STUDIES ON SCATTERING AND THERMODYNAMICS OF BLACK HOLES IN F(R) THEORY AND EINSTEIN'S THEORY

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Studies on scattering and thermodynamics of black holes in f(R) theory and Einstein's theory
PhD thesis in the field of Black Hole Physics

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Front cover : Penrose diagram, from internet

To my beloved parents

CERTIFICATE

Certified that the work presented in this thesis is a bonafide research work done by Mr. Saneesh Sebastian, under our guidance in the Department of Physics, Cochin University of Science and Technology, Kochi- 682022, India, and has not been included in any other thesis submitted previously for the award of any other degree.

Kochi-22
December, 2014

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DECLARATION

I hereby declare that the work presented in this thesis is based on the original research work done by me under the guidance of Prof. V. C. Kuriakose (Rtd.) and Prof. Ramesh Babu T., Department of Physics, Cochin University of Science and Technology, Kochi- 682022, India, and has not been included in any other thesis submitted previously for the award of any other degree.

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Contents

Preface

One of the interesting consequences of Einstein's General Theory of Relativity is the black hole solutions. Until the observation made by Hawking in 1970s, it was believed that black holes are perfectly black. The General Theory of Relativity says that black holes are objects which absorb both matter and radiation crossing the event horizon. The event horizon is a surface through which even light is not able to escape. It acts as a one sided membrane that allows the passage of particles only in one direction i.e. towards the center of black holes. All the particles that are absorbed by black hole increases the mass of the black hole and thus the size of event horizon also increases. Hawking showed in 1970s that when applying quantum mechanical laws to black holes they are not perfectly black but they can emit radiation. Thus the black hole can have temperature known as Hawking temperature.

Astronomical observations made in 1998 that universe is expanding with acceleration cannot be explained in Einstein's General Theory of Relativity. So after this discovery people have been thinking of other types of theories of gravitation to explain the concepts like dark energy and dark matter etc. There are different approaches to modify Einstein's General Theory of Relativity and f(R) theory is one among them and perhaps the simplest modification. We study black holes in this model and also in Einstein's General Theory of Relativity

The thesis is organized as follows: **Chapter 1** gives an introduction to the research studies presented in the thesis. We discuss the various attempts made to extend this theory with special reference to f(R) theory. We also discuss the concepts like scattering of fields in black hole spacetimes quasinormal modes, thermodynamics of black holes and the area spectrum.

In **Chapter 2** we study scattering of scalar field by a black hole in f(R) gravity. We take a spherically symmetric, static and asymptotically flat black hole solution in f(R) gravity. Using scattering method we have obtained the absorption cross section and also calculated the Hawking temperature of the black hole via tunneling method. Comparing with Schwarzschild black hole, absorption cross section of the present black hole is about one half of that of the Schwarzschild black hole.

In **Chapter 3**, we study the quasinormal modes of electromagnetic and scalar perturbations of the extended black hole in f(R) theory. The black hole space-time is perturbed by electromagnetic and scalar waves and the behavior of the resulting quasinormal modes are evaluated. The present study shows that the imaginary part of complex quasinormal modes for both cases increase showing damping of oscillations. Thus the black hole is stable against both scalar and electromagnetic perturbations. The damping time increases with increasing mass of the field in the case of scalar field.

The first part of **Chapter 4** contains the study of the thermodynamics and spectroscopy of a $2+1$ black hole called MSW black hole introduced by Mandal, Sengupta and Wadia. The thermodynamics of MSW black hole is studied and area spectrum is obtained using adiabatic invariant method. We have obtained the heat capacity, temperature and mass of MSW black hole. The heat capacity is found to be negative as is the case for most black holes. That is, the MSW black hole is thermodynamically unstable. This black hole does not show any phase transition. The period of Euclidean time is found out and adiabatic invariant is calculated. Using Bohr-Sommerfeld quantization rule we have obtained the area spectrum, here the circumference spectrum and is found to be quantized and equally spaced. In the second part of the chapter we discuss the massless Dirac quasinormal modes of MSW black hole. The complex quasinormal mode frequencies are found out and tabulated. The modulus value of imaginary part of the frequencies increases with increasing mode number showing that the radiation is damping which in turn shows that the black hole is stable against massless Dirac field perturbations. Real value of the frequencies increases in this case contradictory to the normal $3+1$ black hole cases.

In **Chapter 5**, we have studied the thermodynamics of a regular black hole proposed by Hayward and the area spectrum is studied using adiabatic invariant method. We have obtained the expression for temperature and heat capacity. Heat capacity curve has a discontinuity at certain value of entropy showing that a phase transition might have occurred. For low values of entropy the heat capacity is positive showing that the black hole is thermodynamically stable. For higher values of entropy the heat capacity becomes negative and hence the black hole is thermodynamically unstable. The spectrum of Hayward black hole is obtained, it is found that the spectrum is quantized but not equispaced and the values of the of the energy levels depend on horizon radius r_+.

In **Chapter 6** we discuss the overall summary of the present studies and future perspectives.

Publications related to the work presented in the thesis:

In refereed journals

- Spectroscopy and Thermodynamics of MSW black hole
  Saneesh Sebastian and V. C. Kuriakose, Modern Physics Letters A, Vol: 28 No. 33 (2013) 1350149

- Scattering of scalar field by an extended black hole in f(R) gravity
  Saneesh Sebastian and V. C. Kuriakose, Modern Physics Letters A, Vol: 29 No. 2 (2014) 1450005

- Dirac quasinormal modes of MSW black hole
  Saneesh Sebastian and V. C. Kuriakose, Modern Physics Letters A, Vol: 29 No.5 (2014) 1450019

- Scalar and electromagnetic quasinormal modes of an extended black hole in f(R) gravity
  Saneesh Sebastian and V. C. Kuriakose, Accepted for publication in Modern Physics Letters A

- Thermodynamics and Spectroscopy of a regular black hole
  Saneesh Sebastian and V. C. Kuriakose, To be submitted

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- "Scattering of Scalar field around an extended black hole in f(R) gravity," Workshop on astronomy research: Opportunities and Challenges. 2013, MACFAST, Thiruvalla, Kerala

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1 Introduction

Einstein's theories of relativity revolutionized our concept of space and time. In 1905 he published the special theory of relativity in which he introduced the concept of space-time continuum and that the speed of light would be the ultimate speed with which any signal could travel. Thus, Einstein put the velocity of light as the ultimate attainable velocity. The velocity of light is constant irrespective of the velocity of the observer. From the time of Newton it was believed that different forces of interaction including gravity act instantaneously among different bodies, but the special theory of relativity restricts that the velocity with which any interaction can travel be less than or equal to the velocity of light. Thus, it becomes necessary to modify the Newton's theory of gravity. There are other reasons also for modifying Newton's theory of gravity. According to special theory of relativity the mass and energy are equivalent and thus the gravitational field affects all forms of energy (including radiation whose rest mass is zero) also. The bending of light near the sun is an example of this. Thus, in general, the source of gravity is the energy-momentum tensor of the matter present in the universe. Einstein solved all these problems in a geometric way in his general theory of relativity. It completely modified the then existing theory of gravitation (i.e. Newton's law of gravitation). According to Einstein, gravitational force is not a force at all. Each body curves the spacetime continuum proportional to its energy-momentum content. All bodies move in curved background. This curvature appears to us as the gravitational force[1]. The General Theory of Relativity was completed in the final form in 1915. Einstein's General Theory of Relativity gives a relation, connecting the curvature of the spacetime and the energy-momentum tensor of matter present in the universe this relation gives the field equation of Einstein's General Theory of Relativity. In General Theory of Relativity spacetime plays an active role other than setting the stage for interactions. The spacetime is dynamic in General Theory of Relativity. A month after the publication of the General Theory of Relativity, a German Mathematician, Karl Schwarzschild[2] found an exact solution of the Einstein's field equation in the vacuum near a spherically symmetric massive body, now known as Schwarzschild solution. This was the first solution of Einstein's field equations. Using this solution a number of new phenomena such as bending of light in gravitational field, the perihelion precession of the planet mercury etc. were found out. The theory also predicts two singularities one at, $r = \frac{2GM}{c^2}$ with r is the distance of event horizon, M is the mass of the body, G is the gravitational constant and c as the velocity of light, and other singularity is at the origin from which no information can come out. The singularity of Schwarzschild solution is now known as Schwarzschild black hole. This is the first black hole solution in General Theory of Relativity. Cosmology as a subject was born after Friedmann[3, 4] in 1922 obtained a solution of General Theory of Relativity and this solution indicates that the universe is expanding. At first, Einstein opposed this, but he accepted it after the discovery made by Hubble that the galaxies show redshift. Thus, Friedmann-Robertson-Walker[5, 6] metric describes an expanding universe, which is homogeneous and isotropic. The standard theory which explains the formation and evolution of universe based on Einstein's Relativity is called the Big Bang model. The basic idea of Big Bang is that all galaxies show redshifts in their spectra, if we assume this is due to the velocity of the galaxy with which they are moving away from us, then we can assume a time at which all galaxies might have merged to a single point in the past. Thus the universe is formed due to a Big Bang which occurred in the past, This idea was due to Lamaitre[7]. The discovery of cosmic microwave background gives a strong support for the Big Bang theory. Later charged solution and rotating solution of Einstein's General Theory of Relativity were found out by Reissner and Nordstrom[8, 9] and Roy Kerr[10] respectively.

1.1 Black holes

Using the words of Kip S Thorne, the well known relativist[11]:

"black hole: a hole in space with a definite edge into which any thing can fall and out of which nothing can escape, a hole with a gravitational force so strong that even light is caught and held in its grip, a hole that curves space and warps time....."

The black hole idea entered into human thought after Newton. Combining both corpuscular theory of light and universal law of gravitation, John Michell in 1783 was the first who said about black holes. Later in 1797 Pierre de Simon Laplace wrote a paper on it and he mathematically showed that a star having sufficient mass could absorb all light that it emits and proved that high luminosity stars would be dark ones because of gravitation and he called them as dark stars. But after some decades, the wave theory of light became dominated over the corpuscular theory. Thomas Young and Fresnel explained the interference and diffraction of light by the wave theory and the phenomenon of polarization shows that the waves are transverse. The studies of Maxwell on electromagnetism completely removed the corpuscular nature of light. It is interesting to note that from Newton's law of gravitation one can get an expression for Schwarzschild radius without using General Theory of Relativity. The escape velocity v of a particle is $v = \sqrt{\frac{2GM}{r}}$ and if v becomes c, the velocity of light in this formula, we get the expression for the Schwarzschild radius. But conceptually sound argument for the existence of black holes came after the formulation of General Theory of Relativity in 1915. Schwarzschild solution is a simple, static and spherically symmetric one. It has a singularity at the origin (r=0). In 1932 Indian Astrophysicist Chandrasekhar showed that there exists an upper limit for the mass of the core of a star to end up in white dwarf stage and using quantum statistics and general relativity he showed the upper mass limit to be about $1.44M_{\odot}$. This limiting mass of the core of the star is now known as Chandrasekhar limit. If such a limit exists then a question arises, "What happens to a core of the star having mass greater than $1.44M_{\odot}$?". The answer came after the discovery of neutron by Chadwick. Like the electron degeneracy pressure balances the gravitational inward pull in the White dwarfs there exist stars made up of neutrons and the gravitational inward pull is balanced by the neutron degeneracy pressure. Landau[12] predicted that such stars would exist in nature, since the neutron degeneracy pressure is greater than electron degeneracy pressure. Stars having mass more than Chandrasekhar limit may end up their lives as neutron stars or as black holes. Due to the inaccuracies of Quantum Chromodynamics (QCD), the upper bound of mass for which the neutron degeneracy pressure that balances the gravitational inward pull is not accurately known but it is assumed to be 2-3$M_\odot$. This limit is known as Oppenhiemer-Volkoff limit[13]. There are stars having mass greater than above this limit and in such cases even neutron degeneracy pressure cannot balance the strong gravitational inward pull and in such cases the core of the star may implode becoming a black hole. All mass of the star squeezed into a single point where both space and time stop[14]. The word black hole was coined by Wheeler[15] in 1968. A paper on continued gravitational collapse came in 1939 by Oppenhiemer and Snyder[16]. For the first time, this paper showed that a black hole could be formed as result of continued gravitational collapse. A solid evidence came from a balloon born experiment. The object named Cygnus X-1 which includes a double star system HDE 226868 emitting hot X-rays may be a strong candidate for a black hole. It was the first evidence that black hole might be existing in our universe. The mass of Cygnus X-1 is calculated to be $\sim 10M_\odot$ and has a radius of 10km. After the discovery of X-Ray telescopes, a number of black hole candidates with solar mass and with super massive black holes have been found out. Nowadays it is believed that all galaxies contain super massive black holes at their centers. The Milky way, our own galaxy, contains an object called Sagittarius A^* at the center[17]. It has a very small radius but with a mass of about $10^6 M_\odot$. It is believed that this is a super massive black hole. The energy source of Quasars and other active galactic nuclei are believed to be of this type of super massive black holes which lying at the center of quasars and active galactic nuclei.

1.1.1 Characteristics of black holes

Solutions of Einstein equation possess singularities. For most cases we cannot remove the central singularity. The Schwarzschild solution is also a solution of the above form. The Schwarzschild line element is given by,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{1.1}$$

It is well known that it has a co-ordinate singularity at $r = 2M$ and a real singularity at $r = 0$. The coordinate singularity at $r = 2M$ divides the space time that contains the black hole into two. For one region $r{>}2M$ and for the other r lies in between zero and $2M$. In Schwarzschild coordinate there exists no connection between the two regions, but we can explore the two regions using other coordinates such as Eddington-Finkelstein coordinates or more general Kruskal-Szekeres coordinates. Using these new coordinates, it is possible to remove the singularity at $r = 2M$ but the singularity at $r = 0$ cannot be removed using any coordinate transformations. The Kretschmann scalar[29] is infinite at $r = 0$ and the curvature becomes infinite there. It is a static, spherically symmetric and asymptotically flat solution of Einstein's equation. The Schwarzschild black hole is one with no rotations and no charge. The Reissner-Nordstrom black hole is an example of charged black hole. The Kerr solution leads us to rotating black holes. Kerr-Newman solution gives a black hole solution with charge and rotation. Kerr-Newman black holes are the most general type of black holes. The No-Hair theorem[1] limits the characteristic of black holes as mass, charge and angular momentum (M, Q, J).

1.2 Need for extended theories of gravity

The standard Big Bang cosmology (SBBC)[20, 21] fits exactly to the astronomical observations until recently. The need to modify the Einstein's theory came from both the study of cosmology and quantum field theory. In cosmology the presence of primordial singularity is one where the general theory breaks down. There are other issues such as flatness, horizon and monopole problems in SBBC which lead to think for other gravitational theories. The detailed studies show that the universe has two accelerating phases, the first one is the inflation[22] and the other one is the late time acceleration observed in 1998[23, 24]. The first accelerated phase occurred prior to the radiation dominated era called inflation and late time acceleration started after the matter dominated era. Both acceleration phases cannot be explained using General Theory of Relativity. The data from CMBR and supernovae surveys show that the universe contain $\sim 4\%$ ordinary baryonic mass, $\sim 20\%$ dark matter and $\sim 76\%$ dark energy[23, 25–27]. Dark energy is introduced to explain the late time accelerating expansion but the nature of dark energy is not known yet. One way to solve these problems is by introducing one or more scalar fields into the Einstein's theory. The galaxy rotation curve is the other challenge of General Theory of Relativity and when the virial theorem is applied to galaxies and clusters of galaxies they contain more matter than we can see. This is the dark matter problem. The modified Newtonian dynamics popularly known as MOND is an attempt to explain the galaxy rotation curve without invoking the hypothetical dark matter[28]. Standard Model of particle physics also does not fit with the cosmological observations[29]. The case of vacuum energy is the simple example which deviates from cosmological observation. The calculated value of vacuum energy is one twenty orders larger than the observed value. The General Theory of Relativity (GTR) is a classical theory and which cannot be applied to singularities and other high energy regime such as early stages of the universe. Thus we seek for a quantum description of gravity. Due to the absence of a successful quantum gravity, we must seek other gravitational theories which must give general relativity as the classical approximation. The attempts in these directions lead to the modification of general theory of relativity and these calculations now lead to new formulations and are in general called Extended Theories of Gravity (ETGs). The ETGs are based on corrections and enlargement of GTR. There are a number of extended theories of gravity in the literature, such as Gauss-Bonnet theory, massive gravity, the f(R) theory, Born-Infield theory etc.. In higher dimensions, the Lovelock gravity works comparatively successful. The loop quantum gravity is an attempt to quantize gravity[30]. Horava-Lifshitz theory is also an attempt to quantize gravity[31]. In all unified theories such as super string theory, super gravity and grand unified theories (GUT), one has effective actions which couple non minimally to the geometry and also may contain higher order curvature invariant terms. These are due to the first order or higher order loop corrections in high curvature limit. These evidences show that in General Theory of Relativity we must introduce changes in such a way that we obtain non-minimal coupling in higher curvature limit.

Another motivation to modify GTR is to incorporate the Mach's principle, into the theory[32]. The Brans-Dicke theory[33] is an attempt in this direction. According Mach's principle the local inertial frame is determined by the average motion of distant astronomical objects. In Brans-Dicke theory, a variable gravitational 'constant' equivalent to a scalar field non-minimally coupled to the geometry makes it a good approximation for the Mach principle. When quantum corrections are introduced, the higher order terms of R such as R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ must be added to the gravitational Lagrangian. The study towards quantum gravity shows the presence of curvature invariants in the formulation. The Gauss-Bonnet invariant is one example of curvature invariant. There are many proposals of different modified gravity in literature[34–38].

