Functional Analysis: Spectral Theory

SPECTRAL ANALYSIS OF BOUNDED SELF-ADJOINT OPERATORS

- A LINEAR ALGEBRAIC APPROACH

Thesis submitted to the Cochin University of Science and Technology for the award of the degree of

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under the Faculty of Science
by

KIRAN KUMAR V.B

Department of Mathematics
Cochin University of Science and Technology
Cochin - 682 022

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Certificate

This is to certify that the thesis entitled **'Spectral Analysis of Bounded Self-adjoint operators - A Linear Algebraic Approach'** submitted to the Cochin University of Science and Technology by Mr. Kiran Kumar V.B for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bona fide record of studies carried out by him under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

Dr. M. N. Narayanan Namboothiri
(Research Guide)
Assistant Professor
Department of Mathematics
Cochin University of Science and Technology
Cochin-682 022, Kerala.

Cochin-22
30-07-2012.

Declaration

I, Kiran Kumar V.B, hereby declare that this thesis entitled '**Spectral Analysis of Bounded Self-adjoint operators - A Linear Algebraic Approach** ' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

Kiran Kumar V.B
Research Scholar
Registration No. 3521
Department of Mathematics
Cochin University of Science and Technology
Cochin-682 022, Kerala.

Cochin-22
30-07-2012.

Dedicated to

My Parents and Teachers

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Contents

Notations and symbols

$\mathbb{H}$	-	Separable Hilbert space.
$B(\mathbb{H})$	-	Set of all bounded operators on $\mathbb{H}$.
$K(\mathbb{H})$	-	Set of all compact operators on $\mathbb{H}$.
$\sigma(A)$	-	Spectrum of the operator A.
$\sigma_e(A)$	-	Essential spectrum of the operator A.
$\|A\|_e$	-	Essential norm.
$A(x)$	-	Analytic family of operators.
$A(f)$	-	Toeplitz operator with symbol f.
$L(f)$	-	Toeplitz-Laurent operator with symbol f.
$A_n(f)$	-	$n \times n$ Toeplitz matrix with symbol f.

Chapter 1

Introduction

Self-adjoint operators on Hilbert spaces have an extremely detailed theory and are of great importance in modern analysis due to its immense applications (see [30, 59] and the references there reported). For instance, the fundamental equations of quantum mechanics involve certain self-adjoint and unitary operators. Interestingly, the Hamiltonian in quantum mechanics is an unbounded self-adjoint operator on a Hilbert space. The study of spectrum and the related properties of these operators are inevitable and has numerous applications, most notably the mathematical formulation of quantum mechanics. The interplay between the physical phenomena in quantum mechanics and the spectra of the linear operators associated with it, is not surprising. Most of the physical phenomena in quantum mechanics can be understood by knowing the spectra of the corresponding linear operators. For example, the point spectrum of the Hamiltonian corresponds to the energy levels of the bound states of the system. The rest of the spectrum plays an important role in scattering theory of the system. This interplay signifies the fundamental question that, 'How to approximate spectra of linear operators on separable Hilbert spaces?'. This question in operator theory goes back to Szegö [63] and has received some attention since then.

Several attempts have been made to make use of the finite dimensional theory in the computation of spectrum of bounded operators in an infinite dimensional space through an asymptotic way. This approach found success in getting good estimates in the case of some self-adjoint operators. Significant efforts have been done by many mathematicians to build up a general theory for the approximation of spectrum of bounded self-adjoint operators on an infinite dimensional Hilbert space. To quote some of the recent contributions in this direction are due to W.B Arveson [3], Albrecht Böttcher *et al.*[19, 15], E.B Davies [34, 35], I. Gohberg *et al.*[37], Hagen.R *et al.*[40], V.S Varadarajan and S.R.S Varadhan [69], A.Hansen [41, 42] etc. The list is nevertheless incomplete.

This thesis discusses the linear algebraic approach used to study the spectrum of a bounded self-adjoint operator A on a separable complex Hilbert space $\mathbb{H}$. The finite dimensional compressions A_n of A are considered here. The asymptotic values of spectrum of A_n are used to study the nature of spectrum of A. The spectral gap prediction problem is addressed first. Also the holomorphic family of operators $A(x)$ are considered to study the linear algebraic techniques under this holomorphic perturbation. The approximation techniques used in [19], are translated into the case of a one parameter family of operators, in a uniform way. This is an attempt to answer the question of stability in the spectral approximation and spectral gap predictions under a holomorphic perturbation.

The discrete version of Borg-type theorems and its several modifications are proved in this thesis. Borg theorem deals with the uniqueness question of potentials associated with the Schrodinger operator. Historically this uniqueness theorem comes in the discipline of inverse spectral theory and found its applications in many branches of science. Here we use the block Toeplitz-Laurent operators to prove the discrete versions of Borg-type theorems. The symbol corresponding to these block Toeplitz-Laurent operators are used to compute the spectrum. It should be noted that in many practical situations, the symbol function is not explicitly known; only its Fourier coefficients are available. Also it is a difficult task to recapture the symbol from its Fourier coefficients. Some of the results proved here, partly overlap with the known theorems in operator theory. The pure linear algebraic approach is the main novelty of the results proved here.

The pre-conditioners arising from the Frobenius optimal approximants were used in the special case of Toeplitz matrices by Stefano Serra Capizzano and Tyrtyshnikove (see [60, 65]). In this thesis, we extend this notion of pre-conditioners in the setting of operators acting on a separable Hilbert space. Here the new notions of convergence of Completely Positive maps are introduced, using the notion of eigenvalue clustering. The new versions of Korovkin-type theorems are proved with these new notions of convergence. The classical Korovkin-type theorems are used as valuable tools in the constructive approximation theory. The noncommutative versions were proved in the case of algebras of operators. The theorems proved in this thesis, are the infinite dimensional versions of the results in [60]. We hope that these developments are useful from a spectral theory point of view, since the asymptotic of pre-conditioners contain much spectral information of the operator under concern.

1.1 Basic definitions and preliminary results

We begin with some basic definitions and useful results in our context. Although these are classical notions, we present them just for the sake of completeness. Denote by $\mathbb{B}(\mathbb{H})$, the set of all bounded operators on $\mathbb{H}$.

Definition 1.1.1. A complex number λ is said to be in the **resolvent set** $\rho(A)$ of A, if the operator $\lambda I - A$ is bijective with a bounded inverse. $R_\lambda(A) = (\lambda I - A)^{-1}$ is called the **resolvent** of A at λ. If λ is not an element of $\rho(A)$, then λ is said to be in the **spectrum** of A, denoted by $\sigma(A)$.

Definition 1.1.2. The spectral radius $r(A)$ of A is defined as

$$r(A) = \sup\{|\lambda|\,,\ \lambda \in \sigma(A)\}.$$

For bounded self-adjoint operator A, $r(A) = \|A\|$, and we recall the following result.

Theorem 1.1.1. *If $A \in \mathbb{B}(\mathbb{H})$ and A is self-adjoint, then, $\sigma(A)$ is contained in the interval $[m, M]$, where m, M are given by*

$$m = \inf_{\|x\|=1} \langle A(x), x\rangle \quad \textit{and} \quad M = \sup_{\|x\|=1} \langle A(x), x\rangle \tag{1.1}$$

The above bounds for the spectrum of a bounded self-adjoint operator can not be shrunk further, as observed in the next theorem. Also these bounds give the norm of the operator.

Theorem 1.1.2. *Let m, M are as in (1.1) for the bounded self-adjoint operator A. Then both m and M are spectral values of A. Moreover, we have*

$$r(A) = \|A\| = \max(|m|\,,|M|) = \sup_{\|x\|=1} |\langle A(x), x\rangle|\,.$$

The spectrum of a bounded self-adjoint operator acting on an infinite dimensional Hilbert space, may or may not have eigenvalues. The subset of the spectrum consisting of discrete eigenvalues of finite multiplicity, is called the discrete spectrum and denoted by $\sigma_d(A)$. The remaining part of the spectrum is called the essential spectrum and denoted by $\sigma_e(A)$. This part of the spectrum is invariant under compact perturbations and it contains all spectral values, which are not discrete eigenvalues of finite multiplicity. The definition given below is in the case of an arbitrary bounded operator.

Definition 1.1.3. **(Essential spectrum)** We say that λ lies in the essential spectrum $\sigma_e(A)$ of a bounded operator A, if $\lambda I - A$ is not a Fredholm operator.

Since the set $\mathbb{K}(\mathbb{H})$ of all compact operators on $\mathbb{H}$ is a norm closed two-sided ideal in the Banach algebra $\mathbb{B}(\mathbb{H})$, the quotient algebra

$$\mathbb{L}(\mathbb{H}) = \mathbb{B}(\mathbb{H})/\mathbb{K}(\mathbb{H})$$

is a Banach algebra with respect to the quotient norm

$$\|\pi(A)\| = inf\left\{\|A + K\| : K \in \mathbb{K}(\mathbb{H})\right\},$$

where $\pi : \mathbb{B}(\mathbb{H}) \to \mathbb{L}(\mathbb{H})$ is the quotient map. $\mathbb{L}(\mathbb{H})$ is called the Calkin algebra.

Remark 1.1.1. In the case of a bounded self-adjoint operator A, the notation $\|A\|_{ess}$ will be used to denote the quotient norm and will be called essential norm, since

$$\|A\|_{ess} = \max(|\nu|, |\mu|),$$

where μ and ν are the upper and lower bounds of the essential spectrum respectively.

Theorem 1.1.3. *The bounded operator A on $\mathbb{H}$ is Fredholm if and only if A is invertible in the Calkin algebra. Hence $\sigma_e(A) = \sigma(\pi(A))$.*

From the above theorem, it is clear that $\sigma_e(A)$ is a closed subset of $\sigma(A)$ and in the case of self-adjoint operators, it is contained in a closed interval, say $[\nu, \mu]$ which is a sub interval of $[m, M]$.

Remark 1.1.2. There are other notions of essential spectrum in the case of non separable Hilbert spaces, associated with ideals other than $K(\mathbb{H})$. However we will consider only the separable Hilbert spaces.

We end this section by recalling the Spectral Mapping Theorem, which is useful for us.

Theorem 1.1.4. ***(Spectral Mapping Theorem)*** *Let A be a self-adjoint operator and let $\phi(.)$ be a bounded continuous function on $\sigma(A)$. Then $\sigma(\phi(A)) = \phi(\sigma(A))$, where $\phi(A)$ can be defined using appropriate functional calculus.*

1.2 Outline of the thesis

What follows is a brief description of the contents of the thesis. The thesis focus on the linear algebraic techniques used in the spectral theory of bounded self-adjoint operators on a separable Hilbert space. The thesis is divided into six chapters including this introduction.

This introductory chapter is followed by a chapter on spectral gap problems. The usage of truncation method in the computation of spectrum of a bounded self-adjoint operator A on a separable complex Hilbert space $\mathbb{H}$ is discussed in that chapter. The sequence of eigenvalues of the finite dimensional truncations $A_n = P_n A P_n$, where P_n is a sequence of finite dimensional orthogonal projections on $\mathbb{H}$, are considered to approximate the spectrum of A. It was already observed in [19] that the bounds of essential spectrum and the discrete eigenvalues lying outside these bounds, can be approximated by this method. The major problem considered here is to predict the existence of gaps that may occur between the bounds of the essential spectrum of A, using the eigenvalues of A_n. When considering self-adjoint operators coming from Chemistry or Mathematical Physics [59], one is interested in the spectral gaps because they represent the region of instability of the associated eigenvalue problem $Au = \lambda u$. Also the intervals between these bounds, containing only discrete eigenvalues, are treated as spectral gaps. Locating such eigenvalues in between a spectral gap is another interesting problem.

In the third chapter, the discrete version of classical Borg theorem for Schrodinger operator with periodic potential is proved. The discrete versions of Borg-type theorem (see [36, 59]) are proved here, using the rich theory of Toeplitz-Laurent matrices [18, 38].

Consider the one dimensional Schrodinger operator $\tilde{A}(u) = -\ddot{u} + V \cdot u$ with real valued periodic potential $V(\cdot)$, defined on a suitable subspace of $L^2(\mathbb{R})$: the spectrum is the union of closed intervals and in some cases, these intervals may be separated by open intervals (spectral gaps). The Borg theorem states that the spectrum has no gaps if and only if the periodic potential $V(\cdot)$ is constant almost everywhere. In this chapter, the families of finite difference approximations of the operator $\tilde{A}$ are considered, depending on two parameters n, that is the number of periodicity intervals possibly infinite, and p, the precision of the approximation in each interval. It is shown that the approach, with fixed p, leads to families of sequences $\{A_n(p)\}$, where every matrix $A_n(p)$ can be interpreted as a block Toeplitz matrix generated by a $p \times p$ matrix-valued symbol f: in other words, every $A_n(p)$ with finite n is a finite section of the bi-infinite Toeplitz-Laurent operator $A_\infty(p) = L(f)$. The specific feature of the symbol f, which is a linear trigonometric polynomial, allows to identify the distribution of the collective spectra of the matrix-sequence $\{A_n(p)\}$ and in particular provide a simple way for proving a discrete version of Borg theorem. Also, the Borg-type theorems in the case of a more general block Toeplitz-Laurent operator and in the case of a periodic Jacobi operator are proved in this chapter.

In the next chapter, the approximation theorems in [19] are studied under an analytic perturbation of the operator, using the perturbation techniques due to Kato [43]. The observations in [19] are studied when the operator is subjected to analytic perturbation, following the definitions by Kato in [43]. It is shown that the bounds of the essential spectrum and the discrete eigenvalues those lie outside the bounds of essential spectrum of a holomorphic family of operators $A(x)$ can be approximated uniformly in any compact neighborhood of x. Also, the family of block Toeplitz operators arising from a particular kind of matrix valued symbols is considered. The perturbation results for the eigenvalues of matrices (see [7]) are applied to the matrix valued symbol and achieved some estimates.

Finally, we extend the notion of pre-conditioners used in the case of Toeplitz matrices (see [60, 65]), into the setting of operators acting on separable Hilbert spaces, and study with the help of certain noncommutative versions of Korovkin-type theorems. This is interesting in spectral theory point of view, because the pre-conditioners play a crucial role in the approximation of spectrum. Stefano Serra Capizzano and Tyrtyshnikove used the classical Korovkin theorem [45] to deal with pre-conditioners of Toeplitz as well as block Toeplitz matrices (see [60, 65]). In this chapter, some of the noncommutative Korovkin-type theorems are used to translate the results in [60, 65], to a more general context of infinite dimensional bounded linear operators. The notion of strong, weak and uniform clustering of matrix sequences are introduced. These concepts were used to study the problem of pre-conditioners for the Toeplitz matrix sequences by Stefano Serra Capizzano and Tyrtyshnikove (see [60, 65]).

The theme of this chapter is to study pre-conditioners of infinite dimensional bounded linear operators on separable Hilbert spaces. This problem is analyzed using the concept of Completely Positive-maps (CP-maps). The noncommutative Korovkin-type theorems are proved with respect to strong, weak and uniform clustering, analogous to the Korovkin-type theorems by Stefano Serra Capizzano in [60]. These noncommutative Korovkin-type theorems are used to study the above mentioned infinite dimensional pre-conditioners.

The thesis ends with a concluding chapter, which lists down some of the problems that are to be addressed in future.

The main results of the thesis can be classified as three different approaches to the spectral approximation problems. The truncation method and its perturbed versions are part of the classical linear algebraic approach to the subject. The usage of block Toeplitz-Laurent operators and the matrix valued symbols is considered as a particular example where the linear algebraic techniques are effective in simplifying problems in inverse spectral theory. The abstract approach to the spectral approximation problems via pre-conditioners and Korovkin-type theorems is an attempt to make the computations involved, well conditioned. However, in all these approaches, linear algebra comes as the central object.

Chapter 2

Spectral Gap Problems

The usage of linear algebraic techniques in the computation of spectrum of a bounded self-adjoint operator A on a separable Hilbert space $\mathbb{H}$, are discussed in this chapter. The eigenvalues of truncations of a bounded self-adjoint operator are used to study the behavior of its spectrum.

It was already observed in [19] that the bounds of essential spectrum and the discrete eigenvalues lying outside these bounds, are possible to approximate by this method. The usage of algebraic techniques in this problem was done earlier in [3]. The major problem that is considered here is to predict the gaps that may occur between the bounds of the essential spectrum using the eigenvalues of truncations. An interval I is called spectral gap if there exist real sets J_1, J_2 containing the spectrum of A such that $\sup J_1 \leqslant \inf I < \sup I \leqslant \inf J_2$. We are interested in the gaps lie between the bounds of essential spectrum of A. Also the intervals between these bounds, containing only discrete eigenvalues, are treated as spectral gaps. Locating such eigenvalues in between a spectral gap, is another interesting problem, to be handled linear algebraically. Historically, gap related problems have been studied with special attention for Schrodinger operators (see e.g. [26, 34, 35, 59]).

The chapter is organized as follows. We begin with a preliminary section in which a survey of the algebraic and linear algebraic developments in this area due to [3] and [19] are presented. In the second section, the results which predict the existence of spectral gaps, using the eigenvalues of truncations, are proved. A new method to detect the spectral gaps is proposed in the third section, which is an analogue of the study by E.B Davies, Levitin and Shargorodsky (see [34],[35], [51],[52]). Also some computational issues are addressed there.

2.1 The Truncation method

Let A be a bounded self-adjoint operator on the separable Hilbert space $\mathbb{H}$ and let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathbb{H}$. Denote by P_n, the projection of $\mathbb{H}$ onto the finite dimensional subspace, $L_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Consider the finite dimensional truncations of A, that is $A_n = P_n A P_n$. Now if $(a_{i,j}) = (\langle Ae_j, e_i\rangle)$ is the matrix representation of A associated to the orthonormal basis $\{e_1, e_2, \ldots\}$, then the $n \times n$ matrix $(a_{i,j})_{1 \leqslant i,j \leqslant n}$ coincides with the matrix representation of A_n restricted to the image of P_n.

The following basic question is addressed here. What is the relation between the eigenvalue sequence of the matrices $(a_{i,j})_{1 \leqslant i,j \leqslant n}$, and spectrum of A. Whether the spectrum can be approximated using the eigenvalue sequence of truncations. There are some disappointing examples in which the eigenvalues of truncations give little information about the spectrum. For instance, in the case of the right shift operator on the sequence space $l^2(\mathbb{Z})$, the eigenvalue sequence of the truncations is the constant sequence 1, while the spectrum is the whole closed unit disc. For a self-adjoint example, one can consider the operator A on $l^2(\mathbb{N})$, defined as follows.

$$A(x_n) = (x_{\pi(n)}), \tag{2.1}$$

where π is a suitably chosen permutation on $\mathbb{N}$. The essential properties required for the permutation π, are discussed in [3], due to which the truncation method fails to approximate the spectrum.

Some developments in this area are reported below. The major contributions are due to W.B Arveson, who generalized the notion of band limited matrices in [3], and achieved some success in the case of some special class of operators. We brief up the definitions and some results below which will play a very important role in the approximation of spectrum of self-adjoint operators. The notation A_n is used to denote the matrix $(a_{i,j})_{1\leqslant i,j\leqslant n}$.

Definition 2.1.1. A **filtration** of a Hilbert space $\mathbb{H}$ is a sequence of finite dimensional subspaces of $\mathbb{H}$, $\{L_n; n \in \mathbb{N}\}$ such that

$$L_n \subset L_{n+1} \text{ and closure of } \bigcup_n L_n \text{ is } \mathbb{H}.$$

Example 2.1.1. A typical example for filtration in a Hilbert space with an orthonormal basis is the following. Let $\{e_n : n \in \mathbb{Z}\}$ be the bilateral orthonormal basis for $\mathbb{H}$ and let $\{L_n\}$ be defined by

$$L_n = \text{span}\{e_{-n}, e_{-n+1}, \dots e_n\}.$$

Then $\{L_n; n \in \mathbb{Z}\}$ is a filtration.

Definition 2.1.2. Let $\{L_n : n \in \mathbb{N}\}$ be a filtration. And P_n be the projection onto L_n. The **degree** of a bounded operator A on $\mathbb{H}$ is defined by

$$\deg(A) = \sup_{n \geqslant 1} \text{ rank}(P_n A - A P_n).$$

Corresponding to each filtration, a Banach $*-$algebra of operators can be defined, which is named as Arveson's class, defined as follows.

Definition 2.1.3. A is an operator in the **Arveson's class** if

$$A = \sum_{n=1}^{\infty} A_n, \text{ where } \deg(A_n) < \infty \text{ for every n and convergence is in the}$$

operator norm, in such a way that

$$\sum_{n=1}^{\infty} (1 + \deg(A_n)^{\frac{1}{2}}) \|A_n\| < \infty$$

In case the filtration is the span of finite number of elements in the basis as defined in example (2.1.1), the following gives a concrete description of operators in the Arveson's class.

Theorem 2.1.1. *[3] Let $\{L_n; n \in \mathbb{Z}\}$ be the filtration defined in example (2.1.1). Also let $(a_{i,j})$ be the matrix representation of a bounded* *operator A, with respect to* $\{e_n\}$*, and for every* $k \in \mathbb{Z}$ *let*

$$d_k = \sup_{i \in \mathbb{Z}} |a_{i+k,i}|$$

be the sup norm of the k^{th} *diagonal of* $(a_{i,j})$*. Then A will be in the Arveson's class whenever the series* $\sum_k |k|^{1/2} d_k$ *converges.*

In particular, any operator whose matrix representation $(a_{i,j})$ is band-limited, in the sense that $a_{i,j} = 0$ whenever $|i-j|$ is sufficiently large, must be in the Arveson's class. Before stating the spectral inclusion theorems for arbitrary self-adjoint operators and for operators in the Arveson's class, recall the notion of essential points and transient points.

Definition 2.1.4. Essential point: A real number λ is an essential point of A, if for every open set U containing λ,

$$\lim_{n\to\infty} N_n(U) = \infty, \text{where } N_n(U) \text{ is the number of eigenvalues of } A_n, \text{ in } U.$$

Definition 2.1.5. Transient point: A real number λ is a transient point of A if there is an open set U containing λ, such that $\sup N_n(U)$ with n varying on the set of all natural numbers, is finite.

Remark 2.1.1. It should be noted that a number can be neither transient nor essential.

Denote $\Lambda = \{\lambda \in R; \lambda = \lim \lambda_n, \lambda_n \in \sigma(A_n)\}$ and Λ_e as the set of all essential points. The following spectral inclusion results for a bounded self-adjoint operator A is of high importance throughout this thesis.

Theorem 2.1.2. *[3] $\sigma(A) \subseteq \Lambda \subseteq [m, M]$ and $\sigma_e(A) \subseteq \Lambda_e$.*

Equality in one of the above inclusion for self-adjoint operators in the Arveson's class, was also proved in [3]. The precise result is the following.

Theorem 2.1.3. *[3] If A is a bounded self-adjoint operator in the Arveson's class, then $\sigma_e(A) = \Lambda_e$ and every point in Λ is either transient or essential.*

Remark 2.1.2. The above two theorems enable us to confine our attention to the limiting set Λ and the essential points Λ_e, in the task of computation of spectrum and essential spectrum of a bounded self-adjoint operator respectively. Now the following issues may arise. The limiting set Λ may contain points which do not belong to the spectrum. Such points are called spurious eigenvalues. In the case of an operator in the Arveson's class, the essential points will give all information about essential spectrum, while the transient points may be misleading. Here we loose only information about eigenvalues of finite multiplicity. But this is very important if such points exist between the lower and upper bounds of essential spectrum, since they lead to the existence of spectral gaps between these bounds.

Things can be more difficult in the case of an arbitrary bounded self-adjoint operator. There may exist essential points, which are not spectral values. The operator given by the equation (2.1) is of that kind. Anyway the inclusion in Theorem (2.1.2) helps us to determine the spectrum, with an additional assumption of connectedness of the essential spectrum. The details of this claim are given below, which is a brief review of the article [19] with some slight modifications. This will play a key role in the forthcoming sections.

2.1.1 Linear algebraic approach:

Recall that, for a bounded self-adjoint operator A, $\sigma(A)$ is contained in the interval $[m, M]$ and $\sigma_e(A)$ in $[\nu, \mu]$ where m, M, ν, μ, are bounds of $\sigma(A)$ and $\sigma_e(A)$ respectively. The following definitions and preliminary results are needed further.

Definition 2.1.6. Consider the singular number s_k, k natural number, $s_k(A) = \inf \{\|A - F\|; F \in \mathbb{B}(\mathbb{H}), \operatorname{rank} F \leqslant k-1\}$ is the k^{th} **approximation number** of A.

Clearly we have $\|A\| = s_1(A) \geqslant s_2(A) \geqslant \ldots \geqslant 0$

Theorem 2.1.4. *[37] $\lim_{k\to\infty} s_k(A) = \|A\|_{ess}$ where $\|A\|_{ess}$ is the essential norm.*

Theorem 2.1.5. *[19]* $\lim_{n\to\infty} s_k(A_n) = s_k(A)$.

Remark 2.1.3. For $|A| = (A^*A)^{1/2}$, in case A is a finite matrix, the approximation numbers are the eigenvalues of $|A|$. That is $s_k(A) = \lambda_k(|A|)$, where $\lambda_k(|A|)$ is the k^{th} eigenvalue of $|A|$.

Theorem 2.1.6. *[37] The set $\sigma(|A|) - [0, \|A\|_{ess}]$ is at most countable, $\|A\|_{ess}$ is the only possible accumulation point, and all the points* *of the set are eigenvalues with finite multiplicity of* $|A|$. *Furthermore if*

$$\lambda_1(|A|) \geqslant \lambda_2(|A|) \geqslant \ldots \geqslant \lambda_N(|A|)$$

are those N eigenvalues (N can be infinity), then

$$s_k(A) = \begin{cases} \lambda_k(|A|), \, if \, N = \infty \text{ or } 1 \leqslant k \leqslant N \\ \|A\|_{ess}, \, if \, N < \infty \, and \, k \geqslant N+1 \end{cases} \tag{2.2}$$

Corollary 1.

$$\lim_{n\to\infty} \lambda_k(|A_n|) = \lim_{n\to\infty} s_k(A_n) = s_k(A) = \begin{cases} \lambda_k(|A|) \; if \, N = \infty \, or \, 1 \leqslant k \leqslant N \\ \|A\|_{ess} \; if \, N < \infty \, and \, k \geqslant N+1 \end{cases}$$

Remark 2.1.4. The above result will play a key role in the approximation of spectrum. Considering the positive operator $A - mI$, it can be deduced that the set $\sigma(A) \cap (\mu, M]$ is at most countable and that consists of eigenvalues of finite multiplicity by Theorem (2.1.6). Also μ is the only possible accumulation point. Let these eigenvalues be

$$\lambda_R^+(A) \leqslant \ldots \leqslant \lambda_2^+(A) \leqslant \lambda_1^+(A).$$

Similarly by considering the operator $MI - A$, it can be observed that $\sigma(A) \cap [m, \nu)$ consists of at most countably many eigenvalues of finite multiplicity with only possible accumulation point ν. Let

$$\lambda_1^-(A) \leqslant \lambda_2^-(A) \leqslant \ldots \leqslant \lambda_S^-(A)$$

be those eigenvalues. Also the numbers R and S can be infinity. Arrange the eigenvalues of A_n as

$$\lambda_1(A_n) \geqslant \lambda_2(A_n) \geqslant \ldots \geqslant \lambda_n(A_n).$$

From here onwards, the above notations will be used.

Now we prove the following result from [19] which is the major tool that is used frequently in this thesis.

Theorem 2.1.7. *For every fixed integer k we have*

$$\lim_{n\to\infty} \lambda_k(A_n) = \begin{cases} \lambda_k^+(A), & \text{if } R = \infty \text{ or } 1 \leqslant k \leqslant R \\ \mu, & \text{if } R < \infty \text{ and } k \geqslant R+1 \end{cases}$$

$$\lim_{n\to\infty} \lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-(A), & \text{if } S = \infty \text{ or } 1 \leqslant k \leqslant S \\ \nu, & \text{if } S < \infty \text{ and } k \geqslant S+1 \end{cases}$$

In particular,

$$\lim_{k\to\infty}\lim_{n\to\infty} \lambda_k(A_n) = \mu \text{ and } \lim_{k\to\infty}\lim_{n\to\infty} \lambda_{n+1-k}(A_n) = \nu.$$

Proof. The following observations are made first.

