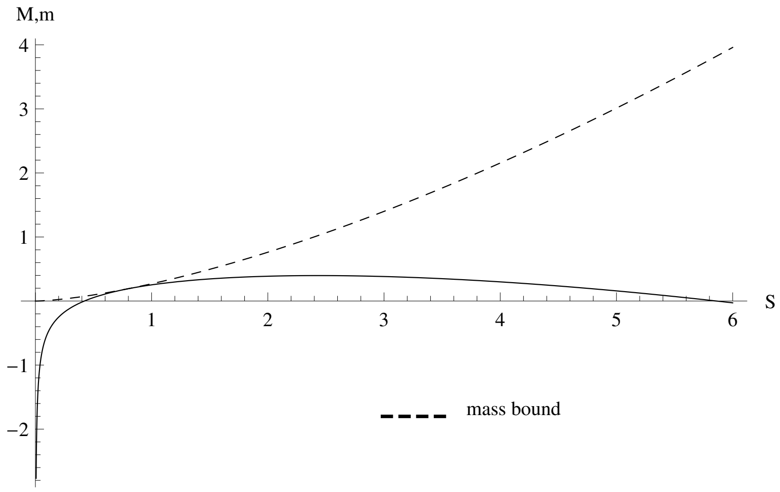

Figure 2.12: Variation of mass of the Park black hole with entropy. Change in mass bound value with entropy is depicted using dotted lines.

has a mass given by the upper mass bound value given in (2.65) i.e.,

$$M_{\text{bound}} = \frac{(2\Lambda_W - \omega)}{4} r_+^3. \tag{2.66}$$

Using Bekenstein-Hawking area law, we can rewrite (2.66) as,

$$M_{\text{bound}} = \frac{(2\Lambda_W - \omega)}{4} \left(\frac{S}{\pi}\right)^{3/2}. \tag{2.67}$$

From the usual definition of temperature and that of heat capacity, we can arrive at,

$$T = \frac{3(2\Lambda_W - \omega)\sqrt{S}}{8\pi^{3/2}}, \tag{2.68}$$

and

$$C = 2S. \tag{2.69}$$

Since the heat capacity is positive, the black hole having a mass given by (2.66) is thermodynamically stable. From this fact we can say that the occurrence of infinite discontinuity as well as negative temperature may be due to the existence of a restriction given by (2.65) for the mass parameter.

We have analyzed the usual thermodynamics of both these cases and found that there exist many abnormal behaviors like the existence of an upper mass bound, negative temperature, infinite discontinuity in temperature and heat capacity. Employing the ideas of usual thermodynamics, it is not possible to explain these peculiar behaviours of the Park black hole system.

2.2 Black holes in Massive gravity

From the theoretical point of view, massive gravity is the theory of gravity that modifies Einstein's general relativity by adding a nonzero mass to the graviton. As we have discussed in the Chapter 1, Massive gravity has a prolonged history, from early 1930s onwards [13]. The much awaited breakthrough was achieved when de Rham, Gabadadze, and Tolley proposed [19, 20] and constructed, a theory of massive gravity which is free from Boulware-Deser ghost [17, 18] by wrapping all the ghostly terms into a total derivatives that has no contribution to the equations of motion. In this section, we will discuss black hole solution in two different massive gravity theories, which are free from any ghosts. Among them first one is de Rham, Gabadadze, and Tolley thoery (dRGT) [19, 20] and the other is New massive gravity (NMG) [33].

2.2.1 dRGT massive gravity

The massive gravity model proposesd by de Rham, Gabadadze, and Tolley can be described using the action [19, 20],

$$S = \int d^D x \left[\frac{M_{pl}^2}{2} \sqrt{-g} \left(R + m^2 U(g, H) \right) \right], \tag{2.70}$$

where the first term is the usual Einstein-Hilbert action and the second term is arising from the contributions of mass of the graviton m, and from the nonlinear higher derivative term U corresponding to the massive graviton. It is given by,

$$U = U_2 + \alpha_3 U_3 + \alpha_4 U_4, \tag{2.71}$$

where,

$$\begin{aligned} U_2 &= [K]^2 - [K^2] \\ U_3 &= [K]^3 - [K][K^2] + 2[K^3] \\ U_4 &= [K]^4 - 6[K]^2[K^2] + 8[K^3][K] - 6[K^4]. \end{aligned} \tag{2.72}$$

In the above set of equations, the tensor K^μ_ν is defined as,

$$K^\mu_\nu = \delta^\mu_\nu - \sqrt{\partial^\mu \phi^\alpha \partial_\nu \phi^\beta f_{\alpha\beta}}, \tag{2.73}$$

where ϕ^α and ϕ^β are the corresponding Stučkelberg field and $f_{\alpha\beta}$ is a fixed symmetric tensor usually called as the reference metric.

In the unitary gauge, defined as $\phi^a = x^a$, the term $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the gravitational analogue of the Proca field of massive electrodynamics [36]. By introducing the Stückelberg field ϕ^a, which can be considered as background field plus a pion contribution, $\phi^a = x^a + \pi^a$ [37], and replacing the Minkowski metric by,

$$g_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} + H_{\mu\nu},$$

where $H_{\mu\nu}$ is the covariantized metric perturbation, one can restore the diffeomorphism invariance. As given in [36, 37], two new coefficients α and β are introduced which relate the coefficients α_3 and α_4 in (2.71) by,

$$\alpha_3 = -\frac{(-\alpha + 1)}{3}, \tag{2.74}$$

and

$$\alpha_4 = \frac{-\beta}{2} + \frac{(-\alpha + 1)}{12}. \tag{2.75}$$

In empty space, the equation of motion is given as,

$$G_{\mu\nu} + m^2 X_{\mu\nu} = 0, \tag{2.76}$$

where $X_{\mu\nu}$ is the effective energy-momentum tensor contributed by the graviton mass m, which is given by,

$$\begin{aligned} X_{\mu\nu} &= -\frac{1}{2}\Big[K g_{\mu\nu} - K_{\mu\nu} + \alpha\Big(K^2_{\mu\nu} - K K_{\mu\nu} + \frac{1}{2} g_{\mu\nu}([K]^2 - [K^2])\Big) \\ &\quad +6\beta\Big(K^3_{\mu\nu} - K K^2_{\mu\nu} + \frac{1}{2} K_{\mu\nu}([K]^2 - [K^2]) \\ &\quad -\frac{1}{6} g_{\mu\nu}([K]^3 - 3[K][K^2] + 2[K^3])\Big)\Big]. \end{aligned} \tag{2.77}$$

Now applying the Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$ in (2.76), we arrive at the constraint equation,

$$m^2 \nabla^\mu X_{\mu\nu} = 0. \tag{2.78}$$

The parameters of the action, namely α and β can be chosen in different ways so that one ends up with different black hole solutions. Particularly for the choice $\beta = \alpha^2$, the space of solution is much wider than the general case discussed in [115]. As a result, this choice leads to much richer family of solutions compared to the general choice of α and β. Here, we concentrate on a particular family of the ghost-free theory of massive gravity [36–38], where,

$$\beta = -\frac{\alpha^2}{6}. \tag{2.79}$$

For this special choice, introduced in [38], (2.78) is automatically satisfied for a certain diagonal and time-independent metrics in spherical polar coordinates. One can consider this as a limitation and these exact analytic black hole solutions are obtained only for a specific choice of the two free parameters of massive gravity given by (2.79). But such a choice of parameters is peculiar because on this background the kinetic terms for both the vector and scalar fluctuations vanish in the decoupling limit. Hence one would expect infinitely strong interactions for these modes. To account for this issue, we would consider these solutions as just as an example for demonstrating how non-singular solutions could emerge as well as how their thermodynamic properties and interactions behave in the presence and absence of the massive parameters.

Now using (2.76), (2.77) and (2.78), a spherically symmetric and time independent metric in de Sitter space can be obtained, by choosing,

$$m^2 X_{\mu\nu} = \lambda g_{\mu\nu}, \tag{2.80}$$

where λ is a constant. The solutions of (2.76) that satisfies the condition given in (2.80) with a positive but otherwise arbitrary α is given by,

$$ds^2 = -\kappa^2 dt^2 + \left(\frac{\alpha}{\alpha+1}dr \pm \kappa\sqrt{\frac{2}{3\alpha}}\frac{\alpha}{\alpha+1}mr dt\right)^2 + \frac{\alpha^2}{(\alpha+1)^2}r^2 d\Omega^2. \tag{2.81}$$

Here κ is a positive integration constant. The solution thus obtained is free of any singularities. Now by coupling this ghost-free massive gravity theory to the Maxwell's theory of electromagnetism, one can obtain the

Reissner- Nordström solution in dS space as,

$$ds^2 = -dt^2 + \left(\tilde{\alpha} dr \pm \sqrt{\frac{r_g}{\tilde{\alpha}} + \frac{2\tilde{\alpha}^2}{3\alpha} m^2 r^2 - \frac{\tilde{Q}^2}{\tilde{\alpha}^4 r^2}} dt\right)^2 + \tilde{\alpha}^2 r^2 d\Omega^2, \quad (2.82)$$

here $\tilde{\alpha} \equiv \alpha/(\alpha+1)$, m is mass of graviton, α is the curvature parameter and the electromagnetic field is given by,

$$E = \frac{\tilde{Q}}{r^2} \qquad \text{and} \qquad B = 0. \quad (2.83)$$

To rewrite the above charged dS solution in arbitrary space-time in the static slicing, one can make the following transformations for spatial and temporal coordinates respectively as,

$$r \to \frac{r}{\tilde{\alpha}}, \quad (2.84)$$

and,

$$dt \to dt + f'(r) dr, \quad (2.85)$$

where,

$$f'(r) \equiv -\frac{g_{01}}{g_{00}} = \pm \frac{\sqrt{\frac{r_g}{r} + \frac{2}{3\alpha} m^2 r^2 + \frac{\tilde{Q}^2}{\tilde{\alpha}^2 r^2}}}{k - \frac{r_g}{r} - \frac{2}{3\alpha} m^2 r^2 + \frac{\tilde{Q}^2}{\tilde{\alpha}^2 r^2}}. \quad (2.86)$$

Once these transformations are performed, one can easily rewrite (2.82) in a familiar form as,

$$\begin{aligned} ds^2 &= -\left(k - \frac{r_g}{r} - \frac{2}{3\alpha} m^2 r^2 + \frac{\tilde{Q}^2}{\tilde{\alpha}^2 r^2}\right) dt^2 \\ &\quad + \frac{dr^2}{k - \frac{r_g}{r} - \frac{2}{3\alpha} m^2 r^2 + \frac{\tilde{Q}^2}{\tilde{\alpha}^2 r^2}} + r^2 d\Omega^2. \end{aligned} \quad (2.87)$$

Due to the above transformations, the Stückelberg field becomes,

$$\phi^0 = t + f(r), \quad (2.88)$$

$$\phi^r = r\left(\frac{\alpha+1}{\alpha}\right), \tag{2.89}$$

and electromagnetic field is,

$$E = \frac{\tilde{Q}}{\tilde{\alpha}r^2} \qquad \text{and} \qquad B = 0. \tag{2.90}$$

where the actual charge should be redefined as $Q \equiv \tilde{Q}/\tilde{\alpha}$.

Using (2.87) and solving for $f(r) = 0$ at the horizon limit and incorporating the Bekenstein-Hawking area law, $S = \frac{A}{4}$, one can easily obtain the mass of the black hole as,

$$M = \frac{3\pi^2 Q^2\alpha + \tilde{\alpha}S\left(3k\pi\alpha - 2m^2 S\right)}{6\tilde{\alpha}^2\pi^{3/2}\sqrt{S}\alpha}. \tag{2.91}$$

From the first law of thermodynamics, $\delta M = T\delta S + \Phi\delta Q$, temperature T can be calculated as,

$$T = \frac{\tilde{\alpha}S\left(k\pi\alpha - 2m^2 S\right) - \pi^2 Q^2\alpha}{4\tilde{\alpha}^2\pi^{3/2}S^{3/2}\alpha}. \tag{2.92}$$

Using the classical thermodynamic relation $C = T\left(\frac{\partial S}{\partial T}\right)$, one can obtain the heat capacity of the black hole as,

$$C = \frac{2S\left(\pi^2 Q^2\alpha + \tilde{\alpha}S(2m^2 S - k\pi\alpha)\right)}{\tilde{\alpha}^2 S(k\pi\alpha + 2m^2 S) - 3\pi^2 Q^2\alpha}. \tag{2.93}$$

It is interesting to note that, the charged black hole solution in massive gravity (2.87) with m as the mass of the graviton can behave as Reissner-Nordström solution in de-Sitter and anti de-Sitter space-time in massive gravity with respect to the choice of curvature parameter α. We can see that (2.87) will reduce to RNdS solution in massive gravity for $\alpha > 0$, for $\alpha < 0$ we can obtain RNAdS solution and the RN solution in Einstein's general relativity for mass of the graviton $m = 0$.

Before going in to the details of phase transition structure of charged black holes in massive gravity, let us consider the physics of the curvature parameter α in details. From the black hole space-time metric (2.87), and comparing it with the charged dS or AdS black hole in Einstein's gravity theory, one can easily identify the cosmological constant term in massive gravity as,

$$\Lambda = \frac{m^2}{\alpha}. \tag{2.94}$$

We know that in most of the black hole thermodynamic studies, the cosmological constant is treated as a fixed parameter. But it has been suggested that it is better to treat Λ as a thermodynamic variable [116–118]. In many studies, this cosmological constant is treated as the thermodynamic variable, the pressure [119, 120]. Accordingly the cosmological constant generated pressure can be written as,

$$P = -\frac{\Lambda}{8\pi G}. \tag{2.95}$$

From (2.94), one can rewrite the pressure generated by cosmological constant in massive gravity as,

$$P = -\frac{m^2}{8\pi\alpha}. \tag{2.96}$$

Hence the choice of curvature parameter α will determine whether the space-time is de Sitter or anti de Sitter. The presence of negative pressure, by choosing the curvature parameter as positive, point towards the accelerated expansion of the present universe. Further studies in this direction may lead to a better understanding of this phenomenon.

Charged de-Sitter black hole in massive gravity

Let us consider the case where the curvature parameter is taken to be positive ($\alpha > 0$), then the black hole solution given by (2.87) reduces to charged de-Sitter solution in massive gravity (RNdS). Now the variation of temperature and specific heat with entropy of RNdS black hole is plotted for different space-time curvature in figs.(2.13) and (2.14).

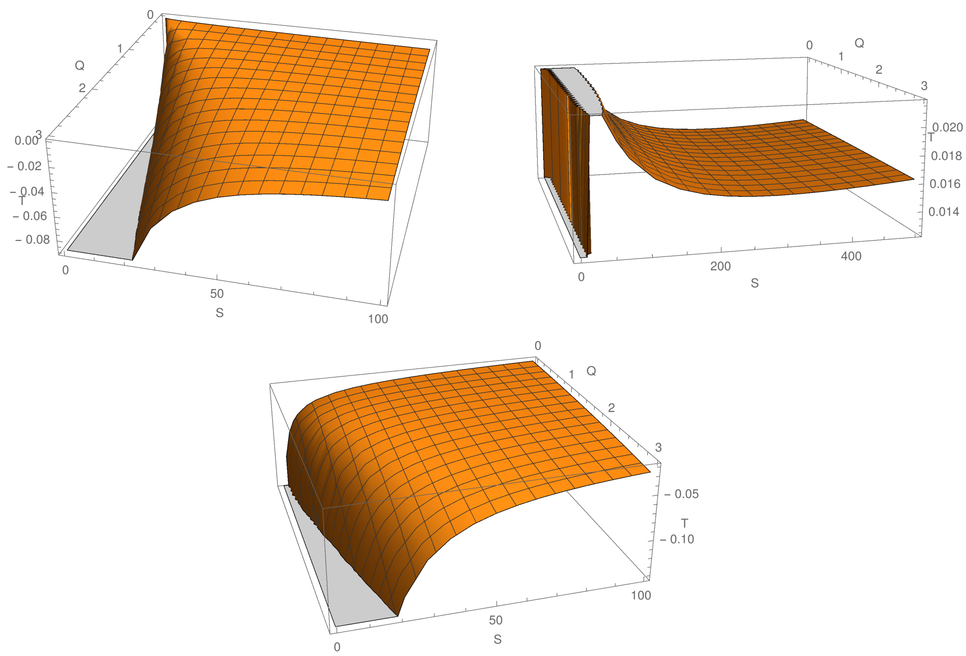

Figure 2.13: 3D Variation of temperature against entropy and charge for de-Sitter black holes for flat, spherical and hyperbolic topology of space-time in massive gravity

From fig.(2.13) we can see that for the flat case ($k = 0$), the temperature is always negative, and hence has no curiosity. For the spherically symmetric space-time case ($k = 1$), the temperature initially enters in to a physically insignificant region (with negative temperature) and lies in a positive region for intermediate sized black holes. For black holes with larger horizon radius it again goes to the negative temperature regions. For the hyperbolic space-time case ($k = -1$), the behaviour exactly resembles that of flat space-time. So for the RNdS case in massive gravity for spherically symmetric space-time, there exists a window at which the black hole has positive temperatures and hence lies in a physically significant region.

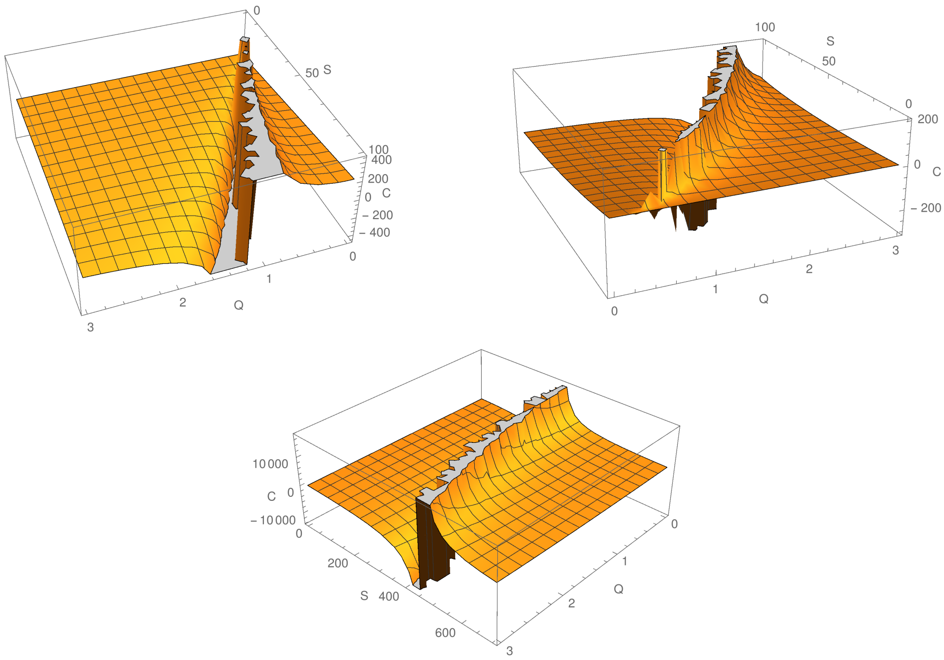

Figure 2.14: 3D Variation of heat capacity against entropy and charge for de-Sitter black holes for flat, spherical and hyperbolic topology of space-time in massive gravity

The variation of heat capacity depicted in fig.(2.14) shows that, black hole undergoes phase transitions for all the cases, $k = 0, 1, -1$. For flat and hyperbolic space-time cases, the black hole lies in a thermodynamically unstable phase and undergoes an infinite discontinuity and become thermodynamically stable. Similarly for spherically symmetric space-time, the black hole goes from a thermodynamically stable region to an unstable region.