1.3 $f(R)$ gravity theories

f(R) theory is the simplest modification of General Theory of Relativity and extensively studied during the past decade[39]. The new cosmological observation shows that universe went through two phases of acceleration, the first is inflation and the other is the late time acceleration. The inflation is believed to have occurred before the radiation dominated era. Using the phenomenon of inflation, we can explain the horizon and flatness problems in Big Bang cosmology. It is also required to explain a nearly flat spectrum of temperature anisotropies observed in Cosmic Microwave Background Radiation. After matter domination era the second accelerating phase (late time acceleration) started. In Einstein gravity this acceleration phase is driven by a mysterious fluid possessing negative pressure component called the dark energy. The existence of dark energy has been speculated by a number of ways such as Supernova studies, baryon acoustic oscillation, large scale structure and cosmic microwave background. The Lagrangian density of general relativity is given by $f(R) = R - 2\Lambda$ where R is the Ricci scalar and Λ is the cosmological constant. This action leads to the Λ Cold Dark Matter (ΛCDM) model. We can modify the General Theory of Relativity by modifying either energy-momentum part or modifying the curvature part and in f(R) theory we modify the curvature part. In f(R) theory, the Lagrangian density changes in such a manner that the Ricci scalar is changed to some general function of Ricci scalar[40]. There are two formalisms in deriving the field equation in f(R) gravity: The first one is the metric formalism in which the f(R) action is varied using the metric tensor $g_{\mu\nu}$ and obtain metric f(R) gravity with fourth order field equation. The other formalism is by varying the f(R) action with respect to the affine connection which is independent of the metric tensor and obtain Palatini f(R) gravity. Unlike the metric formalism, here the field equation is second order as in General Relativity. Considering a simple modification of the action, $f(R) = R + \alpha R^2$ model with a positive α can explain the cosmic acceleration, first proposed by Starobinsky in 1980 for explaining the cosmological inflation[41]. The metric formalism of f(R) gravity is equivalent to the more general Brans-Dicke theory with the Brans-Dicke parameter $\omega_{BD} = 0$. Unlike in the original Brans-Dicke theory, in f(R) gravity there exists a scalar degree of freedom (called the scalaron) with gravitational origin. The mass of the scalaron is as light as the present day Hubble parameter H_0.

In the present study we discuss only the metric f(R) gravity. The general action for f(R) gravity can be written as ,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + \int d^4x L_m(g_{\mu\nu}, \Psi_m), \tag{1.2}$$

where $\kappa^2 = 8\pi G$, g is the determinant of the metric $g_{\mu\nu}$ and L_m is the Lagrangian depending on $g_{\mu\nu}$ and matter field Ψ_m. In Einstein Gravity, the action will be of the form,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R + \int d^4x L_m(g_{\mu\nu}, \Psi_m). \tag{1.3}$$

The difference is that the Ricci scalar R changes to some regular function of R thus the name f(R) gravity.

Varying the above action with respect to the metric tensor we get the field equation for general f(R) as,

$$f'(R)R_{\mu\nu}(g) - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) = \kappa^2 T_{\mu\nu}(m), \tag{1.4}$$

where $f'(R)$ is the first derivative of f(R) with respect to R. The last two terms make the field equation fourth order. Since there are no experimental evidences for the existence of dark matter and dark energy, by suitably defining the f(R) we may fit the observational data with high accuracy. Early solutions involving constant f(R) include de la Cruz et al. [42]. As any extended theories of gravity, f(R) theory has its own limitations. We cannot find out a single f(R) model which explains both galaxy rotation curves and acceleration of the universe. There are differences between Einstein's theory and f(R) theory and we must require more observational data for finding out which theory is the correct one

1.4 Quasinormal modes

The quasinormal modes are popularly known as the characteristic sound of black hole. The study shows that if we perturb a black hole from its equilibrium it will execute nearly harmonic oscillations which are called the quasinormal modes. The frequency and damping time are related to only black hole parameters. According to Chandrasekhar, black holes are the most perfect objects in nature and mass, charge, and angular momentum are the only known black hole parameters. The characteristics of quasinormal modes (such as mode frequencies and damping time) depend only on these black hole parameters. In the case of Schwarzschild black hole the mode frequencies and the damping time depend only on the mass of black hole. The resulting quasinormal modes are independent of the initial perturbations that excited the black hole. The quasinormal oscillations are much different from the oscillations of a star. In the case of star, the oscillations are originated and are carried by the fluid that are contained in the star. In the case of black holes there is no fluid or matter to execute these oscillations. The horizon is in general of the form of a membrane. The oscillations are not carried out by the horizon but the oscillations are executed by the spacetime outside the horizon. It is not surprising because in General Theory of Relativity the spacetime is a dynamic quantity just like the fluid in the star. There are differences between the normal mode system and these oscillations which are not truly stationary but are damping exponentially and appearing only for a limited period of time. Due to these reasons they are called the quasinormal modes. In fact, through quasinormal modes the black hole spacetime is radiating energy to infinity as gravitational waves. In general, the normal frequencies are not damping and it is possible to expand the field variables in terms of normal modes. The stationary mode expansion reflects the no energy loss assumption. This is not the case with a black hole. Since the black hole is made up of spacetime, there is always energy loss in terms of gravitational waves in spacetime. Thus it is impossible to expand the field in terms of quasinormal modes. Studies of quasinormal modes started with the various perturbation studies of black holes[59]. The perturbative study becomes important because according to Einstein's theory of relativity, a collapsing star may or may not form a stable system. Thus, in order to prove that the black hole is a stable system, studies on quasinormal modes are important. A black hole can be perturbed in a number of ways such as the scattering of particles by the black hole[66], test particle falling into or passing closely by the black holes[60–62] or slightly non-spherical gravitational collapse in forming black holes [63–65]. Usually the black holes are perturbed by scalar, electromagnetic, Dirac and gravitational fields. In each case, the response of black holes has three stages: the first one is an initial wave burst coming directly from the source and depends on the initial wave form of the original wave field. The second part involves damped oscillations called quasinormal frequencies which are independent of the initial parameters of the wave but depending on the background black hole space time. Because of radiation damping, the normal frequencies are complex. The last stage is the power law tail which arises because of back scattering of long range gravitational field. The black hole perturbation study started in 1950s by Regge and Wheeler[72]. Since the Einstein equations are highly nonlinear in metric tensor it is difficult to find out the perturbations in the general form and hence Regge and Wheeler studied perturbations only at the linearized level. Vishveshwara[66] studied the quasinormal modes for the first time and coined the word quasinormal modes, and later Chandrasekhar and Detweller[67] studied the quasinormal modes of Schwarzschild black hole. Since then a number of papers have appeared in literature studying quasinormal modes of different black holes and for different types of perturbations. The main study tool for quasinormal modes, namely the third order WKB approximation method[68–70] was developed initially by Schutz and Will and is finally modified by Sai Iyer and Will and then a sixth order WKB approximation method[71] is developed.

1.4.1 Basic equations for quasinormal modes

The different classical wave field perturbations lead to the same form of final equation for quasinormal modes. It is like a Schrodinger equation with the wave function Ψ as the variable[72]:

$$\frac{d^2\Psi}{dr_*^2} + \left(\omega^2 - V\right)\Psi = 0, \tag{1.5}$$

where ω is the black hole ringing frequency and V is the effective potential of the Schwarzschild black hole. Here we use the tortoise coordinate r_* which can be defined as $r_* = r + 2Mln\left(\frac{r}{2M} - 1\right)$. Using tortoise coordinate one can push the event horizon at $2M$ to $-\infty$. i.e.,

$r \to \infty; r_* \to \infty$

$r \to 2M; r_* \to -\infty.$

The form of the potential depends on what kind of wave field is perturbing the black hole. For massless integer spin field the effective potential can be written as[73],

$$V = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \sigma\frac{2M}{r^3}\right), \tag{1.6}$$

where σ is given by,

$$\sigma = \begin{cases} +1 & scalar\ field \\ \\ 0 & electromagnetic\ field \\ \\ -3 & odd\ gravitational\ perturbation \end{cases} \tag{1.7}$$

In the case of massive field, the square of the mass will appear in Eq.(1.6). The shape of the potential barrier is similar to potential barrier of scattering in quantum mechanics.

1.4.2 Importance of studying quasinormal modes

Since the black hole can be detected using gravitational waves, the quasi normal modes are imprinted on the gravitational waves. Thus quasinormal modes give a direct method to detect black holes in our universe. The formation of black holes and collision of the black holes etc are the events that large amount of energy is radiated as gravitational waves. Using the gravitational wave detectors which may start working in near future we may be able to detect the quasinormal modes of both solar mass black holes and super massive black holes(mass of order $10^6 M_\odot$). Using the detected quasinormal modes we can get black hole parameters, mass, electric charge and angular momentum. Even though the black holes are electromagnetically silent, their activities (such as collision of black holes and black hole merges etc.) can be detected through gravitational waves since all the above processes will release a large amount of gravitational energy into gravitational waves. The black holes are the testing fields of quantum gravity since they are the centers of very strong gravity. In the laboratories on earth it is impossible to study the gravitational effects on a microscopic scale and in particle physics the effects of gravity is neglected usually. But in black holes and in neutron stars there exist strong gravity. Hence analyzing the quasinormal spectrum we get the information about how the gravity affects on a macroscopic scale. Thus black holes play the same role in developing quantum gravity as the role played by hydrogen atom in developing the quantum theory.

1.5 Gravitational lensing

Gravitational lensing is also one of the important success of general theory of relativity. The theory of gravitational lensing rests on the deflection of light in the presence of a strong gravitational field. This is a direct consequence of principle of equivalence. Since the light travels in geodesics, when a strong gravity changes the geometry from Euclidean to non Euclidean then the geodesics change from normal straight lines to curved paths thus the light rays follow curved paths, i.e., light bends in the presence of a strong gravitational field. In gravitational lensing there are three important factors, the first one is the source which may be a distant galaxy or a Quasar, the second one is the lens. According to the nature of the lens the gravitational lensing is termed as micro lensing or macro lensing. In microlensing the lens will be a black hole or a neutron star but in macro lensing it will be a galaxy or a quasar. There are a number of dense galactic lenses in universe. The third factor is the observer.

Orest Chwolson[74] was the first who wrote about the gravitational lensing. But it was Einstein[75] who published a famous paper on gravitational lensing in 1936. The gravitational lensing was confirmed experimentally in 1979 by the observation of "Twin QSO" SBS 0957+561. Unlike the optical lenses which have maximum bending occurs near the aperture, in gravitational lensing the maximum bending occurs closest to the gravitational lens and minimum bending occurs farthest from the center of the gravitational lens. Thus a gravitational lens has a focal line in place of a single focal point as in the case of optical lenses. Thus, there occurs a ring around the massive objects(i.e. the lens) other than forming a point image of the source (i.e. the light source) if the source, gravitational lens and the observer are on a straight line. If the observer, source and lens are not on a straight line we get an arc. The ring obtained thus is called the Einstein ring. The gravitational lens are some times called the gravitational mirage.

1.6 Thermodynamics and Spectroscopy of black holes

1.6.1 Thermodynamics

In 1973 Bardeen, Carter and Hawking published the four laws of black hole mechanics, which may be regarded as the starting point of black hole thermodynamics, together with a conceptual work of Bekenstein on second law of thermodynamics and black holes. Hawking later introduced the area theorem of black hole, i.e., the area of the black hole never decreases in any process involving black holes, which is analogous to the thermodynamic law that entropy never decreases in any thermodynamic process. Bekenstein later introduced that the area of the event horizon is a measure of black hole entropy. The four laws of black hole mechanics can be written as follows: [54]

The zeroth law : The surface gravity κ of a stationary black hole is constant over the event horizon of stationary black holes.

If we take the black hole surface gravity as a measure of temperature, then the zeroth law of black hole is analogous to the zeroth law of thermodynamics. The zeroth law of thermodynamics may be stated in the following form:

If two systems are both in thermal equilibrium with a third then they are in thermal equilibrium with each other, which implies that that temperature is constant throughout the body in thermal equilibrium.

The first law: The first law of black hole mechanics is expressed in terms mass M, the area A, the angular momentum J and the charge Q as,

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ, \tag{1.8}$$

where κ is the surface gravity, Ω is the angular velocity and Φ, is the electromagnetic potential. Assuming κ is the analogue of temperature and area is the analogue of entropy then the first law of black hole mechanics is analogous to the first law of thermodynamics, which states that if T is the temperature and S is the entropy, J is the angular momentum and Q is the charge then,

$$dE = TdS + \Omega dJ + \Phi dQ, \tag{1.9}$$

where Ω and Φ have the same meaning as above. Hence we take M as the analogue of E of thermodynamics.

The second law: The area A of the event horizon of a black hole does not decrease with time i.e., $\delta A \geq 0$ which is analogous to the second law of thermodynamics which states that in any process entropy never decreases. If binary black holes merge the final area of the event horizon A_3 of the merged black hole should be greater than the sum of the areas of the two black holes A_1 and A_2, i.e., $A_3 > A_1 + A_2$.

The third law: By any finite sequence of operations it is impossible by any procedure, no matter how idealized, to reduce the surface gravity κ to zero. If we attribute κ as temperature then this is a restatement of the third law of thermodynamics which states that by any finite number of processes the temperature cannot be reduced to zero.

Since the four laws of black hole mechanics have striking similarities with the four laws of thermodynamics, Bekenstein argued that the black hole horizon area is a measure of black hole entropy[55, 56]. (like entropy, the area theorem restricts the area of the event horizon never to decrease). Thus assuming area as the entropy, surface gravity as the temperature we can define the thermodynamic quantities of black hole as,

$$T = \frac{\partial M}{\partial S}, \tag{1.10}$$

where T is the temperature of the black hole, M is the mass of black hole and S is the entropy. The entropy S can be calculated as,

$$S = \frac{A_h}{4l_p^2}, \tag{1.11}$$

where A_h is the area of the horizon and l_p is the Planck length. Finally the heat capacity can be calculated using the relation,

$$C = T\left(\frac{\partial S}{\partial T}\right), \tag{1.12}$$

where C is the constant volume heat capacity.

1.6.2 Hawking radiation

If black hole has a non zero entropy, then it should have a non-zero temperature. If it has a nonzero temperature it must have an emission spectrum and such a situation has been proved to exist by Hawking[18] and the black hole temperature is known as Hawking temperature and the emitted radiation is known as Hawking radiation. The calculation of Hawking temperature is made using quantum field theory in curved space time but it can be simply explained using tunneling mechanism. Quantum field theory shows that in free space always there is pair creation and annihilation of particles. It will not violate the energy conservation because of the time-energy uncertainty principle. Let us consider this process at the event horizon then one of the particles may fall into the black hole and the other may escape to infinity. An observer at infinity finds that this radiation is coming from the black hole. The particle that goes into the black hole always has negative energy. Thus as the radiation process proceeds the mass of the black hole decreases. The area of the event horizon of black hole also decreases. One may think that this may violate the Hawking area theorem. But since the area represent entropy, the total entropy including both black hole and outside it always remaining constant or increases and this is called the generalized second law of black hole thermodynamics. Hawking also showed that the particles that come out of the black hole have a spectrum characteristic of black body with a temperature equal to[19],

$$T_H = \frac{\hbar}{8\pi kM}, \tag{1.13}$$

where M is the mass of the black hole. Most black holes are thermodynamically unstable because as they radiate, temperature increases and as the temperature increases they radiate more particles also and they may or may not be in thermal equilibrium with the surroundings. They absorb the cosmic microwave background radiation if its temperature permits.

1.6.3 Spectroscopy of black hole entropy/area

It was Bekenstein who proposed that black hole area is related to entropy[55]. In his further studies he showed that the black hole area is quantized and thus entropy is also quantized[56]. He obtained a value $8\pi l_p^2$ (where l_p is the Planck length) as the unit of quantum of area. Later Hod proposed[57] that the area quantum is related to the the imaginary part of asymptotic quasinormal modes and obtained a value $4ln3l_p^2$. Maggiore[58] proposed that the modulus value of complex frequency must be taken and he got the same value as obtained by Bekenstein. The studies of Hod and Maggiore showed that the area spectrum of black hole in general is related to the asymptotic value of quasinormal modes of the black hole. Using adiabatic invariant method we can calculate the area spectrum of black hole. The adiabatic invariant integral is quantized using Bohr-Sommerfeld quantization rule. The Bohr-Sommerfeld quantization rule is given by,

$$\oint pdq = n\hbar. \tag{1.14}$$

According to Bekenstein for a harmonic oscillator with slowly time varying frequency ω, the energy E and ω form an adiabatic invariant I and is given by,

$$I = \frac{2\pi E}{\omega}. \tag{1.15}$$

In the case of black holes we use the black hole mass M in place of energy E. ω is directly related to κ_r, the surface gravity of the event horizon of the black hole. Thus for a particle tunneling the black hole horizon, we can use the adiabatic invariant integral as,

$$I = \int \frac{2\pi dM}{\kappa_{r_+}}, \tag{1.16}$$

Using Bohr-Sommerfeld quantization rule,

$$I = \int \frac{2\pi dM}{\kappa_{r_+}} = n\hbar, \tag{1.17}$$

where κ_{r_+} is the surface gravity at the horizon r_+. We write M and κ in terms of horizon radius r_+, and we finally get the area spectrum in terms of the Planck constant $\hbar$.