$|A - mI| = A - mI,\ P_n(A - mI)P_n = A_n - mI_n,$ and $|A_n - mI_n| = A_n - mI_n.$

Hence from the above corollary, we have

$$\lim_{n\to\infty} \lambda_k(A_n - mI_n) = \begin{cases} \lambda_k(A - mI), & \text{if } R = \infty \text{ or } 1 \leqslant k \leqslant R \\ \|A - mI\|_{ess}, & \text{if } R < \infty \text{ and } k \geqslant R+1 \end{cases} \tag{2.3}$$

Similarly, by considering the operator $MI - A$, we get

$$\lim_{n\to\infty} \lambda_k(MI_n - A_n) = \begin{cases} \lambda_k\left(MI - A\right), \text{ if } S = \infty \text{ or } 1 \leqslant k \leqslant S \\ \|MI - A\|_{ess}, \quad \text{if } S < \infty \text{ and } k \geqslant S+1 \end{cases} \tag{2.4}$$

Also we have the following identities

$$\|A - mI\|_{ess} = \mu - m, \quad \|MI - A\|_{ess} = M - \nu. \tag{2.5}$$

$$\lambda_k(A_n - mI_n) = \lambda_k(A_n) - m, \quad \lambda_k(MI_n - A_n) = M - \lambda_{n+1-k}(A_n). \tag{2.6}$$

$$\lambda_k(A - mI) = \lambda_k^+(A) - m, \quad \lambda_k(MI - A) = M - \lambda_k^-(A). \tag{2.7}$$

Substituting them in equations (2.3) and (2.4), we get

$$\lim_{n\to\infty} \lambda_k(A_n) = \begin{cases} \lambda_k^+\left(A\right), \text{ if } R = \infty \text{ or } 1 \leqslant k \leqslant R \\ \mu, \quad \text{if } R < \infty \text{ and } k \geqslant R+1 \end{cases}$$

$$\lim_{n\to\infty} \lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-\left(A\right), \text{ if } S = \infty \text{ or } 1 \leqslant k \leqslant S \\ \nu, \quad \text{if } S < \infty \text{ and } k \geqslant S+1 \end{cases}$$

Hence the proof. □

Remark 2.1.5. The above results are also true if we replace A_n by some other sequence A_{1n} with the property that

$$\|A_n - A_{1n}\| \to 0 \text{ as } n \to \infty$$

In order to justify this, we need only to recall an important inequality concerning the eigenvalues of self-adjoint matrices A, B (see page no.63

of [7])

$$|\lambda_k(A) - \lambda_k(B)| \leqslant \|A - B\|. \tag{2.8}$$

Remark 2.1.6. By Theorem (2.1.7), all the discrete spectral values lying outside the bounds of essential spectrum and the upper and lower bounds of the essential spectrum can be approximated. Note that, the theorem points out exactly the particular sequence that converges to a discrete spectral value. But how fast does the convergence take place, is still not known. Looking at some concrete situations, one may hope for a better rate of convergence.

Even the rate of convergence is not estimated, it can be proved that the order of convergence is the same as the order of convergence of approximation numbers. The following theorem gives a vague idea about the rate of convergence.

Theorem 2.1.8. *If $s_k(A_n) - s_k(A) = O(\theta_n)$, where θ_n goes to 0 as n tends to ∞, then*

$$\lambda_k(A_n) = \begin{cases} \lambda_k^+(A) + O(\theta_n), & \text{if } R = \infty \text{ or } 1 \leqslant k \leqslant R \\ \mu + O(\theta_n), & \text{if } R < \infty \text{ and } k \geqslant R+1 \end{cases}$$

$$\lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-(A) + O(\theta_n), & \text{if } S = \infty \text{ or } 1 \leqslant k \leqslant S \\ \nu + O(\theta_n), & \text{if } S < \infty \text{ and } k \geqslant S+1 \end{cases}$$

where R and S are the same notations used in Theorem(2.1.7).

Proof. Let N be the number of eigenvalues lying in $\sigma(|A|) - [0, \|A\|_{ess}]$. From equation (2.2), and the remarks that was made just before Theorem

(2.1.6), we have the following identity.

$$s_k(A_n) - s_k(A) = \begin{cases} \lambda_k(|A_n|) - \lambda_k(|A|), \text{ if } N = \infty \text{ or } 1 \leqslant k \leqslant N \\ \lambda_k(|A_n|) - \|A\|_{ess}, \text{ if } N < \infty \text{ and } k \geqslant N+1 \end{cases}$$

Since by hypothesis, $s_k(A_n) - s_k(A) = O(\theta_n)$, we get

$$\lambda_k(|A_n|) - \lambda_k(|A|) = O(\theta_n), \quad \text{if } N = \infty \text{ or } 1 \leqslant k \leqslant N,$$

$$\lambda_k(|A_n|) - \|A\|_{ess} = O(\theta_n), \quad \text{if } N < \infty \text{ and } k \geqslant N+1.$$

Applying this to the positive operators $A - mI$, and $MI - A$, with the notations used in Theorem (2.1.7), we get the following conclusions.

$$\lambda_k(A_n - mI_n) = \begin{cases} \lambda_k(A - mI) + O(\theta_n), \text{ if } R = \infty \text{ or } 1 \leqslant k \leqslant R \\ \|A - mI\|_{ess} + O(\theta_n), \text{ if } R < \infty \text{ and } k \geqslant R+1 \end{cases}$$

and

$$\lambda_k(MI_n - A_n) = \begin{cases} \lambda_k(MI - A) + O(\theta_n), \text{ if } S = \infty \text{ or } 1 \leqslant k \leqslant S \\ \|MI - A\|_{ess} + O(\theta_n), \text{ if } S < \infty \text{ and } k \geqslant S+1 \end{cases}$$

Using the identities (2.5), (2.6) and (2.7), we get the desired conclusions

$$\lambda_k(A_n) = \begin{cases} \lambda_k^+(A) + O(\theta_n), \text{ if } R = \infty \text{ or } 1 \leqslant k \leqslant R \\ \mu + O(\theta_n), \qquad \text{ if } R < \infty \text{ and } k \geqslant R+1 \end{cases}$$

$$\lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-(A) + O(\theta_n), \text{ if } S = \infty \text{ or } 1 \leqslant k \leqslant S \\ \nu + O(\theta_n), \qquad \text{ if } S < \infty \text{ and } k \geqslant S+1 \end{cases}$$

Hence the proof. $\square$

The above theorem is the first result regarding the rate of convergence in the approximations done in Theorem (2.1.7). So far there is no evidence of remainder estimation and the error estimation in these approximations in the case of an arbitrary self-adjoint operator to the best of our knowledge. The subsequent theorem taken from [19] denies the existence of spurious eigenvalues under the assumption of connectedness of essential spectrum.

Theorem 2.1.9. *[19] If A is a self-adjoint operator and if $\sigma_e(A)$ is connected, then $\sigma(A) = \Lambda$.*

Remark 2.1.7. It is worthwhile to notice that the connectedness of essential spectrum enables us to compute the spectrum using finite dimensional truncations. Thus, if we can not determine the spectrum fully by the truncations, then the essential spectrum is not connected. In short, if there is a spurious eigenvalue, then there exists a gap in the essential spectrum.

Remark 2.1.8. The converse of the above observation need not be true. That is the existence of a spectral gap does not lead to the existence of a spurious eigenvalue. For example, if we take A to be be the projection operator on to some closed subspace of $\mathbb{H}$, then the eigenvalues of truncations are 0 and 1 only. There we have $\Lambda = \sigma(A) = \{0, 1\}$. Hence no spurious eigenvalues, but still there is a gap.

In summary, the upper and lower bounds of the essential spectrum can be computed by using the sequence of eigenvalues of finite dimensional truncations. Also the discrete eigenvalues lying below and above these bounds can be computed. The above results pinpointing the particular sequence of eigenvalues that converges to a particular eigenvalue of the operator. Now the remaining part is the computation of essential spectrum. The problem is whether it is possible to locate the gaps in the essential spectrum using these truncations. If it is possible, then the spectrum is fully determined up to some discrete eigenvalues that may have trapped between these gaps.

2.2 Gaps in the essential spectrum

The following theorem is an attempt to predict the existence of spectral gaps, using the finite dimensional truncations. The notation $\#S$ is used to denote the number of elements in the set S.

Theorem 2.2.1. *Let A be a bounded self-adjoint operator and $\lambda_{n1}(A_n) \geqslant \lambda_{n2}(A_n) \geqslant ... \geqslant \lambda_{nn}(A_n)$ be the eigenvalues of A_n arranged in decreasing order. For each positive integer n, let $\{w_{nk} : k = 1, 2, ...n\}$ be a set of numbers such that $0 \leqslant w_{nk} \leqslant 1$ and $\sum\limits_{k=1}^{n} w_{nk} = 1$. If there exists a $\delta > 0$ and $K > 0$ such that*

$$\#\left\{\lambda_{nj}; \left|\sum_{k=1}^{n} w_{nk}\lambda_{{}_{nk}} - \lambda_{nj}\right| < \delta\right\} < K \tag{2.9}$$

and in addition if $\sigma_e(A)$ and $\sigma(A)$ has the same upper and lower bounds, then $\sigma_e(A)$ has a gap.

Proof. Consider the set $S = \left\{\sum\limits_{k=1}^{n} w_{nk}\lambda_{nk}, n = 1, 2, 3 \ldots\right\}$. Observe that $\lambda_{nn} \leqslant \sum_{k=1}^{n} w_{nk}\lambda_{nk} \leqslant \lambda_{n1}$. Also since each λ_{nj}'s lying in the interval $[m, M]$, the set S is contained in the interval $[m, M] = [\nu, \mu]$.

Case 1. Assume that S is a finite set, say $S = \{a_1, a_2, a_3 \ldots a_m\}$. In this case, the value of the sum $\sum_{k=1}^{n} w_{nk}\lambda_{nk}$ equals some of the numbers a_i's for infinitely many n. Let $a_1, a_2, a_3 \ldots a_p$ be those numbers. That is

$$\sum_{k=1}^{n} w_{nk}\lambda_{nk} = a_i \text{ for infinitely many n and } i = 1, 2, \ldots p.$$

From this and by the condition (2.9), for each $i = 1, 2, \ldots p$, we have

$$N_n(a_i - \delta, a_i + \delta) = \#\{\lambda_{nj}; |a_i - \lambda_{nj}| < \delta\} < K \text{ for infinitely many n.}$$

Hence $N_n(a_i - \delta, a_i + \delta)$ will not go to infinity as n goes to infinity. Therefore no number in the interval $(a_i - \delta, a_i + \delta)$ is an essential point. Since the essential spectrum is contained the set of all essential points, by Theorem (2.1.2), there is no essential spectral values in this interval. Also since each a_i lies between the bounds of essential spectrum, we can choose an appropriate $\epsilon > 0$ such that $(a_i - \epsilon, a_i + \epsilon)$ lies between the bounds and contained in the interval $(a_i - \delta, a_i + \delta)$. Then the interval $(a_i - \epsilon, a_i + \epsilon)$ is a spectral gap.

Case 2. Now we consider the case when S is an infinite set. Here S has at least one limit point in $\mathbb{R}$. If w_0 is a limit point of the set S, then we have $\nu \leqslant w_0 \leqslant \mu$.

Now the interval $\left(w_0 - {}^{\delta}\!/_{2}, w_0 + {}^{\delta}\!/_{2}\right)$ will contain infinitely many points from the set S. Corresponding to these points, there are infinitely many A_n's for which the number of eigenvalues in $\left(w_0 - {}^{\delta}\!/_2, w_0 + {}^{\delta}\!/_2\right)$ is bounded by K due to (2.9). Hence the sequence $N_n\left(w_0 - \frac{\delta}{2}, w_0 + \frac{\delta}{2}\right)$ will not go to infinity, since a subsequence is bounded by K. Hence no point in the interval $\left(w_0 - {}^{\delta}\!/_2, w_0 + {}^{\delta}\!/_2\right)$ is an essential point. Since the essential spectrum is contained the set of all essential points, by Theorem (2.1.2), $\left(w_0 - {}^{\delta}\!/_2, w_0 + {}^{\delta}\!/_2\right)$ contains no essential spectral values. Hence, as in the case 1, we can choose an $\epsilon > 0$, such that the interval $(w_0 - \epsilon, w_0 + \epsilon)$ is a spectral gap between the bounds of the essential spectrum and the proof is completed. □

Remark 2.2.1. The proof of the above theorem gives some information regarding the gap size. Since the interval $\left(w_0 - {}^{\delta}\!/_2, w_0 + {}^{\delta}\!/_2\right)$ contains no essential spectral values, it is a spectral gap if it lies between the bounds of the essential spectrum. In that case the gap size may be greater than δ. In the case 1, it could be greater than 2δ.

Remark 2.2.2. There is the possibility for the presence of discrete eigenvalues inside the spectral gaps detected using the above theorem.

Remark 2.2.3. The special case which is more interesting is when we choose $w_{nk} = \frac{1}{n}$, for all n. In that case, we are actually looking at the averages of eigenvalues of truncations and these averages can be computed using the trace at each level.

In the Theorem (2.2.1), the weighted mean of the eigenvalues at each level and its deviation is analyzed. Now some special choices of the weighting method are discussed below to predict the existence of spectral gaps, using the Theorem (2.2.1).

Special Choice I

Here is an instance where these weights w_{nk} arises naturally associated with a self-adjoint operator on a Hilbert space. Let $A_n = \sum_{k=1}^{n} \lambda_{n,k} Q_{n,k}$ be the spectral resolution of A_n. Define $w_{nk} = \langle Q_{n,k} e_1, e_1 \rangle$. Then $0 \leqslant w_{nk} \leqslant 1$ and $\sum_{k=1}^{n} w_{nk} = 1$. Now

$$\sum_{k=1}^{n} w_{nk} \lambda_{nk} = \sum_{k=1}^{n} \lambda_{nk} \langle Q_{n,k} e_1, e_1 \rangle = \langle A_n e_1, e_1 \rangle = \langle A e_1, e_1 \rangle = a_{11}.$$

Therefore by Theorem (2.2.1), if there exists a $\delta > 0$ and a $K > 0$, such that

$$\# \{\lambda_{nj}; |a_{11} - \lambda_{nj}| < \delta\} < K$$

then there exists a gap in the essential spectrum of A. That means the spectral gap prediction is done by looking at the first entry in the matrix representation of A. That is, if the first entry in the matrix representation of A, is not an essential point, then there exists a gap in the essential spectrum.

Remark 2.2.4. All points of the form $\langle Ae_i, e_i \rangle = a_{ii}$ are in the numerical range which lies between the bounds of the essential spectrum, in the case that the bounds coincide with the bounds of the spectrum. Hence in that case, if a_{ii} is not an essential point for some i, then that will lead to the existence of a spectral gap. That means if any one of the diagonal entries in the matrix representation of A is not an essential point, then there exists a gap in the essential spectrum as indicated in the above special choice of w_{nk}.

The following is an example where the first entry a_{11} is a transient point and the spectral gap prediction is valid.

Example 2.2.1. Define a bounded self-adjoint operator A on $l^2(\mathbb{N})$, as follows.

$$A(x_n) = (x_{n-1} + x_{n+1}) + (v_n x_n), x_0 = 0;$$

where the periodic sequence $(v_n) = (1, 2, 3, 1, 2, 3, \ldots)$. The matrix representation of A, associated to the standard orthonormal basis, is tridiagonal. The diagonal entries are the entries in the periodic sequence (v_n) and upper and lower diagonal will be 1. In the next chapter, we will see that such matrices can be identified as the block Toeplitz operator with corresponding matrix valued symbol given by

$$\tilde{f}(\theta) = \begin{bmatrix} 1 & 1 & e^{i\theta} \\ 1 & 2 & 1 \\ e^{-i\theta} & 1 & 3 \end{bmatrix}.$$

By our special choice above, Theorem (2.2.1) guarantees that if $\langle A(e_1), e_1 \rangle = 1$ is a transient point, then $\sigma_e(A)$ has a gap. The proof for the fact that 1 is a transient point, is given in the Example (3.4.1).

The operator considered in the above example comes as a discrete version of Schrodinger operator, which arises naturally in many practical problems. In general, the discrete Schrodinger operator is defined on $l^2(Z)$ as follows.

$$A(y) = (y_{n-1} + y_{n+1}) + (v_n y_n) \text{ for every } y = (\ldots y_1, y_2, y_3 \ldots) \in l^2(Z)$$

where $v = (\ldots v_1, v_2, \ldots)$ is a fixed bounded sequence. The corresponding truncations are

$$(A)_{2n+1} = \begin{bmatrix}
v_{-n} & 1 & 0 & 0 & & . & & & & \\
1 & v_{-n+1} & 1 & 0 & 0 & & . & & & \\
0 & 1 & . & 1 & 0 & 0 & & . & & \\
0 & 0 & 1 & . & 1 & 0 & 0 & & . & \\
 & 0 & 0 & 1 & . & 1 & 0 & 0 & & . \\
. & & 0 & 0 & 1 & . & 1 & 0 & 0 & \\
 & . & & 0 & 0 & 1 & . & 1 & 0 & 0 \\
 & & . & & 0 & 0 & 1 & . & 1 & 0 \\
 & & & . & & 0 & 0 & 1 & . & 1 \\
 & & & & . & & 0 & 0 & 1 & v_n
\end{bmatrix}$$

We consider the cases where the bounds of $\sigma(A)$ and $\sigma_e(A)$ coincide. Let a_n be the averages of the 2n+1 terms $(v_{-n} \ldots v_1, v_2, \ldots v_n)$ of the sequence v. If we choose $w_{nk} = \frac{1}{n}$, by Theorem (2.2.1), if there exists a $\delta > 0$ such that

$$\#\left\{\lambda_{nj}; |a_n - \lambda_{nj}| < \delta\right\} < K \text{ for some fixed K,}$$

then there exists a gap in the essential spectrum.

Remark 2.2.5. The Borg-type theorems will be proved in the next chapter, which will ensure that if the potential function is periodic and non constant, then the operator will have gaps in the essential spectrum [39]. For eg. if $x = (\ldots a, b, a, b, a, b, \ldots)$, with $a < b$, then the interval (a, b) is a gap.

Special Choice II

By invoking Theorem (2.1.7), there exist sequences of eigenvalues of truncations, which converges to the bounds of the essential spectrum. That is

$$\text{there exist } \lambda_{n_l}, \lambda_{n_m} \text{ such that } \lim_{n_l\to\infty} \lambda_{n_l} = \nu, \text{ and } \lim_{n_m\to\infty} \lambda_{n_m} = \mu.$$

Now fix a number $t \in (0,1)$, and define w_{nk} as follows.

$$w_{nk} = \begin{cases} t, \, if \, k = l, \\ 1-t, \, if \, k = m, \\ 0, \text{ otherwise,} \end{cases}$$

Then we have the following conclusions. If there exists a $\delta > 0$ and $K > 0$ such that

$$\# \{\lambda_{nj}; |t\lambda_{nl} + (1-t)\lambda_{nm} - \lambda_{nj}| < \delta\} < K,$$

then there is a gap in the essential spectrum $\sigma_e(A)$.

Proof of the above assertion is only a repetition of the arguments used in the proof of Theorem (2.2.1). Theorem (2.2.1) can not be applied directly because the crucial assumption that $\sigma(A)$ and $\sigma_e(A)$ have the same bounds, is missing here. Notice that this assumption was used only to ensure that the sum $\sum_{k=1}^{n} w_{nk}\lambda_{nk}$ lying between the bounds of essential spectrum. But the above choice of w_{nk} guarantees that the sum $\sum_{k=1}^{n} w_{nk}\lambda_{nk}$ converges to some number between the bounds of essential spectrum. And that number will create a gap in the essential spectrum, as observed in the proof of Theorem (2.2.1).

Remark 2.2.6. The above observations show that we may be able to predict the existence of spectral gaps, relaxing the assumptions of Theorem (2.2.1). But the freedom for choosing the weights w_{nk} to be arbitrary, is lost here.

It is not clear whether the converse of Theorem (2.2.1) is true for an arbitrary self-adjoint operator. The converse is proved below in the case of operators in the Arveson's class.

Theorem 2.2.2. *Let A be a bounded self-adjoint operator in the Arveson's class. And suppose that there exists a gap in the essential spectrum. Then there exists a set of numbers $\{w_{nk} : k = 1, 2, ...n\}$ such that $0 \leqslant w_{nk} \leqslant 1$ and $\sum\limits_{k=1}^{n} w_{nk} = 1$ and a $\delta > 0$ such that*

$$\# \left\{ \lambda_{nj}; \left| \sum_{k=1}^{n} w_{nk} \lambda_{n_k} - \lambda_{nj} \right| < \delta \right\} < K,$$

for some $K > 0$.

Proof. Let (a, b) be a gap in the essential spectrum. Then there exists sequences of eigenvalues of truncations $\lambda_{n_l}, \lambda_{n_m}$ such that

$$\lim_{n_l \to \infty} \lambda_{n_l} = a \text{ and } \lim_{n_m \to \infty} \lambda_{n_m} = b.$$

Fix a $t \in (0, 1)$ and define the sequence w_{nk} as

$$w_{nk} = \begin{cases} t, \, if \, k = l, \\ 1 - t, \, if \, k = m, \\ 0, \text{ otherwise}, \end{cases}$$

Since the number $c_t = ta + (1-t)b$ is in the interval (a, b), it is not an essential point. Also since A is in the Arveson's class, all such points are transient points by Theorem (2.1.3). Hence there exists a $\delta_1 > 0$ such that $\sup N_n(c_t - \delta_1, c_t + \delta_1) < K_1$ for some $K_1 > 0$. Also

$$\sum_{k=1}^{n} w_{nk}\lambda_{nk} = t\lambda_{n_l} + (1-t)\,\lambda_{n_m} \to ta + (1-t)b = c_t \, as \, n \to \infty.$$

Therefore there exists an N such that

$$\left| c_t - \sum_{k=1}^{n} w_{nk}\lambda_{nk} \right| < {}^{\delta_1}\!/_2 \text{ for all } n > N.$$

Now if for some $n > N$, $\left| \sum_{k=1}^{n} w_{nk}\lambda_{{}_{nk}} - \lambda_{nj} \right| < {}^{\delta_1}\!/_2$, then $|c_t - \lambda_{nj}| < \delta_1$. Therefore,

$$\#\left\{ \lambda_{nj};\left| \sum_{k=1}^{n} w_{nk}\lambda_{{}_{nk}} - \lambda_{nj} \right| < \frac{\delta_1}{2} \right\} < N_n(c_t - \delta_1, c_t + \delta_1) < K_1, \, \forall\, n > N.$$

Now choosing $K = \sup\{K_1, N\}$ and $\delta = \frac{\delta_1}{2}$, the proof is completed. $\square$

Remark 2.2.7. In the above proof, the sequence $\{w_{nk}\}$ and the bound K will depend on the particular choice of $t \in (0, 1)$.

2.3 Gap prediction methods

The concepts of second order relative spectra and quadratic projection method, which are almost synonyms of the other, were used in the spectral pollution problems and in determining the eigenvalues in the gaps by E.B. Davies, Levitin, Shagorodsky, etc.(see [34],[35], [51],[52]). Analogous to them, a new method is proposed in this section, to use in the spectral gap prediction problems. In short, the spectral gap prediction problems are reduced into the determination of nonzero values of a particular function. This particular function can be approximated by a sequence of functions uniformly. And this sequence of functions comes directly from the eigenvalues of truncations of the operator under concern.

The idea is to open the gap by translating and squaring the operator and identify each numbers in the interval (ν, μ) as the lower bound of essential spectrum of a positive definite operator. And there the truncation methods, in particular, Theorem (2.1.7) are applied to compute this lower bound. The idea of squaring the operator to get information about its spectrum was used before. First, we shall briefly mention the work done by E.B.Davies in [34] and [35], which is of great interest, where he considered functions which are related to the distance from the spectrum.

2.3.1 Analytical approach

In his paper published in 1998 [34], E.B.Davies considered the function F defined by

$$F(t) = \inf \left\{ \frac{\|A(x) - tx\|}{\|x\|} : 0 \neq x \in \mathbb{L} \right\} \tag{2.10}$$

where $\mathbb{L}$ is a subspace of $\mathbb{H}$. Then he observed the following (Lemma 1 and its corollary in [34]).

- F is Lipschitz continuous and satisfies $|F(s) - F(t)| \leqslant |s - t|$, for all $s, t \in \mathbb{R}$.

- $F(t) \geqslant d(t, \sigma(A)) = dist(t, \sigma(A))$
- If $0 \leqslant F(t) \leqslant \delta$, then $\sigma(A) \cap [t-\delta, t+\delta] \neq \emptyset$.

From these observations, he obtained some bounds for the eigenvalues in the spectral gap of A, and found it useful in some concrete situations. For the efficient computation of the function F, he considered family of operators $N(s)$ on the given finite dimensional subspace $\mathbb{L}$, defined by

$$N(s) = A_{\mathbb{L}}{}^* A_{\mathbb{L}} - 2sPA_{\mathbb{L}} + s^2 I_{\mathbb{L}} \tag{2.11}$$

where P is the projection onto $\mathbb{L}$ and the notation $A_{\mathbb{L}}$ means A restricted to $\mathbb{L}$. The eigenvalues of these finite dimensional operators form sequence of real analytic functions. He used these sequence to approximate the function F and thereby obtain information about the spectral properties of A. The main result is stated below (special case of Theorem 9 in [34], under the assumption that A is bounded).

Theorem 2.3.1. *Suppose $\{\mathbb{L}_n\}_{n=1}^{\infty}$ is an increasing sequence of closed subspaces of $\mathbb{H}$. If F_n the functions associated with $\mathbb{L}_n$ according to (2.10), then F_n decreases monotonically and converge locally uniformly to $d(., \sigma(A))$. In particular, $s \in \sigma(A)$ if and only if*

$$\lim_{n\to\infty} F_n(s) = 0.$$

In his paper on spectral pollution [35] in 2004, he tried to link the above method with various techniques that were known in the past due to Lehmann [49], Behnke and Goerisch [11], Zimmerman and Mertin [72]. There he tried to resolve the problem of spurious eigenvalues in a spectral gap. He considered the function

$$G(t) = \inf\{\|A(x) - tx\| : \ x \in \mathbb{H}, \|x\| = 1\}$$

and wanted to evaluate G numerically and to locate spectrum of A, using the fact that $G(t) = d(t, \sigma(A))$. He introduced the approximating sequence of functions as

$$G_n(t) = \inf\{\|A(x) - tx\| : \ x \in \mathbb{L}_n, \|x\| = 1\}$$

where $\mathbb{L}_n$ is an increasing sequence of subspaces whose union is dense in $\mathbb{H}$, and used them to obtain some results as listed below (page no. 422-425 of [35]).

- Given $\epsilon > 0$, there exists an N_ϵ such that $n \geqslant N_\epsilon$ implies

$$G(t) \leqslant G_n(t) \leqslant G(t) + \epsilon \ \text{ for all } \ t \in \mathbb{R}$$

- $\sigma(A) \cap [t - G_n(t), t + G_n(t)] \neq \emptyset$ for every $t \in \mathbb{R}$.

Using these and with some assumptions on G_n, he obtained some bounds for the eigenvalues between the bounds of essential spectrum. He also produced some numerical evidence for the implementation of these techniques in bounding the eigenvalues of some particular operators.

Levitin and Shargorodsky considered the problem of spectral pollution in [52]. They suggested the usage of second order relative spectra, to deal the problem. For the sake of completion, the definition is given below.

Definition 2.3.1. [52] Let $\mathbb{L}$ be a finite-dimensional subspace of $\mathbb{H}$. A complex number z is said to belong to the second order spectrum $\sigma_2(A, \mathbb{L})$ of A relative to $\mathbb{L}$ if there exists a nonzero u in $\mathbb{L}$ such that

$$\langle (A - zI)u, (A - \bar{z}I)v \rangle = 0, \text{ for every } v \in \mathbb{L}$$

They proved the following. Consider a disc in the complex plane with diameter is an interval on the real line which intersect with the spectrum of A. Every such discs will have nonempty intersection with the second order relative spectrum (Lemma 5.2 of [52]). They also provided some numerical results in case of some Multiplication and Differential operators, which indicated the effectiveness of second order relative spectra in avoiding the spectral pollution. In [51], Boulton and Levitin used the quadratic projection method to avoid spectral pollution in the case of some particular Schrodinger operators. Before introducing the new method, we list down a couple of theorems from [54] which considered operators with disconnected essential spectrum and useful in our context.

Lemma 2.3.1. [54] Let A be a bounded self-adjoint operator with the essential spectrum, $\sigma_e(A) = [a, b] \bigcup \{c\}$ where $a < b < c$. Assume that b is not an accumulation point of the discrete spectra of A. Then a,b,c can be computed by truncation method.

Next theorem will give information about one endpoint of the spectral gap, provided the other end point is known.