Charged anti de-Sitter black hole in massive gravity

In this section we will consider the case in which the curvature parameter is taken as negative ($\alpha < 0$). For this case the the black hole solution (2.87) reduces to charged anti de Sitter solution in massive gravity (RNAdS). Here also variation of temperature and specific heat against entropy for different space-time cases are depicted in figs.(2.15) and (2.16).

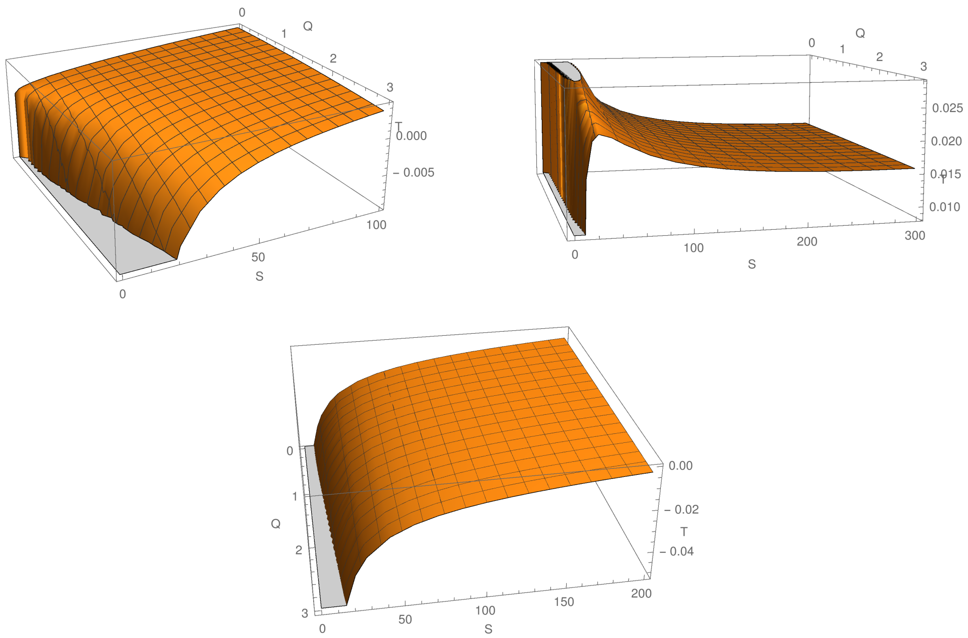

Figure 2.15: 3D Variation of temperature against entropy and charge for anti de-Sitter black holes for flat, spherical and hyperbolic topology of space-time in massive gravity

Temperature variation in fig.(2.15), implies that for each space-time cases, the black hole changes from a negative temperature region to a region where temperature become positive. But for the spherically symmetric space-time case, the temperature goes to a maximum positive value and falls down to lower values. Hence a temperature window-like behaviour is shown by these black holes too. From the fig.(2.16) we can infer that the for the flat case, the specific heat goes from negative values to positive values and hence the black hole system changes from a thermodynamically unstable phase to a stable phase without any phase transitions.

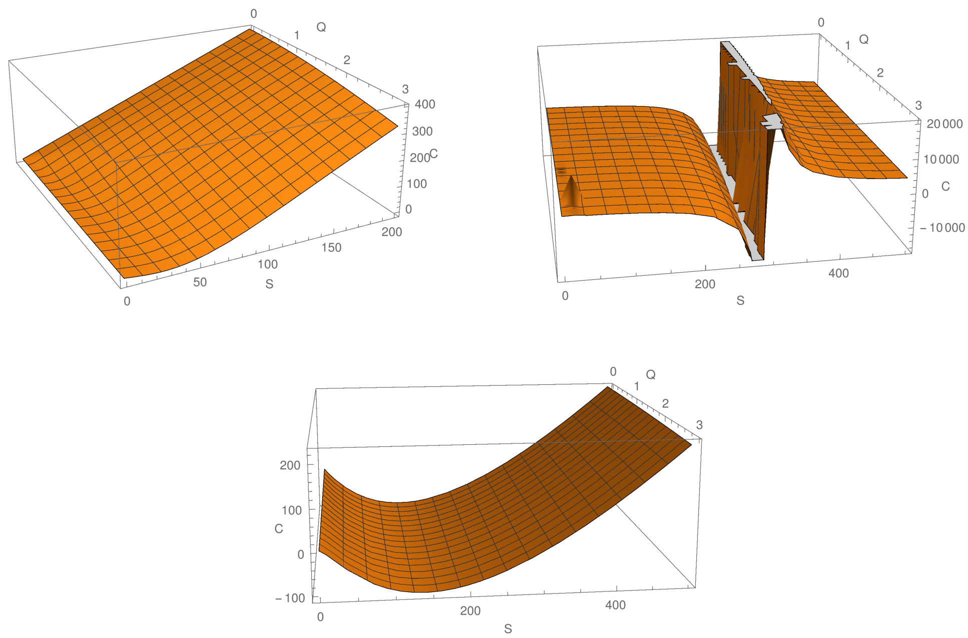

Figure 2.16: 3D Variation of heat capacity against entropy and charge for anti de-Sitter black holes for flat, spherical and hyperbolic topology of space-time in massive gravity

This behaviour exactly resembles that of RNdS black holes. This resemblance exists in the hyperbolic space-time case too. For the spherically symmetric case, the black hole initially lies in a thermodynamically unstable phase and transit to a stable phase. Later it undergoes a phase transition in which the stable black hole becomes an unstable one.

Charged black holes in Einstein's general relativity

Now let us consider the situation that the graviton has no mass ($m = 0$), then the black hole system given by (2.87) will reduce to a charged black hole solution in Einstein's general relativity, i.e., Reissner-Nordström black hole solution. For $m = 0$ case, one can write down the metric $f(r)$ from (2.87) as,

$$f(r) = k - \frac{r_g}{r} + \frac{\tilde{Q}^2}{\tilde{\alpha}^2 r^2}. \tag{2.97}$$

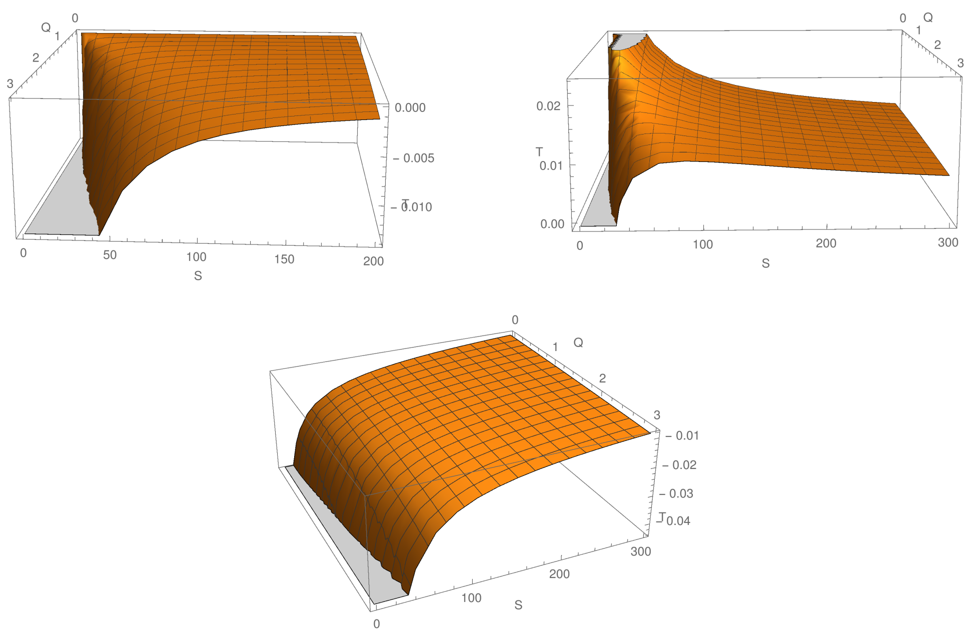

Figure 2.17: 3D Variation of temperature against entropy and charge for black holes in Einstein's general relativity for flat, spherical and hyperbolic topology of space-time

It is evident from the above equation that, it exactly matches with the Reissner-Nordström black hole solution in Einstein's general relativity. Solving the above equation (2.97), and using the relation $Q \equiv \tilde{Q}/\tilde{\alpha}$, one can easily write the mass of the black hole as,

$$M = \frac{\pi Q^2 + kS}{2\sqrt{\pi}\sqrt{S}}. \tag{2.98}$$

From the usual thermodynamic relations $T = \frac{\partial M}{\partial S}$ and $C = T\frac{\partial S}{\partial T}$ one can write temperature and heat capacity respectively as,

$$T = \frac{-\pi Q^2 + kS}{4\sqrt{\pi}S^{3/2}}, \tag{2.99}$$

and

$$C = \frac{2S(-\pi Q^2 + kS)}{3\pi Q^2 - kS}. \tag{2.100}$$

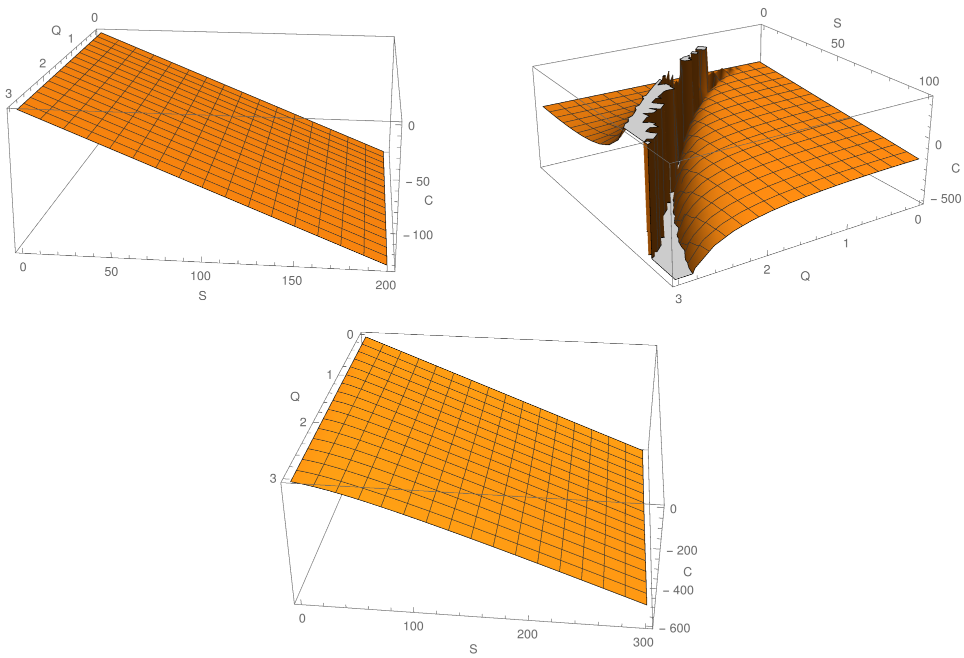

Figure 2.18: 3D Variation of heat capacity against entropy and charge for black holes in Einstein's general relativity for flat, spherical and hyperbolic topology of space-time

The variations of both temperature and heat capacity are plotted. From the figs.(2.17) and (2.18), we can see that for flat case ($k = 0$) as well as for the hyperbolic ($k = -1$) case temperature always lies in the negative value region and hence in physically insignificant situation. Now for the spherically symmetric space-time case ($k = 1$), black hole initially lies in a negative temperature region and as the black hole horizon radius increases it attains a maximum temperature value. After the maximum value of temperature is attained it lies in the positive temperature region itself. We can see that when mass of the graviton becomes zero, it exactly reproduces the results of RN black holes. So the limiting case of RNdS or RNAdS black hole in massive gravity coincides with RN solution in Einstein's theory and their thermodynamic behaviour gives the proof for the same.

We have analysed the thermodynamic behaviour of charged dS and AdS black holes in dRGT massive gravity in different space-time curvature using both analytical and graphical methods. The analysis showed that these black holes undergo phase transitions by changing the heat capacity signs.

2.2.2 New massive gravity

BTZ black hole

After the investigations on general relativity in three dimensional space-time by Deser, Jackiw, 't Hooft and Witten [121–124], (2+1) models gained much attention. In the history of research they are used as a laboratory tool to provide a fundamental platform to perform the studies on thermodynamic aspects of the gravitational system, though it has been widely believed that these models are physically unrealistic. Black hole solution to this lower dimensional gravitation theory came as a surprise when Bañados, Teitelboim and Zanelli obtained the BTZ solution in 1992

[125]. (2+1) black holes are now known as the toy model black holes. Here we will discuss the (2+1) black hole solutions in New massive gravity.

Three dimensional higher derivative gravity model, the New Massive Gravity (NMG), was proposed by Bergshoeff, Hohm and Townsend in 2009 [33]. The action can be written as a higher curvature term added to the usual Einstein-Hilbert action,

$$S_{\mathrm{NMG}} = S_{\mathrm{EH}} + S_{\mathrm{R}}, \tag{2.101}$$

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\,(R-2\lambda)\;, \tag{2.102}$$

$$S_{\mathrm{R}} = -\frac{1}{16\pi G m^2}\int d^3x\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2\right), \tag{2.103}$$

where m^2 is a mass parameter with mass dimension and G is a three dimensional Newton constant. The equation of motion is given by,

$$G_{\mu\nu}+\lambda g_{\mu\nu}-\frac{1}{2m^2}K_{\mu\nu}=0\;, \tag{2.104}$$

where $G_{\mu\nu}$ is the Einstein tensor given by,

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$$

and,

$$\begin{aligned} K_{\mu\nu} &= 2\Box R_{\mu\nu}-\frac{1}{2}\nabla_\mu\nabla_\nu R-\frac{\Box R}{2}g_{\mu\nu}+4R_{\mu\rho\nu\sigma}R^{\rho\sigma} \\ &- \frac{3R}{2}R_{\mu\nu}-R^2_{\rho\sigma}g_{\mu\nu}+\frac{3R^2}{8}g_{\mu\nu}. \end{aligned} \tag{2.105}$$

Now let us choose the parameters in such a way that, one can obtain the BTZ black hole solution [126]. For that, we write

$$m^2=\frac{\Lambda^2}{4(-\lambda+\Lambda)}, \qquad \Lambda=-\frac{1}{l^2}, \tag{2.106}$$

where Λ is the cosmological constant. From this, the BTZ solution can be extracted as ,

$$ds^2{}_{\mathrm{BTZ}}=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\phi^2, \tag{2.107}$$

$$f(r) = -M + \frac{r^2}{\ell^2} \ , \tag{2.108}$$

where M is the integration constant corresponding to the ADM mass. Horizon radius r_+ can be determined using the condition, $f(r_+) = 0$. Then the horizon is located at,

$$r_+ = l\sqrt{M}. \tag{2.109}$$

Now one can calculate the thermodynamic quantities using the above relations. ADM mass of the black hole can be written as,

$$M = \frac{r_+^2}{l^2}. \tag{2.110}$$

Hawking temperature is obtained from the relation, $T = \frac{\kappa}{2\pi}$, as,

$$T_H = \frac{\sqrt{M}}{2\pi l}. \tag{2.111}$$

From the ADM mass, entropy of the BTZ black hole can be calculated using either Cardy formula or Wald's formula. We adopt Wald's method to calculate the entropy as,

$$S = 2\pi \oint_h dx \sqrt{\gamma} \frac{\delta \mathcal{L}}{\delta \mathcal{R}_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}, \tag{2.112}$$

where h is the spatial cross section of the event horizon, γ is the determinant of the induced metric on h, $\mathcal{L}$ is the Lagrangian in the action (2.103) and $\epsilon_{\mu\nu}$ is the binormal to h. So we obtain entropy as,

$$S = \frac{\pi r_+}{2G} \left(1 - \frac{1}{2m^2 l^2}\right). \tag{2.113}$$

It is interesting to note that the entropy (2.113) is simply Bekenstein-Hawking entropy renormalized by a factor. From the above relation one can conclude that in order to get a non-zero central charge we need to rely on the condition $m^2 l^2 \geq 1/2$. We will explore this condition in details in the preceding sections. The thermodynamic quantities of the BTZ black hole, heat capacity and on shell free energy can be calculated using Abott-Deser-Tekin (ADT) approach [127–129]. Using this method one can obtain, on shell free energy as,

$$F_{\text{bh}}^{\text{on}} = -\frac{M}{8G}\Big(1 - \frac{1}{2m^2l^2}\Big), \tag{2.114}$$

thermodynamic energy as,

$$E = \frac{M}{8G}\Big(1 - \frac{1}{2m^2l^2}\Big), \tag{2.115}$$

and the heat capacity as,

$$C = \frac{\partial E}{\partial T} = \frac{\pi l\sqrt{M}}{2G}\Big(1 - \frac{1}{2m^2l^2}\Big). \tag{2.116}$$

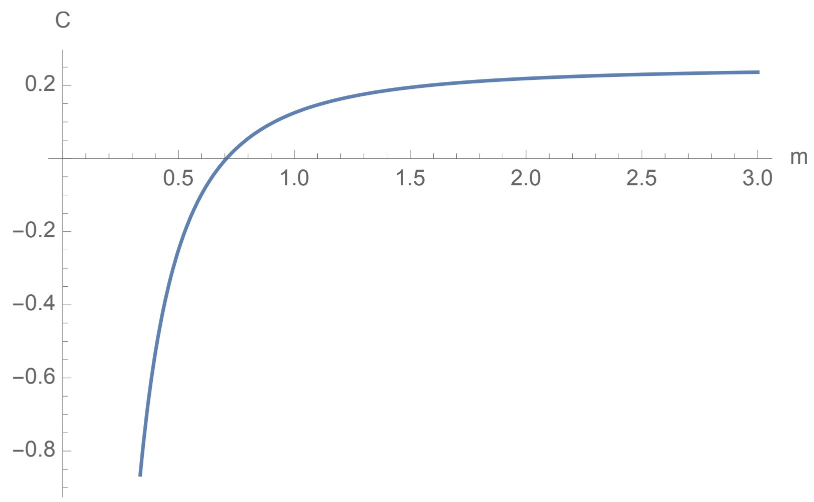

Figure 2.19: Variation of heat capacity of the BTZ black hole against the changes in mass of the graviton with $l = 1$, $G = 1$ and $M = 1$.

In order to investigate the thermodynamic stability of the black hole space-time, we will further explore the above equation. Therefore, we have plotted the variation of heat capacity with entropy of the BTZ black hole in fig.(2.19). From this figure it is evident that, there exists a point where the heat capacity changes sign continuously, showing a transition between thermodynamically stable and unstable phases. We can see that there exists one such point, where heat capacity changes from thermodynamically unstable phase to stable phase in a continuous manner rather than in a usual discontinuous way as seen in many cases where black hole exhibits phase transition. From (4.11), the BTZ black hole is stable when $m^2l^2 \geq \frac{1}{2}$ and unstable when $0 < m^2l^2 < \frac{1}{2}$.

