STUDIES ON COLLECTIVE EXCITATIONS OF QUASI-TWO-DIMENSIONAL BOSE-EINSTEIN CONDENSATE

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Studies on collective excitations of quasi-two-dimensional Bose-Einstein condensate
PhD thesis in the field of Bose-Einstein condensation

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Front cover : Bose-Einstein condensate, from internet
Courtesy: old.quantumconsciousness.org and Dave Cantrell

To my Sensei

who taught me the spirit of a fighter

CERTIFICATE

Certified that the work presented in this thesis is a bonafide research work done by Mr. Anoop P D, under my guidance in the Department of Physics, Cochin University of Science and Technology, Kochi- 682022, India and has not been included in any other thesis submitted previously for the award of any other degree.

Kochi-22
June, 2015

Dr. Ramesh Babu T
(Supervising Guide)

DECLARATION

I hereby declare that the work presented in this thesis is based on the original research work done by me under the guidance of Dr. Ramesh Babu T, Professor (Retd.), Department of Physics, Cochin University of Science and Technology, Kochi- 682022, India and has not been included in any other thesis submitted previously for the award of any other degree.

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Anoop P D

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Preface

Bose-Einstein condensation has been a field of intense theoretical and experimental research for the past few decades. Being a macroscopic window into the quantum realm, properties and dynamics of this peculiar phase have raised considerable interest to low temperature physicists.

Collective excitations in three dimensional as well as reduced dimensional Bose-Einstein condensate(BEC) have been investigated rigorously in the literature. But the theoretical studies often neglected the influence of factors like gravity, anharmonicity in the trap potential, beyond mean field effects etc. In this thesis, an attempt has been made to study the effect of such parameters on the collective excitations of a quasi-two-dimensional BEC. A quasi-two-dimensional BEC is one, in which the trap frequency in one of the three dimensions is made large enough to freeze the particle dynamics (except the zero point oscillation) in that dimension. The significance of this work is that by the introduction of these modifications in the potential, we are making the trapped system closer to an experimental one. This leads to a more accurate theoretical description of the experimental results. In this work, a variational procedure is adopted for the study of the dynamics of the condensate. As the trial wave function for the variational procedure, a Gaussian wave function with a few variational parameters is chosen. This thesis is divided in to five chapters.

Chapter 1 introduces the phenomenon of Bose-Einstein condensation and the theoretical frame work of this phenomenon. The thermodynamics of trapped and untrapped ideal Bose gas are discussed in detail. The effect of inter-particle interaction on the dynamics of BEC is further analyzed. The derivation of the mean field nonlinear Schrodinger equation called Gross-Pitaevskii (GP) equation is also elaborated in this chapter.

Chapter 2 deals with the variations of the collective excitation frequencies of a quasi-two-dimensional BEC due to the influence of gravity and anharmonicity. The variational study reveals that the center of the condensate is coupled with its internal degrees of freedom. It is observed that the two excitation frequencies corresponding to the width of the condensate get modified due to the effect of these modifications in the potential. One of the excitation frequencies, which is a breathing mode in a pure harmonic trap, is destabilized due the additional effects. The other excitation frequency is seen to be independent of the effect of gravity, if the inter-particle interaction is negligibly small. Plots showing the variation of excitation frequencies against the condensate particle number (N) are prepared for both attractively and repulsively interacting particles.

Chapter 3 - The mean field approach to BEC is exactly valid only for the low values of number density and scattering length. But, ingenious methods which exploit phenomena like Feshbach resonance have made it possible to tune the scattering length and have thus resulted in condensate experiments, requiring analysis which goes beyond the mean field realm. Hence an extension of GP equation into the beyond mean field regime is necessary. In this chapter, the influence of beyond mean field effects, namely, effective range interaction and quantum fluctuation on the collective excitations of quasi-two-dimensional BEC are studied. Analysis reveals that the beyond mean field effects do not result in a coupling of the center of the condensate and its internal degrees of freedom. Plots showing the variation of excitation frequencies against the condensate particle number reveals that the breathing mode oscillation is destabilized by the beyond mean field effects. In a pure harmonic trap, for attractively interacting condensates, a collapse of the condensate can be observed at certain particle number. It is followed by a reappearance of the condensate at a higher value of N, i.e. there exist a 'forbidden window' in the condensate particle number. When the beyond mean field effects are included, it is observed that the reappearance of the condensate does not happen.

Chapter 4 deals with BEC at finite temperature. Finite temperature results in the inclusion of the effect of thermal cloud on the condensate. As a first approximation, a static thermal cloud is considered. In addition to the thermal cloud, the effects of gravity, anharmonicity and beyond mean field effects are also included in the potential. Plots showing the variation of excitation frequencies against the condensate particle number(N) and temperature are prepared. It is observed that the dynamics of the center of the condensate are not influenced by the thermal cloud. But the oscillations of the width of the condensate are found to depend on the thermal cloud.

In **Chapter** 5, a summary of the entire work is given and the scope for future work is suggested.

List of Publications

1. "Effect of gravity on the collective excitations of a quasi two-dimensional Bose-Einstein Condensate in an anharmonic trap", Anoop P D and Ramesh Babu T, Commun. Theor. Phys., 56, 2011, 669-671.

2. "Collective excitations of a quasi two-dimensional BEC: Beyond mean field theory", Anoop P D and Ramesh Babu T, Eur. Phys. J. D, 67, 2013, 67.

1 Introduction

1.1 Classification of quantum particles

In the formulation of classical statistical mechanics, a system of identical particles is distinguishable. But when it comes to the quantum many body systems, the characteristic of distinguishability is lost. This indistinguishability confines the choice of wavefunction for a quantum many body system into just two, a symmetric wavefunction or an antisymmetric wavefunction which can be expressed as

$$\Psi(....\eta_{\mathbf{i}}, ...\eta_{\mathbf{j}}, ..) = \pm\Psi(....\eta_{\mathbf{j}}, ...\eta_{\mathbf{i}}, ..) \tag{1.1}$$

where $\eta_{\mathbf{i}}$ is the label of $\mathbf{i}$th particle which represents the particle's spin as well as position. System of particles for which the '+' sign holds are known as bosons (which have zero or integral spin), whose thermodynamic behavior is governed by Bose-Einstein statistics. While the particles for which the '$-$' sign holds comes under the class of fermions(which possess half integral values of spin). The fermionic particles obey Pauli's exclusion principle which puts a natural limit on the number of such particles which can occupy a particular quantum level. Where as the population of bosonic particles does not follow exclusion principle so that, when the temperature of a system of bosons decreases, more and more of them start to occupy the lowest energy level. Below a critical temperature T_c, a large fraction of the particles occupy the ground state or the lowest quantum state. This is the phenomenon of Bose-Einstein Condensation (BEC). A further decrease of temperature towards absolute zero results in the migration of more and more particles to the ground state. Since atoms are composed of fermionic particles, an atom consisting of even number of fermions behaves as a boson, while that containing odd number of fermions has the characteristics of a fermion.

Proposed by Einstein in the year 1925 [1], based on the ideas of Satyendra Nath Bose on photon distribution [2], BEC provides a bridge between the macroscopic and quantum realms. But at the time of its theoretical proposition, Einstein's contemporaries considered BEC to be just a theoretical fantasy thanks to the non-interacting nature assigned to the system and the extremely low temperature needed for attaining this state. This attitude towards BEC changed with the discovery of superfluid helium-4. Liquid Helium is unique in the sense that it doesn't solidify at normal pressures even at absolute zero temperature. Below a particular temperature called 'lambda point', liquid Helium is found to become a superfluid with many striking features. Most notable among them is its ability to flow through narrow capillaries experiencing almost zero friction. F. London proposed that the superfluid behavior of helium-4 below a critical temperature is due to the Bose condensation of helium-4 atoms [3]. London showed that the transition temperature of helium-4 is in good agreement with that calculated using Einstein's formula of transition temperature for BEC. This was followed by the 'Two-fluid model' of helium-4 by L Tisza and its further development by Landau [4, 5]. These theoretical studies on helium-4 served as the basis for the development of BEC theoretical framework. These theories backed by the new experimental techniques for attaining extremely low temperatures finally led to the realization of BEC. The attainment of BEC necessitates temperature as low as $10^{-9}K$. In these temperature ranges, the equilibrium phases of most substances are solid. But, for maintaining a gaseous state of the system, it is necessary to minimize the inter particle collisions which may lead to the formation of molecules. This necessitates the use of a weakly interacting bose gas. These constraints made spin polarized hydrogen atoms the most preferred candidate for early experiments for the realization of BEC [6]. Actually, Hecht in 1959 itself had pointed out the advantages of spin polarized hydrogen as a suitable choice for BEC experiments [7]. It is observed that when the electronic spin of two hydrogen atoms are aligned, their attractive interaction is not strong enough to form a bound state and this keeps them in gaseous state even at absolute zero temperature. The development of efficient and elegant cooling mechanisms such as laser cooling and evaporative cooling in the 1980's made it possible to attain temperatures low enough for the realization of BEC. But the experiments using spin polarized hydrogen did not meet with complete success as the 3-body collisions resulted in spin flips of hydrogen atoms which in turn resulted in the formation of molecules. Since the laser cooling techniques were greatly developed in the case of alkali metals, further BEC experiments were concentrated on them.

Experimental realization of BEC was achieved finally in the year 1995 by JILA group led by E Cornell and C Wieman in the dilute gases of rubidium-87 [8]. Soon Ketterle's groups at MIT succeeded in producing BEC of sodium-23 and Hulet's groups at Rice University in lithium-7 [9, 10]. Since then condensates have been produced in many other atoms which include dilute gases of hydrogen, potassium, chromium etc.[11, 12, 14, 15]. The idea of BEC, though in a more generalized sense, is now understood to be the basic idea behind many phenomena. Superconductivity, exhibited by some materials at low temperatures, is understood to be from the formation electron-electron bound states which are bosonic in nature. The superfluid phases exhibited by liquid helium-3, which is a fermionic atom, is also found to be resulting from the condensation of pairs of atoms. It is postulated that the interior of neutron stars consists of neutron superfluid formed from nucleon pair condensation.

1.2 Theoretical frame work

1.2.1 Bose-Einstein condensation in ideal Bose gas

By an ideal gas, we mean a collection of $N \gg 1$ non-interacting particles each of mass m in a large box of volume $V = L^3$. The system is considered in the thermodynamic limit, where both the number of particles, N and the dimension of the box L tend to infinity such that their ratio, N/V, which is the density, remains a finite constant. When the statistics of a system of particles is taken into account, it is convenient to opt for a grand canonical partition function formulation and in the case of bosons it can be given as [13]

$$Q = \prod_i \frac{1}{1 - \exp\left[-\beta(\varepsilon_i - \mu)\right]} \tag{1.2}$$

where $\beta = \frac{1}{kT}$, ε_i is the energy of the ith single particle state and μ is the chemical potential of the system . It is determined using the constraint that the total number of particles of the system is conserved which is given by

$$N = \sum_i n_i \tag{1.3}$$

where

$$n_i = \frac{1}{\exp\left[\beta(\varepsilon_i - \mu)\right] - 1}. \tag{1.4}$$

The chemical potential, μ must be less than or equal to ε_0, otherwise, as it is obvious from the Eq.(1.2), the system will have negative occupation numbers. In the thermodynamic limit, the ground state energy of a non-interacting system is taken to be zero i.e. $\varepsilon_0 = 0$. The average particle number of the system can be given as

$$\langle N \rangle = \sum \frac{1}{\exp\left[\beta(\varepsilon_i - \mu)\right] - 1} \tag{1.5}$$

Identifying $z = \exp(\mu/kT)$ as the fugacity of the system, it can be seen that fugacity, $z < 1$ above the critical temperature and equal to unity in the condensed state. As the temperature increases from the critical temperature, chemical potential falls below the energy of the lowest single particle state (ε_0), whereas, with decrease in the temperature, chemical potential as well as the occupation number of the states rises. But as can be seen from Eq.(1.4), the highest value possible for chemical potential is ε_0. This puts an upper limit to the population of the excited states. If the population of all the excited states summed is less than the total number of particles in the system, the remaining particles will obviously be populating the ground state which, in this case, will be having an arbitrarily large occupation number. This is the mechanism of Bose-Einstein condensation. Thus a BEC can be considered as a state of matter which consists of a dilute gas of weakly interacting bosons which has existence only in the regime of extremely low temperature. At such low temperatures, the interaction between bosons is mainly due to s-wave scattering. In this case, the knowledge of a single parameter, the scattering length (a), alone is sufficient to describe the inter-particle interaction.

If the variation of Bose-Einstein distribution function in Eq. (1.4) with ε_i is small compared to the level spacing of the system, the summation in Eq.(1.3) can be replaced by integration. Thus

$$N = \sum f(\varepsilon_i) \approx \int d\varepsilon g(\varepsilon) f(\varepsilon) \tag{1.6}$$

where $g(\varepsilon)$ gives the density of states. In the case of a three dimensional, homogeneous system,

$$g(\varepsilon) = \frac{2m^{3/2}V}{4\pi^2\hbar^2}\varepsilon^{1/2} \tag{1.7}$$

From the expression in Eq.(1.4) for the distribution function, it is clear that, in the limit of $\mu \to 0$, the ground state of the system gets macroscopically populated, which can be given as $N_0 = \frac{z}{1-z}$. It is also obvious from Eq.(1.7) that $g(0) = 0$. Hence, if Eq.(1.7) is used in Eq.(1.6), the ground state population of the system will not be properly accounted. Consequently, the expression for the total number of particles in the system should be written as

$$N = N_0 + \int d\varepsilon g(\varepsilon) f(\varepsilon) \tag{1.8}$$

where the integral,

$$N_{exc} = \int d\varepsilon g(\varepsilon) f(\varepsilon) = g_{3/2}(z) V \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \tag{1.9}$$

denotes the number of particles in the excited state and the Bose function $g_{3/2}(z)$ can be given as

$$g_{3/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}} \tag{1.10}$$

The value of N_{exc} attains its maximum when $\mu = 0$. Hence, if the system is cooled, N_{exc} will decrease as the chemical potential will increase from zero. Also, the Bose function has a maximum value for the fugacity, $z = 1$ and it is $g_{3/2} \approx \zeta(3/2) = 2.612$. This puts a cap on the maximum number of particles in the excited states N_{exc}, which can be given as

$$N_{exc}^{max} = 2.612 V \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} \tag{1.11}$$

When the temperature is high, $N_{exc}^{max} \gg N_0$, the number of particles in the ground state. But when the temperature is decreased, a critical temperature is reached, at which $N_{exc}^{max} \approx N$. If the temperature is decreased any further, the excited states can accommodate only a fraction of the total particles in the system, resulting in an increased population of the ground state. The critical temperature below which this happens can be given as

$$T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{N}{2.612V}\right)^{2/3} \tag{1.12}$$

The variation of condensate population with temperature can be given as

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}. \tag{1.13}$$

1.2.2 Bose-Einstein condensation in a harmonic trap

Realization of BEC in the laboratory necessitates the use of some kind of a 'container' inside which the bose gas can be kept out of the influence of surroundings. A material container cannot be used for the purpose as the collision of the atoms with the walls will heat up the system, making the attainment of BEC impossible. Hence potential traps are employed for the same purpose. To a first approximation, traps generally employed in BEC experiments can be considered to be harmonic in nature. Here we are considering a non-interacting Bose gas in an anisotropic harmonic potential

$$V = \frac{1}{2}m\left[\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right] \tag{1.14}$$

where ω_x, ω_y and ω_z are the frequencies of trapping potential in x, y and z directions respectively. In this case, the density of states is found to be

$$g(\varepsilon) = \frac{\varepsilon^2}{2\hbar^3\omega_x\omega_y\omega_z} \tag{1.15}$$

The highest temperature at which the BEC phenomena occur in a system is referred to as the Bose-Einstein condensation temperature represented as T_c. For a non-interacting gas, the number of particles in the excited state is given by

$$N_{exc} = \int_0^{\infty} d\varepsilon g(\varepsilon) f^0(\varepsilon) \tag{1.16}$$

where

$$f^0(\varepsilon_i) = \frac{1}{\exp\left(\frac{\varepsilon_i}{kT}\right) - 1} \tag{1.17}$$

since N_{exc} attains its maximum value at $\mu = 0$. Assuming that at T_c, all the particles populate the excited states, an expression for T_c for particles trapped in different potentials can be obtained from the above equation. For the case of harmonic oscillator,

$$kT_c = \hbar\bar{\omega}\left[\frac{N}{\zeta(3)}\right]^{1/3} \tag{1.18}$$

where $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$, N is the total number of particles in the system and $\zeta(3) = \sum_{n=1}^{\infty} n^{-3}$ is the Reimann zeta function of order 3.

For a three dimensional harmonic trap, the expression for number of particles in the excited states can be obtained as [66]

$$N_{exc} = C_3 \int_0^{\infty} d\varepsilon \varepsilon^2 \frac{1}{\exp\left(\frac{\varepsilon}{kT}\right) - 1} \tag{1.19}$$

The integral gives rise to [66]

$$N_{exc} = N\left[\frac{T}{T_c}\right]^3 \tag{1.20}$$

Hence the number of particles in the condensate, $N_0 = N - N_{exc}$ becomes

$$N_0 = N\left[1 - \left(\frac{T}{T_c}\right)^3\right] \tag{1.21}$$

1.3 Path to Bose-Einstein condensation

The essential step involved in the attainment of the state of BEC is to reduce the energy of the bosonic system under consideration. This involves the cooling of the system to ultra low temperatures. Cooling the atomic bose gas to attain the BEC phase involves essentially three steps.

1.3.1 Trapping of atoms

Confinement of the atoms to a finite region of space is a prerequisite for the cooling of atoms in the laboratory. But because of the reasons discussed in Sec.(1.2.2), the trapping of neutral atoms is achieved by using magnetic traps. If an atom having magnetic moment μ is placed in a magnetic field B, then the magnetic contribution to the energy of the atom is $-\mu B$. For atoms with positive magnetic moment, they are acted upon by a force which tends to drive them to the regions of high field. While for atoms with negative magnetic moment, the drive will be towards the low field region. These atoms are known as high field seekers and low field seekers respectively. A general theorem asserts that a local maximum of magnetic field cannot be created in a current free region. But there is no such restriction for local minimum of $|\mathbf{B}|$. Thus, by using the magnetic traps, 'low field seeker' atoms can be trapped. Magnetic traps are very shallow. Their depth is much less than one kelvin. Hence, in order to trap the atoms in a magnetic field, they must be cooled to a very, very low temperature.

1.3.2 Laser cooling

The term laser cooling represents a number of methods that exploit laser to reduce the temperature of an atomic system. These comprise Doppler cooling, Sisyphus cooling etc. The simplest among these methods is the 'Doppler cooling method'. Consider an atom subjected to two counter propagating laser beams. The frequency of the beams will be carefully adjusted so that, when the atom is moving towards any one beam, due to Doppler effect, the frequency of the beam matches with that of an atomic transition. This results in the atoms absorbing more photons coming in one direction than in the other. Since the momentum imparted by the net photons will be in the direction opposite to the motion of the atom, it experiences a frictional force. As the re-emission of the photon will be in random direction, the kinetic energy, and consequently the temperature of the atomic system, reduces as this process continues [16]. This reduces the temperature of the atoms to approximately 10μK. This temperature actually represents the 'Doppler cooling limit', the lowest temperature that can be attained by this method. But this temperature is far too high for the appearance of BEC phase. Hence this stage is followed by another one.

1.3.3 Evaporative cooling

The basic principle behind this technique is that, if the energy of particles escaping from a system is greater than the average energy of particles in the system, then the remaining system suffers a decrease of temperature. The use of this technique in the case of rarefied gases was first proposed by Hess [17]. Here, atoms precooled by laser cooling are trapped by a magnetic field. If it is possible to make an 'escape window' at some higher potential region in the trap, then the atoms having energy equal to or greater than the potential energy of this position will be able to escape from the trap. In BEC experiments, such a hole is made in the trap by the application of radio frequency radiation. By tuning the frequency of the radiation, the position of the 'escape window' can be adjusted. When high energy atoms are lost from the trap, cooling of the remaining system will happen and more and more atoms come to occupy the ground state, giving rise to BEC. Evaporative cooling can result in the reduction of temperature to 50 nK-100 nK range [18].

1.4 Effect of interaction

The equations (1.18) and (1.21) for transition temperature and condensate fraction respectively get modified due to the effect of interaction among the atoms. At temperatures of the order of $10^{-6} - 10^{-9}$ K, atoms occupy their electronic ground state. At these temperatures, the only relevant transitions are those between the different hyperfine states of the electronic ground state. In the dilute atomic vapors in BEC experiments, typical atomic separations are of the order of 10^2 nm. At such large separations, two body interactions dominate over the three or higher body interactions. Although the interactions are weak due to the dilute nature of the system, the ultra low temperatures at which the condensed state can be achieved, make the effect of these weak interactions critical. Also, since the velocities of the atoms in the cloud are small, only s-wave scattering need to be considered. The effective two-body interaction coupling between the atoms of equal mass can be given as

$$g = \frac{4\pi a\hbar^2}{m} \tag{1.22}$$

where 'a' is the s-wave scattering length and 'm' is the mass of the atom.

1.5 Condensate dynamics - mean field theory

In order to describe the dynamics of condensate, usually the mean field theory is assumed. In this, each of the particle in the system is considered to be moving in an average field created by all the other atoms. For temperatures away from the transition temperature, the mean field theory gives an excellent description of condensate dynamics. Based on the mean field theory, a nonlinear Schrodinger equation, which describes the dynamics of the condensate near absolute zero temperature, can be derived. This equation, known as the Gross-Pitaevskii(GP) equation, and its extensions serve as the backbone in the study of the condensed phase and is given as

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},\mathbf{t}) + g|\psi|^2\right)\psi, \tag{1.23}$$

where $V_{ext}(\mathbf{r},t)$ is the external trapping potential, $g = 4\pi a\hbar^2/m$ governs the interatomic interaction and $\psi(\mathbf{r},t)$ is the condensate wavefunction. The nonlinearity of the equation arises from interparticle interaction governed by the parameter 'g' which is the same as that in Eq.(1.22).

1.5.1 Derivation of GP equation

In the second quantized notations, the Hamiltonian of the bosonic system of atoms can be given as

$$\begin{aligned} H &= \int \psi^\dagger(\mathbf{r},t)\left(\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},t)\right)\psi(\mathbf{r},t)dr \\ &+ \frac{1}{2}\int \psi^\dagger(\mathbf{r},t)\psi^\dagger(\mathbf{r}',t)V_{int}(\mathbf{r}'-\mathbf{r}) \\ &\quad \psi(\mathbf{r},t)\psi(\mathbf{r}',t)d^3r'd^3r \end{aligned} \tag{1.24}$$

Here $\psi(\mathbf{r},t)$ and $\psi^\dagger(\mathbf{r},t)$ are the creation and annihilation operators, $V_{ext}(\mathbf{r},t)$ is the external trapping potential and $V_{int}(\mathbf{r}'-\mathbf{r})$ is the inter-particle interaction potential. The dilute nature of the system allows its collision dynamics to be governed by s-wave scattering length alone. Another consequence of the dilute nature of the system is that the s-wave scattering length will be much smaller than the interparticle spacing. This allows us to replace the interatomic potential by a mean field contact potential,

$$V_{int}(\mathbf{r}'-\mathbf{r}) = g\delta^3(\mathbf{r}'-\mathbf{r}), \tag{1.25}$$

where g is the interparticle interaction parameter defined above [19]. Here positive values of scattering length correspond to repulsive interaction among the condensate atoms, whereas a negative 'a' value represents attractive interaction. The Heisenberg equation of motion corresponding to the above Hamiltonian can be obtained as

$$i\hbar\frac{\partial\psi}{\partial t} = [\psi(\mathbf{r},t), H] \tag{1.26}$$

Since $\psi(\mathbf{r},t)$ and $\psi^\dagger(\mathbf{r},t)$ satisfy the following commutation relations,

$$\begin{aligned} &\left[\psi(\mathbf{r},t), \psi^\dagger(\mathbf{r}',t)\right] = \delta^3(\mathbf{r}-\mathbf{r}') \\ &\left[\psi(\mathbf{r},t), \psi(\mathbf{r}',t)\right] = 0, \left[\psi^\dagger(\mathbf{r},t), \psi^\dagger(\mathbf{r}',t)\right] = 0 \end{aligned} \tag{1.27}$$

Eq.(1.26) can be written as

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},\mathbf{t}) + g\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t)\right]\psi(\mathbf{r},t). \quad (1.28)$$

When the temperature of the system is close to absolute zero, that is, when the condensate fraction is large, the condensate part in the field operator can be replaced by a classical field. Hence the quantum field operator can be represented as the sum of the classical field term for the condensate and another term which will take into account the quantum fluctuation in the system [20, 21]. Thus

$$\psi(\mathbf{r},t) = \Psi(\mathbf{r},t) + \phi(\mathbf{r},t). \quad (1.29)$$

where $\Psi(\mathbf{r},t)$ is the classical order parameter of the condensed phase and $\phi(\mathbf{r},t)$ is the quantum fluctuation. It must be kept in mind that $< \psi(\mathbf{r},t) >= \Psi(\mathbf{r},t)$.

Using Eq.(1.29) in Eq.(1.28) and taking in to account the above condition, the equation governing the dynamics of the condensate is obtained as

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},t)\right]\Psi + g\langle\psi^\dagger\psi\psi\rangle \\ &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},t) + g(n_0 + 2n_T)\right]\Psi \\ &+ g\tilde{m}\Psi^* + g\langle\phi^\dagger\phi\phi\rangle \end{aligned} \quad (1.30)$$

where $n_0 = |\Psi|^2$ is the condensate density, $n_T = \langle\phi^\dagger\phi\rangle$ is the non -condensate density, $\tilde{m} = \langle\phi\phi\rangle$ is the off-diagonal non condensate density and $\langle\phi^\dagger\phi\phi\rangle$ is the three field correlation function.