Theorem 2.3.2. *[54] Let A be a bounded self-adjoint operator and*

$\sigma_e(A)$ = [a,b] $\bigcup$ [c,d], where $a < b < c < d$. Assume that b is known and not an accumulation point of the discrete spectra of A. Then c can be computed by truncation method.

2.3.2 The new method

To predict the existence of a gap in the essential spectrum, we need to know whether a number λ in (ν, μ) belongs to the spectrum or not. If it is not a spectral value, then there exists an open interval between (ν, μ) as a part of the compliment of the spectrum, since the compliment is an open set. We observe that the spectral gap prediction is possible by computing values of the following function.

Definition 2.3.2. Define the nonnegative valued function f on the real line $\mathbb{R}$ as follows.

$$f(\lambda) = \nu_\lambda = \inf \sigma_e((A - \lambda I)^2).$$

The primary observation is that we can predict the existence of a gap inside the essential spectrum by evaluating the function and checking whether it attains a nonzero value. The nonzero values of this function give the indication of spectral gaps.

Theorem 2.3.3. *The number λ in the interval (ν, μ) is in the gap if and only if $f(\lambda) > 0$. Also one end point of the gap will be $\lambda \pm \sqrt{f(\lambda)}$.*

Proof. Using the spectral mapping theorem, we observe that $f(\lambda)$ is the square of the distance of λ to the essential spectrum of A. The details are given below.

$$\inf \sigma_e((A-\lambda I)^2) = d(0, \sigma_e(A-\lambda I)^2) = d(0, \sigma_e(A-\lambda I))^2 = d(\lambda, \sigma_e(A))^2$$

Hence λ is in the essential spectrum of A if and only if $f(\lambda) = 0$, since essential spectrum is a closed set. Therefore the number λ in the interval (ν, μ) is in the gap if and only if $f(\lambda) > 0$. Now if λ is in the gap, then one of the end points will be at a distance $\sqrt{f(\lambda)}$ from λ. Hence that end point will be $\lambda \pm \sqrt{f(\lambda)}$. $\square$

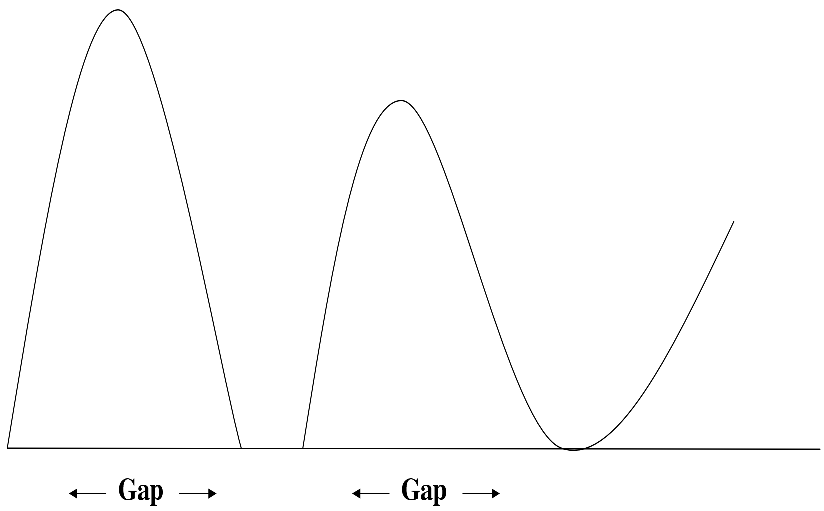

Figure 2.1: Graph of $f(\lambda)$

The advantage of considering $f(\lambda)$ is that, it is the lower bound of the essential spectrum of the operator $(A-\lambda I)^2$, which we can compute by using the finite dimensional truncations with the help of Theorem (2.1.7).

So the computation of $f(\lambda)$, for each λ, is possible. This enables us to predict the gap using truncations. Also here we are able to compute one end point of a gap. The other end point is possible to compute as discussed in Theorem (2.3.2).

Coming back to the Arveson's class, we observe that the essential points and hence the essential spectrum is fully determined by the zeros of the function in the definition (2.3.2)

Corollary 2. If A is a bounded self-adjoint operator in the Arveson's class, then λ is an essential point if and only if $f(\lambda) = 0$.

Proof. This follows easily from Theorems (2.1.3) and (2.3.3). $\square$

When one wishes to apply the above results to determine the gaps in the essential spectrum of a particular operator, one has to face the following problems. To check for each λ in (ν, μ), is a difficult task in the computational point of view. Also taking truncations of the square of the operator may lead to difficulty. Note that $(P_nAP_n)^2$ and $P_nA^2P_n$ are entirely different. So we may have to do more computations to handle the problem.

Another problem is the rate of convergence and estimation of the remainder term. For each λ in (ν, μ) the value of the function $f(\lambda)$ has to be computed. This computation involves truncation of the operator $(A-\lambda I)^2$ and the limiting process of sequence of eigenvalues of each truncation. The rate of convergence of these approximations and the remainder estimate are the questions of interest.

Below, the function $f(.)$ is approximated by a double sequence of functions, which arise from the eigenvalues of truncations of operators.

Theorem 2.3.4. *Let $f_{n,k}$ be the sequence of functions defined by $f_{n,k}(\lambda) = \lambda_{n+1-k}\left(P_n (A - \lambda I)^2 P_n\right)$. Then $f(.)$ is the uniform limit of a subsequence of $\{f_{n,k}(.)\}$ on all compact subsets of the real line.*

Proof. By Theorem (2.1.7), we have for each λ,

$$f(\lambda) = \lim_{k\to\infty} \lim_{n\to\infty} f_{n,k}(\lambda), \text{ where } f_{n,k}(\lambda) \;=\; \lambda_{n+1-k}\left(P_n (A-\lambda I)^2 P_n\right).$$

Now the quantity $\Delta = |f_{n,k}(\lambda) - f_{n,k}(\lambda_0)|$ can be estimated as follows.

$$\begin{aligned}
\Delta &= \left|\lambda_{n+1-k}\left(P_n (A-\lambda I)^2 P_n\right) - \lambda_{n+1-k}\left(P_n (A-\lambda_0 I)^2 P_n\right)\right| \\
&\leqslant \left\|P_n (A-\lambda I)^2 P_n - P_n (A-\lambda_0 I)^2 P_n\right\| \\
&\leqslant \left\|(A-\lambda I)^2 - (A-\lambda_0 I)^2\right\| = \left\|(\lambda^2 - \lambda_0{}^2)I - 2(\lambda_0 - \lambda)A\right\| \leqslant M\,|\lambda - \lambda_0|\,,
\end{aligned}$$

where $M = 2(|\mu| + \|A\|)$. The first inequality follows from (2.8) and the second one from the fact that $\|P_n\| = 1$. Hence we have

$$|f_{n,k}(\lambda) - f_{n,k}(\lambda_0)| \leqslant M\,|\lambda - \lambda_0|\,. \tag{2.12}$$

Since the constant M above is independent of n, k or λ, $\{f_{n,k}(.)\}$ forms an equicontinuous family of functions, also it is point wise bounded. Hence $\{f_{n,k}(.)\}$ has a subsequence which converges uniformly on all compact subsets by Arzela-Ascoi theorem. Hence the proof is complete. □

The following result makes the computation of $f(\lambda)$ much easier for a particular class of operators. When the operator is truncated first and square the truncation rather than truncating the square of the operator, the difficulty of squaring a bounded operator is reduced. The computation needs only to square the finite matrices. Denote $A - \lambda I$ by the symbol A_λ.

Theorem 2.3.5. *If* $\|P_n A - AP_n\| \to 0 \, as \, n \to \infty$, *then*

$$\lim_{k\to\infty} \lim_{n\to\infty} \lambda_{n+1-k} \left(P_n \left(A - \lambda I\right)^2 P_n\right) = \lim_{k\to\infty} \lim_{n\to\infty} \lambda_{n+1-k} \left(P_n \left(A - \lambda I\right) P_n\right)^2 .$$

Proof. Observe the following chain of equalities;

$$\left\|P_n \left(A_\lambda\right)^2 P_n - \left(P_n \left(A_\lambda\right) P_n\right)^2\right\| = \left\|P_n \left(A_\lambda\right) \left(A_\lambda\right) P_n - \left(P_n \left(A_\lambda\right) P_n\right) \left(P_n \left(A_\lambda\right) P_n\right)\right\|$$
$$= \left\|P_n \left(A_\lambda\right) \left(A_\lambda\right) P_n - \left(A_\lambda\right) P_n \left(A_\lambda\right) P_n + \left(A_\lambda\right) P_n P_n \left(A_\lambda\right) P_n - P_n \left(A_\lambda\right) P_n \left(A_\lambda\right) P_n \right\|$$

using ${P_n}^2 = P_n$ and adding and subtracting $\left(A_\lambda\right) P_n \left(A_\lambda\right) P_n$. And notice that the latter is equal to

$$\left\|\left[P_n \left(A_\lambda\right) - \left(A_\lambda\right) P_n\right] \left(A_\lambda\right) P_n - \left[P_n \left(A_\lambda\right) - \left(A_\lambda\right) P_n\right] P_n \left(A_\lambda\right) P_n\right\| =$$

$$\left\|\left[P_n \left(A_\lambda\right) - \left(A_\lambda\right) P_n\right] \left[\left(A_\lambda\right) P_n - P_n \left(A_\lambda\right) P_n\right]\right\| \leqslant 2 \left\|A_\lambda\right\| \left\|P_n \left(A_\lambda\right) - \left(A_\lambda\right) P_n\right\| =$$

$$2 \left\|A_\lambda\right\| \left\|P_n A - AP_n\right\| \to 0,$$

as the dimension n tends to infinity. The proof is completed by applying (2.12) to the matrices $\left(P_n \left(A - \lambda I\right)^2 P_n\right)$ and $\left(P_n \left(A - \lambda I\right) P_n\right)^2$. $\square$

Remark 2.3.1. The function $f(.)$ that is considered here is directly related to the distance from the essential spectrum, while Davies' function was related with the distance from the spectrum. Here the approximation results in [19], especially Theorem (2.1.7) are used to approximate the function. But it is still not known to us whether these results are useful in a computational point of view. The methods due to Davies were applied in the case of some Schrodinger operators with a particular kind of potentials in [52] and [51]. We hope that a combined use of both methods may give a better understanding of the spectrum.

Chapter 3

The Borg-type theorems

In this chapter, the indications are given for the possibility of classical Borg-type theorems in the case of discrete Schrodinger operator with periodic potential. Moreover we convert other results as those regarding the spectral distribution, in the spirit of Szegö theorems [38]. The main tools are the use of finite differences for identifying the analogous discrete operators and a formulation of the discrete problem in terms of block Toeplitz sequences with $p \times p$ matrix-valued symbols. Also, different possibilities for the generalized versions of the theorem are discussed. As it is remarked earlier, the usage of the matrix valued symbol to compute the spectrum of the operator may not be possible in many practical situations. In many such cases, only the Fourier coefficients will be available from which the symbol has to be recaptured, which is again a difficult problem.

Consider the one dimensional Schrodinger operator $\tilde{A}(u) = -\ddot{u} + V \cdot u$ with real valued periodic potential $V(\cdot)$, defined on a suitable subspace of

$L^2(\mathbb{R})$. The Borg theorem states that there are no spectral gaps if and only if the periodic potential $V(\cdot)$ is constant almost everywhere. Here the families of finite difference approximations of the operator $\tilde{A}$ are considered depending on two parameters n, that is the number of periodicity intervals possibly infinite, and p, the precision of the approximation in each interval. In this chapter, it is shown that the approach, with fixed p, leads to families of sequences $\{A_n(p)\}$, where every matrix $A_n(p)$ can be interpreted as a block Toeplitz matrix generated by a $p \times p$ matrix-valued symbol f: in other words, every $A_n(p)$ with finite n is a finite section of the bi-infinite Toeplitz-Laurent operator $A_\infty(p) = L(f)$. The specific feature of the symbol f, which is a linear trigonometric polynomial, allows to identify the distribution of the collective spectra of the matrix-sequence $\{A_n(p)\}$ and in particular provide a simple way for proving a discrete version of Borg theorem, in which the discrete operator $L(f)$ has no gaps if and only if the potential $V(\cdot)$ is constant.

Some of the results here, partly overlap with known theorems from operator theory due to Flaschka (see [36]). The main novelty here is the purely linear algebraic approach.

The chapter is organized as follows. In the first section, the gap related problems of Schrodinger operators with periodic potential are described briefly. Section 3.2 is devoted to describe the process of approximation of the Schrodinger operator, that leads to the families of matrix-sequences $\{A_n(p)\}$ and to the Toeplitz-Laurent operator $A_\infty(p) = L(f)$. The next section deals with basic notions, definitions and preliminary results. Section 3.4 contains the main results on a discrete Borg theorem via block Toeplitz-Laurent operators. In the next two sections, the possibility of these results into more general block Toeplitz-Laurent operators and periodic Jacobi matrices are discussed.

3.1 Description of the problem

As already mentioned in the previous chapter, gap related problems have been studied with special attention for Schrodinger operators in the past (see e.g. [26, 34, 35, 59]). Consider the one dimensional Schrodinger operator with periodic potential, $\tilde{A}(u) = -\ddot{u} + V \cdot u$, defined on a suitable subspace of $L^2(\mathbb{R})$. It can be proved that the spectrum is the union of closed intervals (see [59] for the proof). In some cases these intervals may be separated by nonempty open intervals: it is evident that all these nonempty open sets are spectral gaps. For instance looking at the Mathieu operator, which is defined by the potential $V(x) = \beta \cos(x)$ for a certain nonzero constant β, it is known that all the spectral gaps are open; (see page 298 of [59]). A summary of general and elegant classical results regarding the Schrodinger operator with periodic potential is reported below (see page 297 of [59] for the proof).

Theorem 3.1.1. *Take the one dimensional Schrodinger operator*

$$\tilde{A}(u) = -\ddot{u} + V \cdot u \tag{3.1}$$

with periodic potential V, defined on a suitable subspace of $L^2(\mathbb{R})$:

- *There are no gaps in the spectrum if and only if the potential function reduces to a constant (Borg theorem; see e.g. [29]).*
- *If there exists exactly one gap, then the potential function is elliptic.*

- *The potential function is real analytic if there exist finitely many gaps.*

Here families of finite difference approximations of the operator $\tilde{A}$ are considered, depending on two parameters n, the number of periodicity intervals, and p the precision of the approximation in each interval. Here it is shown that the approach, with fixed p, leads to families of sequences $\{A_n(p)\}$, where every matrix $A_n(p)$ that can be interpreted as a block Toeplitz matrix generated by a $p \times p$ matrix-valued symbol. Indeed, the parameter p is the periodicity index appearing in the diagonal of the approximating matrices, where the periodicity is induced by that of the potential V and in fact the entries on the diagonal are, up to a proper scaling related to the finesse discretization parameter $1/(p+1)$, exact samplings of the potential $V(\cdot)$ in equispaced points.

3.2 From continuous to discrete

In this section, a simple (in fact the simplest) finite differences approximation is proposed for the Schrodinger operator with periodic potential $V(x)$. Without loss of generality, assume that the periodicity width is 1 that is $V(x+1) = V(x)$ for every $x \in \mathbb{R}$. Now approximate the equation (3.1) in the interval $[-n, n]$ with $n \in \mathbb{N} \cup \{\infty\}$ by using p equispaced points in each interval $[j, j+1] \subset [-n, n]$ by using the standard difference

$$\frac{-u(x_{i+1,(j)}) + 2u(x_{i,(j)}) - u(x_{i-1,(j)})}{h^2}$$

with $h = h(p) = 1/p$, $x_{s,(j)} = j + sh(p)$, $j = -n, \ldots, n-1$, $s = 0, \ldots, p$. In this way, letting $n = \infty$, this can be treated as an operator acting on the sequence space $l^2(\mathbb{Z})$, defined as follows.

$$\text{For } \{u_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}),\; A(\{u_n\}_{n\in\mathbb{Z}}) = \frac{-(u_{n-1} + u_{n+1}) + 2u_n}{h^2} + v_n u_n, \tag{3.2}$$

where the sequence $\{v_n\}_{n\in\mathbb{Z}}$ is obtained as the values of the periodic function V at p equispaced points in an interval of length 1. And the periodicity of V will imply that the sequence $\{v_n\}_{n\in\mathbb{Z}}$ is also periodic with period p. Now, the matrix representation of this bounded operator with respect to the standard basis in $l^2(\mathbb{Z})$, is obtained as a tridiagonal matrix that, up to the scaling factor h^2, coincides with

$$A_n(p) = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & \ddots & \ddots & & \\ & -1 & 2 + h^2 V(x_{s,(j)}) & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \tag{3.3}$$

Given the periodicity this matrix can be re-written as

$$A_n(p) = \begin{bmatrix} \ddots & \ddots & & & & & & & \\ \ddots & \ddots & \ddots & & & & & & \\ & -1 & 2+v_0 & -1 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & -1 & 2+v_{p-1} & -1 & & & \\ & & & & -1 & 2+v_0 & -1 & & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \ddots \end{bmatrix}, \tag{3.4}$$

with $v_j = h^2 V(x_{s,(j)})$, $j = 0, \ldots, p-1$.

When n is finite then the resulting matrix of size np is just a truncation of the bi-infinite matrix reported above.

Along the same lines, consider the variable coefficient one dimensional Schrodinger operator $\tilde{A}(u) = -\frac{d}{dx}\left(a \cdot \frac{d}{dx}u\right) + V \cdot u$, with $a(\cdot)$ being positive and periodic with the same period as $V(\cdot)$. In that case, the very same type of infinite difference approximation will lead to a bi-infinite symmetric matrix of the form

$$A_{n,\alpha}(p) = \begin{bmatrix} \ddots & \ddots & & & & & & & \\ \ddots & \ddots & \ddots & & & & & & \\ & -\alpha_{p-1} & \gamma_0 & -\alpha_0 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & -\alpha_{p-2} & \gamma_{p-1} & -\alpha_{p-1} & & & \\ & & & & -\alpha_{p-1} & \gamma_0 & -\alpha_0 & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & \end{bmatrix}, \tag{3.5}$$

with $\gamma_s = \alpha_s + \alpha_{(s+1) \bmod p} + h^2 V(x_{s;(j)})$,

$$\alpha_s = a(x_{s+1/2}, j), x_{s+1/2,j} = j + h(p)(s + 1/2).$$

Observe that resulting structure, up to the sign, represents the case of general p-periodic Jacobi matrices.

3.3 Preliminary results and notation

This section is divided into two parts. In the first, a few results concerning the spectra of Toeplitz-Laurent operators with matrix-valued symbols are briefly recalled. And in the second, the definition of spectral distribution and results regarding the case of Toeplitz sequences coming from sections of infinite Toeplitz operators are given.

The connections among these ingredients will become evident in Section 3.4, since the approximation of the Schrodinger operator with periodic potential using finite differences as in Section 3.2 leads to matrix-sequences $\{A_n(p)\}$, where p is a parameter associated with the precision of the approximation. The results in Section 3.3.1 allow us to prove the main results on a discrete version of Borg theorem, while the results in Section 3.3.2 are of interest for the distributional analysis.

3.3.1 Toeplitz operators and sequences

Given a $p \times p$ matrix-valued integrable function f defined on $(-\pi, \pi)$, the $p \times p$ matrices f_j, $j \in \mathbb{Z}$, represent the Fourier coefficients of f defined as

$$f_j(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-\hat{\imath} j \theta} dx, \quad j = 0, \pm 1, \pm 2, \ldots .$$

Then for n being a nonnegative integer number or ∞ we define $T_n(f)$ the Toeplitz matrix or operator of size n generated by f via the relations

$$(T_n(f))_{i,j} = f_{i-j}, \quad i, j = 1, \ldots, n.$$

Here the integration of the matrix valued function turns out to be the entry wise integration. When $n = \infty$ the Toeplitz operator $T_n(f)$ is simply written as $T(f)$ while the symbol $L(f)$ denotes the doubly infinite Toeplitz matrix with $(L(f))_{i,j} = f_{i-j}$, $i, j \in \mathbb{Z}$. Furthermore by $\{T_n(f)\}$ we indicate the Toeplitz matrix-sequence generated by f, with $T_n(f)$ of finite order.

Let f be a continuous and Hermitian $p \times p$ matrix-valued function on the unit circle, and let $\lambda_1(f(.)) \geqslant \cdots \geqslant \lambda_p(f(.))$ denote its eigenvalues. Then it is well known that the essential spectrum of $L(f)$ and the essential spectrum of $T(f)$ coincide with the union of the ranges of the eigenvalues $\lambda_1(f(\cdot)) \geqslant \cdots \lambda_p(f(\cdot))$, that is

$$\sigma_{\text{ess}}(L(f)) = \sigma_{\text{ess}}(T(f)) = \bigcup_{j=1}^{p} \left[\inf_{\theta}(\lambda_j(f(\theta))), \sup_{\theta}(\lambda_j(f(\theta))) \right]. \tag{3.6}$$

For the latter result which is crucial for the discrete version of the Borg theorem, see Proposition 2.29(a) of the book [15].

3.3.2 Spectral Distributions

Here the definition of spectral distribution concerning matrix-sequences of increasing size are reported and a distribution result for block Toeplitz sequences, are given in the spirit of Weyl.

Definition 3.3.1. Let $\mathcal{C}_0(\mathbb{C})$ be the set of continuous functions with bounded support defined over the complex field, N be a positive integer, and ψ be a $p \times p$ matrix-valued measurable function defined on a set $G \subset \mathbb{R}^N$ of finite and positive Lebesgue measure $\mu(G)$. A matrix-sequence $\{A_n\}$ is said to be *distributed* (*in the sense of the eigenvalues*) *as the pair* (ψ, G), or to *have the eigenvalue distribution function* ψ ($\{A_n\} \sim_\lambda (\psi, G)$), if for every F in $\mathcal{C}_0(\mathbb{C})$, the following limit relation holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F\left(\lambda_j(A_n)\right) = \frac{1}{\mu(G)} \int_G \frac{1}{p} \sum_{s=1}^{p} F(\lambda_s(\psi(t)))\, dt, \qquad t = (t_1, \ldots, t_N). \tag{3.7}$$

Remark 3.3.1. Here G will be often equal to $(-\pi,\pi)^d$ so that $e^{i\overline{G}} = \mathbb{T}^d$ with $i^2 = -1$ and $\mathbb{T}$ denoting the complex unit circle.

Concerning the spectral distribution of Toeplitz matrix-sequences, the main result is the Theorem of Szegö (see [17]), that was reported in its most general version due to Tilli [64].

Theorem 3.3.2 (Szegö-Tilli)**.** *Let f be a $p \times p$ matrix-valued integrable function defined on $(-\pi, \pi)$ and let $\{T_n(f)\}$ be the block Toeplitz sequence generated by f. Assume that f is Hermitian almost everywhere on its definition set. Then*

$$\{T_n(f)\} \sim_\lambda (f, (-\pi, \pi)),$$

that is, for every function F continuous with bounded support we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(T_n(f))) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{p} \sum_{s=1}^{p} F(\lambda_s(f(\theta)))d\,\theta.$$

with $\lambda_j(A)$ denoting the eigenvalues of the square matrix A.

We end this section by stating the **Cauchy interlacing theorem** which plays a crucial role in the further proofs (see page 59 of [7] for the proof).

Theorem 3.3.3. *Let A be an $n \times n$ Hermitian matrix, and let B be its principal sub matrix. If $\lambda_j(A)$ and $\lambda_j(B)$ are eigenvalues of A and B respectively, arranged in non increasing order, then*

$$\lambda_1(A) \geqslant \lambda_1(B) \geqslant \lambda_2(A) \geqslant \lambda_2(B) \geqslant \ldots \geqslant \lambda_{n-1}(B) \geqslant \lambda_n(A).$$

3.4 The discrete Borg Theorem

Our aim is to suggest a pure linear algebraic approach to discrete version of the celebrated Borg theorem. First of all we make a simple but crucial observation concerning the matrix $A_n(p)$ in (3.3) when $n = \infty$. Given the periodicity, well emphasized in the expression reported in (3.4), and taking into account the definition of the Toeplitz-Laurent operator $L(f)$ in Section 3.3.1, it is clear that $A_n(p) = L(f)$ where $f = f_p$ has size p and can be chosen from the finite set $\{g_0, \ldots, g_{p-1}\}$ with

$$g_j = 2I_p - H_p + \mathrm{diag}_{i=0,\ldots,p-1}\left(w_{(j+i) \bmod p}\right) - e^{i\theta}E_{1,p} - e^{-i\theta}E_{p,1}, \quad (3.8)$$

$j = 0, \ldots, p-1$, $E_{s,t}$, $1 \leqslant s, t \leqslant p$, being the dyadic matrix having 1 in position (s,t) and zero otherwise, H_p being the matrix having 1 in position (s,t) with $|s-t| = 1$ and zero otherwise, I_p being the identity matrix of size p.

That means upto some scaling by constant, and translation by the identity, the symbols have the following form.

$$g_j(\theta) = \begin{bmatrix} v_{j+1} & 1 & & & e^{i\theta} \\ 1 & v_{j+2} & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{-i\theta} & & & 1 & v_{j+p} \end{bmatrix}.$$

In this way the matrix g_j, $j = 0, \ldots, p-1$, will have on its diagonal the vector

$$\pi^j(\{v_0, \ldots, v_{p-1}\}),$$

where π is the periodic permutation defined by

$$\pi(\{\alpha_0, \alpha_1, \ldots, \alpha_{p-1}\}) = \{\alpha_{p-1}, \alpha_0, \ldots, \alpha_{p-2}\}.$$

If Π is the square matrix of size p that represents such a permutation, then it is easy to see that

$$\operatorname{diag}(g_j) = \Pi \operatorname{diag}(g_{j-1}) \Pi^T.$$

However the matrix g_j is not similar to g_{j-1} via the same transformation Π. On the other hand, without using any linear algebra argument, we can immediately see that these symbols share somehow the same spectrum in the sense that, for every $s, t = 0, \ldots, p-1$, we observe

$$\bigcup_{k=1}^{p} \left[\inf_\theta(\lambda_k(g_s(\theta))), \sup_\theta(\lambda_k(g_s(\theta)))\right] = \bigcup_{k=1}^{p} \left[\inf_\theta(\lambda_k(g_t(\theta))), \sup_\theta(\lambda_k(g_t(\theta)))\right]. \tag{3.9}$$

As already mentioned the latter statement is immediate due to (3.6). Now we are ready to prove that the operator $A_\infty(p)$ has no gaps if and only if $v_0 = \ldots = v_{p-1}$. Equivalently, for the sake of notational simplicity, we can consider for $\hat{A}_\infty(p) = -A_\infty(p) + 2I$, with I denoting the identity operator acting on $l^2(\mathbb{Z})$.

Theorem 3.4.1. *The essential spectrum of $A = \hat{A}_\infty(p)$ is connected if and only if the p-periodic potential $(v_j)_{j\in\mathbb{Z}}$ is constant.*

Proof. **Case1: period** $p = 2$. Let $(-v_j) = (\ldots a, b, a, b, \ldots)$. Then

$$A = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & . & & & & \\ \ddots & b & 1 & 0 & 0 & & . & & & \\ \ddots & 1 & a & 1 & 0 & 0 & & . & & \\ \ddots & 0 & 1 & b & 1 & 0 & 0 & & . & \\ & 0 & 0 & 1 & a & 1 & 0 & 0 & & . \\ . & & 0 & 0 & 1 & b & 1 & 0 & 0 & \\ & . & & 0 & 0 & 1 & a & 1 & 0 & \ddots \\ & & . & & 0 & 0 & 1 & b & 1 & \ddots \\ & & & . & & 0 & 0 & 1 & a & \ddots \\ & & & & . & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

which can be put in block form as

$$A = \begin{bmatrix} \ddots & \ddots & & & & & & & & \\ \ddots & A_0 & A_{-1} & & & & & & & \\ & A_1 & A_0 & A_{-1} & & & & & & \\ & & A_1 & A_0 & A_{-1} & & & & & \\ & & & A_1 & A_0 & A_{-1} & & & & \\ & & & & A_1 & A_0 & A_{-1} & & & \\ & & & & & A_1 & A_0 & A_{-1} & & \\ & & & & & & A_1 & A_0 & A_{-1} & \\ & & & & & & & A_1 & A_0 & \ddots \\ & & & & & & & & \ddots & \ddots \end{bmatrix}$$

where

$$A_0 = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Now this is in the block Toeplitz-Laurent form, whose matrix-valued symbol is

$$f(\theta) = \begin{bmatrix} a & 1+e^{i\theta} \\ 1+e^{-i\theta} & b \end{bmatrix}$$

The eigenvalues are given by the following functions

$$\lambda_{1,2}(\theta) = \frac{a+b \pm \left((a-b)^2 + 8\cos\theta + 8\right)^{1/2}}{2}.$$

Now we observe the following simple facts.