We know that, the Banados-Teitelboim-Zanelli (BTZ) black hole system [126] , there are two distinct solutions, the BTZ black hole of $M \geq 0$ and the thermal soliton of the global AdS_3 whose mass is $M = -1$ [130–132]. Since we have already considered the black hole case, we will dig in to the thermodynamics of thermal soliton of the global AdS_3 with mass, $M = -1$. In this case, the free energy of the thermal solitons can be calculated as,

$$F^{\text{on}}_{\text{sol}} = -\frac{1}{8G}\left(1 - \frac{1}{2m^2l^2}\right), \tag{2.117}$$

In order to analyze the phase transition between BTZ black hole and thermal soliton, let us plot the free energies of both black hole and soliton as a function of temperature. The variation of the same is plotted in fig.(2.20).

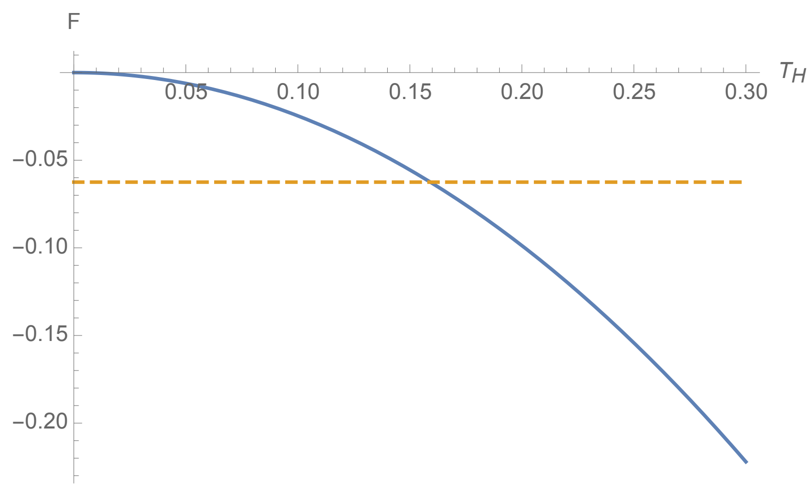

Figure 2.20: variation of free energy of BTZ black hole and thermal solitons in AdS_3 against the changes in temperature. The solid line represent the behaviour of BTZ black hole, while the dashed line represents the thermal soliton case.

From the figures it is evident that the BTZ black hole undergoes a phase transition to thermal soliton of the global AdS_3 at the critical temperature given by,

$$T_c = \frac{1}{2\pi l} \tag{2.118}$$

From fig.(2.20), a phase transition may occur at $T = T_c$ between BTZ black hole and thermal AdS_3 soliton. From the same figure we can see that, for $T < T_c$, free energy of thermal soliton is lower than that of the black hole. So it can be inferred that the thermal soliton is more probable below the critical temperature. On the other hand for $T > T_c$, the BTZ black hole is more probable than the thermal soliton.

Here we have analyzed the thermodynamic behaviour and phase transition structure of BTZ black hole in New massive gravity and found that the there exists a phase transition from BTZ black hole to a thermal soliton. Analysing the heat capacity behaviour indicates that the system undergoes a transition from thermodynamically stable phase to unstable phase in a continuous manner. Further studies are needed to exactly identify the order of transition and other thermodynamic details.

2.3 Discussion and conclusion

In this chapter we have investigated the thermodynamics of black holes in modified theories of gravity, mainly concentrating on Hořava-Lifshitz gravity and Massive gravity. Different black hole solutions in these modified theories have been considered, that include Kehagias-Sfetsos black hole, Lü-Mei-Pope black hole, Park black hole, charged dRGT black hole and BTZ black hole. The thermodynamic behaviour of these black hole systems are studied analytically as well as graphically with a great emphasis on the phase transition structure existing among them. According to ordinary thermodynamics, heat capacity of the black hole is directly related to thermodynamic stability of black holes and phase transition. Positive heat capacity indicates the thermodynamically stable phase where as the negative heat capacity indicates the unstable phase. Any transition between these phases can be termed as thermodynamic phase transition. For the KS black hole, it is found that they are thermodynamically stable for a certain range of values of the entropy as well as the black hole system undergoes a phase transition from thermodynamically stable phase to unstable phase as we observe the heat capacity variation with respect to entropy and found that there is an infinite discontinuity at the transition point. In the case of LMP black hole, from our thermodynamic study in different space-time cases, the heat capacity of the system is found to be diverging, thereby indicating the presence of a phase transition. Even though there exists phase transition, the exact order of the transition remains undetermined. In these two cases, the possibility of first order phase transition has been ruled out due to the non existence of any infinity discontinuity in temperature. Park black hole studies showed that the system exhibits a number of abnormal thermodynamic behaviors like the existence of an upper mass bound, negative temperature, infinite discontinuity in temperature and heat capacity. Heat capacity divergence points out the existence of phase transition in the system. But exactly determining the order of this transition remained as an unsolved problem. Whereas in the case of Massive gravity black holes too, the same problem remains as unsolved. In the case of charged dRGT black holes, the system undergoes a phase transition by changing the sign of heat capacity. In this case one expects that thermodynamics of black holes would be the same as in general relativity, taking into account that massive gravity differs from general relativity by a non-derivative coupling to a fiducial metric [1]. But our studies show that, even though the results agree with general relativity when massive parameter tends to zero, there are significant changes in the phase transition structure. For the (2+1) black hole case in New massive gravity, the BTZ black hole undergoes a phase transition to thermal AdS_3 soliton. Whereas its heat capacity also shows phase transition by showing changes in the sign heat capacity in a continuous manner.

Ordinary thermodynamic studies in all these cases are found to be inadequate to exactly determine the order of the phase transition existing in these systems. Hence our study points in the direction of considering a new method to be incorporated into the classical thermodynamics to answer these puzzling scenarios as well as to well explain these typical behaviours of certain thermodynamic quantities.

[1]The covariant formulation of the dRGT theory has a physical metric and four Stučkelberg scalar fields. The pullback of the metric in the space of Stučkelberg scalar fields into the physical spacetime is called the fiducial metric.

3

Entropy spectrum of black holes in modified theories of gravity

From the discovery of Einstein's general relativity, black hole gained attention of the entire scientific community. Rigorous studies are done in the direction of exploring different aspects of black hole dynamics. But the calculation of entropy of the black hole in semiclassical and quantum regime remains as a puzzling scenario. We will address a closely related problem, calculation of the entropy spectrum of these black holes in modified theories of gravity, particulrly in Hořava-Lifshitz gravity and massive gravity. Eventhough many black hole solutions have been proposed in these theories, we will restrcit our discussion to Kehagias-Sfetsos black hole solution and BTZ black hole solution.

3.1 Entropy spectrum of Kehagias-Sfetsos black hole in Hořava gravity

As a renormalizable theory of gravity, Hořava-Lifshitz gravity, is an ultraviolet completion of general relativity and it reduces to Einstein's gravity with a nonvanishing cosmological constant in IR limit. Kehagias and Sfetsos obtained a static spherically symmetric black hole solution in the IR modified Hořava-Lifshitz theory. In this part of our work, entropy of the KS black hole is quantized via the adiabatic invariance and the Bohr-Sommerfeld quantization rule. As we discussed in the introduction chapter, considering the properties of black hole such as adiabaticity and oscillating velocity of black hole horizon, one can write the action as proposed in Jiang-Han [72] method as,

$$I = \oint p_i dq_i = \int_{q_i^{in}}^{q_i^{out}} p_i^{out} dq_i + \int_{q_i^{out}}^{q_i^{in}} p_i^{in} dq_i, \tag{3.1}$$

where p_i^{in} or p_i^{out} is the conjugate momentum and q_i^{in} or q_i^{out} are the corresponding coordinates and $i = 0, 1, 2....$ It is also to be considered that $q_1^{in} = r_h^{in}(q_1^{out} = r_h^{out})$ and $q_0^{in}\left(q_0^{out}\right) = \tau$ where r_h is the horizon radius and τ is the Euclidean time with a periodicity $\frac{2\pi}{\kappa}$ in which κ is the surface gravity which is given by,

$$\kappa = \frac{1}{2}\frac{df(r)}{dr}\bigg|_{r_h} . \tag{3.2}$$

Considering the Hamilton's equation $\dot{q}_i = \frac{\partial H}{\partial p_i}$, where H is the Hamiltonian of the system, the integral given by (3.1), adiabatic covariant action can be evaluated by considering the contour integration over a closed path from q_i^{out} (outside the event horizon) to q_i^{in}(inside the event horizon). Thus action given by (3.1) can be rewritten as,

$$\begin{aligned} \int_{q_i^{out}}^{q_i^{in}} p_i^{in} dq_i &= \int_{\tau^{out}}^{\tau^{in}} \int_0^H dH' d\tau + \int_{r_h^{out}}^{r_h^{in}} \int_0^H \frac{dH'}{\dot{r}_h} dr_h \\ &= 2\int_{r_h^{out}}^{r_h^{in}} \int_0^H \frac{dH'}{\dot{r}_h} dr_h . \end{aligned} \tag{3.3}$$

where r^{out} and r^{in} represent the horizon location before and after shrinking and $\dot{r}_h = \frac{dr_h}{d\tau}$ is the oscillating velocity of black hole horizon. From the tunneling picture, when a particle tunnels in or out, the black hole horizon will shrink or expand due to the loss or gain of black hole mass [133]. Since the tunneling and oscillation take place at the same time, one can write [134],

$$\dot{r}_h = -\dot{r} \ . \tag{3.4}$$

where $\dot{r}$ is the velocity of the tunneling particle. Since the two contour integrals in the (3.1) are equal we can simplify it as,

$$\oint p_i dq_i = 4 \int_{r^{out}}^{r^{in}} \int_0^H \frac{dH'}{\dot{r}_h} dr_h \ . \tag{3.5}$$

To evaluate this adiabatic invariant quantity for the black hole, let us consider a general class of static and spherically symmetric space-time of the form,

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ , \tag{3.6}$$

where the horizon $r = r_h$ is given by $N(r)^2 = f(r) = 0$. To euclideanize this metric, we consider the transformation in time coordinate $t \to -i\tau$. Hence,

$$ds^2 = N(r)^2 d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ . \tag{3.7}$$

Now we will incorporate the tunneling method to the scenario. In the case of black holes, since the tunneling across the event horizon occurs radially, only the radial paths will be considered here. Let a photon travel across the black hole horizon. Then the radial null geodesic can be obtained by setting $ds^2 = 0$ and $d\Omega^2 = 0$ in (3.7). It is then obtained as,

$$\dot{r} = \frac{dr}{d\tau} = \pm i \sqrt{N(r)^2 f(r)} \ . \tag{3.8}$$

In the tunneling picture there are both incoming and outgoing paths. In our discussions we will focus on the outgoing paths. On the other hand a spherically symmetric black hole solution, Kehagias-Sfetsos black hole solution was obtained in the Hořava-Lifshitz gravity theory, given by the metric,

$$f_{\rm KS}(r) = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}. \tag{3.9}$$

From the above equation it is evident that $f_{\rm KS} = N^2_{\rm KS}(r)$, and thus, from (3.8) we get

$$\dot{r} = +i f_{\rm KS}(r) \ . \tag{3.10}$$

The adiabatic invariant will be,

$$\oint p_i dq_i = -4i \int_{r^{out}}^{r^{in}} \int_0^H \frac{dH'}{f_{\rm KS}(r)} dr \ . \tag{3.11}$$

Using the near horizon approximation, $f_{\rm KS}(r)$ can be Taylor expanded to get,

$$f_{\rm KS}(r) = f_{\rm KS}(r)\Big|_{r_h} + (r - r_h)\frac{df_{\rm KS}(r)}{dr}\Big|_{r_h} + \cdots \tag{3.12}$$

Since there is a pole at horizon r_h, we can consider a contour integral over a half loop going above the pole from right to left. Using the Cauchy's theorem, we can evaluate the integral in (3.11) to obtain,

$$\oint p_i dq_i = 4\pi \int_0^H \frac{dH'}{\kappa} = 2\hbar \int_0^H \frac{dH'}{T} \ , \tag{3.13}$$

where temperature of the black hole is connected with the surface gravity of the black hole via $T = \frac{\hbar\kappa}{2\pi}$ relation. The Smarr formula [135] expresses the mass of a black hole in terms of its geometrical and dynamical parameters (angular momentum, electromagnetic potential, area, etc). Using this Smarr formula in the case of KS black hole, one can arrive at,

$$dH' = dM = TdS \ . \tag{3.14}$$

Therefore (3.13) can be rewritten as,

$$\oint p_i dq_i = 2\hbar S \ . \tag{3.15}$$

If one imposed an ad hoc Bohr-Sommerfeld quantization rule given by,

$$\oint p_i dq_i = 2\pi n\hbar, \qquad n = 1, 2, 3, \cdots \tag{3.16}$$

one would find the entropy spectrum as,

$$S = n\pi. \qquad n = 1, 2, 3, \cdots \tag{3.17}$$

So the black hole entropy is quantized with a spacing of the entropy spectrum given by

$$\Delta S = S_{(n+1)} - S_{(n)} = \pi. \tag{3.18}$$

Thus, we see that both entropy and area spectra of KS black hole are quantized and are equally spaced and they are independent of the black hole parameters. Eventhough the values of equispacing obtained in the present study are different from the values obtained using QNMs approach for LMP black holes [136] and KS black holes [137] in HL theory, the equispaced property is maintained and their order of magnitudes are the same. The exclusion of coordinate dependency in the theory makes this result more relevant than the other calculations.

3.2 Entropy spectrum of BTZ black hole in massive gravity

New Massive Gravity (NMG) was proposed by Bergshoeff, Hohm and Townsend in 2009 [33] as a higher derivative model of gravity in (2+1) dimensions, and the field equations of this massive gravity theory are solved by the black hole metrics discovered by Banados, Teitelboim and Zanellli [126]. These solutions are commonly known as the BTZ solutions, and is given by,

$$ds^2_{\text{BTZ}} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \tag{3.19}$$

where,

$$f_{\text{BTZ}}(r) = -M + \frac{r^2}{\ell^2}, \tag{3.20}$$

here M is the integration constant that can be directly related to the mass of the BTZ black hole. As we have discussed and derived the thermodynamics of the black hole solution in chapter 2, we can write down the thermodynamic quantities of the black hole solution from (3.20). Then the ADM mass of the black hole can be written as,

$$M = \frac{r_+^2}{l^2}.$$

Hawking temperature is obtained from the relation, $T = \dfrac{\kappa}{2\pi}$, as,

$$T_H = \frac{\sqrt{M}}{2\pi l}.$$

From the ADM mass, entropy of the BTZ black hole can be calculated using Wald's method as,

$$S = \frac{\pi r_+}{2G}\left(1 - \frac{1}{2m^2 l^2}\right).$$

Now we will quantize the entropy of the BTZ black hole via the adiabatic invariance and Bohr-Sommerfeld quantization rule. According to this method proposed by Jiang and Han, the adiabatic invariant integral can be written as given in (3.5) as,

$$\oint p_i dq_i = 4 \int_{r^{out}}^{r^{in}} \int_0^H \frac{dH'}{\dot{r}_h} dr_h \ .$$

By considering the radial null geodesic path formulation we can arrive at the oscillating velocity of the black hole horizon as,

$$\dot{r}_h = -i f_{\text{BTZ}}(r) \ . \tag{3.21}$$

Hence the adiabatic invariant integral will reduce to,

$$\oint p_i dq_i = -4i \int_{r^{out}}^{r^{in}} \int_0^H \frac{dH'}{f(r)} dr \ . \tag{3.22}$$

Using the near horizon approximation, $f(r)$ can be Taylor expanded as,

$$f(r) = f(r)_{r_h} + (r - r_h) \frac{df(r)}{dr}\bigg|_{r_h} + ... \tag{3.23}$$

Since there exists a pole at $r = r_h$ one has to consider a contour integral in such a way that the half loop is going above the pole from right to left, to evaluate the adiabatic invariant integral (3.22). Using the Cauchy integral theorem, we can arrive at,

$$\oint p_i dq_i = 4\pi \int_0^H \frac{dH'}{\kappa} = 2\hbar \int_0^H \frac{dH'}{T_H} \tag{3.24}$$

Now we can write the Smarr formula for the BTZ black hole in new massive gravity as,

$$dM = dH = TdS. \tag{3.25}$$

Then (3.24) becomes,

$$\oint p_i dq_i = 2\hbar S. \tag{3.26}$$

The semi classical quantization rule, the Bohr-Sommerfeld quantization rule, can be written as,

$$\oint p_i dq_i = 2\pi n\hbar, \qquad n = 1, 2, 3, ... \tag{3.27}$$

Comparing (3.26) and (3.27), one can arrive at the entropy spectrum of the bTZ black hole,

$$S = n\pi \ . \tag{3.28}$$

So the black hole entropy is quantized with a spacing of the entropy spectrum given by,

$$\Delta S = S_{n+1} - S_n = \pi. \tag{3.29}$$

We can see that the entropy spectrum of BTZ black hole in new massive gravity is quantized. The spectrum is characterized by equally spacing and it is independent of the black hole parameters of the system. Since 2+1 black hole system and their properties act as the basic tool to enhance the thermodynamic understanding of the system, we will further extend this discussion to another class of BTZ black hole solutions existing in massive gravity theories other than in new massive gravity. The most interesting cases in this respect are the charged BTZ black hole solution present in massive gravity. They are the solution to Einstein-Maxwell massive gravity theory.

Entropy spectrum of charged black hole in Einstein-Maxwell solution in the context of massive gravity.

