PROPERTIES OF EQUILIBRIUM DISTRIBUTIONS OF ORDER n

Thesis Submitted to the

Cochin University of Science and Technology

for the Award of the Degree of

Doctor of Philosophy

under the Faculty of Science

by

Preeth M.

Department of Statistics
Cochin University of Science and Technology
Cochin - 682022
India

June 2014

CERTIFICATE

Certified that the thesis entitled **Properties of Equilibrium Distributions of Order n** is a bonafide record of work done by Mr. Preeth M. under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

Cochin- 22,
June 2014.

Dr. N. Unnikrishnan Nair
Supervising Teacher,
Department of Statistics,
Cochin University of
Science and Technology.

CERTIFICATE

Certified that all the relevant corrections and modifications suggested by the audience during pre-synopsis seminar and recommended by the Doctoral committee of the candidate has been incorporated in the thesis.

Cochin- 22,
June 2014.

Dr. N. Unnikrishnan Nair
Supervising Teacher,
Department of Statistics,
Cochin University of
Science and Technology.

DECLARATION

The thesis entitled **Properties of Equilibrium Distributions of Order n** contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Cochin- 22, **Preeth M.**
June 2014.

Acknowledgments

I hereby acknowledge the people whose involvement, direct or indirect, helped in this thesis seeing the light of the day.

I wish to express my deep sense of respect and gratitude to my supervising guide Dr. N. Unnikrishnan Nair, who has been a constant source of inspiration during the course of my Ph.D. work. He has always been patient towards my shortcomings and kept encouraging me to work in a better way. Without his help and support, perhaps, I would have not been able to write this thesis.

I take this opportunity to record my sincere respect and heartiest gratitude to Dr.Asha Gopalakrishnan, Professor and Head, Department of Statistics, CUSAT for her wholehearted support. I am obliged to Dr.K.R.Muraleedharan Nair, Professor and Formerly Dean, Faculty of Science, CUSAT, Dr.V.K.Ramachandran Nair, Professor, Dr.N.Balakrishna, Professor, Dr. P.G.Sankaran, Professor and Dr.S.M.Sunoj, Associate Professor, Department of Statistics, CUSAT for their valuable suggestions and help to complete this endeavour.

I offer my regards to all non-teaching staff of the Department of Statistics, CUSAT for their kind cooperation.

I owe a debt of gratitude to all my teachers, research scholars and friends who influenced me, a gratitude that can never be paid back in totality, and perhaps be paid back only in part, if I can impart some of what I have learned over the years.

I express my gratitude to all my family members who have provided me their helping hands in each and every walk of my life.

Above all, I bow before the grace of the Almighty.

Preeth M.

Contents

Chapter 1

Introduction

The notion of equilibrium distribution was introduced by Cox (1962) as the asymptotic distribution of the forward or backward recurrence times in a renewal process. Since then, in different contexts, it has been given various interpretations as the distribution of the product of two independent random variables, one of which is size biased and the other is uniform (Cox and Lewis (1966)), as a weighted distribution (Jain et al. (1989)) and also as a stationary excess operator (Pakes (1996)) . The technical details of these interpretations are explained in Chapter 2. A basic paper from Gupta (1979), containing several interesting properties of the equilibrium random variable, laid the foundation for subsequent researches on its role in reliability theory. During the past five decades the study of the theoretical properties of equilibrium distributions and their applications in other disciplines have kept it as a fertile area of research.

Several applications of equilibrium distributions include the areas of characterization of distributions (Gupta (1979), Hitha and Nair (1989), Nair and Hitha (1989), Gupta and Kirmani (1990), Sen and Khattree (1996), etc.), criteria for ageing (Deshpande et al. (1986), Averous and Meste (1989), Bon and Illayk (2002), Abouammoh and Qamber (2003), Nair and Sankaran (2010), etc.), formulation of maintenance policies (Bhattacharjee et al. (2000)), income analysis (Kleiber and Kotz (2003)), concepts of system improvements (Ebrahimi (1989)), life length studies (Bluementhal (1967), Scheaffer (1972)), estimation problems in survival studies (Zelen and

Feinleib (1969), Zelen (1974)), queues and insurance (Denuit et al. (1998), Kaas et al. (1994)), tests of hypothesis (Abouammoh et al. (1993, 2000)), system availability (Mi (1998)), moment inequalities (Mugdadi and Ahmad (2005)) and system reliability (Jean-Louis and Abbas (2005)). An important turning point in the development of this topic is the introduction of equilibrium distributions of higher orders (Harkness and Shantaram (1969)). Several applications of the higher order equilibrium distribution, that include new partial orderings and ageing concepts (Fagiuoli and Pellerey (1993)), moment properties (Nanda et al. (1996b)), interpretations of Bondesson's functions (Stein and Dattero (1999)), link between shapes of failure rates and mean residual life functions (Navarro and Hernandez (2004)), applications to reliability (Gupta (2007)) and characterizations (Pakes and Navarro (2007), etc.), have appeared in the literature. Moreover, Willmot et al. (2005) discussed the applications of higher order equilibrium distributions to insurance claim modelling.

From a perusal of the literature, it appears that equilibrium distribution and its properties are most studied in the context of reliability modeling and analysis. Various aspects investigated in this respect can be summarized as follows. The relationships between various concepts in reliability for the equilibrium distribution and the baseline distribution is most important among them. These in turn provide the basis of many characterizations of lifetime models. Secondly, most of the ageing concepts can be either interpreted or characterized by appropriate properties of the equilibrium distribution. Further, many new ageing concepts are evolved by comparing the ageing patterns of the baseline distribution and the corresponding equilibrium counterpart. Equilibrium distributions of higher orders have been proposed by a process of iteration that brings in new models whose characteristics can be expressed in terms of the original model. Many such relationships provide new methodology for establishing simple proofs in several cases and also enable statistical inference and analysis. The role of equilibrium distributions is fundamental in deriving proofs of properties of stochastic orders connecting reliability functions.

The present work is intended to discuss various properties and reliability aspects of higher order equilibrium distributions in continuous, discrete and multivariate cases, which contribute to the study on equilibrium distributions. At first, we have to study and consolidate the existing literature on equilibrium distributions. For this we need some basic concepts in reliability. These are being discussed in the next chapter, which include univariate discrete and continuous cases.

In Chapter 3, some identities connecting the failure rate functions and moments of residual life of the univariate, non-negative continuous equilibrium distributions of higher order and that of the baseline distribution are derived. These identities are then used to characterize the generalized Pareto model, mixture of exponentials and gamma distribution. An approach using the characteristic functions is also discussed with illustrations. Moreover, characterizations of ageing classes using stochastic orders has been discussed. Part of the results of this chapter have been reported in Nair and Preeth (2009).

Various properties of equilibrium distributions of non-negative discrete univariate random variables are discussed in Chapter 4. Then some characterizations of the geometric, Waring and negative hyper-geometric distributions are presented. Moreover, the ageing properties of the original distribution and nth order equilibrium distributions are compared. Part of the results of this chapter have been reported in Nair, Sankaran and Preeth (2012).

Chapter 5 is a continuation of Chapter 4. Here, several conditions, in terms of stochastic orders connecting the baseline and its equilibrium distributions are derived. These conditions can be used to redefine certain ageing notions. Then equilibrium distributions of two random variables are compared in terms of various stochastic orders that have implications in reliability applications.

In Chapter 6, we make two approaches to define multivariate equilibrium distributions of order n. Then various properties including characterizations of higher order equilibrium distributions are presented. Part of the results of this chapter have been reported in Nair and Preeth (2008).

The Thesis is concluded in Chapter 7. A discussion on further studies on equilibrium distributions is also made in this chapter.

Chapter 2

Equilibrium Distributions - A Review

2.1 Introduction

The study on equilibrium distributions has gained the interest of researchers from various fields ever since it was introduced by Cox (1962). Over the past fifty years, many results have been put forward by several researchers on this topic which sparked off applications to numerous areas such as characterization of distributions, criteria for ageing, formulation of maintenance policies, income analysis, insurance etc.. The objective of this chapter is to make a brief survey of the important results on equilibrium distributions that are relevant to the present study.

As mentioned in Chapter 1, since the applications of equilibrium distributions is oriented towards reliability modeling and analysis, we need some background materials from reliability theory for the development of our results in the subsequent chapters. These are being discussed in the next two sections.