If $a \leqslant b$, then $\left((a-b)^2+16\right)^{1/2} \geqslant b-a$ and $-\left((a-b)^2+16\right)^{1/2} \leqslant a-b$.

Hence we get

$$\frac{a+b-\left((a-b)^2+16\right)^{1/2}}{2} \leqslant a \leqslant b \leqslant \frac{a+b+\left((a-b)^2+16\right)^{1/2}}{2}$$

And if $a \geqslant b$, then

$$\left((a-b)^2+16\right)^{1/2} \geqslant a-b \text{ and } -\left((a-b)^2+16\right)^{1/2} \leqslant b-a.$$

Hence

$$\frac{a+b-\left((a-b)^2+16\right)^{1/2}}{2} \leqslant b \leqslant a \leqslant \frac{a+b+\left((a-b)^2+16\right)^{1/2}}{2}$$

We conclude that the range of $\lambda_1(\theta)$ is

$$[a, s_+], \text{ if } a \geqslant b, \text{ and } [b, s_+], \text{ if } a \leqslant b,$$

while for the range of $\lambda_2(\theta)$ is

$$[s_-, b], \text{ if } a \geqslant b, \text{ and } [s_-, a], \text{ if } a \leqslant b,$$

where the notation

$$s_\pm = \frac{a + b \pm \left((a-b)^2 + 16\right)^{1/2}}{2}.$$

Now the union of ranges of these functions will be precisely the essential spectrum of A (refer to (3.6)). That is,

$$\sigma_e(A) = [s_-, b] \cup [a, s_+], \text{ if } a \geqslant b,$$

$$\sigma_e(A) = [s_-, a] \cup [b, s_+], \text{ if } a \leqslant b.$$

The latter concludes the proof because we clearly see that there exist no gaps if and only if $a = b$.

Case2: period $p > 2$.

Let the p-periodic sequence be $(-v_j) = (\ldots, a_1, \ldots a_p, a_1, a_2, \ldots)$. If the sequence is constant that is $a_1 = a_2 = \cdots = a_p = a$, then the operator A reduces to a standard Toeplitz-Laurent operator with (scalar-valued) symbol $f(\theta) = a + 2\cos(\theta)$. Therefore, in the light of (3.6), the spectrum is the closed interval $[a-2, a+2]$ and of course there are no gaps. Conversely, if the p-periodic sequence is not constant, then there exists at least one pair $(v_s, v_{(s+1) \bmod p})$ such that $v_s \neq v_{(s+1) \bmod p}$. Without loss of generality we take $-v_s = a_p$ so that necessarily $-v_{(s+1) \mod p} = a_1$: it is important to make clear that we have this degree of freedom due to the possibility of choosing the generating function among p different symbols.

Now

$$A = \begin{bmatrix} \ddots & \ddots & & & & & & & & \\ \ddots & A_0 & A_{-1} & & & & & & & \\ & A_1 & A_0 & A_{-1} & & & & & & \\ & & A_1 & A_0 & A_{-1} & & & & & \\ & & & A_1 & A_0 & A_{-1} & & & & \\ & & & & A_1 & A_0 & A_{-1} & & & \\ & & & & & A_1 & A_0 & A_{-1} & & \\ & & & & & & A_1 & A_0 & A_{-1} & \\ & & & & & & & A_1 & A_0 & \ddots \\ & & & & & & & & \ddots & \ddots \end{bmatrix}$$

where

$$A_0 = \begin{bmatrix} a_1 & 1 & & & & \\ 1 & a_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & a_p \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ 1 & & & & \end{bmatrix} = A_1^T = E_{p,1}.$$

Consequently, the matrix-valued symbol is

$$f(\theta) = \begin{bmatrix} a_1 & 1 & & & & e^{i\theta} \\ 1 & a_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ e^{-i\theta} & & & & 1 & a_p \end{bmatrix}.$$

We now consider two special principal minors of this matrix-valued symbol of size $p-1$: the first is obtained by deleting the first row and the first column from $f(\theta)$ and the second is obtained by deleting the last row and the last column from $f(\theta)$. Both minors P_1 and P_2 are constant matrices since they do not contain terms depending on θ:

$$P_1 = \begin{bmatrix} a_1 & 1 & & & & \\ 1 & a_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & a_{p-1} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} a_2 & 1 & & & & \\ 1 & a_3 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & a_p \end{bmatrix}.$$

Now these matrices cannot have the same eigenvalues because they have different trace. Indeed

$$\mathrm{trace}(P_1) - \mathrm{trace}(P_2) = a_1 - a_p \tag{3.10}$$

and the latter is different from zero because $a_p = -v_s \neq -v_{(s+1) \mod p} = a_1$. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_{p-1}$ be the eigenvalues of P_1 (all distinct because P_1 is a Jacobi matrix, see [22]) and let $\beta_1 < \beta_2 < \cdots < \beta_{p-1}$

(all distinct because also P_2 is a Jacobi matrix). Now due to (3.10) there exists $i \in \{1, \ldots, p-1\}$ such that $\alpha_i \neq \beta_i$. Without loss of generality assume $\alpha_i < \beta_i$ (the other case is treated identically). Then by Cauchy interlacing theorem we find

$$\lambda_1(\theta) \leqslant \alpha_1, \beta_1 \leqslant \lambda_2(\theta) \leqslant \cdots \leqslant \lambda_i(\theta) \leqslant \alpha_i < \beta_i \leqslant \lambda_{i+1}(\theta) \leqslant \cdots \leqslant \lambda_p(\theta).$$

Thus it is clear that the ranges of the eigenvalue functions will not intersect the interval (α_i, β_i) which will be a gap. In conclusion we have determined the existence of at least one gap and the theorem is proved. □

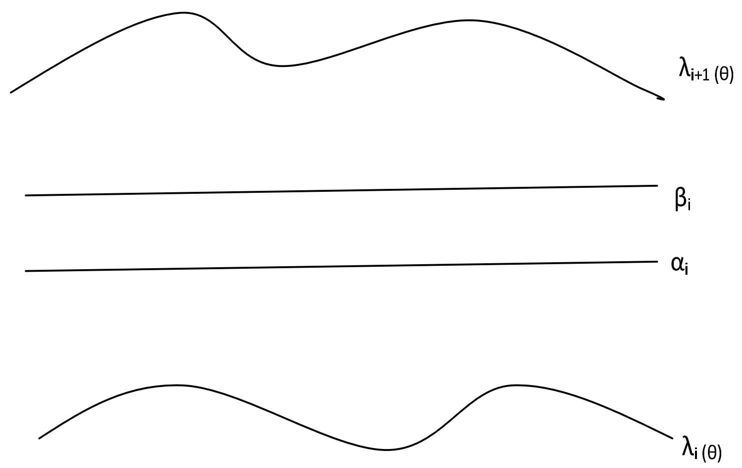

Figure 3.1: Graphical visualization

Example 3.4.1. Recall the operator introduced in Example (2.2.1).

That is an operator with non constant periodic potential and therefore the essential spectrum is disconnected by the above theorem. It can be noticed that the essential spectrum is given by the following equation.

$$\sigma_{\text{ess}}(A) = \bigcup_{j=1}^{3} \left[\inf_{\theta}(\lambda_j(\tilde{f}\,(\theta)), \sup_{\theta}(\lambda_j(\tilde{f}\,(\theta)) \right]$$

where $\lambda_j(\tilde{f}\,(\theta))$ are the eigenvalues of $\tilde{f}\,(\theta)$. A straightforward numerical computation of the eigenvalue functions gives

$$\sigma_{\text{ess}}(A) = [-0.2143, 0.3249] \cup [1.4608, 2.5392] \cup [3.6751, 4.2143]\,.$$

Also since A is in the Arveson's class (since it is represented by a band limited matrix), the point 1 lies in the gap, is a transient point. Hence the prediction of the existence of gap, in Theorem (2.2.1), is valid in this example.

3.5 Generalized version

In this section, the spectral gap issues of some block Toeplitz-Laurent operators are studied. The operators under concern are some perturbations of discrete Schrodinger operator on $l^2(Z)$, that was considered in the previous sections. The discrete version of Borg theorem proved in the last section is generalized here.

The following question is addressed here. In Theorem (3.4.1), if more nonzero entries are in the off diagonal, does the same result remain valid. When doing that, an additional assumption on the diagonal entries has to be made. That is the assumption that they follow a monotone order periodically. Also, this assumption can not be relaxed as it can be seen by an example.

Theorem 3.5.1. *Let A be the bounded operator defined by the block Toeplitz-Laurent matrix*

$$A = \begin{bmatrix} \ddots & \ddots & & & & & & & & \\ \ddots & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} & \dots & & \\ & A_1 & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} & \dots & \\ & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} & \dots \\ & & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} \\ \dots & A_N & \dots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & \\ & \dots & A_N & \dots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & \\ & & \dots & A_N & \dots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} \\ & & & \dots & A_N & \dots & A_2 & A_1 & A_0 & \ddots \\ & & & & & & & & \ddots & \ddots \end{bmatrix}$$

where

$$A_0 = \begin{bmatrix} b_1 & 1 & & & & a_0 \\ 1 & b_2 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ a_0 & & & & 1 & b_p \end{bmatrix}, \quad A_k = \begin{bmatrix} & & & a_k \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} = {A_{-k}}^T,$$

such that $b_1 \leqslant b_2 \ldots \leqslant b_p$ and $\sum_k |a_k| < \infty$. If A has connected essential spectrum, then $b_1 = b_2 \ldots = b_p$.

Proof. For the case $p = 2$, the matrix-valued symbol associated with the operator A is given by

$$\tilde{f}(\theta) = \begin{bmatrix} b_1 & 1 + f(\theta) \\ 1 + f(\bar{\theta}) & b_2 \end{bmatrix},$$

where $f(\theta) = \sum_k a_k e^{ik\theta}$. For $p > 2$, the matrix-valued symbol associated with the operator A is

$$\tilde{f}(\theta) = \begin{bmatrix} b_1 & 1 & & & f(\theta) \\ 1 & b_2 & 1 & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ f(\bar{\theta}) & & & 1 & b_p \end{bmatrix}.$$

Therefore from equation (3.6), we have

$$\sigma_{\text{ess}}(A) = \bigcup_{j=1}^{p} \left[\inf_{\theta}(\lambda_j(\tilde{f}(\theta)), \sup_{\theta}(\lambda_j(\tilde{f}(\theta)) \right]. \tag{3.11}$$

Now consider the sub matrices

$$P_1 = \begin{bmatrix} b_1 & 1 & & & \\ 1 & b_2 & 1 & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & 1 & b_{p-1} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} b_2 & 1 & & & \\ 1 & b_3 & 1 & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & 1 & b_p \end{bmatrix}.$$

If any of their eigenvalues are different, say $\lambda_j(P_1) < \lambda_j(P_2)$, then by Cauchy Interlacing theorem, $\lambda_j(\tilde{f}(\theta)) \leqslant \lambda_j(P_1) < \lambda_j(P_2) \leqslant \lambda_{j+1}(\tilde{f}(\theta))$, for every θ. But from (3.6), this will give us the contradiction that essential spectrum of A is not connected. Hence all the eigenvalues of P_1 and P_2 are same. Therefore

$$\text{trace}(P_1) - \text{trace}(P_2) = b_1 - b_p = 0$$

Hence $b_1 = b_2 \ldots = b_p$. □

Remark 3.5.1. The assumption $\sum_k |a_k| < \infty$, is used to make sure the convergence in the expression of $\tilde{f}(\theta)$. The assumption on the diagonal entries, $b_1 \leqslant b_2 \ldots \leqslant b_p$ is used in the last line of the proof. The assumption is not required for the case $p = 2$.

Remark 3.5.2. The converse of the Theorem (3.5.1) is not true in general. There may have gaps even if the diagonal entries of the block Toeplitz-Laurent operator are same. For if A is the block Toeplitz-Laurent operator arising from the matrix valued symbol

$$\tilde{f}(\theta) = \begin{bmatrix} b & 1 + f(\theta) \\ 1 + f\bar{(\theta)} & b \end{bmatrix}.$$

where f is a non negative function, then the eigenvalue functions of $\tilde{f}(\theta)$ are

$$\lambda_1(\theta) = b - 1 - f(\theta),\ \lambda_2(\theta) = b + 1 + f(\theta)$$

Since $f(\theta) \geqslant 0,\ \lambda_1(\theta) \leqslant b - 1 < b + 1 \leqslant \lambda_2(\theta)$. Hence spectrum of A will have a gap, since the ranges of the eigenvalue functions never intersect.

Example 3.5.1. The assumption $b_1 \leqslant b_2 \ldots \leqslant b_p$ can not be dropped in the above theorem, if $p > 2$. For if we consider the block Toeplitz-Laurent operator arising from the matrix valued symbol

$$\tilde{f}\,(\theta) = \begin{bmatrix} 1 & 1 & 0 & 10cos\,(\theta) \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 10cos\,(\theta) & 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalue functions of $\tilde{f}\,(\theta)$ are

$$\lambda_{1,2}(\theta) = 2 + 5cos(\theta) \pm \sqrt{25cos^2(\theta) - 10cos(\theta) + 2}$$

$$\lambda_{3,4}(\theta) = 1 - 5cos(\theta) \pm \sqrt{25cos^2(\theta) + 1}$$

We list the values of these functions at certain points in the table below.

θ	$\lambda_1(\theta)$	$\lambda_2(\theta)$	$\lambda_3(\theta)$	$\lambda_4(\theta)$
0	11.123	2.877	1.099	-9.099
π	3.083	-9.083	11.099	.901

From the table, it is clear that the ranges of the above continuous functions intersect. Hence their union is a connected interval. Therefore the essential spectrum of the operator has no gaps, even the periodic potential does not reduce to a constant.

3.6 Periodic Jacobi matrices

In this section, we look at the possibility for Borg-type theorems in the case of periodic Jacobi matrices, by identifying them as block Toeplitz-

Laurent operators. The double infinite, p-periodic, $p \geqslant 2$, real Jacobi matrix is defined below.

$$J = \begin{bmatrix} \ddots & \ddots & & & & & \\ \ddots & b_1 & a_1 & & & & \\ & a_1 & b_2 & a_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & b_p & a_p & \\ & & & & a_p & b_1 & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix}, a_{n+p} = a_n > 0; b_{n+p} = b_n : \tag{3.12}$$

The standard convention $a_n > 0$, is followed here which differs by sign from (3.5): indeed the bi-infinite matrix reported in (3.5) is easily converted into a Jacobi matrix multiplying it by -1. An important observation is that $J = L(f_k)$, where in the case $p \geqslant 3$, the symbols are given by

$$f_k(\theta) = \begin{bmatrix} b_{k+1} & a_{k+1} & 0 & & & e^{i\theta} a_{k+p-1} \\ a_{k+1} & b_{k+2} & a_{k+2} & & & 0 \\ 0 & a_{k+2} & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & a_{k+p-1} \\ e^{-i\theta} a_{k+p-1} & 0 & & & a_{k+p-1} & b_{k+p} \end{bmatrix}, k = 0, 1, \dots p-1,$$

(for the case $p = 2$, an example is given later in this chapter). Denote by $\lambda_j(f_k), j = 1, 2, \dots p$ the eigenvalues of f_k, arranged in the decreasing order

$$\lambda_1(f_k(\cdot)) \geqslant \cdots \lambda_p(f_k(\cdot)),$$

and put

$$\lambda_{j,k}^{+} = \max_{\theta} \lambda_j(f_k(\theta)), \quad \lambda_{j,k}^{-} = \min_{\theta} \lambda_j(f_k(\theta))$$

The spectrum of the original matrix J is (see Equation (3.6) earlier in this chapter)

$$\sigma(J) = \bigcup_{j=1}^{p} \left[\lambda_{j,k}^{-}, \lambda_{j,k}^{+}\right]. \tag{3.13}$$

and the right hand side does not depend on k. Consider the following sub matrices of the symbols of order $p-1$.

$$J_k = \begin{bmatrix} b_{k+1} & a_{k+1} & 0 & & & \\ a_{k+1} & b_{k+2} & a_{k+2} & & & 0 \\ 0 & a_{k+2} & . & . & & \\ & & . & . & . & \\ & & & . & . & a_{k+p-2} \\ & 0 & & & a_{k+p-2} & b_{k+p-1} \end{bmatrix}, k = 0, 1, \dots p-1,$$

and put

$$\sigma(J_k) = \{\mu_{1,k} > \mu_{2,k} > \dots \mu_{p-1,k}\}.$$

The Cauchy interlacing properties for eigenvalues of Hermitian matrices lead to the following inequalities

$$\lambda_1(f_k(\theta)) \geqslant \mu_{1,k} \geqslant \lambda_2(f_k(\theta)) \geqslant \dots \lambda_{p-1}(f_k(\theta))\mu_{p-1,k} \geqslant \lambda_p(f_k(\theta)),$$
$$\lambda_1(f_k(\theta)) \geqslant \mu_{1,k+1} \geqslant \lambda_2(f_k(\theta)) \geqslant \dots \lambda_{p-1}(f_k(\theta))\mu_{p-1,k+1} \geqslant \lambda_p(f_k(\theta)).$$

Lemma 3.6.1. Suppose that J has no spectral gaps. Then all J_k have the same spectrum.

Proof. Assuming the contrary, we would have $\mu_{j,k'} > \mu_{j,k''}$ so by the interlacing properties there is a gap in the spectrum. □

Remark 3.6.1. Let $b_j = 0, a_1 = a_3 = \ldots, a_2 = a_4 = \ldots,$ but $a_1 \neq a_2$. Put $p = 4$. Then

$$\det(J_0 + \lambda) = \det(J_1 + \lambda) = \begin{vmatrix} \lambda & a_1 & 0 \\ a_1 & \lambda & a_2 \\ 0 & a_2 & \lambda \end{vmatrix} = \lambda^3 - (a_1^2 + a_2^2)\lambda,$$

so $\sigma(J_0) = \sigma(J_1)$. From a simple computation with the help of Equation (3.6), we see that

$$\sigma(J) = \left[-(a_1 + a_2)^2, -(a_1 - a_2)^2\right] \cup \left[(a_1 - a_2)^2, (a_1 + a_2)^2\right].$$

Hence it is evident that J has spectral gaps since $a_1 \neq a_2$, and the converse to above Lemma (3.6.1) is false.

Lemma 3.6.2. Suppose that J has no gaps. Then $b_1 = b_2 = \ldots = b_p$.

Proof. By Lemma (3.6.1), $tr(J_0) = tr(J_1) = \ldots = tr(J_{p-1})$ and so

$$\sum_{j=1}^{p-1} b_j = \sum_{j=2}^{p} b_j = \ldots \sum_{j=p}^{2p-2} b_j$$

which implies $b_1 = b_p, b_2 = b_{p+1} = b_1, b_3 = b_{p+2} = b_2$ etc. as claimed. □

With no loss of generality we put $b_j = 0$. Assume also that the period p is an even number (otherwise take 2p as the period, see also Remark below). We proceed with the simple case of $p = 2$.

Example 3.6.1. Let $p = 2$ and $b_j = 0$. The symbol now is

$$f(\theta) = \begin{bmatrix} 0 & a_1 + e^{-i\theta} a_2 \\ a_1 + e^{i\theta} a_2 & 0 \end{bmatrix}, \ \lambda_{1,2}(f(\theta)) = \pm \left| a_1 + e^{-i\theta} a_2 \right|$$

Again we get

$$\sigma(J) = \left[-(a_1 + a_2)^2, -(a_1 - a_2)^2 \right] \cup \left[(a_1 - a_2)^2, (a_1 + a_2)^2 \right].$$

Hence there is no spectral gap for J if and only if $a_1 = a_2$, so $p = 1$.

The following result is a version of Borg theorem for Jacobi matrices.

Theorem 3.6.1. *Let $p = 2m + 2$ and J has no gaps in the spectrum. Then $a_1 = a_2 = \ldots = a_p$.*

Proof. Since J has no gaps, the diagonal sequence is constant by Lemma (3.6.2), which we assume to be 0 without loss of generality. We use the notation $D(\lambda; a_1, a_2, \ldots a_n)$ for the determinant. That is

$$D(\lambda; a_1, a_2, \ldots a_n) = \begin{vmatrix} \lambda & a_1 & 0 & \cdots & 0 \\ a_1 & \lambda & a_2 & & \\ 0 & a_2 & \ddots & & \\ \vdots & & \ddots & & a_n \\ 0 & & 0 & a_n & \lambda \end{vmatrix}.$$

By expanding over the last row and induction we see that

$$D(\lambda; a_1, a_2, \ldots a_n) = \lambda^{n+1} - \lambda^{n-1} \sum_{j=1}^{n} {a_j}^2 + \ldots$$

Here we are not bothered about the other terms in the above sum, since we will only equate the coefficients of higher powers of λ. By Lemma (3.6.2),

$$\det(J_k+\lambda) = D\left(\lambda; a_{k+1}, \ldots a_{k+2m}\right) = \lambda^{2m+1} - \lambda^{2m-1}\sum_{j=k+1}^{k+2m} {a_j}^2 + \ldots$$

does not depend on k so

$$\sum_{j=k+1}^{k+2m} {a_j}^2 = \sum_{j=k+2}^{k+2m+1} {a_j}^2 \Rightarrow a_{k+1} = a_{k+2m+1},\ k = 0, 1, \ldots p-1.$$

Hence $a_1 = a_3 = \ldots = a_{2m+1}, a_2 = a_4 = \ldots = a_{2m+2}$, so $p = 2$. By Example(3.6.1), $a_1 = a_2 = \ldots = a_p$, as claimed. Hence the proof. $\square$

Remark 3.6.2. For the odd period $p = 2m-1$, the argument is simple. Since

$$\begin{vmatrix} 0 & a_1 & 0 & \cdots & 0 \\ a_1 & 0 & a_2 & & \\ 0 & a_2 & \ddots & & \\ \vdots & & \ddots & & a_{2m-1} \\ 0 & & 0 & a_{2m-1} & 0 \end{vmatrix} = (-1)^m \prod_{j=1}^{m} a_{2j-1}^2,$$

$\det(J_k) = (-1)^m \prod\limits_{j=1}^{m} a_{k+2j-1}^2$,for $k = 0, 1, \ldots p-1$. But the left hand side is independent from k. Therefore we have

$$\prod_{j=1}^{m} a_{2j-1}^2 = \prod_{j=1}^{m} a_{2j}^2 = \prod_{j=1}^{m} a_{2j+1}^2 \ldots = \prod_{j=1}^{m} a_{2j+p-1}^2.$$

Using this and $a_{k+p} = a_k$, we can conclude that so $a_1 = a_2 = \ldots a_p$.

Example 3.6.2. Coming back to the symbol associated with discrete Schrodinger operator, the following observations can be made.

$$f(\theta) = \begin{bmatrix} 0 & 1 & & & & e^{i\theta} \\ 1 & 0 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ e^{-i\theta} & & & & 1 & 0 \end{bmatrix} = Z_{e^{-i\theta}} + Z^*_{e^{-i\theta}},$$

where

$$Z_\omega = \begin{bmatrix} 0 & 0 & & & & \omega \\ 1 & 0 & 0 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & \\ 0 & & & & 1 & 0 \end{bmatrix}.$$

The following matrix multiplications can be used to obtain a decomposition of Z_ω.

$$\begin{bmatrix} 1 & 0 & & & & \\ 0 & \delta^{-1} & 0 & & & \\ & 0 & \delta^{-2} & . & & \\ & & . & . & . & \\ & & & . & . & \\ 0 & & & & 0 & \delta^{-p+1} \end{bmatrix} \begin{bmatrix} 0 & 0 & & & & 1 \\ 1 & 0 & 0 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & \\ 0 & & & & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & & & & \\ 0 & \delta & 0 & & & \\ & 0 & \delta^2 & . & & \\ & & . & . & . & \\ & & & . & . & \\ 0 & & & & 0 & \delta^{p-1} \end{bmatrix}$$

$$= \delta^{-1} \begin{bmatrix} 0 & 0 & & & & \delta^p \\ 1 & 0 & 0 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \\ 0 & & & & 1 & 0 \end{bmatrix}.$$

Therefore we have

$$Z_\omega = \delta D_\delta^{-1} Z_1 D_\delta, \text{ where } D_\delta = diag_{j=0,1,\ldots p-1}(\delta^j), \text{ with } \delta^p = \omega.$$

Also, recall that $Z_1 = F_p D_\delta F_p^*$, where $\delta = e^{i\frac{2\pi}{p}}$ and F_p is the Fourier matrix of size p.

$$\text{That is } F_p = \sqrt{\frac{1}{p}}(e^{-i\frac{2\pi jk}{p}})_{j,k=0}^{p-1}.$$

Therefore we get the Jordan decomposition of the symbol as

$$\begin{aligned} f(\theta) &= Z_{e^{-i\theta}} + Z_{e^{-i\theta}}^* = e^{-i\theta/p} D_{e^{-i\theta/p}}^{-1} Z_1 D_{e^{-i\theta/p}} + e^{i\theta/p} D_{e^{-i\theta/p}}^* Z_1^* D_{e^{-i\theta/p}} \\ &= D_{e^{-i\theta/p}}^* F_p [e^{-i\theta/p} D_\delta + e^{i\theta/p} D_\delta^*] F_p^* D_{e^{-i\theta/p}} \\ &= D_{e^{-i\theta/p}}^* F_p diag\left(2\cos\left(\frac{2\pi j - \theta}{p}\right)\right) F_p^* D_{e^{-i\theta/p}} \end{aligned}$$

With reference to the previous notations we observe that for fixed j and p large, we have

$$\lambda_j^+ - \lambda_j^- = \lambda_{p-j}^+ - \lambda_{p-j}^- = O(p^{-2})$$

while, for indices j in a fixed neighborhood of $p/2$, with size independent of p, we have $\lambda_j^+ - \lambda_j^- = O(p^{-1})$. Finally, for all indices j, we obtain $\lambda_j^+ - \lambda_j^- = O(p^{-1})$.

3.6.1 A specific example

The connection between the number of spectral gaps and the period, has to be studied in detail. We give an example below which gives some hints of the connection. We say that J has essential period p if p is the minimal positive integer for which $a_{n+p} = a_n > 0, b_{n+p} = b_n$, for all integer n. Here we give a specific example that supports the general statement that J of essential period p implies that the spectrum is the union of p disjoint intervals. That is the number of spectral gaps is exactly $p-1$.

Consider $f_0(\theta)$ with $a_1 = a_2 = \dots a_p = 1$ and $b_1 = b_2 = \dots b_{p-1} = 0$ and $b_p = 1$. The last relation implies that the essential period is p. Hence the symbol $f_0(\theta)$ is

$$f_0(\theta) = \begin{bmatrix} 0 & 1 & & & & e^{-i\theta} \\ 1 & 0 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ e^{i\theta} & & & & 1 & 1 \end{bmatrix}.$$

Observe that the eigenvalues of $f_0(\theta)$ are separated by those of

$$H_{p-1} = \begin{bmatrix} 0 & 1 & & & & 0 \\ 1 & 0 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ 0 & & & & 1 & 0 \end{bmatrix}.$$

and those of

$$\tilde{H_{p-1}} = \begin{bmatrix} 0 & 1 & & & & 0 \\ 1 & 0 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ 0 & & & & 1 & 1 \end{bmatrix}.$$

with $H_{p-1}, \tilde{H_{p-1}}$ being principal minors of $f_0(\theta)$. In order to prove that there exist exactly $p-1$ gaps, it is enough to prove that the eigenvalue $\lambda_j(\tilde{H_{p-1}})$ is strictly greater than $\lambda_j(H_{p-1})$ for every $j = 1, 2, \ldots p-1$. The matrices $H_{p-1}, \tilde{H_{p-1}}$ are the generators of sine-transform algebras with different boundary conditions and their Jordan form can be explicitly computed.