1.7 Scattering of waves by $f(R)$ black holes

It is highly motivating to study the scattering of waves by black holes because it is possible to calculate a number of black hole parameters and to study the existence of complicated diffraction effects at horizon[43]. As an example, we discuss massless scalar field near a black hole in f(R) theory. There are a number of f(R) models available in the literature[44, 45] and we pick up an f(R) model which gives a static spherically symmetric and asymptotically flat black hole solution. Such an f(R) is given by $f(R) = \sqrt{R + 6C_2}$ [45]. This choice of f(R) gives a black hole like solution with C_2 in general related to the cosmological constant. Suitably defining the co-efficients we can obtain a black hole solution in f(R) gravity with cosmological constant zero, that is, asymptotically flat space time

Even though we are not aware of the existence of a massless scalar field, for simplicity we consider black hole scattering of massless scalar field. The evolution of scalar field is given by the Klein-Gordon equation in the curved back ground i.e.,

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\Phi = 0. \tag{1.18}$$

We take the wave function in the following form since the metric is spherically symmetric,

$$\Phi_{lm} = \frac{\Psi_l(r,t)}{r}Y_{lm}(\theta,\phi). \tag{1.19}$$

Thus the radial part of the Klein-Gordon equation becomes,

$$\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - V_l(r)\right)\Psi_l(r,t) = 0, \tag{1.20}$$

where r_* is the tortoise coordinate and is defined as,

$$\frac{dr_*}{dr} = \left(1 - \frac{2\alpha m}{r^2}\right)^{-1}. \tag{1.21}$$

The absorption cross section is a function of scattering matrix element S_l and is given by[53],

$$\sigma_{abs} = \frac{\pi}{\omega^2}\sum_{l=0}^{\infty}(2l+1)\left[1 - S_l^2\right], \tag{1.22}$$

and the scattering matrix element S_l is given by,

$$S_l = (-1)^{l+1}\frac{A_l}{B_l}exp(2i\delta_l), \tag{1.23}$$

where A_l is the amplitude of the incident wave and B_l is the amplitude of the reflected wave and δ_l is the phase shift of the scattered wave. The absorption cross section of different black holes can be obtained using this method. We can extend this method for massive fields also. It is shown that the absorption cross section is inversely related to the Hawking temperature. This is conceptually true as the black hole has large event horizon the temperature will be small and large event horizon radius indicates the large absorption cross section. For charged black holes and de Sitter black holes it is also possible to study the absorption cross section with charge and cosmological constant.

1.8 Regular black holes

The regular black holes are objects that contain event horizon as normal black holes but have no central singularity. It is one of the drawbacks of Einstein's theory that all solutions contain singularities. At singularity the General Theory of Relativity breaks down. Thus a complete theoretical analysis of black hole becomes impossible. Contrary to this, regular black holes have no singularity at the center thus the different regions of regular black holes can be studied using General Theory of Relativity. Like normal black holes regular black holes are also solutions of Einstein field equation. The first regular black hole solution was found out by Bardeen[76]. But the physical source associated with this solution was clarified later by Ayon-Beato and Garcia[77] who interpreted it as a gravitational field of a nonlinear magnetic monopole of self gravitating magnetic field. In the frame work of Einstein's General Theory of Relativity, it is possible to find out singularity free solutions coupled to the nonlinear electrodynamics which obeys Maxwell's equations in the weak field limit and geometric behavior approaches to the Reissner-Nordstrom geometry. Later Hayward[78] proposed a black hole with regular center. The Hayward black hole for a distant observer is similar to the Schwarzschild black hole with horizon. The central singularity is replaced by a regular de Sitter space time. Even though the regular black hole has no central singularity but has a horizon and it has almost all properties of black holes such quasinormal modes, area-entropy relation etc. Regular black hole with nonlinear electric source is obtained in a recent paper[79]

1.9 Outline of the thesis

In this thesis we study the scattering and thermodynamics of different black holes in a modified gravity called the f(R) theory and Einstein's General Theory of Relativity. In f(R) theory we have studied a particular type of black hole (the extended black hole) and obtain the scattering wave functions in different regions and find out the absorption cross section. Scalar and electromagnetic quasinormal modes are also studied. In Einstein's theory, a 2+1 black hole called MSW black hole is considered and its thermodynamics, spectroscopy and Dirac quasinormal modes are studied finally the thermodynamics and spectroscopy of a regular black hole is studied. In **Chapter 1**, we give a general introduction of the thesis.

In **Chapter 2**, we study the scattering of extended black hole in f(R) gravity. We find out the scattered wave function in different regions in the vicinity of the horizon, away and far away from the horizon. The Hawking temperature of this black hole is calculated using tunneling method developed by Parikh and Wilczek. By evaluating the amplitude of the incident wave and the scattered wave we calculate the absorption cross section. It is found that Hawking temperature is inversely related to the absorption cross section

Chapter 3 contains studies on the scalar and electromagnetic perturbations of the extended black hole in f(R) theory. The electromagnetic quasinormal modes are studied and the study shows that the imaginary part of the quasinormal mode increases with mode number and shows exponential damping, thus the black hole is stable against the perturbation. Both massive and massless scalar fields show the same property. In all these studies third order WKB approximation method is used. Thermodynamics and spectroscopy of the extended black hole are studied. The various thermodynamic quantities are evaluated and the area spectrum is studied using adiabatic invariant integral method. The spectrum is discrete and equispaced.

Chapter 4 deals with the study of thermodynamics, spectroscopy and Dirac Quasinormal modes of $2 + 1$ dimensional MSW black hole The different thermodynamic quantities are found out and are plotted. Area spectrum is also found out using adiabatic invariant integral method. The Dirac quasinormal modes are also evaluated and found that this black hole is stable against the Dirac perturbation.

In **Chapter 5**, we study the thermodynamics and spectroscopy of regular Hayward black hole. This black hole is known as regular because of this type of black holes have regular center and have event horizon. The different thermodynamic quantities are found out and plotted and studied their behavior. The area spectrum of the regular black hole is also studied. The overall conclusion and future perspectives are given in **Chapter 6**.

2

Scattering of scalar field by an extended black hole in $f(R)$ gravity

2.1 Introduction

The end of the 20th century met with new problems in gravitational physics. The 1998 discovery that universe is expanding with acceleration showed that Einstein's general relativity should be modified in order to explain the cosmic acceleration. This led to extended theories of gravity. f(R) theory is one of the many kinds of extended theories of gravity. It is a direct modification of Einstein's theory. Since it is a generalization of Einstein's theory, all solutions of General Theory of Relativity are solutions of f(R) gravity also, but there are other solutions in this theory. Here we study one such solution which represents a black hole. We select here an f(R) which gives a valid black hole solution. This solution depends up on the mass of the central body and a length parameter. This solution evolves as the inverse second power of r, where r is the radial distance from the central object.

The recent advances in cosmology and particle physics led physicists to think of more accurate theories of gravity. In normal energy and weak field limit, the General Theory of Relativity is experimentally proved to be correct. But in strong gravity field such as in black holes and in the early universe, the General Theory of Relativity breaks down. In General Theory of Relativity we assume minimal coupling between gravity and matter. It may not be the case always and we can also assume non-minimal coupling between gravity and matter field. It is also impossible to explain the recently observed acceleration of the universe without invoking the hypothetical dark energy in General Theory of Relativity. Galaxy rotation curve is the other situation where General Theory of Relativity fails. In order to explain the rotation curve of many galaxies, mainly spiral galaxies, we may require the presence of the hypothetical dark matter. Both dark energy and dark matter have no observational evidences till now. These difficulties of modern gravitational physics may be solved using extended theories of gravity. These are the modifications of Einstein's general theory of relativity.

f(R) theory is one of the extended theories of gravity. It is a direct modification of Einstein's theory. f(R) theory is equivalent to one scalar field coupled to the Einstein gravity[29]. Such scalar field can explain the observed cosmic acceleration as well as inflation. The inflation is required to explain the the horizon, monopole and flatness problems in cosmology. There exist a number of works in literature discussing the cosmological aspects as well as black holes in f(R) gravity. There are different black hole solutions in f(R) field equations. Here we discuss a static spherically symmetric solution and the singularity of this particular solution. Since it is a black hole solution in f(R) gravity we call it as an extended black hole.

Before 1970s it is believed that black holes are objects which are truly black. No information can escape from black holes in any form. Applying the quantum field theory in curved space time to the black hole, Hawking[18] showed that the black holes are not truly black. Black hole possesses temperature and entropy and the black hole emits radiation like any other thermal objects. This temperature of the black hole is now known as Hawking temperature. It leads to particle production by black holes. These semi-classical studies on black holes which are strong centers of gravity system may lead to a path to quantize gravity. We may also get valid information about black hole by studying the black hole scattering of different quantum fields by a black hole. The studies on black hole scattering become an important field of study after Hawking showed that black hole could emit, scatter and absorb quantum radiation. According to Hawking, the evaporation rate is related to the total absorption cross section. Scattering of a number of black holes are presented in the book by Futterman et al.[43]. The scattering theory mainly rests on reflection from the horizon. Kuchiev studied the reflection of waves from the horizon in detail[80]. The reflection from the horizon is mainly a property of the event horizon. This means that the ingoing wave (or ingoing particle wave packet) has a finite probability to get reflected back to the outer space other than going into the black hole. The absorption cross section is one fourth area of the event horizon in the infrared limit.

The Penrose process is also a black hole scattering process[81]. Using this process it is possible to extract energy from a rotating black hole. It is interesting to study how a black hole scatters in an extended black hole spacetime. In this chapter we study the scattering of massive scalar field by an extended black hole in f(R) gravity.

2.2 Theory of black hole scattering

We know that the equations of motion can be derived from suitable actions. This is also true in the case of General Theory of Relativity and all Extended Theories of Gravity. In Einstein's General Theory of Relativity the field equations are derivable from an action principle. Comparing with actions in classical mechanics, the Lagrangian density in general theory of relativity is represented by the Ricci scalar R. The field equation of f(R) theory can also be derived from an action where, as the name indicated, the Lagrangian density is a general function of R, i.e., $f(R)$. The action for the gravity in general f(R) theory can be written as,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R), \tag{2.1}$$

where g is the determinant of the metric and f is a general function of the Ricci scalar R. κ is a proportionality constant to correct the dimensions. Using this action we can find out a number of solutions for f(R) theory with different symmetry properties. But for a general study we use spherically symmetric solution. We start with a static spherically symmetric solution of the form [45],

$$ds^2 = -e^{2\beta(r)} B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2. \tag{2.2}$$

Following [45], we assume that β is constant, and we take a function of f(R) in such a way that we get a valid black hole solution. Then the scalar curvature becomes,

$$R = -\frac{d^2B(r)}{dr^2} - \frac{4}{r}\frac{dB(r)}{dr} - 2\frac{B(r)}{r^2} + \frac{2}{r^2}. \tag{2.3}$$

We define a Lagrangian L as,

$$L \equiv L(\beta,\ \frac{d\beta}{dr}, B, \frac{dB}{dr}, R, \frac{dR}{dr}). \tag{2.4}$$

Varying Lagrangian with respect to β we get,

$$\frac{Rf'(R)-f(R)}{f'(R)}-2\frac{(1-B(r)-r(\frac{dB(r)}{dr}))}{r^2}+$$
$$\frac{2B(r)f''(R)}{f'(R)}\left[\frac{d^2R}{dr^2}+\left(\frac{2}{r}+\frac{dB(r)}{2B(r)dr}\right)\frac{dR}{dr}+\frac{f'''(R)}{f''(R)}\left(\frac{dR}{dr}\right)^2\right]=0 \quad (2.5)$$

The variation with respect to $B(r)$ leads to the equation of motion as,

$$\frac{-d\beta(r)}{dr}\left(\frac{2}{r}+\frac{f''(R)dR}{f'(R)dr}\right)+\frac{f''(R)d^2R}{f'(R)dr^2}+\frac{f'''(R)}{f'(R)}\left(\frac{dR}{dr}\right)^2=0, \quad (2.6)$$

which reduces to,

$$f'''\left(\frac{dR}{dr}\right)^2+f''\left(\frac{d^2R}{dr^2}\right)=0. \quad (2.7)$$

since β is assumed to be a constant. The above equation can be rewritten as,

$$\frac{d^2f'(R)}{dr^2}=0, \quad (2.8)$$

solution of which is given by,

$$f'(R)=ar+b, \quad (2.9)$$

where a and b are integration constants. A specific f(R) solution can be obtained when $b=0$ and then $B(r)$ takes the form[45],

$$B(r)=\left(1-\frac{C_1}{r^2}+C_2r^2\right). \quad (2.10)$$

Following[45] we choose f(R) as $f(R)=a\sqrt{R+6C_2}$. Deriving first the field equation for this f(R) and then substituting the metric ansatz, we get a solution of the form,

$$ds^2=-B\left(r\right)dt^2+\frac{dr^2}{B\left(r\right)}+r^2d\Omega^2. \quad (2.11)$$

Comparing with the Schwarzschild-de Sitter solution, C_2 term in the solution represents a cosmological constant term. For our study we assume a static spherically symmetric solution and which is asymptotically flat, thus we set $C_2 = 0$ and $C_1 = 2\alpha m$, to get,

$$ds^2 = -(1 - \frac{2m\alpha}{r^2})dt^2 + \frac{dr^2}{(1 - \frac{2m\alpha}{r^2})} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2, \quad (2.12)$$

where all the symbols except α have the usual meaning. α is a constant parameter having the dimension of length. The metric has singularities at both $r = \sqrt{2\alpha m}$ and $r = 0$. Since α is a length parameter, Eq. (2.12) forms a black hole solution with horizon at $r = \sqrt{2\alpha m}$. In order to prove that there exists a null surface at $r = \sqrt{2\alpha m}$, we take this metric to the Eddington-Finkelstein coordinates. We will now invert the metric into tortoise co-ordinate as,

$$ct = \pm r \pm \sqrt{2\alpha m}\ arctanh[\frac{r}{\sqrt{2\alpha m}}] + constant, \quad (2.13)$$

where the minus sign is for the ingoing photon and positive sign is for the outgoing photon. Proceeding as in GTR, we first obtain a new coordinate p with p as,

$$ct = -r - \sqrt{2\alpha m}\ arctanh[\frac{r}{\sqrt{2\alpha m}}] + p. \quad (2.14)$$

With the null coordinate p, the metric in Eq.(2.5) can be simplified and can be written as,

$$ds^2 = \left(1 - \frac{2\alpha m}{r^2}\right) dp^2 - 2dpdr - r^2 d\Omega^2. \quad (2.15)$$

Now defining a time like coordinate t' as $ct' = p - r$ so that we get the Eddington-Finkelstein coordinate system as,

$$ds^2 = c^2 \left(1 - \frac{2\alpha m}{r^2}\right) dt'^2 - \frac{4m\alpha c}{r^2} dt' dr - \left(1 + \frac{2\alpha m}{r^2}\right) dr^2 - r^2 d\Omega^2 \quad (2.16)$$

The Eddington-Finkelstein coordinate metric for the extended black hole has the same form as Schwarzschild metric and thus this metric can have a regular null surface at $r = \sqrt{2\alpha m}$ showing that the above metric has a horizon at $r = \sqrt{2\alpha m}$ and this forms a black hole. We are considering here the scattering of scalar particles by the extended black hole. We now place the black hole immersed in a massive scalar field, the field equation in this back ground can be described by the Klein-Gordon equation[82, 83],

$$(\Box + \mu^2)\Phi = 0. \tag{2.17}$$

This is the general form of Klein-Gordon equation representing the massive scalar field of mass μ and Φ is the general wave function depending on both space and time co-ordinates. In curved space-time this equation can be written in the following form,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) + \mu^2\Phi) = 0. \tag{2.18}$$

Since the black hole is spherically symmetric we exploit the spherical symmetry and by writing this equation in spherical polar coordinates, we find[84],

$$\frac{1}{r^2(sin\theta)}[(\partial_t(r^2(sin\theta))\frac{1}{(1-\frac{2m\alpha}{r^2})}\partial_t) - (\partial_r r^2 sin\theta$$
$$(1-\frac{2m\alpha}{r^2})\partial_r) - (\partial_\theta r^2 sin\theta\frac{1}{r^2}\partial_\theta) - (\frac{1}{sin\theta}\partial_\phi{}^2)]\Phi = 0. \tag{2.19}$$

The above equation contains temporal, spatial and angular parts and for the present study we need only require temporal and radial parts of this equation. In order to separate this equation into radial and the angular parts, as we are considering a spherically symmetric system, we write Φ as,

$$\Phi(r,t) = exp(-i\epsilon t)\phi_l(r)Y_{lm}(\theta,\phi), \tag{2.20}$$

where ϵ is the energy, l and m are the angular momentum and its projection respectively. We consider only the radial part and it takes the following form

$$[\frac{1}{r^2}\partial_r(r^2-2\alpha m)\partial_r+\frac{r^2\epsilon^2}{(r+\sqrt{2\alpha m})(r-\sqrt{2\alpha m})}-\frac{l(l+1)}{r^2}+\mu^2]\phi_l(r)=0. \tag{2.21}$$

It is a second order differential equation in the variable r. The temporal part gives the square of the energy of the field ϵ. The radial part also consists of angular momentum l and mass μ of the field. We now focus our study to three different regions of black hole spacetime namely, the vicinity of the horizon and away from the horizon and far away from the horizon, We study these three situations below.