The 3-dimensional action of Einstein-Maxwell massive gravity with an abelian $U(1)$ gauge field is given by,

$$S = -\frac{1}{16\pi} \int d^3x\sqrt{-g} \left[R - 2\Lambda + L(F) + m^2 \sum_i^4 c_i U_i(g,f) \right], \quad (3.30)$$

where R, $L(F)$ and Λ respectively are , the scalar curvature, an arbitrary Lagrangian of electrodynamics and the cosmological constant. The Maxwell invariant is given as $F = F_{\mu\nu}F^{\mu\nu}$, Faraday tensor as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the the gauge potential as A_μ. The c_i's are some constants and U_i's are symmetric polynomials of the eigenvalues of the $d \times d$ matrix $K^\mu_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$ which can be written as

$$\begin{aligned} U_1 &= [K], \quad U_2 = [K]^2 - \left[K^2\right], \quad U_3 = [K]^3 - 3[K]\left[K^2\right] + 2\left[K^3\right], \\ & U_4 = [K]^4 - 6\left[K^2\right][K]^2 + 8\left[K^3\right][K] + 3\left[K^2\right]^2 - 6\left[K^4\right]. \end{aligned}$$

Considering (3.30) and employing the variational principle, we can arrive at the field equations as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} L(F) - 2 L_F F_{\mu\rho} F_{\nu}^{\rho} + m^2 \chi_{\mu\nu} = 0, \tag{3.31}$$

$$\partial_\mu \left(\sqrt{-g} L_F F^{\mu\nu} \right) = 0, \tag{3.32}$$

where $\chi_{\mu\nu}$ is the massive term given by,

$$\begin{aligned} \chi_{\mu\nu} \;\; = \;\; & -\frac{c_1}{2} \left(U_1 g_{\mu\nu} - K_{\mu\nu} \right) - \frac{c_2}{2} \left(U_2 g_{\mu\nu} - 2U_1 K_{\mu\nu} + 2K_{\mu\nu}^2 \right) \\ & -\frac{c_3}{2} (U_3 g_{\mu\nu} - 3U_2 K_{\mu\nu} + U_1 K_{\mu\nu}^2 - 6K_{\mu\nu}^3) - \\ & \frac{c_4}{2} (U_4 g_{\mu\nu} - 4U_3 K_{\mu\nu} + 12 U_2 K_{\mu\nu}^2 - 24 U_1 K_{\mu\nu}^3 + 24 K_{\mu\nu}^4). \end{aligned} \tag{3.33}$$

In order to obtain the linearly charged three dimensional black hole solutions, consider the metric ansatz as,

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2. \tag{3.34}$$

To obtain the exact linearly charged BTZ solutions, we would make the choice of parameters as [138, 139],

$$f_{\mu\nu} = \mathrm{diag}(0, 0, c^2 h_{ij}), \tag{3.35}$$

$$\mathcal{U}_1 = \frac{(d-2)\, c}{r}, \qquad \mathcal{U}_2 = \mathcal{U}_3 = \mathcal{U}_4 = 0, \tag{3.36}$$

$$L(\mathcal{F}) = -\mathcal{F}, \tag{3.37}$$

and as a result of such choice one can arrive at the solution as,

$$f\,(r)_{\mathrm{BTZ}} = -\Lambda r^2 - M - 2q^2 \ln\left(\frac{r}{l}\right) + m^2 r C, \tag{3.38}$$

where M and q are related to the mass and charge of the black hole respectively where the m represents the massive term contribution and C is an integration constant

Now one can calculate the thermodynamic quantities using the above metric. Hawking temperature can be calculated from the surface gravity on the outer horizon r_+ as,

$$T = -\frac{\Lambda r_+}{2\pi} - \frac{q^2}{2\pi r_+} + \frac{m^2 C}{4\pi}. \tag{3.39}$$

For the three dimensional case, entropy of the system takes the form [41, 46, 140–143]

$$S = \frac{\pi}{2} r_+. \tag{3.40}$$

ADM mass of the black hole can be written as,

$$M = -\Lambda r_+^2 + m^2 r_+ C - 2q^2 \ln\left(\frac{r_+}{l}\right). \tag{3.41}$$

The electric potential Φ can be obtained from the relation $\Phi = \left(\frac{\partial M}{\partial Q}\right)_S$,

$$\Phi = -q \ln\left(\frac{r_+}{l}\right) \tag{3.42}$$

Similar to the above section, we will explore and quantize the entropy of the BTZ black hole via the adiabatic invariance, Bohr-Sommerfeld quantization rule and Cauchy integral theorem. Using these, one can easily arrive at (3.24), given as,

$$\oint p_i dq_i = 4\pi \int_0^H \frac{dH'}{\kappa} = 2\hbar \int_0^H \frac{dH'}{T_H}$$

Now we can write the Smarr formula for the linearly charged BTZ black hole as,

$$dM = dH = TdS - \Phi dQ \tag{3.43}$$

Then (3.24) become,

$$\oint p_i dq_i = 2\hbar S\left[1 + \frac{\Phi}{2Q} \ln\left(\frac{\mathcal{A}(\mathcal{A}\Lambda - Cm^2\pi)}{4}\right)\right] \tag{3.44}$$

where $\mathcal{A}$ is the circumference of the 2+1 BTZ black hole system. Comparing (3.44) and (3.27), one can arrive at the entropy spectrum as,

$$S = \frac{n\pi}{2\hbar\left[1 + \frac{\Phi}{2Q}\ln\left(\frac{\mathcal{A}(\mathcal{A}\Lambda - Cm^2\pi)}{4}\right)\right]} \tag{3.45}$$

It is interesting to note that the entropy of the BTZ black holes in Einstein Maxwell massive gravity is quantized. From the above relation it is evident that the entropy spectrum depends on the value of the black hole parameters. Considering the absence of electric charge, one can find that the relation (3.45) reduces to chargeless BTZ black hole case with some numerical differences given by the relation (3.28).

Another kind of BTZ solutions in massive gravity can be find out when Einstein-Born-Infeld field is coupled to the context of massive gravity [139]. These are nonlinearly charged BTZ black hole solutions in massive gravity, given as

$$f\left(r\right) = -\Lambda r^2 - m_0 + 2\beta^2 r^2\left(1 - \Gamma\right) + q^2\left[1 - 2\ln\left(\frac{r}{2l}\left(1 + \Gamma\right)\right)\right] + m^2 Cr, \tag{3.46}$$

where β is the nonlinearity parameter which arises from the Born-Infeld Lagrangian, and Γ is given by the relation $\Gamma = \sqrt{1 + \frac{q^2}{r^2\beta^2}}$. If one uses the formalism of adiabatic invariance proposed by Majhi and Vagenas to calculate the entropy spectrum of nonlinearly charged BTZ black hole, it miserably fails in this attempt. So a new method should be found out in the case case of nonlinear systems.

3.3 Discussion and conclusion

The black hole spectroscopy is fascinatingly described by combining the black hole property of adiabaticity and the oscillating velocity of the black hole horizon through the tunneling mechanism proposed by Majhi and Vagenas and later modified by Jiang and Han. Unlike Kunstatter's method, there is no need to use quasinormal frequency of the black hole, instead the oscillating velocity of the black hole horizon has been introduced. To calculate this oscillating velocity of black horizon, we employed the tunneling framework. As a result, the adiabatic invariant quantity that is invariant with respect to coordinate transformation is formulated as $\oint p_i dq_i$. In this chapter, we have investigated the quantization of the entropy of black holes in Hořava-Lifshitz gravity and Massive gravity models via Jiang and Han's method. We used KS black hole in Hořava gravity and different BTZ black hole solutions in massive gravity to study the black hole spectroscopy in modified theories of gravity. We have found that the entropy spectrum in all cases are quantized. In the case of KS black hole, they are equally spaced and does not depend on any of the black hole parameters. Whereas in the case of chargeless BTZ black hole in New Massive Gravity case also, the entropy spectrum is quantized, equally spaced and is independent of black hole parameters. But on the other hand, in the case of linearly charged BTZ black holes in massive gravity, the entropy spectrum is quantized and the spectrum depends on the black hole parameters such as the circumference of the BTZ black hole. In all these black hole cases, the entropy spectrum of black holes are found to be equally spaced. Hence we can say that for the modified theories of gravity, quantization of entropy behaves as more fundamental than other quantizations like area and circumference [144].

4

Geometrothermodynamics of black holes in Modified theories of gravity

The application of differential geometric ideas to classical thermodynamics using a metric on the space of states, pioneered by the works of Weinhold and Ruppeiner [78, 79], proved it as an alternative way to characterize the phase transitions. But many puzzling anomalies remain unexplained in these formulations when one applying them to different known classical systems. A possible resolution in this direction was suggested by Hernando Quevedo [97–99] who emphasized the importance of preserving Legendre invariance in the construction of thermodynamic metrics. This method gave physically consistent explanations for the anomalies existed in the thermodynamic systems. The exact phase transition structure of these systems are described in Davies' [57] phase transition picture, geometrothermodynamics is able to reproduce the same. The order of transition can also be confirmed from the same. As a result of this formalism one can exactly explain the interactions existing in the system. We have already noted the phase transitions in black hole systems of modified theories of gravity from the studies regarding the black hole thermodynamic discussed in chapter 2. But these calculations failed to determine the order of phase transition. Even though one can eliminate the possibility of first order phase transition from the non existence of any discontinuity in the temperature, but infinite discontinuity in heat capacity can be second or higher orders in nature. Here in GTD method, thermodynamic interaction can be reflected from the curvature of the metric defined on equilibrium spaces via the auxiliary metric given in (1.80). If thermodynamic curvature of the system is free of singularities, then GTD interprets it as the non-existence of any singular points at the level of the heat capacity and no (second order) phase transitions occur in the system. Also the same curvature reproduce all the abnormalities existing in the thermodynamic system. This chapter is devoted for the study of geometrothermodynamics of black holes and their thermodynamics discussed in chapter 2.

4.1 Geometrothermodynamics of black holes in Hořava-Lifshitz gravity

Park black hole

Before going in to the details of thermodynamic geometry, let us revisit the thermodynamics of Park black hole in a slightly different way. In chapter 2 we have considered Park-de Sitter solution in details. Now let us construct the Park black hole solution in such a way that both de Sitter and anti de Sitter Park black hole can be analysed in a single scenario. From the thermodynamics discussed in chapter 2, we can write down the Park solution [112] as,

$$N^2 = f_{\text{Park}} = 1 + (\omega - \Lambda_W)r^2 - \sqrt{r[\omega(\omega - 2\Lambda_W)r^3 + \beta]} \ , \tag{4.1}$$

where β is an integration constant related to the black hole mass. Now let us consider (4.1) in a detailed way. For $r \gg [\beta/|\omega(\omega - 2\Lambda_W)|]^{1/3}$, we can arrive at two solutions. First one is the asymptotically AdS case with $\Lambda_W < 0$ and $\omega > 0$,

$$f = 1 + \frac{|\Lambda_W|}{2}\left|\frac{\Lambda_W}{\omega}\right| r^2 - \frac{2M}{\sqrt{1 + 2|\Lambda_W/\omega|}}\frac{1}{r} + \mathcal{O}(r^{-4}) \ , \tag{4.2}$$

and the second one is the asymptotically dS case with $\Lambda_W > 0$ and $\omega < 0$,

$$f = 1 - \frac{\Lambda_W}{2}\left|\frac{\Lambda_W}{\omega}\right| r^2 - \frac{2M}{\sqrt{1+2|\Lambda_W/\omega|}}\frac{1}{r} + \mathcal{O}(r^{-4}) \ . \tag{4.3}$$

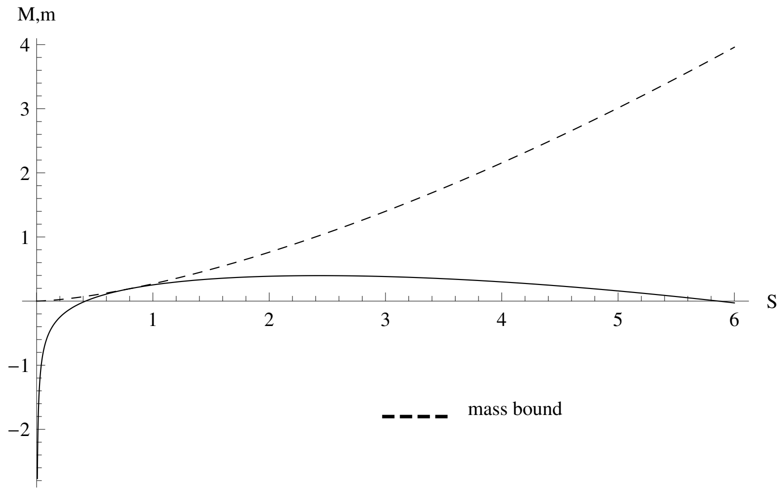

Figure 4.1: Plots of mass vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$.

Thermodynamics of Park black hole has been studied in [112]. Now we will further investigate the different behaviors of these potentials. In general Park black hole solution has two horizons, one cosmological horizon and the other black hole horizon. By considering the black hole horizon r_+, mass of the Park black hole can be written as,

$$M = \frac{1 + 2(\omega - \Lambda_W)r_+^2 + \Lambda_W^2 r_+^4}{4\omega r_+}. \tag{4.4}$$

Using the Bekenstein-Hawking area law,

$$S = \frac{A}{4} = \pi r_+^2, \tag{4.5}$$

and the relation,

$$\Lambda = \frac{(d-1)(d-2)}{2l^2}, \tag{4.6}$$

which connects the radius of curvature l of dS or AdS space with Λ the cosmological constant (where d is the dimension of space-time), one can arrive at the mass-entropy relation,

$$M = \frac{4S^2 - 4l^2\pi S + l^4\pi(\pi + 2S\omega)}{4\; l^4\pi^{3/2}\omega\sqrt{S}}. \tag{4.7}$$

Particularly in the dS case, also there exists an upper mass bound given by,

$$M_{\text{bound}} = \frac{(\frac{4}{l^2} - \omega)}{4}\left(\frac{S}{\pi}\right)^{3/2}. \tag{4.8}$$

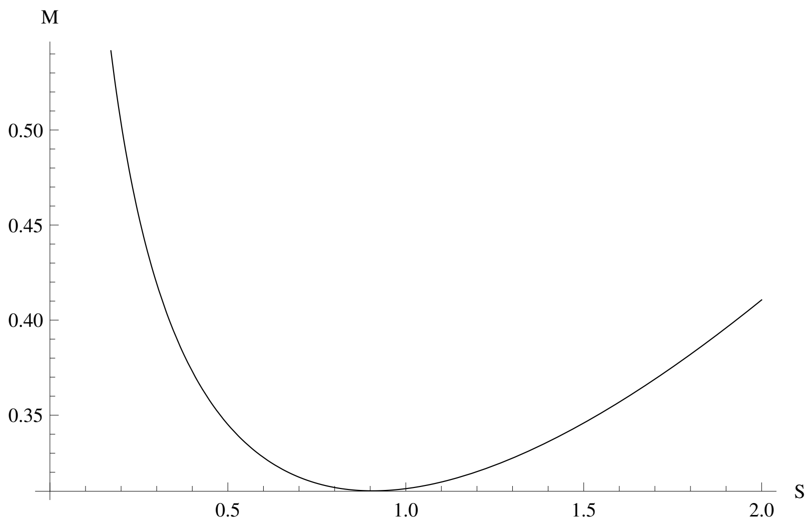

Figure 4.2: Plots of mass vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$.

Thermodynamics regarding this upper mass bound is well discussed in the 2nd chapter. Now other thermodynamic quantities like temperature, heat capacity and free energy can be obtained from the usual definitions of these quantities in ordinary thermodynamics,

$$\begin{aligned} T &= \left(\frac{\partial M}{\partial S}\right), \\ C &= T\left(\frac{\partial S}{\partial T}\right), \\ F &= M - T\ S. \end{aligned} \tag{4.9}$$

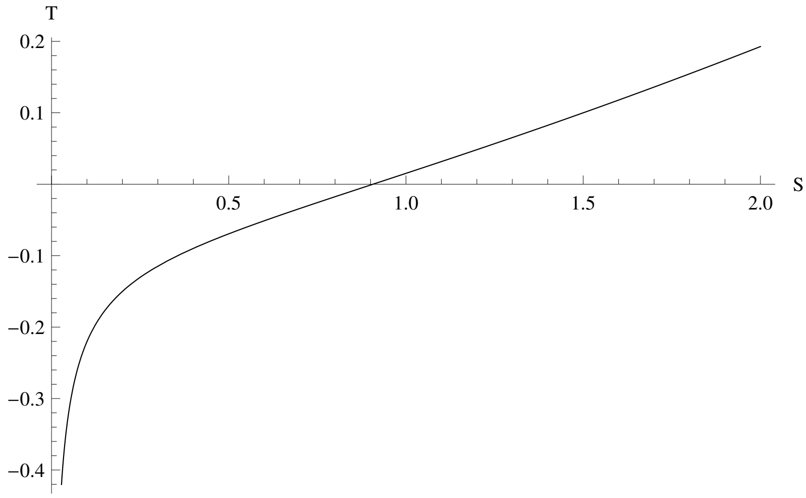

Figure 4.3: Plots of temperature vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$.

Here the temperature of the black hole is obtained as,

$$T = \frac{12S^2 - 4l^2\pi S + l^4\pi(\pi - 2S\omega)}{8\ l^4\pi^{3/2}S^{3/2}\omega}, \tag{4.10}$$

the heat capacity as,

$$C = \frac{N}{D}\ , \tag{4.11}$$

where,

$$N = 2S(12S^2 - 4l^2\pi S + l^4\pi(\pi - 2S\omega)(l^2(\pi + S\omega) - 2S),$$

and

$$D = -24S^3+4l^2S^2(7\pi+3S\omega)+2l^4\pi S(-5\pi+4S\omega)+l^6\pi(\pi^2+5\pi S\omega-2S^2\omega^2).$$

and the free energy as,

$$F = \frac{K}{J} \ , \tag{4.12}$$

where,

$$K = -16S^3-4l^2S^2(-6\pi+S\omega)-12l^4\pi S(\pi+S\omega)+l^6\pi(2\pi^2+7\pi S\omega+2S^2\omega^2),$$

and

$$J = 8l^4\pi^{3/2}\sqrt{S}\omega(l^2(\pi+S\omega)-2S).$$

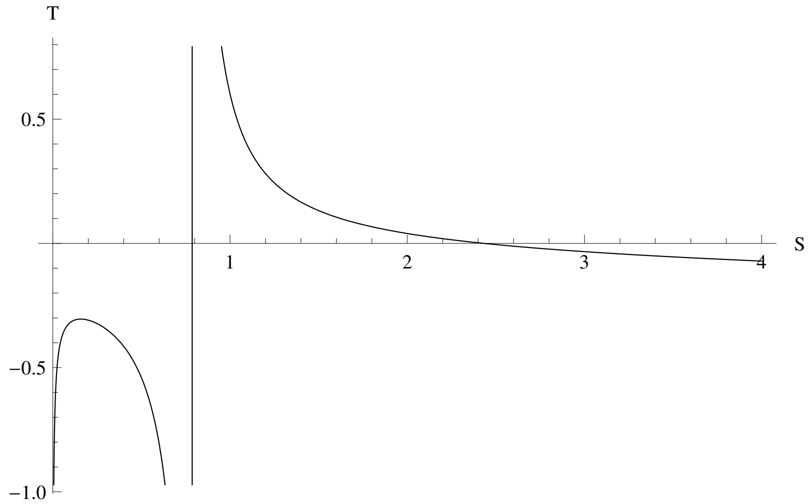

Figure 4.4: Plots of temperature vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$.

We have plotted the variation of mass against the entropy in figs.4.1 and 4.2 for dS and AdS case respectively. Similarly, in figs.4.4 and 4.3 temperature variations are plotted. For the dS case (fig.4.4), we can see that there is an infinite discontinuity in temperature and for a certain range of S values temperature becomes negative also.

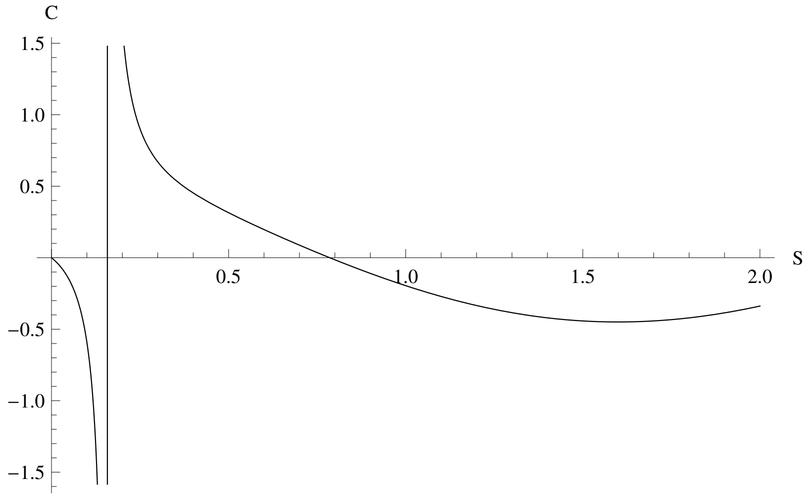

Figure 4.5: Plots of specific heat vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$.