In both case we observe that the matrices are real symmetric and irreducible so that the use of the first and of the third Gershgorin theorem implies that the eigenvalues belong to the open interval $(-2, 2)$. Let $\lambda(X)$ be a generic eigenvalue of $X \in \{H_{p-1}, \tilde{H_{p-1}}\}$ and let

$$v(X) = (v_1(X), v_2(X), \ldots v_{p-1}(X))$$

be the corresponding normalized eigenvector. The corresponding eigenvalue equation is

$$X.v(X) = \lambda(X).v(X); X \in \{H_{p-1}, \tilde{H_{p-1}}\}.$$

Setting $\lambda(X) = 2cos(\psi)$, we get the following system of linear difference equations.

$$v_{i-1} + v_{i+1} = 2cos(\psi)v_i, i = 1, 2, \ldots p-1.$$

with boundary conditions given by

$$v_0(H_{p-1}) = v_{p-1}(H_{p-1}) = 0 \text{ and } v_0(\tilde{H_{p-1}}) = 0, v_{p-1}(\tilde{H_{p-1}}) = v_p(\tilde{H_{p-1}})$$

respectively. The general solution for this is $v_j = ae^{ij\psi} + be^{-ij\psi}$. Applying boundary conditions for the case of H_{p-1}, we get $a = -b$ and then

$$a(e^{ip\psi} - e^{-ip\psi}) = 0$$

which implies $e^{ip\psi} - e^{-ip\psi} = 2isin(p\psi) = 0$. Hence $p\psi = j\pi$, and therefore $\lambda_j(H_{p-1}) = 2cos(\frac{j\pi}{p})$. In the case of $\tilde{H_{p-1}}$, the use of boundary conditions shows that

$$\lambda_j(\tilde{H_{p-1}}) = 2\cos\left(\frac{\pi\,(2j-1)}{2p-1}\right) > 2\cos\left(\frac{\pi j}{p}\right) = \lambda_j(H_{p-1}),\ j = 1, 2, \ldots p-1.$$

where the strict inequality is true simply because $\frac{\pi(2j-1)}{2p-1} < \frac{\pi j}{p}$ for every $j = 1, 2, \ldots p-1$. Hence the number of spectral gaps is $p-1$.

3.6.2 Remarks

Some observations on the sequence $\{A_n(p)\}$, are made here. Clearly each $A_n(p)$ can be viewed as $T_n(g_j) + R_{n,j}$, where g_j, j=0, $\ldots p-1$, are those reported in(p-symbols) and where the correction term $R_{n,j}$ is Hermitian for every n and j and has rank bounded uniformly by p. Therefore in the light of general perturbation results (see e.g. Proposition 2.3 in [61]) and in the light of Theorem 3.3.2, we have

$$\{A_n(p)\} \sim_\lambda (g_j, (-\pi, \pi)), \qquad j = 0, \ldots, p-1, \tag{3.14}$$

which clearly implies that, for every k, the range of $\lambda_k(g_s)$ coincides with the range of $\lambda_k(g_t)$, for $s, t = 0, \ldots, p-1$. It is evident that (3.14) improves relation (3.9). Moreover, from (3.14), for every function F continuous with bounded support, we obtain that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{p}\sum_{k=1}^{p}F(\lambda_k(g_s(\theta)))d\,\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{p}\sum_{k=1}^{p}F(\lambda_k(g_t(\theta)))d\,\theta, \qquad (3.15)$$

$s, t = 0, \ldots, p-1$, which means that the eigenvalues of every g_s, $s = 0, \ldots, p-1$, induce the same measure on the real line.

Chapter 4

Perturbation and Approximation of spectrum

Perturbation theory of operators incorporates a good deal of spectral theory. In this chapter, the approximation theory of spectrum is discussed in the case of a one parameter family of operators. The basic ingredients are the perturbation theory of linear operators developed by Kato in his celebrated book [43] written in 1966, and the linear algebraic techniques in the approximation of spectrum in [19]. All the results in [19] are considered here under a holomorphic perturbation of operators. It is shown that the bounds of the essential spectrum and the discrete spectral values outside the bounds of a holomorphic family of operators $A(x)$, can be approximated uniformly on all compact subsets by the sequence of eigenvalue functions of $A(x)_n$. The known results for a bounded self-adjoint operator in [19], are translated into the case of a holomorphic family of operators. Also an attempt is made to study the effect of holomorphic perturbation of operators in the prediction of spectral gaps. Finally, some known results on the bounds for eigenvalues of perturbed matrices are used to analyze the symbol of the block Toeplitz-Laurent operator that was used to prove the Borg-type theorems in the last chapter (see [7],[27] for reference).

This chapter is organized as follows. It begins with a brief introduction to the perturbation theory of linear operators, followed by the discussion on the approximation of spectrum of a one parameter family of operators, by truncation method. In the third section we consider the spectral gaps under perturbation. In the last section the symbol associated with some block Toeplitz-Laurent operators, is analyzed using the known results regarding the eigenvalues of perturbed matrices.

4.1 Introduction to Perturbation theory

The basic problem that is addressed here is to investigate the behavior of the spectrum of an operator, when we make a small perturbation of the operator. To deal such problems, one may have to consider a family of operators of the form

$$A(x) = A + xC \tag{4.1}$$

where x varies in some subset of the complex plane, say D_0. The operator $A(0) = A$ is the unperturbed operator and xC is the perturbation. In general, we may suppose that $A(x)$ is an operator-valued function, which is holomorphic in a given domain D_0 of the complex plane. That is for each $y \in D_0$, the following limit

$$\lim_{x \to y} \frac{\|A(x) - A(y)\|}{|x - y|}$$

should exist and must be finite. The aim is to study the changes in the behavior of spectrum, under these perturbations. The approximation techniques used in the case of a single operator, is generalized into a holomorphic family of operators. It is observed that all the main results are preserved under a holomorphic perturbation in a uniform way. The important thing to be noticed is that here the eigenvalues of truncations are sequence of functions. But the holomorphic assumption will help us to guarantee that each functions involved, are continuous. So we will be dealing with sequence of functions, instead of numbers.

4.2 Spectrum under perturbation

Let $A(x)$ be a holomorphic family of operators with domain D_0 in the complex plane. Recall the inequality of approximation numbers. For each $x \in D_0$,

$$\|A(x)\| = s_1(A(x)) \geqslant s_2(A(x)) \geqslant \ldots \geqslant s_k(A(x)) \geqslant \ldots \geqslant 0. \quad (4.2)$$

Before proving the generalized approximation results for holomorphic family of operators, we state Dini's theorem (for the proof, see page 150 of [58]) which is used in the subsequent theorems.

Theorem 4.2.1. *Suppose K is a compact subset of a metric space,* *and*

- $\{f_n\}$ *is a sequence of continuous functions on K,*
- $\{f_n\}$ *converges point wise to a continuous function f on K,*
- $f_n(x) \geqslant f_{n+1}(x)$ *for every x in K,* $n = 1, 2, 3, \ldots$

Then f_n *converges to f uniformly on K.*

Theorem 4.2.2. $s_k(A(.))$ *converges to* $\|A(.)\|_{ess}$ *as* $k \to \infty$*, uniformly on all compact subsets of* D_0*.*

Proof. Consider the sequence of functions $f_k(x) = s_k(A(x))$. Then for each x, by Theorem (2.1.4),

$$f_k(x) = s_k(A(x)) \to \|A(x)\|_{ess}.$$

Also since

$$\begin{aligned} |f_k(x) - f_k(y)| &= |s_k(A(x)) - s_k(A(y))| \\ &\leqslant \|A(x) - A(y)\|, \end{aligned}$$

and since $A(x)$ is holomorphic, we observe that each functions in the sequence are continuous. Hence using the monotonicity of the sequence of functions in (4.2), we conclude that the convergence is uniform in each compact subsets, by Theorem(4.2.1). □

Now consider the truncations $A(x)_n = P_n A(x) P_n$ and singular numbers $s_k(A(x)_n) = \inf\{\|A(x)_n - F_n\|,\ \mathrm{rank}(F_n) \leqslant k-1\}$.

Remark 4.2.1. A similar interlacing theorem as Theorem (3.3.3), holds in the case of singular values. If $s_j(A)$ and $s_j(B)$ are singular values of the matrices A and B respectively, arranged in non increasing order, then

$$\begin{array}{lllll} s_1(A) & \geqslant & s_1(B) & \geqslant & s_3(A), \\ s_2(A) & \geqslant & s_2(B) & \geqslant & s_4(A), \\ \ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ s_{n-2}(A) & \geqslant & s_{n-2}(B) & \geqslant & s_n(A), \\ s_{n-1}(A) & \geqslant & s_{n-1}(B) & \geqslant & 0. \end{array}$$

With these tools, the following theorem can proved.

Theorem 4.2.3. *$s_k(A(x)_n)$ converges to $s_k(A(x))$ as $n \to \infty$, for each k, and the convergence is uniform on all compact subsets of D_0.*

Proof. Our first observation is that the sequence of functions

$$f_{n,k}(x) = s_k(A(x)_n)$$

form an equicontinuous family of functions. This follows from the following inequality.

$$\begin{aligned} &|f_{n,k}(x) - f_{n,k}(y)| = |s_k(A(x)_n) - s_k(A(y)_n)| \leqslant \|A(x)_n - A(y)_n\| \\ &\leqslant \|A(x) - A(y)\|. \end{aligned}$$

Also, from the interlacing theorem for singular values remarked above, we have

$$f_{n,k}(x) = s_k(A(x)_n) \geqslant s_k(A(x)_{n-1}) = f_{n-1,k}(x),$$

for each k and for every $x \in D_0$. Hence the sequence of singular value functions form a monotone sequence of functions. And by Theorem (2.1.5),

$$f_{n,k}(x) = s_k(A\,(x)_n) \to s_k(A\,(x)) \text{ as } n \to \infty,$$

for each k and for all x $\in D_0$. Now by Theorem (4.2.1), the convergence is uniform on all compact subsets of D_0 and the proof is completed. □

For the rest of this chapter, we assume that $A(x)$ is self-adjoint for each x. Let $\nu(x), \mu(x)$ be the lower and upper bounds of $\sigma_e(A(x))$ respectively. Also let

$$\lambda_R^+(A(x)) \leqslant \ldots \leqslant \lambda_2^+(A(x)) \leqslant \lambda_1^+(A(x))$$

be the discrete eigenvalues of $A(x)$ lying above $\mu(x)$ and

$$\lambda_1^-(A(x)) \leqslant \lambda_2^-(A(x)) \leqslant \ldots \leqslant \lambda_S^-(A(x))$$

be the eigenvalues lying below $\nu(x)$. Here R and S can be infinity. The quantities $\lambda_{1,n}(x) \geqslant \lambda_{2,n}(x) \geqslant \ldots \geqslant \lambda_{n,n}(x)$ denote the eigenvalues of $A(x)_n$ in non increasing order.

Theorem 4.2.4.

$$\lim_{n\to\infty} \lambda_{k,n}(x) = \begin{cases} \lambda_k^+(x), & \text{if } R = \infty \text{ or } 1 \leqslant k \leqslant R, \\ \mu(x), & \text{if } R < \infty \text{ and } k \geqslant R+1, \end{cases}$$

$$\lim_{n\to\infty} \lambda_{n+1-k,n}(x) = \begin{cases} \lambda_k^-(x), & \text{if } S = \infty \text{ or } 1 \leqslant k \leqslant S, \\ \nu(x), & \text{if } S < \infty \text{ and } k \geqslant S+1. \end{cases}$$

In particular,

$$\lim_{k\to\infty}\lim_{n\to\infty} \lambda_{k,n}(x) = \mu(x) \text{ and } \lim_{k\to\infty}\lim_{n\to\infty} \lambda_{n+1-k,n}(x) = \nu(x).$$

Furthermore, in each of the cases given above, the convergence is uniform on all compact subsets of D_0.

Proof. For each fixed $x \in D_0$, these limits exist by Theorem (2.1.7). Now observe the fact that the sequence of eigenvalue functions,

$$f_{n,k}(x) = \lambda_{k,n}(x)$$

form an equicontinuous family of functions, from the following inequalities.

$$\begin{aligned} |f_{n,k}(x) - f_{n,k}(y)| &= |\lambda_{k,n}(x) - \lambda_{k,n}(y)| \leqslant \|A(x)_n - A(y)_n\| \\ &\leqslant \|A(x) - A(y)\|. \end{aligned}$$

Also by Cauchy's interlacing theorem for eigenvalues,

$$\lambda_{1,n+1}(x) \geqslant \lambda_{1,n}(x) \geqslant \lambda_{2,n+1}(x) \geqslant \ldots \lambda_{n,n+1}(x) \geqslant \lambda_{n,n}(x) \geqslant \lambda_{n+1,n+1}(x),$$

for each $x \in D_0$. In particular, for each k and for every $x \in D_0$,

$$f_{n+1,k}(x) = \lambda_{k,n+1}(x) \geqslant \lambda_{k,n}(x) = f_{n,k}(x).$$

Hence $f_{n,k}(.)$ forms a monotone sequence of continuous functions that converges point wise. Therefore by Theorem(4.2.1), the convergence is uniform on all compact subsets of D_0. Hence the proof is completed. □

Remark 4.2.2. Using Theorem (4.2.4), we can approximate the discrete spectrum of a holomorphic family of operators, lying outside the bounds of essential spectrum by the eigenvalue functions of truncations uniformly on all compact subsets. This also reveals the following fact. If one wishes to study the effect of perturbation in the spectrum of an operator, it suffices to study the effect in the eigenvalues of truncations of the perturbed operator. Hence the sequence of eigenvalue functions contain much of the discrete spectral information.

It was observed in [19] that norm of ${A_n}^{-1}$ is uniformly bounded if A is invertible and the essential spectrum is connected. The perturbed version of this result is proved below.

Corollary 3. Let $A(x)$ be a holomorphic family of bounded self-adjoint operators such that $\sigma_e(A(x))$ is connected for all x in the domain D_0. Then

$$\lim_{n\to\infty} \left\|(A(x)_n - \lambda I_n)^{-1}\right\| = \left\|(A(x) - \lambda I)^{-1}\right\| \text{ for every } \lambda \in \mathbb{C} - \mathbb{R}.$$

Also the convergence is uniform on all compact subsets of D_0.

Proof. By Theorem (2.1.9), $\sigma(A(x)) = \Lambda(A(x))$. Hence we can easily observe the following.

$d(z, \sigma(A(x)_n)) \to d(z, \Lambda(A(x)) = d(z, \sigma(A(x)))$ for every complex number z.

Therefore, for every non real λ,

$$\left\|(A(x)_n - \lambda I_n)^{-1}\right\| = \frac{1}{d(\lambda, \sigma(A(x)_n))} \to \frac{1}{d(\lambda, \sigma(A(x)))} = \left\|(A(x) - \lambda I)^{-1}\right\|.$$

Also the convergence is uniform on all compact subsets of D_0 as observed in the previous theorems. $\square$

4.3 Gaps under perturbation

Now we look at the spectral gaps that may occur between the bounds of the essential spectrum of a holomorphic family of self-adjoint operators. Recall that the gaps remain invariant under a compact perturbation of the operator. The question that is addressed here is how stable these gaps, under a more general perturbation. Also the stability of the spectral gap predictions under a holomorphic perturbation, is another question to be addressed here.

The **stability theorem of bounded invertibility** is stated below and it will be used to achieve some invariance for the gaps. The theorem is stated in a more general form in [43]. We need only the following special case.

Theorem 4.3.1. *Let A and B are bounded operators and A is invertible. If $\|A^{-1}\| \, \|B\| < 1$, then $A+B$ is also invertible.*

The following theorem is an immediate consequence of the stability theorem stated above.

Theorem 4.3.2. *Let (a, b) be a gap in $\sigma_e(A(0))$ which contains no discrete spectral value in it. Then for all small enough $\varepsilon > 0$, there exists a $\delta > 0$ such that $(a + \varepsilon, b - \varepsilon)$ is a gap in the essential spectrum of the analytic family of operators $A\,(x)$ for every x with $|x| < \delta$.*

Proof. First we note that, $A - \lambda I$ is invertible for every λ in the interval (a, b), since it contains no spectral value.

Therefore,

$$\sup\left\{\left\|(A-\lambda I)^{-1}\right\|; \lambda \in (a+\varepsilon, b-\varepsilon)\right\} = M_\epsilon < \infty \text{ for a fixed } \varepsilon > 0.$$

Now using the continuity assumption, corresponding to minimum of $\{\frac{1}{M_\epsilon}, \epsilon\}$, there exists a $\delta > 0$, such that

$$\|(A(x) - A(0)\| < \min\{\frac{1}{M_\epsilon}, \epsilon\} \text{ for every x with } |x| < \delta.$$

Now for $|x| < \delta$, observe that

$$\left\|(A-\lambda I)^{-1}\right\| \|A(x) - A(0)\| < M_\epsilon . \frac{1}{M_\epsilon} < 1$$

for every λ in the interval $(a+\varepsilon, b-\varepsilon)$.

Hence by Theorem (4.3.1), if $|x| < \delta$, then

$$A(x) - \lambda I = A(x) - A(0) + A(0) - \lambda I$$

is invertible for every λ in the interval $(a+\varepsilon, b-\varepsilon)$. Therefore the interval $(a+\varepsilon, b-\varepsilon)$ does not intersect with $\sigma(A\,(x))$, for every x with $|x| < \delta$.

Now, since $\|A(x) - A(0)\| < \epsilon$, $(a+\varepsilon, b-\varepsilon)$ will lie between the bounds of $\sigma_e(A(x))$, for every x with $|x| < \delta$. We conclude that $(a+\varepsilon, b-\varepsilon)$ is a spectral gap in $\sigma_e(A(x))$ for all x, with $|x| < \delta$. □

Remark 4.3.1. In Theorem (4.3.2), ϵ must be small enough so that the interval $(a+\varepsilon, b-\varepsilon)$ should makes sense. This theorem indicates that to some extend, the gaps are stable under small norm perturbation. Once we get $(a+\varepsilon, b-\varepsilon)$ is a gap, we may remove that interval and look at the rest of the interval (a, b) and continue the search for gaps.

Let's look at an example to support the above theorem.

Example 4.3.1. Define a two parameter family of matrix valued symbols as follows

$$f(x,\theta) = \begin{bmatrix} a_1(x) & 1 & & & & e^{-i\theta} \\ 1 & a_2(x) & 1 & & & \\ & 1 & a_3(x) & 1 & & \\ & & 1 & a_4(x) & 1 & \\ & & & \ddots & \ddots & \ddots \\ e^{i\theta} & & & & 1 & a_p(x) \end{bmatrix},$$

where $a_1(.), a_2(.) \dots a_p(.)$ are analytic functions defined on complex domains which have nonempty intersection with real line. Also the functions are real valued on the real line, and θ varying in the interval $[0, 2\pi]$. Note that

$$f(x,\theta) = A_0(x) + A_1 e^{i\theta} + A_{-1}e^{-i\theta}, \text{ where}$$

$$A_0(x) = \begin{bmatrix} a_1(x) & 1 & & & \\ 1 & a_2(x) & 1 & & \\ & 1 & \cdot\ \cdot & & \\ & & \cdot\ \cdot\ \cdot & & \\ & & \cdot\ \cdot & 1 & \\ & & & 1 & a_p(x) \end{bmatrix},$$

$$A_1 = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} = {A_{-1}}^T.$$

Consider the one parameter family of block Toeplitz-Laurent operators arising from these symbols, which are represented by the following doubly infinite matrices,

$$A(x) = \begin{bmatrix} \ddots & \ddots & & & & & & \\ \ddots & A_0(x) & A_{-1} & & & & & \\ & A_1 & A_0(x) & A_{-1} & & & & \\ & & A_1 & A_0(x) & A_{-1} & & & \\ & & & A_1 & A_0(x) & A_{-1} & & \\ & & & & A_1 & A_0(x) & A_{-1} & \\ & & & & & A_1 & A_0(x) & \ddots \\ & & & & & & \ddots & \ddots \end{bmatrix}$$

Thus we get an analytic family of bounded operators, $A(x)$ which are self-adjoint for all real x in the domain. Now, by Theorem (3.4.1), the essential spectrum of $A(x_0) : x_0$ real, has no gaps if and only if $a_1(x_0) = a_2(x_0) \ldots = a_p(x_0)$. Hence if there is a gap in $\sigma_e(A(0))$ then $a_i(0) \neq a_{i+1}(0)$ for some i. Using the continuity of a_i and a_{i+1}, we can find a $\delta > 0$ such that $a_i(x) \neq a_{i+1}(x)$ for all x with $|x| < \delta$. Hence there is a gap in the essential spectrum of $A(x)$ for all such x lying in the interval $(-\delta, +\delta)$.

Remark 4.3.2. To check whether two complex analytic functions are identical, it suffices to check on a sequence with one limit point in the set. This can be used to detect spectral gaps in the above example.

Using Theorem (2.2.1) and Theorem (4.3.2), we arrive at the following conclusions. The gap predictions that we have done for a single operator, are remain valid for a family of operators. The advantage is that we can predict gaps of a family of operators, with assumptions only on the unperturbed operator. We give the precise statement below.

Corollary 4. Let A(x) be a holomorphic family of operators with $A(0) = A$, and $\lambda_{n1}(A_n) \geqslant \lambda_{n2}(A_n) \geqslant ... \geqslant \lambda_{nn}(A_n)$ be the eigenvalues of A_n arranged in decreasing order. For each positive integer n, let

$\{w_{nk} : k = 1, 2, ...n\}$ be a set of numbers such that $0 \leqslant w_{nk} \leqslant 1$

and $\sum\limits_{k=1}^{n} w_{nk} = 1$. If there exists a $\delta > 0$ and $K > 0$ such that

$$\#\left\{\lambda_{nj};\left|\sum_{k=1}^{n} w_{nk}\lambda_{nk} - \lambda_{nj}\right| < \delta\right\} < K,$$

and in addition, if we assume that $\sigma_e(A)$ and $\sigma(A)$ coincide, then $\sigma_e(A(x))$ has a gap for each x in a sufficiently small neighborhood of 0.

4.4 Perturbation of matrices

In this section, we recall some classical results on the eigenvalues of perturbed matrices and use them to get some estimates, which may be useful in the practical implementation of the Borg-type theorems, proved in the last chapter. Recall that in the Borg-type theorems, the matrix valued symbol,

$$\tilde{f}(\theta) = \begin{bmatrix} b_1 & 1 & & & & f(\theta) \\ 1 & b_2 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ f(\bar{\theta}) & & & & 1 & b_p \end{bmatrix}.$$

had a crucial role to play. But computing the eigenvalue functions of this symbol, may not be easy in general. Hence the application of the theorems may be difficult in the computational aspects. The symbol may be identified as a one parameter family of matrices which is the perturbation of a matrix with constant entries. Also the following lemma gives the bounds for the eigenvalues of perturbed matrices (see [7],[27] and references there in). Hence we try to strengthen our results by inputting these bounds to our matrix valued symbol.

Lemma 4.4.1. Let $H = \begin{pmatrix} H_1 & E \\ E^* & H_1 \end{pmatrix}$ and $\tilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_1 \end{pmatrix}$,

$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_p$ and $\tilde{\lambda_1} \geqslant \tilde{\lambda_2} \geqslant \ldots \tilde{\lambda_p}$ be the eigenvalues respectively.

Then

$$\left|\lambda_j - \tilde{\lambda_j}\right| \leqslant \|E\| \tag{4.3}$$

$$\left|\lambda_j - \tilde{\lambda_j}\right| \leqslant \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}, \text{ where } \eta = \min_{\mu_i \in \sigma(H_1), \mu_i \in \sigma(H_2)} |\mu_i - \tilde{\mu_i}|. \tag{4.4}$$

Theorem 4.4.1. *Let A be the operator considered in Theorem (3.5.1). If $\lambda_1 \geqslant \lambda_2 \geqslant \dots \lambda_p$ are eigenvalues of the matrix*

$$\begin{bmatrix} b_1 & 1 & & & & \\ 1 & b_2 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ & & & & 1 & b_p \end{bmatrix},$$

and $\sigma_{\mathrm{ESS}}(A)$ has no gap, then

$$|\lambda_j - \lambda_{j+1}| \leqslant 2 \|f\|_\infty \text{ for every } j = 1, 2 \dots p-1.$$

Furthermore, if we assume that $b_1 \leqslant b_2 \dots \leqslant b_p$, then

$$|\lambda_j - \lambda_{j+1}| \leqslant 2 \text{ for every } j = 1, 2 \dots p-1.$$

Proof. Apply above lemma with

$$H(.) = \begin{bmatrix} b_1 & 1 & & & & f(\theta) \\ 1 & b_2 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ f\bar{(\theta)} & & & & 1 & b_p \end{bmatrix}, \tilde{H} = \begin{bmatrix} b_1 & 1 & & & & \\ 1 & b_2 & 1 & & & \\ & 1 & . & . & & \\ & & . & . & . & \\ & & & . & . & 1 \\ & & & & 1 & b_p \end{bmatrix}$$

and

$$E = \begin{bmatrix} & & f(\theta) \\ & & \\ & & \\ & & \\ & & \end{bmatrix}.$$

Then we get

$$|\lambda_j - \lambda_j(H(\theta)| \leqslant \|E\| = \|f\|_\infty \text{ by (4.3).}$$

Combining with (3.6), we get

$$\sigma_{\text{ess}}(A) = \bigcup_{j=1}^{p} \left[\inf_\theta(\lambda_j(H(\theta))), \sup_\theta(\lambda_j(H(\theta)))\right] \subseteq \bigcup_{j=1}^{p} \left[\lambda_j - \|f\|_\infty, \lambda_j + \|f\|_\infty\right].$$

Therefore if $|\lambda_j - \lambda_{j+1}| > 2\|f\|_\infty$ for some j, then there exists a gap in the essential spectrum. Hence we proved the first assertion.

Now in addition, if we assume that $b_1 \leqslant b_2 \ldots \leqslant b_p$, then since the essential spectrum of A is connected, by Theorem (3.4.1), $b_1 = b_2 \ldots = b_p$. This implies that $\tilde{H}$ is a tridiagonal Toeplitz matrix with b_1 on diagonal and 1 as off diagonal entry. The eigenvalues of such matrices are explicitly known and they are $b_1 + 2cos(\frac{\pi j}{p+1})$. So the second assertion follows by the following computation.

$$\begin{aligned} |\lambda_j - \lambda_{j+1}| &= 2\left|cos(\frac{\pi j}{p+1}) - cos(\frac{\pi(j+1)}{p+1})\right| \\ &= 2\left|2sin(\frac{\pi(2j+1)}{2(p+1)})sin(\frac{\pi}{2(p+1)})\right| \leqslant 4\left|sin(\frac{\pi}{2(p+1)})\right|. \end{aligned}$$

Since $\left|sin(\frac{\pi}{2(p+1)})\right| \leqslant \frac{1}{2}$, for $p \geqslant 2$, we get the desired conclusion. $\square$

We can apply the same technique in the case of general Jacobi matrices to predict gaps in the essential spectrum.

Corollary 5. Let J be the Jacobi matrix defined by (3.12), and let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_p$ are eigenvalues of the matrix

$$\begin{bmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & a_2 & & & \\ & a_2 & . & . & & \\ & & . & . & . & \\ & & & . & . & a_{p-1} \\ & & & & a_{p-1} & b_p \end{bmatrix}.$$

Then $\sigma_{\text{ess}}(J)$ has a gap, if $|\lambda_j - \lambda_{j+1}| > 2\,|a_{p-1}|$ for some j.

Proof. The proof is an imitation of the proof of Theorem (4.4.1), however we provide all the details. Apply Lemma (4.4.1) with

$$H(.) = \begin{bmatrix} b_1 & a_1 & & & & a_{p-1}e^{i\theta} \\ a_1 & b_2 & a_2 & & & \\ & a_2 & . & . & & \\ & & . & . & . & \\ & & & . & . & a_{p-1} \\ a_{p-1}e^{-i\theta} & & & & a_{p-1} & b_p \end{bmatrix}, \tilde{H} = \begin{bmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & a_2 & & & \\ & a_2 & . & . & & \\ & & . & . & . & \\ & & & . & . & a_{p-1} \\ & & & & a_{p-1} & b_p \end{bmatrix},$$

and

$$E = \begin{bmatrix} & & a_{p-1}e^{i\theta} \\ & & \\ & & \\ & & \end{bmatrix}.$$

Then we get

$$|\lambda_j - \lambda_j(H(\theta)| \leqslant \|E\| = \left\|a_{p-1}e^{i\theta}\right\|_\infty = |a_{p-1}| \text{ by (4.3).}$$

Combining with (3.13), we get

$$\sigma_{\text{ess}}(J) = \bigcup_{j=1}^{p} \left[\inf_\theta(\lambda_j(H(\theta))), \sup_\theta(\lambda_j(H(\theta)))\right] \subseteq \bigcup_{j=1}^{p} \left[\lambda_j - |a_{p-1}|\,, \lambda_j + |a_{p-1}|\right].$$

Therefore if $|\lambda_j - \lambda_{j+1}| > 2\,|a_{p-1}|$ for some j, then there exists a gap in the essential spectrum. Hence the proof. □

Remark 4.4.1. The last couple of results help us to reduce the computations in predicting spectral gaps, for operators arising from the matrix valued symbols. We need to check only the eigenvalues of a matrix with constant entries. The proof also gives us the spectral inclusion

$$\sigma_{\text{ess}}(A) \subseteq \bigcup_{j=1}^{p} \left[\lambda_j - \|f\|_\infty\,, \lambda_j + \|f\|_\infty\right].$$

which is very important, since the right hand side includes only the eigenvalues of a constant matrix. Whether equality holds in this inclusion, is still not clear to us.