2.3 Solution of the wave equation in the vicinity of the horizon

We use WKB approximation for solving the radial scattering equation. Here we assume that reflection of the wave from the horizon takes place, that is, a particle entering into or near the horizon gets reflected back and then we assume[82] a solution of the form,

$$\phi_l(r)=exp(-i\int k(r)dr), \tag{2.22}$$

where k is the wave vector of the field. Using Eq.2.15 in Eq.2.14 we get,

$$[\frac{\epsilon^2}{1-\frac{2\alpha m}{r^2}}-\frac{l(l+1)}{r^2}+\mu^2]+[-k^2(1-\frac{2\alpha m}{r^2})]=0. \tag{2.23}$$

On rearranging we find,

$$k^2=(1-\frac{2\alpha m}{r^2})^{-2}[\epsilon^2-(\frac{l(l+1)}{r^2}+\mu^2)(1-\frac{2\alpha m}{r^2})]. \tag{2.24}$$

Since we are considering a situation where r approaches r_h and hence we need only consider the ϵ^2 term in the square bracket. Then solving for k we get,

$$k = (1 - \frac{2\alpha m}{r^2})^{-1}\epsilon. \tag{2.25}$$

and,

$$(1 - \frac{2m\alpha}{r^2}) = \frac{1}{r^2}(r^2 - 2\alpha m) = \frac{1}{r^2}(r + \sqrt{2\alpha m})(r - \sqrt{2\alpha m}). \tag{2.26}$$

When $r \to r_h$, k can be written in a compact form as,

$$k = \frac{\xi}{(r - r_h)}, \tag{2.27}$$

where ξ is,

$$\xi = \frac{r_h^2 \epsilon}{r + r_h}. \tag{2.28}$$

Hence from our above assumption, the wave function as $r \to r_h$ becomes,

$$\phi_l = e^{i \int \frac{\xi}{r - r_h}}, \tag{2.29}$$

which can be integrated to the following form,

$$\phi_l = e^{i\xi ln(r - r_h)}. \tag{2.30}$$

Thus the wave function in the vicinity of the event horizon may be written as,

$$\phi_l = e^{\pm i\xi ln(r - r_h)}. \tag{2.31}$$

Now we will consider the scalar wave approaching the black hole horizon and using Eq.(2.24) the wave function in the vicinity of the black hole horizon can be written, assuming the field gets reflected at the black hole horizon as,

$$\phi_l = e^{-i\xi ln(r - r_h)} + |R| e^{+i\xi ln(r - r_h)}, \tag{2.32}$$

where R represents the reflection coefficient. If $R \neq 0$, there is a definite probability for the incident wave to get reflected at the horizon.

2.4 Solution of the wave equation in a region r greater than r_h.

In this section we consider a situation where the field is sufficiently away from the event horizon. We also assume a situation that the energy ϵ and mass μ of the field are very small and they can be neglected. For s-wave, Eq.(2.14) now takes the form,

$$\frac{1}{r^2}\partial_r((r^2-2\alpha m)\partial_r\phi_0)=0, \tag{2.33}$$

which can be simplified to,

$$(r^2-2\alpha m)\phi_0''=-2r\phi_0', \tag{2.34}$$

from which we get,

$$ln\phi_0'=-ln(r^2-2\alpha m)+lnC. \tag{2.35}$$

We can remove the logarithm on both sides, such that we get,

$$\phi_0'=\frac{C}{(r^2-2\alpha m)}, \tag{2.36}$$

the denominator of the above equation can be written as $(r-r_h)(r+r_h)$ and for sufficiently long distances from the horizon this can be simplified as $2r(r-r_h)$, hence

$$\phi(r)=\int\frac{C}{r(r-r_h)}dr+K, \tag{2.37}$$

and the wave function is obtained as,

$$\phi_0=Cln(\frac{r-r_h}{r})+k, \tag{2.38}$$

where $K and\ k$ are the integration constants. Comparing the solutions for regions 1 and 2, region 1 is in the vicinity of horizon and region

2 is slightly away from the horizon, we have in region 1 the wave function as,

$$\phi_l = e^{\pm i\xi ln(r-r_h)}, \tag{2.39}$$

which contains both incoming and outgoing waves. For s-wave, we can rewrite the solution for region 1, as,

$$\phi_0 = 1 - i\xi ln(r - r_h) + R(1 + i\xi ln(r - r_h)), \tag{2.40}$$

where R is the reflection coefficient[80]. This can be further simplified as,

$$\phi_0 = -i\xi ln(r - r_h)(1 - R) + (1 + R). \tag{2.41}$$

Thus in region 2 also, we can write the s wave, thus

$$\phi_0 = \alpha ln\frac{r - r_h}{r} + \beta, \tag{2.42}$$

where $\alpha = i\xi(1 - R)$ and $\beta = (1 + R)$.

In the next section we will study the behavior of scalar field in a region sufficiently far away from the horizon.

2.5 Solution of the wave equation far away from the horizon

We have the radial part of Klein-Gordon equation as,

$$\left(1 - \frac{2\alpha m}{r^2}\right)\phi''(r) + \frac{2}{r}\phi'(r) + (\epsilon^2 - \mu^2)\phi(r) = 0 \tag{2.43}$$

In this region, the radial part of the wave equation can be written as,

$$\phi_l'' + \frac{2}{r}\phi_l' + p^2\phi_l = 0, \tag{2.44}$$

with p is the linear momentum associated with scalar field and $p^2 = \epsilon^2 - \mu^2$.

Using Frobenius method we can solve the above equation. The solution is given by,

$$\phi_l = \frac{1}{r}(A_l e^{iz} + B_l e^{-iz}), \tag{2.45}$$

with $z = pr$. This can be again simplified as,

$$\phi_l = \frac{1}{r}(aF(r) + bG(r)), \tag{2.46}$$

where $F(r) = sin(pr)$ and $G(r) = cos(pr)$. At the boundary between Region 2 and Region 3, where $pr \ll 1$, we can expand both sine and cosine terms and need to take only first terms; then $F(r) \approx pr$ and $G(r) \approx 1$ and the s - wave solution is given by,

$$\phi_0 = ap + \frac{b}{r}, \tag{2.47}$$

and in Region 2, ϕ_0 is given by,

$$\phi_0 = -\alpha\frac{r_h}{r} + \beta \tag{2.48}$$

From region 1 we get the wave function finally as,

$$\phi_0 = i\xi(1 - |R|)\frac{r_h}{r} + (1 + |R|). \tag{2.49}$$

Comparing Eq.(2.47) and Eq.(2.49) we can obtain a and b as,

$$a = \frac{1+R}{p}, b = i\xi r_h(1 - R). \tag{2.50}$$

Now we will calculate the absorption coefficient, assuming the wave gets reflected at the horizon.

2.6 Absorption cross section

The absorption cross section can be evaluated using the above data. We now consider the solution of Eq.(2.32) given by Eq.(2.38). The scattering matrix element defined as the ratio of the coefficients of the incoming and outgoing waves[53], is given by,

$$S_l = (-1)^{l+1}\frac{A_l}{B_l}e^{2i\delta_l}, \tag{2.51}$$

where δ_l is the phase shift corresponding to angular momentum l.

We take the low energy limit so that $l = 0$. Comparing Eq.(2.45), Eq.(2.46), and we can obtain, for s - waves, $A_0 = \frac{a+ib}{2i}$ and $B_0 = \frac{-a+ib}{2i}$. Substituting the values of a and b given by Eq.(2.50), we get S_0 as,

$$S_0 = \frac{(1+R) - \xi p r_h(1-R)}{(1+R) + \xi p r_h(1-R)}. \tag{2.52}$$

Defining $\eta = \frac{1-R}{1+R}$, the equation of S_0 takes the form,

$$S_0 = \frac{1 - \xi p r_h \eta}{1 + \xi p r_h \eta}. \tag{2.53}$$

The absorption cross section σ_{abs}, is then given by,

$$\sigma_{abs} = \frac{\pi}{p^2}(1 - S_0^2). \tag{2.54}$$

Using the above equations and also using the relation $p = \epsilon v$ we obtain finally,

$$\sigma_{abs} = \frac{2\pi^2 \epsilon r_h^3}{v}. \tag{2.55}$$

Since r_h is inversely proportional to the Hawking temperature T_H, we can see that absorption cross section is inversely related to Hawking temperature.

2.7 Hawking temperature via tunneling

Now we will determine the Hawking temperature using tunneling mechanism[85]. This mechanism has been used by many authors[86–88] for determining the Hawking radiation of black holes in Einstein gravity. Even though the complete properties of Hawking radiation can be obtained using quantum field theory in curved space-time, the tunneling mechanism gives a simple understandable picture. According to this picture the radiation arises by a process similar to electron-positron pair creation in a constant electric field. Using tunneling picture of black hole radiation we can have a direct semi-classical derivation of black hole radiation. There are two different schemes in tunneling approach, first is the radial null geodesic method and the other is Hamilton-Jacobi method. Here we use the radial null geodesic method. The metric in Eq.(2.5) can be transformed in to Painleve[89] like coordinate system to remove the co-ordinate singularity at $r = \sqrt{2\alpha m}$ in the original extended metric as,

$$ds^2 = -\left(1 - \frac{2\alpha m}{r^2}\right) dt^2 + 2\sqrt{\frac{2\alpha m}{r^2}} dtdr + dr^2 + r^2 d\Omega^2. \tag{2.56}$$

The radial null geodesics are

$$\frac{dr}{dt} = \dot{r} = \pm 1 - \sqrt{\frac{2\alpha m}{r^2}}. \tag{2.57}$$

The typical wavelength of the radiation is of the order of the size of the black hole and hence when the outgoing wave is traced back towards the horizon its wavelength as measured by local observers, is blue shifted. Near the horizon, radial wave number approaches to infinity and we can use WKB approximation to study for the particle tunneling. We start with the action,

$$S = \int p(r) dr. \tag{2.58}$$

Using Hamilton's equation of motion $dp(r) = \frac{dH}{\dot{r}}$, we can write the imaginary part of the action as,

$$ImS = Im \int_{r_{in}}^{r_{out}} p_r dr = Im \int_{m}^{m-\omega} \int_{r_{in}}^{r^{out}} \frac{drdH}{\dot{r}}. \tag{2.59}$$

Using Eq.(2.58), the imaginary part of the action S can be written as,

$$ImS = Im\int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{drd\,(-\omega)}{\left(1-\sqrt{\frac{2\alpha(m-\omega)}{r^2}}\right)}. \tag{2.60}$$

where ω is the frequency of the out going particle. Eq.(2.59) can be integrated to obtain,

$$ImS = \frac{4\pi\sqrt{2\alpha}}{3}\,(m-\omega)^{\frac{3}{2}}\,. \tag{2.61}$$

Since m is much greater than ω we can apply the binomial expansion to get,

$$ImS = \frac{4\pi\sqrt{2\alpha m}\;m}{3}\left(1-\frac{3\omega}{2m}\right). \tag{2.62}$$

Now the semi-classical emission rate can be written as,

$$\Gamma \sim e^{-2ImS} \sim exp\left(\frac{4\pi\sqrt{2\alpha m}\;m}{3}\left(\frac{3\omega}{2m}\right)\right) \tag{2.63}$$

from which we get,

$$T_H = \frac{1}{2\pi\sqrt{2\alpha m}}. \tag{2.64}$$

There exist higher order corrections of ω due to the conservation of energy, but for the first order calculation they are neglected. It is also possible to find out the frequency dependent transmission coefficient or graybody factors in this tunneling scenario.

2.8 Conclusion

In this chapter we have studied the scattering properties of extended black holes in F(R) theory. We have obtained the scattered wave equation in regions near the horizon, away from horizon and far away from the horizon. We have calculated the wave function at different regions(i.e., in the vicinity of the horizon away from horizon and far away from horizon) and reflection coefficient is obtained and the absorption cross section is calculated. We have also calculated the Hawking temperature of the black hole via tunneling method.

3

Scalar, electromagnetic quasinormal modes, thermodynamics and spectroscopy of an extended black hole in $f(R)$ gravity

3.1 Introduction

Quasinormal modes are the characteristic ringing frequency of black holes. Black holes emit these waves of frequency when it is perturbed by an external field. The quasinormal modes are the characteristics of particular black hole spacetime and is independent of the initial form of the perturbing field. The quasinormal wave pattern is imprinted in the gravitational waves emitted by the black hole as the gravitational wave detectors start operating, the detection of the quasinormal mode frequencies may become a direct evidence for black holes. Actually the oscillations are complex, thus these oscillations are damped in most cases. These oscillation amplitudes are a measure of the black hole stability. If the amplitudes are damped exponentially then the black hole is stable against that particular perturbation and if the amplitude increases exponentially the black hole is unstable against the particular perturbation. Thus the study of quasinormal modes is a direct method to study the stability of black holes. The quasinormal mode perturbations have three different stages, the first is an initial wave burst coming directly from the source which may have any wave form. Second stage consists of quasinormal oscillations emitted by the black hole which is independent of initial wave form but only depends on the particular black hole spacetime, for a stable black hole the quasinormal modes are oscillations which damps exponentially and last stage is a power law tail originating from the back scattering of long range gravitational field.

The black hole quasinormal oscillations are different from oscillations of stars. In stars, the oscillations occur in the fluid which constitutes the star. In the case of black holes there is no fluid and only the horizon. The horizon is some what like a one sided membrane which is not the carrier of oscillations. It is the spacetime outside the horizon is the carrier of these oscillations. This is not surprising because in general relativity the spacetime is dynamic instead of playing a passive role for setting up a stage for interactions. These oscillations are not stationary as in other cases such as stars but exponentially damping, thus the name quasinormal modes.

The extended theories of gravity open a new door to study extended black holes. The properties of these new family of black holes form an area of study. The quasinormal modes in Gauss-Bonnet gravity are studied by Konoplya et.al[90, 91]. It is interesting to study how the quasinormal modes behavior in f(R) black holes. In all cases the main study tool is the WKB approximation method developed by Schutz, Will and Iyer[68–70]

In this chapter we study the quasinormal modes thermodynamics and spectroscopy of extended black holes.

It is widely believed that black hole has a characteristic temperature proportional to its surface gravity and an entropy related to the area of the horizon of black hole. In a nutshell the black hole is a thermal object. It was Bekenstein who introduced the concept of black hole entropy. He also stated that the black holes have quantized entropy. Since the black hole area is related to the black hole entropy area also get quantized. Bekenstein obtained a value $8\pi l_p^2$ where l_p is the Planck length. The quasinormal oscillations of black hole is an important property of black hole and there exists a connection between area quantization and quasinormal modes. Hod[57] proposed that the area spectrum of black hole is related to the imaginary part of the quasinormal frequencies and he obtained a value $4ln3l_p^2$. Later Maggiore[58] proposed that we must consider both real and imaginary parts of quasinormal frequencies in order get the minimum value of the area quantum and he obtained the same value obtained by Bekenstein. Using this method Kunstatter[46] found the area spectrum of higher dimensional black holes. The quasinormal modes, area spectrum, and black hole entropy are discussed in the light of canonical quantum gravity by Dreyer etal.[48]. The statistical interpretation of black hole entropy is discussed in Ropotenko etal.[49]

3.2 Static spherically symmetric solution in f(R) theory

Starting with the f(R) action as,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R), \tag{3.1}$$

where g is the determinant of the metric tensor and f is the general function of Ricci scalar R. We start with a static spherically symmetric solution[45]. We take $f(R) = a\sqrt{R + 6C_2}$ and this particular form of f(R) is taken as it gives a valid black hole solution. We take this f(R) as positive, where C_2 is an integration constant. This is chosen either as positive or zero and all terms in the radical are taken as positive. There exist other solutions of different f(R) theories. Static cylindrically symmetric interior solutions are found out by Sharif et.al[50]. Godel solution is studied by Santos[51] and thermodynamics of evolving Lorentzian wormholes at apparent horizon in f(R) theory of gravity is studied by Saiedi[52]. We write the static spherically symmetric solution as[45],

$$ds^2 = -e^{2\beta(r)}B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2. \tag{3.2}$$

With constant β, we get the solution as,

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \tag{3.3}$$

where $B(r)$ is given by,

$$B(r) = 1 - \frac{C_1}{r^2} + C_2 r^2. \tag{3.4}$$

Comparing with Schwarzschild-de Sitter solution, the C_2 term in the solution which is the coefficient of r^2 can represent the cosmological constant term. In the present study we choose $C_1 = 2\alpha m$ and $C_2 = 0$. We choose $C_2 = 0$ because we restrict our study to a space time which is asymptotically flat[44]. We select C_1 as $2\alpha m$ such that the metric should be positively related to the mass only and in such cases we get the correct Newtonian limit and α is chosen as a length parameter. Thus the extended metric in f(R) gravity is given by[45],

$$ds^2 = -\left(1 - \frac{2\alpha m}{r^2}\right) dt^2 + \left(1 - \frac{2\alpha m}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2, \tag{3.5}$$

where m is the black hole mass and α is a length parameter of the metric. The black hole mass is related to the metric linearly and thus we assume a linear mass term with one length parameter. This length parameter can be adjusted to obtain an effective rotation curve of galaxies and gravitational lensing. f(R) is chosen as above because only this form of f(R) will give a static spherically symmetric black hole solution. f(R) models with R^{-1}, R^2 etc. have been studied earlier[92, 93].