The last two terms together are called anomalous terms. The dynamics of the thermal cloud is governed by the equation obtained by subtracting Eq.(1.30) from Eq.(1.28) and is given by

$$i\hbar\frac{\partial\phi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},t)\right]\phi(\mathbf{r},t) + g\left[\psi^\dagger\psi\psi - \langle\psi^\dagger\psi\psi\rangle\right]. \tag{1.31}$$

A steady state equation governing the dynamics of the system can be obtained in a way equivalent to the derivation of time independent Schrodinger equation from time dependent equation. Assuming the external potential to be independent of time, the quantum field operator can be expressed as

$$\psi(\mathbf{r},t) = \psi(\mathbf{r})\exp\left(-i\mu t/\hbar\right) \tag{1.32}$$

where μ is the chemical potential of the system.
Using Eq.(1.32) in Eq.(1.28), we get the time independent nonlinear Schrodinger equation for the quantum field operator.

$$\mu\psi(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + g\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\right]\psi(\mathbf{r}). \tag{1.33}$$

Separating the condensate and thermal cloud parts in $\psi(\mathbf{r})$ as we have done in the time dependent case,

$$\psi(\mathbf{r}) = \Phi(\mathbf{r}) + \tilde{\Phi}(\mathbf{r}) \tag{1.34}$$

we get the steady state dynamical equation for the condensate part as

$$\begin{aligned}\mu\Phi(\mathbf{r}) &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + g(n_0 + 2n_T)\right]\Phi(\mathbf{r}) \\ &+ g\tilde{m}\Phi^* + g\langle\tilde{\Phi}^\dagger\tilde{\Phi}\tilde{\Phi}\rangle\end{aligned} \tag{1.35}$$

Depending on the temperature of the system, the significance of different terms in the above dynamical equations varies. Different mean field approximations, which are valid for different ranges of temperatures, can be introduced.

1.5.2 Hartree-Fock-Bogoliubov (HFB) model

In the HFB model, the three field correlation function is ignored [35]. This modifies the Eqs.(1.30), (1.35) and (1.31) to get

$$\begin{aligned} i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} &= \Bigg[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},t) \\ &+ g\,(n_0+2n_T)\Bigg]\Psi(\mathbf{r},t) + g\tilde{m}\Psi^*(\mathbf{r},t) \end{aligned} \tag{1.36}$$

$$\begin{aligned} i\hbar\frac{\partial\phi(\mathbf{r},t)}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0+n_T)\right]\phi(\mathbf{r},t) \\ &+ g\left(\Psi^2+\tilde{m}\right)\phi^\dagger(\mathbf{r},t) \end{aligned} \tag{1.37}$$

and

$$\mu\Phi(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + g(n_0+2n_T)\right]\Phi(\mathbf{r}) + gm\Phi^*(\mathbf{r}). \tag{1.38}$$

where we use the approximations

$$\begin{aligned} &\phi^\dagger\phi \approx \langle\phi^\dagger\phi\rangle \\ &\phi\phi \approx \langle\phi\phi\rangle \\ &\phi^\dagger\phi\phi \approx 2\langle\phi^\dagger\phi\rangle\phi + \phi^\dagger\langle\phi\phi\rangle \end{aligned} \tag{1.39}$$

Using a transformation involving quasiparticle creation and annihilation operators, Eq.(1.37) can be diagonalized. For this, we consider

$$\phi(\mathbf{r},t) = \sum_i \left[u_i(\mathbf{r},t)a_i - v_i^*(\mathbf{r},t)a_i^\dagger\right] \tag{1.40}$$

where $a_i^\dagger$ and a_i are respectively the quasiparticle creation and annihilation operators and u_i and v_i are amplitudes. The quasiparticle operators also satisfy the bosonic commutation relations.

Applying the Bogoliubov transformation in Eq.(1.40) to Eq.(1.37) generates the following system of HFB equations for the amplitudes [35]:

$$\begin{aligned} i\hbar\frac{\partial u_i}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0+n_T)\right]u_i \\ &+ g\left(\Psi^2+\tilde{m}\right)v_i, \end{aligned} \tag{1.41}$$

$$\begin{aligned} -i\hbar\frac{\partial v_i}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0+n_T)\right]v_i \\ &+ g\left(\Psi^{*2}+\tilde{m}^*\right)u_i \end{aligned} \tag{1.42}$$

where

$$\begin{aligned} n_0 &= |\Psi|^2, \\ n_T &= \sum_i \left[|u_i|^2 N_i + |v_i|^2(N_i+1)\right], \\ \tilde{m} &= \sum_i u_i v_i^*(1+2N_i). \end{aligned} \tag{1.43}$$

Here, $N_i = \langle a_i^\dagger a_i\rangle$ is the occupation number of the ith quasi particle state. Stationary state equations for u_i and v_i can be obtained in a way very similar to that in the previous section by assuming the trapping potential to be independent of time. In that case we get

$$\begin{aligned} \epsilon_i u_i(\mathbf{r}) &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) - \mu + 2g(n_0+n_T)\right]u_i(\mathbf{r}) \\ &+ g\left(\Phi^2+\tilde{m}\right)v_i(\mathbf{r}). \end{aligned} \tag{1.44}$$

$$-\epsilon_i v_i(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + 2g(n_0 + n_T)\right] v_i(\mathbf{r}) + g\left((\Phi^*)^2 + \tilde{m}^*\right) u_i(\mathbf{r}). \quad (1.45)$$

where the quasi particle amplitudes $u_i(\mathbf{r})$ and $v_i(\mathbf{r})$ satisfy the normalization condition

$$\int (u_i^* u_j - v_i^* v_j)\, d^3r = \delta_{ij} \quad (1.46)$$

1.5.3 Hartree-Fock-Bogoliubov-Popov (HFBP) model

Despite its success in the prediction of excitation frequencies of BEC, HFB model has its shortcomings. For example, it predicts an energy gap in the excitation spectrum which is quite unphysical [35]. In order to overcome the problems of HFB model, Popov used an approximation so that a gapless spectrum is obtained [36]. The Popov approximation involves neglecting the off-diagonal non-condensate density, $\tilde{m}$ [37, 38]. This results in the set of equations known as HFBP equations given by

$$i\hbar\frac{\partial\Psi(r,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(x,t) + g(n_0 + 2n_T)\right]\Psi, \quad (1.47)$$

$$i\hbar\frac{\partial u_i}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0 + n_T)\right] u_i + g\Phi^2 v_i, \quad (1.48)$$

$$-i\hbar\frac{\partial v_i}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0 + n_T)\right] v_i + g(\Phi^*)^2 u_i. \quad (1.49)$$

The steady state equations can be obtained as

$$\mu\Phi(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r},t) + g(n_0 + 2n_T)\right]\Phi(\mathbf{r}), \quad (1.50)$$

$$\begin{aligned} \epsilon_i u_i(\mathbf{r}) &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} - \mu + 2g(n_0 + n_T)\right] u_i(\mathbf{r}) \\ &+ g\Phi^2 v_i(\mathbf{r}). \end{aligned} \tag{1.51}$$

$$\begin{aligned} -\epsilon_i v_i(\mathbf{r}) &= \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0 + n_T)\right] v_i(\mathbf{r}) \\ &+ g(\Phi^*)^2 u_i(\mathbf{r}) \end{aligned} \tag{1.52}$$

The HFBP theory is the most accurate of the mean field theories in the sense that its results agree well with the experimental observations [39, 40]. It is observed that HFBP theory explains dilute gas BEC systems for temperatures of the order of $0.6T_c$. With further decrease in temperature, results of the theory deviate from experimental observations. This is because of the fact that at temperatures close to T_c, beyond mean field effects become more significant.

1.5.4 Hartree-Fock (HF) model

The excitation generated in BEC systems ranges from low to high energies. When the high energy excitations are more significant, the quasi particle amplitude v_j will be negligible. Under this condition, the Eqs.(1.48) and (1.51) get modified as

$$i\hbar\frac{\partial u_i}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0 + n_T)\right] u_i \tag{1.53}$$

and for the time independent situation, we have:

$$\epsilon_i u_i(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2g(n_0 + n_T)\right] u_i(\mathbf{r}), \tag{1.54}$$

where

$$n_T = \sum_i |u_i|^2 N_i.$$

1.5.5 Gross-Pitaevskii (GP) equation

When the temperature of the bose system reaches absolute zero, theory predicts that the anomalous terms vanish. At this temperature, all the particles in the system will attain the ground state, exhausting the thermal cloud. That is, at this temperature, the thermal fluctuation term in the quantum field operator vanishes. Hence, the quantum field operator can be replaced by the classical order parameter. This gives rise to the 'Gross-Pitaevskii' equation introduced earlier in Eq.(1.23). Here the order parameter is normalized to the total number of particles of the system.

$$\int |\psi(\mathbf{r},t)|^2 d^3r = N. \tag{1.55}$$

A time independent form of GP equation can also be obtained as

$$\mu\Phi(\mathbf{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + g|\Phi(r)|^2\right)\Phi(\mathbf{r}). \tag{1.56}$$

1.6 Collective excitations

Collective excitations and quasiparticles are closely analogous terminologies, which represent the phenomena occurring in a macroscopic system consisting of a large number of interacting particles. For example, in the case of an electron in a semiconductor, its motion will be hindered in a complicated way by all the particles in its vicinity. Such an electron can be picturised in an alternative way as an 'almost' free electron with a different effective mass moving in a force-free space. In short, the effect of interactions can be taken into account by a change in the effective mass of the particle. The 'almost electron' is called an 'electron quasiparticle'. Usually the term quasiparticle is given to those excitations, which are built around a core particle, whereas collective excitations are aggregate behavior of the individual particles constituting the system. Another loose way of distinguishing between collective excitations and quasiparticles is based on the particles involved. If they are related with fermions, they are called quasiparticles, whereas those related with bosons are called collective excitations. But, none of these classifications are unique or universally agreed upon.

Many experiments on BEC have studied intensely the collective excitations of the condensates trapped in harmonic potentials. The reason for this choice is the fact that they can be measured very accurately and hence is a reliable method for computing the system parameters of such ultracold bosonic systems [41]. Earlier experiments on the collective excitations of BEC used condensates at very low temperature ranges where the system consists entirely of the condensate, and the collective oscillation modes were induced by a modulation of the external trapping potential [42, 43]. The collective modes were excited by applying a small time-dependent perturbation of a given frequency to the trap potential and the resulting dynamics in terms of shape oscillations of the condensate were observed [41]. These measurements led to the identification of two low-lying eigen modes of different symmetry. At temperatures above T_c, when the condensate is not present, the same calculation on the thermal cloud gave the excitations of a normal Bose gas. When excitations of the condensate was done using time dependent modulation of the trapping potential, which included a spatial displacement of the potential minimum [44], coupling between oscillations of shape and center of mass was observed. Experimentally observed frequencies [43, 44, 45] of the collective excitations show good agreement with the theoretical predictions [46, 47, 48, 49, 50]. It is observed that the mean field theory is remarkably successful in such predictions [51, 52, 53, 66]. Thus time dependent GP equation and its modifications to finite temperature will be sufficient for the theoretical analysis of condensate excitations. Analytical and numerical procedures had been followed by several people to solve the Gross-Pitaevskii equations. Numerical procedures used include the Crank-Nicolson finite difference method [54, 55, 56, 57] for computing ground state solution, time splitting method [58, 59, 60, 61]etc. Runge Kutta method also has been used [62, 63, 64] to find numerical solution of 1-dimensional and 3-dimensional GP equations. Analytical treatment of GP equation involves the minimization of energy functional followed by the computation of ground state solution and the collective excitation frequencies of the condensate by assuming a Gaussian ansatz for the choice of the variational wavefunction [65].

1.7 Finite temperature condensates-effect of interaction

Effect of interaction and finite temperatures is a topic of significance in the case of BEC as these conditions make the theoretical models close to the experimentally realized ones. For temperatures close to T_c, the interaction energy per particle of the condensate corresponds to the temperature [66],

$$T_0 = \frac{15^{2/5}}{7}\left(\frac{Na}{\overline{a}}\right)^{2/5}\frac{\hbar\omega}{k}, \tag{1.57}$$

where $\overline{a}$=$\sqrt{\hbar/m\bar{\omega}}$ and $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. But when the temperature is much greater than T_0, properties of the excitations are rather independent of the interaction of the condensate with the excitations. The expression in Eq.(1.57) when expressed in terms of the transition temperature for non-interacting system, T_c, can be given by [66]

$$T_0 \approx 0.45\left(\frac{N^{1/6}a}{\overline{a}}\right)^{2/5} T_c. \tag{1.58}$$

The dimensionless quantity in the parenthesis in Eq.(1.58) determines the effect of interactions on the properties of the thermal excitations in a harmonic trap. The quantity is usually less than one and its dependence on the number of particles of the condensate is small. This minimizes its effect on the thermodynamical properties of the condensate. At temperature $T > T_0$, the number of particles in the thermal cloud can be approximately given as [66]

$$N_{exc} = N\left[t^3 + \frac{\zeta(2)}{\zeta(3)}t\frac{\mu}{kT_c}\right] \tag{1.59}$$

which, on substituting for μ becomes

$$N_{exc} = N\left[t^3 + 2.15\left(\frac{N^{1/6}a}{\overline{a}}\right)^{2/5} t^2\left(1-t^3\right)^{2/5}\right], \tag{1.60}$$

where $t = T/T_c$.

Similarly, the expression for energy

$$E = NkT_c \left[3\frac{\zeta(4)}{\zeta(3)} t^4 + \frac{5+16t^3}{7} \frac{\mu(T)}{kT_c} \right] \tag{1.61}$$

can be approximately given as

$$E = NkT_c \left[2.70t^4 + 1.12 \left(\frac{N^{1/6} a}{\bar{a}} \right)^{2/5} \left(1 + 3.20t^3\right) \left(1 - t^3\right)^{2/5} \right], \tag{1.62}$$

where $\zeta(\alpha)$ is the Riemann zeta function [66]. In the Eq.(1.59), the first term, Nt^3, gives the number of particles in the excited states in the case of a non-interacting gas. The second term governs the effect of interaction. In condensate experiments, this term is small which implies the weak influence of interactions on the thermal cloud.

1.7.1 Finite temperature models for condensates

Although, the mean field models discussed above, namely, the HFB and HFBP models, can be applied to the study of condensates at nonzero temperatures, they cannot give an accurate description of the condensate at temperatures $T > 0.6T_c$. The shortcoming of HFB and HFBP models is that they are applicable only in the collisionless regime, where the collisional mean free path of the excited particles is much greater than the wavelength of the excitations. This assumption necessitates the presence of a low density thermal cloud surrounding the condensate. But for temperatures, $T > 0.6T_c$, we have a collision dominated regime, which can be dealt with using the hydrodynamic formalism [66]. There are a number of mean field and beyond mean field theories which are applicable in the high temperature region. Of these, the models which deserve special attention are the following [67].

1. Zaremba-Griffin-Nikuni (ZGN) theory: This theory essentially follows a meanfield approach as in Eq.(1.30) under Popov's approximation. It differs from the HFBP formalism in the retention of the three-field correlation term $< \psi^\dagger \psi \psi >$. This term takes into account the interaction between the condensate atoms and the thermal cloud. In the ZGN formalism, a semiclassical approximation is applied to the noncondensate, which can be represented by a phase space distribution function. This function is described by a quantum kinetic equation which is coupled to the condensate through meanfield and interparticle collisions. Proceeding this way, ZGN theory leads to a system of two-fluid hydrodynamic equations for the trapped Bose gases. In the limiting case, the ZGN formalism can be shown to be consistent with Landau's two-fluid model for liquid helium [74, 75, 76].

2. Number conserving approaches: In number conserving approaches the total number of atoms in the system is conserved, but the number of excited particles is not. In these methodologies, number of condensed atoms is not a separate variable. it is determined from $N_0 = N - N_{exc}$ where $N_{exc} = \sum_k N_k$[81, 82].

3. Projected Gross-Pitaevskii equation formalism: This model was proposed by Davis et. al. [77, 78, 79, 80]. The idea behind this formalism is the assumption that the low lying energy modes of the Bose system are largely populated. This

makes a classical field interpretation of the system possible, where the field is evolving according to the modification of Gross-Pitaevskii equation with a projection operator in the region of low temperature. The results using PGPE formalism are close to that of Bogoliubov formalism.

1.8 Objective of this work

There have been numerous papers in both experimental and theoretical regime in the field of Bose-Einstein condensation. These studies have generated a large amount of results regarding the properties and dynamics of BEC. But there still remains open questions in the field of finite temperature BEC, dimensional reduction of BEC etc. In this work we try to make the theoretical analysis of collective excitations more relevant by introducing usually neglected corrections into the potential acting on BEC. We also analyze the effect of finite temperature on the collective excitations. In chapter 2, we study the effect of anharmonicity in the trapping potential and gravity on the collective excitations of a 'quasi-two-dimensional(2D)' BEC. In Chapter 3, as a step towards the finite temperature regime, the influence of beyond mean field effects on the collective excitations of BEC is analyzed. In this case also, the test bed is a quasi-2D BEC. In chapter 4, a quasi-2D condensate at a finite temperature is analyzed. The effect of finite temperature is taken in to account by including a static thermal cloud term in the potential of the condensate. In addition, the effect of gravity and a quartic anharmonicity along with beyond mean field effects are also included in the potential. The influence of the thermal cloud and the other additional effects on the behavior of collective excitation frequencies are analyzed. In chapter 5, the overall conclusion of the work is discussed. This chapter also envisages the prospects for future works.

2

Effect of Gravity and Anharmonicity on the Collective Excitations of a Quasi-Two-Dimensional Bose-Einstein Condensate

2.1 Introduction

Bose-Einstein condensation and condensates (BEC) have been a field of rigorous research for both experimentalists and theoreticians for the past few decades. Being a macroscopic quantum system, investigations on its peculiar properties have resulted in interesting conclusions. The inherent dynamics of BEC can be studied to the level of reasonable accuracy using time dependent GP equation. Experimental and theoretical research investigations in the field of Bose-Einstein condensation and condensate (BEC) had a boost since 1995, when the first condensate was realized by Anderson et.al. [8] and soon by others in the trapped vapors of alkali metals [9, 10]. Since then a large amount of research work has been done on different aspects of BEC. Of these, collective excitations of BEC in different trap potentials and influence of dimensionality on BEC are two of the important topics of research in this area [25, 26, 27].

A large number of papers have extensively investigated the collective excitations in a BEC using various theoretical methods [28, 46]. But only a few have taken into account the influence of earth's attractive force on the condensate [29]. Here, we are analyzing the effect of gravity on the collective excitations generated in a quasi-two-dimensional (2D) BEC 'caged' in an anharmonic potential trap. A quasi-2D BEC is the one for which the trapping frequency in one of the three dimensions, ω_z(say) is so large that $\hbar\omega_z$ is much greater than the mean field interparticle interaction ($n_0 g$), where n_0 and g are number density and the coupling strength respectively [22]. A 'quasi- reduced dimensional condensate' is chosen as the test bed for this analysis thanks to the richness of the dynamics it exhibits. It is anticipated that the 'quasi reduction' of dimension, along with the effect of gravity and the anharmonicity of the trap potential, will result in interesting new behavior of the collective excitations. The analysis involves a variational procedure to compute the collective excitation frequencies of the system [28, 33]. Consequently a Gaussian wavefunction containing a few variational parameters seems to be a simple and ideal choice.

2.2 Equations governing the condensate

The necessary dynamics of BEC at near absolute zero temperature is well described by the non-linear Schrodinger equation, also known as Gross-Pitaevskii (GP) equation [30], which is given in the equation (1.28) as

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + g|\Psi|^2\right)\Psi \tag{2.1}$$

The 3-dimensional GP equation can be obtained from the energy functional,

$$E[\Psi] = \int d^3r\left(\frac{\hbar^2}{2m}|\nabla\Psi|^2 + V_{ext}|\Psi|^2 + \frac{g}{2}|\Psi|^4\right) \tag{2.2}$$

by a variational procedure

$$\frac{\delta E}{\delta \Psi^*} = i\hbar \frac{\partial \Psi}{\partial t} \tag{2.3}$$

For simplicity, we are considering an axially symmetric harmonic trap such that $\omega_x = \omega_y = \omega_0$. Also, we are choosing $\omega_z \gg \omega_0$ so that in the z-direction, the particles are prevented from having any dynamics other than the zero point oscillations. Thus, the condensate is essentially a '*quasi-two-dimensional*' one. Then $\Psi(\mathbf{r})$ can be approximately written as

$$\Psi(\mathbf{r}) = \Phi(\boldsymbol{\rho})\varphi_0(z) \tag{2.4}$$

where $\boldsymbol{\rho} = (x, y)$ and

$$\varphi_0(z) = \left(\frac{1}{\pi \ell^2}\right)^{\frac{1}{4}} \exp\left(-z^2/2\ell^2\right)$$

is the normalized ground state wavefunction of the trap in the z-direction with $\ell = \sqrt{\hbar/(m\omega_z)}$ as the width of the Gaussian. The normalization condition is

$$\int d^2\rho |\Phi(\boldsymbol{\rho})|^2 = N \tag{2.5}$$

$$\int dz |\varphi_0(z)|^2 = 1 \tag{2.6}$$

Now, for a quasi-two-dimensional condensate, the GP equation can be obtained from Eq.(2.2) by using the separated form of wavefunction given in Eq.(2.4). The energy functional for quasi-two-dimensional BEC can be obtained as

$$\begin{aligned} E[\Phi] &= \int d^3r \left(\frac{\hbar^2}{2m}|\nabla(\Phi(\boldsymbol{\rho})\varphi_0(z)|^2\right. \\ &+ \left. V_{ext}|\Phi(\boldsymbol{\rho})\varphi_0(z)|^2 + \frac{g}{2}|\Phi(\boldsymbol{\rho})\varphi_0(z)|^4\right) \end{aligned} \tag{2.7}$$

As we are considering a quasi-2D trap,

$$\begin{aligned} E[\Phi] &= \left[\frac{\hbar^2}{2m}\int d^2\rho|\nabla(\Phi(\boldsymbol{\rho}))|^2\int dz|\varphi_0(z)|^2\right. \\ &+ \int d^2\rho|\Phi(\boldsymbol{\rho})|^2\int dz|\frac{\partial}{\partial z}\varphi_0(z)|^2 \\ &+ V_{ext}\int d^2\rho|\Phi(\boldsymbol{\rho})|^2\int dz|\varphi_0(z)|^2 \\ &+ \left.\frac{g}{2}\int d^2\rho|\Phi(\boldsymbol{\rho})|^4\int dz|\varphi_0(z)|^4\right] \end{aligned} \tag{2.8}$$

Since $\varphi_0(z)$ is normalized, after the integration, we get

$$\begin{aligned} E[\Phi] &= \int d^2\rho\left(\frac{\hbar^2}{2m}|\nabla_2\Phi(\boldsymbol{\rho})|^2\right. \\ &+ \left. V_2|\Phi(\boldsymbol{\rho})|^2+\frac{g^{2d}}{2}|\Phi(\boldsymbol{\rho})|^4\right)+\frac{N}{2\ell^2}. \end{aligned} \tag{2.9}$$

From the above expression for energy functional, using the variational principle as in Eq.(2.3), we can get the quasi-two-dimensional GP equation as

$$i\hbar\frac{\partial\Phi(\boldsymbol{\rho},t)}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla_2^2+V_2(\boldsymbol{\rho})+g^{(2d)}|\Phi|^2\right)\Phi(\boldsymbol{\rho},t). \tag{2.10}$$

The trapping can be oriented in such a way that the gravity is acting along the x-direction. With the effect of gravity and anharmonicity, the external potential acting on the condensate has the form,

$$V_2=\frac{1}{2}m\left[\omega_x^2(x-x_a)^2+\omega_y^2y^2\right]+\lambda\left(x^4+y^4\right) \tag{2.11}$$

where $x_a = -g_a/\omega_x^2$ with g_a as the acceleration due to gravity and $g^{(2d)} = \sqrt{8\pi}a\hbar^2/(m\ell)$ is the interaction term for the quasi-2D boson condensate. It is obtained from the interaction term in three dimension (g^{3d}), by performing the z integration in the following way:

$$\begin{aligned} g^{(2d)} &= g\int_{-\infty}^{\infty} |\varphi_0(z)|^4 dz \\ &= g\frac{1}{\sqrt{2\pi}\ell} = \frac{4\pi a\hbar^2}{m}\frac{1}{\sqrt{2\pi}\ell} \\ &= \frac{\sqrt{8\pi}a\hbar^2}{m\ell}. \end{aligned} \tag{2.12}$$

Thus the dimensional reduction of the condensate has resulted in the inter-particle interaction term, which depend on the trap frequency in the z-direction [31]. From the expression for $g^{(2d)}$, it is clear that the inter particle interaction can be controlled to some extent by the change in the trap frequency along the z-direction. The λ dependent term in Eq.(2.11) is the quartic anharmonicity of the trap and λ is a controllable parameter ($0 \leq \lambda \leq 1$). To perform the variational calculation, we choose the condensate wave function to be of the form [28, 32]

$$\begin{aligned} \Phi(x,y,t) = \quad & A(t)\prod_{\eta=x,y}\exp\left(-\frac{|\eta-\eta_0(t)|^2}{2w_\eta^2}\right. \\ & \left. +i\left[\eta\alpha_\eta(t)+\eta^2\beta_\eta(t)\right]\right), \end{aligned} \tag{2.13}$$

where $A(t)$ is the amplitude, $w_\eta(t)$ is the width, $\alpha_\eta(t)$ is the slope and $\beta_\eta(t)$ is the curvature of the Gaussian distribution at time t. The quasi-two-dimensional GP equation can also be obtained by the minimization of the action associated with the following Lagrangian density.

$$\mathcal{L} = \frac{i\hbar}{2}\left(\Phi\frac{\partial\Phi^*}{\partial t} - \Phi^*\frac{\partial\Phi}{\partial t}\right) + \frac{\hbar^2}{2m}|\nabla_2\Phi|^2 + V_2|\Phi|^2 + \frac{g^{(2d)}}{2}|\Phi|^4 \tag{2.14}$$

This actually provides us an alternate way of finding the solution of the Eq.(2.10). Now, substituting Eq.(2.13) in Eq.(2.14) and integrating the Lagrangian density over the 2D space, we can obtain the Lagrangian [33]:

$$L = \int_{-\infty}^{\infty} d^2\rho \mathcal{L} = L_1 + L_2 + L_3 + L_4, \tag{2.15}$$

and the different terms constituting the Lagrangian can be obtained as

$$\begin{aligned} L_1 &= \int_{-\infty}^{\infty} d^2\rho \frac{i\hbar}{2}\left[\Phi\frac{\partial \Phi^*}{\partial t} - \Phi^*\frac{\partial \Phi}{\partial t}\right] \\ &= N\hbar \sum_\eta \left[\eta_0 \dot{\alpha_\eta} + \left[\eta_0^2 + \frac{w_\eta^2}{2}\dot{\beta}_\eta\right]\right], \end{aligned} \tag{2.16}$$

$$\begin{aligned} L_2 &= \int_{-\infty}^{\infty} d^2\rho \frac{\hbar^2}{2m}|\nabla_2\Phi|^2 \\ &= \frac{N\hbar^2}{2m}\left[(\alpha_\eta + 2\eta_0\beta_\eta)^2 + \frac{1}{2}\left(\frac{1}{w_\eta^2} + 4\beta_\eta^2 w_\eta^2\right)\right], \end{aligned} \tag{2.17}$$

$$\begin{aligned} L_3 &= \int_{-\infty}^{\infty} d^2\rho V_2|\Phi|^2 \\ &= \int_{-\infty}^{\infty} d^2\rho \left[\frac{1}{2}m\left[\omega_x^2(x-x_a)^2 + \omega_y^2 y^2 + \omega_z^2 c_1\right]\right. \\ &+ \left.\lambda\left(x^4 + y^4 + c_2\right)\right]|\Phi|^2 \\ &= \frac{1}{2}m\omega_0^2\left(\eta_0^2 + \frac{w_\eta^2}{2}\right) + mg_a x_0 + \frac{mg_a^2}{2\omega_0^2} + \frac{1}{4}\hbar\omega_z \\ &+ \frac{\lambda}{2}\left[\eta_0^4 + 3\eta_0^2 w_\eta^2 + \frac{3}{4}w_\eta^4 + \frac{3}{16}\frac{\lambda\hbar^2}{m^2\omega_z^2}\right], \end{aligned} \tag{2.18}$$

$$L_4 = \int_{-\infty}^{\infty} d^2\rho \frac{g^{(2d)}|\Phi|^4}{2} = \frac{g^{(2d)}N^2}{4\pi w_x w_y}. \tag{2.19}$$

Thus

$$\begin{aligned} L &= N \sum_{\eta=x,y} \left\{ \hbar \left[\eta_0 \dot{\alpha_\eta} + \left(\eta_0^2 + \frac{w_\eta^2}{2} \right) \dot{\beta_\eta} \right] \right. \\ &+ \frac{\hbar^2}{2m} \left[(\alpha_\eta + 2\eta_0 \beta_\eta)^2 + \frac{1}{2} \left(\frac{1}{w_\eta^2} + 4\beta_\eta^2 w_\eta^2 \right) \right] \\ &+ \frac{1}{2} m\omega_0^2 \left(\eta_0^2 + \frac{w_\eta^2}{2} \right) + m g_a x_0 + \frac{m g_a^2}{2\omega_0^2} + \frac{1}{4}\hbar\omega_z \\ &+ \left. \frac{\lambda}{2} \left[\eta_0^4 + 3\eta_0^2 w_\eta^2 + \frac{3}{4} w_\eta^4 + \frac{3}{16} \frac{\lambda\hbar^2}{m^2\omega_z^2} \right] \right\} + \frac{g^{(2d)} N^2}{4\pi w_x w_y} \end{aligned} \quad (2.20)$$

The differential equations governing the variational parameters can then be obtained from the above Lagrangian, using Euler-Lagrange equations.