2.2 Reliability concepts for continuous lifetime distributions

Let X be a non-negative random variable representing lifetime of a system or a device having absolutely continuous distribution function

$$F(x) = P(X \leq x),\ x > 0$$

with respect to the Lebesgue measure. Let $f(x)$ be the probability density function (*pdf*) of X. Then the survival function of X is denoted by $S(x)$ and is defined as

$$\begin{aligned} S(x) &= P(X > x), \\ &= 1 - F(x),\ x > 0. \end{aligned} \tag{2.1}$$

In other words, the survival function, $S(x)$ is the probability of a device or a system of components performing its purpose adequately for the period of time $(0, x)$ under the operating conditions encountered. In the context, involving lifetimes of systems or devices, it is referred to as the reliability function. $S(x)$ is a non-increasing continuous function with

$$\lim_{x \to 0} S(x) = 1$$

and

$$\lim_{x \to \infty} S(x) = 0.$$

2.2.1 Hazard rate

An important function that characterizes lifetime distributions is the hazard rate. It is denoted by $h(x)$ and is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X < x + \Delta x | X > x)}{\Delta x},\ x > 0.$$

The hazard rate specifies the instantaneous rate of failure of a device in the next small interval of time Δx, given that the device has survived up to time x. Thus $h(x)\Delta x$ is the approximate probability of failure in the interval $[x, x+\Delta x)$, given survival up to time x. In actuarial studies $h(x)$ is known under the name of *force of mortality*. The reciprocal of the hazard rate for the normal distribution is known as *Mill's ratio*. In extreme value theory, it is called *intensity function*. The hazard rate is also known as *conditional failure* rate in reliability and the *age-specific failure rate* in epidemiology. When X is absolutely continuous, the hazard rate is expressed as

$$\begin{aligned} h(x) &= \frac{f(x)}{S(x)}, \\ &= -\frac{d}{dx} \log S(x). \end{aligned} \tag{2.2}$$

Integrating (2.2) with respect to x, we obtain

$$S(x) = \exp\left[-\int_0^x h(u)\ du\right], \tag{2.3}$$

which shows that $h(x)$ characterizes the distribution of X. The *pdf* of X can also be represented as

$$f(x) = h(x)\ \exp\left[-\int_0^x h(u)\ du\right]. \tag{2.4}$$

2.2.2 Mean residual life function

Mean residual life function plays an important role in reliability, survival analysis and various other areas. It is often referred as *life expectancy* or *expectation of life* in demography. The mean residual life function (*mrl*) of X, $m(x)$, is defined as the mean of the the residual life $(X - x|X > x)$. More explicitly,

$$\begin{aligned} m(x) &= E(X - x|X > x), \\ &= \frac{1}{S(x)} \int_x^\infty (u - x)\ f(u)\ du, \\ &= \frac{1}{S(x)} \int_x^\infty S(u)\ du, \end{aligned} \tag{2.5}$$

which characterizes the distribution of X through (Cox (1962))

$$S(x) = \frac{m(0)}{m(x)} \exp\left[- \int_0^x \frac{1}{m(u)} du \right]. \tag{2.6}$$

Differentiating (2.6) with respect to x, we obtain the density function that is expressed in terms of the mean residual life function

$$f(x) = \frac{\mu \, \left(\frac{d}{dx} m(x) + 1 \right)}{m^2(x)} \exp\left[- \int_0^x \frac{1}{m(u)} du \right]. \tag{2.7}$$

Mean residual life summarizes the entire residual life distribution, whereas the hazard rate relates only to the risk of immediate failure. The hazard rate and mrl of X are linked through the relation (Muth (1977))

$$h(x) = \frac{1 + \frac{d}{dx} m(x)}{m(x)}. \tag{2.8}$$

Moreover,

$$\begin{aligned} \mu &= \lim_{x \to 0} m(x), \\ &= \int_0^\infty S(u) \, du. \end{aligned}$$

Calabria and Pulcini (1987) established that

$$\lim_{x \to \infty} m(x) = \lim_{x \to \infty} \frac{1}{h(x)},$$

provided the latter limit exists, finite and strictly positive. They also deduced that

$$\lim_{x \to \infty} \frac{d}{dx} m(x) = 0, \tag{2.9}$$

or equivalently, that

$$\lim_{x \to \infty} m(x) \; h(x) = 1.$$

It is to be noted that the $m(x)$ is constant and $m(x)\; h(x) = 1$ for the exponential distribution and conversely.

2.2.3 Variance residual life function

Another function which has also generated interest in the recent years is the variance residual life function. It is denoted by $\sigma^2(x)$ and is defined as

$$\begin{aligned}\sigma^2(x) &= E\left[(X-x)^2|X>x\right] - m^2(x)\\ &= \frac{1}{S(x)}\int_x^{\infty}(u-x)^2 f(u)\ du - m^2(x),\\ &= \frac{2}{S(x)}\int_x^{\infty}(u-x)S(u)\ du - m^2(x),\\ &= \frac{2}{S(x)}\int_x^{\infty}\int_t^{\infty} S(u)\ du\ dt - m^2(x),\end{aligned} \tag{2.10}$$

obtained by integrating by parts on each of the steps. Abouammoh et al. (1990) showed that the variance residual life together with mean residual life function characterizes the distribution of X through the identity

$$S(x) = \exp\left[-\int_0^x \frac{\frac{d}{du}\sigma^2(u)}{\sigma^2(u)-m^2(u)}\ du\right]. \tag{2.11}$$

2.2.4 Stochastic orders

Stochastic orders have been used during the last few decades in many diverse areas of probability and statistics such as reliability theory, queuing theory, survival analysis, biology, economics, insurance, actuarial science, operations research, and management science. Stochastic orders are used to compare distributions in terms of their characteristics. Definitions of the stochastic orders given below, unless otherwise specified, can be seen in Shaked and Shanthikumar (2007).

Let X be a random variable having the characteristics discussed above and Y be another non-negative random variable with *pdf* $g(x)$, survival function $T(x)$, finite mean λ, hazard rate function $k(x)$ and mean residual life function $r(x)$. In the sequel, the phrase *for all* x means that for all x in the union of supports of X and Y; $\uparrow x$ ($\downarrow x$) means that it is increasing (decreasing) in $x > 0$. In the present work, increasing (decreasing) is used in the week sense, that is, non-decreasing (non-increasing).

Definition 2.2.1. *X is said to be smaller than Y in the usual stochastic order ($X \leq_{st} Y$) if and only if $S(x) \leq T(x)$ for all x.*

Definition 2.2.2. *X is said to be smaller than Y in Laplace transform order ($X \leq_{Lt} Y$) if and only if*

$$E(e^{-sX}) \geq E(e^{-sY}), \text{ for all } s > 0.$$

This is equivalent to

$$\int_0^\infty e^{-su} S(u)\, du \leq \int_0^\infty e^{-su} T(u)\, du.$$

Definition 2.2.3. *X is said to be smaller than Y in hazard rate order ($X \leq_{hr} Y$) if and only if $h(x) \geq k(x)$ for all x or equivalently,*

$$\frac{T(x)}{S(x)} \uparrow x$$

or equivalently for absolutely continuous distributions (Shaked and Shanthikumar (2007, p. 17)),

$$\frac{f(x)}{S(x+y)} \geq \frac{g(x)}{T(x+y)}, \text{ for all } x,\ y > 0.$$

These are again equivalent to any of the following (Mukherjee and Chatterjee (1992))

$$(X - x | X > x) \leq_{st} (Y - x | Y > x),$$

$$(X - x | X > x) \leq_{hr} (Y - x | Y > x)$$

and (Theorem 5.A.22 of Shaked and Shanthikumar (2007))

$$(X - x | X > x) \leq_{Lt} (Y - x | Y > x).$$

Definition 2.2.4. *X is said to be smaller than Y in likelihood ratio order ($X \leq_{lr} Y$) if and only if*

$$\frac{g(x)}{f(x)} \uparrow x.$$

This is equivalent to (Hu et al. (2001))

$$(X - x | X > x) \leq_{lr} (Y - x | Y > x).$$

Definition 2.2.5. *X is said to be smaller than Y in harmonic mean residual life order ($X \leq_{hmrl} Y$) if and only if*

$$\left[\frac{1}{x}\int_0^x \frac{1}{m(u)}\,du\right]^{-1} \leq \left[\frac{1}{x}\int_0^x \frac{1}{r(u)}\,du\right]^{-1}$$

for all x.

Definition 2.2.6. *X is said to be smaller than Y in variance residual life order ($X \leq_{vrl} Y$) if and only if $\sigma_X^2(x) \leq \sigma_Y^2(x)$ for all x, where $\sigma_X^2(x)$ and $\sigma_Y^2(x)$ are the variance residual life functions of X and Y respectively.* This is equivalent to (Fagiuoli and Pellerey (1993))

$$\frac{\int_x^\infty \int_t^\infty S(u)\,du\,dt}{\int_x^\infty \int_t^\infty T(u)\,du\,dt} \downarrow x$$

and (Hu et al. (2001)) $(X - x|X > x) \leq_{vrl} (Y - x|Y > x)$.

Definition 2.2.7. *X is said to be smaller than Y in increasing convex order ($X \leq_{icx} Y$) if and only if*

$$\int_x^\infty S(u)\,du \leq \int_x^\infty T(u)\,du \text{ for all } x.$$

Definition 2.2.8. *X is said to be smaller than Y in increasing concave order ($X \leq_{icv} Y$) if and only if*

$$\int_0^x S(u)\,du \leq \int_0^x T(u)\,du \text{ for all } x.$$

Definition 2.2.9. *X is said to be smaller than Y in mean residual life order ($X \leq_{mrl} Y$) if and only if $m(x) \leq r(x)$ for all x.* This is equivalent to any one of the following

$$\frac{\int_x^\infty T(u)\,du}{\int_x^\infty S(u)\,du} \uparrow x,$$

$$\frac{1}{S(x)}\int_{x+y}^\infty S(u)\,du \leq \frac{1}{T(x)}\int_{x+y}^\infty T(u)\,du, \text{ for all } x, y > 0$$

due to Shaked and Shanthikumar (2007, p. 82),

$$(X - x|X > x) \leq_{hmrl} (Y - x|Y > x),$$

$$(X - x|X > x) \leq_{icx} (Y - x|Y > x)$$

and

$$(X - x|X > x) \leq_{mrl} (Y - x|Y > x).$$

The last two relations are due to Hu et al. (2001).