3.3 Evolution of quasinormal modes-Electromagnetic perturbation

We are studying the evolution of Maxwell field in this extended space time in f(R) gravity. The Maxwell's equation can be written as,

$$F^{\mu\nu}_{;\nu} = 0, \;\; F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \tag{3.6}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and A_μ is the vector potential and A_μ can be expressed as four dimensional vector spherical harmonics[94],

$$A_\mu(t,r,\theta,\phi) = \sum_{l,m} \left(\begin{bmatrix} 0 \\ 0 \\ \frac{a^{lm}}{sin\theta}\partial_\phi Y_{lm} \\ -a^{lm}(t,r) sin\theta \partial_\phi Y_{lm} \end{bmatrix} + \begin{bmatrix} f^{lm}(t,r) Y_{lm} \\ h^{lm}(t,r) Y_{lm} \\ k^{lm}(t,r)\partial_\theta Y_{lm} \\ k^{lm}(t,r)\partial_\phi Y_{lm} \end{bmatrix} \right), \tag{3.7}$$

where l and m are angular momentum quantum number and azimuthal quantum number respectively. The first column has a parity of $(-1)^{l+1}$ and the second column has $(-1)^l$. We define the tortoise coordinates as $\frac{dr_*}{dr} = \left(1 - \frac{2\alpha m}{r^2}\right)^{-1}$ such that after some mathematical

Table 3.1: Quasinormal modes-electromagnetic perturbation

l	n	Re(E)	Im(E)
2	0	1.142700	-0.343905
	1	0.992173	-1.082550
	2	0.767162	-1.877770
	3	0.475127	-2.699680
3	0	1.67599	-0.348490
	1	1.567820	-1.070310
	2	1.389310	-1.835120
	3	1.160460	-2.628530
	4	0.881184	-3.440780
4	0	2.19324	-0.350460
	1	2.109240	-1.066020
	2	1.961730	-1.812700
	3	1.769050	-2.586900
	4	1.536860	-3.380010
5	0	2.70389	-0.351472
	1	2.635290	-1.064070
	2	2.510170	-1.799570
	3	2.342660	-2.559400
	4	2.140600	-3.338360

steps, we get the Regge-Wheeler[72] equation as,

$$\frac{d^2}{dr_*^2}\Phi(r) + (E^2 - V)\Phi(r) = 0, \tag{3.8}$$

with the potential V given by,

$$V(r) = \left(1 - \frac{2\alpha m}{r^2}\right)\left(\frac{l(l+1)}{r^2}\right), \tag{3.9}$$

where $\Phi(r) = a^{lm}$ with parity $(-1)^{(l+1)}$ and $\Phi(r) = \frac{r^2}{l(l+1)}(-i\omega h^{lm} - \frac{df^{lm}}{dr})$ with parity $(-1)^l$. Using WKB approximation method given in section 1.6 we can evaluate quasinormal modes of electromagnetic

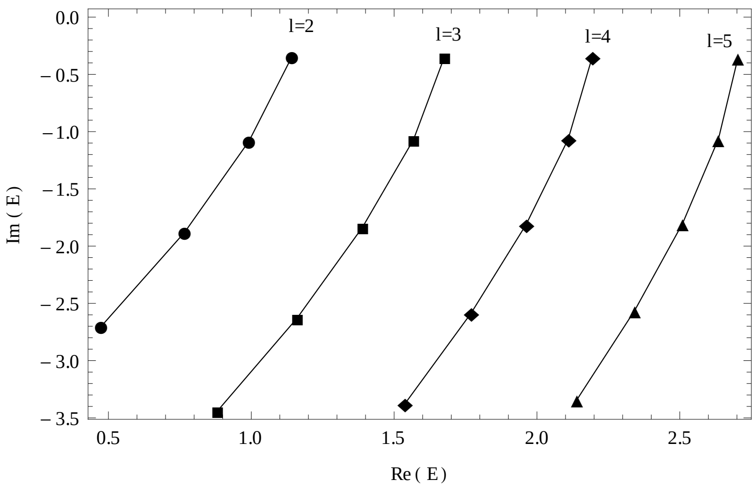

Figure 3.1: Quasinormal modes-electromagnetic perturbation

perturbation. The electromagnetic quasinormal modes are given in Table 3.1

3.4 Quasinormal modes - Scalar perturbation

In this section, we study the quasinormal modes of the extended black hole in f(R) gravity perturbed by scalar fields. The metric is given by Eq.(3.5). Since we are considering the scalar perturbation, we can use the Klein-Gordon equation. We consider here both massless and massive scalar fields. The Klein-Gordon equation for a massive scalar field is,

$$\Box\Psi - u^2\Psi = \frac{1}{\sqrt{-g}}\left(g^{\mu\nu}\sqrt{-g}\Psi_{,\mu}\right)_{,\nu} - u^2\Psi = 0, \tag{3.10}$$

where u is the mass of the scalar field. In order to separate the wave function into radial, temporal, and angular parts, we write Ψ as

$$\Psi = \Phi(r)Y_{lm}(\theta,\phi)e^{-i\omega t}. \tag{3.11}$$

Substituting Eq.(3.11) in Eq.(3.10) and after straight forward calculation, the radial part is given by,

$$\left[\left(1-\frac{2\alpha m}{r^2}\right)^{-1}\omega^2+\left(1-\frac{2\alpha m}{r^2}\right)\partial_r^2-\frac{l(l+1)}{r^2}+\frac{1}{r}\partial_r\left(1-\frac{2\alpha m}{r^2}\right)-u^2\right]\Phi=0. \tag{3.12}$$

Introducing the tortoise coordinate as $dr_* = \frac{dr}{1-\frac{2\alpha m}{r^2}}$, Eq. (3.12) can be written as,

$$\left(\frac{d^2}{dr_*^2}+E^2-V(r)\right)\Phi(r)=0, \tag{3.13}$$

where the potential V(r) is given by

$$V(r)=\left(1-\frac{2\alpha m}{r^2}\right)\left(\frac{l(l+1)}{r^2}+\frac{4\alpha m}{r^4}+u^2\right). \tag{3.14}$$

r_* ranges from $-\infty$ to$+\infty$

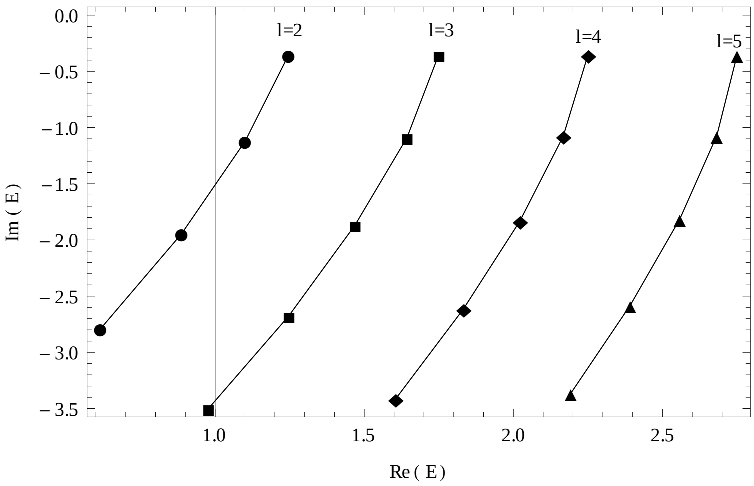

Figure 3.2: Quasinormal modes-scalar perturbation-massive case

Table 3.2: Quasinormal modes-Scalar perturbations

l	n	Re(E)	Im(E)	Re(E)	Im(E)
		$u = 0$	$u = 0$	$u = 0.1$	$u = 0.1$
2	0	1.24391	-0.358055	1.24565	-0.357272
	1	1.09979	-1.12409	1.10026	-1.12267
	2	0.887537	-1.946290	0.887124	-1.94536
	3	0.615554	-2.792610	0.614869	-2.79230
3	0	1.74795	-0.355789	1.749280	-0.355375
	1	1.64228	-1.09191	1.64305	-1.09095
	2	1.46848	-1.87071	1.46859	-1.86973
	3	1.24684	-2.67761	1.24654	-2.67691
	4	0.977252	-3.50226	0.976773	-3.50188
4	0	2.24908	-0.354872	2.25015	-0.354617
	1	2.166320	-1.07915	2.16710	-1.07849
	2	2.021180	-1.83435	2.02154	-1.83355
	3	1.832020	-2.61689	1.83203	-2.61616
	4	1.604590	-3.41799	1.60436	-3.41743
5	0	2.74951	-0.35442	2.75039	-0.354248
	1	2.68161	-1.072870	2.68233	-1.07240
	2	2.55785	-1.814110	2.55831	-1.81348
	3	2.39234	-2.579580	2.39252	-2.57893
	4	2.19296	-3.364040	2.19292	-3.36347

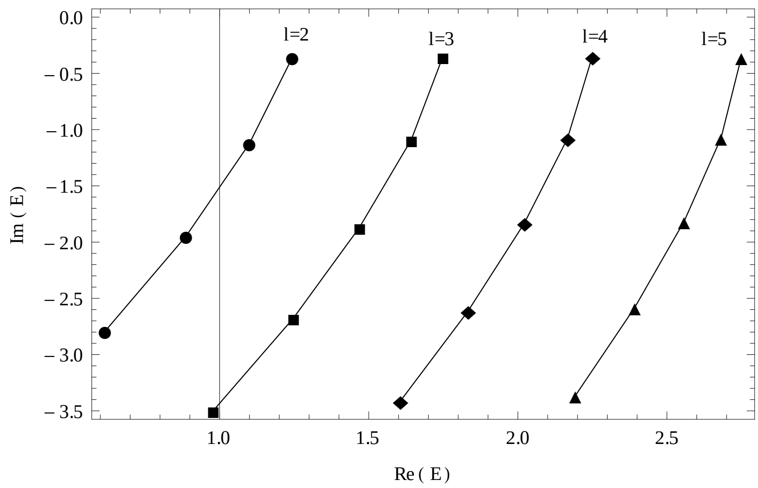

Figure 3.3: Quasinormal modes-scalar perturbation-massless case

3.5 WKB approximation method

Here we study the quasinormal modes using analytic method. In the original work, Chandrasekhar et al. used numerical techniques for their calculations. Using the third order WKB approximation method we can study the quasinormal modes. The WKB method originally developed by Schutz and Will[68] and finally modified by Iyer and Will[69, 70], Now it is more developed to sixth order by Konoplya[71]. We can obtain the frequency of the quasinormal mode in general as,

$$E^2 = [V_0 + (-2V_0'')^{\frac{1}{2}}\Lambda] - i(n+\frac{1}{2})(-2V_0'')^{\frac{1}{2}}(1+\Omega), \tag{3.15}$$

where,

$$\Lambda = \frac{1}{(-2V_0'')^{\frac{1}{2}}}\left[\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4}+a^2\right) - \frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2(7+60a^2)\right], \tag{3.16}$$

and,

$$\begin{aligned} \Omega &= \frac{1}{(-2V_0'')}[\frac{5}{6912}\left(\frac{V_0'''}{V_0''}\right)^4\left(77+188a^2\right) \\ &- \frac{1}{384}\left(\frac{(V_0''')^2(V_0^{(4)})}{(V_0'')^3}\right)\left(51+100a^2\right) \\ &+ \frac{1}{2304}\left(\frac{V_0^{(4)}}{V_0''}\right)^2(67+68a^2) \\ &+ \frac{1}{288}\left(\frac{V_0'''V_0^{(5)}}{(V_0'')^2}\right)(19+28a^2)+\frac{1}{288}\left(\frac{V_0^{(6)}}{V_0''}\right)(5+4a^2)], \end{aligned} \tag{3.17}$$

where, $a = n + \frac{1}{2}$, and,

$$V_0^{(n)} = \frac{d^n V}{dr_0^n}|_{r_*=r_{*max}}. \tag{3.18}$$

The potentials given in Eq.(3.9) and Eq.(3.14) corresponding to electromagnetic and scalar perturbations are substituted in the WKB formula (Eq.(3.15)) and obtained the corresponding complex frequencies. The quasinormal frequencies are tabulated in tables. In Table (3.1) we give the electromagnetic perturbed quasinormal modes and in Table (3.2) the quasinormal modes when the extended black hole is perturbed by the scalar fields.

We have plotted the data in Fig.(3.1), for electromagnetic field case and in Fig.(3.2) and Fig.(3.3) for massless and massive scalar fields. In Fig.(3.1), Fig.(3.2) and Fig.(3.3), the top dots are for $n = 0$. These figures we can see that the real part of quasinormal frequencies decreases with increasing mode number n for a given angular momentum quantum number l. Fig.(3.2) and Fig.(3.3) show scalar field perturbations for massless and massive fields respectively. The mode value of imaginary part of frequencies increases rapidly showing that the oscillation is damping and black hole is stable against electromagnetic and scalar perturbations. The absolute value of imaginary frequency increases with mode number showing damping.

3.6 Thermodynamics of extended black hole

We have the extended black hole metric as,

$$ds^2 = -\left(1 - \frac{2\alpha m}{r^2}\right) dt^2 + \left(1 - \frac{2\alpha m}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{3.19}$$

We substitute,

$$B(r) = \left(1 - \frac{2\alpha m}{r^2}\right). \tag{3.20}$$

In chapter 2 we have showed that this metric has a regular null surface at $r = \sqrt{2\alpha m}$. This forms a horizon. At horizon $B(r) = 0$ putting this into the above equation we get the horizon r_+ as,

$$r_+ = \sqrt{2\alpha m}. \tag{3.21}$$

Now it is possible to express the mass of the extended black hole in terms of horizon radius r_+ as,

$$m = \frac{r_+^2}{2\alpha}. \tag{3.22}$$

The entropy of black hole in f(R) gravity is different from the situations in General Theory of Relativity (in General Theory of Relativity it is one quarter of area of the horizon but here it is multiplied by the derivative of f(R)) and is given by[39],

$$S = \frac{f'(R)A}{4} = \pi a r_+^3. \tag{3.23}$$

In the above equation S is the entropy, A is the area and $f'(R)$ is the first derivative of $f(R)$ with respect to the Ricci scalar R and a is given by Eq.(2.9). We can define the entropy in terms of black hole mass as,

$$S = \pi a(2\alpha m)^{\frac{3}{2}}. \tag{3.24}$$

After obtaining the relation between mass and entropy we now turn

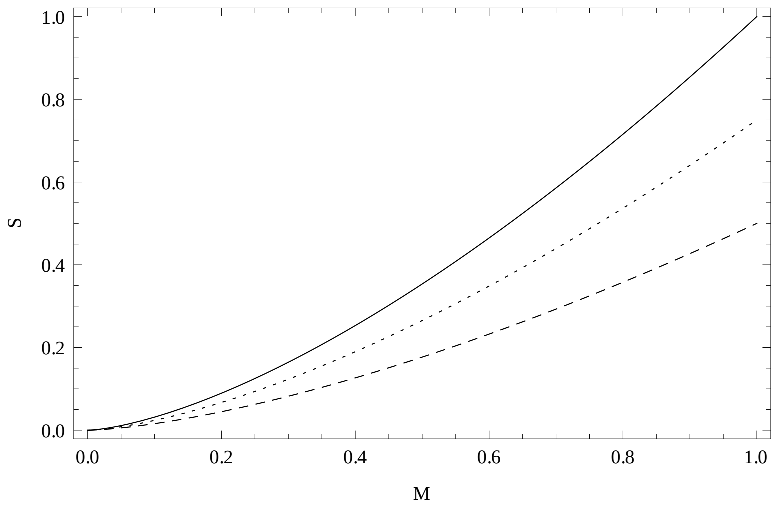

Figure 3.4: A graph showing entropy mass relation

to find out the temperature of the black hole. We have the thermodynamic relation between the energy of the system, entropy and temperature. In the case of black hole system the energy means the ADM mass of the black hole. Thus,

$$T = \left(\frac{\partial m}{\partial S}\right). \tag{3.25}$$

This can be evaluated as,

$$T = \frac{\partial m}{\partial r_+}\frac{\partial r_+}{\partial S}, \tag{3.26}$$

using Eq. (3.22) and Eq. (3.23) we can find,

$$T = \frac{1}{3\pi a\alpha r_+}. \tag{3.27}$$

Since a and α are constants, we put $a = \frac{2}{3\alpha}$, the temperature of black hole become,

$$T = \frac{1}{2\pi\sqrt{2\alpha m}}. \tag{3.28}$$

This result is consistent with the previous result(Eq.(2.64)) The relation between temperature and entropy is plotted in the graph (Fig.(3.4) and Fig.(3.5)) below. The curves are in general similar to black holes in General Theory of Relativity but on a close examination it can be seen that the curve is more steeper because of the presence of an $f'(R)$ term in the definition of entropy. We have obtained the entropy and

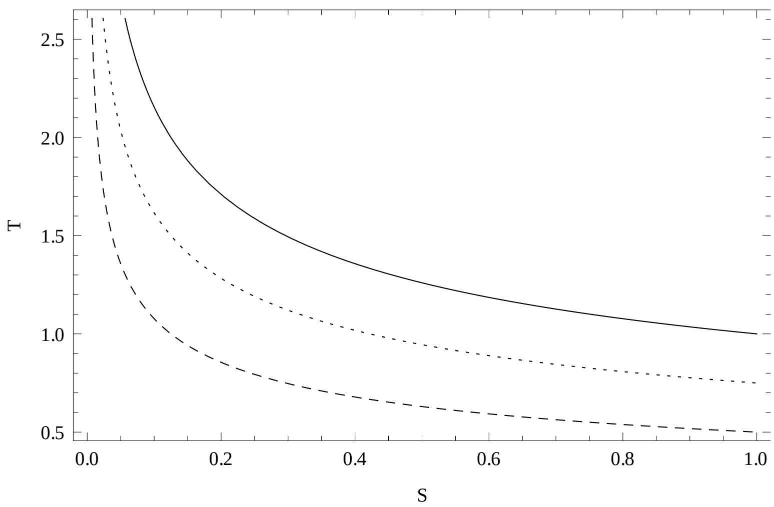

Figure 3.5: A graph showing temperature entropy relation

temperature in terms of horizon radius and mass. Now it is possible to calculate the specific heat of the black hole. The specific heat is actually the specific heat at constant volume. The specific heat is defined by the thermodynamic relation,

$$C = T\frac{\partial S}{\partial T}, \tag{3.29}$$

It is difficult to evaluate this differential but we can use the chain rule as,

$$\frac{\partial S}{\partial T} = \frac{\partial S}{\partial r_+}\frac{\partial r_+}{\partial T}, \tag{3.30}$$

using Eq.(3.23) and Eq.(3.27) we finally obtain the specific heat as,

$$C = -3\pi a r_+^3 = -3S. \tag{3.31}$$

As in the cases of most of the black holes, the specific heat of this black hole is also negative. In the case of Schwarzschild black hole the specific heat is $-2S$ here it is $-3S$. In both cases it is negative which shows that as the black hole radiates its energy through Hawking radiation process, the temperature of the black hole increases. This leads to radiate more and in a nutshell the black hole is thermodynamically unstable. This is the common behavior of a number of black holes. Since there is no discontinuity in the plot of heat capacity versus entropy or temperature there is no sign of phase transitions for this particular black holes.