2.2.1 Euler-Lagranges equations for x_0 and y_0

From the Lagrangian in Eq.(2.20), we get

$$\frac{\partial L}{\partial \dot{x_0}} = 0 \quad (2.21)$$

and

$$\begin{aligned} \frac{\partial L}{\partial x_0} &= N \left[\hbar \left[\dot{\alpha_x} + 2x_0 \dot{\beta_x} \right] + \frac{2\hbar^2}{m} \left[\alpha_x + 2x_0 \beta_x \right] \beta_x \right. \\ &+ \left. m\big(\omega_0^2 x_0 + g\big) + \lambda \left[2x_0^3 + 3x_0 w_x^2 \right] \right] \end{aligned} \quad (2.22)$$

Hence the Euler-Lagrange's (E-L) equation for x_0 can be obtained as

$$\begin{aligned} \hbar \left[\dot{\alpha_x} + 2x_0 \dot{\beta_x} \right] &+ \frac{2\hbar^2}{m} \left[\alpha_x + 2x_0 \beta_x \right] \beta_x \\ &+ m \left(\omega_0^2 x_0 + g \right) + \lambda \left[2x_0^3 + 3x_0 w_x^2 \right] = 0 \end{aligned} \quad (2.23)$$

and a similar equation for y_0 can also be obtained as:

$$\begin{aligned} \hbar\left[\dot{\alpha_y}+2y_0\dot{\beta_y}\right] &+ \frac{2\hbar^2}{m}\left[\alpha_y+2y_0\beta_y\right]\beta_y \\ &+ m\omega_0^2 y_0+\lambda\left[2y_0^3+3y_0w_y^2\right]=0 \quad (2.24) \end{aligned}$$

For α_η, we get

$$\frac{\partial L}{\partial\dot{\alpha_\eta}}=N\sum\hbar\eta_0 \quad (2.25)$$

and

$$\frac{\partial L}{\partial\alpha_\eta}=N\frac{\hbar^2}{m}\left[\alpha_\eta+2\eta_0\beta_\eta\right] \quad (2.26)$$

Using E-L equation, we can write

$$\dot{\eta_o}-\frac{\hbar}{m}\left[\alpha_\eta+2\eta_0\beta_\eta\right]=0 \quad (2.27)$$

From the equation (2.27),

$$\alpha_\eta=\frac{m}{\hbar}\dot{\eta_o}-2\eta_0\beta_\eta \quad (2.28)$$

Similarly, taking the derivative of Eq.(2.27)

$$\ddot{\eta_0}=\frac{\hbar}{m}\left[\dot{\alpha_\eta}+2\eta_0\dot{\beta_\eta}+2\dot{\eta_0}\beta_\eta\right] \quad (2.29)$$

Multiplying Eq.(2.27) by $2\eta_0$, we get

$$2\eta_o\dot{\eta_o}=\frac{2\hbar}{m}\eta_o\left[\alpha_\eta+2\eta_0\beta_\eta\right] \quad (2.30)$$

For β_η,

$$\frac{\partial L}{\partial\dot{\beta_\eta}}=N\sum_\eta\hbar\left[\eta_0^2+\frac{w_\eta^2}{2}\right] \quad (2.31)$$

$$\frac{\partial L}{\partial\beta_\eta}=N\sum_\eta\frac{\hbar^2}{2m}\left[4\eta_0\left(\alpha_\eta+2\eta_0\beta_\eta\right)+4\beta_\eta w_\eta^2\right] \quad (2.32)$$

Hence the E-L equations for β_η can be obtained as:

$$[2\eta_o\dot{\eta}_o + w_\eta\dot{w}_\eta] - \frac{2\hbar}{m}\left[(\alpha_\eta + 2\eta_0\beta_\eta)\,\eta_o + \beta_\eta w_\eta^2\right] = 0 \tag{2.33}$$

Using Eq.(2.30) in Eq.(2.33), we get

$$w_\eta\dot{w}_\eta - \frac{2\hbar}{m}\beta_\eta w_\eta^2 = 0 \tag{2.34}$$

From the Eqs.(2.34) and (2.29), we can write

$$\beta_\eta = \frac{m}{2\hbar}\frac{\dot{w}_\eta}{w_\eta} \tag{2.35}$$

$$\dot{\alpha}_\eta + 2\eta_0\dot{\beta}_\eta = \frac{m}{\hbar}\left[\ddot{\eta}_0 - \frac{2\hbar}{m}\dot{\eta}_o\beta_\eta\right] \tag{2.36}$$

Similarly Eq.(2.33) gives us,

$$\alpha_\eta + 2\eta_0\beta_\eta = \frac{m}{2\hbar\eta_0}\left[2\eta_0\dot{\eta}_0 + w_\eta\dot{w}_\eta\right] - \frac{\beta_\eta w_\eta^2}{\eta_o}. \tag{2.37}$$

Using the x-component of Eq.(2.36), Eq.(2.37) and Eq.(2.35) in Eq.(2.23), we get the dynamical equation governing the x-component of the condensate center (x_0) as

$$\ddot{x}_0 + \omega_0^2 x_0 = -\left[\zeta\left(2x_0^3 + 3x_0w_x^2\right) + g_a\right]. \tag{2.38}$$

Similarly for y_0,

$$\ddot{y}_0 + \omega_0^2 y_0 = -\zeta\left[2y_0^3 + 3y_0w_y^2\right] \tag{2.39}$$

where $\zeta = \lambda/m$.

2.2.2 Euler-Lagranges equations for w_x and w_y

From the Lagrangian,

$$\frac{\partial L}{\partial \dot{w}_x} = 0 \tag{2.40}$$

and

$$\begin{aligned}\frac{\partial L}{\partial w_x} &= N\left[\hbar\dot{\beta}_x w_x + \frac{\hbar^2}{2m}\left[4\beta_x^2 w_x - \frac{1}{w_x^3}\right] + \frac{1}{2}m\omega_0^2 w_x\right.\\ &+ \left.\frac{\lambda}{2}\left[3w_x^3 + 6x_0^2 w_x\right] - \frac{g^{(2d)}N^2}{4\pi w_x^2 w_y}\right]\end{aligned} \tag{2.41}$$

The E-L equation for w_x can be written as:

$$\begin{aligned}\hbar\dot{\beta}_x w_x &+ \frac{\hbar^2}{2m}\left[4\beta_x^2 w_x - \frac{1}{w_x^3}\right] + \frac{1}{2}m\omega_0^2 w_x\\ &+ \frac{\lambda}{2}\left[3w_x^3 + 6x_0^2 w_x\right] - \frac{g^{(2d)}N}{4\pi m w_x^2 w_y} = 0.\end{aligned} \tag{2.42}$$

Similarly, using

$$\frac{\partial L}{\partial \dot{w}_y} = 0 \tag{2.43}$$

$$\begin{aligned}\frac{\partial L}{\partial w_y} &= N\left[\hbar\dot{\beta}_y w_y + \frac{\hbar^2}{2m}\left[4\beta_y^2 w_y - \frac{1}{w_y^3}\right] + \frac{1}{2}m\omega_0^2 w_y\right.\\ &+ \left.\frac{\lambda}{2}\left[3w_y^3 + 6y_0^2 w_y\right] - \frac{g^{(2d)}N^2}{4\pi w_y^2 w_x}\right]\end{aligned} \tag{2.44}$$

the E-L equations for w_y is obtained as

$$\begin{aligned}\hbar\dot{\beta}_y w_y &+ \frac{\hbar^2}{2m}\left[4\beta_y^2 w_y - \frac{1}{w_y^3}\right] + \frac{1}{2}m\omega_0^2 w_y\\ &+ \frac{\lambda}{2}\left[3w_y^3 + 6y_0^2 w_y\right] - \frac{g^{(2d)}N}{4\pi m w_y^2 w_x} = 0.\end{aligned} \tag{2.45}$$

Using Eqs.(2.34 - 2.37), (2.42) and (2.45), the dynamical equations for the width of the condensate can be written as

$$\ddot{w}_x + \omega_0^2 w_x = \frac{\hbar^2}{m^2 w_x^3} + \frac{g^{(2d)} N}{2\pi m w_x^2 w_y} - \zeta \left[3w_x^3 + 6x_0^2 w_x\right] \qquad (2.46)$$

$$\ddot{w}_y + \omega_0^2 w_y = \frac{\hbar^2}{m^2 w_y^3} + \frac{g^{(2d)} N}{2\pi m w_x w_y^2} - \zeta \left[3w_y^3 + 6y_0^2 w_y\right] \qquad (2.47)$$

From Eqs.(2.38) and (2.39), it can be seen that in the absence of gravity and anharmonicity, the center of the condensate oscillates with the frequency of the harmonic trap potential. This implies the absence of any kind of coupling between the motion of the center of the condensate and its internal degrees of freedom. But gravity, in the presence of anharmonicity, introduces such a coupling which results in the dependence of the oscillation of the center on the width of the condensate [34]. For $\lambda > 0$ the equilibrium points for Eqs.(2.38) and (2.39), for y_0 and x_0, can be obtained perturbatively. The equilibrium solutions for Eqs.(2.38) and (2.39) correspond to the stationary states of the condensate. Hence they will satisfy the equations,

$$\omega_0^2 x_0 = -\left[\zeta \left[2x_0^3 + 3x_0 w_x^2\right] + g_a\right] \qquad (2.48)$$

and

$$\omega_0^2 y_0 = -\zeta \left[2y_0^3 + 3y_0 w_y^2\right]. \qquad (2.49)$$

For y_0, it is seen that for $\lambda > 0$, the stable equilibrium point is $y_0^{(0)} = 0$. For finding the equilibrium solution for x_0, we perform a power series expansion of the Eq.(2.48) in ζ as

$$\omega_0^2\left[x_0^{(0)}+\zeta x_0^{(1)}+\cdots\right]+\zeta\left[2\left(x_0^{(0)}+\zeta x_0^{(1)}+\cdots\right)^3\right.$$
$$\left.+\quad 3\left(x_0^{(0)}+\zeta x_0^{(1)}+\cdot\cdot\right)\left(w_x^{(0)}+\zeta w_x^{(1)}+\cdot\cdot\right)^2\right]+g_a=0 \quad (2.50)$$

Comparing the zeroth order terms in ζ, we get

$$\omega_0^2 x_0^{(0)}+g_a=0 \quad (2.51)$$

or

$$x_0^{(0)}=-\frac{g_a}{\omega_0^2}. \quad (2.52)$$

Hence, the equilibrium solutions of the x and y components of the center of the condensate are 0 and $-g_a/\omega_0^2$ respectively. Thus it can be seen that gravity has caused a shift in the center of the condensate in the direction in which it is acting. It is also observed that the shift is inversely proportional to the square of the trapping frequency in that direction. From Eq.(2.46), it can also be seen that the effect of gravity intrudes into the equation only through the anharmonic term. Thus, if the trap is purely harmonic, gravity cannot affect the width of the condensate.

Now, for the coupled equations (2.46) and (2.47), we consider perturbative solutions, which give the width of the condensate. Introducing scaled variables, $\tilde{x}_0 = x_0/a_\perp$, $\tilde{\zeta} = \zeta a_\perp^2/\omega_0^2$, $\tilde{g}^{2d} = g^{(2d)}m/2\pi\hbar^2$, $\tau = \omega_0 t$ and $\tilde{w}_\eta = w_\eta/a_\perp$, where $a_\perp = \sqrt{\hbar/m\omega_0}$. Then the equations for w_x and w_y become

$$\ddot{\tilde{w}}_x+\tilde{w}_x=\frac{1}{\tilde{w}_x^3}+\frac{\tilde{g}^{2d}N}{\tilde{w}_x^2\tilde{w}_y}-\tilde{\zeta}\left[3\tilde{w}_x^3+6\left(\frac{g_a}{a_\perp\omega_0^2}\right)^2\tilde{w}_x\right] \quad (2.53)$$

$$\ddot{\tilde{w}}_y+\tilde{w}_y=\frac{1}{\tilde{w}_y^3}+\frac{\tilde{g}^{2d}N}{\tilde{w}_y^2\tilde{w}_x}-\tilde{\zeta}3\tilde{w}_y^3. \quad (2.54)$$

Since the contribution of the effect of anharmonicity causing the asymmetry between equations (2.53) and (2.54) is considered to be small, we assume the equilibrium solution of the above equations to be $\tilde{w}_0$. Expanding equations (2.53) and (2.54) around this equilibrium value, we get

$$\begin{aligned} \frac{d^2\left(\delta\tilde{w}_x+\tilde{w}_0\right)}{d\tau^2} &+ \left(\delta\tilde{w}_x+\tilde{w}_0\right) = \frac{1}{\left(\delta\tilde{w}_x+\tilde{w}_0\right)^3} \\ &+ \frac{\tilde{g}^{(2d)}N}{\left(\delta\tilde{w}_x+\tilde{w}_0\right)^2\left(\delta\tilde{w}_y+\tilde{w}_0\right)} \\ &- \tilde{\zeta}\left[6\tilde{x}_0^2\left(\delta\tilde{w}_x+\tilde{w}_0\right)+3\left(\delta\tilde{w}_x+\tilde{w}_0\right)^3\right] \end{aligned} \tag{2.55}$$

and

$$\begin{aligned} \frac{d^2\left(\delta\tilde{w}_y+\tilde{w}_0\right)}{d\tau^2} &+ \left(\delta\tilde{w}_y+\tilde{w}_0\right) = \frac{1}{\left(\delta\tilde{w}_y+\tilde{w}_0\right)^3} \\ &+ \frac{\tilde{g}^{(2d)}N}{\left(\delta\tilde{w}_y+\tilde{w}_0\right)^2\left(\delta\tilde{w}_x+\tilde{w}_0\right)} \\ &- \tilde{\zeta}\left[6\tilde{y}_0^2\left(\delta\tilde{w}_y+\tilde{w}_0\right)+3\left(\delta\tilde{w}_y+\tilde{w}_0\right)^3\right] \end{aligned} \tag{2.56}$$

Considering only terms up to the first order in $\delta\tilde{w}_\eta = \tilde{w}_\eta - \tilde{w}_0$, we arrive at the linearized equations

$$\begin{aligned} \frac{d^2\delta\tilde{w}_x}{d\tau^2}+\delta\tilde{w}_x &= -\frac{3}{\tilde{w}_0^4}\delta\tilde{w}_x-\frac{2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\delta\tilde{w}_x-\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\delta\tilde{w}_y \\ &- \tilde{\zeta}\left[6\tilde{x}_0^2\left(\delta\tilde{w}_x+\tilde{w}_0\right)+3\left(3\tilde{w}_0^2\delta\tilde{w}_x+\tilde{w}_0^3\right)\right] \end{aligned} \tag{2.57}$$

$$\begin{aligned} \frac{d^2\delta\tilde{w}_y}{d\tau^2}+\delta\tilde{w}_y &= -\frac{3}{\tilde{w}_0^4}\delta\tilde{w}_y-\frac{2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\delta\tilde{w}_y-\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\delta\tilde{w}_x \\ &- \tilde{\zeta}\left[6\tilde{y}_0^2\left(\delta\tilde{w}_y+\tilde{w}_0\right)+3\left(3\tilde{w}_0^2\delta\tilde{w}_y+\tilde{w}_0^3\right)\right] \end{aligned} \tag{2.58}$$

In order to find the frequencies of the collective excitations, we use the procedure for finding normal modes, by taking,

$$\delta\tilde{w}_x = A_1 \exp(iWt) \tag{2.59}$$

and

$$\delta\tilde{w}_y = A_2 \exp(iWt) \tag{2.60}$$

where W is the frequency of the collective excitations.
Using the expressions in Eqs.(2.59), (2.60), (2.57) and (2.58), we get

$$\begin{aligned} -W^2 A_1 \frac{\exp(iWt)}{\omega_0^2} &= -\left[\left(1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\right) A_1 + \frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4} A_2\right] \exp(iWt) \\ &- \tilde{\zeta}\left[6\tilde{x}_0^2\left(A_1 \exp(iWt) + \tilde{w}_0\right)\right. \\ &+ \left. 3\left(3\tilde{w}_0^2 A_1 \exp(iWt) + \tilde{w}_0^3\right)\right] \end{aligned} \tag{2.61}$$

Rearranging the terms, we can get

$$\begin{aligned} &\left[1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}\left[6\tilde{x}_0^2 + 9\tilde{w}_0^2\right] - \frac{W^2}{\omega_0^2}\right] A_1 \exp(iWt) \\ + &\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4} A_2 \exp(iWt) + \tilde{\zeta}\left[6\tilde{x}_0^2\tilde{w}_0 + 3\tilde{w}_0^3\right] = 0. \end{aligned} \tag{2.62}$$

In a similar way, we have

$$\begin{aligned} &\left[1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}\left[6\tilde{y}_0^2 + 9\tilde{w}_0^2\right] - \frac{W^2}{\omega_0^2}\right] A_2 \exp(iWt) \\ + &\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4} A_1 \exp(iWt) + \tilde{\zeta}\left[6\tilde{y}_0^2\tilde{w}_0 + 3\tilde{w}_0^3\right] = 0. \end{aligned} \tag{2.63}$$

Equations (2.62) and (2.63) can be expressed in matrix form as

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \tag{2.64}$$

where

$$T_{11} = 1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}\left[6\tilde{x}_0^2 + 9\tilde{w}_0^2\right] - \frac{W^2}{\omega_0^2} \tag{2.65}$$

$$T_{22} = 1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}\left[6\tilde{y}_0^2 + 9\tilde{w}_0^2\right] - \frac{W^2}{\omega_0^2} \tag{2.66}$$

and

$$T_{12} = T_{21} = \frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4}. \tag{2.67}$$

For a non trivial solution, the determinant of the coefficient matrix should vanish and we get

$$\begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} = 0. \tag{2.68}$$

Eq.(2.68) gives us

$$\begin{aligned} & \frac{W^4}{\omega_0^4} - \frac{W^2}{\omega_0^2}\left[2\left[1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}9\tilde{w}_0^2\right] + 6\tilde{\zeta}\left[\tilde{x}_0^2 + \tilde{y}_0^2\right]\right] \\ + & \left[1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}\left[6\tilde{x}_0^2 + 9\tilde{w}_0^2\right]\right] \\ \times & \left[1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}\left[6\tilde{y}_0^2 + 9\tilde{w}_0^2\right]\right] - \left[\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\right]^2 = 0 \end{aligned} \tag{2.69}$$

Solving the above equation gives,

$$
\begin{aligned}
2W^2 = & \left[2\left[1+\frac{3+2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}+\tilde{\zeta}9\tilde{w}_0^2\right]+6\tilde{\zeta}\left[\tilde{x}_0^2+\tilde{y}_0^2\right]\right]\omega_0^2 \\
& \pm\left[\left[2\left[1+\frac{3+2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}+\tilde{\zeta}9\tilde{w}_0^2\right]+6\tilde{\zeta}\left[\tilde{x}_0^2+\tilde{y}_0^2\right]\right]^2\omega_0^4\right. \\
& -4\left[\left[1+\frac{3+2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}+\tilde{\zeta}\left[6\tilde{x}_0^2+9\tilde{w}_0^2\right]\right]\times\right. \\
& \left[1+\frac{3+2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}+\tilde{\zeta}\left[6\tilde{y}_0^2+9\tilde{w}_0^4\right]\right]\omega_0^4 \\
& \left.\left.-\left[\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\right]^2\tilde{\omega}_0^4\right]\right]^{1/2}
\end{aligned}
\tag{2.70}
$$

and after simplification, Eq.(4.33) becomes:

$$
\begin{aligned}
W^2 = & \left[\left[\left[1+\frac{3+2\tilde{g}^{(2d)}N}{\tilde{w}_0^4}+\tilde{\zeta}9\tilde{w}_0^2\right]+3\tilde{\zeta}\left[\tilde{x}_0^2+\tilde{y}_0^2\right]\right]\right. \\
\pm & \left.\left[9\tilde{\zeta}^2\left[\tilde{x}_0^2-\tilde{y}_0^2\right]^2+\left(\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\right)^2\right]^{1/2}\right]\omega_0^2
\end{aligned}
\tag{2.71}
$$

Since the equilibrium values of $\tilde{x}_0$ and $\tilde{y}_0$ have already been obtained as

$$\tilde{x}_0 = -\frac{g_a}{a_\perp\omega_0^2} \tag{2.72}$$

and

$$\tilde{y}_0 = 0, \tag{2.73}$$

we can write the frequencies of the collective excitations generated as

$$W_\pm = \omega_0 k_\pm \tag{2.74}$$

where

$$k_{\pm}^2 = \left[1 + \frac{3 + 2\tilde{g}^{(2d)}N}{\tilde{w}_0^4} + \tilde{\zeta}9\tilde{w}_0^2\right] + 3\tilde{\zeta}\left[\frac{g_a}{a_\perp \omega_0^2}\right]^2$$
$$\pm\sqrt{9\tilde{\zeta}^2\left[\frac{g_a}{a_\perp \omega_0^2}\right]^2 + \left(\frac{\tilde{g}^{(2d)}N}{\tilde{w}_0^4}\right)^2} \qquad (2.75)$$

Of these two collective excitation frequencies, W_+ turns out to be a modification in the breathing mode oscillation frequency which exists in the case of a quasi-2D condensate trapped inside a purely harmonic trap [33]. It is very clear that the collective frequencies have been modified by the influence of anharmonicity and gravity. But, if $\tilde{g}^{(2d)}$ is equal to zero, from the equations (2.74) and (2.75), it is interesting to note that the other excitation frequency W_- is independent of the effect of gravity. Thus the inter particle interaction has also got some role in determining the effect of gravity on the collective excitation frequencies.

2.3 Numerical analysis

In the present case, the trap frequencies are chosen to be $\omega_z/2\pi =$ 790 Hz, in the dynamics frozen direction and $\omega_\rho/2\pi = 10$ Hz, where $\omega_\rho = \omega_x = \omega_y$ [22]. Plots showing the variation of the excitation frequencies with the number of particles in the condensate corresponding to different values of the anharmonicity parameter ζ are prepared for both attractively and repulsively interacting condensates. For attractively interacting condensate [lithium atoms, (^{7}Li), scattering length, a = -1.5nm], Fig.(2.1) shows the plots of W_-/ω_0 as a function of N for the anharmonicity parameter $\zeta = 0.1$. It is observed that the excitation frequency shows a sharp increase followed by a reappearance, if the trap is purely harmonic and the effect of gravity is neglected. This sharp increase towards infinity represents the collapse of the condensate. But when these additional effects are included, it is obvious from Fig.(2.1) that the excitation frequency has a constant value of $2\omega_0$ up to a certain value of condensate particle number, beyond which it ceases to exist. Thus the higher values of N does not support W_-. When the value of ζ is decreased, the excitation frequency exhibits a reappearance. Thus there exists a 'forbidden window' in the N values which does not support the W_- excitation. It is observed that the reappeared frequencies converge to its values in the case of a pure harmonic trap. It is to be noted that the width of the 'forbidden window' decreases with decrease in the value of ζ [Figs.(2.2) and (2.3)]. The plots of W_+/ω_0 as a function of N is shown in Figs.(2.4), (2.5), (2.6) and (2.7) for the ζ values of $0.1, 0.01, 0.0001$ and 0.00001 respectively. It is clear from these figures that the breathing mode, W_+ is destabilized due to the modifications introduced in the potential. Figure(2.4) represents a collapse of the condensate, for $N = N_{cr} \simeq 1.2 \times 10^3$. When ζ is decreased to 0.01, the collapse is followed by a reappearance of the excitation frequency at a value marginally greater than N_{cr}, where it exhibits again a sharp rise in its frequency. Beyond this point, excitation frequency is nonexistent. When the value of ζ is decreased to 0.0001, as N increases beyond N_{cr}, the reappeared excitation frequency is seen to converge to the frequency of the breathing mode i.e. $2\omega_0$. But the breathing mode is supported by the system only up to a certain value of $N \simeq 2 \times 10^3$. When the value of the anharmonicity parameter is further decreased to $\zeta = 0.00001$ [Fig.(2.7)], it is found that the breathing mode is supported to higher values of condensate particle number. If the effect of anharmonicity is neglected, it is found that the W_+ excitation frequency still exhibits the sharp rise, which is followed by its convergence to the frequency of $2\omega_0$, and it maintains this frequency to arbitrarily large values of N.