Definition 2.2.10. *X is said to be smaller than Y in stop loss moment order of degree n ($X \leq_{n-sl} Y$) if and only if*

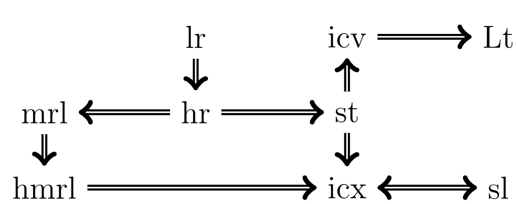
$$E\left[(X - x)_+^n\right] \leq E\left[(Y - x)_+^n\right],$$

for all x. It may be noted that $1 - sl$ is the stop loss function order,

$$E\left[(X - x)_+\right] \leq E\left[(Y - x)_+\right],$$

where $(X - x)_+ = \max(X - x, 0)$.

The following implications exist among the various stochastic orders discussed above which can be seen in Hu et al. (2001) and Shaked and Shanthikumar (2007).



2.2.5 Ageing concepts

By the term ageing of a device, we mean the phenomenon whereby its residual life is affected by its age in some probability sense. This description covers the states, positive ageing, negative ageing and non-ageing a device can undergo. Non-ageing means that the age of a component has no effect on the distribution of residual lifetime of the component. Positive ageing (also known as averse ageing) describes the situation where residual lifetime tends to decrease with increasing age of a component. On the other hand, negative ageing has an opposite effect on the residual lifetime. Concepts of ageing describe how a component or system improves or deteriorates with age. Many classes of life distributions are categorized or defined in the literature according to their ageing properties. The following are some of the concepts discussed in Barlow and Proschan (1981) and Lai and Xie (2006) and others specifically mentioned.

Definition 2.2.11. *The distribution of X is said to be increasing $(decreasing)$ failure rate (IFR/DFR) if and only if*

$$\frac{S(x+t)}{S(x)} \downarrow x \ (\uparrow x),$$

for $t > 0$, which is equivalent to $-\log S(x)$ is convex $(concave)$ or equivalent to $h(x)$ is being increasing $(decreasing)$ for each $x \geq 0$. This is again equivalent to any one of the following (Shaked and Shanthikumar (2007))

$$(X - y|X > y) \leq_{st} (\geq_{st}) \ (X - x|X > x),$$

$$(X - y|X > y) \leq_{hr} (\geq_{hr}) \ (X - x|X > x),$$

$$(X - y|X > y) \leq_{icv} (\geq_{icv}) \ (X - x|X > x)$$

and

$$(X - y|X > y) \leq_{Lt} (\geq_{Lt}) \ (X - x|X > x),$$

for all $y \geq x > 0$.

Definition 2.2.12. *The distribution of X is said to be increasing $(decreasing)$ failure rate average $(IFRA \ / \ DFRA)$ if and only if*

$$-\frac{1}{x}\log S(x) \uparrow x \ (\downarrow x).$$

This is equivalent to

$$\frac{1}{x}\int_0^x h(u) \ du \uparrow x \ (\downarrow x).$$

Definition 2.2.13. *The distribution of X is said to be increasing $(decreasing)$ failure* *of second order (IFR(2) /DFR(2)) if and only if for* $t \geq 0$

$$\frac{1}{S(x)} \int_x^{x+t} S(u)\ du \downarrow x\ (\uparrow x).$$

Definition 2.2.14. *The distribution of* X *is said to be bathtub (BT) shaped failure rate if and only if there exists* $0 < x_1 \leq x_2 < \infty$ *such that*

1. $h(x)$ *is strictly decreasing in* $0 \leq x \leq x_1$,
2. $h(x)$ *is a constant for* $x_1 \leq x \leq x_2$ *and*
3. $h(x)$ *is strictly increasing in* $x_2 \leq x < \infty$.

Definition 2.2.15. *The distribution of* X *is said to be upside down bathtub (UBT) shaped failure rate if and only if there exists* $0 < x_1 \leq x_2 < \infty$ *such that*

1. $h(x)$ *is strictly increasing in* $0 \leq x \leq x_1$,
2. $h(x)$ *is a constant for* $x_1 \leq x \leq x_2$ *and*
3. $h(x)$ *is strictly decreasing in* $x_2 \leq x < \infty$.

Definition 2.2.16. *The distribution of* X *is said to be decreasing (increasing) mean residual life (DMRL / IMRL) if and only if* $m(x)$ *is decreasing (increasing) in* $x \geq 0$ (Bryson and Siddiqui (1969)). In other words, the older the device is, the smaller (larger) is its mean residual life. This is equivalent to any one of the following (Shaked and Shanthikumar (2007))

$$(X - y|X > y) \leq_{mrl} (\geq_{mrl})\ (X - x|X > x),$$

$$(X - y|X > y) \leq_{hmrl} (\geq_{hmrl})\ (X - x|X > x)$$

and

$$(X - y|X > y) \leq_{icx} (\geq_{icx})\ (X - x|X > x),$$

for all $y \geq x > 0$.

Definition 2.2.17. *The distribution of* X *is said to be new better (worse) than used (NBU / NWU) if and only if* $S(x + u) \leq (\geq) S(x)S(u)$*, for all* $x, u \geq 0$. This means that a device of any particular age has a stochastically smaller remaining lifetime than a new device (Barlow and Proschan (1981)).This is equivalent to (Shaked and Shanthikumar (2007))

$$(X - x|X > x) \leq_{st} (\geq_{st})\ X.$$

Definition 2.2.18. *The distribution of* X *is said to be new better (worse) than used in expectation (NBUE / NWUE) if and only if* $m(x) \leq (\geq)\mu$. This means that a device of any particular age has a smaller (larger) mean remaining lifetime than a new device (Barlow and Proschan (1981)). This is equivalent to $X \leq_{hmrl} X + Y$, for any non-negative random variable Y independent of X, having finite positive mean (Shaked and Shanthikumar (2007)).

Definition 2.2.19. *The distribution of* X *is said to be new better (worse) than used in convex ordering (NBUCX / NWUCX) if and only if*

$$\frac{1}{S(x)} \int_y^\infty S(x+u)\ du \leq (\geq) \int_y^\infty S(u)\ du,$$

for all $x, y \geq 0$ (Cao and Wang (1991)). This is equivalent to (Fagiuoli and Pellerey (1993))

$$(X - x|X > x) \leq_{icx} (\geq_{icx}) X.$$

Definition 2.2.20. *The distribution of* X *is said to be new better (worse) than used in concave ordering (NBUCV / NWUCV) if and only if*

$$\int_0^x S(u+y)\ du \leq (\geq) S(y) \int_0^x S(u)\ du,$$

for $x, y \geq 0$ (Cao and Wang (1991)). This is equivalent to (Fagiuoli and Pellerey (1993))

$$(X - x|X > x) \leq_{icv} (\geq_{icv}) X.$$

NBUCV is also called new better than used of second order (*NBU* (2)).

Definition 2.2.21. *The distribution of* X *is said to be used better (worse) than aged (UBA / UWA) if and only if for all* $x, t \geq 0$,

$$S(x+t) \geq\ (\leq)\ S(x)\ \exp\left[\frac{-t}{m(\infty)}\right].$$

Definition 2.2.22. *The distribution of X is said to be used better (worse) than aged in expectation (UBAE / UWAE) if and only if $m(x) \geq (\leq)\ m(\infty)$ for all $x \geq 0$, provided $0 < m(\infty) < \infty$.*

Definition 2.2.23. *The distribution of X is said to be new better (worse) than used in failure rate (NBUFR / NWUFR) if and only if $h(x) > (<) h(0)$ for $x \geq 0$* (Deshpande et al. (1986)).

Definition 2.2.24. *The distribution of X is said to be new better than used in failure rate average (NBAFR or NBUFRA) if*

$$h(0) \leq \frac{1}{x} \int_0^x h(u)\ du,$$

for all $x \geq 0$ (Loh (1984)).

Definition 2.2.25. *The distribution of X is said to be decreasing (increasing) mean residual life in harmonic average (DMRLHA / IMRLHA) if and only if*

$$\left[\frac{1}{x} \int_0^x \frac{1}{m(u)}\ du \right]^{-1}$$

is decreasing (increasing) in $x \geq 0$ (Deshpande et al. (1986)).

Definition 2.2.26. *The distribution of X is said to be decreasing (increasing) variance of residual life (DVRL / IVRL) if and only if $\sigma^2(x)$ is decreasing (increasing) in $x \geq 0$* (Launer (1984)). This is equivalent to

$$(X - y | X > y) \leq_{vrl} (\geq_{vrl})(X - x | X > x),$$

for $y \geq x \geq 0$ (Hu et al. (2001)).