Chapter 5

Preconditioners and Korovkin-type theorems

The sequences of matrices related to operators acting on infinite dimensional Hilbert spaces, were considered so far. But usually these matrix sequences need not be simpler enough to make the computations easy. If a different matrix sequence can be used to do the same approximation, without loosing much spectral data, then it will be helpful to make the computations easier. In many cases it will be convenient to look at a different sequence of matrices associated with the original sequence that we deal with, which are comparatively simpler to handle. When doing so, the results obtained with the new sequence must be sufficient for the spectral information of the original operator under concern. The advantage is that we are able to confine our attention to "simpler" matrices without loosing much spectral data. In a heuristic language, such new sequences of matrices are called preconditioners. Recall that in the numerical linear algebra literature, preconditioners are used to reduce the condition number and to make the iteration methods much more efficient.

The notion of preconditioners, introduced in this chapter, is an extension of the notions available in the case of Toeplitz matrices, to the setting of operators acting on separable Hilbert spaces. Also we introduce the new notions of convergence induced by strong, weak and uniform eigenvalue clustering of matrix sequences with growing order. It is observed that the asymptotic of the preconditioners with respect to the new notions of convergence, contains much of the spectral information of a bounded self-adjoint operator. Hence we hope that the usage of preconditioners may be useful in the spectral approximations that we discussed in the previous chapters.

The classical as well as noncommutative Korovkin-type theorems deal with the convergence of positive linear maps with respect to modes of convergences such as norm convergence and weak operator convergence. In this chapter, new versions of Korovkin-type theorems are proved with respect to the notions of convergence induced by strong, weak and uniform eigenvalue clustering of matrix sequences with growing order. Such modes of convergence were originally considered for the special case of Toeplitz matrices by Stefano Serra Capizzano and Tyrtyshnikove (see [60, 65]). Also the Korovkin-type approach, in the setting of preconditioning large linear systems with Toeplitz structure is well known (see [60]). Here we translate such finite dimensional approach into the infinite dimensional context of operators acting on separable Hilbert spaces. The asymptotic of these preconditioners are obtained and analyzed using the concept of completely positive maps (CP-maps). It is observed that any two limit points of the same sequence of preconditioners are the same modulo compact operators.

The chapter is organized as follows. It begins with an introduction to the basic notions and the classical as well as non commutative versions of Korovkin-type theorems. In the second section, the new notion of preconditioners in an operator theory setting and new modes of convergence of positive linear maps in the strong, weak and uniform distribution sense are introduced. The new versions of non commutative Korovkin-type theorems are proved in the next section. The special case of Toeplitz operators and the example of Frobenius optimal maps are considered in the fourth section. Also, the stronger versions of the finite dimensional results in [60] are proved. Finally, a discussion on the possible applications of the main results to the spectral approximation problems is presented.

5.1 Preliminaries

The notions of preconditioners and convergence of matrix sequences using the clustering of eigenvalues, is used in connection with the Frobenius optimal approximation of matrices of large size which has been widely considered in the numerical linear algebra literature for the design of efficient solvers of complicated linear systems of large size. More specifically, the approximation is constrained in spaces of low complexity: as examples of high interest in several important applications (see [24], [38] and references therein), we may mention algebras of matrices associated to fast transforms like Fourier, Trigonometric, Hartley, Wavelet transforms ([44], [68]) or we may mention spaces with prescribed patterns of sparsity. In the context of general linear systems, accompanied with the minimization in Frobenius norm, these techniques were originally considered and studied by Huckle (see [9] and references there reported), while the specific adaptation in the Toeplitz context started with the work of Tony Chan in [66].

More recently, a unified structural analysis was introduced by Stefano Serra Capizzano, in connection with the Korovkin theory, which represents a nice branch of the theory of functional approximation. More precisely, the analysis of clustering of the preconditioned systems which gives a measure for the approximation quality is reduced to classical Korovkin test set of a finite number of very elementary symbols associated to equally elementary Toeplitz matrices. Here the same approach is considered in an operator theory setting. The Korovkin-type approach used in the finite dimensional case in the setting of preconditioning large linear systems with Toeplitz structure is translated in the infinite dimensional context of operators acting on separable Hilbert spaces.

We begin with the classical Korovkin's theorem.

Theorem 5.1.1. *Let $\{\Phi_n\}$ be a sequence of positive linear maps on $C[0,1]$. If*

$$\Phi_n(f) \to f \text{ for every } f \text{ in the set } \{1, x, x^2\},$$

then

$$\Phi_n(f) \to f \text{ for every } f \text{ in } C[0,1].$$

Here the convergence is the uniform convergence of sequence of functions. For the noncommutative versions of this theorem, we need the notion of *completely positive maps* and *Schwarz maps.*

Definition 5.1.1. Let $\mathbb{A}$ and $\mathbb{B}$ be C^* algebras with identities $1_{\mathbb{A}}$ and $1_{\mathbb{B}}$ respectively and $\Phi : \mathbb{A} \to \mathbb{B}$ be a positive linear map such that $\Phi(1_{\mathbb{A}}) \leqslant (1_{\mathbb{B}})$. For each positive integer n, let $\Phi_n : M_n(\mathbb{A}) \to M_n(\mathbb{B})$ be defined as $\Phi_n(a_{i,j}) = (\Phi(a_{i,j}))$ for every matrix $(a_{i,j}) \in M_n(A)$. If Φ_n is positive for each n, then Φ is called *completely positive.*

Remark 5.1.1. Let $CP\,(\mathbb{A}, \mathbb{B})$ denote the class of all completely positive maps (CP-maps) Φ from $\mathbb{A}$ to $\mathbb{B}$ such that $\Phi(1_{\mathbb{A}}) \leqslant (1_{\mathbb{B}})$. All such maps will have norm less than or equal to 1. Then it is well known that $CP\,(\mathbb{A}, \mathbb{B})$ is compact and convex in the Kadison's B.W topology [1], if $\mathbb{B}$ is a sub algebra of $B(\mathbb{H})$. Convergence in the B.W topology is defined as follows.

$$\Phi_\alpha \to \Phi \text{ in the B.W topology means } \Phi_\alpha(A) \to \Phi(A),$$

in the weak operator topology for every A in the C^* algebra $\mathbb{A}$.

Remark 5.1.2. Let $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ be three C^* algebras. If Φ is in $CP\,(\mathbb{A}, \mathbb{B})$ and Ψ in $CP\,(\mathbb{B}, \mathbb{C})$, then the composition $\Psi \circ \Phi$ is in $CP\,(\mathbb{A}, \mathbb{C})$.

One of the most fundamental result, the Stinespring dilation Theorem [62] is stated below.

Theorem 5.1.2. *Let $\mathbb{A}$ and $\mathbb{B}$ be C^* algebras with identities $1_{\mathbb{A}}$ and $1_{\mathbb{B}}$ respectively. Let $\Phi : \mathbb{A} \to \mathbb{B}$ be a completely positive linear map such that $\Phi(1_{\mathbb{A}}) \leqslant (1_{\mathbb{B}})$. Assume that $\mathbb{B}$ is a sub algebra of $\mathbb{B}(\mathbb{H})$ for some Hilbert space $\mathbb{H}$. Then there exists a representation π of $\mathbb{A}$ on a Hilbert space $\mathbb{K}$ and a bounded linear map V from $\mathbb{H}$ to $\mathbb{K}$ such that $\Phi\,(a) = V^* \pi\,(a)\, V$ for every $a \in \mathbb{A}$.*

Remark 5.1.3. It is known that if either $\mathbb{A}$ or $\mathbb{B}$ is commutative then every positive linear map is completely positive.

Recall that any positive linear map $\Phi : \mathbb{A} \to \mathbb{B}$ with $\Phi(1_{\mathbb{A}}) \leqslant (1_{\mathbb{B}})$, satisfies the well known inequality of Kadison, namely,

$$\Phi(a^2) \geqslant \Phi(a)^2, \text{ for every } a \text{ such that } a = a^*.$$

Definition 5.1.2. A positive linear map Φ from a C^* algebra $\mathbb{A}$ to a C^* algebra $\mathbb{B}$ is called a *Schwarz map* if $\Phi(a^*a) \geqslant \Phi(a^*)\Phi(a)$ for all a in $\mathbb{A}$.

Remark 5.1.4. It can be easily seen that every completely positive map of norm less than 1, is a *Schwarz map*. Also, a Schwarz map is clearly positive and contractive. If the C^*-algebra $\mathbb{A}$ is commutative, then a positive contractive map is a Schwarz map.

Remark 5.1.5. In the case of an arbitrary C^*-algebra $\mathbb{A}$, a positive linear map Ψ with $\Psi(1) \leqslant 1$ was called a *Jordan-Schwarz* map in [6], since it satisfies the following inequality.

$$\Phi(a^* \circ a) \geqslant \Phi(a^*) \circ \Phi(a) \text{ for all } a \text{ in } \mathbb{A},$$

where $\circ$ is the Jordan product defined by $a \circ b = \frac{1}{2}(ab + ba)$.

Definition 5.1.3. [6] a $*-$ closed and norm-closed subspace of a C^*-algebra $\mathbb{A}$, which is also closed with respect to the Jordan product called a J^*-sub algebra of $\mathbb{A}$.

The non commutative versions of the classical Korovkin's theorem have been obtained by various researchers for positive maps, Schwarz maps and CP-maps, in the settings C^*-algebras and W^*-algebras. A short survey of these developments can be found in [56]. For example, the following is such a version proved in [48].

Theorem 5.1.3. *Let the sequence of Schwarz maps Φ_n, from $\mathbb{A}$ to $\mathbb{B}$ be such that $\Phi_n(1_{\mathbb{A}}) \leqslant 1_{\mathbb{B}}$. Then the set*

$$C = \{a \in \mathbb{A}; \Phi_n(a) \to a, \Phi_n(a^*a + aa^*) \to (a^*a + aa^*)\}$$

is a C^*-algebra.

The following theorem, taken from [47], can be thought of as an exact analogue of classical Korovkin's theorem where the convergence is the weak operator convergence.

Theorem 5.1.4. *Let $\{\Phi_\lambda\}$ be a net of CP-maps on $\mathbb{B}(\mathbb{H})$, where $\mathbb{H}$ is a separable, complex Hilbert space. If*

$$\Phi_\lambda(A) \to A \text{ for every } A \text{ in the set } \{I, S, SS^*\}$$

where S is the unilateral right shift operator on $\mathbb{H}$, then

$$\Phi_\lambda(A) \to A, \text{ for all } A \text{ in } \mathbb{B}(\mathbb{H}),$$

where the mode of convergence is the weak operator convergence.

The concept of generalized Schwarz map was introduced by Uchiyama in [67]. Below, the definition and an important inequality (Theorem 2.1

in [67]) are given, which will play a crucial role in the proof of new versions of Korovkin-type theorems.

Consider a binary operation $\circ$ in a C^*-algebra $\mathbb{A}$, satisfying the following properties for every $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathbb{A}$.

1. $(\alpha x + \beta y) \circ z = \alpha(x \circ z) + \beta(y \circ z)$.
2. $(x \circ y)^* = y^* \circ x^*$.
3. $x^* \circ x \geqslant 0$.
4. There is a real number M such that $\|x \circ y\| \leqslant M \|x\| \|y\|$.
5. $(x \circ y) \circ z = x \circ (y \circ z)$
6. $(x \circ y) = (y \circ x)$ and $x \circ x = x^2$ if $x = x^*$.

Remark 5.1.6. Note that $\circ$ is bilinear and that the ordinary product satisfies (5) and the Jordan product (6). Conversely if $\circ$ satisfies (6), then $\circ$ is the Jordan product.

Definition 5.1.4. A linear map Φ on a C^*-algebra $\mathbb{A}$ is called a *generalized Schwarz map* with respect to the binary operation $\circ$, if Φ satisfies $\Phi(x^*) = \Phi(x)^*$ and $\Phi(x^*) \circ \Phi(x) \leqslant \Phi(x^* \circ x)$ for every $x \in \mathbb{A}$.

Remark 5.1.7. Note that a generalized Schwarz map Φ is not necessarily positive. However, under point wise product in function spaces and with usual product of operators or matrices, all Schwarz maps are positive.

Theorem 5.1.5. *[67] Let Φ be a generalized Schwarz map on a C^* algebra $\mathbb{A}$ with respect to $\circ$, and for $f, g \in \mathbb{A}$, if we let*

$$\begin{aligned} X = &\quad \Phi(f^* \circ f) - \Phi(f)^* \circ \Phi(f) \quad \geqslant 0 \\ Y = &\quad \Phi(g^* \circ g) - \Phi(g)^* \circ \Phi(g) \quad \geqslant 0 \\ Z = &\quad \Phi(f^* \circ g) - \Phi(f)^* \circ \Phi(g) \end{aligned}$$

Then we have

$$| \phi(Z)| \leqslant | \phi(X)|^{1/2}. | \phi(Y)|^{1/2}, \text{ for all state } \phi \text{ on } \mathbb{A} \tag{5.1}$$

$$\text{In particular, } \| Z \| \leqslant \| X \|^{1/2} \| Y \|^{1/2} \tag{5.2}$$

Remark 5.1.8. The above inequality holds for Schwarz maps with respect to usual product and for contractive positive maps with respect to the Jordan product.

Now we introduce the notion of preconditioners and new modes of convergence in an operator theory setting.

5.2 Pre-conditioners and convergence of CP-maps

Let $\{P_n\}$ be a sequence of orthogonal projections on $\mathbb{H}$ such that

$$\dim(P_n(\mathbb{H})) = n < \infty, \text{ for each } n = 1, 2, 3 \ldots$$

$$\text{and } \lim_{n \to \infty} P_n(x) = x, \text{ for every x in } \mathbb{H}.$$

Let $\{U_n\}$ be a sequence of unitary matrices over $\mathbb{C}$, where U_n is of order n for each n. For each $A \in \mathbb{B}(\mathbb{H})$, consider the following truncations $A_n = P_n A P_n$, which can be regarded as $n \times n$ matrices in $M_n(\mathbb{C})$, by restricting the domain to the range of P_n. For each n, we define the commutative algebra M_{U_n} of matrices as follows.

$$M_{U_n} = \{A \in M_n(\mathbb{C})\,; {U_n}^* A U_n \text{ complex diagonal}\}$$

Recall that $M_n(\mathbb{C})$ is a Hilbert space with the Frobenius norm,

$$\|A\|_2^2 = \sum_{j,k=1}^{n} |A_{j,k}|^2$$

induced by the classical Frobenius scalar product,

$$\langle A, B\rangle = \text{trace } (B^* A)$$

with trace(.) being the trace of its argument. That is the sum of all its diagonal entries. Observe that M_{U_n} is a closed convex set in $M_n(\mathbb{C})$ and hence, corresponding to each $A \in M_n(\mathbb{C})$, there exists a unique matrix $P_{U_n}(A)$ in M_{U_n} such that

$$\|A - X\|_2^2 \geqslant \|A - P_{U_n}(A)\|_2^2 \text{ for every } X \in M_{U_n}.$$

We recall the following two lemmas, which reveal some fundamental properties of the map P_{U_n} for each n.

Lemma 5.2.1. [60] With A,B $\in M_n(\mathbb{C})$ and α, β complex numbers, we have

$$P_{U_n}(A) = U_n \sigma\left({U_n}^* A U_n\right) {U_n}^* \tag{5.3}$$

where $\sigma(X)$ is the diagonal matrix having X_{ii} as the diagonal elements.

$$P_{U_n}(\alpha A + \beta B) = \alpha P_{U_n}(A) + \beta P_{U_n}(B) \tag{5.4}$$

$$P_{U_n}(A^*) = P_{U_n}(A)^* \tag{5.5}$$

$$\text{Trace} P_{U_n}(A) = \text{Trace}(A) \tag{5.6}$$

$$\|P_{U_n}(A)\| = 1 \text{ (Operator norm)} \tag{5.7}$$

$$\|P_{U_n}(A)\|_F = 1 \text{(Frobenius norm)} \tag{5.8}$$

$$\|A - P_{U_n}(A)\|_F{}^2 = \|A\|_F{}^2 - \|P_{U_n}(A)\|_F{}^2 \tag{5.9}$$

Lemma 5.2.2. [10] If A is a Hermitian matrix, then the eigenvalues of $P_{U_n}(A)$ are contained in the closed interval $[\lambda_1(A), \lambda_n(A)]$, where $\lambda_j(A)$ are the eigenvalues of A arranged in a non decreasing way. Hence if A is positive definite, then $P_{U_n}(A)$ is positive definite as well.

Now we define the generalized notion of preconditioners on $\mathbb{B}(\mathbb{H})$ as follows.

Definition 5.2.1. For each $A \in \mathbb{B}(\mathbb{H})$, $\Phi_n : \mathbb{B}(\mathbb{H}) \to \mathbb{M}_n(\mathbb{C})$ is defined as

$$\Phi_n(A) = P_{U_n}(A_n),$$

where $P_{U_n}(A_n)$ is as in Lemma (5.2.1), for each positive integer n.

We may call $\Phi_n(A)$, the *preconditioners* of A. One of the straightforward but crucial implications of Lemma (5.2.1) is the following theorem.

Theorem 5.2.1. *The maps $\{\Phi_n\}$ in the Definition (5.2.1), is a se-*

quence of completely positive maps on $\mathbb{B}(\mathbb{H})$ *such that*

- $\|\Phi_n\| = 1,$ *for each* n.
- Φ_n *is continuous in the strong topology of operators for each n.*
- $\Phi_n(I) = I_n$ *for each n where I is the identity operator on* $\mathbb{H}$.

Proof. From Lemma (5.2.2), it follows that $P_{U_n}(.)$ is a positive linear map for each n. Since M_{U_n} is a commutative Banach algebra, $P_{U_n}(.)$ is a completely positive map for each n. Hence Φ_n is a completely positive map, since it is the composition of CP-maps ($P_{U_n}(.)$ and the maps which send A to P_nAP_n and pull back). Now continuity in the strong operator topology follows easily from the definition and the remaining part of the theorem follows from the following observations.

$$\|\Phi_n\| = \sup_{\|A\|=1, A\in\mathbb{B}(\mathbb{H})} \|\Phi_n(A)\| = \sup_{\|A\|=1, A\in B(H)} \|P_{U_n}(A_n)\| = 1$$

by the identity (5.7) in Lemma (5.2.1). The last part of the theorem follows easily from the identity (5.3) of Lemma (5.2.1). □

In the next section, Korovkin-type theorems for completely positive maps are proved with respect to various types of clustering of eigenvalues. The completely positive maps that arises from preconditioners discussed above can be considered as an example.

The different notions of convergence of sequence of positive linear maps on $B(\mathbb{H})$ in a *distributional* sense are introduced below. These definitions are motivated from the different notions of convergence for preconditioners in the Toeplitz case done in [60]. To avoid confusion with the classical notion of strong, weak and operator norm convergence, we address them by ***strong cluster,*** ***weak cluster***, and ***uniform cluster*** to mean the strong, weak and uniform convergence respectively used in [60].

Definition 5.2.2. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ Hermitian matrices. We say that $A_n - B_n$ converges to 0 in ***strong cluster*** if for any $\epsilon > 0$, there exist integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all the singular values $\sigma_j(A_n - B_n)$ lie in the interval $[0, \epsilon)$ except for at most $N_{1,\epsilon}$ (independent of the size n) eigenvalues for all $n > N_{2,\epsilon}$.

If the number $N_{1,\epsilon}$ does not depend on ϵ, we say that $A_n - B_n$ converges to 0 in ***uniform cluster***. And if $N_{1,\epsilon}$ depends on ϵ, n and is of $o(n)$, we say that $A_n - B_n$ converges to 0 in ***weak cluster***.

The following powerful lemma is due to Tyrtyshnikov (see Lemma (3.1) in [65]).

Lemma 5.2.3. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ Hermitian matrices. If $\| A_n - B_n \|_F{}^2 = O(1)$, then we have convergence in strong cluster. If $\| A_n - B_n \|_F{}^2 = o(n)$ then the convergence is in weak cluster.

Using the above notions, we introduce the new notions of convergence of positive linear maps on $B(\mathbb{H})$.

Definition 5.2.3. Let $\{\Phi_n\}$ be a sequence of positive linear maps on $B(\mathbb{H})$ and P_n be a sequence of orthogonal projections on $\mathbb{H}$ with rank n such that P_n converges strongly to the identity map. We say that $\{\Phi_n(A)\}$ converges to A in the ***strong distribution sense***, if the sequence of matrices $\{P_n\Phi_n(A)P_n\} - \{P_nAP_n\}$ converges to 0 in strong cluster as per Definition (5.2.2).

Similarly We say that $\{\Phi_n(A)\}$ converges to A in the ***weak distribution sense (uniform distribution sense respectively)***, if the sequence of matrices $\{P_n\Phi_n(A)P_n\}-\{P_nAP_n\}$ converges to 0 in weak cluster (uniform cluster respectively) as per Definition (5.2.2).

Remark 5.2.1. The above definitions make sense only in the case when A is a self-adjoint operator in $B(\mathbb{H})$. In the non self-adjoint case, one may have to translate things in to the language of $\epsilon-$discs instead of intervals. But we are dealing with the self-adjoint case only. Also the definitions depend on the choice of P_n's.

Remark 5.2.2. In the case of nets, the definitions are the same with convergence in terms of directed set.

Consider a sequence of CP-maps $\{\Phi_n\}$ in $B(\mathbb{H})$ with $\|\Phi_n\| \leqslant 1$. By the compactness of $CP(\mathbb{B}(\mathbb{H}))$, in the Kadison's B.W topology, $\{\Phi_n\}$ has limit points. Let Ω be the set of all limit points of $\{\Phi_n\}$. Next we discuss some properties of the limit points Φ in Ω . The relation between $\Phi(A)$ and A for $A \in \mathbb{B}(\mathbb{H})$ are considered here.

Lemma 5.2.4. Let $\Phi \in \Omega$ and let $\{\Phi_{n_\alpha}\}$ be a subnet of $\{\Phi_n\}$ such that Φ_{n_α} converges to Φ in the Kadison's B.W topology. Then for each m, the truncations $\Phi_{m,n_\alpha}(A) = P_m\Phi_{n_\alpha}(A)P_m$ converges uniformly in norm to

$P_m\Phi(A)P_m$. That is $\lim\limits_{\alpha} \|\Phi_{m,n_\alpha}(A) - P_m\Phi(A)P_m\| = 0$.

Proof. This follows immediately since P_m is of finite rank and therefore, on range(P_m), weak, strong and operator norm topologies coincide. □

Remark 5.2.3. For each $A \in \mathbb{B}(\mathbb{H})$, note that $A_{n_\alpha} - \Phi_{n_\alpha}(A)$ converges in the strong operator topology to $A - \Phi(A)$. Hence,

$$P_m A_{n_\alpha} P_m - P_m\Phi_{n_\alpha}(A)P_m \text{ converges to } P_m A P_m - P_m\Phi(A)P_m$$

in the norm topology for each m.

The above observations can be used to deduce the following result.

Theorem 5.2.2. *Let $A \in \mathbb{B}(\mathbb{H})$ be self-adjoint and $\Phi_n(A) - A$ converges to 0 in uniform distribution sense as in Definition (5.2.3). Then $A - \Phi(A)$ is finite rank.*

Proof. By assumption $P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha} - A_{n_\alpha}$ converges to 0 in uniform cluster, as in the Definition (5.2.2). Hence for each $\epsilon > 0$, there exist a β_ϵ in the directed set and N such that

$$\#(\sigma(A_{n_\alpha} - P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha}) \cap \mathbb{R} - (-\epsilon, +\epsilon)) \leqslant N, \text{ whenever } \alpha > \beta_\epsilon.$$

Therefore by Cauchy interlacing theorem,

$$\#(\sigma(P_m(A_{n_\alpha} - P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha})P_m) \cap \mathbb{R} - (-\epsilon, +\epsilon)) \leqslant N, \text{ if } \alpha > \beta_\epsilon \text{ and } n_\alpha \geqslant m.$$

As we apply limit over α, since by the above remark, $P_m(A_{n_\alpha} - P_{U_{n_\alpha}}(A_{n_\alpha}))P_m$ converges to $P_m(A - \Phi(A))P_m$ in the operator norm topology for every m,

$$\#(\sigma(P_m(A - \Phi(A))P_m) \cap \mathbb{R} - (-\epsilon, +\epsilon)) \leqslant N, \text{ for every } m.$$

Therefore $\mathbb{R} - (-\epsilon, +\epsilon)$ contains no essential points of $A - \Phi(A)$ and hence by Arveson's Theorem (2.1.2), it contains no essential spectral values of $A - \Phi(A)$. That is the essential spectrum $\sigma_e(A - \Phi(A))$ is contained in the interval $(-\epsilon, +\epsilon)$ for all $\epsilon > 0$. This implies that $\sigma_e(A - \Phi(A)) = \{0\}$. Hence $A - \Phi(A)$ is compact and it has at most N eigenvalues. Hence it is finite rank by spectral theorem. □

The above theorem is very important from a spectral theory point of view. Since $\Phi(A)$ is one of the limits of the preconditioners of A, the above theorem says that the change in the operator, when we move to preconditioners is not more than a finite rank perturbation if you look at a uniform limit point. We list down the properties preserved by this change.

Remark 5.2.4. Under the assumptions that $A \in \mathbb{B}(\mathbb{H})$ is self-adjoint and $\Phi_n(A)$ converges to A in uniform distribution sense as in Definition (5.2.3), the following results are easy consequences of the above theorem:

- A is compact if and only if $\Phi(A)$ is compact.
- A is Fredholm if and only if $\Phi(A)$ is Fredholm.
- A is Hilbert Schimidt if and only if $\Phi(A)$ is Hilbert Schimidt.
- A is of finite rank if and only if $\Phi(A)$ is of finite rank.

- A has a gap in the essential spectrum $\sigma_e(A)$ of A if and only if $\sigma_e(\Phi(A))$ has a gap.

In the next theorem, we observe that if the mode of convergence is strong, then the change to preconditioners, amounts a compact perturbation.

Theorem 5.2.3. *Let $A \in \mathbb{B}(\mathbb{H})$ be self-adjoint and $\Phi_n(A)$ converges to A in strong distribution sense as in Definition (5.2.3). Then $A - \Phi(A)$ is compact.*

Proof. The proof is not much different from the proof of Theorem (5.2.2). All the arguments are same, except the fact that here the number of eigenvalues of $(A_{n_\alpha} - P_{n_\alpha}\Phi_{n_\alpha}(A)P_{n_\alpha})$, outside $(-\epsilon, +\epsilon)$, is not bounded by a constant, but by a number $N_{1,\epsilon}$, which depends on ϵ. Hence we can conclude that $A - \Phi(A)$ is compact, and can have countably infinite number of eigenvalues. □

Remark 5.2.5. Under the assumptions that $A \in \mathbb{B}(\mathbb{H})$ is self-adjoint and $\Phi_n(A)$ converges to A in strong distribution sense as in Definition (5.2.3), the following results are easy consequences of the above theorem:

- A is compact if and only if $\Phi(A)$ is compact.
- A is Fredholm if and only if $\Phi(A)$ is Fredholm.
- A has a gap in the essential spectrum $\sigma_e(A)$ of A if and only if $\sigma_e(\Phi(A))$ has a gap.

Remark 5.2.6. Hence $(\Phi(A))$ contains much of the essential spectral information of A.

5.2.1 Modified preconditioners

It is interesting to observe that the notion of preconditioners can be modified by replacing *'diagonal transformation'* by *'block diagonal transformation'* which is obtained by applying the pinching functions defined below.

Definition 5.2.4. [7] Let P_{n_k} be a family of m_n pairwise orthogonal projections in $M_n(C)$, such that $\sum_{k=1}^{m_n} P_{n_k} = I_n$, the identity matrix. Then the operation of taking A to $\sum_{k=1}^{m_n} P_{n_k} A P_{n_k}$ is called the pinching function.

Let $M_{U_n} = \{A \in M_n(\mathbb{C})\,; {U_n}^* A U_n \ \text{ is block diagonal}\}$, where the block diagonal is obtained for each A in $M_n(C)$ by applying pinching function to A for each n. The modified preconditioner on $M_n(C)$ takes values

$$\Psi_n(A) = \sum_{k=1}^{m_n} P_{n_k} A P_{n_k} \text{ for every } A \in M_n(C). \tag{5.10}$$

From Stinespring's theorem, it is clear that the maps Ψ_n's are CP-maps. Now if we define $P_{U_n}(A)$ in a similar way with M_{U_n} replaced by M_{U_n}, we can formulate an analogue of Lemma (5.2.1).