For repulsively interacting condensate [sodium atoms, ^{23}Na], the behavior of W_-/ω_0 as a function of N is shown in Fig.(2.8) corresponding to $\zeta = 0.1$. It is clear from this figure that the effect of the modifications in the potential on the collective excitation frequency W_- is small for particle numbers usual to a BEC. Figure(2.9) is the equivalent plot for $\zeta = 0.01$. Figures (2.10) and (2.11) correspond to the variation of W_+/ω_0 against N in the case of ^{23}Na for $\zeta = 0.1$ and 0.01 respectively. In a purely harmonic potential, W_+ corresponds to a breathing mode oscillation as mentioned earlier in the case of attractively interacting condensate. It is observed that anharmonicity introduced in the potential along with the effect of gravity, does not significantly affect this excitation frequency.

2.4 Conclusions

Through our investigations on a quasi-2D BEC in an anharmonic trap in the presence of gravity, we could deduce that the anharmonicity and gravity result in a coupling between the oscillation of the center of the condensate and its internal degrees of freedom, which is absent if the trap is purely harmonic and if the effect of gravity is neglected. We could conclude that gravity has got an effect on the oscillation frequency of the width of the condensate only if anharmonicity is present. It is also noted that inter-particle interaction does have an effect on the influence of gravity on the collective excitation frequencies. It is observed that the anharmonicity in the potential and gravity result in a modified behavior of both the collective excitation frequencies. The notable difference is that in the case of attractively interacting condensates, the effect of anharmonicity and gravity is prominent on both the excitation frequencies. But when it comes to repulsively interacting condensates, the effect of the modifications in the potential on the excitation frequencies is rather negligible at low values of N.

2.5 Collective excitation frequencies ($W_{\pm}$) against condensate particle number(N)

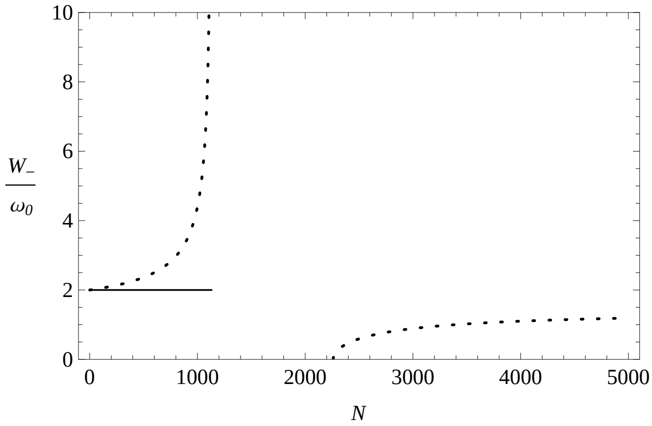

Figure 2.1: Collective excitation frequency W_- as a function of the number of condensed particles N for ^{7}Li in the case of a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.1$) where the effect of gravity is included (solid line).

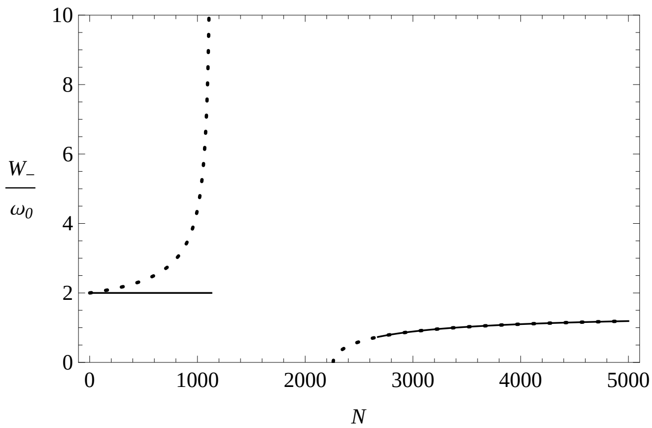

Figure 2.2: Variation of collective excitation frequency W_- with the number of condensed particles N for ^{7}Li in a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.00001$) where the effect of gravity is included (solid line).

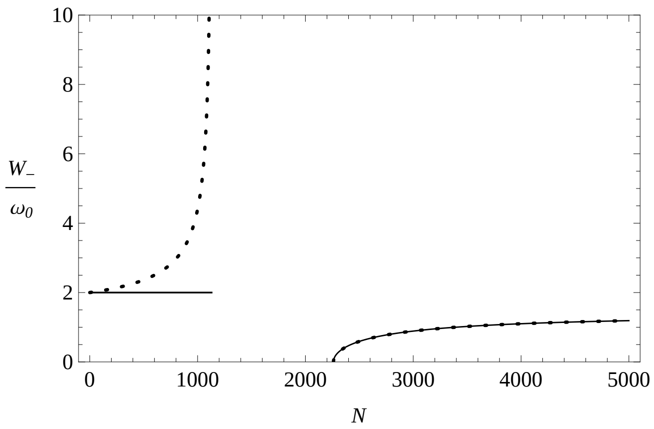

Figure 2.3: Collective excitation frequency W_- against the number of condensed particles N for ^{7}Li in the case of a pure harmonic trap (dashed line) and in a harmonic trap where the effect of gravity is included (solid line).

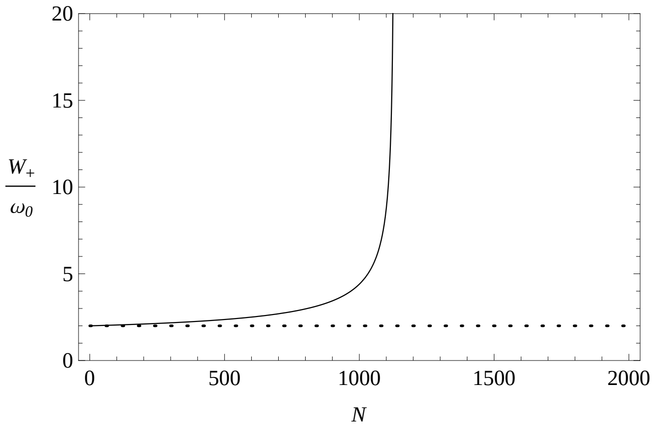

Figure 2.4: Collective excitation frequency W_+ as a function of the number of condensed particles N for ^{7}Li: In pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.1$) where the effect of gravity is included (solid line).

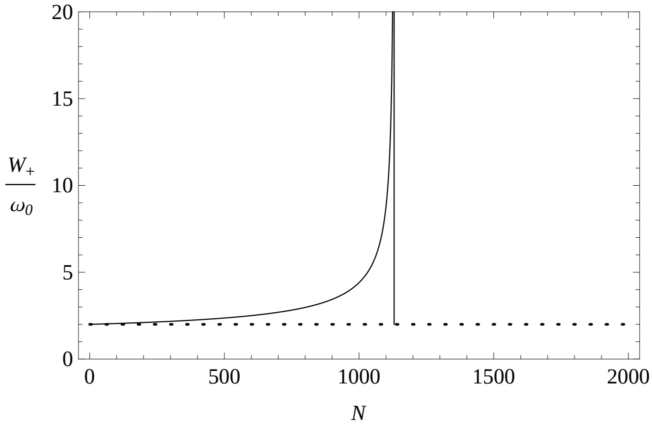

Figure 2.5: Variation of collective excitation frequency W_+ with the number of condensed particles N for ^{7}Li in the case of a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.01$) under the influence of gravity (solid line).

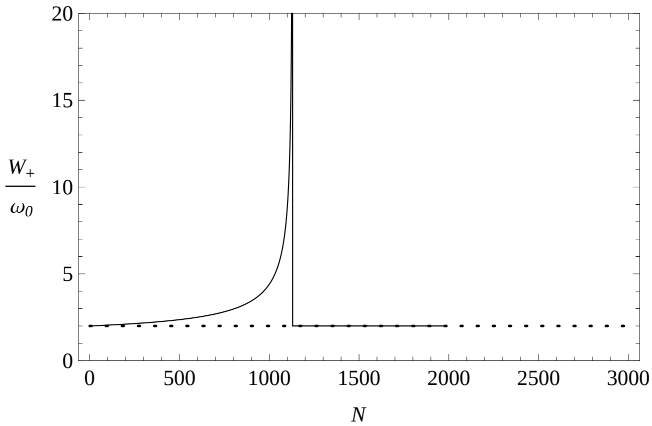

Figure 2.6: Collective excitation frequency W_+ plotted against number of condensed particles N for ^{7}Li in the case of a pure harmonic trap (dashed line) and in an anharmonic trap ($\zeta = 0.0001$) where the effect of gravity is included (solid line).

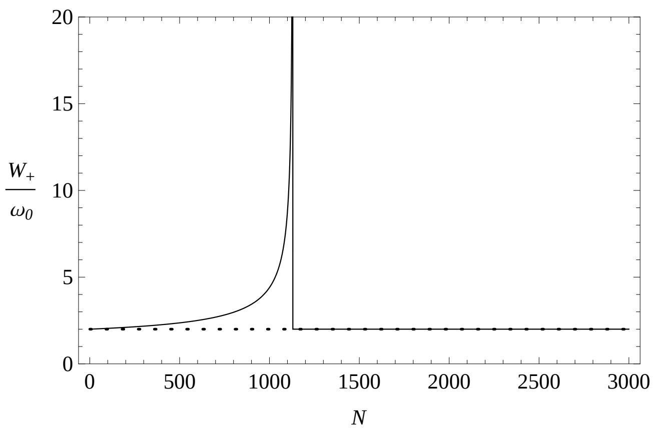

Figure 2.7: Collective excitation frequency W_+ as a function of the number of condensed particles N for ^{7}Li in a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.00001$) where the effect of gravity is included (solid line).

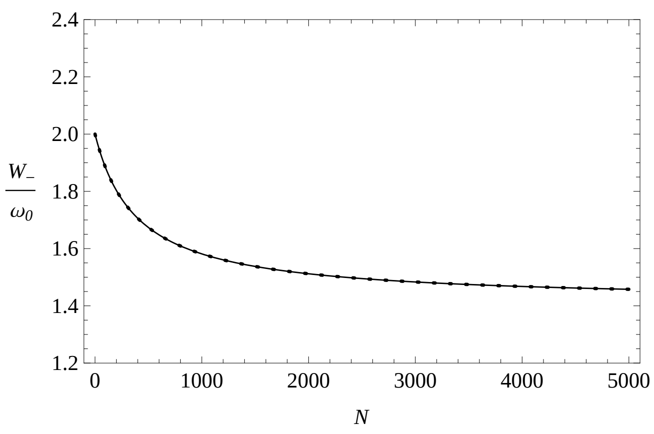

Figure 2.8: Variation of collective excitation frequency W_- with condensed particle number N for ^{23}Na in a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.1$) where the effect of gravity is included (solid line).

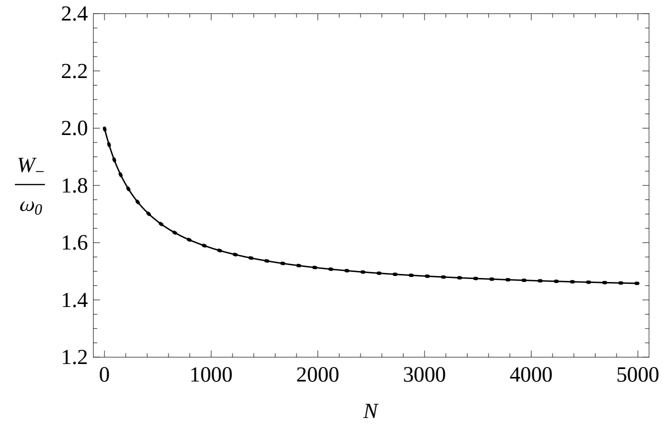

Figure 2.9: Collective excitation frequency W_- versus number of condensed particles N for ^{23}Na in the case of a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.01$): effect of gravity is included (solid line).

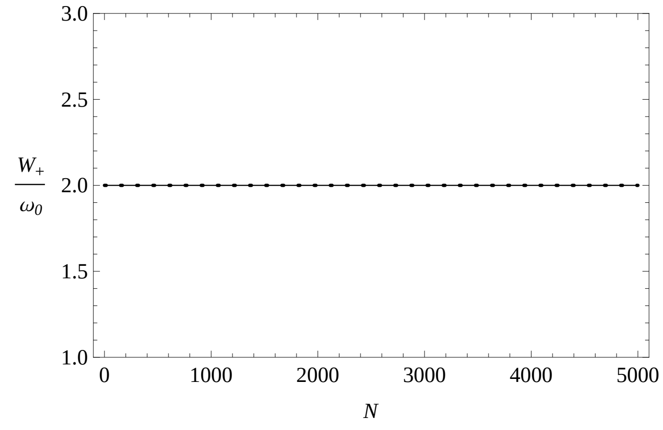

Figure 2.10: Collective excitation frequency W_+ versus number of condensed particles N for ^{23}Na in the case of a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.1$): effect of gravity is included (solid line).

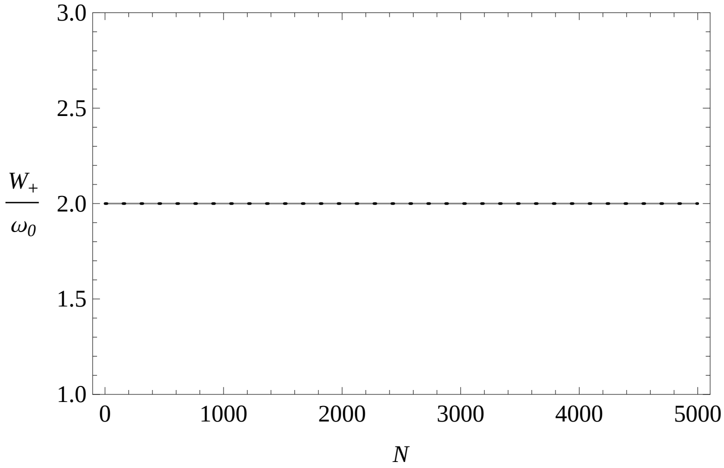

Figure 2.11: Collective excitation frequency W_+ versus number of condensed particles N for ^{23}Na in the case of a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.01$) under the effect of gravity (solid line).

3 Collective Excitations of a Quasi-Two-Dimensional BEC: Beyond Mean Field Theory

3.1 Introduction

The Gross-Pitaevskii (GP) equation, which is essentially a mean field equation, has been remarkably successful in predicting the properties of condensates near absolute zero temperature. The essential reason behind this success is traced to the low value of number density and scattering length of the experimentally realized condensates [68]. But, the development of ingenious methods, which exploit phenomena like Feshbach resonance, have resulted in condensate experiments requiring analysis which goes beyond the mean field realm [69]. It is also observed that the pseudopotential approximation fails in the tight confinement regime [71]. Hence a generalization of GP equation to include beyond mean field effects is relevant and essential. The collective excitations in BEC in the mean field regime have been analyzed rigorously in the literature and a large amount of valuable information has already been derived [25, 26, 28, 33, 46, 73]. Hence, the aim of the present analysis is to study the way these excitation frequencies get modified by the beyond mean field effects. As in the earlier work, here also we choose a quasi-2D condensate for our analysis. An obvious choice for the condensate wavefunction is a Gaussian [33]. In order to step out of the pseudopotential regime, a potential that takes into account the energy dependence of the scattering amplitude is obtained using an effective range expansion [13]. In the case of repulsively interacting system, the effect of quantum fluctuation is also taken in to account [72].

3.1.1 Effective range expansion

When scattering in the low energy realm is analyzed, a single parameter, namely the scattering length, is enough and sufficient to characterize the scattering phenomenon. The reason behind such a simplification is the fact that, at such low energies, only s-wave ($l = 0$ and hence independent of the shape of the potential) scattering is of significance.

But when the energy of interacting particles increases, more and more partial waves will contribute to the scattering process and this makes the scattering phenomena dependent on both the energy and the scattering angle. But when the energy is only slightly higher than the zero-energy, the scattering would still be dominated by the s-waves, but the scattering cross-section now will be energy dependent. This energy dependence is characterized by the parameter 'effective range'. In the case of a BEC, only s-wave scattering between the particles is of importance, thanks to its dilute nature. Hence the scattering amplitude can be expressed in terms of the s-wave phase shift $\delta_0(k)$ as [68]

$$f(k,\theta) = \frac{1}{k\cot\left[\delta_0\left(k\right)\right] - \imath} \tag{3.1}$$

The 'effective range expansion' of $k\cot\left[\delta_0\left(k\right)\right]$ gives

$$k\cot\left[\delta_0\left(k\right)\right] = -\frac{1}{a} + \frac{1}{2}r_e k^2 + \cdots\cdots \tag{3.2}$$

where r_e is the effective range. Up to the second order of k, the expression for the real part of the scattering amplitude is given by,

$$\mathrm{Re}\left[f(k,\theta)\right] = -a + a^2\left(a - \frac{1}{2}r_e\right)k^2 \tag{3.3}$$

The effective range contribution to the potential in a BEC can be given as [68]

$$g_{eff}^{(3d)} = \frac{4\pi a^2\left(a - \frac{1}{2}r_e\right)\hbar^2}{3m} \tag{3.4}$$

The interesting point is that $g_{eff}^{(3d)}$ does not vanish even when $r_e = 0$. This is because of the fact that the real part of the scattering amplitude has a k^2 dependence even when r_e is equal to zero. Effective range, r_e and s-wave scattering length, 'a' are related in general. For hard sphere potential,

$$r_e = \frac{2}{3}a. \tag{3.5}$$

Using this expression for effective range in Eq.(3.4), we get

$$g_{eff}^{(3d)} = \frac{8\pi a^3\hbar^2}{3m}. \tag{3.6}$$

3.1.2 Quantum fluctuations

According to Heisenberg's uncertainty principle,

$$\Delta E\Delta t \approx \frac{\hbar}{2}. \tag{3.7}$$

This implies that the law of conservation of energy can be violated for extremely short span of time. The consequence of this is the creation of particle-antiparticle pairs of virtual particles which have transient existence. The effect of this on the energy of the system is large enough to be measurable. In short, quantum fluctuation can be defined as the transient appearance of energetic particles out of empty space, which results in a temporary change in the energy of a space point. It must be kept in mind that the quantum fluctuation is entirely different from the thermal fluctuation, which are arbitrary deviation of a system from its average equilibrium state.

3.2 Modified GP equation

The three dimensional Gross-Pitaevskii (GP) equation for a BEC in a potential trap can be given as [30]:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + g^{(3d)}|\Psi|^2\right)\Psi, \tag{3.8}$$

where m is the mass of the condensate particle and $V_{ext}(\mathbf{r})$ is the external trapping potential. In the present case $V_{ext}(\mathbf{r})$ is chosen to be harmonic and $g^{(3d)} = 4\pi a\hbar^2/m$ is the coupling parameter which takes in to account the inter-particle interaction of the condensate atoms, where a is the s-wave scattering length. The 3-dimensional GP equation which takes into account the modifications in the potential corresponding to quantum fluctuation and effective range expansion can be obtained from the following energy functional [68],

$$E[\Psi] = \int d^3r \left(\frac{\hbar^2}{2m}|\nabla\Psi|^2 + V_{ext}|\Psi|^2 + \frac{g^{(3d)}}{2}|\Psi|^4 \right.$$
$$\left. + \frac{2}{5}g_1^{(3d)}|\Psi|^5 + \frac{g_2^{(3d)}}{4}|\Psi|^2\nabla^2\left(|\Psi|^2\right)\right), \tag{3.9}$$

using the variational procedure. The resulting modified GP equation is

$$i\hbar\frac{\partial\Psi}{\partial t} = \Bigg[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + g^{(3d)}|\Psi|^2 + g_1^{(3d)}|\Psi|^3 + \frac{g_2^{(3d)}}{2}\nabla^2\left(|\Psi|^2\right)\Bigg]\Psi \tag{3.10}$$

where

$$g_1^{(3d)} = \frac{32a^{3/2}}{3\sqrt{\pi}}g^{(3d)}, g_2^{(3d)} = \frac{8\pi a^3\hbar^2}{3m} \tag{3.11}$$

and $g_1^{(3d)}$ governs the effect of quantum fluctuation and $g_2^{(3d)}$ takes into account the contribution from effective range expansion to the potential. It must be kept in mind that in the case of attractively interacting condensates the expression for $g_1^{(3d)}$ is meaningless [72]. Hence, in such a case, only the effective range modification to the potential is considered. As in chapter 2, the trap frequencies are chosen such that $\omega_x = \omega_y = \omega_0$ and $\omega_z \gg \omega_0$ making the condensate a quasi-2D one. Under this condition, as it has been done earlier, the variational wave function of the condensate can be written as $\Psi(\mathbf{r}) = \Phi(\boldsymbol{\rho})\varphi_0(z)$ where $\boldsymbol{\rho} = (x, y)$ and

$$\varphi_0(z) = \left(\frac{1}{\pi\ell^2}\right)^{\frac{1}{4}} \exp\left(-z^2/2\ell^2\right) \tag{3.12}$$

with $\ell = \sqrt{\hbar/(m\omega_z)}$ as the width of the Gaussian and

$$\Phi(x, y, t) = A(t)\prod_{\eta=x,y}\exp\Bigg(-\frac{|\eta-\eta_0(t)|^2}{2w_\eta^2} + i\left[\eta\alpha_\eta(t) + \eta^2\beta_\eta(t)\right]\Bigg) \tag{3.13}$$

is the Gaussian trial wavefunction.

Making the trap quasi-2D results in the modification of the parameters $g^{(3d)}$, $g_1^{(3d)}$ and $g_2^{(3d)}$ and consequently of the GP equation. The expression for $g^{(2d)}$, interparticle interaction in the quasi-2D case has already been obtained to be

$$g^{(2d)} = \frac{\sqrt{8\pi} a\hbar^2}{m\ell} \tag{3.14}$$

Similarly the expression for $g_1^{(2d)}$, the term governing the effect of quantum fluctuation in the case of a quasi-2D BEC, can be obtained as follows.

3.2.1 Quasi-2D reduction of quantum fluctuation term

The contribution of quantum fluctuation for quasi-2D case can be obtained by performing integration over the z component in the following way:

$$\begin{aligned} g_1^{(2d)} \int_{-\infty}^{\infty} |\Phi|^5 d^2\rho = g_1^{(3d)} \int_{-\infty}^{\infty} |\Psi|^5 d^3 r \\ = \ g_1^{(3d)} \int_{-\infty}^{\infty} |\Phi|^5 d^2\rho \left(\frac{1}{\pi\ell^2}\right)^{5/4} \int_{-\infty}^{\infty} \exp\left(-5z^2/2\ell^2\right) dz \\ = \ \left(\frac{1}{\pi\ell^2}\right)^{5/4} \frac{2\sqrt{\pi}\ell}{\sqrt{10}} g_1^{(3d)} \end{aligned} \tag{3.15}$$

Using the expression for $g_1^{(3d)}$ from Eq.(3.11), we get

$$\begin{aligned} g_1^{(2d)} &= \left(\frac{1}{\pi\ell^2}\right)^{5/4} \frac{64\ell}{3\sqrt{10}} a^{1/4} g^{(3d)} \\ &= \frac{256}{3\sqrt{10}} \frac{\hbar^2 a^{5/2}}{m\pi^{1/4}\ell^{3/2}}. \end{aligned} \tag{3.16}$$

3.2.2 Quasi-2D reduction of effective range interaction term

Following the same procedure as in the case of Eq.(3.15), we can get the quasi-2D form for effective range interaction term,

$$
\begin{aligned}
& g_2^{(2d)} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^2 \nabla_2^2 |\Phi(\boldsymbol{\rho})|^2 d^2\rho = g_2^{(3d)} \int_{-\infty}^{\infty} |\Psi|^2 \nabla^2 (|\Psi|^2) d^3 r \\
= \;& g_2^{(3d)} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})\varphi_0(z)|^2 \left[|\varphi_0(z)|^2 \nabla_2^2 |\Phi(\boldsymbol{\rho})|^2 \right. \\
+ \;& \left. |\Phi(\boldsymbol{\rho})|^2 \frac{\partial^2}{\partial z^2} |\varphi_0(z)|^2 \right] d^3 r \\
= \;& g_2^{(3d)} \left[\int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^2 \nabla_2^2 |\Phi(\boldsymbol{\rho})|^2 d^2 \boldsymbol{\rho} \int_{-\infty}^{\infty} |\varphi_0(z)|^2 dz \right. \\
+ \;& \left. \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^4 \int_{-\infty}^{\infty} |\varphi_0(z)|^2 \frac{\partial^2}{\partial z^2} |\varphi_0(z)|^2 dz \right]. \qquad (3.17)
\end{aligned}
$$

Using the expression for $g_2^{(3d)}$ and integrating over the z-dimension, we obtain

$$
\begin{aligned}
& g_2^{(2d)} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^2 \nabla_2^2 |\Phi(\boldsymbol{\rho})|^2 d^2 \boldsymbol{\rho} = \frac{8\pi a^3 \hbar^2}{3m} \left[\frac{1}{\sqrt{2\pi}\ell} \right. \\
& \left. \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^2 \nabla_2^2 |\Phi(\boldsymbol{\rho})|^2 d^2\rho \; - \frac{1}{\sqrt{2\pi}\ell^3} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^4 d^2\rho \right] \\
= \;& \frac{4\sqrt{2\pi} a^3 \hbar^2}{3m\ell} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^2 \nabla_2^2 |\Phi(\boldsymbol{\rho})|^2 d^2\rho \\
- \;& \frac{4\sqrt{2\pi} a^3 \hbar^2}{3m\ell^3} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^4 d^2\rho. \qquad (3.18)
\end{aligned}
$$

Thus the dimensional reduction of $g_2^{(3d)}$ terms results in two terms

$$
\begin{aligned}
g_2^{(3d)} \int_{-\infty}^{\infty} |\Psi|^2 \nabla^2 (|\Psi|^2) \;=\; & g_2^{(2d)} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^2 \nabla_2^2 |\Phi(\boldsymbol{\rho})|^2 d^2\rho \\
+ \;& g'^{(2d)} \int_{-\infty}^{\infty} |\Phi(\boldsymbol{\rho})|^4 d^2\rho \qquad (3.19)
\end{aligned}
$$

where

$$g_2^{(2d)} = \frac{4\sqrt{2\pi}a^3\hbar^2}{3m\ell}, \qquad g'^{(2d)} = -\frac{4\sqrt{2\pi}a^3\hbar^2}{3m\ell^3}. \tag{3.20}$$

Here, $g'^{(2d)}$ is the additional parameter that arises along with $g_2^{(2d)}$ during the dimensional reduction and it has the same dimension as that of $g^{(2d)}$. The quasi-2D GP equation is found to have the form

$$\begin{aligned} i\hbar\frac{\partial\Phi(\boldsymbol{\rho})}{\partial t} &= \left(-\frac{\hbar^2}{2m}\nabla_2^2 + V_2(\boldsymbol{\rho}) + (g^{(2d)} + g'^{(2d)})|\Phi(\boldsymbol{\rho})|^2\right. \\ &+ \left. g_1^{(2d)}|\Phi(\boldsymbol{\rho})|^3 + \frac{g_2^{(2d)}}{2}\nabla_2^2\left(|\Phi(\boldsymbol{\rho})|^2\right)\right)\Phi(\boldsymbol{\rho}) \end{aligned} \tag{3.21}$$

where

$$V_2(\boldsymbol{\rho}) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right),$$

and

$$g^{(2d)} = \frac{\sqrt{8\pi}a\hbar^2}{m\ell}$$

is the term governing inter-particle interaction. Now

$$g_1^{(2d)} = \frac{256 a^{5/2}\hbar^2}{3\sqrt{10}m\pi^{1/4}\ell^{3/2}}$$

is the term corresponding to quantum fluctuation and

$$g'^{(2d)} = -\frac{4\sqrt{2\pi}a^3\hbar^2}{3m\ell^3}$$

,

$$g_2^{(2d)} = \frac{4\sqrt{2\pi}a^3\hbar^2}{3m\ell}$$

are the terms governing the contribution of effective range interaction.