These two anomalous behaviors are due to the existence of mass bound given by (4.8). For the AdS case (fig.4.3) temperature changes continuously without any discontinuities. In figs.4.5 and 4.6 we have plotted specific heat of Park black hole with entropy, while in figs.4.7 and 4.8, the variation of free energy against entropy is plotted. From fig.(4.5) we can see that the Park-dS black hole undergoes a phase transition from thermodynamically unstable state to a thermodynamically stable state. In fig.(4.7), free energy changes from positive to negative, supportingly the black hole changes from unstable to stable state via phase transition. But for Park-AdS black hole, from fig.(4.6) and fig.(4.8), we can see that black hole undergoes a continuous transition from initial thermodynamically unstable phase to a stable phase and no phase transition takes place.

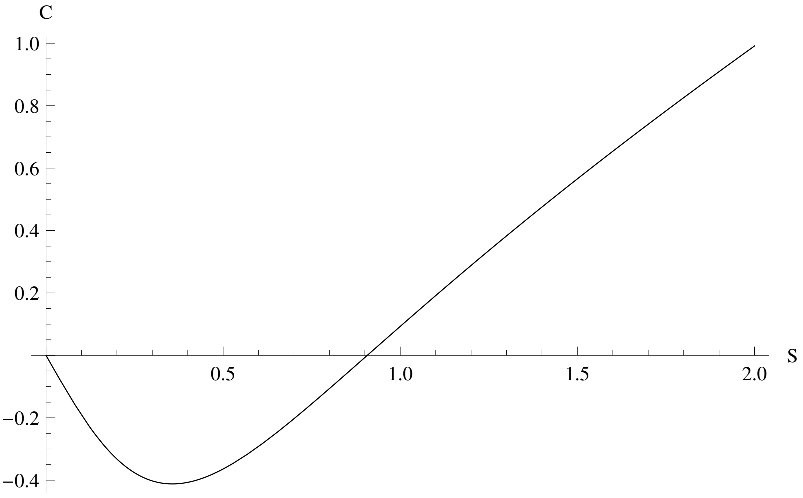

Figure 4.6: Plots of specific heat vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$.

So among Park-dS and Park-AdS black hole, only the dS case shows a phase transition. Now we will further investigate these abnormalities shown by the black hole. We are aiming at a good explanation of these observations in terms of different thermodynamic geometric methods.

Let us start our discussion by employing the newly developed thermodynamic geometry approach geometrothermodynamics to the Park black holes in Hořava-Lifshitz gravity. Like ordinary thermodynamics, in GTD it is the fundamental equation from which all the thermodynamic information can be derived. In order to incorporate the differential geometry in to the thermodynamics and to construct a thermodynamic phase space case we will consider l and ω as the other extensive variables of the present thermodynamic system. We will first consider Weinhold [78] and then Ruppeiner [79] way of incorporating differential geometry in to ordinary thermodynamics. There were detailed discussions of these methods and many discrepancies have been found out. Here we will try to figure out to what extend these methods can reproduce the interaction existing in the

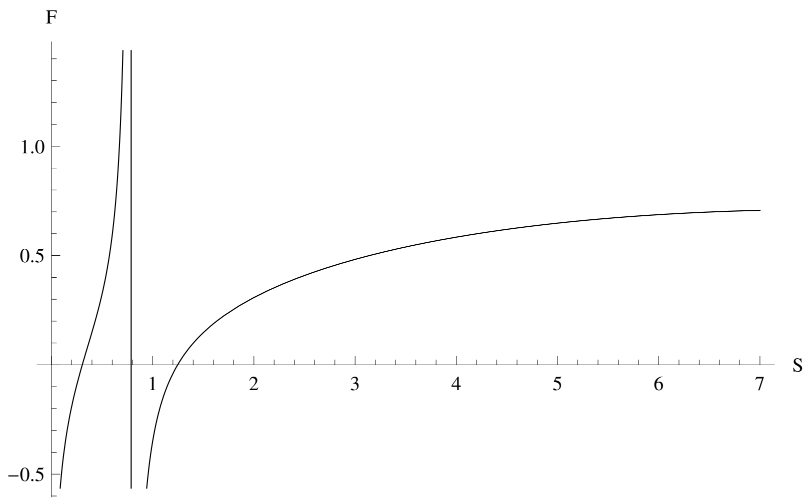

Figure 4.7: Plots of free energy vs. entropy for the dS black hole with $l = 1$ and $\omega = -2$.

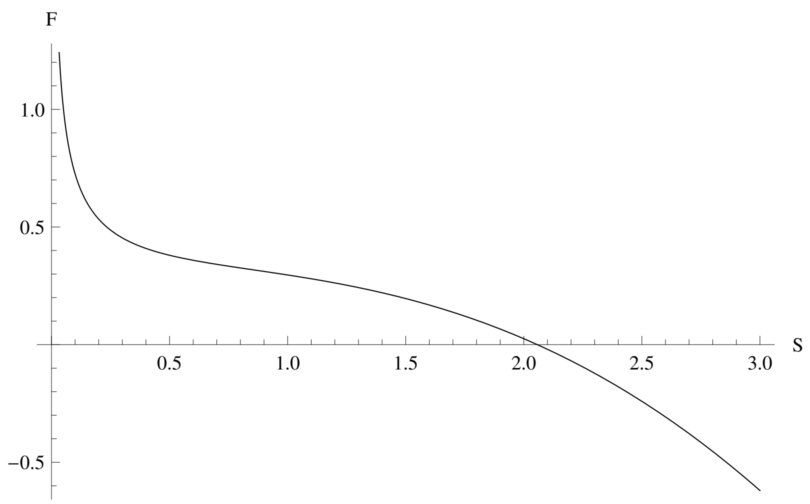

Figure 4.8: Plots of free energy vs. entropy for the AdS black hole with $l = -1$ and $\omega = 2$.

system as well as their phase transition structure. Therefore the Weinhold metric for the Park black hole can be written from the general formula given in (1.70) as,

$$g^W = \begin{bmatrix} M_{SS} & M_{Sl} & M_{S\omega} \\ M_{lS} & M_{ll} & M_{l\omega} \\ M_{\omega S} & M_{\omega l} & M_{\omega\omega} \end{bmatrix},$$

where $M_S = \partial M/\partial S$, etc. We can arrive at the scalar curvature corresponding to the above Weinhold metric as,

$$R^W = \frac{A(S,l,\omega)}{3[l^2\pi - 4S]^3[8l^2\pi S - 36S^2 + l^4\pi(5\pi - 4\omega S)]^2}. \tag{4.13}$$

where $A(S,l,\omega)$ is a complicated expression with no physical interest and having an overall positive sign. From the above expression, R^W diverges at the points $S = 0.785$, $S = 2.06$ for dS case and at $S = 1.171$ for AdS case. (From here, through out this section we are choosing $l = 1$ and $\omega = -2$ for dS case and $l = -1$ and $\omega = 2$ for the AdS case. Imaginary as well as negative roots are not considered in this discussion). The point $S = 0.785$ or $r_+ = 0.5$ corresponds to the infinite discontinuity of temperature and free energy, and the point at which specific heat becomes zero. Moreover the mass bound is saturated at this point. But Weinhold's metric miserably fails to explain any physical singularities in the AdS case.

Now we will consider the Ruppeiner geometry. Since the Ruppeiner metric is conformally related to Weinhold metric (1.70) and from the general expression given in (1.71), one can obtain the Ruppeiner metric for the Park black hole case as,

$$g^{\mathcal{R}} = \frac{1}{T}\begin{bmatrix} M_{SS} & M_{Sl} & M_{S\omega} \\ M_{lS} & M_{ll} & M_{l\omega} \\ M_{\omega S} & M_{\omega l} & M_{\omega\omega} \end{bmatrix}.$$

The curvature of this metric is given by,

$$R^{\mathcal{R}} = \frac{B(S,l,\omega)}{[l^2\pi - 4S]^3[8l^2\pi S - 36S^2 + l^4\pi(5\pi - 4\omega S)]^2} \times \frac{1}{[4l^2\pi S - 12S^2 + l^4\pi(\pi - 2\omega S)][-2S + l^2(\pi + \omega S)]^3}. \quad (4.14)$$

where $B(S,l,\omega)$ is a long complicated expression with less physical interest and having an overall positive sign. For dS and AdS cases, $R^{\mathcal{R}}$ possess singularities at points $S = 0.785, 2.43$ and $S = 0.906$ respectively. The point, $S = 0.785$ is well explained by Weinhold's metric. But the point, $S = 2.43$ or $r_+ = 0.879$ is the new one that corresponds to zero value of temperature and specific heat. For the AdS case, the point $S = 0.906$ or $r_+ = 0.537$ corresponds to the zeros in mass, temperature and specific heat.

Like any physical quantity or formalism related to general relativity not depending on the choice of coordinates, thermodynamics must also be independent of the choice of coordinates used to describe the system. Hence Legendre invariance must be preserved in any geometric description. As we have mentioned in the introduction chapter, the main reason for the less acceptance of Weinhold's and Ruppeiner's metrics is that they are not Legendre invariant. Hence we will consider the newly proposed geometrothermodynamics formalism, where Legendre invariance is preserved, to explain the thermodynamic behavior of these systems.

For geometrothermodynamic calculations of Park black hole, we will construct a 7-dimensional thermodynamic phase space $\mathcal{T}$ which is constituted by the coordinates $Z^A = \{M, S, l, \omega, T, \iota, \vartheta\}$, where S, l, ω are extensive variables and T, ι, ϑ are their dual intensive variables. Then the fundamental Gibbs 1-form defined on $\mathcal{T}$ can be written as,

$$\Theta = dM - TdS - \iota dl - \vartheta d\omega \quad (4.15)$$

The equilibrium phase space $\mathcal{E}$ can be defined as a simple maping φ :

$\{S, l, \omega\} \to \{M(S, l, \omega), S, l, \omega, T(S, l, \omega), \iota(S, l, \omega), \vartheta(S, l, \omega)$. The Quevedo metric is given from (1.80),

$$g^{\text{GTD}} = (SM_S + lM_l + \omega M_\omega) \begin{bmatrix} -M_{SS} & 0 & 0 \\ 0 & M_{ll} & M_{l\omega} \\ 0 & M_{\omega l} & M_{\omega\omega} \end{bmatrix}.$$

Here the absence of cross terms in GTD metric is justified from the choice of auxiliary metric (1.80). Partial derivative with respect to the corresponding coordinate are represented by the subscripts. The curvature scalar corresponding to the above metric is found to be,

$$\begin{aligned} R^{\text{GTD}} &= \frac{D(S, l, \omega)}{[l^2\pi - 2s]^3[4l^2\pi s + 12s^2 + l^4\pi(3\pi - 2\omega s)]^2} \\ &\times \frac{1}{[-20l^2\pi s + 28s^2 + l^4\pi(3\pi - 2\omega s)]^3}. \end{aligned} \tag{4.16}$$

in which $D(S, l, \omega)$ is a complicated expression with less physical interest and having an overall positive sign. At points $S = 0.785$ and at $S = 0.477$ and 2.1 for dS and AdS respectively, the Legendre invariant scalar curvature becomes zero or shows infinite discontinuities. The point $S = 0.785$ or $r_+ = 0.5$ is the same point where the phase transition takes place. To get an exact idea regarding this, we will consider fig.(4.9), which shows the correspondence between the divergence of scalar curvature R^{GTD} and specific heat C. It is very interesting to note that the point $S = 0.477$ or $r_+ = 0.386$ in AdS case corresponds to the point of inflection in the curves of temperature, specific heat and free energy, where the convex nature of curve changes to concave nature or vice versa. Similarly the point $S = 2.1$ or $r_+ = 0.817$ coincides with the point of free energy curve where it becomes zero. So using geometrothermodynamics and hence by constructing the Legendre invariant metric, we are able to reproduce the behavior of thermodynamic potentials and their interactions. The correspondence of divergence and zeros of thermodynamic potentials with

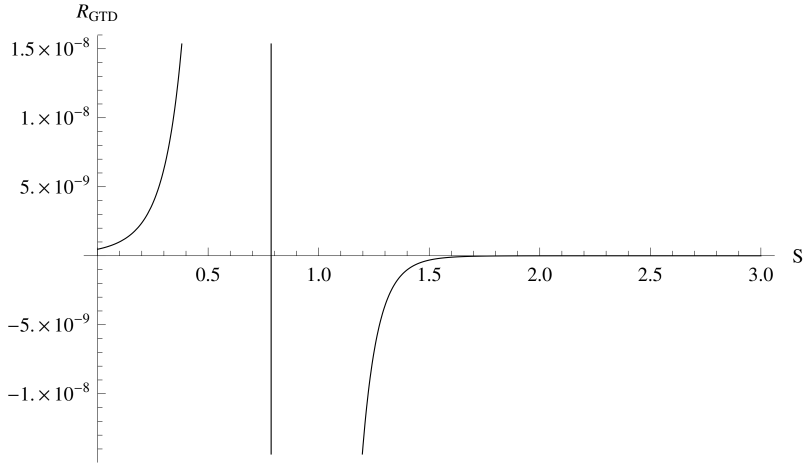

Figure 4.9: Plots scalar curvature vs. entropy for the dS black hole with $l = 1$, $\omega = -2$

the divergence of Legendre invariant scalar curvature leads to the complete understanding of Park black hole thermodynamics.

Here in this chapter, we have investigated the thermodynamics as well as thermodynamic geometry of Park black hole. We have considered both dS and AdS cases. We have analyzed the usual thermodynamics of both these cases and found that there exist many abnormal behaviors like existence of an upper mass bound, negative temperature, infinite discontinuity in temperature, heat capacity and free energy, etc. We have incorporated the geometric ideas in to the usual thermodynamics by means of different thermodynamic geometric methods. We have analyzed first the thermodynamic geometry based on Weinhold's metric and Ruppeiner's metric and the GTD. We have found that the corresponding thermodynamic scalar curvature possesses many singularities, and these singularities are in accordance with the behaviors of mass, temperature, specific heat and free energy. As we have mentioned in this work, these two methods depend entirely on the choice of thermodynamic potential to describe the system. Even though this particular choice gives almost good results, but the lack of Legendre invariance leads us to consider a much more general geometrothermodynamic method. The potential independence of the results or in other words the Legendre invariance is assured in this metric. When we use GTD to explain the thermodynamics, we find that it possesses a true curvature singularity. And the singularity corresponds to the points where the mass bound gets saturated, temperature shows infinite discontinuity and specific heat also shows infinite discontinuity. Park dS black holes undergo a second order phase transition from a thermodynamically unstable state to thermodynamically stable state while in the AdS case, there exists no such behaviors. So GTD reproduces the thermodynamics of Park black hole irrespective of the choice of the potential used to describe the system. When we consider the GTD metric, it is found to be finite and smooth at the regions where the black hole is stable. But when black hole shows changes from thermodynamically stable to unstable phase, this metric possesses true singularities, and as mentioned above, this corresponds to the second order phase transition shown by the black hole. Here GTD explains the second order phase transition, existence of negative temperature, point of inflection and the upper mass bound of Park black hole.

Lü-Mei-Pope black hole

Now we will apply this idea of geometrothermodynamics to the LMP black hole system to study the phase transition behaviour exhibited by this black hole and the abnormalities existing in different thermodynamic variables of the system. It is interesting to note from the thermodynamic studies of the same system in previous chapters, there exist many anomalous behaviours, which include the negative temperature and existence of a temperature window. As in classical thermodynamics, in GTD it is the fundamental equation from which all the thermodynamic information can be extracted. With the choice of extensive variable as, $E^i = \{S, a\}$, and their corresponding intensive variables as $I^i = \{T, A\}$. Here mass M corresponds to the thermodynamic potential. So we are using mass representation to explain the LMP black hole system. We have introduced all the coordinates as, $Z^A = \{M, S, a, T, A\}$. From this we will construct a 5-dimensional thermodynamic phase space $\mathcal{T}$. Thus the fundamental 1-form defined on $\mathcal{T}$ can be written as,

$$d\Theta = dM - T\,dS - a\,dA\ . \tag{4.17}$$

The thermodynamic metric on $\mathcal{E}$ can be computed from the pullback $g = \varphi^*(G)$ that yields

$$g_{\text{GTD}} = (SM_S + aM_a)\begin{bmatrix} -M_{SS} & 0 \\ 0 & M_{aa} \end{bmatrix}.$$

Here subscripts represent partial derivative with respect to the corresponding coordinate. Then the Legendre invariant scalar curvature corresponding to the above metric can be calculated as,

$$R_{\text{GTD}} = \frac{48a^2r^3\left(35r^6\Lambda_W^3 - 5r^4\Lambda_W^2 + 9r^2\Lambda_W + 9\right)}{(r^2\Lambda_W + 3)^2\,(3r^2\Lambda_W + 1)^2\,(5r^2\Lambda_W - 3)^3}\ . \tag{4.18}$$

We have plotted the variation of scalar curvature with entropy in fig.(4.10). From this figure as well as from the above equation, it can be confirmed that this scalar curvature diverges at two points, where one of the divergence accords with that of heat capacity given by (2.41). Usual thermodynamic as well as geometrothermodynamic calculations were unable to address the additional singularity exhibited by the scalar curvature. Hence the geometrothermodynamics exactly reproduces the phase transition structure of the LMP black hole. Along with this we made a consistency check on the above formalism using classical Ehrenfest scheme.