Definition 2.2.27. *The distribution of X is said to be harmonically new better (worse) than used (HNBUE / HNWUE) if and only if*

$$\int_x^\infty S(u)\ du \leq (\geq) \mu e^{\frac{-x}{\mu}},$$

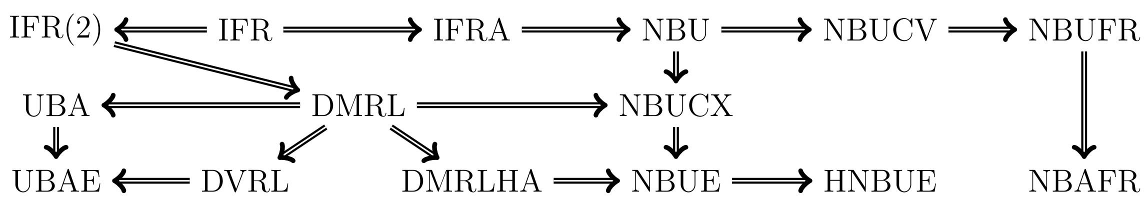
for all $x \geq 0$.

Definition 2.2.28. *The distribution of X is said to be generalized increasing (decreasing) mean residual life (GIMRL / GDMRL) if and only if for all $x \geq 0$*

$$\frac{1}{S(y+x)} \int_y^{\infty} S(u)\, du$$

is increasing (decreasing) in $y \geq 0$ (Bon and Illayk, 2002).

The following implications exist among the ageing classes discussed above.



2.3 Concepts in discrete time

The work on reliability theory when X is discrete is much less voluminous than in the continuous case, as most of the lifetimes discussed in literature are continuous in nature. However, discrete data can also arise naturally in a variety of situations like the following:-

(a) Measuring devices are inaccurate so that the number of items failed are collected at units of intervals of times.

(b) The observations are the number of cycles completed prior to failure. For example, the number of copies produced by a copier before it fails or the number of miles for which a car tyre was operational.

(c) When the continuous data are grouped.

Let X be non-negative integer valued random variable, representing the failure time (*time* is a general term used to specify a realization of X, irrespective of whether it is the actual completed units of time or number of cycles prior to failure etc.) with probability mass function (pmf)

$$f(x) = P(X = x),\ x = 0, 1, 2, \ldots,$$

distribution funtion

$$F(x) = P(X \leq x)$$

and survival function

$$\begin{aligned} S(x) &= 1 - F(x) \\ &= P(X > x) \\ &= \sum_{u=x+1}^{\infty} f(u). \end{aligned}$$

It is to be noted that

$$f(x) = S(x-1) - S(x). \tag{2.12}$$

Let $\mu(< \infty)$ be the mean of X, then

$$\mu = \sum_{u=0}^{\infty} S(u). \tag{2.13}$$

The hazard rate function of X is defined as

$$\begin{aligned} h(x) &= P(X = x | X \geq x), \\ &= \frac{f(x)}{S(x-1)}, \\ &= 1 - \frac{S(x)}{S(x-1)},\ S(x-1) > 0. \end{aligned} \tag{2.14}$$

Notice that $h(x)$ determines the distribution of X uniquely through (Gupta (1979))

$$S(x) = \prod_{u=0}^{x} [1 - h(u)], \tag{2.15}$$

or

$$f(x) = h(x) \prod_{u=0}^{x-1} [1 - h(u)].$$

Another related concept is the mean residual life function, which is defined as

$$\begin{aligned} m(x) &= E(X-x|X>x), \\ &= \frac{1}{S(x)}\sum_{u=x+1}^{\infty}(u-x)f(u), \\ &= \frac{1}{S(x)}\sum_{u=x}^{\infty}S(u),\ \ S(x)>0, \end{aligned} \tag{2.16}$$

that satisfies the identity (Nair and Hitha (1989))

$$h(x+1)=\frac{m(x+1)-m(x)+1}{m(x+1)}. \tag{2.17}$$

Further,

$$\mu = S(0)m(0).$$

Gupta (1979) showed that $m(x)$ also characterizes the distribution of X through

$$S(x)=S(0)\frac{m(0)}{m(x)}\prod_{u=0}^{x-1}\left[1-\frac{1}{m(u)}\right]. \tag{2.18}$$

The variance residual life is

$$\begin{aligned} V(x) &= E\left[(X-x)^2|X>x\right]-m^2(x), \\ &= \frac{1}{S(x)}\sum_{u=x+1}^{\infty}(u-x)^2f(u)-m^2(x). \end{aligned} \tag{2.19}$$

The above definition verifies that

$$\begin{aligned} &[V(x)+m^2(x)]S(x)-[V(x+1)+m^2(x+1)]S(x+1) \\ &\qquad =\sum_{u=x+1}^{\infty}(u-x)^2f(u)-\sum_{u=x+2}^{\infty}(u-x-1)^2f(u), \\ &\qquad =\sum_{u=x+1}^{\infty}(u-x-1)f(u), \\ &\qquad =2S(x)m(x)-S(x), \end{aligned}$$

which, on division by $S(x)$, gives

$$V(x) - V(x+1) = h(x+1)\ [m(x+1)\ (m(x)-1) - V(x+1)]. \tag{2.20}$$

Substituting (2.14) and rearranging terms,

$$S(x+1) = S(x)\left[1 - \frac{V(x) - V(x+1)}{m(x+1)\ (m(x)-1) - V(x+1)}\right].$$

Hence by iteration,

$$S(x+1) = \prod_{u=0}^{x}\left[1 - \frac{V(u) - V(u+1)}{m(x+1)\ (m(x)-1) - V(u+1)}\right]. \tag{2.21}$$

Thus (2.21) confirms that $V(x)$ together with $m(x)$, characterizes the distribution of X. Further, the r^{th} factorial stop loss moment (partial moment) is defined as

$$\begin{aligned} \alpha_r(x) &= E(X-x)_+^{(r)}, \\ &= E\left((X-x)_+ \cdots (X-x-r+1)_+\right), \end{aligned} \tag{2.22}$$

where

$$(X-x)_+ = \max(X-x, 0)$$

and

$$n^{(r)} = n(n-1)\cdots(n-r+1)$$

is the descending factorial of order r. The properties of $\alpha_r(x)$, recurrence relations connecting them for various discrete families and some applications to reliability studies are reported in Nair *et al.* (2000). We also note the identities

$$m(x) = \frac{\alpha_1(x)}{\alpha_1(x) - \alpha_1(x+1)}$$

and

$$1 - h(x) = \frac{\alpha_1(x+1) - \alpha_1(x+2)}{\alpha_1(x) - \alpha_1(x+1)}.$$

2.3.1 Stochastic orders

In addition to X, let Y be another non-negative discrete random variable with pmf $g(x)$, survival function $T(x)$, mean $\lambda < \infty$, hazard rate function $k(x)$ and mean residual life function $r(x)$. In studying the partial orderings between X and Y, we need the following definitions. We use the notation $\uparrow x$ ($\downarrow x$) for *increasing* (*decreasing*) *in* $x \in \mathcal{N} = \{0, 1, 2, \dots\}$. The conditions rendering each partial orders have to be valid for all $x = 0, 1, 2, \dots$, unless otherwise stated. For detailed discussion of these stochastic orders we refer to Shaked and Shanthikumar (2007).

Definition 2.3.1. *X is said to be smaller than Y in stochastic order ($X \leq_{st} Y$), if and only if $S(x-1) \leq T(x-1)$.*

Definition 2.3.2. *X is said to be smaller than Y in hazard rate order ($X \leq_{hr} Y$), if and only if $h(x) \geq k(x)$.* This is equivalent to

$$\frac{S(x)}{T(x)} \downarrow x.$$

Definition 2.3.3. *X is said to be smaller than Y in likelihood ratio order ($X \leq_{lr} Y$), if and only if $f(x)/g(x)$ is non-increasing in x over the union of supports of X and Y.*

Definition 2.3.4. *X is said to be smaller than Y in mean residual life order ($X \leq_{mrl}$ Y, if and only if $m(x) \leq r(x)$.* This is equivalent to

$$\frac{\sum_{u=x}^{\infty} S(u)}{\sum_{u=x}^{\infty} T(u)} \downarrow x.$$

Definition 2.3.5. *X is said to be smaller than Y in harmonic mean residual life order ($X \leq_{hmrl} Y$), if and only if*

$$\frac{1}{\mu}\sum_{u=x}^{\infty} S(u) \leq \frac{1}{\lambda}\sum_{u=x}^{\infty} T(u).$$

Definition 2.3.6. *X is said to be smaller than Y in increasing convex order ($X \leq_{icx}$*

Y), if and only if

$$\sum_{u=x}^{\infty} S(u) \leq \sum_{u=x}^{\infty} T(u).$$

Definition 2.3.7. *X is said to be smaller than Y in probability generating function order ($X \leq_{pgf} Y$), if and only if $E(s^X) \geq E(s^Y)$ for all $s \in (0,1)$.* This is equivalent to

$$\sum_{u=0}^{\infty} s^u S(u) \leq \sum_{u=0}^{\infty} s^u T(u).$$

Definition 2.3.8. *X is said to be smaller than Y in moment generating function order ($X \leq_{mgf} Y$), if and only if $E(e^{tX}) \leq E(e^{tY})$, for all $t > 0$ provided both expectations are finite.*

2.3.2 Ageing classes

The ageing classes in the discrete case are defined analogously to the continuous case. From Lai and Xie (2006), we give the following definitions.