It is important to note the relative values of these parameters as that will help in understanding the significance of different contributions due to beyond mean field corrections. In the case of sodium (^{23}Na), $g^{(2d)} : g'^{(2d)} : g_1^{(2d)} : g_2^{(2d)} \simeq 1 : 10^{-6} : 10^{-9} : 10^{-17}$.

In order to obtain the Eq.(3.21), we start from the following Lagrangian density.

$$\begin{aligned}\mathcal{L} &= \frac{i\hbar}{2}\left(\Phi\frac{\partial\Phi^*}{\partial t} - \Phi^*\frac{\partial\Phi}{\partial t}\right) + \frac{\hbar^2}{2m}|\nabla_2\Phi|^2 \\ &+ V_2|\Phi|^2 + \left[\frac{g^{(2d)}}{2} + \frac{g'^{(2d)}}{4}\right]|\Phi|^4 \\ &+ \frac{2}{5}g_1^{(2d)}|\Phi|^5 + \frac{g_2^{(2d)}}{4}|\Phi|^2\nabla_2^2|\Phi|^2. \end{aligned} \tag{3.22}$$

Using the Gaussian trial function in Eq.(3.13) in the Eq.(3.22) and integrating and using the variational procedure, we can get the Euler-Lagrange's equations of motion for the dynamics of the collective excitations of the system [33]. The terms in the Lagrangian, apart from the contributions due to thermal fluctuation and effective range interaction, have been worked out in Eqs.(2.16), (2.17), (2.18) and (2.19) in chapter 2.

The quantum fluctuation term in the Lagrangian density is

$$\mathcal{L}_1 = \frac{2}{5}g_1^{(2d)}|\Phi|^5 \tag{3.23}$$

and the corresponding Lagrangian can be obtained as

$$\begin{aligned} L_1 &= \int_{-\infty}^{\infty}\mathcal{L}_1 d^2\rho \\ &= \frac{2}{5}g_1^{(2d)}\int_{-\infty}^{\infty}|A(t)|^5\sum_{\eta=x,y}\exp\left[-\frac{5}{2}\frac{|\eta-\eta_0(t)|^2}{w_\eta^2}\right]d^2\rho \\ &= \frac{4\pi}{25}g_1^{(2d)}|A(t)|^5 w_x w_y. \end{aligned} \tag{3.24}$$

Since

$$\pi|A(t)|^2 w_x w_y = N$$

we can write

$$L_1 = \frac{4g_1^{(2d)} N^{5/2}}{25(\pi w_x w_y)^{3/2}}. \tag{3.25}$$

Similarly the term in the Lagrangian corresponding to the effective range interaction is obtained as

$$\begin{aligned} L_2 &= \int_{-\infty}^{\infty} \mathcal{L}_2 d^2\rho| \\ &= \frac{1}{4}|A(t)|^4 g_2^{(2d)} \int_{-\infty}^{\infty} \sum_{\eta=x,y} |\Phi|^2 \nabla_2^2 |\Phi|^2 d^2\rho \\ &= \frac{g_2^{(2d)} N^2}{8\pi w_x^2 w_y^2} \left[\frac{w_x}{w_y} + \frac{w_y}{w_x}\right]. \end{aligned} \tag{3.26}$$

Hence the total Lagrangian of the system can be written as

$$\begin{aligned} L &= N \sum_{\eta=x,y} \left\{ \hbar \left[\eta_0 \dot{\alpha_\eta} + \left(\eta_0^2 + \frac{w_\eta^2}{2}\right) \dot{\beta}_\eta\right] \right. \\ &+ \frac{\hbar^2}{2m} \left[(\alpha_\eta + 2\eta_0\beta_\eta)^2 + \frac{1}{2}\left(\frac{1}{w_\eta^2} + 4\beta_\eta^2 w_\eta^2\right)\right] \\ &+ \left. \frac{1}{2} m\omega_0^2 \left(\eta_0^2 + \frac{w_\eta^2}{2}\right) \right\} + \frac{g^{(2d)} N^2}{4\pi w_x w_y} + \frac{g'^{(2d)} N^2}{8\pi w_x w_y} \\ &+ \frac{4g_1^{(2d)} N^{5/2}}{25\pi^{3/2}(w_x w_y)^{3/2}} + \frac{g_2^{(2d)} N^2}{8\pi w_x^2 w_y^2}\left[\frac{w_x}{w_y} + \frac{w_y}{w_x}\right]. \end{aligned} \tag{3.27}$$

3.2.3 Equations governing the condensate dynamics

The E-L equations corresponding to different variational parameters are obtained as

$$\ddot{\eta_0} + \omega_0^2 \eta_0 = 0, \tag{3.28}$$

$$\begin{aligned}\ddot{w}_x + \omega_0^2 w_x &= \frac{\hbar^2}{m^2 w_x^3} + \frac{g^{(2d)}N}{2\pi m w_x^2 w_y} + \frac{g'^{(2d)}N}{4\pi m w_x^2 w_y} \\ &+ \frac{12 g_1^{(2d)} N^{3/2}}{25 m \pi^{3/2} w_x^{5/2} w_y^{3/2}} \\ &+ \frac{g_2^{(2d)}N}{4\pi m}\left[\frac{3}{w_x^4 w_y} + \frac{1}{w_x^2 w_y^3}\right] \qquad (3.29)\end{aligned}$$

$$\begin{aligned}\ddot{w}_y + \omega_0^2 w_y &= \frac{\hbar^2}{m^2 w_y^3} + \frac{g^{(2d)}N}{2\pi m w_x w_y^2} + \frac{g'^{(2d)}N}{4\pi m w_x w_y^2} \\ &+ \frac{12 g_1^{d} N^{3/2}}{25 m \pi^{3/2} w_x^{3/2} w_y^{5/2}} \\ &+ \frac{g_2^{(2d)}N}{4\pi m}\left[\frac{3}{w_x w_y^4} + \frac{1}{w_x^3 w_y^2}\right]. \qquad (3.30)\end{aligned}$$

Introducing the scaled variables,

$$\tilde{x}_0 = x_0/a_\perp,\ \tilde{\zeta} = \zeta a_\perp^2/\omega_0^2,\ \tau = \omega_0 t,\ \tilde{w}_\eta = w_\eta/a_\perp,\ \tilde{g}^{(2d)} = \frac{g^{(2d)}m}{2\pi\hbar^2},$$

$$\tilde{g'}^{(2d)} = \frac{g'^{(2d)}m}{4\pi\hbar^2},\ \tilde{g_1}^{(2d)} = \frac{12 g_1^{(2d)} m}{25\pi^{3/2}\hbar^2 a_\perp},\ \tilde{g_2}^{(2d)} = \frac{g_2^{(2d)} m}{4\pi\hbar^2 a_\perp^2}$$

where $a_\perp = \sqrt{\hbar/m\omega_0}$. The scaled form of the equations is

$$\ddot{\tilde{\eta}}_0 + \omega_0^2 \tilde{\eta}_0 = 0, \qquad (3.31)$$

$$\begin{aligned}\ddot{\tilde{w}}_x &+ \tilde{w}_x = \frac{1}{\tilde{w}_x^3} + \frac{\tilde{g}^{(2d)}N}{\tilde{w}_x^2 \tilde{w}_y} + \frac{\tilde{g'}^{(2d)}N}{\tilde{w}_x^2 \tilde{w}_y} \\ &+ \frac{\tilde{g_1}^{(2d)} N^{3/2}}{\tilde{w}_x^{5/2} \tilde{w}_y^{3/2}} + \tilde{g_2}^{(2d)} N\left[\frac{3}{\tilde{w}_x \tilde{w}_y^4} + \frac{1}{\tilde{w}_x^3 \tilde{w}_y^2}\right], \qquad (3.32)\end{aligned}$$

$$\ddot{\tilde{w}}_y + \tilde{w}_y = \frac{1}{\tilde{w}_y^3} + \frac{\tilde{g}^{(2d)}N}{\tilde{w}_y^2\tilde{w}_x} + \frac{\tilde{g'}^{(2d)}N}{\tilde{w}_y^2\tilde{w}_x} + \frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_y^{5/2}\tilde{w}_x^{3/2}} + \tilde{g_2}^{(2d)}N\left[\frac{3}{\tilde{w}_y\tilde{w}_x^4} + \frac{1}{\tilde{w}_y^3\tilde{w}_x^2}\right] \quad (3.33)$$

where $\tilde{\eta_0} = (\tilde{x_0}, \tilde{y_0})$ corresponds to the center of the condensate. It can be seen from Eq.(3.31) that the oscillations of the center of the condensate are unaffected by the beyond mean field modifications introduced in the potential.

The symmetry of the equations (3.32) and (3.33) enables us to assume a common equilibrium solution $\tilde{w_0}$ for them. Since $\tilde{g}^{(2d)} \gg \tilde{g'}^{(2d)}$, $\tilde{g_1}^{(2d)}$ and $\tilde{g_2}^{(2d)}$, we can approximately write

$$\tilde{w_0} \simeq (1 + \tilde{g}^{(2d)}N)^{1/4}. \quad (3.34)$$

Using $\tilde{w}_\eta = \tilde{w}_0 + \delta\tilde{w}_\eta$ in Eqs. (3.32) and (3.33), we get

$$\begin{aligned}
\delta\ddot{\tilde{w}}_x + \tilde{w}_0 + \delta\tilde{w}_x = {} & \frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-3}}{\tilde{w}_0^3} + \tilde{g}^{(2d)}N\frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
& + \tilde{g'}^{(2d)}N\frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
& + \tilde{g_1^{(2d)}}N^{3/2}\frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-5/2}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-3/2}}{\tilde{w}_0^4} \\
& + \tilde{g_2^{(2d)}}N\left[\frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-3}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-2}}{\tilde{w}_0^5}\right. \\
& \left. + 3\frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-1}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-4}}{\tilde{w}_0^5}\right] \quad (3.35)
\end{aligned}$$

$$
\begin{aligned}
\delta\ddot{\tilde{w}}_y &+ \tilde{w}_0+\delta\tilde{w}_y = \frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-3}}{\tilde{w}_0^3} + \tilde{g}^{(2d)}N\frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
&+ \tilde{g'}^{(2d)}N\frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
&+ \tilde{g_1^{(2d)}}N^{3/2}\frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-5/2}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-3/2}}{\tilde{w}_0^4} \\
&+ \tilde{g_2^{(2d)}}N\left[\frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-3}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-2}}{\tilde{w}_0^5}\right. \\
&+ \left. 3\frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-1}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-4}}{\tilde{w}_0^5}\right]. \qquad (3.36)
\end{aligned}
$$

Linearizing the equations (3.35) and (3.36), we get

$$
\begin{aligned}
\delta\ddot{\tilde{w}}_x+\delta\tilde{w}_x &= -\frac{3\delta\tilde{w}_x}{\tilde{w}_0^4} - \frac{(\tilde{g}^{(2d)}+\tilde{g'}^{(2d)})N}{\tilde{w}_0^4}[2\delta\tilde{w}_x+\delta\tilde{w}_y] \\
&- \frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5}\left[\frac{5}{2}\delta\tilde{w}_x+\frac{3}{2}\delta\tilde{w}_y\right] \\
&- \frac{\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6}[6\delta\tilde{w}_x+14\delta\tilde{w}_y] \qquad (3.37)
\end{aligned}
$$

$$
\begin{aligned}
\delta\ddot{\tilde{w}}_y+\delta\tilde{w}_y &= -\frac{3\delta\tilde{w}_y}{\tilde{w}_0^4} - \frac{(\tilde{g}^{(2d)}+\tilde{g'}^{(2d)})N}{\tilde{w}_0^4}[2\delta\tilde{w}_y+\delta\tilde{w}_x] \\
&- \frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5}[\frac{5}{2}\delta\tilde{w}_y+\frac{3}{2}\delta\tilde{w}_x] \\
&- \frac{\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6}[6\delta\tilde{w}_y+14\delta\tilde{w}_x] \qquad (3.38)
\end{aligned}
$$

From the equations (3.37) and (3.38) we can obtain the collective excitation frequencies by the same procedure as used in

Sec.(1.3). Thus, by using

$$\delta\tilde{w}_x = A_1 \exp(iWt) \tag{3.39}$$

$$\delta\tilde{w}_y = A_2 \exp(iWt) \tag{3.40}$$

in the equations (3.37) and (3.38), we arrive at the equations,

$$\begin{aligned} &\left[1 + \frac{3 + 2\left[\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right] N}{\omega_0^4} + \frac{5}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{6\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} - \frac{W^2}{\omega_0^2}\right] A_1 \\ + &\left[\frac{\left[\tilde{g}^{(2d)} + g'^{(2d)}\right] N}{\omega_0^4} + \frac{3}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{14\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6}\right] A_2 = 0 \end{aligned} \tag{3.41}$$

$$\begin{aligned} &\left[1 + \frac{3 + 2\left[\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right] N}{\omega_0^4} + \frac{5}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{6\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} - \frac{W^2}{\omega_0^2}\right] A_2 \\ + &\left[\frac{\left[\tilde{g}^{(2d)} + g'^{(2d)}\right] N}{\omega_0^4} + \frac{3}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{14\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6}\right] A_1 = 0 \end{aligned} \tag{3.42}$$

In the matrix form, the Eqs.((3.41) and (3.42)) can be written as

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \tag{3.43}$$

where

$$\begin{aligned} T_{11} = T_{22} &= \left[1 + \frac{3 + 2\left[\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right] N}{\omega_0^4}\right. \\ &+ \left.\frac{5}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{6\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} - \frac{W^2}{\omega_0^2}\right] \end{aligned} \tag{3.44}$$

$$T_{12} = T_{21} = \left[\frac{\left[\tilde{g}^{(2d)} + g'^{(2d)}\right] N}{\omega_0^4} + \frac{14\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} + \frac{3}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} \right] \tag{3.45}$$

For a non-trivial solution, the determinant of the coefficient matrix should vanish. Thus

$$\begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} = 0 \tag{3.46}$$

Using Eqs.(3.44) and (3.45) in Eq.(3.46), we can obtain

$$\begin{aligned} \frac{W^4}{\omega_0^4} \quad &- \quad \frac{W^2}{\omega_0^2}\left\{ 2\left[1 + \frac{3 + 2\left(\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right) N}{\tilde{\omega}_0^4} + \frac{5}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{6\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} \right] \right\} \\ &+ \quad \left[1 + \frac{3 + 2\left(\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right) N}{\omega_0^4} + \frac{5}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{6\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} \right]^2 \\ &- \quad \left[\frac{\left(\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right) N}{\omega_0^4} + \frac{3}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{14\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} \right]^2 = 0 \end{aligned} \tag{3.47}$$

Therefore

$$\begin{aligned} W^2 \quad &= \quad \left[\left[1 + \frac{3 + 2\left(\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right) + N}{\tilde{\omega}_0^4} + \frac{5}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{6\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} \right] \right. \\ &\pm \quad \left. \left[\frac{\left(\tilde{g}^{(2d)} + \tilde{g}'^{(2d)}\right) N}{\omega_0^4} + \frac{3}{2}\frac{\tilde{g}_1^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{14\tilde{g}_2^{(2d)} N}{\tilde{w}_0^6} \right] \right] \omega_0^2 \end{aligned} \tag{3.48}$$

Using $\tilde{w}_0^4 \simeq 1 + \tilde{g}^{(2d)} N$, the collective excitation frequencies can be obtained as

$$W_+ = \sqrt{\left[4 + \frac{3\tilde{g'}^{(2d)} N}{\tilde{w}_0^4} + \frac{4\tilde{g}_1{}^{(2d)} N^{3/2}}{\tilde{w}_0^5} + \frac{20\tilde{g}_2{}^{(2d)} N}{\tilde{w}_0^6} \right]}\omega_0 \tag{3.49}$$

and

$$W_- = \sqrt{\left[\frac{4 + (2\tilde{g}^{(2d)} + \tilde{g'}^{(2d)})N}{\tilde{w}_0^4} - \frac{\tilde{g_1}^{(2d)} N^{3/2}}{\tilde{w}_0^5} - \frac{8\tilde{g_2}^{(2d)} N}{\tilde{w}_0^6}\right]}\omega_0. \tag{3.50}$$

Here, W_+ is a modification of breathing oscillatory mode, which corresponds to $m_l = 0$, where m_l is the angular momentum quantum number in the direction of z-axis, whereas W_- is a modification of the superposition of modes with $m_l = +2$ and $m_l = -2$ [33]. It is to be noted that the excitation frequency, W_+ is independent of the effect of interparticle interaction.

3.3 Numerical results and discussion

We have investigated the behavior of the collective excitation frequencies of quasi 2-D BEC in beyond meanfield regime as a function of the number of condensed particles (N). The corresponding graphs are plotted for both attractively and repulsively interacting condensate atoms.

3.3.1 Behavior of collective excitation frequencies for lithium(^{7}Li)

Figure (3.1) shows the plot of W_-/ω_0 as a function of N for an attractively interacting condensate [lithium atoms(^{7}Li) with scattering length, a = -1.5nm] within and beyond mean field regime. Numerical values of the interaction parameters can be obtained from their expressions in Sec.(3.2.3). For ^{7}Li

$$\tilde{g}^{(2d)} = \frac{g^{(2d)}m}{2\pi\hbar^2} = -8.1 \times 10^{-4},\ \tilde{g'}^{(2d)} = \frac{g'^{(2d)}m}{4\pi\hbar^2} = 2.7339 \times 10^{-10},$$

$$\tilde{g_1}^{(2d)} = \frac{12g_1^{(2d)}m}{25\pi^{3/2}\hbar^2 a_\perp} = 1.4298 \times 10^{-7},$$

$$\tilde{g_2}^{(2d)} = \frac{g_2^{(2d)}m}{4\pi\hbar^2 a_\perp^2} = -5.640 \times 10^{-11}.$$

It is very clear from Fig.(3.1) that the collective excitation frequency shows a sharp increase towards infinity for $N = N_0 \simeq 1240$. This is a signature of the collapse of the condensate [33]. The excitation frequency again reappears at $N = 2N_0$. But the inclusion of the beyond mean field effects in the potential modifies the behavior of the excitation frequencies. Although the collapse occurs at the same particle number, reappearance is not observed in this case. The reason is obvious from the Eq.(3.50). Even though the parameters $g'^{(2d)}, g_1^{(2d)}$ and $g_2^{(2d)}$ are much less than $g^{(2d)}$, the denominators of these terms restrict the excitation frequency W_- from attaining real value for all $N > N_0$.

It has been reported earlier that, in the mean field regime, W_+ corresponds to a breathing oscillation of frequency $2\omega_0$ [33]. But the introduction of the modifications in the potential results in drastic change in the behavior of this excitation frequency [Fig.(3.3)], which indicates a collapse of the condensate similar to that in Fig.(3.1) for $N > N_0 \simeq 1240$. The explanation for this behavior is same as that for the case of Fig.(3.1).

3.3.2 Behavior of collective excitation frequencies for sodium(^{23}Na)

For repulsively interacting condensate of sodium, (^{23}Na), s-wave scattering length has a positive value, $a = 2.8$nm and

$$\tilde{g}^{(2d)} = \frac{g^{(2d)}m}{2\pi\hbar^2} = 2.994 \times 10^{-3}, \; \tilde{g'}^{(2d)} = \frac{g'^{(2d)}m}{4\pi\hbar^2} = -1.4053 \times 10^{-8},$$

$$\tilde{g_1}^{(2d)} = \frac{12 g_1^{(2d)} m}{25 \pi^{3/2} \hbar^2 a_\perp} = 3.8118 \times 10^{-6},$$

$$\tilde{g_2}^{(2d)} = \frac{g_2^{(2d)} m}{4 \pi \hbar^2 a_\perp^2} = 2.8995 \times 10^{-9}.$$

In this case, the modifications in the potential do not show large influence on the variation of W_- with N for small values of N [Fig.(3.5)]. But for large N, while the mean field excitation frequency converges to approximately $1.416\omega_0$, in the beyond mean field regime, it reaches a minimum and then shows a gradual but slow increase. The other collective excitation frequency W_+ shows an increase with N in the modified potential [Fig.(3.7)]. This is in contrast with its behavior in the mean field regime, where it is independent of N. To confirm the generality of the above results, similar plots have been constructed in the case of other condensates also - [rubidium (^{87}Rb) with scattering length, a = 5.4nm and potassium (^{39}K) with scattering length, a = -1.75nm] - and the qualitative behavior is found to be the same [Figs.(3.2), (3.4), (3.6) and (3.8)].

3.4 Conclusions

In this chapter, we have analyzed the modifications in the collective excitation frequencies of a quasi-2D BEC due to beyond mean field effects. Unlike the previous work, where the effect of gravity and anharmonicity on collective excitations were studied, in this case it is found that the oscillation frequency of the center of the condensate is unaffected by these modifications. But the collective excitations corresponding to the width of the condensate are thoroughly modified. For an attractively interacting condensate, reappearance of the collective excitation frequency W_- is prevented by the beyond mean field effects, whereas, for a repulsively interacting condensate, W_- shows an increase with N after reaching a minimum. This is in contrast its behaviour in the mean field regime, where it approach a constant value for large N values. It is also found that, in the presence of beyond mean field modifications of the potential, breathing mode of frequency $2\omega_0$ is not supported by both attractively and repulsively interacting systems. Instead, W_+ shows an increase with N. While the attractively interacting condensate exhibits a sharp increase, indicating a collapse of the condensate, such a drastic behaviour is absent in a repulsively interacting system.

3.5 Plots of collective excitation frequencies against condensate particle number

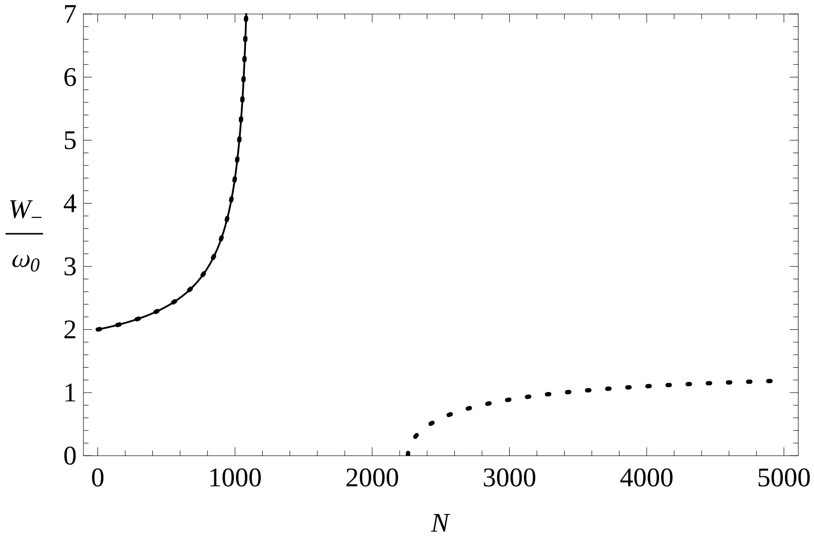

Figure 3.1: Variation of collective excitation frequency W_- as a function of the condensed particle number N for ^{7}Li: In the mean field regime(dashed line) and beyond mean field regime(solid line).