3.7 Spectroscopy of extended black hole in f(R) theory

In this section we study the spectra of this extended black hole in f(R) theory. There are a number of methods to study the spectra of black holes but the one using asymptotic quasinormal modes and the adiabatic invariant method are the commonly used techniques. Here we use adiabatic invariant method for obtaining the spectra of this black hole. We find the area or entropy spectrum by adiabatic invariant integral and then quantize it. The first law of black hole thermodynamics can be written as,

$$dM = \frac{1}{4}T_H dA \tag{3.32}$$

with M is black hole mass and T_H is the Hawking temperature and A is the area of the black hole horizon, which is same as the first law of thermodynamics with the quarter of area representing the entropy. The Hawking temperature is related to the surface gravity κ_r through the relation,

$$T_H = \frac{\hbar\kappa_r}{2\pi}, \tag{3.33}$$

and κ_r is given by the derivative with respect to the radial coordinate of g_{00} thus,

$$\kappa_r = \frac{1}{2}\left|\frac{dB(r)}{dr}\right|_{r_H}. \tag{3.34}$$

We now use the period of motion of out going wave which is related to the vibrational frequency of the perturbed black hole, to quantize the area of the extended black hole in f(R) gravity. We know that the gravity system is periodic with respect to the Euclidean time in Kruskal coordinate. Particle motion in this periodic gravity system also owns a period which is shown to be inverse of Hawking temperature. In order to find out the area spectrum of the extended black hole via periodicity method[104], we substitute the extended metric into Klein-Gordon equation. The Klein-Gordon equation is given by,

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi - \frac{m^2}{\hbar^2}\Phi = 0. \tag{3.35}$$

We can obtain the solution of wave equation from the Hamilton-Jacobi equation, given by,

$$g^{\mu\nu}\partial_\mu S\partial_\nu S + m^2 S = 0, \tag{3.36}$$

where S is the action. The wave function Φ is related to the action S through the equation,

$$\Phi = exp\left[\frac{i}{\hbar}S(t,r,\theta,\phi)\right]. \tag{3.37}$$

Since our black hole is static and spherically symmetric, the action can be decomposed into the following form,

$$S(t,r,\theta,\phi) = -Et + W(r) + J(\theta,\phi). \tag{3.38}$$

Near the horizon J vanishes and W can be written as[104],

$$W(r) = \frac{i\pi E}{B'(r_H)}, \tag{3.39}$$

where we consider the out going wave near horizon. In this case the wave function outside the horizon can be expressed in the form,

$$\Phi = exp\left[\frac{-i}{\hbar}Et\right]\psi(r_H), \tag{3.40}$$

where,

$$\psi(r_H) = exp\left[-\frac{\pi E}{\hbar B'(r_H)}\right], \tag{3.41}$$

and from this equation we can easily find that Φ is a periodic function and period can be obtained from the following calculations,

$$\hbar\omega = E, \tag{3.42}$$

and $\frac{2\pi}{\tau} = \omega$, thus,

$$\tau = \frac{2\pi\hbar}{E}. \tag{3.43}$$

We have the adiabatic invariant I by definition[105],

$$I = \int \frac{2\pi dM}{\kappa_{r_+}} = n\hbar. \tag{3.44}$$

We have obtained earlier mass of the extended black hole as, $m = \frac{r_+^2}{2\alpha}$ and differentiating we get $dm = \frac{r_+}{\alpha}$, the surface gravity at the horizon is given by,

$$\kappa_{r_+} = \left| \frac{dB(r)}{dr} \right|_{r=r_+}. \tag{3.45}$$

We have $B(r) = \left[1 - \frac{2\alpha m}{r^2}\right]$ then,

$$\left| \frac{dB(r)}{dr} \right|_{r_+} = \frac{4\alpha m}{r_+^3}. \tag{3.46}$$

Now the adiabatic invariant integral becomes,

$$I = \int 2\pi \frac{r_+}{\alpha} \frac{r_+^3}{4\alpha m} dr_+. \tag{3.47}$$

Substituting for m we get finally,

$$\frac{\pi r_+^3}{\alpha} = n\hbar, \tag{3.48}$$

or the area $A/4$ we get it as,

$$\frac{A_n}{4} = \pi a r_+^3 = n\alpha\hbar. \tag{3.49}$$

In f(R) gravity the entropy is $f'(R)A$[39] thus,

$$f'(R) = ar + b \tag{3.50}$$

with $b = 0$. The spacing of area spectrum as ΔA can be written as,

$$\Delta f'(R)A = f'(R)A_{n+1} - f'(R)A_n = \alpha\hbar, \tag{3.51}$$

ie, $\Delta A = \alpha\hbar$. Using adiabatic invariant integral method we obtain the area spectrum of extended black hole in f(R) gravity. The result shows that the area spectrum is discrete and equispaced with a separation of $\alpha\hbar$. Clearly the area spectrum as well as quantum area are related directly to the length parameter α.

3.8 Conclusion

In this chapter we have studied the quasinormal modes of extended black hole in f(R) gravity. The black hole space-time is perturbed by electromagnetic and scalar waves and behavior of the resulting quasinormal modes are evaluated. The present study shows that the imaginary part of complex quasinormal modes for both cases increase showing damping of oscillations. Thus the black hole is stable against both scalar and electromagnetic perturbations. For scalar perturbation the damping time increases with increasing mass of the scalar field. We also have studied the thermodynamics and spectroscopy of an extended black hole in f(R) gravity. We have obtained the mass in terms of horizon radius r_+. The entropy of the black hole is also calculated. Unlike the black hole in general relativity here the entropy is related to the third power of r_+, the horizon radius. The temperature is also calculated which shows that the temperature is related inversely to the horizon radius which is the same behavior of the black holes in general relativity. The heat capacity is also studied it is negative and shows no discontinuities indicating that there is no phase transition for this black hole. It is thermodynamically unstable because the heat capacity is negative. The area spectrum of the black hole is also studied. The study shows that the area spectrum of the black hole is discrete and has a spacing $\alpha\hbar$ and depends on the length parameter α.

4

Thermodynamics, Spectroscopy and Quasinormal modes of MSW black hole

4.1 introduction

$2+1$ black holes are the simplest toy model of the more general and more complicated $3+1$ black holes. Since $2+1$ black holes are in reduced dimensions, the Einstein equations are exactly solvable. Both $2+1$ black holes and $3+1$ black holes share a number of properties and hence it is important to study simple $2+1$ black hole in Einstein gravity. The BTZ black hole is the most studied $2+1$ black hole. The name of BTZ comes from its proposers, Bandos, Tietelboim and Zenelli[95], which has a three dimensional geometry of its own. BTZ black hole solution possesses a negative cosmological constant. The quasinormal modes of different perturbations, scalar, electromagnetic and gravitational perturbations of this black hole have been studied analytically earlier[100]. The Dirac perturbation is more complicated and is studied in the present work using WKB approximation method. Charged dilaton black holes are the other kind of $2+1$ black holes. The low dimensional charged dilaton black holes have a relation to the low dimensional string theory, thus it acts as a tool for studying low dimensional string theory. Mandal et al.[96] have obtained a one parameter family of a black hole solution to the classical dilaton system in three dimensions. This black hole is called MSW black hole in our work.

The black hole solution basically contains a singularity surrounded by an event horizon. The black hole horizon is mostly like a one sided membrane. It allows the passage of particles as well as radiation only in one direction. The black hole horizon has its own temperature and entropy. The four laws of black hole mechanics proposed by Bardeen, Carter and Hawking[54] have striking similarity with the four laws of thermodynamics. It was then showed by Hawking that the area of the black hole never decreases. Then Bekenstein[55] argued that the area is a measure of entropy of the particular black hole. Bekenstein also showed that the black hole area is quantized[56].

There exist a number of works on thermodynamics and spectroscopy of different types of black holes. Periodicity and area spectrum of a Schwarzschild and Kerr black hole are obtained by Zeng et al.[106]. Entropy and area spectrum of charged black hole are studied by Wei et al.[97] and area spectrum of BTZ black hole is obtained by periodicity method by Larranaga[98].

The stability analysis of black holes leads to the quasinormal frequencies. These frequencies are complex in most cases, in the first part of this chapter we study the thermodynamics and spectroscopy of MSW black hole and the second part of this chapter discusses Dirac quasinormal modes of MSW black holes. It is believed that the presence of black hole can be inferred using quasinormal waves. Thus recently, the study of quasinormal modes becomes important as a number of gravitational wave detectors are expected to start operating soon. Dirac quasinormal mode of Schwarzschild black hole was studied in a classic paper by Cho[99]. Different perturbations of BTZ black hole are studied by Cardoso et al.[100]. Area spectrum of rotating BTZ black hole from Quasinormal modes is found out in [101]. Non-rotating BTZ black hole area spectrum from quasi-normal modes is discussed in[102]. Since it is a $2 + 1$ black hole, electromagnetic, gravitational and scalar perturbations can be analytically studied but Dirac perturbation is complicated and we use the third order WKB approximation method[68–70] to study the Dirac perturbation of MSW black holes.

4.2 The MSW black hole

The metric for MSW black hole is given by ($c = G = 1$ system of units)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \gamma^2 r d\phi^2, \tag{4.1}$$

where $f(r)$ is given by[103]

$$f(r) = 8\Lambda\beta r - 2M\sqrt{r}. \tag{4.2}$$

This metric has a zero at (ie, the horizon)

$$r_+ = \frac{M^2}{16\Lambda^2\beta^2}, \tag{4.3}$$

where Λ is the cosmological constant, β is a constant factor, M is the mass of the black hole.

4.3 Thermodynamics of MSW black hole

The black hole mass as a function of r_+ from Eq. (4.3) is,

$$M = 4\Lambda\beta\sqrt{r_+}. \tag{4.4}$$

The variation of M versus r_+ is shown in Fig.(4.1) for the different

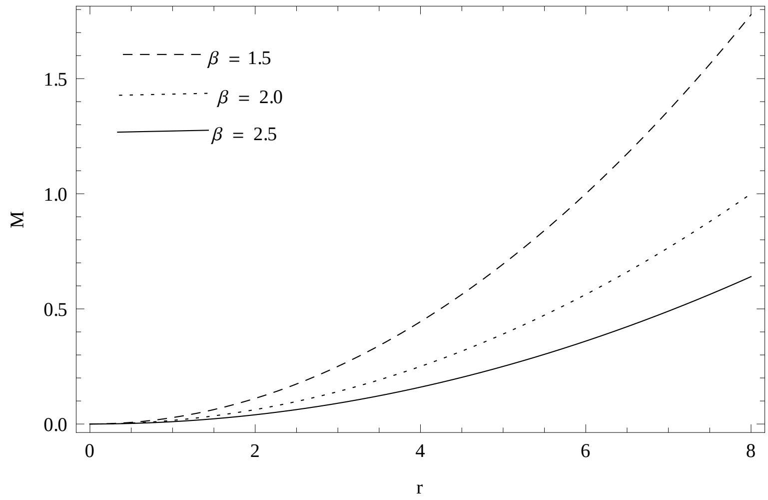

Figure 4.1: Variation of mass with horizon radius for different values of β

values of β and it has a parabolic nature. The entropy S of this $2+1$ dimensional black hole can be obtained as,

$$S = \frac{A}{4} = \frac{\pi r_+}{2}. \tag{4.5}$$

Here A is the area, since it is a $2 + 1$ black hole area implies the circumference of the horizon. Since r_+ can be expressed in terms of M, we can also express the entropy in terms of black hole mass as,

$$S = \frac{\pi M^2}{32\Lambda^2\beta^2}. \tag{4.6}$$

Fig.(4.2) is a plot between entropy and mass and it shows a parabolic nature. The black hole temperature can now be obtained since entropy-mass relation is known. The general expression for temperature T is,

$$T = (\frac{\partial M}{\partial S}). \tag{4.7}$$

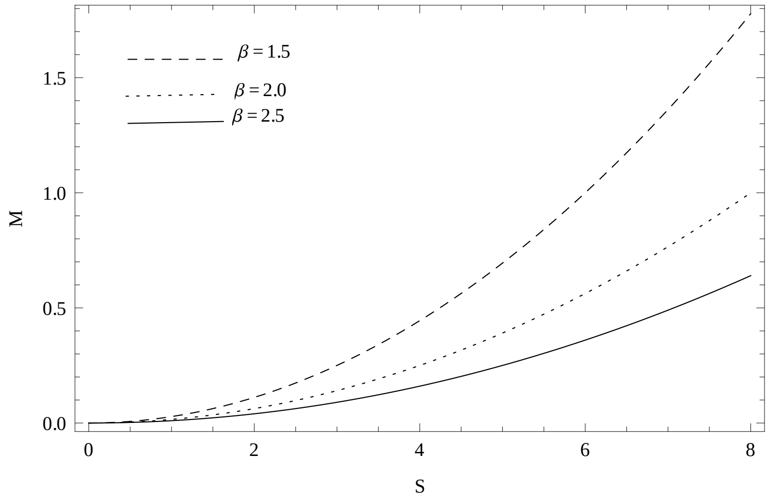

Figure 4.2: variation of entropy with mass for different values of β

The heat capacity is given by the general expression,

$$C = T(\frac{\partial S}{\partial T}). \tag{4.8}$$

Eq.(4.6) can be inverted such that we get r_+ in terms of S as,

$$r_+ = \frac{2S}{\pi}. \tag{4.9}$$

From Eq.(4.7) we can find out the temperature T as,

$$T = \frac{4\Lambda\beta}{\pi\sqrt{r_+}}. \tag{4.10}$$

Fig.(4.3) shows variation of entropy with temperature. Entropy increases rapidly as temperature decreases. The specific heat of black hole can be obtained from Eq.(4.8). The specific heat is obtained as,

$$C = -\pi r_+ = -2S. \tag{4.11}$$

The specific heat is negative and is twice the entropy of the black hole. The negative heat capacity indicates that the black hole is

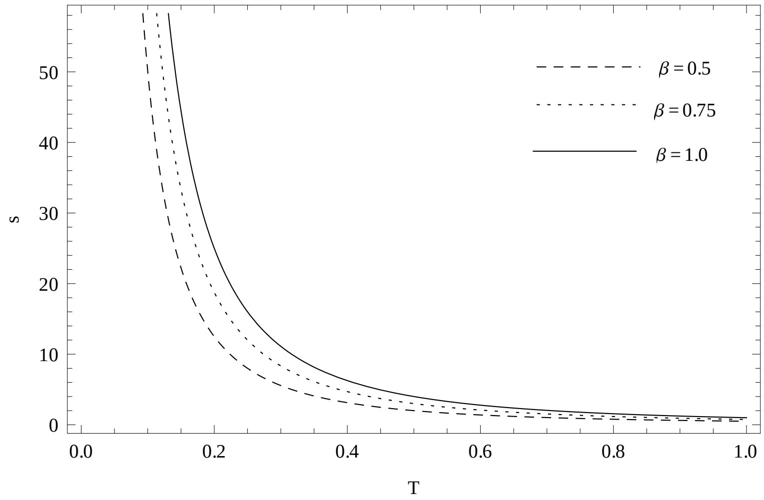

Figure 4.3: variation of entropy with temperature for different values ofβ

thermodynamically unstable. For a large number of black holes heat capacity is negative showing that black holes are thermodynamically unstable.