Definition 2.3.9. *The distribution of X is said to be increasing (decreasing) failure rate (IFR/DFR) if and only if $h(x)$ is increasing (decreasing) in x.* This is equivalent to (Barlow and Proschan (1981))

$$\frac{S(x+1)}{S(x)} \downarrow x \ (\uparrow x).$$

Definition 2.3.10. *X is said to be $IFR(2)$ if and only if for all $y \in \mathcal{N}$*

$$a_y(x) = \frac{1}{S(x)} \sum_{u=x}^{x+y} S(u) \downarrow x.$$

Definition 2.3.11. *The distribution of X is said to be NBU (new better than used) if and only if and only if $S(x+y) \leq S(x)S(y)$ for all $x, y \in \mathcal{N}$.*

Definition 2.3.12. *The distribution of X is said to be new better than used in expectation (NBUE) if and only if $E(X - x|X > x) \leq E(X)$ for all x.* This is equivalent to $m(x) \leq \mu$. *The distribution of X is said to be new better than used in expectation*

$(NBUE)$ *if and only if* $E(X-x|X>x) \leq E(X)$ *for all* x. This is equivalent to $m(x) \leq \mu$.

Definition 2.3.13. *X is said to be new better (worse) than used in failure rate-NBUFR (NWUFR) if* $h(x) \geq (\leq) h(0)$ *for every* $x \geq 0$.

Definition 2.3.14. *The distribution of X is said to be decreasing (increasing) mean residual life (DMRL/IMRL) if and only if $m(x)$ is decreasing (increasing) in x for all x.*

Definition 2.3.15. *The distribution of X is said to be decreasing (increasing) variance residual life (DVRL/IVRL) if and only if $V(x)$ is decreasing (increasing) in x for all x.*

Fagiuoli and Pellerey (1994) discussed more advanced concepts of ageing.

Definition 2.3.16. *We say that the distribution of X is $IFR(2)$ $(DFR(2))$ if for all* $t \geq 0$

$$\sum_{u=x-1}^{x-1+t} \frac{S(u)}{S(x)} \downarrow (\uparrow) x.$$

Definition 2.3.17. *The distribution of X is $NBU(2)$ if for all t and* $x \geq 0$

$$S(x-1)\sum_{u=0}^{t} S(u-1) \geq \sum_{u=x}^{x+t} S(u-1).$$

They have also proved that

$$NBU(2) \Longrightarrow NBUFR.$$

2.4 Origin and interpretations of equilibrium distributions

Suppose that we have a set of components with continuous, independent and identically distributed life times $L_1, L_2, L_3, \ldots$ with *pdf* $f(x)$, such that $f(x) \to 0$ as

$x \to \infty$, survival function $S(x)$ and finite mean μ and that the first component is replaced upon failure by second, second by third and so on. Then the sequence of points $S_n = L_1 + L_2 + \ldots + L_n$ constitute a renewal process. At a fixed time $t > 0$, if $N(t) = \sup\{n : \; S_n \leq t\}$, the random variables $U_t = t - S_{N(t)}$ and $V_t = S_{N(t)+1} - t$ are called the age and residual life of the component working at the time t. Then both the age U_t and residual life V_t of the component in use at the time t, have the same asymptotic distribution with *pdf*

$$f_1(x) = \frac{S(x)}{\mu}, \; x \geq 0, \tag{2.23}$$

as $t \to \infty$ (Cox (1962)). This distribution is called the *equilibrium distribution* or *stationary excess distribution* of the lifetime of the component. This distribution is always J shaped and has a unique mode at $x = 0$. The exponential distribution has been shown to be the only one for which the distribution of failure time coincides with its equilibrium distribution.

An alternative interpretation of the equilibrium distribution was also given again by Cox (1962). In the above discussed process, even if the life times $L_1, L_2, L_3, \ldots$ are not independent, the residual life of a component has the equilibrium distribution (2.23), at a randomly chosen time point.

If X denote the life time of a component or system, Cox and Lewis (1966) showed that the distribution (2.23) can also be obtained as the distribution of the product of two independent random variables, L and U, where L is a random variable corresponding to the length biased distribution of X with *pdf*

$$\frac{x \; f(x)}{\mu}, \; x \geq 0 \tag{2.24}$$

and U is distributed as uniform over $(0, 1)$. This shows that the random variable corresponding to (2.23), say X_1, is smaller than L in the usual stochastic order. Length biased distribution is a special case $w(x) = x$, of the weighted distribution of X having *pdf*

$$f_W(x) = \frac{w(x) \; f(x)}{E(w(X))}, \; x > 0, \tag{2.25}$$

where $w(x) > 0$ and $E(w(X)) < \infty$. Weighted distributions arise when the observations generated from a stochastic process are recorded according to the weight function, $w(x)$ (Rao (1965)). Brown (2006) established that L is identically distributed as $(X|X > X_1)$.

The equilibrium distribution is also a special case of the weighted distribution with the weight function,

$$w(x) = \frac{1}{h(x)},$$

which was noted independently by Jain et al. (1989) and Gupta and Kirmani (1990).

Pakes (1996) viewed the distribution function of (2.23) as the image of an operator called stationary excess operator ($\mathscr{S}$) on distribution functions, which is defined as

$$\begin{aligned} \mathscr{S}(F(x)) &= \frac{1}{\mu}\int_0^x (1 - F(u))\, du, \\ &= \frac{1}{\mu}\int_0^x S(u)\, du. \end{aligned} \tag{2.26}$$

2.4.1 Higher order equilibrium distributions

For a non-negative random variable X with survival function $S(x)$, Harkness and Shantaram (1969) extended (2.23) by defining the equilibrium distribution of order n recursively through the sequence $(1 - S_n)$ of absolutely continuous distribution functions

$$1 - S_n(x) = \begin{cases} \frac{1}{\mu_{n-1}}\int_0^x S_{n-1}(u)\, du, & x > 0,\ n = 1, 2, 3, \dots \\ 0, & x \leq 0, \end{cases} \tag{2.27}$$

where

$$\mu_n = \int_0^\infty S_n(u)\, du \ (< \infty)$$

is the mean of equilibrium distribution of order n and $S_n(x)$ is the survival function of the equilibrium distribution of order n. Note that $S_0(x) = S(x)$ is the survival function of X, $\mu_0 = \mu = E(X)$ and that $S_1(x)$ is the asymptotic survival function of both U_t and V_t and consequently μ_1 is their mean value. In the sequel, we denote by X_n, the random variable with distribution (2.27), so that $X_0 = X$. If the distribution function F is finite on $[a,b]$, where $b = \inf\{x|S(x) = 0\}$, Harkness and Shantaram (1969) showed that the limiting distribution is

$$\lim_{n\to\infty} S_n\left(\frac{x}{n}\right) = e^{-\frac{x}{b}},\ x \geq 0,$$

the exponential distribution.

Pakes (1996) also obtained (2.27) by extending the stationary excess operator, $\mathscr{S}$ defined in (2.26) to its n-fold iterates as

$$\mathscr{S}_n(1 - S(x)) = \mathscr{S}(\mathscr{S}_{n-1}(1 - S(x))),\ n = 1, 2, 3, \ldots, \tag{2.28}$$

where $\mathscr{S}_0(1 - S(x)) = 1 - S(x)$ (Pakes and Navarro (2007)).

Pakes (1996) further established that X_n can be represented as a product of two independent random variables as, $X_n \equiv BL_n$, where B is random variable having beta law, $\beta(1, n)$ with density function

$$g(t) = (n + 1)\ (1 - t)^n,\ 0 < t < 1$$

and L_n is the weighted version of the random variable X having weight function $w(x) = x^n$. Later on, Brown (2006) obtained the same result by stating that B is the minimum of n independent and identically distributed uniform random variables over $(0, 1)$.

Pakes and Navarro (2007) showed that we can get the lower order equilibrium distributions from the higher orders by using the inverse stationary excess operator defined as follows:

Definition 2.4.1. *The inverse stationary excess operator defined as*

$$\mathscr{S}_{-n}(1 - S(x)) = 1 - \frac{\mathcal{D}^n(1 - S(x))}{\mathcal{D}^n(1 - S(0))},$$

over a class $\Delta_n,\ n = 1, 2, , 3, \ldots$ *of distribution functions satisfy the following conditions:*

1. $\mathcal{D}^n(1 - S(x))$ *exists for* $x > 0$

2. $0 < |\mathcal{D}^n(1 - S(0))| < \infty$ *and*

3. $\mathcal{D}^n(1 - S(x))/\mathcal{D}^n(1 - S(0)) \to 0$ *as* $x \to \infty$

where $\mathcal{D}^n(1 - S(x))$ *is the nth derivative of* $1 - S(x)$ *at* x.

Thus by using the inverse operator, we get the distribution of X from that of X_n as

$$\mathscr{S}_{-n}\mathscr{S}_n(1 - S(x)) = \mathscr{S}_n\mathscr{S}_{-n}(1 - S(x)) = 1 - S(x),$$

provided $1 - S_n(x) \in \Delta_n$.