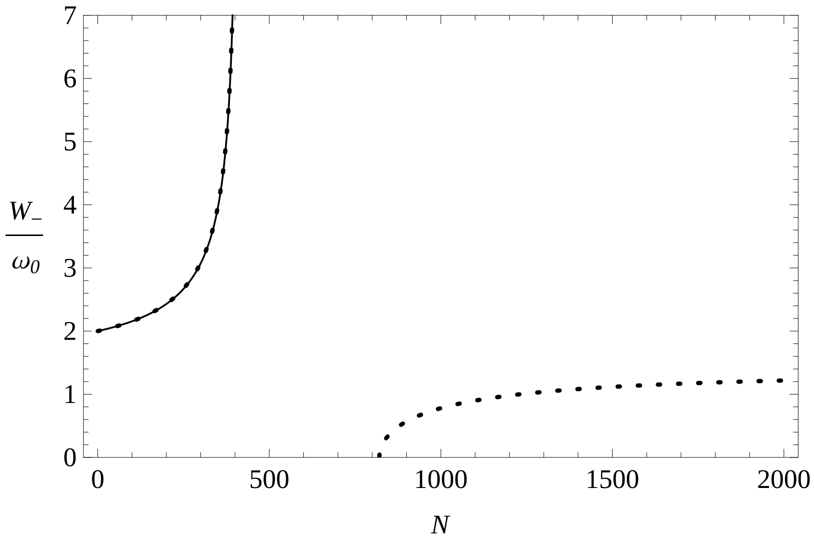

Figure 3.2: Collective excitation frequency W_- versus the number of condensed particles N for ^{39}K: In the mean field regime(dashed line) and beyond mean field regime(solid line).

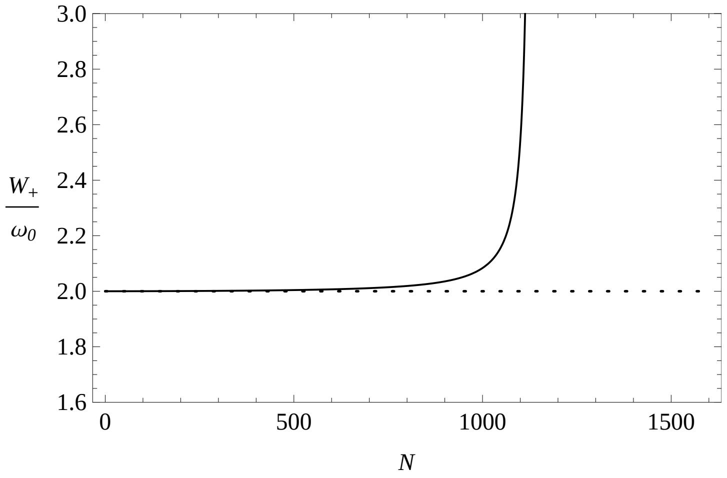

Figure 3.3: Collective excitation frequency W_+ plotted against number of condensed particles N for ^{7}Li: Dashed line refers to mean field regime and solid line refers to beyond mean field regime.

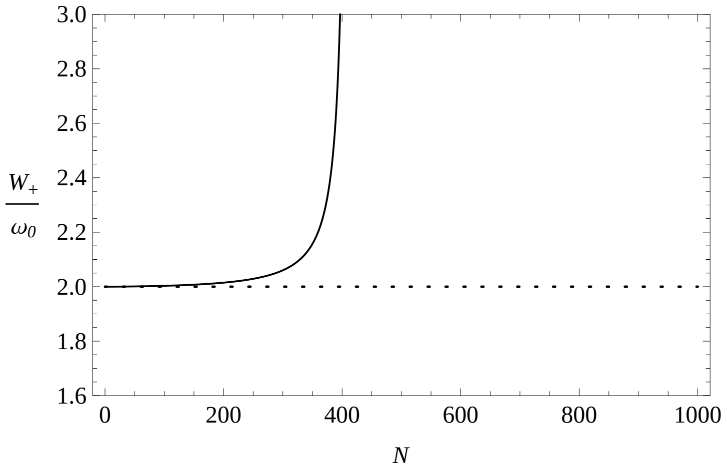

Figure 3.4: Collective excitation frequency W_+ as a function of number of particles in the condensate N for ^{39}K: In the mean field regime(dashed line) and beyond mean field regime(solid line).

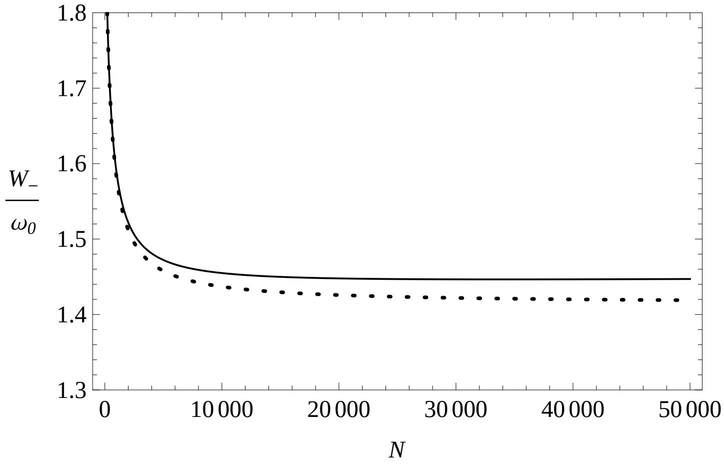

Figure 3.5: Variation of collective excitation frequency W_{-} with number of condensed particles N for ^{23}Na: Dashed line refers to mean field regime and solid line refers to beyond mean field regime.

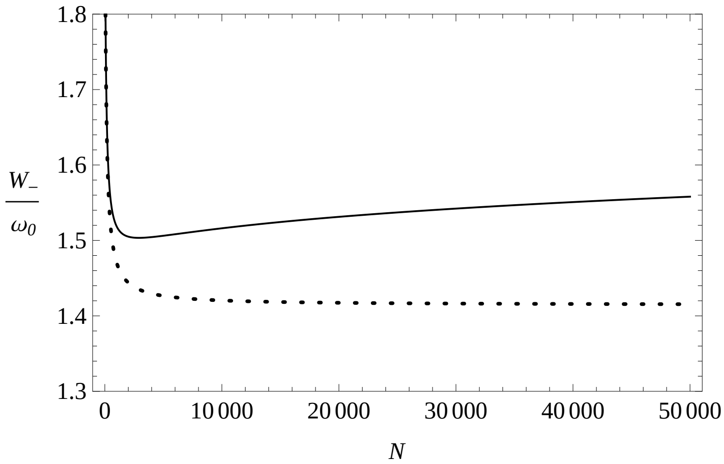

Figure 3.6: Plot showing variation of collective excitation frequency W_{-} with condensed particle number N for ^{87}Rb: In the mean field regime(dashed line) and beyond mean field regime(solid line).

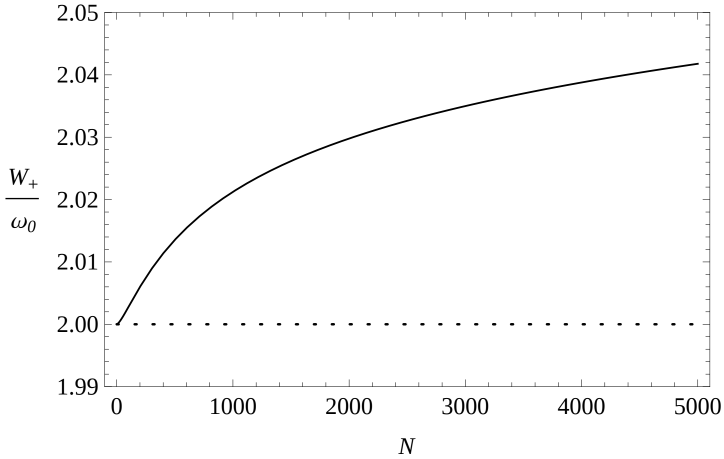

Figure 3.7: Collective excitation frequency W_+ plotted against number of condensed particles N for ^{23}Na: In the mean field regime(dashed line) and beyond mean field regime(solid line).

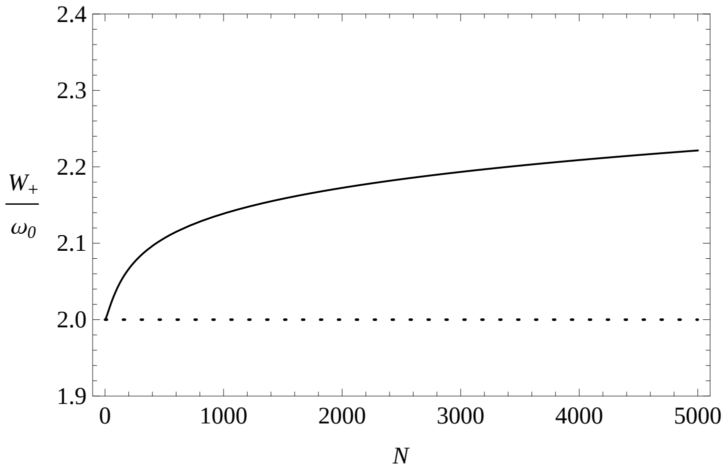

Figure 3.8: Collective excitation frequency W_+ versus number of condensed particles N for ^{87}Rb: In the mean field regime(dashed line) and beyond mean field regime(solid line).

4

Collective Excitations of a Quasi-Two-Dimensional BEC: Effect of Thermal Cloud

4.1 Introduction

The Gross-Pitaevskii (GP) equation, which is often used in the analysis of BEC, is actually a zero temperature equation. This equation is exactly valid only for a condensate at absolute zero temperature. But, it is a well established fact that absolute zero temperature is unattainable by any experimental means. All the Bose-Einstein condensates realized in laboratory have been attained at ultra low, yet finite, temperatures [8, 9]. And the experiments are focussed in the direction of attaining this phase at a higher temperature than at a lower one. The essential difference between a BEC at absolute zero (or very low) and at a finite temperature is that, in the case of the latter condensate, a finite number of atoms at higher energy levels will be co-existing along with the condensate particles. This constitutes the *Thermal Cloud*. When the temperature of the system is very low, the number of atoms in the thermal cloud is very small. Under this condition, the GP equation in its usual form gives a good mean field description of the condensate dynamics. But, if the temperature is high enough to 'feed' the thermal cloud appreciably, a modification of the GP equation becomes necessary for a faithful description of the system.

Experimentally, it is already observed that mode coupling between the condensate and the thermal cloud is negligible [70]. This motivates the usage of a static distribution for the thermal cloud density [23, 24].

$$\rho_n(\mathbf{r}) = R(T)\exp[-V(\mathbf{r})/kT] \tag{4.1}$$

where

$$V(\mathbf{r}) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

and $R(T)$ is the normalization constant which is a function of temperature.

The Lagrangian density corresponding to the GP equation modified to include the effect of thermal cloud can be written as [23]

$$\begin{aligned} \mathcal{L} &= \frac{i\hbar}{2}\left(\Psi\frac{\partial\Psi^*}{\partial t} - \Psi^*\frac{\partial\Psi}{\partial t}\right) + \frac{\hbar^2}{2m}|\nabla_2\Psi|^2 + V_2|\Psi|^2 \\ &+ \frac{g^{(2d)}}{2}|\Psi|^4 + \frac{4\pi a\hbar^2}{m}\rho_n(\mathbf{r})|\Psi|^2 \end{aligned} \tag{4.2}$$

The wavefunction Ψ and the thermal cloud density ρ_n satisfy the normalization conditions

$$\int d^3r|\Psi(\mathbf{r})|^2 = N_0 \tag{4.3}$$

and

$$\int d^3r\rho_n(\mathbf{r}) = N_n \tag{4.4}$$

where N_0 is the condensate particle number and N_n is the number of particles in the thermal cloud.

In this chapter, we analyze the effect of thermal cloud on the collective excitation of quasi-2D BEC. This analysis has already been done elsewhere for one, two and three dimensional traps [23]. But an extension of the work to quasi-2D traps is new. It is a fact that an exact two dimensional BEC is impossible to realize in the laboratory. Here comes the significance of investigations of the quasi-2D condensates as this is one of the physically realizable versions of low dimensional BEC. The difference in the analysis of an exact two dimensional and quasi-2D BEC is that, in the latter case, the interaction terms in the potential undergo changes. As it was done earlier, the condensate wavefunction is written as $\Psi(\mathbf{r}) = \Phi(\boldsymbol{\sigma})\varphi_0(z)$ where $\boldsymbol{\sigma} = (x, y)$ and

$$\varphi_0(z) = \left(\frac{1}{\pi\ell^2}\right)^{\frac{1}{4}} \exp\left(-z^2/2\ell^2\right)$$

is the normalized ground state wavefunction of the trap in the z-direction. The interparticle interaction term in three dimension,

$$g^{(3d)} = \frac{4\pi a\hbar^2}{m} \tag{4.5}$$

becomes

$$g^{(2d)} = \frac{\sqrt{8\pi}a\hbar^2}{m\ell} \tag{4.6}$$

in a quasi two dimensional system, where a is the s-wave scattering length and $\ell = \sqrt{\hbar/m\omega_z}$. In a similar fashion, the term in the Lagrangian density corresponding to the effect of the thermal cloud

$$\mathcal{L}_T = \frac{4\pi a\hbar^2}{m}\rho_n(\mathbf{r})|\Psi(\mathbf{r})|^2 \tag{4.7}$$

has to be reduced to the case of a quasi-2D trap by integrating out the z-coordinate from it.

$$\begin{aligned}
\mathcal{L}_T &= \frac{4\pi a\hbar^2}{m}\sqrt{\frac{1}{\pi\ell^2}}R(T)\exp[-V(\boldsymbol{\sigma})/kT]|\Phi(\boldsymbol{\sigma})|^2 \\
&\quad \int_{-\infty}^{\infty} dz \exp[-m\omega_z^2 z^2/2kT]\exp[-z^2/\ell^2] \\
&= \frac{4\pi a\hbar^2}{m\ell\sqrt{\frac{m\omega_z^2}{2kT}+\frac{1}{\ell^2}}}\rho_n(\boldsymbol{\sigma})|\Phi(\boldsymbol{\sigma})|^2 \\
&= g_T^{(2d)}\rho_n(\boldsymbol{\sigma})|\Phi(\boldsymbol{\sigma})|^2 \qquad (4.8)
\end{aligned}$$

Thus the quasi-two-dimensional reduction has resulted in the thermal cloud term showing dependence on the trapping frequency in the dynamically frozen direction, i.e. ω_z. In order to enable a variational analysis, we assume a Gaussian wavefunction as before. Thus

$$\begin{aligned}
\Phi(x,y,t) &= \sqrt{A(t)}\prod_{\eta=x,y}\exp\Bigg(-\frac{|\eta-\eta_0(t)|^2}{2w_\eta^2(t)} \\
&\quad + i\left[\eta\alpha_\eta(t)+\eta^2\beta_\eta(t)\right]\Bigg) \qquad (4.9)
\end{aligned}$$

Then the Lagrangian corresponding to the thermal cloud term can be obtained by

$$\begin{aligned}
L_T &= \int_{-\infty}^{\infty} d^2\sigma\mathcal{L}_T = g_T^{(2d)} \times \\
&\int_{-\infty}^{\infty} d^2\sigma R(T)A(t)\prod_{\eta=x,y}\exp\left[-\left(\frac{m\omega_\eta^2}{2kt}+\frac{1}{w_\eta^2}\right)\eta^2\right] \qquad (4.10)
\end{aligned}$$

Using the normalization conditions in Eqs.(4.3) and (4.4), we can obtain the expressions for $A(t)$ and $R(T)$ using

$$\int_{-\infty}^{\infty} d^2\sigma A(t)\prod_{\eta=x,y}\exp\left[-\frac{(\eta-\eta_0)^2}{W_\eta^2}\right] = A(t)\pi W_x W_y, \qquad (4.11)$$

$$\int_{-\infty}^{\infty} d^2\sigma R(T) \exp\left[-\frac{m}{2kT}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right)\right] = N_n \tag{4.12}$$

and we get

$$A(t) = \frac{N_0}{\pi w_x w_y} \tag{4.13}$$

and

$$R(T) = \frac{N_n m \omega_x \omega_y}{2\pi kT}. \tag{4.14}$$

Using Eqs.(4.13) and (4.14) in Eq.(4.10), we get

$$L_T = \prod_{\eta=x,y} \sqrt{\frac{m\omega_\eta^2}{\pi kT}} \frac{g_T^{(2d)} N_0 N_n}{\sqrt{\left(\frac{m\omega_\eta^2}{kT} w_\eta^2 + 2\right)}} \tag{4.15}$$

As all the other terms in the Lagrangian have already been evaluated earlier, the total Lagrangian of the system can be written as

$$\begin{aligned} L &= N_0 \sum_{\eta=x,y} \left\{ \hbar \left[\eta_0 \dot{\alpha_\eta} + \left(\eta_0^2 + \frac{w_\eta^2}{2} \right) \dot{\beta}_\eta \right] \right. \\ &+ \frac{\hbar^2}{2m} \left[(\alpha_\eta + 2\eta_0 \beta_\eta)^2 + \frac{1}{2} \left(\frac{1}{w_\eta^2} + 4\beta_\eta^2 w_\eta^2 \right) \right] \\ &+ \left. \frac{1}{2} m\omega_0^2 \left(\eta_0^2 + \frac{w_\eta^2}{2} \right) + \frac{1}{4}\hbar\omega_z \right\} \\ &+ \frac{g^{(2d)} N^2}{4\pi w_x w_y} + \frac{g'^{(2d)} N^2}{8\pi w_x w_y} + \frac{4 g_1^{(2d)} N^{5/2}}{25\pi^{3/2} (w_x w_y)^{3/2}} \\ &+ \frac{g_2^{(2d)} N^2}{8\pi w_x^2 w_y^2} \left[\frac{w_x}{w_y} + \frac{w_y}{w_x} \right] \\ &+ \prod_\eta g_T^{(2d)} N_0 N_n \sqrt{\frac{m\omega_\eta^2}{\pi kT}} \frac{1}{\sqrt{\left(\frac{m\omega_\eta^2}{kT} w_\eta^2 + 2\right)}}. \end{aligned} \tag{4.16}$$

Then the Euler-Lagrange's equations corresponding to the variational parameters can be obtained by the usual procedure. The E-L equations for the center of the condensate is found to be

$$\ddot{x}_0 + \omega_0^2 x_0 = -\left\{\zeta\left[2x_0^3 + 3x_0 w_x^2\right] + g_a\right\} \tag{4.17}$$

$$\ddot{y}_0 + \omega_0^2 y_0 = -\zeta\left[2y_0^3 + 3y_0 w_y^2\right] \tag{4.18}$$

These E-L equations for x_0 and y_0 can be seen to be independent of the effect of the thermal cloud density. Thus, it can be concluded that the effect of quantum fluctuation, thermal cloud and effective range interaction does not result in introducing a coupling between the center of the condensate and its internal degrees of freedom. Now, for obtaining the E-L equations for w_x, we start with

$$\frac{\partial L}{\partial \dot{w}_x} = 0 \tag{4.19}$$

In order to find out $\dfrac{\partial L}{\partial w_x}$ it is reasonably assumed that in the finite temperature BEC, the condensate widths are smaller than that of the thermal cloud, that is $m\omega_\eta^2 w_\eta^2 << kT$. With this approximation, L_T can be written as

$$\begin{aligned} L_T &= g_T^{(2d)} N_0 N_n \prod_{\eta=x,y} \sqrt{\frac{m\omega_\eta^2}{\pi kT}} \frac{1}{\left[\frac{m\omega_\eta^2}{kT} w_\eta^2 + 2\right]^{1/2}} \\ &= g_T^{(2d)} N_0 N_n \left(\frac{m}{2\pi kT}\right) \omega_x \omega_y \left[1 + \frac{m\omega_x^2}{2kT} w_x^2\right]^{-1/2} \left[1 + \frac{m\omega_y^2}{2kT} w_y^2\right]^{-1/2} \\ &\approx g_T^{(2d)} N_0 N_n \left(\frac{m}{2\pi kT}\right) \omega_x \omega_y \left[1 - \frac{m\omega_x^2}{4kT} w_x^2 - \frac{m\omega_y^2}{4kT} w_y^2\right] \end{aligned} \tag{4.20}$$

Assuming an isotropic trap, $\omega_x = \omega_y = \omega_0$ and using

$$\frac{\partial L}{\partial w_x} = -g_T^{(2d)} N_0 N_n \frac{m^2 \omega_0^4}{4\pi (kT)^2} w_x, \tag{4.21}$$

the E-L equation for w_x can be obtained as

$$\begin{aligned}\ddot{w}_x + \omega_0^2 w_x &= \frac{\hbar^2}{m^2 w_x^3} + \frac{g^{(2d)} N}{2\pi m w_x^2 w_y} - \zeta\left[3w_x^3 + 6x_0^2 w_x\right] \\ &+ \frac{g'^{(2d)} N}{4\pi m w_x^2 w_y} + \frac{12 g_1^{(2d)} N^{3/2}}{25 m \pi^{3/2} w_x^{5/2} w_y^{3/2}} \\ &+ \frac{g_2^{(2d)} N}{4\pi m}\left[\frac{3}{w_x^4 w_y} + \frac{1}{w_x^2 w_y^3}\right] \\ &+ g_T^{(2d)} N_n \frac{m^2 \omega_0^4}{2\pi (kT)^2} w_x. \end{aligned} \tag{4.22}$$

Similarly, for w_y

$$\begin{aligned}\ddot{w}_y + \omega_0^2 w_y &= \frac{\hbar^2}{m^2 w_y^3} + \frac{g^{(2d)} N}{2\pi m w_x w_y^2} - \zeta\left[3w_y^3 + 6y_0^2 w_y\right] \\ &+ \frac{g'^{(2d)} N}{4\pi m w_x w_y^2} + \frac{12 g_1^{(2d)} N^{3/2}}{25 m \pi^{3/2} w_x^{3/2} w_y^{5/2}} \\ &+ \frac{g_2^{(2d)} N}{4\pi m}\left[\frac{3}{w_x w_y^4} + \frac{1}{w_x^3 w_y^2}\right] \\ &+ g_T^{(2d)} N_n \frac{m^2 \omega_0^4}{2\pi (kT)^2} w_y. \end{aligned} \tag{4.23}$$

Defining a new parameter $g'^{(2d)}_T = g_T^{(2d)} N_n \frac{m^2 \omega_0^2}{2\pi (kT)^2}$, the equations (4.22) and (4.23) can be rewritten as

$$\begin{aligned}\ddot{w}_x + \omega_0^2\left[1 - g'^{(2d)}_T\right] w_x &= \frac{\hbar^2}{m^2 w_x^3} + \frac{g^{(2d)} N}{2\pi m w_x^2 w_y} - \zeta\left[3w_x^3 + 6x_0^2 w_x\right] \\ &+ \frac{g'^{(2d)} N}{4\pi m w_x^2 w_y} + \frac{12 g_1^{(2d)} N^{3/2}}{25 m \pi^{3/2} w_x^{5/2} w_y^{3/2}} \\ &+ \frac{g_2^{(2d)} N}{4\pi m}\left[\frac{3}{w_x^4 w_y} + \frac{1}{w_x^2 w_y^3}\right] \end{aligned} \tag{4.24}$$

$$\ddot{w}_y + \omega_0^2 \left[1 - {g'}_T^{(2d)}\right] w_y = \frac{\hbar^2}{m^2 w_y^3} + \frac{g^{(2d)} N}{2\pi m w_x w_y^2} - \zeta \left[3w_y^3 + 6y_0^2 w_y\right] + \frac{g'^{(2d)} N}{4\pi m w_x w_y^2} + \frac{12 g_1^{(2d)} N^{3/2}}{25 m \pi^{3/2} w_x^{3/2} w_y^{5/2}} + \frac{g_2^{(2d)} N}{4\pi m}\left[\frac{3}{w_x w_y^4} + \frac{1}{w_x^3 w_y^2}\right] \quad (4.25)$$

Using the following scaled variables,

$$\tilde{x}_0 = x_0/a'_\perp,\ \tilde{\zeta} = \frac{\zeta a'^2_\perp}{\left[1 - {g'}_T^{(2d)}\right]\omega_0^2},\ \tau = \omega_0\sqrt{1 - {g'}_T^{(2d)}}\,t,\ \tilde{w}_\eta = w_\eta/a'_\perp,$$

$$\tilde{g}^{(2d)} = \frac{g^{(2d)} m}{2\pi\hbar^2},\ \tilde{g'}^{(2d)} = \frac{g'^{(2d)} m}{4\pi\hbar^2},\ \tilde{g_1}^{(2d)} = \frac{12 g_1^{(2d)} m}{25\pi^{3/2}\hbar^2 a'_\perp},$$

$$\tilde{g_2}^{(2d)} = \frac{g_2^{(2d)} m}{4\pi\hbar^2 {a'_\perp}^2}$$

where $a'_\perp = \sqrt{\frac{\hbar}{m\omega_0\sqrt{1-{g'}_T^{(2d)}}}}$, equations (4.22) and (4.25) are modified as

$$\ddot{\tilde{w}}_x + \tilde{w}_x = \frac{1}{\tilde{w}_x^3} + \frac{\tilde{g}^{(2d)} N}{\tilde{w}_x^2 \tilde{w}_y} + \frac{\tilde{g'}^{(2d)} N}{\tilde{w}_x^2 \tilde{w}_y} + \frac{\tilde{g_1}^{(2d)} N^{3/2}}{\tilde{w}_x^{5/2} \tilde{w}_y^{3/2}} + \tilde{g_2}^{(2d)} N \left[\frac{3}{\tilde{w}_x \tilde{w}_y^4} + \frac{1}{\tilde{w}_x^3 \tilde{w}_y^2}\right] - \tilde{\zeta}\left[3\tilde{w}_x^3 + 6\tilde{x}_0^2 \tilde{w}_x\right] \quad (4.26)$$

$$\ddot{\tilde{w}}_y + \tilde{w}_y = \frac{1}{\tilde{w}_y^3} + \frac{\tilde{g}^{(2d)} N}{\tilde{w}_y^2 \tilde{w}_x} + \frac{\tilde{g'}^{(2d)} N}{\tilde{w}_y^2 \tilde{w}_x} + \frac{\tilde{g_1}^{(2d)} N^{3/2}}{\tilde{w}_y^{5/2} \tilde{w}_x^{3/2}} + \tilde{g_2}^{(2d)} N \left[\frac{3}{\tilde{w}_y \tilde{w}_x^4} + \frac{1}{\tilde{w}_y^3 \tilde{w}_x^2}\right] - \tilde{\zeta}\left[3\tilde{w}_y^3 + 6\tilde{y}_0^2 \tilde{w}_y\right] \quad (4.27)$$