4.4 Spectroscopy of MSW black holes

In this section we study the spectra of MSW black hole using adiabatic invariant method[104, 105]. We find the area spectrum by evaluating the adiabatic invariant integral which varies very slowly compared to the external perturbations. The adiabatic invariant is quantized using Bohr-Sommerfeld quantization rule. The first law of black hole thermodynamics says that,

$$dM = \frac{T_H dA}{4}. \tag{4.12}$$

We utilize the period of the motion of out going wave, which is shown to be related to the vibrational spectrum of the perturbed black hole, to quantize the area of the MSW black hole. It is well known that the gravity system is periodic with respect to the Euclidean time in Kruskal coordinate. Particles in this periodic gravitating system also own a period which is inverse of Hawking temperature. To find out area spectrum by periodicity method we use the Klein-Gordon equation given by,

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi - \frac{M^2}{\hbar^2}\Phi = 0. \tag{4.13}$$

We define action A such that Φ is related to the action as,

$$\Phi = exp[\frac{i}{\hbar}A(t,r)]. \tag{4.14}$$

Adopting the wave equation ansatz for the scalar field, we get the solution of wave equation. We can also obtain the solution from the Hamilton-Jacobi equation,

$$g^{\mu\nu}\partial_\mu A\partial_\nu A + M^2 A = 0. \tag{4.15}$$

The action $A(t,r)$ can be decomposed into the functions of r and t and be written as[106, 107],

$$A(t,r) = -Et + W(r), \tag{4.16}$$

where E is the energy of the emitted particle measured by an observer at infinity. $W(r)$ in the action can be written near the horizon as[107],

$$W(r) = \frac{i\pi E}{f'(r_+)}, \tag{4.17}$$

where we consider the out going wave near the horizon. In this case, it is obvious that the wave function Φ outside the horizon can be expressed in the form,

$$\Phi = \Psi(r_+)exp[-\frac{iEt}{\hbar}], \tag{4.18}$$

and $\Psi(r_+)$ is given by,

$$\Psi(r_+) = exp[\frac{\pi E}{\hbar f'(r_+)}], \tag{4.19}$$

From Eq.(4.18) we can find that Φ is periodic with a period,

$$\tau = \frac{2\pi\hbar}{E}. \tag{4.20}$$

The gravitating system is periodic in Kruskal coordinate in Euclidean time and the moving particle also gets a periodic motion in this periodic gravity. The period is related inversely to the Hawking temperature as,

$$\tau = \frac{2\pi}{\kappa_r} = \frac{\hbar}{T_H}, \tag{4.21}$$

and hence, the Hawking temperature is given by,

$$T_H = \frac{\hbar\kappa_r}{2\pi}. \tag{4.22}$$

In the near horizon,

$$U(r) = -M + 4\Lambda\beta\sqrt{r_+}. \tag{4.23}$$

As $U(r) \to 0$, the change in the mass dM, becomes,

$$dM = 2\Lambda\beta\frac{dr_+}{\sqrt{r_+}}. \tag{4.24}$$

The adiabatic invariant[105] is defined as,

$$I = \int \frac{2\pi dM}{\kappa_{r_+}}, \tag{4.25}$$

by Bohr-Sommerfeld quantization rule, we can write,

$$I = \int \frac{2\pi dM}{\kappa_{r_+}} = n\hbar, \tag{4.26}$$

where κ_{r_+} is the surface gravity and is given by,

$$\kappa_{r_+} = \frac{dU(r)}{dr}|_{r=r_+}. \tag{4.27}$$

From (4.23),

$$\frac{dU(r)}{dr}|_{r=r_+} = \frac{4\Lambda\beta}{2\sqrt{r_+}}, \tag{4.28}$$

and using Eq.(4.26), we can have,

$$\kappa_{r_+} = \frac{2\Lambda\beta}{\sqrt{r_+}} \tag{4.29}$$

Substituting Eqs.(4.24) and (4.28) in Eq.(4.25) and applying Bohr-Sommerfeld quantization rule, we get,

$$2\pi r_+ = A_n = n\hbar. \tag{4.30}$$

and ΔA is given by,

$$\Delta A = A_{n+1} - A_n = \hbar. \tag{4.31}$$

The result shows that the circumference of an MSW black hole is quantized and it can take only discrete values. It can increase or decrease only in the integral values of $\hbar$. Thus we see that the circumference spectrum of MSW black hole is discrete and the spacing is equidistant. For this system, the circumference spectrum is independent of black hole parameters. Since the entropy of this system is related to circumference spectrum, the entropy spectrum of MSW black hole is also quantized.

4.5 Dirac field in MSW space-time background

In this section we consider the Dirac field in MSW background space-time. The Dirac equation in a general background space-time can be

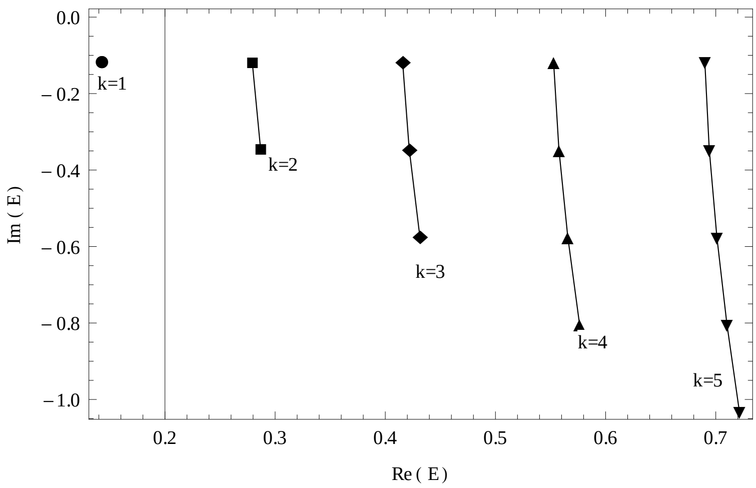

Figure 4.4: Quasinormal frequencies of MSW black hole with $\beta = 0.1$

written as[108],

$$[i\gamma^{\mu}\partial_{\mu} - i\gamma^{\mu}\Gamma_{\mu}]\Phi = m\Phi, \tag{4.32}$$

where m is the mass of the Dirac field, γ^{μ} are the curvature dependent Dirac matrices and are represented in terms of tetrad field as,

$$\gamma^{\mu} = e^{\mu}_{a}\gamma^{a}, \tag{4.33}$$

γ^{a} represent the standard flat space Dirac matrices, which satisfy,

$$\{\gamma^{a}, \gamma^{b}\} = 2\eta^{ab}, \tag{4.34}$$

with γ^{0} given by,

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

and γ^{i} with $i = 1, 2, 3$ is given by,

$$\begin{pmatrix} 0 & -i\sigma^{i} \\ i\sigma^{i} & 0 \end{pmatrix}.$$

Table 4.1: Quasinormal frequencies for $\beta = 0.1$

k	n	Re(E)	Im(E)
1	0	0.142987	-0.113155
2	0	0.279143	-0.114674
	1	0.286690	-0.341522
3	0	0.415674	-0.115149
	1	0.421666	-0.343879
	2	0.431245	-0.571293
4	0	0.552724	-0.115311
	1	0.557569	-0.344896
	2	0.565582	-0.573155
	3	0.576227	-0.800933
5	0	0.690033	-0.115378
	1	0.694058	-0.345414
	2	0.700958	-0.574273
	3	0.710081	-0.802461
	4	0.721375	-1.030450

The tetrad field is defined by,

$$g_{\mu\nu} = \eta^{ab} e^a_\mu e^b_\nu, \tag{4.35}$$

where $\eta^{ab} = diag(-1, 1, 1, 1)$ being the Minkowski metric. Γ_μ, the spin connections are given by,

$$\Gamma_\mu = \frac{1}{8}[\gamma^a, \gamma^b] e^\nu_a e_{b\nu;\mu}, \tag{4.36}$$

where $e_{b\nu;\mu} = \partial_\mu e_{b\nu} - \Gamma^\alpha_{\mu\nu} e_{b\alpha}$ is the covariant derivative of $e_{b\nu}$ with $\Gamma^\alpha_{\mu\nu}$ being Christoffel symbols. With the MSW metric given as above we can take the tetrad as,

$$e^\mu_a = diag\left(-\left(8\Lambda\beta r - 2M\sqrt{r}\right)^{-\frac{1}{2}}, \left(8\Lambda\beta r - 2M\sqrt{r}\right)^{\frac{1}{2}}, 0, r\right) \tag{4.37}$$

The spin connection satisfy the equation,

$$[\Gamma_\mu, \gamma^\nu] = \frac{\partial \gamma^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\rho}\gamma^\rho. \tag{4.38}$$

Table 4.2: Quasinormal frequencies for $\beta = 0.05$

k	n	Re(E)	Im(E)
1	0	0.051064	-0.056245
2	0	0.099745	-0.056991
	1	0.103979	-0.169241
3	0	0.147834	-0.057383
	1	0.151505	-0.170971
	2	0.157294	-0.284075
4	0	0.196101	-0.057550
	1	0.199216	-0.171805
	2	0.204175	-0.285420
	3	0.210956	-0.398999
5	0	0.244513	-0.057627
	1	0.247177	-0.172257
	2	0.251510	-0.286211
	3	0.257315	-0.399993
	4	0.264698	-0.513824

We solve Eq. (4.40) for spin connection Γ_μ and are given by,

$$\begin{aligned}\Gamma_0 &= -\frac{1}{2}\left(4\Lambda\beta r - \frac{m}{2\sqrt{r}}\right)(\gamma_1\gamma_0)\,,\ \Gamma_1 = 0,\\ \Gamma_2 &= 0,\ \Gamma_3 = \frac{1}{2}\left(8\Lambda\beta r - 2m\sqrt{r}\right)^{\frac{1}{2}}(\gamma_1\gamma_3)\,,\end{aligned} \tag{4.39}$$

and hence $\gamma^\mu\Gamma_\mu$ becomes,

$$\gamma^\mu\Gamma_\mu = \gamma_1\left(8\Lambda\beta r - 2M\sqrt{r}\right)^{\frac{1}{2}}\left[\frac{1}{2r} + \frac{\left(4\Lambda\beta - \frac{M}{2\sqrt{r}}\right)}{2\left(8\Lambda\beta r - 2M\sqrt{r}\right)}\right]. \tag{4.40}$$

Therefore the Dirac equation becomes,

$$\begin{aligned}&i\gamma_0\left(8\Lambda\beta r - 2M\sqrt{r}\right)^{\frac{-1}{2}}\partial_t\Phi \qquad (4.41)\\ &+\left[i\gamma_1\left(8\Lambda\beta r - 2M\sqrt{r}\right)^{\frac{1}{2}}\left[\partial_r + \frac{1}{2r}\right] + i\frac{\gamma_3}{r}\partial_\phi - m\right]\Phi = 0.\end{aligned}$$

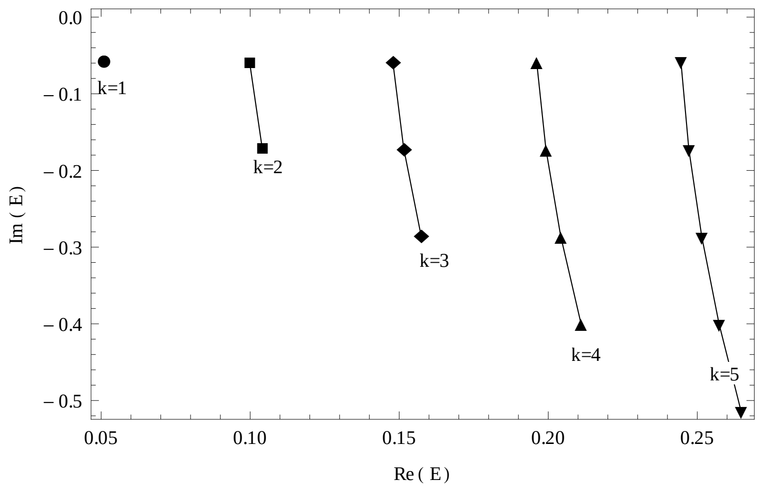

Figure 4.5: Quasinormal frequencies of MSW black hole with $\beta = 0.05$

This equation can be transformed using tortoise coordinates to Schrodiger like equations as,

$$\left(-\frac{d^2}{dr_*^2} + V_{(+)1}\right) F^+ = E^2 F^+, \tag{4.42}$$

and,

$$\left(-\frac{d^2}{dr_*^2} + V_{(+)2}\right) G^+ = E^2 G^+, \tag{4.43}$$

where r_* is the tortoise coordinate, F and G are the two wave functions of the Dirac equation. $V_{(+)1,2}$ are given by,

$$V_{(+)1,2} = \pm \frac{dW_{(+)}}{dr_*} + W_{(+)}^2, \tag{4.44}$$

where W is a function of k related to the potential. The same set of equations repeat with $(+)$ changed to $(-)$. Putting together we get,

$$\left(-\frac{d^2}{dr_*^2} + V_1\right) F = E^2 F, \tag{4.45}$$

and,

$$\left(-\frac{d^2}{dr_*^2} + V_2\right) G = E^2 G. \tag{4.46}$$

Table 4.3: Quasinormal frequencies for $\beta = 0.01$

k	n	Re(E)	Im(E)
3	0	0.0135923	-0.0113035
	1	0.0144644	-0.0334106
	2	0.0158395	-0.0555517
4	0	0.0179184	-0.0113719
	1	0.0187814	-0.0337322
	2	0.0201505	-0.0560795
	3	0.0221412	-0.0784838
5	0	0.0222176	-0.0114231
	1	0.0230410	-0.0339561
	2	0.0243446	-0.0564386
	3	0.0262203	-0.0789680
	4	0.0286961	-0.1015530

The effective potential can be derived from this as,

$$V(r,k) = \left(\frac{\sqrt{\Delta}k}{r^2}\right)^2 + \frac{\Delta}{r^2}\frac{d}{dr}\left(\frac{\sqrt{\Delta}k}{r^2}\right), \tag{4.47}$$

where Δ is related to the metric,

$$\Delta = r^2(8\Lambda\beta r - 2M\sqrt{r}). \tag{4.48}$$

For our metric given by Eqs.(4.1 & 4.2), using Eqs.(4.46 & 4.47) the potential can be now written as,

$$V(r,k) = \frac{k^2}{r^2}\left(8\Lambda\beta r - 2M\sqrt{r}\right) + \frac{k}{2r^2}\left(8\Lambda\beta r - 2M\sqrt{r}\right)^{\frac{1}{2}}\left(-8\Lambda\beta r + 3M\sqrt{r}\right). \tag{4.49}$$

Fig.(4.6) shows the dependence of potential on the angular momentum quantum number k. n is the projection of angular momentum. It is of barrier form and the barrier increases in height as the k value increases.

The potential is plotted for different values of k is given in figure

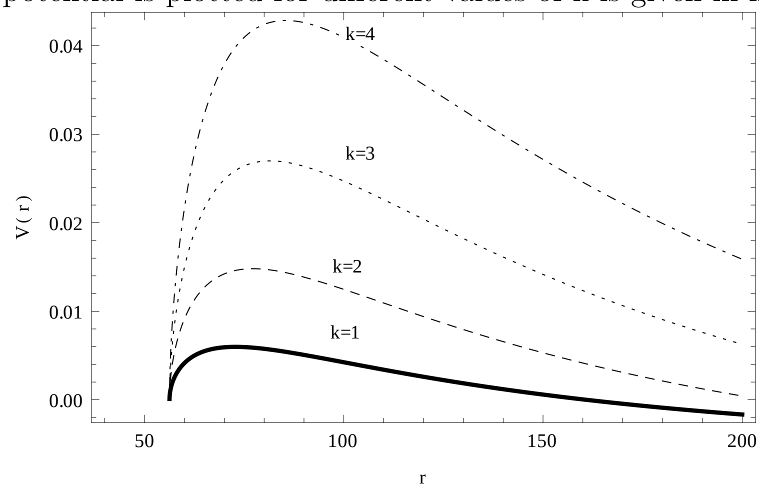

Figure 4.6: Variation of potential with k

4.6 Evaluation of quasinormal modes for massless Dirac field

In order to evaluate the Dirac quasinormal modes we use the WKB approximation method developed by Schutz, Will and Iyer[68–70]. Comparing with other numerical methods this method has been found to be accurate up to around one percent for both real and imaginary parts. The equation for computing the quasinormal modes E is given by,

$$E^2 = [V_0 + (-2V_0'')^{\frac{1}{2}}\Lambda] - i(n+\frac{1}{2})(-2V_0'')^{\frac{1}{2}}(1+\Omega), \tag{4.50}$$

where,

$$\Lambda = \frac{1}{(-2V_0'')^{\frac{1}{2}}}\left[\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4}+\alpha^2\right) - \frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2(7+60\alpha^2)\right], \tag{4.51}$$

and,

$$\begin{aligned}
\Omega &= \frac{1}{(-2V_0'')}[\frac{5}{6912}\left(\frac{V_0'''}{V_0''}\right)^4\left(77+188\alpha^2\right) \\
&- \frac{1}{384}\left(\frac{(V_0''')^2(V_0^{(4)})}{(V_0'')^3}\right)\left(51+100\alpha^2\right) \\
&+ \frac{1}{2304}\left(\frac{V_0^{(4)}}{V_0''}\right)^2(67+68\alpha^2) \\
&+ \frac{1}{288}\left(\frac{V_0'''V_0^{(5)}}{(V_0'')^2}\right)(19+28\alpha^2)+\frac{1}{288}\left(\frac{V_0^{(6)}}{V_0''}\right)(5+4\alpha^2)],
\end{aligned} \tag{4.52}$$

where $\alpha = n + \frac{1}{2}$, and,

$$V_0^{(n)} = \frac{d^nV}{dr_0^n}|_{r_*=r_{*max}}. \tag{4.53}$$

Substituting the potential $V(r)$ obtained from Eq.(4.50) in Eq.(4.51) we get the complex quasinormal modes for massless Dirac field in MSW space-time. The values are listed in tables and plotted. Fig.(4.4) and Fig.(4.5) represent the quasinormal mode frequencies for different values of β. The figures show that the real part of complex frequencies slightly increase, while the magnitude of imaginary part of frequencies increases rapidly with mode number n for the same value of k. This indicates that the higher modes decay faster than the low lying modes. Since the imaginary part of the quasinormal mode increases indicating that the oscillations damp exponentially and die-out very fast. Thus MSW spacetime is stable against perturbation, by a massless Dirac field

4.7 Conclusion

The thermodynamics of 2+1 dimensional MSW black holes are studied and area spectrum is obtained using adiabatic invariant method. We have obtained the heat capacity, temperature and mass of MSW black hole. The different quantities are plotted and studied their behavior. The mass of the black hole is plotted for various values of horizon radius with different parameter β and the graph shows that the horizon radius increases quadratically with mass, the same behavior is shown for mass versus entropy. The variation of temperature with entropy is also plotted and the graph shows that the entropy increases very rapidly with decreasing temperature. The heat capacity is found to be negative as is the case most of the black holes. This black hole does not show any phase transition. The period of Euclidean time is found out and adiabatic invariant is calculated. Using Bohr-Sommerfeld quantization rule we have obtained the area spectrum, here the circumference spectrum and is found to be quantized and equally spaced. In this chapter we have also studied the quasinormal modes of MSW black hole. The complex frequencies are found out and tabulated. Graphs are plotted for various values of the parameter β. The modulus value of imaginary part of the frequencies increases with increasing mode number showing that the radiation is damping which in turn shows that the black hole is stable against massless Dirac perturbations. Real values of the frequencies increase in this case contradictory to the normal $3 + 1$ black holes, where the real part of the quasinormal modes decreases with increasing mode number.