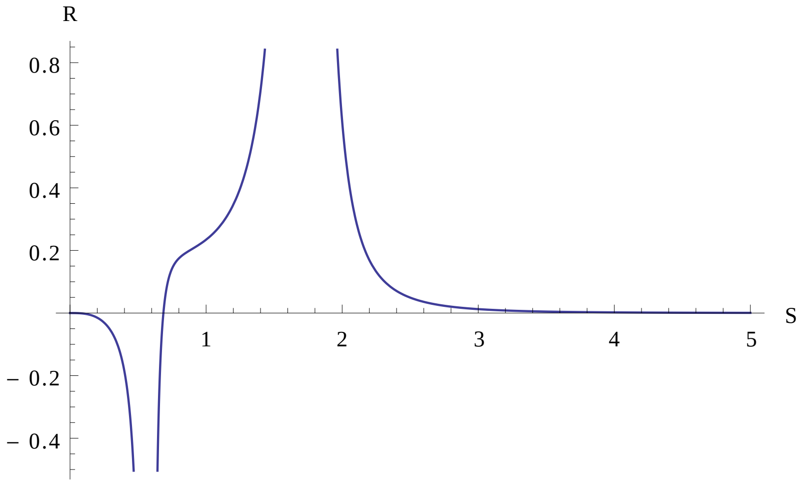

Figure 4.10: variation of scalar curvature with horizon radius for $\Lambda_W = -1$,$a = 1$

Ehrenfest Scheme

Infinite discontinuity in the heat capacity of the black hole does not always indicate a second order phase transition, but it suggests the possibility of a higher order phase transition. In classical thermodynamics, one can confirm the first order phase transition by utilizing Clausius-Clapeyron equations [150, 151]. Similarly a second order transition can be confirmed by checking whether it satisfies Ehrenfest equations [152] or not. The original expressions of Ehrenfest equations in classical thermodynamics are given by,

$$\left(\frac{\partial P}{\partial T}\right)_S = \frac{C_{P_2} - C_{P_1}}{VT(\alpha_2 - \alpha_1)} = \frac{\Delta C_P}{VT\Delta\alpha}, \tag{4.19}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\alpha_2 - \alpha_1}{\kappa_{T_2} - \kappa_{T_1}} = \frac{\Delta\alpha}{\Delta\kappa}, \tag{4.20}$$

where $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$ and $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ are the volume expansion coefficient and isothermal compressibility coefficient respectively. Considering the analogy between the thermodynamic variables and black hole parameters, where pressure (P) is replaced by the negative of the electrostatic potential difference ($-\Phi$), and volume (V) is replaced by charge of the black hole (Q). Thus for black hole thermodynamics, the two Ehrenfest equations (4.19) and (4.20) become,

$$-\left(\frac{\partial\Phi}{\partial T}\right)_S = \frac{1}{QT}\frac{C_{\Phi_2} - C_{\Phi_1}}{(\alpha_2 - \alpha_1)} = \frac{\Delta C_\Phi}{QT\Delta\alpha}, \tag{4.21}$$

$$-\left(\frac{\partial\Phi}{\partial T}\right)_Q = \frac{\alpha_2 - \alpha_1}{\kappa_{T_2} - \kappa_{T_1}} = \frac{\Delta\alpha}{\Delta\kappa}, \tag{4.22}$$

where $\alpha = \frac{1}{Q}\left(\frac{\partial Q}{\partial T}\right)_\Phi$ and $\kappa_T = -\frac{1}{Q}\left(\frac{\partial Q}{\partial \Phi}\right)_T$ are the volume expansion coefficient and the isothermal compressibility coefficient of the black hole system respectively. Here, in the above sets of equations, the subscripts 1 and 2 denote two distinct phases of the system.

In the present study, rather than considering the black hole analogy of Ehrenfest equation, we will introduce the classical Ehrenfest equation directly in to the black hole system under consideration. Using (2.37), (2.38), (2.39) and (2.40), we can arrive at the expressions of specific heat at constant pressure, volume expansion coefficient and isothermal compressibility coefficient respectively as,

$$C_P = \frac{4\pi\sqrt{-\Lambda_W r_h}}{a}\frac{\left(3\Lambda_W r_h^2 - 1\right)}{\left(3\Lambda_W r_h^2 + 1\right)}, \tag{4.23}$$

$$\alpha = \frac{6\pi r_h}{1 - 3r_h^2\Lambda_W}, \tag{4.24}$$

and,

$$\kappa = \frac{16}{7r_h}\frac{1}{1 - 3r_h^2\Lambda_W}. \tag{4.25}$$

From these relations, it is interesting to note that both volume expansion coefficient and isothermal compressibility coefficient have same factor in

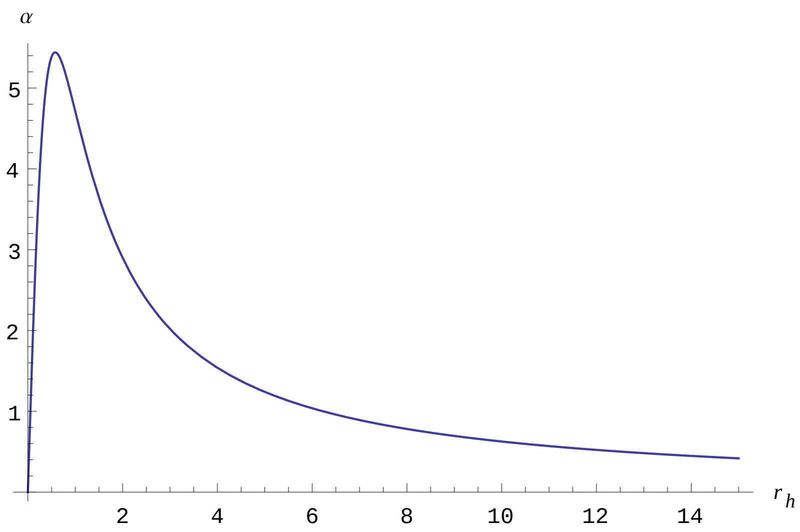

Figure 4.11: variation of volume expansion coefficient with horizon radius for $\Lambda_W = -1$,$a = 1$

the denominator, which implies that both these parameters diverge at the same point. We have plotted the variation of these coefficients with respect to the horizon radius (r_h) in figs.(4.11) and(4.12) respectively. Now we will investigate the nature of phase transition at the critical point of LMP black hole by doing the analytic check of classical Ehrenfest equations (4.19) and (4.20). The values of temperature, pressure and volume at the critical point are respectively given by,

$$T_c = \frac{\sqrt{3}\sqrt{\Lambda_W}}{4\pi}, \tag{4.26}$$

$$P_c = \frac{7\sqrt{3}}{32\sqrt{\Lambda_W}}, \tag{4.27}$$

and,

$$V_c = \frac{16(-\Lambda_W)^{5/4}}{7\sqrt[4]{3}a}. \tag{4.28}$$

Now let's check the validity of Ehrenfest equations at the critical point. From the definition of volume expansion coefficient α, given by (4.24), we obtain,

$$V\alpha = \left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial V}{\partial S}\right)_P \left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial V}{\partial S}\right)_P \left(\frac{C_P}{T}\right) \tag{4.29}$$

then, the R.H.S of first classical Ehrenfest equation (4.19) becomes,

$$\frac{\Delta C_P}{TV\Delta\alpha} = \left[\left(\frac{\partial S}{\partial V}\right)_P\right]_{r_{\text{cri}}}, \tag{4.30}$$

where r_{cri} denotes the critical point. Applying the above equation to LMP black hole system, we obtain,

$$\frac{\Delta C_P}{TV\Delta\alpha} = \frac{21\pi}{24}\frac{1}{-\Lambda_W}. \tag{4.31}$$

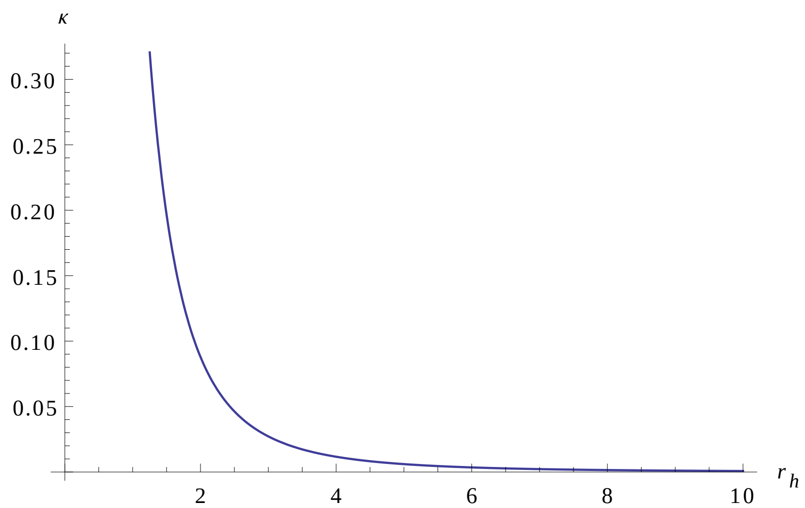

Figure 4.12: variation of isothermal compressibility with horizon radius for $\Lambda_W = -1$,$a = 1$

Now the L.H.S of first classical Ehrenfest equation (4.19) becomes,

$$\left[-\left(\frac{\partial P}{\partial T}\right)_S\right]_{r_{\text{cri}}} = \frac{21\pi}{24}\frac{1}{-\Lambda_W}. \tag{4.32}$$

From (4.32) and (4.31), we can arrive at the conclusion that both L.H.S and R.H.S of first Ehrenfest equation are in good agreement at the critical point r_{cri}. From (4.24) and (4.25), using the thermodynamic identity,

$$\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P = -1, \tag{4.33}$$

we can obtain,

$$V\kappa_T = -\left(\frac{\partial V}{\partial P}\right)_T = \left(\frac{\partial T}{\partial P}\right)_V \left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial T}{\partial P}\right)_V V\alpha. \tag{4.34}$$

Now the R.H.S of (4.20) can be obtained as,

$$\frac{\Delta\alpha}{\Delta\kappa_T} = \left[\left(\frac{\partial P}{\partial T}\right)_V\right]_{r_{\text{cri}}} = \frac{21\pi}{24}\frac{1}{(-\Lambda_W)^{\frac{4}{3}}}. \tag{4.35}$$

Also the L.H.S of (4.20) can be obtained as,

$$\left[\left(\frac{\partial P}{\partial T}\right)_V\right]_{r_{\text{cri}}} = \frac{21\pi}{24}\frac{1}{(-\Lambda_W)^{\frac{4}{3}}}. \tag{4.36}$$

From (4.36) and (4.35), we can obtain the conclusion that second Ehrenfest equation is satisfied at the critical points. Hence both the Ehrenfest equations are in good agreement at the critical point. Using (4.31) and (4.35), the Prigogine-Defay (PD) ratio is

$$\Pi = \frac{\Delta C_P \Delta\kappa_T}{TV(\Delta\alpha)^2} = 1. \tag{4.37}$$

This confirms that the phase transition of LMP black hole in Hořava-Lifshitz gravity is second order in nature.

The complete thermodynamics and phase transition picture of LMP black holes in Hořava-Lifshitz gravity has been investigated using thermodynamic geometry. We have systematically analyzed the thermodynamics and phase transition. From this thermodynamic study, absence of any discontinuity in entropy-temperature relationship eliminates the presence of any first order transition. Then, the heat capacity is found to be diverging, thereby indicating the presence of a phase transition. But the order of phase transition is not revealed. To further clarify the existence of phase transition, geometrothermodynamics is applied. In which the critical point where the heat capacity diverges coincides with the diverging point of Legendre invariant geometrothermodynamic scalar curvature. Hence GTD metric exactly reproduces the phase transition structure of LMP black hole and their corresponding thermodynamic interactions. Now we can confirm that the LMP black hole exhibits second order phase transition similar to Davies type transition. Hence it will be possible to answer whether one can have a quantum field theory at a finite temperature by studying the thermodynamic stability of the black hole, as evident from the specific heat. However, the black hole configuration must be favourable over pure thermal radiation in anti-de Sitter space; that is, have dominant negative free energy. The present black hole solution satisfies these conditions. Hence from this study one can look forward for the implication on the dual field theory which exists on the boundary of the anti-de Sitter space.

The other solution, Kehagias-Sfetsos solution, have been studied in details using the language differential geometry in [147]. In that work it is shown that the curvature corresponding to the Legendre invariant metric for the KS black hole is singular at the phase transition point For the GTD metric case Legendre invariance is tested by checking the consistency of taking either the entropy or mass as the thermodynamic potential.

4.2 Black holes in Massive gravity

dRGT black hole

Now we will investigate the phase transition structure of the charged black hole solution in massive gravity discussed in Chapter 2. We will apply the ideas of GTD in to the charged black hole system in massive gravity. For this, we will construct a 7 dimensional thermodynamic phase space $\mathcal{T}$ using the extensive variables and their dual intensive variables as coordinates. In the dRGT black holes in massive gravity case, coordinates are given by $Z^A = \{M, S, Q, \alpha, T, \phi, a\}$, where S, Q, α are extensive variables and T, ϕ, a are their corresponding dual intensive variables. Now one can write the Gibbs 1-form as,

$$\Theta = dM - TdS - \phi dQ - ad\alpha. \tag{4.38}$$

From the usual mapping one can determine the equilibrium manifold $\mathcal{E}$, and in $\mathcal{E}$ one can define the GTD metric from (1.80) as,

$$g^{\text{GTD}} = (SM_S + QM_Q + \alpha M_\alpha) \begin{bmatrix} -M_{SS} & 0 & 0 \\ 0 & M_{QQ} & M_{Q\alpha} \\ 0 & M_{\alpha Q} & M_{\alpha\alpha} \end{bmatrix}.$$

where subscripts represent partial derivative with respect to the corresponding coordinate. Notice that no cross terms of the form g_{SQ} or $g_{S\alpha}$ which is proportional to M_{SQ} or $M_{S\alpha}$, appear in the GTD metric defined above.

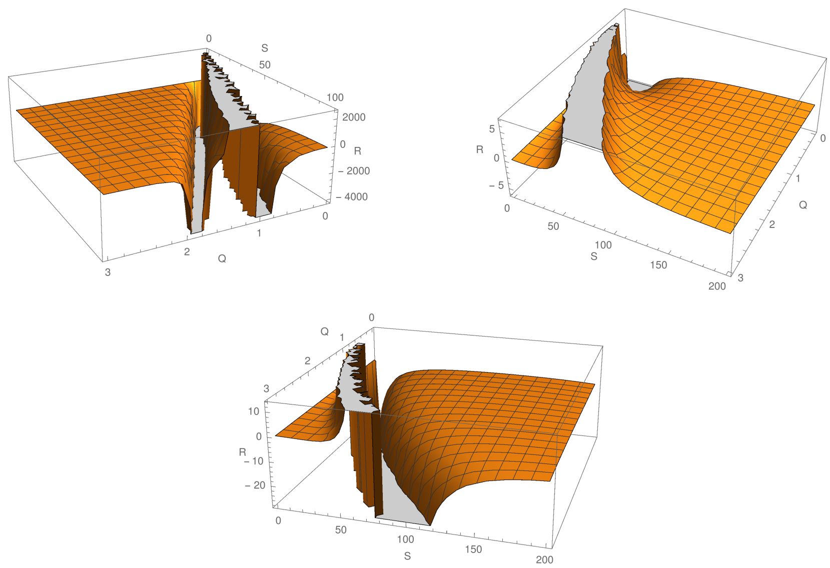

Figure 4.13: 3D Variation of Ricci scalar against entropy and charge de-Sitter black holes for for flat, spherical and hyperbolic topology of space-time in massive gravity

This is due to the special choice of the auxiliary metric defined generally as (1.80). Then, the Legendre invariant scalar curvature corresponding to the above metric is given by,

$$R^{\mathrm{GTD}} = \frac{f(S,Q,\alpha)}{\left(\pi\alpha\left(kS-3\pi Q^2\right)+2m^2S^2\right)^2\left(2m^2S^2-3\pi\alpha\left(kS+3\pi Q^2\right)\right)^3} \tag{4.39}$$

where $f(S,Q,\alpha)$ is a complicated expression of less physical interest. Now we will investigate the thermodynamic behaviour of the black hole system using the scalar curvature. According to the theory of geometrothermodynamics, the points of zero scalar curvature as well as the infinite discontinuities will exactly matches with the singular behaviours of thermodynamic potentials which corresponds to the black hole system.

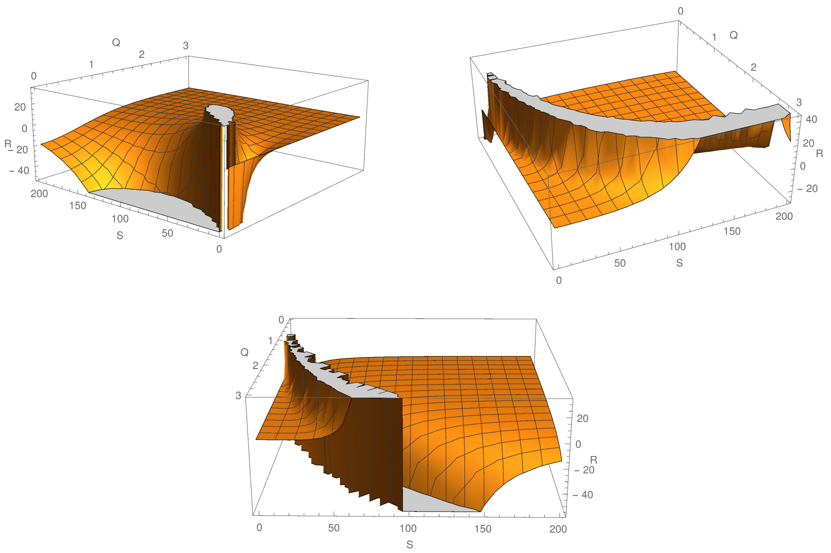

Figure 4.14: 3D Variation of Ricci scalar against entropy and charge anti de-Sitter black holes for for flat, spherical and hyperbolic topology of space-time in massive gravity

Now let us evaluate different charged black hole system in massive gravity, as they vary with respect to mass of the graviton, topology of the solutions and the sign of the curvature parameter.

The variation of Ricci scalar curvature with entropy is depicted in figs.(4.13) and (4.14). Let us first consider the case in which the curvature parameter, α is taken as positive, then as we have discussed earlier, the black hole system will behave like charged de-Sitter black hole system. From the fig.(4.13) it is evident that, all singular points in the thermodynamic parameters like temperature and heat capacity are exactly reproduced by the Ricci scalar either by vanishing or showing infinite discontinuities at the same points. For the second case also Ricci scalar behaves in a similar manner, where the curvature parameter, α is taken as negative, and hence the the black hole system becomes a charged anti de-Sitter black hole system.

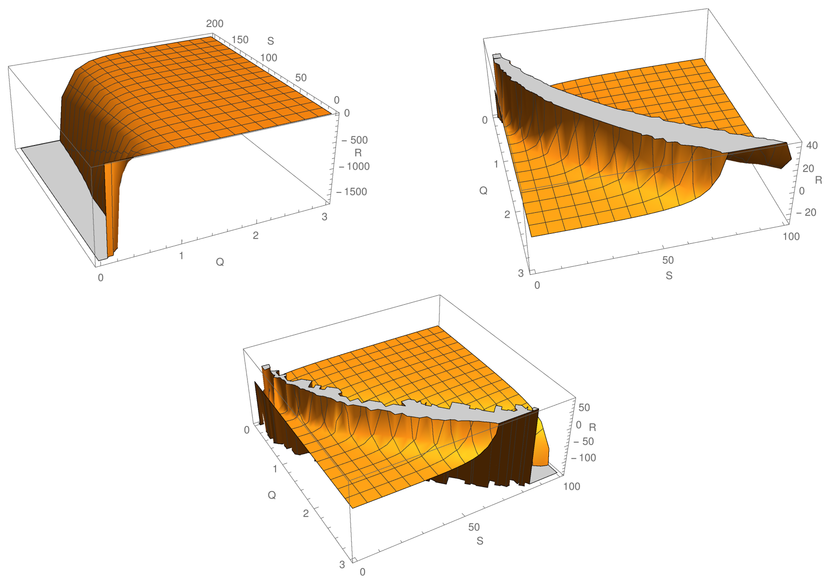

Figure 4.15: 3D Variation of Ricci scalar against entropy and charge de-Sitter black holes for for flat, spherical and hyperbolic topology of space-time in Einstein's gravity

The variation of scalar curvature for this case is plotted in fig.(4.14). Now let us investigate the geometrothermodynamics and the behaviour of Legendre invariant scalar curvature of the black hole system, when the mass of the graviton becomes zero. The corresponding variation of Ricci scalar is depicted in figs.(4.15) and (4.16).