2.4.2 Discrete case

Let us consider a sequence of repeated trials with possible outcomes E_j $(j = 1, 2, \ldots)$, which may not be independent and it is possible to continue the trials indefinitely. Let Ψ be an attribute of finite sequences; that is we suppose that it is uniquely determined whether a sequence $(E_{j_1}, \ldots, E_{j_n})$ has, or has not, the characteristic Ψ and we say that Ψ occurs at the nth place in the sequence $E_{j_1}, E_{j_2}, \ldots$. Suppose further that Ψ is repetitive with the repeated trials; but the number of trials in between two successive occurrences of Ψ is not constant and

$$\sum_{u=0}^{\infty} f(u) = 1,$$

where $f(x) = P(\Psi \text{ occurs for the first time at the } x\text{th trial})$. Let $\tau \geq 0$ be an integer and we start to observe the process after the τth trial. Then Feller (1957) deduced that the residual waiting time for the first occurrence of Ψ after the τth trial takes place at the $(\tau + x)$th trial is distributed with *pmf*

$$f_1(x) = \frac{S(x)}{\mu}, \; x = 0, 1, 2, \ldots, \tag{2.29}$$

as $\tau \to \infty$, where

$$S(x) = \sum_{u=x+1}^{\infty} f(u)$$

and

$$\mu = \sum_{u=0}^{\infty} S(u) < \infty.$$

The distribution (2.29) is called the discrete equilibrium distribution or stationary excess distribution and the corresponding random variable is denoted by X_1. Also, the above discussed process is called stationary or equilibrium point process.

Assuming that the probability of a sampled component possesses a certain life length X is proportional to x. Then the distribution of the total life length L of the sampled component is

$$P(L = x) = \frac{x\ P(X = x)}{\mu}$$

when the conditional distribution of the equilibrium random variable X_1 is uniform over $[0, x]$. Hence the conditional distribution of X is given by (2.29).

Whitt (1985) viewed (2.29) as the image of operator called discrete stationary excess operator defined by

$$\mathscr{S}(f(x)) = \frac{S(x)}{\mu},\ x = 0, 1, 2, \ldots. \tag{2.30}$$

2.4.3 Higher order discrete equilibrium distributions

Consider a sequence of independent and identically distributed stationary point processes with stationary excess distribution (2.29). Let a new process be formed by observing one stationary point process until a point occurs in it, then observing a second point process until a point occurs in it, and so forth. This is a renewal process having (2.29) as its renewal interval distribution, which, can also be obtained by using the stationary excess operator on (2.29). The iterative use of the stationary excess operator on a distribution gives higher order equilibrium distributions (Whitt (1985)). The higher order equilibrium distribution converges in distribution to geometric as the number of iterations increase (Whitt (1985)).

2.5 Characteristics of equilibrium distributions

If $\phi(t)$ and $\phi_1(t)$ are the characteristic functions of the baseline random variable X and its equilibrium counterpart X_1 respectively, Harkness and Shantaram (1969) showed that they are related through

$$\phi_1(t) = \begin{cases} \dfrac{\phi(t)-1}{it\mu}, & t \neq 0, \\ 1, & t = 0, \end{cases} \tag{2.31}$$

which shows that there is a one - one relationship between the distributions of X and X_1. By extending this to the nth order they established that the characteristic function $\phi_n(t)$ of X_n is given by

$$\phi_n(t) = \begin{cases} \dfrac{n!}{\mu_n (it)^n} \left[\phi(t) - \displaystyle\sum_{j=0}^{n-1} \mu_j \frac{(it)^j}{j!} \right], & t \neq 0, \\ 1, & t = 0, \end{cases} \tag{2.32}$$

where $\mu_n = E(X_n)$. Moreover, from (2.31), they deduced the identity,

$$E(X_1^r) = \frac{E(X^{r+1})}{(r+1)\ \mu},$$

connecting the moments of X and X_1. They further extended the identity by connecting the moments of X and X_n as

$$E(X_n^r) = \frac{1}{\binom{n+r}{r}} \frac{E(X^{n+r})}{E(X^n)}, \tag{2.33}$$

for all positive integers n and r. It is to be noted that the relation

$$\mu_n = \frac{E(X^{n+1})}{(n+1)\ E(X^n)} \tag{2.34}$$

given in the Remark 3.1 of Mukherjee and Chatterjee (1992) and Corollary 2.1 of Nanda et al. (1996b) is a special case of (2.33). Gupta (1979) showed that the hazard rate $h_1(x)$ of X_1 characterizes the distribution of X through

$$S(x) = \frac{h_1(x)}{h_1(0)} \ \exp\left[-\int_0^x h_1(u)\ du\right],$$

and that $h_1(x)$ is the reciprocal of the mean residual life of X. That is

$$h_1(x) = \frac{1}{m(x)}. \tag{2.35}$$

Nanda et al. (1996b) showed that

$$\mu\ \mu_1\ \mu_2 \cdots \mu_n = \frac{E(X^{n+1})}{(n+1)!},\ n = 0, 1, 2, \ldots. \tag{2.36}$$

Stein and Dattero (1999) extended (2.34) for the residual lives through

$$m_n(x) = \frac{E[(X-x)^{n+1}|X > x]}{(n+1)\ E[(X-x)^n|X > x]}, \tag{2.37}$$

where $m_n(x)$ is the mean residual life of X_n. They also showed that the hazard rate of X_n is the reciprocal of $m_{n-1}(x)$

$$h_n(x) = \frac{1}{m_{n-1}(x)}. \tag{2.38}$$

Navarro and Hernandez (2004) showed that

$$\eta_n(x) = h_{n-1}(x), \tag{2.39}$$

where $\eta_n(x)$ is the Glaser's η function of X_n, defined as

$$\begin{aligned} \eta_n(x) &= \frac{-f'_n(x)}{f_n(x)}, \\ &= -\frac{d}{dx} \log f_n(x). \end{aligned} \tag{2.40}$$

Gupta (2007) showed that the stop loss moments of X characterizes the distribution of X_n through the relation

$$S_n(x) = \frac{E[(X-x)_+^n]}{E(X^n)},$$

where

$$E[(X-x)_+^n] = \int_x^\infty (u-x)^n \ f(u) \ du$$

is the n-th stop loss moment and $(x)_+ = \max\{x, 0\}$. Nair and Sankaran (2010) defined the mean, of the conditional residual life distribution, given the survival of a component or system after age x with asymptotic survival function as $t \to \infty$,

$$P(V_t > y | U_t > x) = \frac{\displaystyle\int_{x+y}^\infty S(u)du}{\displaystyle\int_x^\infty S(u)du},$$

as renewal mean residual life (RMRL),

$$\begin{aligned} e(x) &= E(V|U > x), \\ &= \frac{E[(X-x)^2 | X > x]}{2 \ m(x)}. \end{aligned}$$

They further established that $e(x)$ characterizes the distribution of X through

$$S(x) = \frac{E(X^2) \ (1 + e'(x))}{2 \ e^2(x)} \exp\left[-\int_0^x \frac{1}{e(u)} du\right].$$

It is clear from the identity (2.37) that

$$e(x) = m_1(x) = \frac{1}{h_2(x)},$$

from which we can say that $h_2(x)$ also characterizes the distribution of X.

2.6 Characterizations involving equilibrium distributions

Several characterizations by way of interrelationships between properties of the original and its equilibrium versions have been proposed in literature. In this connection Gupta (1979) has proved the following results.

(i) The survival distribution is exponential if and only if its equilibrium distribution is also exponential with the same parameter. Consequently

$$h_1(x) = h(x),$$

for all $x > 0$.

(ii) If X has increasing hazard rate and $E(X_1) = E(X)$, then X is exponential and conversely.

(iii) Let the distribution of X belongs to the one parameter exponential family with density function

$$f(x;\theta) = c(\theta)\ h(x)\ e^{\theta x}, \gamma < \theta < \delta.$$

If $E(X_1) = E(X)$ for all θ in some interval I, then X has exponential distribution for all $0 < \gamma < \theta < \delta$.

Nair (1989) strengthened some of the above results, by virtue of the identities connecting the reliability characteristics of X and X_1 in the form

$$h(x) = h_1(x) - \frac{h_1'(x)}{h(x)} \tag{2.41}$$

and

$$m(x) = \frac{m_1(x)}{1 + m_1'(x)}.$$

The main results therein are the following: The mean residual life of X is linear if and only if the mean residual of X_1 is linear. This result gives an extension of the characterization (i) of Gupta (1979) noted above. We have that X is exponential with density function

$$f(x) = \lambda e^{-\lambda x}, \ x > 0 \tag{2.42}$$

or Pareto II (α, a) with

$$f(x) = a\alpha^a (x + \alpha)^{-(a+1)}, \ x > 0 \tag{2.43}$$

or rescaled beta (c, R) with

$$f(x) = \frac{c}{R} \left(1 - \frac{x}{R}\right)^{c-1}, \ 0 \leq x \leq R \tag{2.44}$$

if and only if X_1 is exponential (λ) or Pareto II $(\alpha, a-1)$ or rescaled beta $(c+1, R)$)
.

Jain et al. (1989) characterizes the distribution (2.42) through (2.44) by the relationship between the survival functions of X and X_1. Their result is

$$\frac{S_1(x)}{S(x)} = 1 + cx,$$

for all $x > 0$, where $c > -\mu^{-1}$ is a constant if only if X has a Pareto, exponential or power distribution according as $c > 0, \ c = 0$ or $-\mu^{-1} < c < 0$.