As it was done previously, by introducing $\tilde{w}_\eta = \tilde{w}_0 + \delta\tilde{w}_\eta$ in the equations (4.26) and (4.27), we get

$$\begin{aligned}
\delta\ddot{\tilde{w}}_x &+ \tilde{w}_0 + \delta\tilde{w}_x = \frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-3}}{\tilde{w}_0^3} + \tilde{g}^{(2d)} N \frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
&+ \tilde{g'}^{(2d)} N \frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
&+ \tilde{g_1}^{(2d)} N^{3/2} \frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-5/2}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-3/2}}{\tilde{w}_0^4} \\
&+ \tilde{g}_2 N \left[\frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-3}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-2}}{\tilde{w}_0^5} + 3\frac{\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-1}\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-4}}{\tilde{w}_0^5}\right] \\
&- \tilde{\zeta}\left[3\left[\tilde{w}_0^3 + 3\tilde{w}_0^2\delta\tilde{w}_x\right] + 6\tilde{x}_0^2\left[\tilde{w}_0 + \delta\tilde{w}_x\right]\right] \qquad (4.28)
\end{aligned}$$

$$\begin{aligned}
\delta\ddot{\tilde{w}}_y &+ \tilde{w}_0 + \delta\tilde{w}_y = \frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-3}}{\tilde{w}_0^3} + \tilde{g}^{(2d)} N \frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
&+ \tilde{g'}^{(2d)} N \frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-2}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-1}}{\tilde{w}_0^3} \\
&+ \tilde{g_1}^{(2d)} N^{3/2} \frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-5/2}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-3/2}}{\tilde{w}_0^4} \\
&+ \tilde{g}_2 N \left[\frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-3}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-2}}{\tilde{w}_0^5} + 3\frac{\left[1+\frac{\delta\tilde{w}_y}{\tilde{w}_0}\right]^{-1}\left[1+\frac{\delta\tilde{w}_x}{\tilde{w}_0}\right]^{-4}}{\tilde{w}_0^5}\right] \\
&- \tilde{\zeta}\left[3\left[\tilde{w}_0^3 + 3\tilde{w}_0^2\delta\tilde{w}_y\right] + 6\tilde{y}_0^2\left[\tilde{w}_0 + \delta\tilde{w}_y\right]\right] \qquad (4.29)
\end{aligned}$$

Then, the linearized equations can be written as

$$\begin{aligned}\delta\ddot{\tilde{w}}_x + \delta\tilde{w}_x &= -\frac{3\delta\tilde{w}_x}{\tilde{w}_0^4} - \frac{(\tilde{g}^{(2d)} + \tilde{g'}^{(2d)})N}{\tilde{w}_0^4}[2\delta\tilde{w}_x + \delta\tilde{w}_y] \\ &- \frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5}[\frac{5}{2}\delta\tilde{w}_x + \frac{3}{2}\delta\tilde{w}_y] \\ &- \frac{\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6}[6\delta\tilde{w}_x + 14\delta\tilde{w}_y] \\ &- \tilde{\zeta}\left[6\tilde{x}_0^2 + 9\tilde{w}_0^2\right]\delta\tilde{w}_x \end{aligned} \quad (4.30)$$

$$\begin{aligned}\delta\ddot{\tilde{w}}_y + \delta\tilde{w}_y &= -\frac{3\delta\tilde{w}_y}{\tilde{w}_0^4} - \frac{(\tilde{g}^{(2d)} + \tilde{g'}^{(2d)})N}{\tilde{w}_0^4}[2\delta\tilde{w}_y + \delta\tilde{w}_x] \\ &- \frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5}[\frac{5}{2}\delta\tilde{w}_y + \frac{3}{2}\delta\tilde{w}_x] \\ &- \frac{\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6}[6\delta\tilde{w}_y + 14\delta\tilde{w}_x] \\ &- \tilde{\zeta}\left[6\tilde{y}_0^2 + 9\tilde{w}_0^2\right]\delta\tilde{w}_y \end{aligned} \quad (4.31)$$

Going through the usual procedure of finding the normal modes leads to the following collective excitation frequencies:

$$\begin{aligned}W_\pm^2 &= \left[\left[1 + \frac{3 + 2\left(\tilde{g}^{(2d)} + \tilde{g'}^{(2d)}\right)N}{\tilde{w}_0^4} + \frac{5}{2}\frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5} + \frac{6\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6} + \tilde{\zeta}9\omega_0^2\right]\right. \\ &+ 3\tilde{\zeta}\left[\tilde{x}_0^2 + \tilde{y}_0^2\right] \pm \left[9\tilde{\zeta}^2\left[\tilde{x}_0^2 - \tilde{y}_0^2\right]^2 + \left[\frac{\left(\tilde{g}^{(2d)} + \tilde{g'}^{(2d)}\right)N}{\tilde{w}_0^4}\right.\right. \\ &+ \left.\left.\left.\frac{3}{2}\frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5} + \frac{14\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6}\right]^2\right]\right]\omega_0^2\left[1 - g'^{(2d)}_T\right] \end{aligned} \quad (4.32)$$

Using the equilibrium values of $\tilde{x}_0$ and $\tilde{y}_0$, the collective excitation frequencies become

$$
\begin{aligned}
W_{\pm} &= \left[\left[1+\frac{3+2\left(\tilde{g}^{(2d)}+\tilde{g'}^{(2d)}\right)N}{\tilde{w}_0^4}+\frac{5}{2}\frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5}+\frac{6\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6}+\tilde{\zeta}9\omega_0^2\right]\right. \\
&+ 3\tilde{\zeta}\left[\frac{g_a}{a'_{\perp}\omega_0^2}\right]^2 \pm \left[9\tilde{\zeta}^2\left[\frac{g_a}{a'_{\perp}\omega_0^2}\right]^4+\left[\frac{\left(\tilde{g}^{(2d)}+\tilde{g'}^{(2d)}\right)N}{\omega_0^4}\right.\right. \\
&+ \left.\left.\left.\frac{3}{2}\frac{\tilde{g_1}^{(2d)}N^{3/2}}{\tilde{w}_0^5}+\frac{14\tilde{g_2}^{(2d)}N}{\tilde{w}_0^6}\right]^2\right]^{1/2}\right]\omega_0\sqrt{1-{g'}_T^{(2d)}} \qquad (4.33)
\end{aligned}
$$

The above expression implies that the effect of thermal cloud results in a decrease in the collective excitation frequencies.

4.2 Numerical results and discussion

Since the effect of thermal cloud on the excitation frequencies is on the focus, two sets of plots are prepared - one showing the variation of the excitation frequencies with the temperature, while the number of particles in the system, N kept constant, and the other showing the variation of the excitation frequencies with N at constant temperature.

4.3 Effect of temperature on the collective excitation frequencies

The plots are prepared under the assumption that the temperature of the BEC is greater than T_0, where

$$T_0 = \frac{15^{2/5}}{7}\left(\frac{Na}{\bar{a}}\right)^{2/5}\frac{\hbar\bar{\omega}}{k}, \qquad (4.34)$$

and it gives a measure of the effective potential acting on a thermal excitation. Under this condition, collective excitations in the system can safely be approximated as free particles and the expression for the number of particles in the thermal cloud can be obtained as [66]

$$N_{exc} = N\left[t^3 + 2.15\left(\frac{N^{1/6}a}{\bar{a}}\right)^{2/5} t^2\left[1 - t^3\right]^{2/5}\right] \qquad (4.35)$$

where $t = T/T_c$ and the plots are prepared for temperatures, $T > T_0$. Dashed lines in the plots represent the condensate behaviour in the absence of thermal cloud and bold lines represent the behaviour when the effect of thermal cloud is also taken into account.

4.3.1 Effect of temperature on the Collective excitation frequencies of lithium(^{7}Li)

Figures (4.1) - (4.4) show the plot of W_-/ω_0 as a function of temperature, T for an attractively interacting condensate [lithium atoms(^{7}Li) with scattering length, a = -1.5nm]. As it has already been stated in chapter 3, in the case of attractively interacting condensates, the effect of quantum fluctuation can be omitted. These plots show the variation of W_-/ω_0 with temperature within and beyond meanfield regime. It is observed from the plots that temperature has a significant influence on the excitation frequencies of the system. As can be observed from Figs.(4.1) - (4.4), the number of particles in the system drastically affects the behavior of W_- with temperature. Figs.(4.5) - (4.8) represent the variation of W_+ with temperature. From the figures, it can be concluded that the influence of beyond mean field effects is to reduce the temperature range over which the excitation frequency has physical existence.

It can also be seen from Fig.(4.8) that a decrease in the anharmonicity of the system increases the range of this existence of the excitation frequencies.

In the absence of beyond mean field effects, for the anharmonicity parameter, $\zeta = 0.1$ and $N \geq 10^5$, the excitation frequency W_+ is not supported by the condensate. But when the mean field effects are also taken in to account, it can be observed that only for particle number $N < 10^4$, the excitation frequency W_+ is supported by the condensate, whereas the other excitation frequency, W_- is not supported for $N \geq 10^3$. Thus the beyond mean field effects reduce the range of particle number over which the excitation frequency will be sustained in the condensate.

4.3.2 Variation of collective excitation frequencies with temperature for sodium(^{23}Na)

For repulsively interacting condensate [sodium(^{23}Na)], s-wave scattering length has a positive value, ($a = 2.8$nm). Figs.(4.9) - (4.14) represent the variation of W_- excitation frequency with temperature and Figs.(4.15) - (4.20) are similar plots for the excitation frequency W_+. In this case also, the general observation is that inclusion of beyond mean field effects decreases the range of physical existence of the excitation frequencies. In the absence of beyond mean field effects, the excitation frequency W_- is supported by the condensate even at large values of $N \geq 10^7$. But, when the beyond mean field effects are also taken in to account, it is observed that for $N \geq 10^5$, the excitation frequency W_- is not supported by the condensate. In the case of W_+ also, the limit of particle numbers over which the excitation frequency is supported by the condensate is the same as that for W_-.

4.4 Variation of excitation frequencies with number of particles

Plots which show the variation of the excitation frequencies with number of particles at a constant temperature are also prepared. Two sets of plots are prepared for both attractively and repulsively interacting condensates. The first set extends the effect of gravity and anharmonicity on the collective excitations to the case of a condensate at finite temperature, while the second set represents the influence of beyond mean field effects on the collective excitations of a finite temperature BEC.

4.4.1 Behaviour of collective excitation frequencies as a function of the number of particles: Effect of gravity and anharmonicity

Figures (4.21) - (4.24) represent the variation of collective excitation frequencies against the number of particles in a BEC at finite temperature. Each plot contains the behavior of the system in the presence of gravity and anharmonicity and in the absence of these effects. From Fig.(4.21), it is clear that for an attractively interacting condensate, thermal cloud severely affects the behavior of W_{-}. In a pure harmonic trap at zero temperature, W_{-} exhibits an increase towards infinity, which represents a collapse of the condensate. This is followed by a reappearance of the excitation frequency at higher values of N [Fig.(2.1)]. But for a finite temperature condensate in pure harmonic trap as well as in an anharmonic trap (anharmonicity parameter, $\zeta = 0.001$) under the influence of gravity, although the reappearance is observed, the collapse is absent. When the value of anharmonicity parameter, ζ

is increased to 0.1, even the reappearance is not observed. The behavior of the other excitation frequency, W_+ is similar to the case of a zero temperature BEC as shown in Figs.(2.5)-(2.7) in chapter 2. In the case of repulsively interacting condensates, the behavior of both the excitation frequencies are relatively unaffected by the thermal cloud as well as by anharmonicity and gravity [Figs.(4.23) and (4.24)], although, at very low value of particle numbers, we observe a deviation in the plot due to the influence of the thermal cloud.

4.4.2 Behaviour of collective excitation frequencies as a function of number of particles: Beyond mean field effects

It is observed that, for attractively interacting condensates at finite temperature, the introduction of beyond mean field effects does not significantly affect the behavior of the excitation frequency W_- [Fig.(4.25)]. But it is particularly interesting to note that the thermal cloud has severely affected the excitation frequency. For a condensate in the mean field regime, at zero temperature, collapse and reappearance of the excitation frequency are observed, whereas when the beyond mean field effects are included, reappearance is absent [Fig(3.1)]. But, for a condensate at a finite temperature, the collapse of W_- excitation frequency is absent in the mean field regime. The behavior of W_+ is not found to be affected significantly by the thermal cloud [Fig.(4.26)].

In the case of repulsively interacting condensates also, the behavior of the excitation frequency W_- is not affected significantly by the thermal cloud [Fig.(4.27)]. But it is observed that, the beyond mean field effects result reducing the 'lower cut-off value' of

N below which the excitation frequency do not exist. The behavior of the excitation frequency W_+ is observed to be qualitatively the same as that in the case of a condensate at zero temperature [Fig.(3.7)].

4.5 Conclusions

It can be concluded from the figures [Figs.(4.1) - (4.20)] that the number of particles in the condensate decides the dependence of the collective excitation frequency on the temperature. It can also be seen that, for both W_+ and W_- excitation frequencies, inclusion of beyond mean field effects decreases the temperature range over which they exist, whereas a decrease of the anharmonicity parameter, ζ increases the range of existence of these excitation frequencies. It is particularly interesting to note that beyond mean field effects reduce the range of particle number over which the excitation frequency will be sustained in the condensate. In the presence of thermal cloud, the collapse of the attractively interacting condensate, as it was observed for condensate in a pure harmonic trap at zero temperature, is absent, whereas the excitation frequencies in a repulsively interacting BEC at finite temperature remains unaffected by the modifications in the potential.

4.6 Plots of collective excitation frequencies($W_\pm$) vs temperature(T)

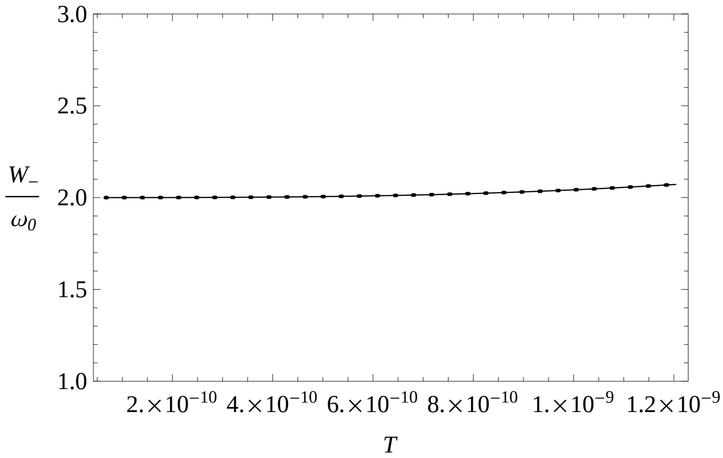

Figure 4.1: Collective excitation frequency W_- as a function of the temperature, T for ^{7}Li in the mean field regime: N=$10^2, \zeta = 0.1$.

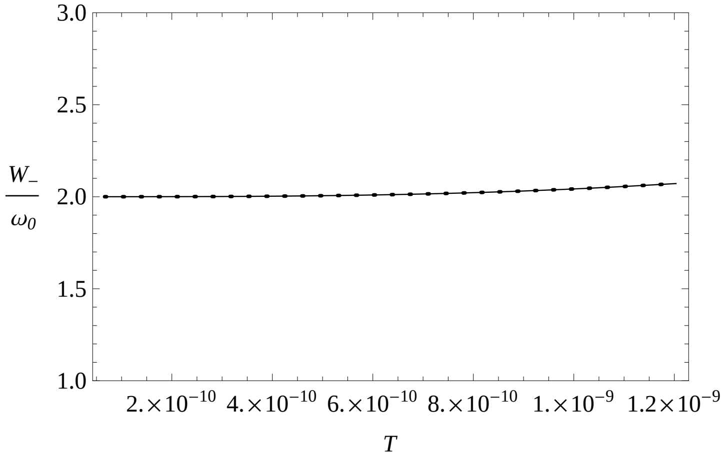

Figure 4.2: Collective excitation frequency W_- against temperature, T for ^{7}Li in the beyond mean field regime: N=10^2 and $\zeta = 0.1$.

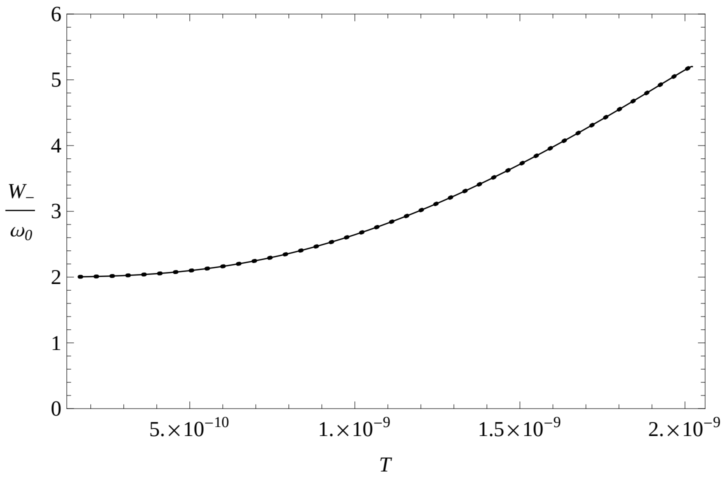

Figure 4.3: Collective excitation frequency W_- as a function of the temperature, T for ^{7}Li in the mean field regime: N=10^3, $\zeta = 0.1$.

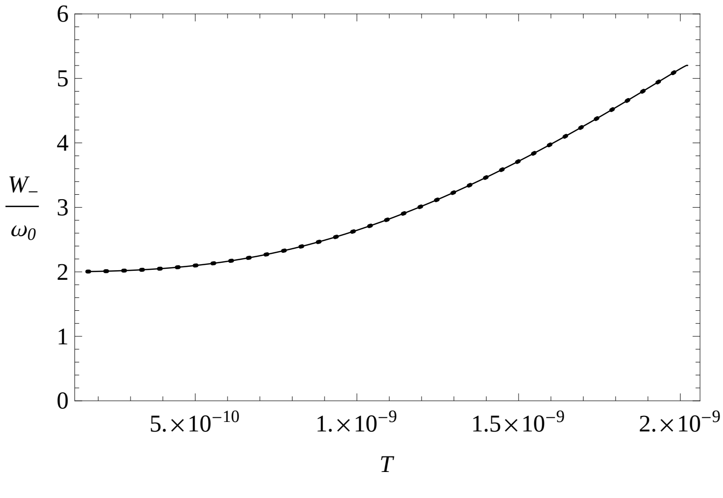

Figure 4.4: Collective excitation frequency W_- as a function of the temperature, T for ^{7}Li(beyond mean field regime): N=10^3 and $\zeta = 0.1$.

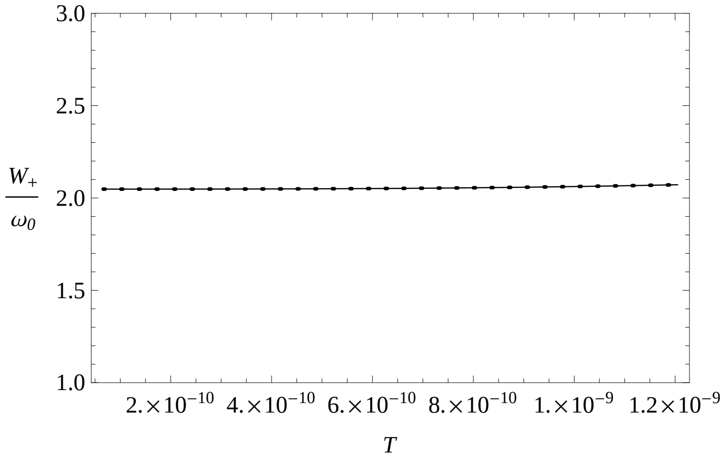

Figure 4.5: Collective excitation frequency W_+ as a function of the temperature, T for ^{7}Li within the mean field regime: N=10^2, $\zeta = 0.1$.

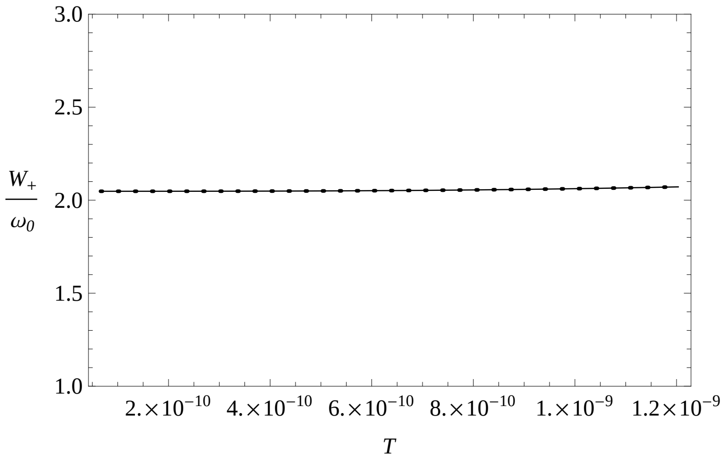

Figure 4.6: Variation of collective excitation frequency W_+ as a function of the temperature, T for ^{7}Li when beyond mean field effects are taken into account: N=10^2 and $\zeta = 0.1$.

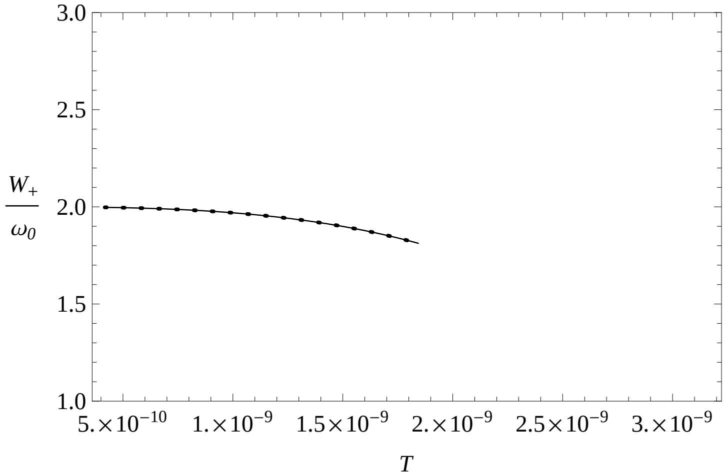

Figure 4.7: Collective excitation frequency W_+ as a function of the temperature, T for ^{7}Li (mean field regime): N=10^4, $\zeta = 0.1$.

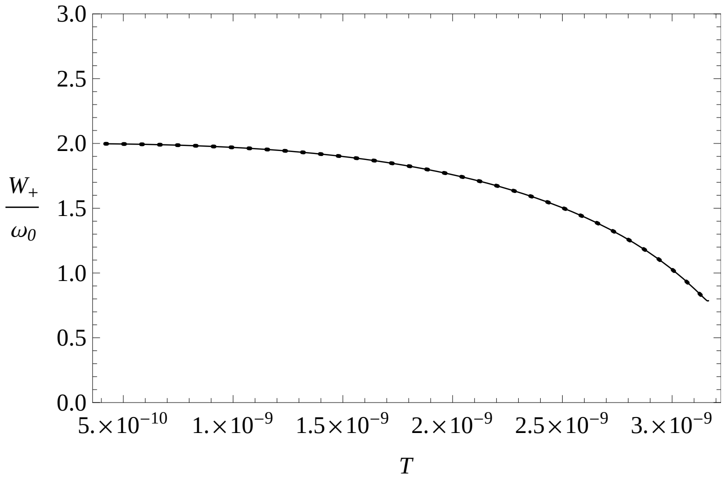

Figure 4.8: Variation of collective excitation frequency W_+ with temperature, T for ^{7}Li (beyond mean field regime): N=10^4, $\zeta = 0.01$.

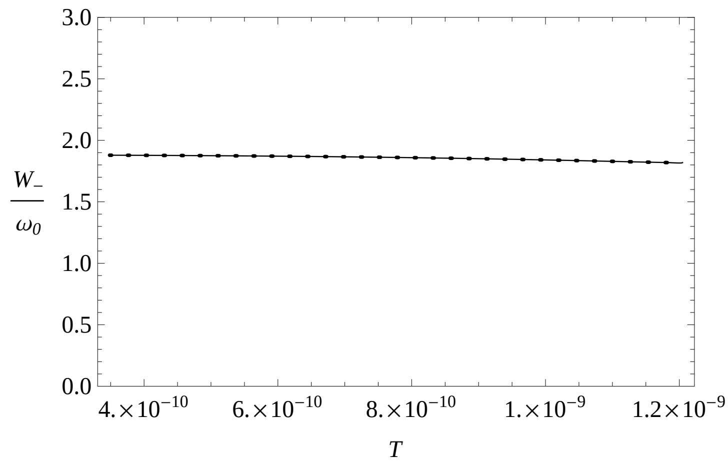

Figure 4.9: Plot of collective excitation frequency W_{-} as a function of the temperature, T for ^{23}Na in the mean field regime: N=10^2, $\zeta = 0.1$.

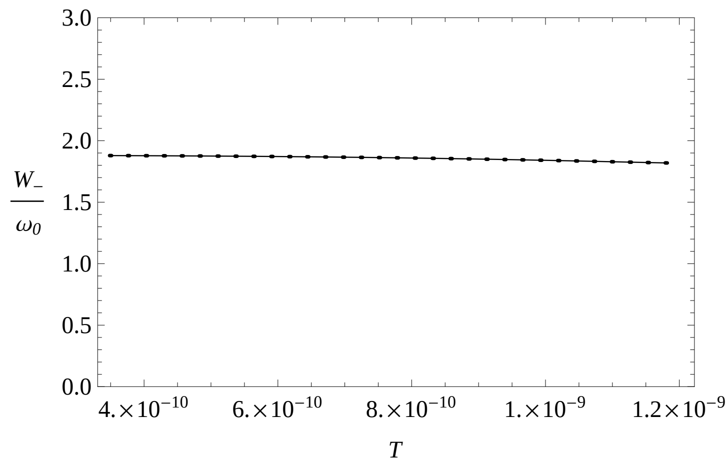

Figure 4.10: Collective excitation frequency W_{-} as a function of the temperature, T for ^{23}Na in the beyond mean field regime: N=10^2 and $\zeta = 0.1$.