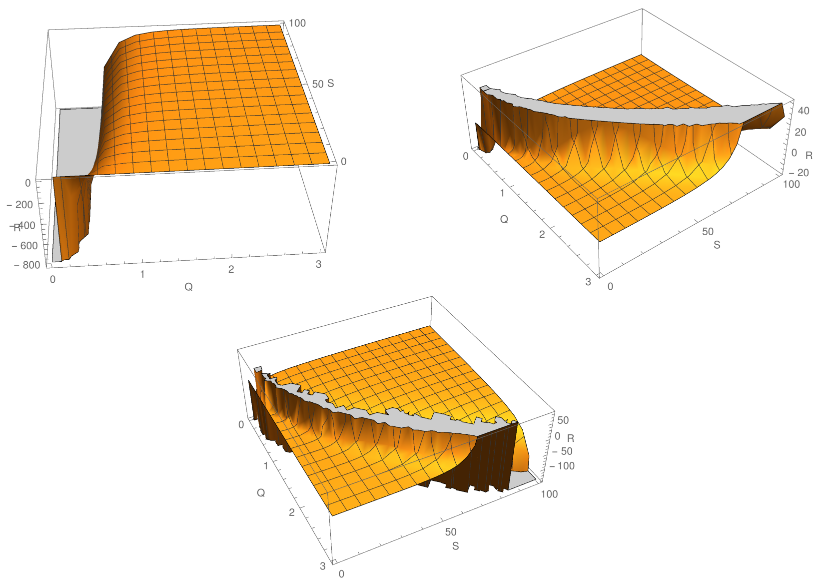

Figure 4.16: 3D Variation of Ricci scalar against entropy and charge anti de-Sitter black holes for for flat, spherical and hyperbolic topology of space-time in Einstein's gravity

Here too, the singularities of the Ricci scalar matches with those of temperature and heat capacity of the RN black hole in Einstein's general relativity. It is interesting to note that the GTD results obtained from the charged black hole solution in massive gravity coincides with the study of RN black hole previously obtained in [148]. Hence the geometrothermodynamics exactly reproduces the phase transition structure of the charged black hole solutions in massive gravity.

We have studied the geometrothermodynamics of the charged de-Sitter and anti de-Sitter black hole solutions in dRGT massive gravity. Phase transition as well as the singular behaviours in the thermodynamic potentials using GTD method has been discussed. We used the Quevedo metric or GTD metric to obtain the Legendre invariant Ricci scalar curvature. We analysed the thermodynamic behaviour using both analytical and graphical methods. The analysis showed that, the singular behaviours in the thermodynamic potentials, including the point where heat capacity diverges, exactly reproduced by the Ricci scalar obtained using the GTD metric. Hence we can say that, GTD metric exactly reproduces the phase transition structure of charged black holes in massive gravity and their corresponding interactions. From this study it is evident that the charged black holes in massive gravity undergoes second order phase transition. The order of phase transition that was unrevealed in usual thermodynamic studies get revealed in this geometric study. One expects that thermodynamics of black holes would be the same as in general relativity. But our studies show that, even though the results agrees with general relativity when massive parameter tends to zero, there are significant changes in the phase transition structure of the system when $m \neq 0$. Comparative studies on analytical and graphical representation of the changes of heat capacity and Ricci scalar against entropy and massive parameter of the black hole system reveal the same result. Also the present study shows that, like all other charged black hole solutions in Einstein's gravity and in all modified gravities, there exists a temperature window, where the black hole temperature lies in a physically significant region with positive temperature. Much attention and rigorous studies are needed to well explain the existence of temperature windows.

BTZ black holes

Now we will apply the same GTD formalism in BTZ black hole in new massive gravity. For that, let us consider a 5 dimensional thermodynamic phase space, constituted by the coordinates $Z^a = \{M, S, l, T, \alpha\}$, where S, l are extensive variables while T and α are their corresponding dual intensive variables. From the ideas of equilibrium manifold, one can obtain

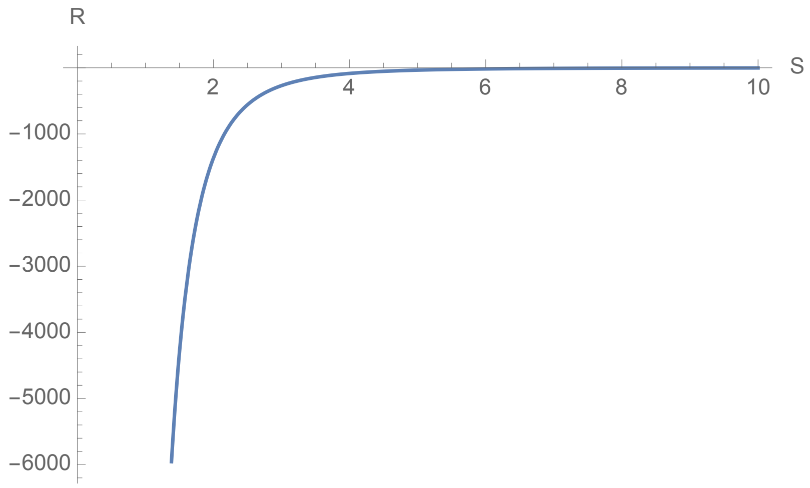

Figure 4.17: Variation of scalar curvature of the BTZ black hole against the changes in entropy with $l = 1$, $G = 1$ and $M = 1$.

the GTD metric as,

$$g^{\mathrm{GTD}} = (SM_S + lM_l) \begin{bmatrix} -M_{SS} & 0 \\ 0 & M_{ll} \end{bmatrix}.$$

Now one can calculate the Legendre invariant scalar curvature corresponding to the above hessian metric in mass representation as,

$$R_{\mathrm{GTD}} = \frac{\pi^4 l^4 \left(m^2 l^2 - 2\right)^4 \left(8m^2 l^2 - 15\right)}{32 G^4 s^4 \left(5m^4 l^4 - 9m^2 l^2 + 6\right)^2} \tag{4.40}$$

We will now explore the thermodynamic behaviour of the system as well as their interactions using this scalar curvature. By plotting the scalar curvature as a function of entropy, the variation is depicted in fig(4.17). From this figure it can be inferred that, the particular equilibrium manifold under consideration is a space of negative curvature for any values of entropy or for any values of horizon radius. Hence the scalar curvature corresponding to the BTZ black hole does not possess any discontinuities or zeros. Then we can say that the BTZ space-time is free of any thermodynamic curvature singularities. As we have already discussed, the non existence of any discontinuity in the variation of heat capacity implies the non existence of phase transition (second order). According to GTD formalism, the regular variation of curvature scalar indicates that no (second order) phase transition occurs. This result does not imply that there is no thermodynamic interactions exists in the case of BTZ black hole, but no second order phase transitions can occur.

In this work we used the formalism of GTD to construct a thermodynamic equilibrium phase space to study the thermodynamic behaviour of the BTZ black hole solution in new massive gravity. This method shows that the thermodynamic curvature corresponding to BTZ black hole is non zero and free of singularities, indicating the absence of second order phase transition. Even though the black hole system shows a continuous transition from a thermodynamically stable to an unstable phase, it is not second order in nature.

4.3 Discussion and conclusion

During the last few decades several attempts have been made in order to incorporate differential geometry ideas in ordinary thermodynamics. Weinhold and Ruppeiner methods are the corner stone in these studies. Explorations made on the relation between the phase space and the metric structures in the space of equilibrium states led to the result that Weinhold's and Ruppeiner's thermodynamic metrics are not invariant under Legendre transformations. This result contradicts ordinary equilibrium thermodynamics which is obviously Legendre invariant. A consistent method is proposed recently, named as Geometrothermodynamics which answers all the discrepancies faced in earlier studies. In this Chapter we have analysed the thermodynamic geometry of different black hole solution in modified theories of gravity, particularly in Hořava-Lifshitz gravity and massive gravity. Different black hole solutions in these theories are considered, which include Kehagias-Sfetsos black hole, Lü-Mei-Pope black hole, Park black hole, charged dRGT black hole and BTZ black hole. Thermodynamic behaviour of these black hole systems are studied both analytically and graphically with immense stress on the phase transition structure in chapter 2. From those studies we have concluded that many solutions exhibit phase transition via change of sign of heat capacity and many abnormalities in different thermodynamic potentials. Ordinary thermodynamics was not adequate enough to answer this scenario. In this chapter we have analysed all these unanswered questions with the help of geometrothermodynamics. Initially, thermodynamics of Park black hole solution is studied once again in a slightly different way by incorporating both de Sitter and anti de Sitter solutions in a single scenario. GTD formalism is applied to all the black hole case by defining an equilibrium manifold and thus defining a metric structure. Scalar curvature corresponding to each case is calculated. Within the GTD formalism, scalar curvature is the geometric object that account for the physics of the system. In the black hole systems, the curvature singularities determines the phase transition structure of the system in such a way that the critical point where scalar curvature diverges matches with that of phase transition points. Singularities in these scalar curvature also reproduce the anomalous behaviours existing in the system. Here the phase transitions shown by the black hole systems in modified theories of gravity have been analysed using GTD approach and concluded that they undergo a second order phase transition. All abnormal behaviours exhibited by each black holes are well coded on the singularities in the scalar curvature.

5 Conclusion and Plan of future works

5.1 Conclusion

In this thesis we have obtained some new results both in the context of thermodynamics and Geometrothermodynamics of black holes in modified theories of gravity, particularly in Hořava-Lifshitz gravity and Massive gravity.

In the second chapter after giving a general introduction to different black hole solution in Hořava-Lifshitz and Massive gravity. Kehagias-Sfetsos black hole, Lü-Mei-Pope black hole, Park black hole, dRGT black hole and (2+1) BTZ black hole are extensively explored by studying their thermodynamic properties with a great emphasize on phase transition structure of the system. From Davies's phase transition idea [57], phase transitions occur in black holes whenever there is a change of sign through a divergency in heat capacities. This is the most basic and at the same time the most discussed definition of second order phase transitions in black holes. By exploring the heat capacity behaviours of KS, LMP, Park and dRGT black holes, we also conclude that they exhibit phase transitions. For the Park black hole solution, it is found that there exist different anomalous behaviours in the form of negative temperature and existence of mass bound. Whereas in the BTZ black hole case in new massive gravity, it does not possess any infinite discontinuity in transitions. Ordinary thermodynamic ideas miserably fail to explain these anomalies.

We have analysed the entropy spectra of these black hole solution in the third chapter by employing the method suggested by Majhi, Vagenas, Jiang and Han, in which they have incorporated the ideas form the adiabatic invariance, tunneling mechanism, Bohr-Sommerfeld quantization rule and near horizon approximations. We have found that the entropy spectrum of KS black hole in Hořava-Lifshitz gravity is equi-spaced and the spectrum is independent of black hole parameters. Whereas in the case of charged BTZ solution in massive gravity, the entropy spectrum depends on the black hole parameters, even though the spectrum is equi-spaced.

We have analyzed the geometric structure of equilibrium manifold for the above mentioned black holes in modified theories of gravity using the new approach proposed by Hernando Quevedo [97–99]. By analysing the Riemannian thermodynamic metric or the GTD metric defined on the equilibrium manifold, these geometric structures give complete description of the critical behaviours exhibited by these black hole systems. Interactions present in the thermodynamic system are reflected from the curvature of the metric defined on equilibrium spaces. If thermodynamic curvature is free of singularities or varying regularly, then GTD interprets it as non-existence of singular points at the level of the heat capacity and no (second order) phase transitions occur in the system. It is interesting to note that the same scalar curvature of the GTD metric reflects all abnormalities shown by the system. From the examination of phase transition structure of all black hole solutions considered in the previous chapters, we found that scalar curvature exactly reproduce the same as per the GTD formalism. In Physics certain systems can exhibit negative temperature behaviour. But most familiar systems cannot achieve negative temperatures, because adding energy always increases their entropy.

In some of the black hole cases we examined, it is evident that there are regions where the temperature becomes negative. In the spin system the temperature can be negative, due to the upper bound of energy spectrum [153]. Recently, a number of black hole solutions which have similar upper bound of black hole mass have been discovered [154–158]. Park black hole is one among them. The most important but most difficult question in this scenario would be, can we have a plausible explanation of why the standard computation of black hole temperature should fail in these cases?. Even though GTD can explain the thermodynamic behaviours of the system, it miserably fails to answer this question.

5.2 Plan of future works

Many questions about the thermodynamic geometry methods and their application in modified theories of gravity remain open so far and it will be considered in future attempts. A more generalized thermodynamic geometry formalism which can exactly give the order of phase transition is yet to be formulated. Prigogine-Defay (PD) ratio gives the exact order of transition and we will be looking forward to incorporate these ideas along with geometrothermodynamics to formulate a unique geometric method to explain the thermodynamic systems.

Developing such a generic geometric framework will enable us to use the geometric quantities like curvature, distance, etc to explore further the thermodynamic relations through different laws and new equation of states. Then one can directly apply these ideas to different scenarios e.g. in cosmology. From the Friedmann equations of standard cosmological model, there are two differential equations for three unknowns, so that to close the system it is necessary to add an equation. Recent studies show that one can use GTD to derive the missing equation by considering GTD fluid [100, 149]. So generating further cosmological models using GTD formalism and testing these GTD cosmological models by investigating whether they reproduce the entire dynamics in all epochs of the evolution of the Universe, the consistency with observations will be a fascinating topic of research.

In summary, we expect that this will not be the end of our exploration, instead the beginning of a better and deeper understanding of thermodynamic geometry of different systems in nature which include the fascinating black holes.

References

[1] I. Newton,*Philosophiae naturalis principia mathematica* (1687).

[2] A. Einstein, *"Die Grundlage der Allgemeinen Relativitäts Theorie"*, Annalen der Physik **49**, 50 (1916).

[3] Schwarzschild, K., *"On the gravitational field of a mass point according to Einstein's theory"*, Proceedings of the Royal Prussian Academy of Sciences, **7**: 189196 (1916).

[4] H. Reissner, *"Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie"*, Ann. Phys. **50**, 106 (1916).

[5] G. Nordström, *"On the Energy of the Gravitational Field in Einstein's Theory"*, Proc. Kon. Ned. Akad. Wet. **20**, 1238 (1918).

[6] R. P. Kerr, *"Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics"*, Phys. Rev. Lett **11**, 237 (1963).

[7] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, *"Metric of a Rotating, Charged Mass"*, J. Math. Phys. **6**, 918 (1965)

[8] S. Capozziello, M. De Laurentis, *"Extended Theories of Gravity"*, Physics Reports **509**, 167321 (2011).

[9] T. Clifton, P. G. Ferreira, A. Padilla, C. Skordis, *"Modified gravity and cosmology"*, Physics Reports **513**, 1189 (2012).

[10] P. Hořava, *"Quantum gravity at a Lifshitz point"*, Phys. Rev. D **79**, 084008 (2009).

[11] P. Hořava, *"Membranes at Quantum Criticality"*, JHEP **03**, 020 (2009).

[12] P. Hořava, *"Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Point"*, Phys. Rev. Lett. **102**,161301 (2009).

[13] M. Fierz and W. Pauli, *"On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field"*, Proc. Roy. Soc. Lond. A **173**, 211 (1939).

[14] H. van Dam and M. J. G. Veltman, *"Massive and mass-less Yang-Mills and gravitational fields"*, Nucl. Phys. B **22**, 397 (1970).

[15] V. I. Zakharov, *"Experimental investigations of the mechanical characteristics of frozen ground at different rates of loading"*, JETP Lett. **12**, 312 (1970) .

[16] A. I. Vainshtein, *"To the problem of nonvanishing gravitation mass"*, Phys. Lett. B **39**, 393 (1972).

[17] D. G. Boulware and S. Deser, *"Inconsistency of finite range gravitation"*, Phys. Lett. B **40**, 227 (1972).

[18] D. G. Boulware and S. Deser, *"Can Gravitation Have a Finite Range?"*, Phys. Rev. D **6**, 3368 (1972).

[19] C. de Rham and G. Gabadadze, *"Generalization of the Fierz-Pauli action"*, Phys. Rev. D **82**, 044020 (2010).

[20] C. de Rham, G. Gabadadze and A. J. Tolley, *"Resummation of Massive Gravity"*, Phys. Rev. Lett. **106**, 231101 (2011).

[21] S. Hassan and R. A. Rosen, *"Resolving the Ghost Problem in Nonlinear Massive Gravity"*, Phys. Rev. Lett. **108**, 041101 (2012).

[22] S. Hassan, R. A. Rosen, and A. Schmidt-May, *"Ghost-free massive gravity with a general reference metric"*, JHEP **02**, 026 (2012).

[23] S. Hassan and R. A. Rosen, *"Bimetric gravity from ghost-free massive gravity"*, JHEP **126**, 1202 (2012).

[24] N. Rosen, *"General Relativity and Flat Space. I"*, Phys. Rev. **57**, 150 (1940).

[25] N. Rosen,In Erice 1975, Proceedings, *"Topics In Theoretical and Experimental Gravitation Physics"*, New York 1977, 273-294.

[26] C. Aragone and S. Deser, *"Consistency problems of hypergravity"*, Nuovo Cim. A **3**, 709 (1971).

[27] C. J. Isham, A. Salam and J. A. Strathdee, *"f-Dominance of Gravity"*, Phys. Rev. D **3**, 867 (1971).

[28] A. H. Chamseddine, A. Salam and J. A. Strathdee, *"Strong gravity and supersymmetry"*, Nucl. Phys. B **136**, 248 (1978).

[29] G. R. Dvali, G. Gabadadze and M. Porrati, *"Metastable gravitons and infinite volume extra dimensions"*, Phys. Lett. B **484**, 112 (2000).

[30] G. R. Dvali, G. Gabadadze and M. Porrati, *"4D Gravity on a Brane in 5D Minkowski Space"*, Phys. Lett. B **485**, 208 (2000).

[31] T. Kaluza, *"On the Problem of Unity in Physics"*, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1921 966 (1921).

[32] O. Klein, *"Quantum theory and five dimensional theory of relativity"*, Z. Phys. **37**, 895 (1926) [Surveys High Energ. Phys. 5 (1986) 241].

[33] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *"Massive Gravity in Three Dimensions"*, Phys. Rev. Lett. **102**, 201301 (2009).

[34] S. Deser, R. Jackiw and S. Templeton, *"Topologically massive gauge theories"*, Annals Phys. **140**, 372 (1982).

[35] S. Deser, R. Jackiw and S. Templeton, *"Three-Dimensional Massive Gauge Theories"*, Phys. Rev. Lett. **48**, 975 (1982).

[36] L. Berezhiani, G. Chkareuli, C. deRham, G. Gabadadze, and A. Tolley, *"On black holes in massive gravity"*, Phys. Rev. D **85**, 044024 (2012).

[37] K. Koyama, G. Niz, and G. Tasinato, *"Strong interactions and exact solutions in nonlinear massive gravity"*, Phys. Rev. D **84**, 064033 (2011).

[38] T. M. Nieuwenhuizen, *"Exact Schwarzschild-de Sitter black holes in a family of massive gravity models"*, Phys. Rev. D **84**, 024038 (2011).