5

Thermodynamics, Spectroscopy of a regular black hole

5.1 Introduction

One of the drawbacks of Einstein's General Theory of Relativity is that in some cases the solutions of the field equation contain singularities. The different black hole solutions and cosmological solution are examples that contain singularities. At the singularities all laws of physics break down. Bardeen[76] proposed a regular space-time with a horizon and without singularity. But the physical source associated with his solution is clarified much later, when Ayon-Beato and Garcia[77] interpreted it as the gravitational field of a nonlinear magnetic monopole of self gravitating magnetic field. Such black holes in general are known as regular black holes. Their metrics and curvature invariants such as R, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\kappa\tau}R^{\mu\nu\kappa\tau}$ are regular every where. This type of black holes violate the strong energy condition somewhere in space-time. Some of them obey the weak energy condition everywhere. The solutions obeying the weak energy condition necessarily have a de Sitter type central region. There are other features that characterize regular black holes due to the nonlinearities of the field equations. As an example, the thermodynamic quantities do not satisfy the Smarr formula. The Komar charge is not satisfied by the regular black holes. Hayward[78] proposed a new type of regular black hole which has no singularity at the center but have a horizon. For large distances it appears like a Schwarzschild black hole but inside the horizon it is like a de Sitter space time. Usually regular black holes are spherically symmetric, static and asymptotically flat and have regular centers. The Einstein tensor is physically reasonable and satisfying the weak energy condition and its components are bounded and fall off appropriately at large distances. The simplest causal structure is similar to the Reissner-Nordstorm black hole with internal singularity replaced by regular centers. Hayward black hole has the following properties, $f(r) \to \left(1 - \frac{2M}{r}\right)$ as $r \to \infty$ and as $r \to 0$, $f(r) \to \left(1 - \frac{r^2}{l^2}\right)$ where l is related to the cosmological constant Λ.

It is widely believed that black hole area is quantized. This assumption was first introduced by Bekenstein[55]. The Bekenstein-Hawking entropy is one quarter of horizon area expressed in Planck units. Bekenstein argued that the black hole entropy is an adiabatic invariant with an equally spaced quantum spectrum. In order to get a statistical interpretation, the entropy must corresponds to the logarithm of an integer, i.e., $S_{BH} = kln(\Omega)$, where Ω is an integer analogues to number of micro-states in statistical mechanics. It is widely believed that the quasinormal modes and the black hole entropy spectrum are related. It was Hod[57] who first showed that the entropy spectrum is related to the imaginary part of the asymptotic quasinormal modes and obtained a value $4ln3l_p^2$. But later Maggiore's[58] study showed that one must take both real and imaginary parts to get the minimum value of entropy and he obtained a value $8\pi l_p^2$ which is same as obtained earlier by Bekenstein. Bohr-Sommerfeld quantization rule implies that adiabatic invariant has equally spaced spectrum in the semi-classical limit. It is interesting to study the thermodynamics and spectrum of the regular black hole using adiabatic invariant method.

5.2 Thermodynamics of Hayward regular black hole

The metric of Hayward regular black hole is given by ($c = G = 1$ system of units),

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{5.1}$$

where $f(r)$ is given by[78],

$$f(r) = 1 - \frac{2Mr^2}{r^3 + 2\beta^2}, \tag{5.2}$$

where M is the mass of the regular Hayward black hole and β is a constant factor depending on cosmological constant. $f(r) = 0 \; at \; r = r_+$, then the black hole mass as a function of r_+ is,

$$M = \left[\left(\frac{\beta}{r_+} \right)^2 + \frac{r_+}{2} \right]. \tag{5.3}$$

The entropy S of this four dimensional regular black hole is given by,

$$S = \frac{A}{4} = \pi r_+^2. \tag{5.4}$$

Putting this into the equation for M, we get the entropy equation,

$$2M \left(\frac{S}{\pi} \right) - \left(\frac{S}{\pi} \right)^{\frac{3}{2}} = 2\beta^2. \tag{5.5}$$

The mass can now be written in terms of entropy as,

$$M = \left(\frac{\pi \beta^2}{S} + \frac{1}{2}\sqrt{\frac{S}{\pi}} \right). \tag{5.6}$$

The mass versus entropy graph is plotted in Fig. 5.1, which shows that the mass decreases very rapidly as the entropy increases. Temperature

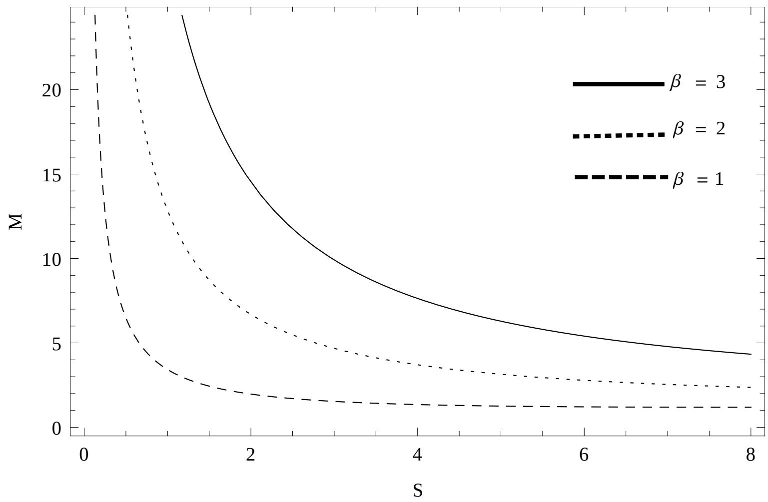

Figure 5.1: Variation of mass with entropy for different values of β

T can be obtained using the formula $T = \left(\frac{\partial M}{\partial S}\right)$, as,

$$T = \frac{-4\pi^{\frac{3}{2}}\beta^2 + S^{\frac{3}{2}}}{4\sqrt{\pi}S^2}. \tag{5.7}$$

The temperature is plotted in Fig. 5.2 for different values of β as from this graph we can see that temperature is positive for above a certain values of S. The specific heat of black hole is given by $C = T\left(\frac{\partial S}{\partial T}\right)$. Now,

$$C = 2S\frac{\left[-4\pi^{\frac{3}{2}}\beta^2 + S^{\frac{3}{2}}\right]}{\left[16\pi^{\frac{3}{2}}\beta^2 - S^{\frac{3}{2}}\right]}, \tag{5.8}$$

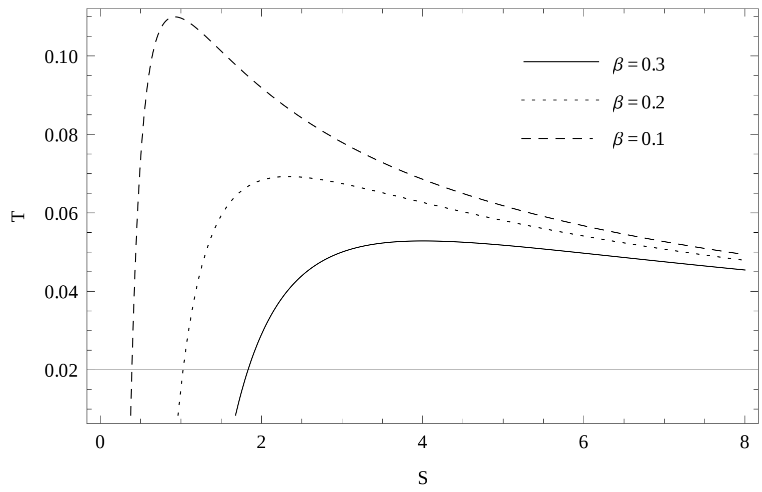

Figure 5.2: Variation of temperature with entropy for different values of β

and it is plotted in Fig. 5.3. Specific heat is positive for small values of S and diverges at a particular values of S and then becomes negative for high values of S

5.3 Spectroscopy of regular black holes

In this section, we study the spectra of Hayward black hole using adiabatic invariant method[104, 105]. We find the area spectrum by evaluating the adiabatic invariant integral which varies very slowly compared to the external perturbations. The adiabatic invariant is quantized using Bohr-Sommerfeld quantization rule. We utilize the period of the motion of out going wave, which is shown to be related to the vibrational spectrum of the perturbed black hole, to quantize the area of the Hayward black hole. It is well known that the gravity system is periodic with respect to the Euclidean time in Kruskal coordinate. Particles in this periodic gravitating system also own a

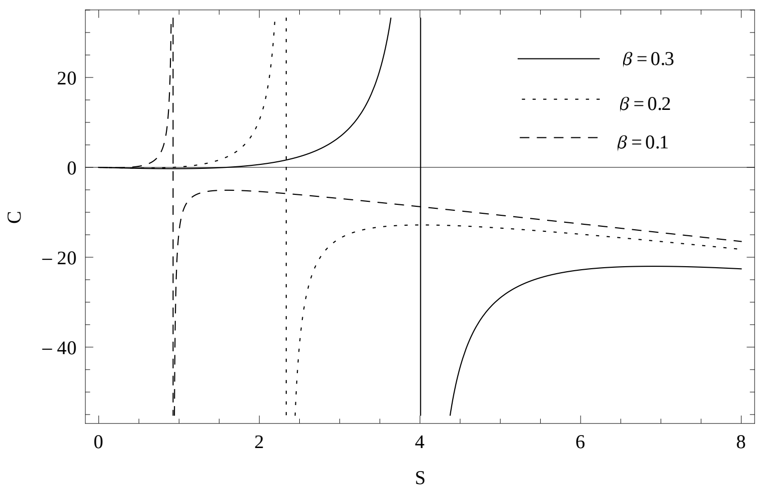

Figure 5.3: Variation of heat capacity with entropy for different values of β

period which is inverse of Hawking temperature[104]. To find out area spectrum by periodicity method we use the Klein-Gordon equation given by,

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi - \frac{M^2}{\hbar^2}\Phi = 0. \tag{5.9}$$

We define action A such that Φ is related to the action as,

$$\Phi = exp\left[\frac{i}{\hbar}A(t,r)\right]. \tag{5.10}$$

Adopting the wave equation ansatz for the scalar field we get the solution of wave equation. We can also obtain the solution from the Hamilton-Jacobi equation,

$$g^{\mu\nu}\partial_\mu A\partial_\nu A + M^2 A = 0. \tag{5.11}$$

The action $A(t, r)$ can be decomposed into functions of r and t and be written as[106, 107],

$$A(t, r) = -Et + W(r), \tag{5.12}$$

where E is the energy of the emitted particle measured by an observer at infinity. $W(r)$ in the action can be written near the horizon as[107],

$$W(r) = \frac{i\pi E}{f'(r_+)}, \tag{5.13}$$

where we consider the out going wave near the horizon. In this case, it is obvious that the wave function Φ outside the horizon can be expressed in the form,

$$\Phi = exp\left[-\frac{iEt}{\hbar}\right]\Psi(r_+), \tag{5.14}$$

and $\Psi(r_+)$ is given by,

$$\Psi(r_+) = exp\left[\frac{\pi E}{\hbar f'(r_+)}\right]. \tag{5.15}$$

From Eq.(5.14) we can find that Φ is periodic with a period,

$$\tau = \frac{2\pi\hbar}{E}. \tag{5.16}$$

The gravitating system is periodic in Kruskal coordinate in Euclidean time and the moving particle also gets a periodic motion in this periodic gravity. The period is related inversely to the Hawking temperature as,

$$\tau = \frac{2\pi}{\kappa_r} = \frac{\hbar}{T_H}, \tag{5.17}$$

where κ_r is the surface gravity. Hence the Hawking temperature is given by,

$$T_H = \frac{\hbar\kappa_r}{2\pi}. \tag{5.18}$$

In the near horizon,

$$f(r) = 1 - \frac{2Mr^2}{r^3 + 2\beta^2}. \tag{5.19}$$

As $f(r) \to 0$, the change in the mass dM, becomes,

$$dM = \left(\frac{1}{2} - \frac{2\beta^2}{r^3}\right) dr. \tag{5.20}$$

The adiabatic invariant[105] is defined as,

$$I = \int \frac{8\pi dM}{\kappa_{r_+}}, \tag{5.21}$$

The surface gravity κ_{r_+} is given by,

$$\kappa_{r_+} = \frac{1}{2}\frac{df(r)}{dr}|_{r=r_+}. \tag{5.22}$$

From (5.19),

$$\frac{df(r)}{dr}|_{r=r_+} = \frac{\left(1 - \frac{4\beta^2}{r_+^3}\right)}{2\left(\frac{r_+}{2} + \frac{\beta^2}{r_+^2}\right)}, \tag{5.23}$$

and using Eq.(5.22), we can have,

$$\kappa_{r_+} = \frac{\left(1 - \frac{4\beta^2}{r_+^3}\right)}{4\left(\frac{r_+}{2} + \frac{\beta^2}{r_+^2}\right)}. \tag{5.24}$$

Substituting Eqs.(5.24) and (5.20) in Eq.(5.21) we get,

$$I = \int \frac{8\pi dM}{\kappa_{r_+}} = \int \frac{4\pi(2dM)}{\kappa_{r_+}}. \tag{5.25}$$

We have the differential mass given by,

$$2dM = \left(1 - \frac{4\beta^2}{r^3}\right) dr, \tag{5.26}$$

and the surface gravity is,

$$\kappa_{r_+} = \frac{\left(1 - \frac{4\beta^2}{r^3}\right)}{4\left(\frac{r_+}{2} + \frac{\beta^2}{r^2}\right)}. \tag{5.27}$$

Substituting Eq.(5.26) and Eq.(5.27) into Eq.(5.25) and applying Bohr-Sommerfeld quantization rule, we get,

$$4\pi r_+^2 - 16\pi\frac{\beta^2}{r_+} = n\hbar \tag{5.28}$$

Thus we get the area spectrum as,

$$4\pi r_+^2 = A_n = n\hbar + 16\pi\frac{\beta^2}{r_+}. \tag{5.29}$$

and the area quantum as,

$$A_{n+1} - A_n = \hbar \tag{5.30}$$

Thus we see that the area spectrum is quantized. The spectrum depends on the horizon radius r_+ and on the black hole parameter β and since entropy spectrum is related to the area spectrum it is also quantized.

5.4 Conclusion

In this chapter we have studied the thermodynamics of a regular black hole proposed by Hayward and the area spectrum is studied using adiabatic invariant method. We have obtained the expression for temperature and heat capacity. The behavior of mass, temperature, heat capacity etc. are plotted and studied for different values of β. The heat capacity curve has a discontinuity at a certain value of entropy showing that a phase transition may occur. For lower values of entropy the heat capacity is positive showing that the black hole is thermodynamically stable. For higher values of entropy the heat capacity becomes negative and hence the black hole is thermodynamically unstable. The area spectrum and entropy spectrum of Hayward black hole are obtained, it is found that the spectra are quantized.

6 Conclusion

6.1 Results and Summary

In the thesis we have studied some aspects of black holes in f(R) theory of gravity and Einstein's General Theory of Relativity.

- In the first two chapters we have studied the properties of an extended black hole in f(R) modified gravity.
- The scattering of scalar field in this background space time studied in the first chapter shows that the extended black hole will scatter scalar waves and have a scattering cross section and applying tunneling mechanism we have obtained the Hawking temperature of this black hole.
- In the following chapter we have investigated the quasinormal properties of the extended black hole.
- We have studied the electromagnetic and scalar perturbations in this spacetime and find that the black hole frequencies are complex and show exponential damping indicating the black hole is stable against the perturbations.
- In the present study we show that not only the black holes exist in modified gravities but also they have similar properties of black hole space times in General Theory of Relativity.

$2+1$ black holes or three dimensional black holes are simplified examples of more complicated four dimensional black holes. Thus these models of black holes are known as toy models of black holes in four dimensional black holes in General theory of Relativity.

- We have studied some properties of these types of black holes in Einstein model (General Theory of Relativity). A three dimensional black hole known as MSW is taken for our study.

- The thermodynamics and spectroscopy of MSW black hole are studied and obtained the area spectrum which is equispaced and different thermodynamical properties are studied. The Dirac perturbation of this three dimensional black hole is studied and the resulting quasinormal spectrum of this three dimensional black hole is obtained.

- The different quasinormal frequencies are tabulated in tables and these values show an exponential damping of oscillations indicating the black hole is stable against the massless Dirac perturbation.

  In General Theory of Relativity almost all solutions contain singularities. The cosmological solution and different black hole solutions of Einstein's field equation contain singularities. The regular black hole solutions are those which are solutions of Einstein's equation and have no singularity at the origin. These solutions possess event horizon but have no central singularity. Such a solution was first put forward by Bardeen. Hayward proposed a similar regular black hole solution.

- We have studied the thermodynamics and spectroscopy of Hayward regular black holes.

- We have also obtained the different thermodynamic properties and the area spectrum.

- The area spectrum is a function of the horizon radius. The entropy-heat capacity curve has a discontinuity at some value of entropy showing a phase transition.

6.2 Towards future

There remains some of the areas in f(R) gravity which are not yet studied.

- The three dimensional black hole solution are not yet discovered in f(R) gravity.

- Unlike Einstein gravity the curvature at empty space does not vanish. Thus the vacuum solutions of f(R) gravity is a topic of interest.

- The extra curvature terms present in vacuum field equation of f(R) gravity may lead to solutions which are completely different from Schwarzschild solution in Einstein gravity.

- The thermodynamics of MSW black hole is studied in the thesis but the thermodynamic geometry is not studied.

- In the case of extended black hole quasinormal modes due to gravitational perturbation and Dirac perturbation are worth while to be studied.

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