In the context of weighted distributions in stochastic modelling Gupta and Kirmani (1990) proved that each of the conditions

(i) $h_1(x) = d \ h(x)$

(ii) $m_1(x) = \frac{1}{d} m(x), \ d > 0$

is necessary and sufficient for X to be Pareto II $(0 < d < 1)$, exponential $(d = 0)$ and rescaled beta $(d > 1)$ in case (i) and X_1 to be Pareto II $(0 < d < 1)$, exponential $(d = 0)$ and rescaled beta $(d > 1)$ in case (ii) respectively. Further if X is NBUE and $E(X_1) = E(X)$, then X has exponential distribution and conversely.

Huang and Lin (1995) extended the characterization result involving exponential distribution of Gupta (1976) and Nair (1989) to the equilibrium distribution of order n as follows.

Theorem 2.6.1. *If X has finite moments of all orders and if for some $r \geq 1$, $E(X_n^r) = E(X^r),\ n = 1, 2, 3, \ldots$ then X is exponential.*

Another interesting result from their work is that if $S(x)$ and $T(x)$ are survival functions of X and Y, their equilibrium distributions are equal if and only if

$$1 - S(x) = c\ (1 - T(x)),\ c > 0.$$

Sen and Khattree (1996) showed that the distribution (2.23) coincides with that of (2.24) if and only if X degenerates at 0 and that $L = X + X_1$ if and only if X is exponential, where L is the random variable corresponding to the length biased distribution, (2.24). Chatterjee and Mukherjee (2000) extended the result given by Gupta and Kirmani (1990) involving the mean residual life as, $m_n(x) = m_{n-1}(x)$ if and only if X is exponential.

There are many similar results for discrete equilibrium distributions. The characterization theorems were pioneered by Gupta (1979) who provided basic identities that were used in the subsequent researches. Some of the main findings of Gupta (1984) are

1. $h_1(x) = h_2(x)$ for all $x = 0, 1, 2, \ldots$ if and only if X is geometric.

2. If X_1 is IFR and $E(X_1) = E(X)$, then X is geometric and conversely.

3. When X has modified power series distribution

$$P(X = x) = \frac{a(x)\ (g(\theta))^x}{f(\theta)},\ x \in T,$$

where T is a subset of non-negative integers, $a(x) > 0,\ g(\theta)$ and $f(\theta)$ are positive finite and differentiable, then the geometric distribution is the only one having $m_1(x) = \mu$ for all θ in some non-degenerate interval I.

Nair and Hitha (1989) derived a characterization of certain distributions in terms of (2.29) as shown in the following theorems

Theorem 2.6.2. *A necessary and sufficient condition for X to be $G(p)$ ($W(a,b)$; $NH(k,m)$) is that X_1 is $G(p)$ ($W(a,b+1)$; $NH(k+1,m-1)$), where $G(p)$ is the geometric distribution with pmf.,*

$$f_G(x) = p\ (1-p)^x,\ x = 0,1,2,\ldots, \tag{2.45}$$

$W(a,b)$ is the Waring distribution with pmf.,

$$f_W(x) = (a-b)\frac{(b)_x}{(a)_{x+1}},\ x = 0,1,2\ldots;\ a > b,\ a,b \in \mathcal{N}, \tag{2.46}$$

where $(b)_x = b(b+1)\ldots(b+x-1)$ is the Pochhammer symbol and $NH(k,m)$ is the negative hyper-geometric distribution with pmf.,

$$f_N(x) = \frac{\binom{-1}{x}\binom{-k}{m-k}}{\binom{-1-k}{m}},\ x = 0,1,2,\ldots,\ k > 0. \tag{2.47}$$

Theorem 2.6.3. *The relationship $h(x) = C_1 h_1(x)$ or $m(x) = C_2 m_1(x)$ hold for all x, if and only if X is geometric for $C_1 = 1$ ($C_2 = 1$), Waring for $C_1 > 1$ ($C_2 < 1$) and negative hyper-geometric for $C_1 < 1$ ($C_2 > 1$).*

Theorem 2.6.4. *A mean residual life function of the form $m(x) = Ax + b$ characterizes the distributions* (2.45), (2.46) *and* (2.47)

The above findings identified the discrete models that possess linear mean residual life as the exponential, Pareto II and rescaled beta in the continuous case. They also obtained identities that connect reliability functions of X and X_1. These are

$$m(x+1) = \frac{m_1(x+1)}{1 + m_1(x+1) - m_1(x)}$$

and

$$h(x) = 1 + h_1(x)\left[1 - \frac{1}{h_1(x-1)}\right],\ x \geq 1.$$

Sen and Khattree (1996) pointed out some differences between the discrete and continuous cases in proving certain analogous results. They considered the weighted distribution

$$P(L^* = x) = \frac{(x+1)\ f(x)}{\mu + 1},$$

instead of the usual length biased model to show that L^* is identically distributed as $X + X_1$ if and only if X is geometric and X_1 is independent of X.

Willmot et al. (2005) discussed the properties of higher order discrete equilibrium distributions from the context of reliability as well as analytic representation for the stop-loss premium or interest in connection with insurance claims modelling. They obtained discrete equilibrium model for the geometric, the Poisson, mixed Poisson, Pascal and the phase-type distributions. A closed form analytic expression for the k^{th} order discrete equilibrium distribution was also obtained as

$$S_k(x) = \frac{\left[(-1)^x \sum_{i=0}^{x} \binom{k+j-1}{j} f(x-j) + \sum_{m=0}^{k-1} (-1)^m \binom{m+n}{n} E\left[\binom{X}{k-1-m}\right]\right]}{E\left[\binom{X}{k}\right]},$$

$x = 0, 1, 2, \ldots,\ k = 1, 2, 3, \ldots$ provided that $E\left[X(X-1)\cdots(X-k+1)\right] < \infty$. Alternatively,

$$S_k(x) = \frac{k}{E\left[X^{(k)}\right]} \sum_{m=x+k}^{\infty} (m-x-1)^{(k-1)} f(m),$$

where $a^{(k)} = a(a-1)\cdots(a-k+1)$. More properties of the distribution will be discussed in Chapter 4.

2.7 Ageing classes involving equilibrium distributions

There are two different applications of equilibrium distributions in studying ageing properties. The first is to interpret the traditional concepts of ageing in terms of properties of equilibrium distributions. These interpretations often act as tools in providing various properties of ageing classes. Secondly, we have some new ageing classes based on comparison between the reliability concepts of the baseline and equilibrium distributions. In the present section we review some important developments in this connection.

Gupta (1979) proved that the IFR (DFR) property is preserved under the formation of equilibrium distribution and also that the converse is not true. Deshpande et al. (1986) argued that stochastic comparison between X_1 and X are meaningful from the point of view of ageing on the premise that the life distribution of the unit, which ages more rapidly, will come off worse in such a comparison. They have established some correspondence between the characteristics of X with those of X_1 as shown below.

Theorem 2.7.1. *1. X is DMRL $\Longleftrightarrow X_1$ is IFR,*

2. *X is DMRLHA $\Longleftrightarrow X_1$ is IFRA,*
3. *X is NBUE $\Longleftrightarrow X_1$ is NBUFR,*
4. *X is HNBUE $\Longleftrightarrow X_1$ is NBUFRA $\Longleftrightarrow X_E \geq_{st} X_1$, where X_E has the exponential distribution with the same mean as X,*
5. *X is NBUE $\Leftrightarrow h_1(x) \geq h_1(0)$,*
6. *DMRL$\Rightarrow$DMRLHA $\Rightarrow$NBUE and*
7. *X is HNBUE $\Leftrightarrow \frac{1}{x}\displaystyle\int_0^x h_1(t)\ dt \geq h_1(0)$.*

Gupta et al. (1987) proved that X is DVRL (IVRL) $\Longleftrightarrow X_1$ is DMRL (IMRL). Gupta and Kirmani (1990) showed that this is again equivalent to $X_1 \leq_{mrl} (\geq_{mrl})X$. They also deduced that $X_1 \leq_{st} (\geq_{st})X \Longleftrightarrow X$ is NBUE (NWUE) and $X_1 \leq_{hr} (\geq_{hr})X \Longleftrightarrow X$ is DMRL (IMRL). Fagiuoli and Pellerey (1993) extended the ageing notions to higher order equilibrium models. They defined $(n+1)-*$ ageing classes of X which in fact is $*$ ageing classes of X_n, where $*$ denotes any of the ageing classes. It should be noted that $1-*$ is the same as $*$. Fagiuoli and Pellerey (1993) also established the following results comparing the random variables X_n and $X_{n,t} = (X_n - t|X_n > t)$.

Theorem 2.7.2. *For all $t \geq 0$*

1. *$X_n \geq_{hr} X_{n,t} \Leftrightarrow X_n$ is IFR,*
2. *$X_n \geq_{st} X_{n,t} \Leftrightarrow X_n$ is NBU,*
3. *$X_n \geq_{icv} X_{n,t} \Leftrightarrow X_n$ is NBUCV,*

4. $X_n \geq_{icx} X_{n,t} \Leftrightarrow X_n$ *is NBUCX.*

Theorem 2.7.3. 1. X_n *is NBUCV* $\Longrightarrow X_{n+1}$ *is NBUFR and*

2. X_{n+1} *is NBU* $\Longrightarrow X_n$ *NBUCX* $\Longrightarrow X_{n+1}$ *is NBUFR.*

Abouammoh et al. (2000) put forward some new ageing classes derived from renewal theory, which are stated below:

Definition 2.7.1. 1. *X is said to have new renewal better (worse) than used parent (NRBU / NRWU) property if and only if* $(X - x|X > x) \leq_{st} (\geq_{st}) X_1$.