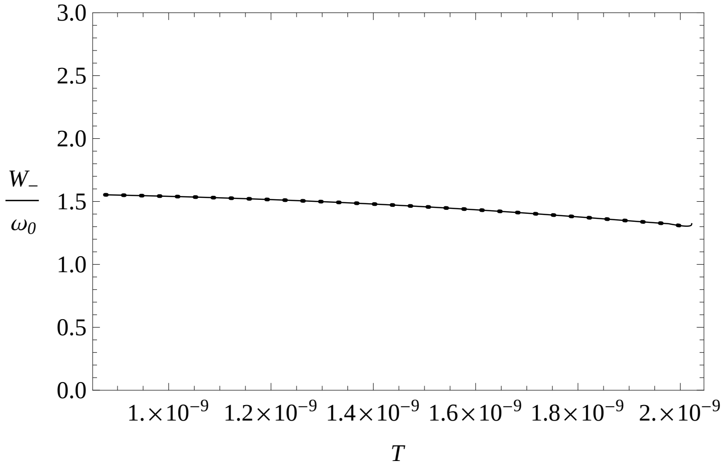

Figure 4.11: Collective excitation frequency W_- versus temperature, T for ^{23}Na (mean field regime): N=10^3, $\zeta = 0.1$.

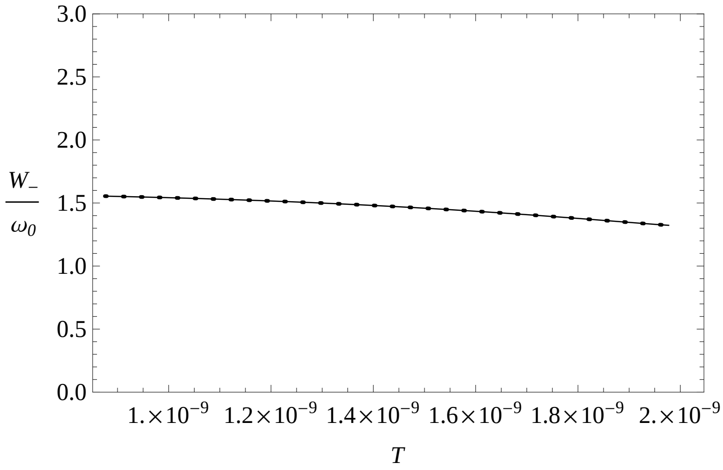

Figure 4.12: variation of collective excitation frequency W_- with temperature, T for ^{23}Na (beyond mean field regime): N=10^3 and $\zeta = 0.1$.

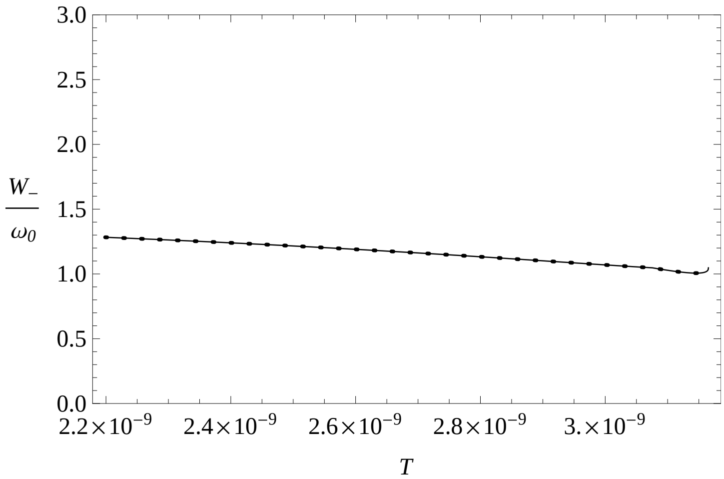

Figure 4.13: Collective excitation frequency W_{-} as a function of the temperature, T for ^{23}Na (mean field regime): N=10^4, $\zeta = 0.1$.

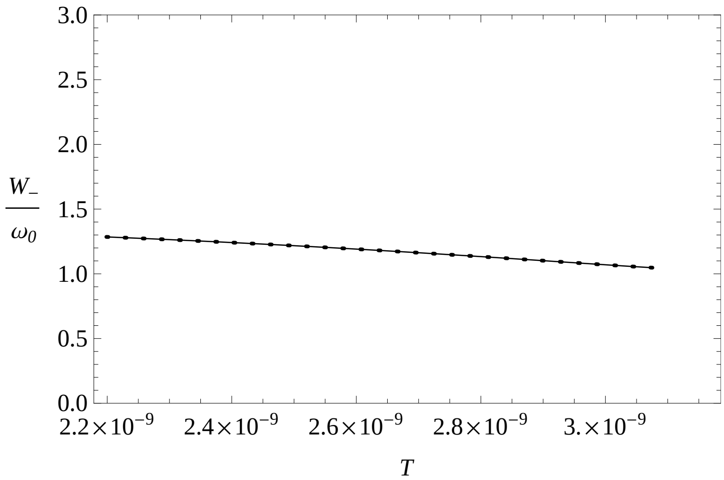

Figure 4.14: Collective excitation frequency W_{-} plotted as a function of the temperature, T for ^{23}Na with N=10^4 and $\zeta = 0.1$ (beyond mean field regime).

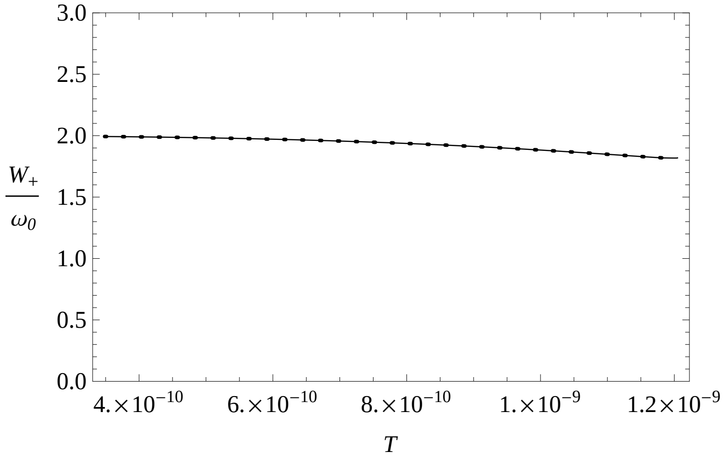

Figure 4.15: Collective excitation frequency W_+ as a function of the temperature, T for ^{23}Na for N=10^2, $\zeta = 0.1$ within the mean field regime.

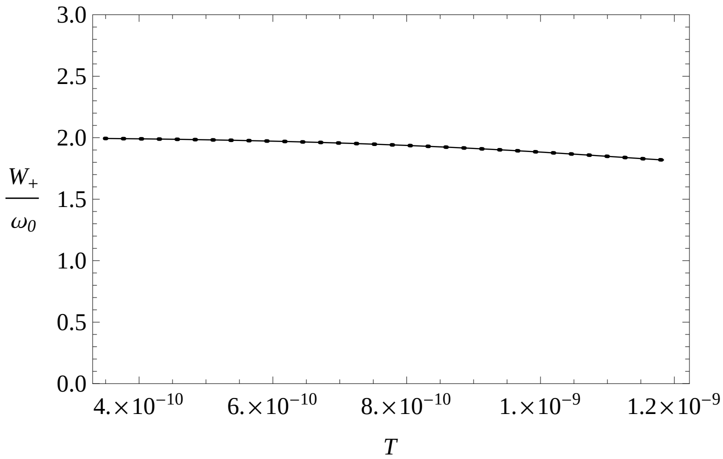

Figure 4.16: Collective excitation frequency W_+ versus temperature, T for ^{23}Na in the beyond mean field regime: N=10^2 and $\zeta = 0.1$.

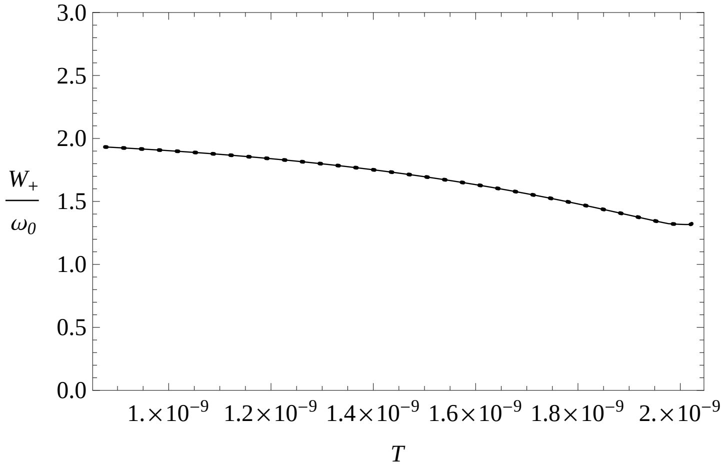

Figure 4.17: Plot showing variation of collective excitation frequency W_+ as a function of the temperature, T for ^{23}Na: N=10^3, $\zeta = 0.1$ (mean field regime).

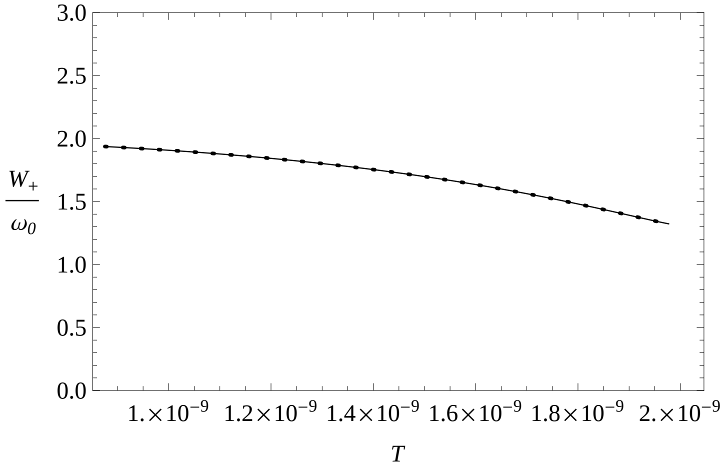

Figure 4.18: Collective excitation frequency W_+ as a function of the temperature, T for ^{23}Na in the beyond mean field regime with N=10^3 and $\zeta = 0.1$.

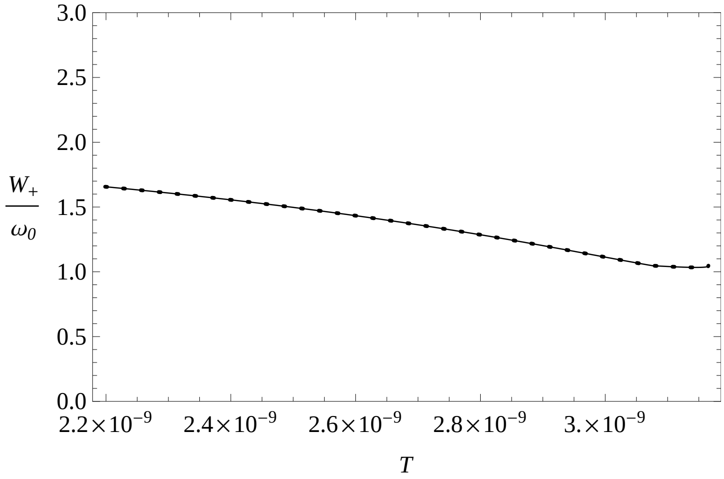

Figure 4.19: Variation of collective excitation frequency W_+ with temperature, T for ^{23}Na (mean field regime): N=10^4, $\zeta = 0.1$ and $g', g_1, g_2 = 0$.

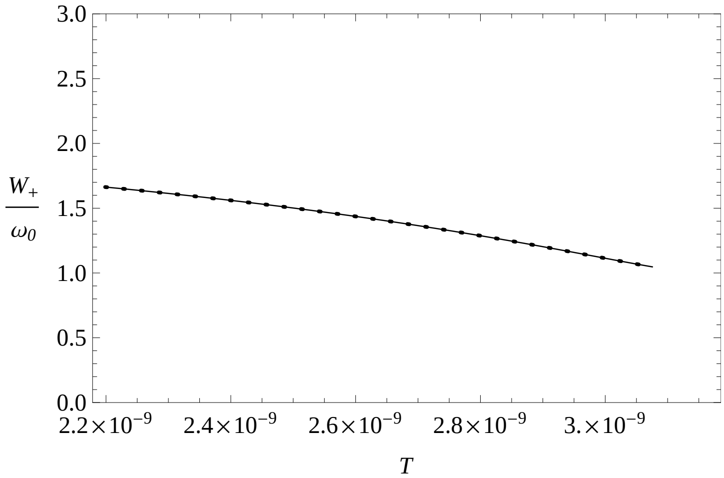

Figure 4.20: Collective excitation frequency W_+ as a function of the temperature, T for (^{23}Na) with N=10^4 and $\zeta = 0.1$ (beyond mean field regime).

4.7 Plots of collective excitation frequencies against number of particles(N)

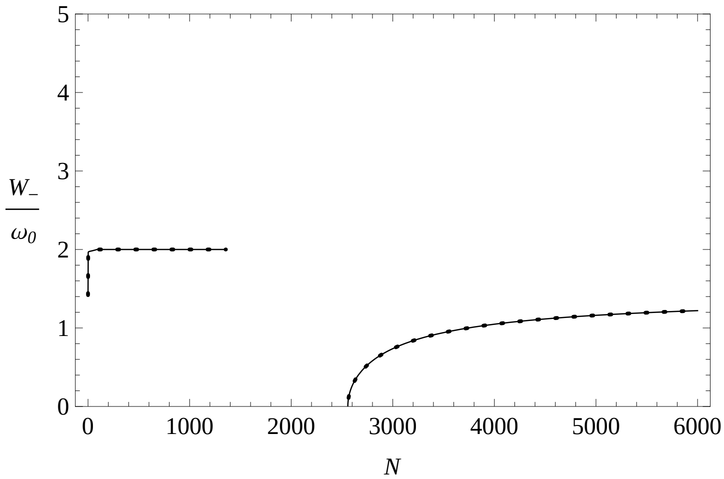

Figure 4.21: Plot of collective excitation frequency W_{-} as a function of the number of particles N for ^{7}Li in the case of a pure harmonic trap (dashed line) and in an anharmonic trap where the effect of gravity is included (solid line)with anharmonicity parameter $\zeta = 0.001$.

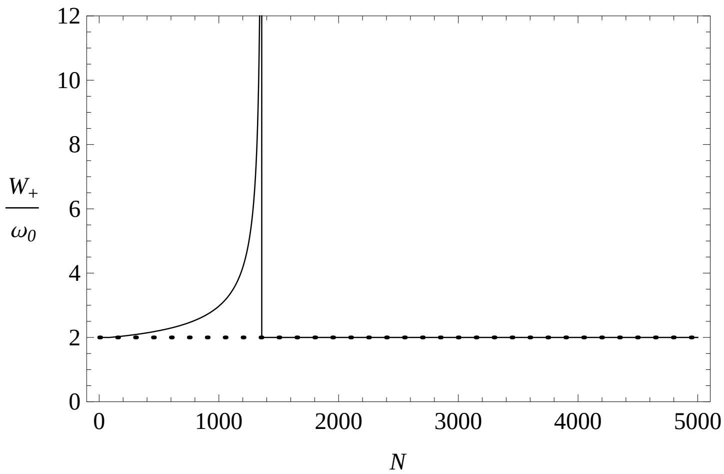

Figure 4.22: Collective excitation frequency W_{+} as a function of the number of particles N for ^{7}Li in the case of a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.1$)under the effect of gravity (solid line).

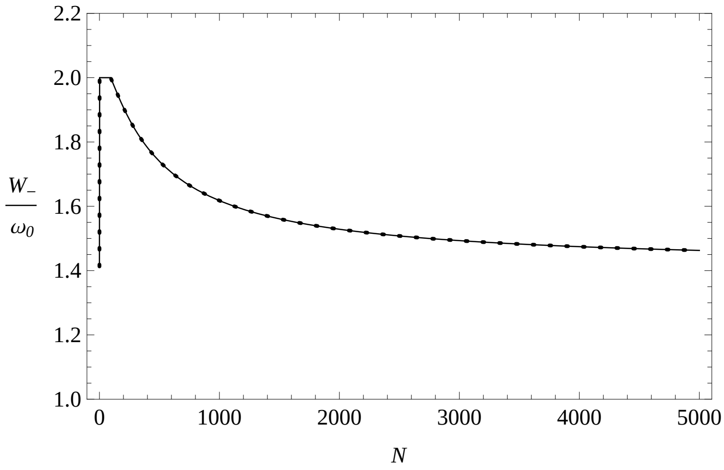

Figure 4.23: Plot showing variation of collective excitation frequency W_- with the number of particles N for ^{23}Na in the case of a pure harmonic trap (dashed line) and in an anharmonic trap (anharmonicity parameter $\zeta = 0.1$)where the effect of gravity is included (solid line).

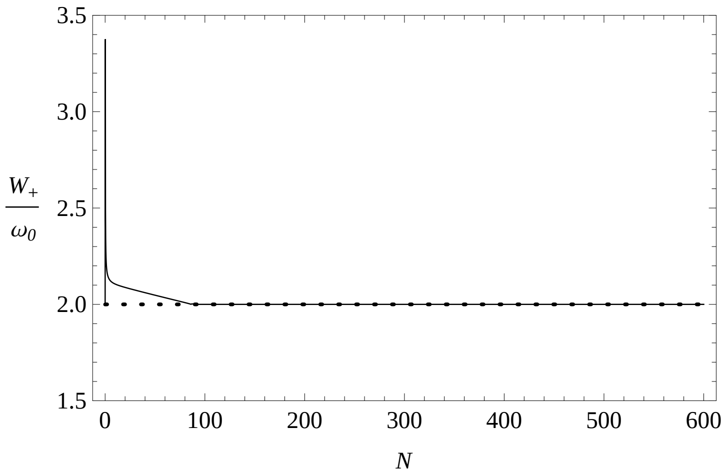

Figure 4.24: Collective excitation frequency W_+ as a function of the number of particles N for ^{23}Na in the case of a pure harmonic trap (dashed line) and in an anharmonic trap where the effect of gravity is included (solid line)with anharmonicity parameter $\zeta = 0.1$.

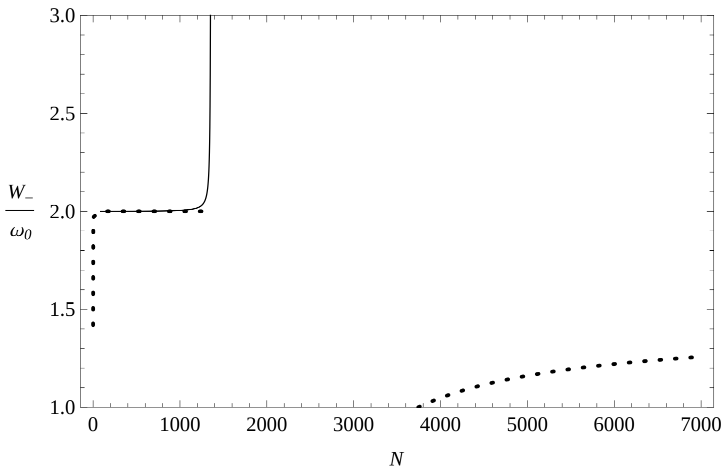

Figure 4.25: Variation of collective excitation frequency W_{-} with the number of particles in the condensate N for ^{7}Li: In the mean field regime (dashed line) and beyond mean field regime (solid line).

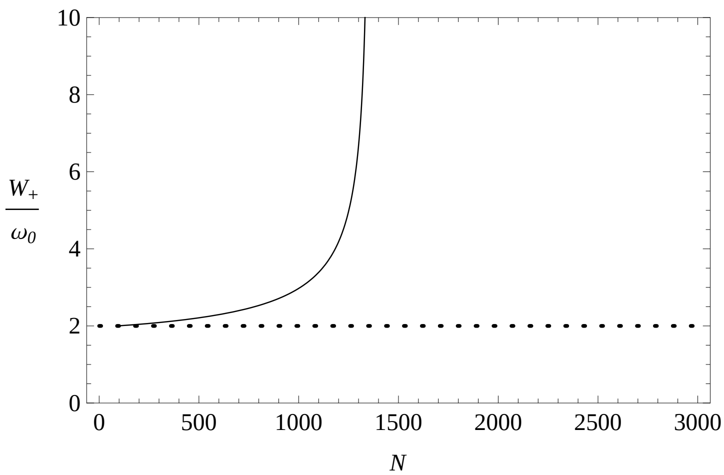

Figure 4.26: Collective excitation frequency W_{+} versus number of condensed particles N (^{7}Li): In the mean field regime (dashed line) and beyond mean field regime (solid line).

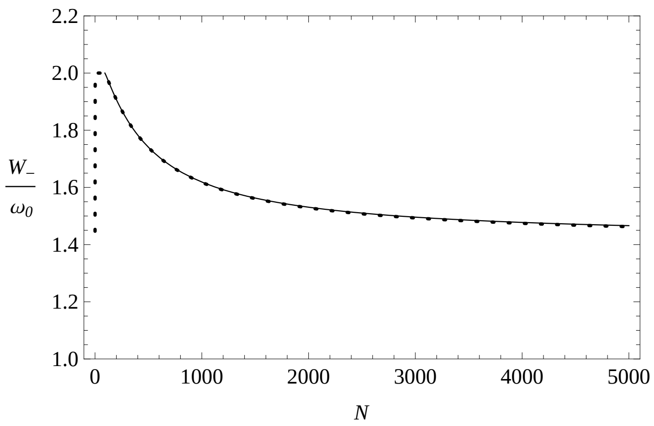

Figure 4.27: Collective excitation frequency W_- variation with number of condensed particles N for ^{23}Na: In the mean field regime (dashed line) and beyond mean field regime (solid line).

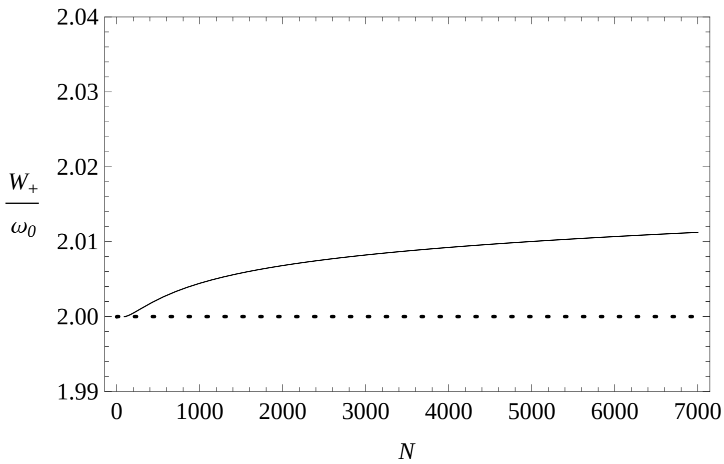

Figure 4.28: Plot of collective excitation frequency W_+ versus number of condensed particles N for ^{23}Na: In the mean field regime (dashed line) and beyond mean field regime (solid line).

5

Conclusions and envisaged works for future

5.1 Observations and conclusions

In this work, attention is given to the study of the behavior of collective excitations in a quasi-two-dimensional BEC. The idea behind the study is to asses the effect of smaller modifications in the potentials that will be present in an actual experimental BEC set up. By modifying the potentials in this way, the theoretical calculations are expected to deliver results that are closer to those from the experiments. In chapter 2, the effect of gravity and anharmonicity on the collective excitations of a quasi-two-dimensional BEC is analyzed. In the absence of these modifications, condensate width exhibits breathing mode oscillations, $m = 0$ and $|m| = 2$ oscillations, which show dependence on the inter particle interaction parameter, $g^{(2d)}$. The center of the condensate is found to oscillate with the frequency of the trap. When the modifications are introduced, it is observed that the oscillations of the center of the condensate have a frequency other than the frequency of the trap. Hence it is concluded that the oscillations of the center get coupled with their internal degrees of freedom. In the case of the oscillations of the width of the condensate, it is observed that the breathing mode is destabilized by the modifications in the potential. The other excitation frequency is seen to be independent of the effect of gravity, if the inter-particle interaction is negligibly small. This shows that the inter-particle interaction plays a role in governing the effect of gravity on the condensate dynamics.

In the following chapter, the variations of the collective excitations of a quasi-2D BEC due to beyond mean field effects are analyzed. It is observed that the beyond mean field effects do not result in a coupling between the center of the condensate and its internal degrees of freedom. For an attractively interacting condensate, reappearance of the collective excitation frequency W_- is observed to be hindered by the beyond mean field effects, whereas for a repulsively interacting condensate, W_- shows an increase with N after reaching a minimum. It is also found that, in the presence of beyond mean field modifications in the potential, breathing mode W_+ of frequency $2\omega_0$ is not supported by both attractively and repulsively interacting systems. Instead, it exhibits an increase with N.

In Chapter 4, the collective excitations in a quasi-2D BEC at finite temperature are investigated. In addition to the thermal cloud, the effects of gravity, anharmonicity and beyond mean field effects are also included in the potential. Variation of excitation frequencies with respect to the condensate particle number(N) and with respect to temperature has been studied. It is observed that the dynamics of the center of the condensate is not influenced by the thermal cloud. But the oscillations of the width of the condensate are found to depend on the thermal cloud. From this observation, it is concluded that the number of particles in the condensate decides the dependence of the collective excitation frequency on the temperature. It is further noticed that, the inclusion of beyond mean field effects decreases the temperature range, over which the excitation frequencies are supported by the condensate, whereas a decrease of the anharmonicity in the potential increases the range of existence of excitation frequencies. Another observation is the reduction in the range of particle number, over which the excitation frequency will be sustained, in the condensate due to the beyond mean field effects. Also, in the presence of thermal cloud, the collapse and revival of the attractively interacting condensate, as it was observed for condensate in a pure harmonic trap, is found to be absent. From the plots, it could also be deduced that there is a certain limit in the particle number, beyond which the excitation frequencies are not supported in the condensate. This limit is reduced due to the presence of modifications in the potential.

5.2 Future Prospects

An extension of the present work to the case of a quasi-one-dimensional BEC is a trivial step to be taken further, as another reduction in the dimension will result in a greater control over the scattering length through the manipulation of trapping potentials. In the present work, we used mean field formalism for the analysis. Though the mean field theory is observed to give appreciably accurate results, beyond mean field theories like ZGN- theory can be expected to increase the accuracy of the results. The validity of the already arrived at conclusions can be established using such beyond mean field theories. To conclude, we have in this thesis concentrated our studies on the collective excitations of quasi-2D

BEC. Analytical as well as numerical studies in this area can be seen extensively in the literature and further studies are going on. Studies already conducted in this thesis are to be extended to the above mentioned points in future.

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