Lemma 5.2.5. With A,B $\in M_n(\mathbb{C})$, we have

$$P_{U_n}(A) = U_n \Psi_n\left({U_n}^* A U_n\right) {U_n}^* \text{ where } \Psi_n \text{ is as in (5.10).}$$

$$P_{U_n}(\alpha A + \beta B) = \alpha P_{U_n}(A) + \beta P_{U_n}(B)$$

$$P_{U_n}(A^*) = P_{U_n}(A)^*$$

$$Trace P_{U_n}(A) = Trace(A)$$

$$\|P_{U_n}(A)\| = 1 \text{ (Operator norm)}$$

$$\|P_{U_n}(A)\|_F = 1 \text{(Frobenius norm)}$$

$$\|A - P_{U_n}(A)\|_F{}^2 = \|A\|_F{}^2 - \|P_{U_n}(A)\|_F{}^2$$

We shall list down some of the properties of the maps $\{\Psi_n\}$ as we did in Theorem (5.2.1).

Theorem 5.2.4. *The maps $\{\Psi_n\}$ is a sequence of completely positive maps on $\mathbb{B}(\mathbb{H})$ such that*

- $\|\Psi_n\| = 1$, *for each n.*
- Ψ_n *is continuous in the strong topology of operators.*
- $\Psi_n(I) = I_n$ *for each n, where I is the identity operator on $\mathbb{H}$.*

Remark 5.2.7. The above mentioned modified version of preconditioners are better than the previous one in the sense that the modified version is closer to the operator in the Frobenius norm and is simpler enough also.

We may construct examples for the modified preconditioners as follows.

Example 5.2.1. Let $\tilde{U_n}$ be unitaries in $M_n(C)$ as in Definition (5.2.2). For each positive integer n, let U_n be unitaries in $\mathbb{B}(\mathbb{H})$ defined as $U_n \bigoplus (I - P_n)$. Observe that there are many interesting, concrete examples of unitaries U_n in [60]. For the sake of completeness, we quote them below.

Let $v = \{v_n\}_{n\in N}$ with $v_n = (v_{nj})_{j\leqslant n-1}$ be a sequence of trigonometric functions on an interval I. Let $S = \{S_n\}_{n\in N}$ be a sequence of grids of n points on I, namely, $S_n = \{x_i^n, i = 0, 1, \dots n-1\}$. Let us suppose that the generalized Vandermonde matrix

$$V_n = (v_{nj}(x_i^n))_{i;j=0}^{n-1}$$

is a unitary matrix. Then, algebra of the form M_{U_n} is a trigonometric algebra if $U_n = V_n{}^*$ with V_n a generalized trigonometric Vandermonde matrix.

We get examples of trigonometric algebras with the following choice of the sequence of matrices U_n and grid S_n. These examples will be considered in section 5.4.2.

$$\begin{aligned}
U_n &= F_n = \left(\frac{1}{\sqrt{n}} e^{ijx_i^n}\right) \ , \quad i,j = 0,1,\dots n-1, \\
S_n &= \left\{x_i^n = \frac{2i\pi}{n}, i = 0,1,\dots n-1\right\} \subset I = [-\pi, \pi] \\
U_n &= G_n = \left(\sqrt{\frac{2}{n+1}} sin(j+1)x_i^n\right) \ , \quad i,j = 0,1,\dots n-1, \\
S_n &= \left\{x_i^n = \frac{(i+1)\pi}{n+1}, i = 0,1,\dots n-1\right\} \subset I = [0, \pi] \\
U_n &= H_n = \left(\frac{1}{\sqrt{n}} \left[sin(jx_i^n) + cos(jx_i^n)\right]\right) \ , \quad i,j = 0,1,\dots n-1, \\
S_n &= \left\{x_i^n = \frac{2i\pi}{n}, i = 0,1,\dots n-1\right\} \subset I = [-\pi, \pi]
\end{aligned}$$

5.3 Korovkin-type Theorems

The noncommutative Korovkin-type theorems are proved in this section. We begin with the noncommutative analogue of the remainder estimate in the classical Korovkin-type theorems, as proved in [60] for the Toeplitz operators.

Lemma 5.3.1. Let $\{A_1, A_2, \ldots A_m\}$ be a finite set of operators in $\mathbb{B}(\mathbb{H})$ and Φ_n be a sequence of positive linear Schwarz maps on $\mathbb{B}(\mathbb{H})$ such that $\|\Phi_n\| \leqslant 1$, for every n and $\|\Phi_n(A) - A\| = O(\theta_n)$, for every A in the set $D = \{A_1, A_2, \ldots A_m, \sum_{k=1}^{m} A_k {A_k}^*\}$, where $\theta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then $\|\Phi_n(A) - A\| = O(\theta_n)$ for every A in the algebra generated by $\{A_1, A_2, \ldots A_m\}$.

Proof. We have by linearity,

$$\Phi_n(\sum_{k=1}^{m} A_k {A_k}^*) = \sum_{k=1}^{m} \Phi_n(A_k {A_k}^*).$$

Also by adding and subtracting the term $\sum_{k=1}^{m} \Phi_n(A_k)\Phi_n(A_k)^*$,

$$\Phi_n\left(\sum_{k=1}^{m} A_k A_k^*\right) - \sum_{k=1}^{m} A_k A_k^* = \left[\Phi_n\left(\sum_{k=1}^{m} A_k A_k^*\right) - \sum_{k=1}^{m} \Phi_n\left(A_k\right)\Phi_n\left(A_k\right)^*\right] + \left[\sum_{k=1}^{m} \Phi_n\left(A_k\right)\Phi_n\left(A_k\right)^* - \sum_{k=1}^{m} A_k A_k^*\right]$$

The norm of left side of the above equation as well as of the last term of the right side are of $O(\theta_n)$. The first term of the right side is

$$\sum_{k=1}^{m}[\Phi_n(A_k {A_k}^*) - \Phi_n(A_k)\Phi_n(A_k)^*]$$

Hence norm of this term is of $O(\theta_n)$. But each term inside this sum are nonnegative operators by Schwarz inequality for positive linear maps. Therefore norm of each term namely $\Phi_n(A_k {A_k}^*) - \Phi_n(A_k)\Phi_n(A_k)^*$ is of $O(\theta_n)$. Also since each Φ_n is a Schwarz map, by applying inequality (5.2) to maps Φ_n for each n and operators A_k and A_l, we get

$$\|\Phi_n({A_k}^* A_l) - \Phi_n(A_k)^*\Phi_n(A_l)\| = O(\theta_n). \tag{5.11}$$

Also, we can manipulate $\|{A_k}^* A_l - \Phi_n(A_k)^*\Phi_n(A_l)\|$ as follows.

$$\begin{aligned}
&\|A_k^* A_l - \Phi_n\left(A_k\right)^* \Phi_n\left(A_l\right)\| = \\
&\|\left(A_k - \Phi_n\left(A_k\right) + \Phi_n\left(A_k\right)\right)^* A_l - \Phi_n\left(A_k\right)^* \left(\Phi_n\left(A_l\right) - A_l + A_l\right)\| \\
&\leqslant \|\left(A_k - \Phi_n\left(A_k\right)\right)^* A_l\| + \|\Phi_n\left(A_k\right)^* \left(\Phi_n\left(A_l\right) - A_l\right)\|
\end{aligned}$$

Now each of the terms in the last sum is of $O(\theta_n)$, since by the assumption on A_k, A_l and since $\|\Phi_n\| \leqslant 1$. Therefore we have

$$\|{A_k}^* A_l - \Phi_n(A_k)^*\Phi_n(A_l)\| = O(\theta_n) \tag{5.12}$$

Also we have the following identity:

$$\begin{aligned}
\|\Phi_n({A_k}^* A_l) - {A_k}^* A_l\| &= \\
\|\Phi_n({A_k}^* A_l) - \Phi_n(A_k)^*\Phi_n(A_l) + \Phi_n(A_k)^*\Phi_n(A_l) - {A_k}^* A_l\|
\end{aligned}$$

Applying (5.11) and (5.12) in the above identity, we get

$$\|\Phi_n({A_k}^* A_l) - {A_k}^* A_l\| = O(\theta_n)$$

Therefore the proof is completed for every operator of the form ${A_k}^* A_l$ and hence in the algebra generated by the finite set $\{A_1, A_2, \ldots A_m\}$. Hence the proof. □

Before proving the more general Korovkin-type theorems, we prove the following lemma, which is useful for us.

Lemma 5.3.2. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ Hermitian matrices such that $\{A_n\} - \{B_n\}$ converges to 0 in strong cluster (weak cluster respectively). Assume that $\{B_n\}$ is positive definite and invertible such that there exists a $\delta > 0$, with

$$B_n \geqslant \delta I_n > 0, \quad \text{for all n.}$$

Then for a given $\epsilon > 0$, there will exist positive integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all eigenvalues of ${B_n}^{-1} A_n$ lie in the interval $(1 - \epsilon, 1 + \epsilon)$ except possibly for $N_{1,\epsilon} = O(1)$ ($N_{1,\epsilon} = o(n)$ respectively) eigenvalues for every $n > N_{2,\epsilon}$.

Proof. First we observe that, since $\{A_n\} - \{B_n\}$ converges to 0 in strong cluster (weak cluster respectively), by definition, for any given $\epsilon > 0$, there exists integers $N_{1,\epsilon}, N_{2,\epsilon}$ such that all eigenvalues of $A_n - B_n$ lie in the interval $(-\epsilon, \epsilon)$ except for at most $N_{1,\epsilon}(N_{1,\epsilon} = o(n)$ respectively) eigenvalues whenever $n \geqslant N_{2,\epsilon}$. Hence by spectral theorem there exist orthogonal projections P_n and Q_n whose ranges are orthogonal such that

$$\operatorname{rank}(P_n) + \operatorname{rank}(Q_n) = n,\ \operatorname{rank}(Q_n) \leqslant N_{1,\epsilon},\ \| P_n(A_n - B_n)P_n \| < \varepsilon$$

$$\text{and} \quad A_n - B_n = P_n(A_n - B_n)P_n + Q_n(A_n - B_n)Q_n$$

Hence for $\epsilon_1 = \epsilon.\delta > 0$, there exists natural numbers $N_{1,\epsilon}, N_{2,\epsilon}$ with the following decomposition.

$$A_n - B_n = R_n + N_n, \text{ for all } n \geqslant N_{2,\epsilon}, \tag{5.13}$$

where the rank of R_n is bounded above by $N_{1,\epsilon}$ and $\|N_n\| \leqslant \epsilon_1$. Now let β be an eigenvalue of ${B_n}^{-1}A_n$ with x being the associated eigenvector of norm one. Then we have

$${B_n}^{-1}A_n(x) = \beta x.$$

Hence,

$$(A_n - B_n)(x) = (\beta - 1)B_n(x).$$

Which implies that

$$\langle (A_n - B_n)(x), x\rangle = (\beta - 1)\langle B_n(x), x\rangle.$$

And

$$\beta - 1 = \frac{\langle (A_n - B_n)(x), x\rangle}{\langle B_n(x), x\rangle}$$

Now from the decomposition (5.13), we have

$$\beta - 1 = \frac{\langle (R_n + N_n)(x), x\rangle}{\langle B_n(x), x\rangle} = \frac{\langle R_n(x), x\rangle}{\langle B_n(x), x\rangle} + \frac{\langle N_n(x), x\rangle}{\langle B_n(x), x\rangle}$$

Now since $\|N_n\| \leqslant \epsilon_1$ and $B_n \geqslant \delta I_n > 0$, the second term in the last sum is less than $\frac{\epsilon_1}{\delta} = \epsilon$. Also since rank of R_n is bounded above by $N_{1,\epsilon} = O(1)(o(n)$ respectively), there are only at most $N_{1,\epsilon}$ linearly independent vectors x for which $R_n(x) \neq 0$, by rank-nullity theorem. Hence, except for at most $N_{1,\epsilon} = O(1)(o(n)$ respectively) eigenvalues,

$$|\beta - 1| \leqslant \epsilon.$$

This means that all eigenvalues of ${B_n}^{-1}A_n$ lie in the interval$(1-\epsilon, 1+\epsilon)$ except possibly for $N_{1,\epsilon} = O(1)(o(n)$ respectively). This completes the proof. □

Now we prove our main result of this chapter, the non commutative versions of Korovkin-type theorems. Here the $\circ$ denotes the Jordan product of operators or matrices.

Theorem 5.3.1. *Let $\{A_1, A_2, \ldots A_m\}$ be a finite set of self-adjoint operators on $\mathbb{H}$ and Φ_n be a sequence of contractive positive maps on $B(\mathbb{H})$, such that $\Phi_n(A)$ converges to A in the strong (or weak respectively) distribution sense, for A in $\left\{A_1, A_2, \ldots A_m, {A_1}^2, {A_2}^2, \ldots {A_m}^2\right\}$. In addition, if we assume that the difference $P_n({A_k}^2)P_n - (P_n(A_k)P_n)^2$ converges to the 0 matrix in strong cluster (weak cluster respectively), for each k, then $\Phi_n(A)$ converges to A in the strong (or weak respectively) distribution sense, for all A in the J^*- sub algebra $\mathbb{A}$ generated by $\{A_1, A_2, A_3, \ldots .A_m\}$.*

Proof. First we consider the following sequence of Hermitian matrices.

$$\begin{aligned} X_n &= P_n\Phi_n(A_k^2)P_n - (P_n\Phi_n(A_k)P_n)^2 \ \geqslant 0 \\ Y_n &= P_n\Phi_n(A_l^2)P_n - (P_n\Phi_n(A_l)P_n)^2 \ \geqslant 0 \\ Z_n &= P_n\Phi_n(A_k \circ A_l)P_n - (P_n\Phi_n(A_k)P_n) \circ (P_n\Phi_n(A_l)P_n) \end{aligned}$$

Since these sequences of matrices are norm bounded, we have

$$\| Y_n \| < \gamma < \infty \ \text{ for all n, for some } \gamma > 0. \tag{5.14}$$

Also if we write

$$\begin{aligned} X_n &= P_n\Phi_n(A_k^2)P_n - (P_n\Phi_n(A_k)P_n)^2 \\ &= [P_n\Phi_n(A_k^2)P_n - P_n(A_k^2)P_n] + [P_n(A_k^2)P_n - (P_n(A_k)P_n)^2] \\ &+ \ [(P_n(A_k)P_n)^2 - (P_n\Phi_n(A_k)P_n)^2], \end{aligned}$$

the first two terms in the above sum, converges to 0 in strong cluster (weak respectively) by assumption. Also since $P_n\Phi_n(A_k)P_n - P_n(A_k)P_n = R_n + N_n$, where R_n and N_n are as in the proof of Lemma (5.3.2), we have the following.

$$\begin{aligned} (P_n\Phi_n(A_k)P_n) - (P_n(A_k)P_n) &= R_n + N_n \\ ((P_n\Phi_n(A_k)P_n) - R_n)^2 &= ((P_n(A_k)P_n) + N_n)^2 \end{aligned}$$

From the above identity, we can deduce that $(P_n(A_k)P_n)^2 - (P_n\Phi_n(A_k)P_n)^2 = R_n' + N_n'$ where R_n' has bounded rank and N_n' has small norm as required for the convergence in strong cluster (weak respectively) to 0. Hence the third term also converges to the 0 matrix in strong cluster (weak respectively). Therefore X_n converges to the 0 matrix in strong cluster (weak respectively).

Now for each fixed x with $\|x\| = 1$, if we consider the state ϕ_x on $B(\mathbb{H})$ defined as

$$\phi_x(A) = \langle A(x), x \rangle$$

then by the inequality (5.1) applied to the contractive positive maps $P_n\Phi_n(.)P_n$, we get

$$|\langle Z_n(x), x \rangle| \leqslant |\langle X_n(x), x \rangle|^{1/2}.|\langle Y_n(x), x \rangle|^{1/2} \tag{5.15}$$

Now let $\delta > 0$, be given and $\epsilon = \delta^2/\gamma$, as in the proof of Lemma (5.3.2), there exists integers $N_{1,\epsilon} = O(1)$ ($N_{1,\epsilon} = o(n)$ respectively) and $N_{2,\epsilon}$ such that we have the following decomposition

$$X_n = N_n + R_n \text{ for all } n > N_{2,\epsilon},$$

with $\|N_n\| < \epsilon$ and rank of R_n is less than $N_{1,\epsilon} = O(1)$ ($N_{1,\epsilon} = o(n)$ respectively). Applying this and (5.14) in the inequality (5.15), we get

$$|\langle Z_n(x), x \rangle| \leqslant \sqrt{\gamma}.[|\langle N_n(x), x \rangle|^{1/2} + |\langle R_n(x), x \rangle|^{1/2}] \text{ (for all } n > N_{2,\epsilon})$$

Since the rank of R_n is bounded above by $N_{1,\epsilon} = O(1)(o(n)$ respectively), there are only at most $N_{1,\epsilon}$ linearly independent vectors x for which $R_n(x) \neq 0$, by rank-nullity theorem. Hence, $|\langle Z_n(x), x \rangle| \leqslant \delta$, except for at most $N_{1,\epsilon} = O(1)(o(n)$ respectively) linearly independent vectors x. Therefore all eigenvalues of Z_n, except for possibly $N_{1,\epsilon} = O(1)(o(n)$ respectively), lie in the interval $(-\delta, \delta)$, whenever $n > N_{2,\epsilon}$. Since $\delta > 0$, was arbitrary, Z_n converges to the 0 matrix in strong cluster (weak respectively).

Now consider

$$P_n\Phi_n\left(A_k \circ A_l\right)P_n - P_n\left(A_k \circ A_l\right)P_n = [P_n\Phi_n\left(A_k \circ A_l\right)P_n -$$

$$\left(P_n\Phi_n\left(A_k\right)P_n\right) \circ \left(P_n\Phi_n\left(A_l\right)P_n\right)] + [\left(P_n\Phi_n\left(A_k\right)P_n\right) \circ \left(P_n\Phi_n\left(A_l\right)P_n\right) -$$

$$\left(P_n\left(A_k\right)P_n\right) \circ \left(P_n\left(A_l\right)P_n\right)] + [\left(P_n\left(A_k\right)P_n\right) \circ \left(P_n\left(A_l\right)P_n\right) - P_n\left(A_k \circ A_l\right)P_n]$$

The first term on the right hand side of the above equation is Z_n and the last term is also in the form of Z_n for the positive contractive maps $P_n(.)P_n$ on $B(\mathbb{H})$. Therefore both the terms converges to the 0 matrix in strong cluster (weak respectively). By simple computation, it can be proved that the middle term also converges to the 0 matrix in strong cluster (weak respectively). Hence the theorem is proved for the operators of the form $A_k \circ A_l$.

The same proof can be repeated for operators of the form $A_j \circ (A_k \circ A_l)$, using the boundedness of $A_k \circ A_l$ and convergence assumption on A_j in strong cluster (weak respectively). Continuing like this inductively, we get the assertion is true for any operator in the form of a polynomial in $\{A_1, A_2, A_3, \ldots .A_m\}$, with respect to the Jordan product.

Now for $A \in \mathbb{A}$, $\epsilon > 0$, let T be the operator in the form of a polynomial in $\{A_1, A_2, A_3, \ldots .A_m\}$, with respect to the Jordan product, such that

$$\| A - T \| < \epsilon/3, \text{ and } \| \Phi_n(A) - \Phi_n(T) \| < \epsilon/3.$$

Consider the following equation:

$$\begin{aligned} P_n\Phi_n(A)P_n - P_nAP_n &= [P_n\Phi_n(A)P_n - P_n\Phi_n(T)P_n] + [P_n\Phi_n(T)P_n - P_nTP_n] \\ &+ [P_nTP_n - P_nAP_n] \end{aligned}$$

Thus the norm of the sum of the first and third terms is less than $2\epsilon/3$. The middle term $P_n\Phi_n(T)P_n - P_nTP_n$ can be split into a term with norm less than $\epsilon/3$ and a term with constant rank independent of the order n (or of o(n) respectively) since T is in the form of a polynomial in $\{A_1, A_2, A_3, \dots .A_m\}$, with respect to the Jordan product. Thus the sequence of matrices $P_n\Phi_n(A)P_n - P_nAP_n$ converges to 0 in strong cluster (or in weak cluster respectively). Hence the proof is completed. □

Remark 5.3.1. Note that even if A_k and A_l are self-adjoint, their composition need not be self-adjoint. But the Jordan product of two self-adjoint elements is self-adjoint. The proof of the above theorem uses this fact.

5.4 Toeplitz case

In this section, we will be dealing with the completely positive maps Φ_n, that we introduced in Definition (5.2.1), and its modifications. We use them to get stronger versions of the Korovkin-type theorems in [60] and provide examples for the results in last section.

Consider the sequence $\{\Phi_n\}$ of Definition (5.2.1). By the compactness of $CP(\mathbb{B}(\mathbb{H}))$, in the Kadison's B.W topology, $\{\Phi_n\}$ has limit points. We note some of the properties of the limit points Φ of $\{\Phi_n\}$, as immediate consequences of Theorem (5.2.2) and Theorem (5.2.3). The following special cases are of interest from a spectral theory point of view.

- A is a Hilbert Schmidt operator on $\mathbb{H}$.
- $A = A(f)$, is the Toeplitz operator where the symbol function $f \in C\,[-\pi, \pi]$ and $\mathbb{H} = L^2\,[-\pi, \pi]$.
- A is a Fredholm or compact operator.

Theorem 5.4.1. *Let $A \in \mathbb{B}(\mathbb{H})$ be self-adjoint and $P_{U_n}(A_n) - A_n$ converges to 0 in uniform cluster as in Definition (5.2.2). Then $A - \Phi(A)$ is finite rank.*

Proof. Follows easily from Theorem (5.2.2), by considering

$$\Phi_n(A) = P_{U_n}(A_n).$$

□

Theorem 5.4.2. *Let $A \in \mathbb{B}(\mathbb{H})$ be self-adjoint and $P_{U_n}(A_n) - A_n$ converges to 0 in strong cluster as in Definition 5.2.2. Then $A - \Phi(A)$ is compact.*

Proof. Follows easily from Theorem (5.2.2), by considering

$$\Phi_n(A) = P_{U_n}(A_n).$$

□

Remark 5.4.1. The analysis of convergence in 'weak cluster', in the sense of Def (5.2.2) is taken up later in this chapter.

5.4.1 Korovkin-type theory for Toeplitz operators

Now we consider the case where $A = A(f)$, is the Toeplitz operator where the symbol function $f \in C[-\pi, \pi]$ and $\mathbb{H} = L^2[-\pi, \pi]$. Some of the results in [60] are generalized and get stronger versions. First we recall the Korovkin-type results in [60]. The notation $A_n(f)$ is used for the finite Toeplitz matrix with symbol f.

Theorem 5.4.3. *Let f be a continuous periodic real-valued function. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in strong cluster, if $P_{U_n}(A_n(p)) - A_n(p)$ converges to 0 in strong cluster for all the trigonometric polynomials p.*

Theorem 5.4.4. *Let f be a continuous periodic real-valued function. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in weak cluster if $P_{U_n}(A_n(p)) - A_n(p)$ converges to 0 in weak cluster for all the trigonometric polynomials p.*

Before proving the general results, we prove the following lemma, the remainder estimate version of classical Korovkin's theorem as proved in [60], which is used to get more general versions of Theorems (5.4.3) and (5.4.4). This is the commutative version of Lemma (5.3.1).

Lemma 5.4.1. Let $\{g_1, g_2, \ldots g_m\}$ be a finite set of continuous periodic functions and Φ_n be a sequence of positive linear maps on $C[0, 2\pi]$

such that $\|\Phi_n\| \leqslant 1$, for every n, and

$$\Phi_n(g) = g + O(\theta_n) \text{ for every g in the set } D = \{g_1, g_2, \dots g_m, \sum_{k=1}^{m} g_k {g_k}^*\},$$

where $\theta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then $\Phi_n(g) = g + O(\theta_n)$ for every g in the algebra generated by $\{g_1, g_2, \dots g_m\}$.

Proof. The proof is obtained by replacing functions in place of operators, in the proof of Lemma (5.3.1). Using linearity of Φ_n's, we write,

$$\begin{aligned} \Phi_n(\textstyle\sum_{k=1}^m g_k {g_k}^*) - \sum_{k=1}^m g_k {g_k}^* &= (\textstyle\sum_{k=1}^m \Phi_n(g_k {g_k}^*) - \sum_{k=1}^m \Phi_n(g_k)\Phi_n(g_k)^*) \\ &+ (\textstyle\sum_{k=1}^m \Phi_n(g_k)\Phi_n(g_k)^* - \sum_{k=1}^m g_k {g_k}^*) \end{aligned}$$

The left side of the above equation as well as the last term of the right side are of $O(\theta_n)$. Hence the first term of the right side

$$\sum_{k=1}^{n} [\Phi_n(g_k {g_k}^*) - \Phi_n(g_k)\Phi_n(g_k)^*]$$

is of $O(\theta_n)$. But each term inside this sum is nonnegative by Schwarz inequality for positive linear maps. Therefore each of its terms, namely $\Phi_n(g_k {g_k}^*) - \Phi_n(g_k)\Phi_n(g_k)^*$ is of $O(\theta_n)$. Also since every positive contractive map in a commutative C^* algebra is a Schwarz map, each Φ_n is a Schwarz map. Therefore by applying inequality (5.2) to the maps Φ_n for each n and functions g_k, g_l, we get

$$\Phi_n({g_k}^* g_l) - \Phi_n(g_k)^* \Phi_n(g_l) = O(\theta_n)$$

Also, we observe the following

$$\Phi_n(g_k)^*\Phi_n(g_l) - {g_k}^* g_l = ({g_k}^* + O(\theta_n))(g_l + O(\theta_n)) - {g_k}^* g_l = O(\theta_n)$$

Using the above two identities, we deduce that

$$\Phi_n({g_k}^* g_l) - {g_k}^* g_l = [\Phi_n({g_k}^* g_l) - \Phi_n(g_k)^*\Phi_n(g_l)] + [\Phi_n(g_k)^*\Phi_n(g_l) - {g_k}^* g_l] = O(\theta_n)$$

Therefore the proof is completed for every function of the form ${g_k}^* g_l$ and hence in the algebra generated by $\{g_1, g_2, \ldots g_m\}$. Hence the proof. □

Now we prove some general versions of Theorems (5.4.3) and (5.4.4). The technique of the proof is the same as in Theorem (5.3.1). However we provide all the details.

Theorem 5.4.5. *Let $\{g_1, g_2, \ldots g_m\}$ be a finite set of real valued continuous 2π periodic functions such that $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in strong cluster, for f in $\{g_1, g_2, \ldots g_m, {g_1}^2, {g_2}^2, \ldots {g_m}^2\}$. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in strong cluster for all f in the C^*- algebra $\mathbb{A}$ generated by $\{g_1, g_2, g_3, \ldots . g_m\}$.*

Proof. For any $k, l = 1, 2, 3 \ldots m$, setting

$$\begin{aligned} X_n &= P_{U_n}(A_n(g_k^2)) - P_{U_n}(A_n(g_k))^2 \ \geqslant 0 \\ Y_n &= P_{U_n}(A_n(g_l^2)) - P_{U_n}(A_n(g_l))^2 \ \geqslant 0 \\ Z_n &= P_{U_n}(A_n(g_k^* \circ g_l)) - P_{U_n}(A_n(g_k))^* \circ P_{U_n}(A_n(g_l)) \end{aligned}$$

(Here $\circ$ denotes the usual point wise product in the case of scalar valued functions and matrix product in the case of matrices.) Observe that X_n, Y_n

and Z_n are all Hermitian matrices of order n. It is clear that all the above sequences of matrices are norm bounded. Then for all n

$$\| Y_n \| < \gamma < \infty \tag{5.16}$$

Also if we write

$$\begin{aligned} X_n = \Phi_n(g_k^2) - \Phi_n(g_k)^2 &= [\Phi_n(g_k^2) - A_n(g_k^2)] + [A_n(g_k^2) - A_n(g_k)^2] \\ &+ [A_n(g_k)^2 - \Phi_n(g_k)^2], \end{aligned}$$

the first term on the right hand side of the above equation converges to 0 in strong cluster by assumption. The second term is

$$A_n(g_k^2) - A_n(g_k)^2 = P_n H(g_k)^2 P_n + Q_n H(g_k)^2 Q_n \tag{5.17}$$

where Q_n's are projections and $H(g_k)$ is the Hankel operator, which is compact, since the symbols are continuous. This equality is due to Widom (page 2 of [70]). Hence $A_n(g_k^2) - A_n(g_k)^2$ can be written as the sum of sequences of matrices that are truncations of compact operators. But the compliment of any neighborhood of 0 contains only finitely many eigenvalues of a compact operator, as its spectral values. Also the truncations of a compact operator on a separable Hilbert space converges to the operator in norm. Therefore we conclude that $A_n(g_k^2) - A_n(g_k)^2$ converges to the 0 matrix in strong cluster.