[39] A. Accioly, J. Helayel-Neto, E. Scatena, J. Morais, R. Turcati, *"Some interesting features of new massive gravity"*, Class. Quant. Grav. **28**, 225008 (2011).

[40] S. W. Hawking, *"Black hole explosions"*, Nature **248**, 30 (1974).

[41] J. D. Bekenstein, *"Black holes and entropy"*, Phys. Rev. D **7**, 2333 (1973).

[42] J. D. Bekenstein, *"Generalized second law of thermodynamics in black-hole physics"*, Phys. Rev. D **9**, 3292-3300 (1974).

[43] J. D. Bekenstein, *"The quantum mass spectrum of the Kerr black hole"*, Lett. Nuovo Cimento Soc. Ital. fis. **11**, 467 (1974).

[44] R. Penrose and R.M. Floyd, *"Extraction of rotational energy from a black hole"*, Nature (Physical Science) **229**, 177 (1971).

[45] D. Christodoulou, *"Reversible and irreversible transformations in black hole physics"*, Phys. Rev. Lett. **25**, 1596 (1970).

[46] J. M. Bardeen, B. Carter and S. W. Hawking, *"The laws of black hole mechanics"*, Commun. Math. Phys. **31**, 161 (1973).

[47] S. W. Hawking and Don N. Page, *"Thermodynamics of black holes in Anti-de Sitter space"*, Commun.Math. Phys. **87**, 577-588 (1983).

[48] P. C. W. Davies, *"Thermodynamic phase transitions of Kerr-Newman black holes in de-Sitter space"*, Class. Quantum. Grav. **6**, 1909 (1989).

[49] L. M. Sokolowski and P. Mazur, *"Second order phase transitions in black-hole thermodynamics"*, J. Phys. A: Math. Gen. **13**, 1113-1120 (1980).

[50] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, *"Holography, thermodynamics, and fluctuations of charged AdS black holes"*, Phys. Rev. D **60**, 104026 (1999).

[51] M. M. Caldarelli, G. Cognola, and D. Klemm, *"Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories"*, Class. Quantum. Grav. **17**, 399 (2000).

[52] B. M. N. Carter and I. P. Neupane, *"Thermodynamics and stability of higher dimensional rotating (Kerr-) AdS black holes"*, Phys. Rev. D **72**, 43534 (2005).

[53] T. K. Dey, S. Mukherji, S. Mukhopadhyay, and S. Sarkar, *"Phase transitions in higher derivative gravity"*, JHEP **04**, 014 (2007).

[54] A. Sheykhi, *"Thermodynamical properties of topological Born-Infeld-Dilaton black holes"*, Int. J. Mod. Phys. D **18**, 25 (2009).

[55] S. Fernando, *"Thermodynamics of Born-Infeld–anti-de Sitter black holes in the grand canonical ensemble"*, Phys. Rev. D **74**, 104032 (2006).

[56] B. R. Majhi and D. Roychowdhury, *"Phase transition and scaling behavior of topological charged black holes in Hořava-Lifshitz gravity"*, Class. Quantum Grav. **29**, 245012 (2012).

[57] P. C. W. Davies, *"The thermodynamic theory of black holes"*, Proc. R. Soc. Loud. A. **353**, 499-521 (1977).

[58] G. J. Stephens and B. L. Hu, *"Notes on black hole phase transition"* (2008).

[59] J. D. Bekenstein, *"Black holes and the second law"*, Lett. Nuovo. Cim **4**, 737 (1972).

[60] S. Hod, *"Bohr's Correspondence Principle and the Area Spectrum of Quantum Black Holes"*, Phys. Rev. Lett. **81**, 4293 (1998).

[61] G.Kunstatter, *"d-Dimensional black hole entropy spectrum from quasinormal modes"*, Phys. Rev. Lett. **90**, 161301 (2003).

[62] M . Maggiore, *"Physical interpretation of the spectrum of black hole quasinormal modes"*, Phys. Rev. Lett. **100**, 141301 (2008).

[63] E. C. Vagenas, *"Area spectrum of rotating black holes via the new interpretation of quasinormal modes"*, JHEP **0811**, 073 (2008).

[64] A. J. M. Medved, *"On the universal quantum area spectrum"*, Mod. Phys. Lett. A **24**, 2601 (2009).

[65] A. J. M. Medved, *"On the Kerr Quantum Area Spectrum"*, Class. Quant. Grav. **25**, 205014 (2008).

[66] B. R. Majhi and Elias C. Vagenas, *"Black hole spectroscopy via adiabatic invariance"*, Phys. Lett. B **701**, 623 (2011).

[67] E. T. Akhmedov, V. Akhmedova, D. Singleton, *"Hawking temperature in the tunneling picture"*, Phys. Lett. B **642**, 124 (2006).

[68] E. T. Akhmedov, V. Akhmedova, T. Pilling, D. Singleton, *"Thermal radiation of various gravitational backgrounds"*, Int. J. Mod. Phys. A **22**, 1705 (2007).

[69] V. Akhmedova, T. Pilling, A. de Gill, D. Singleton, *"Temporal contribution to gravitational WKB-like calculations"*, Phys. Lett. B **666**, 269 (2008).

[70] V. Akhmedova, T. Pilling, A. de Gill, D. Singleton, *"Comments on anomaly versus WKB/tunneling methods for calculating Unruh radiation"*, Phys. Lett. B **673**, 227 (2009).

[71] E. T. Akhmedov, T. Pilling, D. Singleton, *"Subtleties in the quasi-classical calculation of Hawking radiation"* Int. J. Mod. Phys. D **17**, 2453 (2008).

[72] Jiang Q. Q, Han Y, *"On black hole spectroscopy via adiabatic invariance"*, Phys. Rev. Lett. B **718**, 584 (2012).

[73] J. Gibbs, *"The collected works"*, Vol. 1, Thermodynamics, Yale University Press, (1948).

[74] C. Charatheodory, *"Untersuchungen über die Grundlagen der Thermodynamik"*, Gesammelte Mathematische Werke, Band 2, Munich, 1995.

[75] R. Hermann, *"Geometry, physics and systems"*, Marcel Dekker, New York, (1973).

[76] R. Mrugala, *"Geometrical formulation of equilibrium phenomenological thermodynamics"*, Rep. Math. Phys. **14**, 419 (1978).

[77] R. Mrugala, *"Submanifolds in the thermodynamic phase space"*, Rep. Math. Phys. **21**, 197 (1985).

[78] F. Weinhold, *"Metric Geometry of equilibrium thermodynamics" - I, II, III, IV, V*, J. Chem. Phys. **63**, 2479, 2484, 2488, 2496 (1975); **65**, 558 (1976).

[79] G. Ruppeiner, *"Thermodynamics: A Riemannian geometric model"*, Phys. Rev. A **20**, 1608 (1979).

[80] S. Ferrara, G. W. Gibbons, and R. Kallosh, *"Black holes and critical points in Moduli space"*, Nucl. Phys. B **500**, 75 (1997).

[81] D. A. Johnston, W. Janke, and R. Kenna, *"Information geometry, one, two, three (and four)"*, Acta Phys. Polon. B **34**, 4923 (2003).

[82] W. Janke, D. A. Johnston, and R. Kenna, *"Information geometry and phase transition"*, Physica A. **336**, 181 (2004).

[83] J. E. Åman, I. Bengtsson, and N. Pidokrajt, *"Geometry of black hole thermodynamics"*, Gen. Rel. Grav. **35**, 1733 (2003).

[84] J. E. Åman, and N. Pidokrajt, *"Geometry of higher dimensional black hole thermodynamics"*, Phys. Rev. D **73**, 024017 (2006).

[85] J. E. Åman, and N. Pidokrajt, *"Flat information tries in black hole thermodynamics"*, Gen. Rel. Grav. **38**, 1305 (2006).

[86] J. Shen, R. G. Cai, B. Wang, and R. K. Su, *"Thermodynamic geometry and critical behaviour of black holes"*, Int. J. Mod. Phys. A **22**, 11 (2007).

[87] R. G. Cai, and J. H.Cho, *"Thermodynamic curvature of BTZ black hole"*, Phys. Rev. D **60**, 067502 (1999).

[88] T. Sarkar, G. Sengupta, and B. N. Tiwari, *"On the thermodynamic geometry of BTZ black hole"*, JHEP **0611**, 015 (2006).

[89] A. J. M. Medved, *"A commentary on Ruppeiner metrics for black holes"*, Mod. Phys. Lett. A **23**, 2149 (2008).

[90] B. Mirza, and M. Zamaninasab, *"Ruppeiner Geometry of RN black holes: flat or curved?"*, JHEP **06**, 059 (2007).

[91] H. B. Callen, *"Thermodynamics and an Introduction to Thermostatics"*, Wiley, Newyork, (1985).

[92] N. Goldenfeld, *"Lectures on phase transitions and the renormalization group"*, Addison Wesley, (1992).

[93] G. Ruppeiner, *"Thermodynamic curvature and phase transition in Kerr-Newman black holes"*, Phys. Rev. D **78**, 024016 (2008).

[94] J. E. Åman, I. Bengstsson and N. Pidokrajt, *"Flat information geometries in black hole thermodynamics"*, Gen. Rel. Grav. **38**, 1305 (2006).

[95] J. E. Åman, and N. Pidokrajt, *"Geometry of higher dimensional black hole thermodynamics"*, Phys. Rev. D **73**, 024017 (2016).

[96] J. E. Åman, and N. Pidokrajt, *"On explicit thermodynamic fluctuations and extremal limits of Myers-Perry black holes"*, Eur. Phys. J. C **73**, 011 (2013).

[97] H. Quevedo, *"Geometrothermodynamics"*, J. Math. Phys. **48**, 013506 (2007).

[98] H. Quevedo, *"Geometrothermodynamics of black holes"*, Gen. Relativ. Gravit. **40**, 971 (2008).

[99] H. Quevedo and A. Vazquez, in *Recent Developments in Gravitation and Cosmology*, edited by Alfredo Macias, Claus Lämmerzahl, and Abel Camacho, AIP Conf. Proc. No. 977 (AIP, New York, 2008).

[100] A. Bravetti, Ph. D. Thesis, *"Geometrothermodynamics, from ordinary systems to black holes"*, (2014).

[101] H. Quevedo and A. Sanchez, *"Geometrothermodynamics of asymptotically Anti-de Sitter black holes"*, JHEP **034**, 0809 (2008).

[102] M. Akbar, H. Quevedo, K. Saifullah, A. Sanchez and S. Taj, *"Thermodynamic geometry of charged rotating BTZ black holes"*, Phys. Rev. D **83**, 084031 (2011).

[103] S. Taj, H. Quevedo and A. Sanchez, *"Geometrothermodynamics of five dimensional black holes in Einstein-Gauss-Bonnet theory"*, Gen. Rel. Grav. **44**, 61489 (2012).

[104] H. Quevedo, A. Sanchez, S. Taj, and A. Vazquez, *"Phase transitions in geometrothermodynamics"*, Gen. Rel. Grav. **43**, 1153-1165 (2011).

[105] A. Bravetti, *"Second order phase transition and thermodynamic geometry: a general approach"*, arXiv:1208.0399 (2012).

[106] V. I. Arnold, *"Contact geometry: The geometrical method of Gibbs's thermodynamics"*, in Proceedings of the Gibbs Symposium, D. Caldi and G. Mostow, eds., American Mathematical Society, 163 (1990) .

[107] V. I. Arnold, *"Mathematical Methods of Classical Mechanics"*, Springer Verlag, (1980).

[108] W. L. Burke, *"Applied differential geometry"*, Cambridge University Press, Cambridge, UK, (1987).

[109] R. B. Israel, *"Convexity in the theory of lattice gases"*, Princeton University Press, Princeton, NJ, (1979).

[110] A. Kehagias, K. Sfetsos, *"The black hole and FRW geometries of non-relativistic gravity"*, Phys. Lett. B **678**, 123 (2009).

[111] H. Lü, J. Mei, C. N. Pope, *"Solutions to Hořava Gravity"*, Phys. Rev. Lett. **103**, 091301 (2009).

[112] M. I. Park, *"The Black Hole and Cosmological Solutions in IR modified Hořava Gravity"*, JHEP **0909**, 123 (2009).

[113] E. Kiritsis, G. Kofinas, *"Hořava-Lifshitz Cosmology"*, Nucl. Phys. B **821**, 467 (2009).

[114] J. Sadeghi, K. Jafarzade and B. Pourhassan, *"Thermodynamical Quantities of Hořava-Lifshitz Black Hole"*, Int. J. Thoer. Phys. **51**, 3891 (2012).

[115] E. Babichev and R. Brito, *"Black holes in massive gravity"*, Class. Quantum Grav. **32**, 15 154001 (2015).

[116] M. Henneaux and C. Teitelboim, *"The cosmological constant as a canonical variable"*, Phys. Lett. B **143**, 415 (1984).

[117] M. Henneaux and C. Teitelboim, *"The cosmological constant and general covariance"*, Phys. Lett. B **222**, 195 (1989).

[118] C. Teitelboim, *"The cosmological constant as a thermodynamic black hole parameter"*, Phys. Lett. B **158**, 293 (1985).

[119] B. P. Dolan, *"The cosmological constant and black-hole thermodynamic potentials"*, Class. Quantum Grav. **28**, 125020 (2011).

[120] B. P. Dolan, *"Pressure and volume in the first law of black hole thermodynamics"*, Class. Quantum Grav. **28**, 235017 (2011).

[121] S. Deser, R. Jackiw, and G. 't Hooft, *"Three dimensional Einstein gravity: Dynamics of flat space"*, Ann. Phys. **152**, 220 (1984).

[122] G. 't Hooft, *"Non-perturbative 2 particle scattering amplitudes in 2+1 dimensional quantum gravity"*, Commun. Math. Phys. **117**, 685 (1988).

[123] E. Witten, *"2+1 dimensional gravity as an exactly soluble system"*, Nucl. Phys. B**311**, 46 (1988).

[124] E. Witten, *"Topology - changing amplitudes in 2+1 dimensional gravity"*, Nucl. Phys. B**323** 113 (1989).

[125] M. Bañados, C. Teitelboim, and J. Zanelli, *"Black hole in three dimensional space time"*, Phys. Rev. Lett **69**, 1849 (1992).

[126] G. Clement, *"Warped AdS_3 black holes in new massive gravity"*, Class. Quantum Grav. **26**, 105015 (2009).

[127] L. F. Abbott and S. Deser, *"Stability of gravity with a cosmological constant"*, Nucl. Phys. B **195**, 76 (1982).

[128] S. Deser and B. Tekin, *"Gravitational Energy in Quadratic-Curvature Gravities"*, Phys. Rev. Lett. **89**, 101101 (2002).

[129] S. Deser and B. Tekin, *"Energy in generic higher curvature gravity theories"*, Phys. Rev. D **67**, 084009 (2003).

[130] G. T. Horowitz and R. C. Myers, *"AdS-CFT correspondence and a new positive energy conjecture for general relativity"*, Phys. Rev. D **59**, 026005 (1998).

[131] S. Surya, K. Schleich, and D. M. Witt, *"Phase Transitions for Flat Anti-de Sitter Black Holes"*, Phys. Rev. Lett. **86**, 5231 (2001).

[132] M. Kleban, M. Porrati, and R. Rabadan, *"Poincare recurrences and topological diversity"*, JHEP **0410**, 030 (2004).

[133] M. K. Parikh and F. Wilczek, *"Hawking radiation as tunneling"*, Phys. Rev. Lett. **85**, 24 (2000).

[134] J. Y. Zhang and Z. Zhao, *"Discussion on event horizon and quantum ergosphere of evaporating black holes in a tunneling framework"*, Phys. Rev. D, **83** 064028 (2011).

[135] L. Smarr, *"Mass Formula for Kerr Black Holes"*, Phys. Rev. Lett. **30**, 71 (1973).

[136] B. R. Majhi, *"Hawking radiation and black hole spectroscopy in Horava-Lifshitz gravity"*, Phys. Lett. B **686**, 49 (2010).

[137] M. R. Setare and D. Momeni, *"Spacing of the entropy spectrum for KS Black hole in Horava-Lifshitz gravity"*, Mod. Phys. Lett. A. **26**, 151 (2011).

[138] R. G. Cai, Y. P. Hu, Q. Y. Pan and Y. L. Zhang, *"Thermodynamics of black holes in massive gravity"*, Phys. Rev. D **91**, 024032 (2015).

[139] S. H. Hendi, B. Eslam Panah, S. Panahiyan, *"Massive charged BTZ black holes in asymptotically (a)dS spacetimes"*, JHEP **05**, 029 (2016).

[140] S. W. Hawking, *"Gravitational Radiation from Colliding Black Holes"*, Phys. Rev. Lett. **26**, 1344 (1971).
2333.

[141] S. W. Hawking and C. J. Hunter, *"Gravitational entropy and global structure"*, Phys. Rev. D **59**, 044025 (1999).

[142] C. J. Hunter, *"Action of instantons with a nut charge"*, Phys. Rev. D **59**, 024009 (1999).

[143] S. W. Hawking, C. J. Hunter and D. N. Page, *"NUT charge, anti-de Sitter space, and entropy"*, Phys. Rev. D**59**, 044033 (1999).

[144] B. R. Majhi, Ph. D. thesis, *"Quantum tunneling in black holes"*, (2010).

[145] C. Tsallis, *"Nonadditive entropy: The concept and its use"*, Eur. Phys. J. A **40**, 257-266 (2009).

[146] H. Touchette, *"When is a quantity additive, and when is it extensive?"*, Physica A 305, 84-88 (2002).

[147] W. Janke, D. A. Johnston and R. Kenna, *"Geometrothermodynamics of the Kehagias-Sfetsos Black Hole"*, J. Phys. A **43**, 425206 (2010).

[148] M. Akbar, H. Quevedo, K. Saifullah, A. Sanchez and S. Taj, *"Thermodynamic Geometry Of Charged Rotating BTZ Black Holes"*, Phys. Rev. D **83**, 084031 (2011).

[149] H. Quevedo and M. N. Quevedo, *"Cosmological Applications of Geometrothermodynamics"*, Gravitaion and Cosmology, **20**, 208 (2014).

[150] R. Clausius, *"Ueber die bewegende Kraft der Wärme und die Gesetze, welche sich daraus für die Wrmelehre selbst ableiten lassen"*, Annalen der Physik, **155**, 500-524 (1850).

[151] M. C. Clapeyron, *""Mmoire sur la puissance motrice de la chaleur"*, Journal de l'cole polytechnique **23**, 153-190

[152] P. Ehrenfest, *"Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik"*, Z. Physik **45**, 455-457 (1927).

[153] C. Kittel, Elementary Statistical Physics, John Wiley and Sons Inc., New York (1967).

[154] S. Carlip and J. Gegenberg, Phys. Rev. D **44** 424 (1991).

[155] S. Carlip, J. Gegenberg and R. B. Mann, Phys. Rev. D **44** 424 (1991).

[156] M. Banados, Phys. Rev. D **57** 1068 (1998).

[157] M. -I. Park, Phys. Lett. B **663** 259 (2008).

[158] M. -I. Park, Phys. Lett. B **647** 472 (2007).