Since $\Phi_n(g_k) - A_n(g_k) = R_n + N_n$ with R_n and N_n are sequences of matrices with properties that we already mentioned before, the third term can be written as follows.

$$A_n(g_k)^2 - \Phi_n(g_k)^2 = A_n(g_k)^2 - [A_n(g_k) + R_n + N_n]^2 = R_n^{'} + N_n^{'}$$

where $R_n^{'}$ and $N_n^{'}$ are sequences of matrices with bounded rank and small norm respectively. Hence the third term also converges to the 0 matrix in strong cluster. Therefore X_n converges to the 0 matrix in strong cluster.

By the similar arguments in the proof of Theorem (5.3.1), we conclude that Z_n converges to the 0 matrix in strong cluster.

Now consider

$$\begin{aligned} P_{U_n}(A_n(g_k \circ g_l)) - A_n(g_k \circ g_l) &= [P_{U_n}(A_n(g_k \circ g_l)) - P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l))] \\ &+ [P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l)) - A_n(g_k)A_n(g_l)] \\ &+ [A_n(g_k)A_n(g_l) - A_n(g_k \circ g_l)] \end{aligned}$$

By similar arguments above, we see that each term in the right hand side of the above equation converges to the 0 matrix in strong cluster. Hence the theorem is proved for the functions of the form $g_k g_l$. Hence it is true for any function in the algebra generated by $\{g_1, g_2, g_3, \ldots .g_m\}$.

Now for $f \in \mathbb{A}$, $\epsilon > 0$, g be the function in the algebra generated by $\{g_1, g_2, g_3, \ldots .g_m\}$ such that

$$\| A_n(f) - A_n(g) \| < \epsilon/3, \text{ and } \| P_{U_n}(A_n(g)) - P_{U_n}(A_n(f)) \| < \epsilon/3.$$

Consider the following equation:

$$\begin{aligned} A_n(f) - P_{U_n}(A_n(f)) &= [A_n(f) - A_n(g)] + [A_n(g) - P_{U_n}(A_n(g))] \\ &+ [P_{U_n}(A_n(g)) - P_{U_n}(A_n(f))] \end{aligned}$$

Thus the norm of the sum of the first and third terms is less than $2\epsilon/3$. The middle term $A_n(g) - P_{U_n}(A_n(g))$ can be split into a term with norm less than $\epsilon/3$ and a term with constant rank independent of the order n since g is in the algebra generated by $\{g_1, g_2, g_3, \ldots .g_m\}$. Hence the proof is completed. □

Corollary 6. If $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in strong cluster for all f in $\{1, x, x^2\}$, then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in strong cluster for all f in $C[0, 2\pi]$.

Corollary 7. Under the assumption of Theorem (5.4.5), if $f \in \mathbb{A}$ is strictly positive, then for any $\epsilon > 0$, for n large enough, the matrix $P_{U_n}(A_n(f))^{-1}(A_n(f))$ has eigenvalues in $(1-\epsilon, 1+\epsilon)$ except for $N_\epsilon = O(1)$ outliers, at most.

Proof. Since $f \in \mathbb{A}$ is strictly positive, $(A_n(f))$ is positive definite. This implies that $P_{U_n}(A_n(f))$ is positive definite. Hence the proof is completed by Lemma (5.3.2). □

Now we prove the exact analogue of Theorem (5.4.5) in the case of convergence in weak cluster. The proof is more or less is the same but for some obvious modifications. However all the details are provided.

Theorem 5.4.6. *Let $\{g_1, g_2, \ldots g_m\}$ be a finite set of real valued con-*

tinuous 2π periodic functions such that $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in weak cluster for f in $\{g_1, g_2, \ldots g_m, {g_1}^2, {g_2}^2, \ldots {g_m}^2\}$. Then $P_{U_n}(A_n(f))$ converges to $A_n(f)$ in weak cluster for all f in the C^*- algebra $\mathbb{A}$ generated by $\{g_1, g_2, g_3, \ldots .g_m\}$.

Proof. The proof is the same as Theorem (5.4.5), except that the splitting of terms must be as the sum of one with small norm and the other of rank o(n). We give the details below. Applying (5.2) with $\Phi_n = P_{U_n}(A_n(.))$ and X_n, Y_n, Z_n as in the proof of Theorem (5.4.5), if we write

$$\begin{aligned} X_n = \Phi_n(g_k^2) - \Phi_n(g_k)^2 &= [\Phi_n(g_k^2) - A_n(g_k^2)] + [A_n(g_k^2) - A_n(g_k)^2] \\ &+ [A_n(g_k)^2 - \Phi_n(g_k)^2] \end{aligned}$$

the first term on the right hand side of the above equation converges to 0 in weak cluster by assumption. The second term, $A_n(g_k^2) - A_n(g_k)^2$ converges to the 0 matrix in strong cluster by the same argument in the proof of Theorem (5.4.5), and hence it converges in weak cluster. By a simple computation, we get that the third term also converges to the 0 matrix in weak cluster. Hence X_n converges to the 0 matrix in weak cluster.

By a similar argument in the proof of Theorem (5.3.1), we conclude that Z_n converges to the 0 matrix in weak cluster.

Now consider

$$\begin{aligned} P_{U_n}(A_n(g_k \circ g_l)) - A_n(g_k \circ g_l) &= [P_{U_n}(A_n(g_k \circ g_l)) - P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l))] \\ &+ [P_{U_n}(A_n(g_k))P_{U_n}(A_n(g_l)) - A_n(g_k)A_n(g_l)] \\ &+ [A_n(g_k)A_n(g_l) - A_n(g_k \circ g_l)] \end{aligned}$$

By similar arguments above, we see that each term on the right hand side of the above equation converges to the 0 matrix in weak cluster. Hence the theorem is proved for the functions of the form $g_k g_l$. Hence it is true for any function in the algebra generated by $\{g_1, g_2, g_3, \ldots . g_m\}$.

Now for $f \in \mathbb{A}$, $\epsilon > 0$, g be the function in the algebra generated by $\{g_1, g_2, g_3, \ldots . g_m\}$ such that

$$\| A_n(f) - A_n(g) \| < \epsilon/3, \text{ and } \| P_{U_n}(A_n(g)) - P_{U_n}(A_n(f)) \| < \epsilon/3.$$

Consider the following equation:

$$\begin{aligned} A_n(f) - P_{U_n}(A_n(f)) &= [A_n(f) - A_n(g)] + [A_n(g) - P_{U_n}(A_n(g))] \\ &+ [P_{U_n}(A_n(g)) - P_{U_n}(A_n(f))] \end{aligned}$$

Thus the norm of the sum of the first and third terms is less than $2\epsilon/3$. The middle term $A_n(g) - P_{U_n}(A_n(g))$ can be split into a term with norm less than $\epsilon/3$ and a term with rank $o(n)$, since g is in the algebra generated by $\{g_1, g_2, g_3, \ldots . g_m\}$. Hence the proof is completed. □

Corollary 8. With the hypotheses Theorem (5.4.6), if $f \in \mathbb{A}$ is positive, then for any $\epsilon > 0$, for n large enough, the matrix $P_{U_n}(A_n(f))^{-1}(A_n(f))$ has eigenvalues in $(1-\epsilon, 1+\epsilon)$ except $N_\epsilon = o(n)$ outliers, at most.

Proof. Proof follows easily from Lemma (5.3.2). □

Remark 5.4.2. It is to be noted that Theorem (5.4.5), (5.4.6) and the corollaries are much stronger than the corresponding theorems in [60], where it has been assumed that the convergence takes place on the algebra generated by the *test set*. But here it is assumed that the convergence takes place only on the *test set* as in the classical Korovkin-type theorems. However it is not clear whether the assumption of convergence on ${g_k}^2$ for each k can be replaced by convergence on $\sum_{k=1}^n {g_k}^2$ as in the usual case.

5.4.2 The LPO sequences

The behavior of eigenvalues of $P_{U_n}(A_n(f))$ has been studied in [60] when U_n is the sequence of generalized Vandermonde matrices (Example 5.2.1). Recall that the jth row of U_n is a vector of trigonometric functions calculated on the grid point ${x_j}^{(n)}$. From Lemma (5.2.1), it follows that the jth eigenvalue λ_j of $P_{U_n}(A_n(f))$ is $\sigma(U_n A_n(f) {U_n}^*)_{j,j}$. Thus λ_j is the value of the trigonometric function that takes on the jth grid point $x = {x_j}^{(n)}$. Now we consider the function $[L_n[U_n](f)](x)$ obtained by replacing ${x_j}^{(n)}$ by x in $[0, 2\pi]$ in the expression of λ_j. To make it precise, let $v(x)$ denote the vector trigonometric function whose values at grid points $\{{x_j}^{(n)}\}$, form the jth generic row of ${U_n}^*$. We define the linear operator $L_n[U_n]$ on $C[0, 2\pi]$ as follows;

$$L_n[U_n](f) = v(x) A_n(f) v^*(x). \tag{5.18}$$

$L_n[U_n](f)$ is the continuous expressions of the diagonal elements of $U_n A_n(f) {U_n}^*$. And it is clear that $L_n[U_n]$ is a sequence of completely positive linear maps on $C[0, 2\pi]$ of norm less than or equal to 1.

First we recall the notion of uniform and quasi-uniform distribution of grid points from [60].

Definition 5.4.1. A sequence of grids $S_n = \{{x_i}^{(n)}, i = 0, 1, \dots n-1\}$ belonging to an interval I is called quasi-uniform if

$$\sum_{i=1}^{n} \left| \frac{|I|}{n} - ({x_i}^{(n)} - {x_i}^{(n-1)}) \right| = o(1)$$

with $|I|$ being the width of I. If the previous relation holds with $o(1)$ is replaced by $O(1/n)$, then the mesh-sequence S_n is called uniform.

Now we recall two theorems from [60], on the Linear Positive operator sequence $L_n[U_n]$.

Theorem 5.4.7. *Let f be a continuous periodic function and let p a function in the test set $\{1, cosx, sinx\}$. If $L_n[U_n](p) = p + \epsilon_n(p)$ with $\epsilon_n(p)$ going uniformly to zero, then $P_{U_n}(A_n(f))$ converges to $A_n(f)$ in the weak sense.*

Theorem 5.4.8. *Under the same assumption of the previous Theorem, if $\epsilon_n(p) = O(1/n)$ for the three test functions p and if the grid points of the algebra are uniformly distributed, then the convergence is strong.*

It can also be observed that similar stronger versions of Theorems (5.4.7) and (5.4.8) are valid. Before proving the stronger versions of the above two theorems, we need the following results from [60].

Lemma 5.4.2. Let S_n be a sequence of quasi-uniformly distributed grid points on I. Then, for any bounded and Riemann integrable function g, we have

$$\sum_{i=1}^{n} g({x_i}^{(n)}) = \frac{n}{2\pi} \int_{-\pi}^{\pi} g + o(n).$$

If the distribution is uniform and if g is bounded and Lipschitz continuous except, at most for a finite number of discontinuity points, then

$$\sum_{i=1}^{n} g(x_i{}^{(n)}) = \frac{n}{2\pi}\int_{-\pi}^{\pi} g + O(1).$$

Now we state the Szego-Tyrtyshnikov Theorem (see [60]), which is a special case of Theorem (3.3.2).

Theorem 5.4.9. *Let $f \in L^2$ and λ_i^n be the eigenvalues of $A_n(f)$. Then, for any continuous function F with bounded support, we find the following asymptotic formula (the Szego relation)*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\lambda_i^n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(x))dx.$$

With these tools in hand, we prove the generalized version of Theorem (5.4.7).

Theorem 5.4.10. *Let $L_n[U_n](g) = g + \epsilon_n(g)$ for every g in the finite set $\{g_1, g_2, g_3, \ldots .g_m, \sum_{k=1}^{m} g_k^2\}$, where each g_k' s are real valued, continuous functions and $\epsilon_n(g)$ converges uniformly to 0. Then $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in weak cluster for all f in the C^*- algebra $\mathbb{A}$ generated by the finite set $\{g_1, g_2, g_3, \ldots .g_m\}$.*

Proof. First we observe that $L_n[U_n](g) = g + \epsilon_n(g)$ for every g in algebra generated by $\{g_1, g_2, g_3, \ldots .g_m\}$, by Lemma (5.4.1). Also we have from the identity (5.9),

$$0 \leqslant \parallel A_n(f_l) - P_{U_n}(A_n(f_l)) \parallel_F{}^2 = \parallel A_n(f_l) \parallel_F{}^2 - \parallel P_{U_n}A_n(f_l) \parallel_F{}^2 \quad (5.19)$$

for every function f_l in the algebra generated by $\{g_1, g_2, g_3, \ldots .g_m\}$.

Here $\parallel (.) \parallel_F$ denotes the Frobenius norm of matrices. Also the eigenvalues of $P_{U_n}(A_n(.))$,

$$\lambda_i(P_{U_n}(A_n(f_l))) = f_l(x_i^n) + \epsilon_n(f_l), \text{ for every l.}$$

Hence we get the following.

$$\parallel P_{U_n}(A_n(f_l)) \parallel_F{}^2 = \sum_{i=1}^{n} \lambda_i^2(P_{U_n}(A_n(f_l))) = \sum_{i=1}^{n} [(f_l + \varepsilon_n\,(f_l))\,(x_i^n)]^2$$

Hence

$$\parallel P_{U_n}(A_n(f_l)) \parallel_F{}^2 = \sum_{i=1}^{n} f_l^2\,(x_i^n) + o(n).$$

Since $\{x_i{}^{(n)}\}$ is quasiuniformly distributed, by Lemma (5.4.2), we get,

$$\sum_{i=0}^{n-1} \left[f_l(x_i{}^{(n)} + \varepsilon_n(f_l)(x_i{}^{(n)})^2\right] = n/2\pi \int_0^{2\pi} f_l^2 + o(n) \quad (5.20)$$

Also

$$\parallel A_n(f_l) \parallel_F{}^2 = \sum_{i=1}^{n} \lambda_i(A_n(f_l))^2,$$

for every l, and hence by Szego-Tyrtyshnikov Theorem (5.4.9), we find

$$\parallel A_n(f_l) \parallel_F{}^2 = n/2\pi \int_0^{2\pi} f_l^2 + o(n) \quad (5.21)$$

Now from (5.19),(5.20) and (5.21) we get

$$\| A_n(f_l) - P_{U_n}(A_n(f_l)) \|_F{}^2 = o(n)$$

for every function f_l in the algebra generated by $\{g_1, g_2, g_3, \ldots .g_m\}$. Therefore by Tyrtyshnikov's Lemma (5.2.3), $P_{U_n}(A_n(f_l)) - A_n(f_l)$ converges to 0 in weak cluster. Hence by Theorem (5.4.6), $P_{U_n}(A_n(f)) - A_n(f)$ converges to 0 in weak cluster for every f in the C^*- algebra $\mathbb{A}$ generated by $\{g_1, g_2, g_3, \ldots .g_m\}$. Hence the proof is completed. □

We need the notion of Krein algebra and Widom's theorem (see [60, 71]) for further generalizations.

Definition 5.4.2. The set of all essentially bounded functions f such that the sum $\sum_{k\in\mathbb{Z}} |a_k|^2 k < \infty$, where $a'_k s$ are the Fourier coefficients of f, forms an algebra. This is named as Krein algebra and denoted by K.

To state Widom's theorem, we let $\{\sigma_i^{(n)}\}$ be the singular values of $A_n(f)$, $t_i^{(n)} = (\sigma_i^{(n)})^2$ and $f \in K$.

Theorem 5.4.11. *Let G be a function belonging to $C^3 \left[m^2{}_f, M^2{}_f\right]$, where m_f, M_f are essential infimum and supremum of $|f|$. Then*

$$\lim_{n\to\infty} \{\sum_{i=1}^{n} G(t_i^{(n)}) - \frac{n}{2\pi} \int_{-\pi}^{\pi} G(|f(x)|^2) dx\} = c(f, G).$$

Here $c(f, G)$ is a known constant characterized in [71].

Theorem 5.4.12. *With the assumptions in Theorem (5.4.10), if $\epsilon_n(g) = O(1/n)$ for g in the finite set $\{g_1, g_2, \ldots g_m, \sum_{k=1}^{m} g_k{}^2\}$ and if the* *"grid point algebra" are uniformly distributed, then the convergence is in strong cluster, provided the test functions in the set* $\{g_1, g_2, g_3, \ldots .g_m\}$ *are Lipschitz continuous and belong to the Krein algebra.*

Proof. The proof can be obtained by replacing the polynomials p by $\{g_1, g_2, g_3, \ldots g_n\}$ in the proof of Theorem(5.4) in [60]. The idea is to replace the term of o(n) by constants in the equations (5.20) and (5.21). For (5.20), we use the hypothesis $\epsilon_n(g) = O(1/n)$ and that the "grid point algebra" are uniformly distributed. For (5.21), we use Widom's theorem stated above with $G(t) = t$. Hence we will attain

$$\| A_n(f_l) - P_{U_n}(A_n(f_l)) \|_F{}^2 = O(1)$$

This completes the proof due to Lemma (5.2.3). □

5.5 Applications to spectral approximation

In this section, we discuss the possible applications of the considered theory. The results proved in section 5.4.1, are the stronger versions of the results in [60]. Theorem (5.4.5) and (5.4.6) considered arbitrary continuous functions, while in [60], trigonometric polynomials were considered. Also the assumptions are reduced to finite number elements as in the classical Korovkin's theorem. Theorem (5.4.5), (5.4.6), (5.4.10) and (5.4.12) can be used to obtain new preconditioners belonging to different algebras. The corollaries (7) and (8) are expected to be useful for deriving and analyzing good preconditioners for the conjugate gradient method.

In Theorem (5.2.2) and (5.2.3), we considered one of the limit points Φ of Φ_n's in the Kadison's B.W topology. Here $\Phi_n(A)$ are the preconditioners of A, but it is not clear to us whether $\Phi(A)$ is a preconditioner of A for at least one limit point of Φ_n. That is to ask whether $\Phi(A)^{-1}A$ has small condition number. Theorem (5.2.2) guarantees that the change from A to its preconditioners amounts only a finite rank perturbation if you look at a uniform limit point. Hence the essential spectrum is unchanged due to this change. The essential properties that are preserved, are mentioned in the corollary. Also in the case of strong limit point, the essential spectrum is unchanged by Theorem (5.2.3).

The Lemma (5.3.1) and Theorem (5.3.1) are of theoretical interest, since it shares the same spirit of the classical Korovkin's theorem. That is the test on a finite number of elements guarantees the assertion on the whole algebra generated by these finite number of elements. Theorem (5.3.1) partly answers the following question. Suppose the usage of preconditioners works for a finite number of self-adjoint operators on $\mathbb{H}$. Does it work for any operator in the C^*- algebra generated by these operators.

In short, we expect that the results may be applicable in many approximation methods. For instance, the truncation method for the spectral approximation problem of bounded self-adjoint operators that we discussed in the previous chapters.

In such problems, the usage of preconditioners may help us to make the linear system well conditioned and to make the computations simpler. $\Phi_n(A)$ converges to A in the distribution sense, means that the truncations converge in the sense of eigenvalue clustering. So instead of considering A, we can consider its preconditioners $\Phi_n(A)$ and do the approximations. We have already noticed that the essential spectrum remains the same in the case of uniform and strong clusterings. Hence the usage of preconditioners in these problems are justified. The examples for the numerical efficiency of such preconditioners is to be investigated in future.

Chapter 6

Concluding Remarks

Some of the interesting observations regarding the considered theory and a few possibilities for future works are remarked in this chapter. The objective of this study is to discuss the linear algebraic techniques in the spectral theory of bounded self-adjoint operators on a separable Hilbert space. The usage of truncation method in approximating the bounds of essential spectrum and the discrete spectral values outside these bounds is well known. The spectral gap prediction and related results was proved in the second chapter. The discrete versions of Borg-type theorems, proved in the third chapter, partly overlap with some known results in operator theory. The pure linear algebraic approach is the main novelty of the results proved here. The perturbed versions of spectral approximation results proved in the fourth chapter is helpful in the stability analysis of the main results. That is to look at the situation when the operator is subject to a small perturbation. The Korovkin-type theorems proved in the last chapter is of high theoretical interest and face the approximation problem in a more abstract way. Some of the related problems are discussed below.

6.1 Discrete eigenvalues in the gap

It is an interesting problem to locate the presence of discrete eigenvalues that may have trapped between the upper and lower bounds of the essential spectrum. As already mentioned, such an eigenvalue will be lying inside a spectral gap. The following theorem is taken from [54], which can be used to get some information about the discrete spectral values between the gaps.

Theorem 6.1.1. *Let A be a bounded self-adjoint operator and f be a real valued continuous function, supported on the interval* $[a, b] \subseteq [m, M]$*. Define* $f_{m,n}$ *as* $f_{m,n}(t) = f(2^m t - n)$ *for integers m and n. Then* $\sigma_e(A)$ *has a gap if and only if* $f_{m,n}(A)$ *is a compact operator for some m,n with support of* $f_{m,n}$ *lie between* ν *and* μ*.*

Proof. Suppose $f_{m,n}(A)$ is a compact operator for some m and n. Then

$$\sigma_e(f_{m,n}(A)) = \{0\}.$$

But $\sigma_e(f_{m,n}(A)) = f_{m,n}(\sigma_e(A))$. Therefore $f_{m,n}(\sigma_e(A)) = 0$. That is the support of $(f_{m,n})$ will not intersect with $\sigma_e(A)$. Hence support of $(f_{m,n})$ is a gap in the essential spectrum, since m and n is such that support of $(f_{m,n})$ lie between ν and μ.

Conversely suppose that $\sigma_e(A)$ has a gap. So we can choose m,n such that support of $(f_{m,n})$ lies in that gap. Also since

$$\sigma_e(f_{m,n}(A)) = f_{m,n}(\sigma_e(A)) = \{0\},$$

$f_{m,n}(A)$ is a compact operator. □

Remark 6.1.1. Observe the above proof shows that for any real valued function f, which is continuous and supported in a spectral gap of a bounded self-adjoint operator A, $f(A)$ is a compact self-adjoint operator. The above theorem tells that the existence of a spectral gap is equivalent to the existence of a compact self-adjoint operator $f_{m,n}(A)$. Now these gaps may contain some discrete spectral values. In the case where these gaps do not contain any discrete spectral value, since

$$\sigma(f_{m,n}(A)) = f_{m,n}(\sigma(A)) = \{0\},$$

and hence $f_{m,n}(A)$ is the Zero operator.

Remark 6.1.2. In case there are finitely many discrete eigenvalues in the gap, the following observations can be made. Let α be an eigenvalue in the gap. Then $f_{m,n}(\alpha)$ will be an eigenvalue of $f_{m,n}(A)$, whose computation is comparatively easier since $f_{m,n}(A)$ is a compact operator. From these eigenvalues, we may compute the eigenvalues in the gap, provided f is simple enough. To check the compactness of an operator, the essential norm has to be computed as the limit of singular values, using Theorem (2.1.4). Recall that an operator is compact if and only if the essential norm is 0. Hence the approximation numbers $s_k(f_{m,n}(A))$ has to be computed and check whether they come closer to 0, for large values of k.

6.1.1 Further problems on Spectral gap

- If we replace A_n by some approximating class of sequences $B_{n,m}$ (finite rank and small norm perturbations, see [61] for precise definitions), can we state similar results as proved in chapter 2. Researchers like Hansen in [41] used a sequence $A_{1,n}$ which is a rank 1 perturbation of A_n, to compute $\sigma(A)$. If the above question has an affirmative answer, we can replace A_n by $A_{1,n}$ whenever we need. In particular, one can study the effect in the Fourier coefficients by a small change in the function. One can look at the spectrum of a multiplication operator via sequence of Toeplitz-Laurent operators. This may contribute to the literature of recapturing the symbol with the information of its Fourier coefficients.

- We point out that the problem of prediction and estimation of spectral gaps of bounded self adjoint operators with the use of truncations leads to the issues of error estimation. So far there is no evidence of such estimation in the case of an arbitrary self adjoint operator to the best of our knowledge.

- Also under compact perturbation, though the spectral gaps remain the same, discrete eigenvalues may appear or disappear in such gaps. Another problem is to handle such situations linear algebraically.

- The discrete spectral values lying between a gap in the essential spectrum, can be computed using linear algebraic techniques. To see this, let (a, b) be a gap in the essential spectrum of A. Let $\lambda_0 = (a+b)/2$. Since λ_0 is in the gap, $f(\lambda_0) > 0$. Now $f(\lambda_0)$ is the lower bound of the essential spectrum of $(A-\lambda_0 I)^2$, all the discrete spectral values below that can be computed with the use of truncations by Theorem (2.1.7). If β is an eigenvalue in the gap, $(\beta - \lambda_0)^2$ will be an eigenvalue lying below the lower bound of the essential spectrum of $(A - \lambda_0 I)^2$. From these we can compute β.

6.2 Further possibilities on Borg-type theorems

Concerning future work on Borg-type theorems, there are some interesting issues that should be addressed. These operators could be considered in multidimensional domains and in the case of systems of equations (as in [11]). A further intriguing issue could be the following: how to relate the number of gaps to the periodicity index p of the diagonal periodic sequence, the latter question been supported by the fact that no gaps is equivalent to have all equal diagonal entries and by the example reported in the section 3.6.1.

Also the random versions of these operators could be considered. The analogue results of third chapter can be tried for the discretized random Schrodinger operators (see [26]). The gap issues of arbitrary random operators, and the truncation method in that is an interesting problem. There the results may be proved in the language of probability and almost sure convergence. The basic object of high interest is the integrated density of states (see [26]). The problem is to compute the integrated density of states using the linear algebraic techniques.

6.3 Perturbation problems

Note that in the theorems and examples discussed in the fourth chapter, only the continuity is used in most of the cases. This shows that a more general result is possible. That means similar results may be established for some non holomorphic perturbations. The importance of the analyticity assumption is that in certain cases, one can use the rich theory of complex analytic functions for eigenvalue functions and singular value functions.

Also, we only considered perturbation of operators and not the perturbations of their truncations. In the Example (4.3.1) also the perturbed symbol is directly related to the perturbation of operators. The perturbation of truncations and their link with the spectrum of the original operator is another problem yet to be handled.

Another question is whether equality holds in the inclusion

$$\sigma_{\text{ess}}(A) = \bigcup_{j=1}^{p} \left[\inf_{\theta}(\lambda_j(f_s(\theta))), \sup_{\theta}(\lambda_j(f_s(\theta))) \right] \subseteq \bigcup_{j=1}^{p} \left[\lambda_j - \|f\|_\infty , \lambda_j + \|f\|_\infty \right].$$

If so, we are able to determine all possibility of gaps with only looking at eigenvalues of the constant matrix

$$\begin{bmatrix} b_1 & 1 & & & & \\ 1 & b_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & b_p \end{bmatrix}.$$

Also the consequence of the estimate (4.4) is yet to be investigated.

6.4 Applications of Pre-conditioners

The pre-conditioners introduced in the last chapter may be very useful in the spectral approximation problems. One future possibility is to link approximation techniques in the second chapter with the notion of pre-conditioners and the convergence in the distributional sense. To find good examples where these pre-conditioners are effective in determining spectra of operators, is another task to be taken up in future.

The Korovkin-type theorems are proved for convergence in the uniform and strong distribution sense. The weak convergence has to be studied in detail.

Finally we wish to change the setting into the case of unbounded self-adjoint operators. All the basic notions have to be defined appropriately. There we have to deal with operators with unbounded spectrum. Hence the task is very difficult. But there are evidences for the approximation is successful in the case of some Schrodinger operator.

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CURRICULUM VITAE

Name :	Kiran Kumar V.B
Present Address :	Department of Mathematics, Cochin University of Science and Technology, Cochin, Kerala, India – 682 022.
Permanent Address :	Aiswerya House, Naderi P.O. Koyilandy, Calicut Kerala, India – 673620.
Email :	kiranbalu36@gmail.com

Qualifications :	**B.Sc.** (Mathematics), 2006, University of Calicut Kerala, India.
	M.Sc. (Mathematics), 2008, Cochin University of Science and Technology, Cochin, Kerala, India – 682 022.
Research Interests :	Functional Analysis– Spectral Theory, Structured Linear Algebra, Approximation Theory.