2. *X is called new renewal better (worse) than used parent in expectation (NRBUE / NRWUE) if and only if* $E(X - x|X > x) \leq (\geq) E(X_1)$.

3. *X is called harmonic new renewal better (worse) than used parent in expectation (HNRBUE / HNRWUE) if and only if* $X_1 \leq_{st} (\geq_{st}) X_E$*, where* X_E *is and exponential random variable with mean* μ_1.

Abouammoh et al. (2000) further specified that NRBU $\Longrightarrow$ NRBUE $\Longrightarrow$ HNRBUE. Hu et al. (2001) showed that X is IFR $\Leftrightarrow X_1 \leq_{lr} X$. They also extended the above relations to higher orders. Much of the applications of the above three concepts become limited in the light of the findings of Bon and Illayk (2002) that, if the first two moments of X are finite then the life distributions possessing these properties are gathered in the exponential class. Abouammoh and Qamber (2003) studied an ageing class defined as follows:

Definition 2.7.2. *X is said to be new better (worse) than renewal used (NBRU / NWRU) if and only if*

$$\frac{1}{S(x)} \int_t^\infty S(u + x)\ du \leq (\geq) \int_t^\infty S(u)\ du,$$

for every $t, x \geq 0$. This is equivalent to $(X_1 - x|X_1 > x) \leq_{st} (\geq_{st}) X$ and $(X - x|X > x) \leq_{icx} (\geq_{icx}) X$ or

$$\int_{t+x}^\infty S(u)\ du \leq S(x) \int_t^\infty S(u)\ du.$$

We can see from the definition that NBRU is the same as NBUCX.

Chapter 3

On Some Properties of Equilibrium Distributions of Order n

3.1 Introduction

The present chapter is an attempt to supplement the existing literature with additional results that relate the original and nth order equilibrium distributions and facilitate characterizations of certain life distributions. There have been very few attempts in this direction other than those for first order equilibrium distributions. Various reliability characteristics worked out for $S(x)$ and $S_n(x)$ are linked by certain identities that facilitate comparison of the two distributions by properties useful in reliability modeling. Recall that if X is a non-negative and absolutely continuous random variable with survival function $S(x)$ satisfying $E(X^n) < \infty$, the n^{th} order equilibrium distribution of X is defined by

$$S_n(x) = \frac{1}{\mu_{n-1}} \int_x^{\infty} S_{n-1}(u)\, du, \tag{3.1}$$

where

$$\mu_n = \int_0^{\infty} S_n(u)\, du,$$

with $\mu_0 = \mu = E(X)$ and $S_0(x) = S(x)$. The usual equilibrium distribution is $S_1(x)$ and $S_{n+1}(x)$ is the residual lifetime in the equilibrium renewal process with components having lifetime distribution $S_n(x)$.

3.2 Basic results

Let X be a non-negative random variable representing the lifetime of a component or device with absolute continuous survival function $S(x)$ and density function $f(x)$. First we obtain a direct relationship between $S(x)$ and $S_n(x)$ defined in (3.1). Integrating by parts on the right side of

$$S_1(x) = \frac{1}{\mu}\int_x^{\infty} S(u)du,$$

we get

$$\begin{aligned} S_1(x) &= \frac{1}{\mu}\left[-xS(x) + \int_x^{\infty} uf(u)du\right], \\ &= \frac{1}{\mu}\int_x^{\infty}(u-x)\ f(u)\ du. \end{aligned}$$

Again integrating by parts on the right side of the equation for $S_2(x)$, obtained from (3.1), we get

$$\begin{aligned} S_2(x) &= \frac{1}{\mu_1}\int_x^{\infty} S_1(u)\ du, \\ &= \frac{1}{\mu_1}\left[-xS_1(x) + \int_x^{\infty} uf_1(u)\ du\right], \\ &= \frac{1}{\mu\mu_1}\left[-x\int_x^{\infty} S(u)\ du + \int_x^{\infty} uS(u)\ du\right], \\ &= \frac{1}{\mu\mu_1}\int_x^{\infty}(u-x)\ S(u)\ du, \\ &= \frac{1}{2!\ \mu\mu_1}\int_x^{\infty}(u-x)^2 f(u)\ du. \end{aligned}$$

Hence by induction,

$$S_n(x) = \frac{1}{n!\ \mu\mu_1\mu_2\cdots\mu_{n-1}}\int_x^\infty (u-x)^n f(u)\ du.$$

Further, assume that $E(X^n) < \infty$. Then incorporating (2.36) in the above identity,

$$\begin{aligned} S_n(x) &= \frac{1}{E\left(X^n\right)}\int_x^\infty (u-x)^n f(u)\ du\ , \\ &= \frac{E[(X-x)^n|X>x]}{E(X^n)}S(x)\ , \\ &= \frac{r_n(x)}{r_n(0)}S(x), \end{aligned} \tag{3.2}$$

where $r_n(x) = E[(X-x)^n|X>x]$, the n^{th} moment of the residual life of X. We note that expression (3.2) is equivalent to expression (3.7) in Gupta (2007) but with a different method of derivation. Let

$$h_n(x) = \frac{-d\log S_n(x)}{dx}$$

and

$$m_n(x) = \frac{1}{S_n(x)}\int_x^\infty S_n(u)\ du$$

be respectively the failure rate function and the mean residual life function of $S_n(x)$. Using the general relationship between failure rate and mean residual life given in (2.8)

$$h_n(x) = \frac{1+m_n'(x)}{m_n(x)}, \tag{3.3}$$

where prime denotes differentiation. From (2.38) we have the recurrence relation connecting the mean residual lives of the n^{th} and $(n-1)^{th}$ order equilibrium distributions

$$m_{n-1}(x) = \frac{m_n(x)}{1+m_n'(x)}. \tag{3.4}$$

From (2.38) and (3.3),

$$h_{n-1}(x) = h_n(x) - \frac{h_n'(x)}{h_n(x)} \tag{3.5}$$

and

$$h_n(x) = h_{n-1}(x) + \frac{d \log h_n(x)}{dx}. \tag{3.6}$$

Stein and Dattero (1999) considered the Bondesson's functions

$$\begin{aligned} m_n(x) &= \frac{E\left((X-x)^n | X > x\right)}{nE\left((X-x)^{n-1} | X > x\right)}, \\ &= \frac{E\left((X-x)_+^n\right)}{nE\left((X-x)_+^{n-1}\right)}, \end{aligned}$$

for integers $n \geq 1$ and all x such that $S(x) < 1$ and $(X-x)_+ = \max(X-x, 0)$ and a new sequence

$$S_{(n)}(x) = \int_x^\infty S_{(n-1)}(u)\ du$$

with $S_{(1)}(x) = S(x)$. They obtained the identities

$$\begin{aligned} E\left((X-x)_+^n\right) &= n!\ S_{(n+1)}(x), \\ &= n!\ S_{n+1}(x)\ \mu_1 \mu_2 \cdots \mu_n. \end{aligned}$$

From the definition (3.1),

$$\begin{aligned} S_{n+1}(x) &= \frac{1}{\mu_n} \int_x^\infty S_n(u)\ du, \\ &= \frac{S_n(x)\ m_n(x)}{\mu_n}. \end{aligned}$$

This gives

$$\begin{aligned} m_n(x) &= \frac{S_{n+1}(x)\mu_n}{S_n(x)}, \\ &= \frac{r_n(x)}{n\ r_{n-1}(x)}, \end{aligned} \tag{3.7}$$

by virtue of the above relationships.

Further, the moment relation of Nanda *et al.* (1996b), given in (2.36), is a special case of the Stein - Dattero identity (3.7) as x tends to zero.

We now give a new interpretation to $m_n(x)$. Assume that we have a set of components with lifetimes $L_1, L_2, \ldots$ which are independent and identically distributed with distribution function $F(x)$ and finite mean μ. If the first component is replaced upon failure by a second component and so on, $L_1 + L_2 + \cdots + L_n$ constitutes a renewal process. Denoting by U_τ and V_τ the age and residual life of the component in use at time τ, the asymptotic distribution of U_τ or V_τ is called the equilibrium distribution corresponding to $F(x)$. The joint distribution of age, U_τ and remaining life, V_τ in the equilibrium renewal process is specified by the survival function

$$P\left(U_\tau > u, V_\tau > v\right) = \frac{1}{\mu}\int_{u+v}^{\infty} S(t)\ dt\ ,\ \ u, v > 0.$$

Accordingly the conditional distribution of V_τ given $U_\tau > u$ has survival function

$$\begin{aligned} P(V_\tau > v | U_\tau > u) &= \frac{\displaystyle\int_{u+v}^{\infty} S(t)\ dt}{\displaystyle\int_{u}^{\infty} S(t)\ dt}, \\ &= \frac{\displaystyle\int_{v}^{\infty} S(t+u)\ dt}{\displaystyle\int_{u}^{\infty} S(t)\ dt} \end{aligned}$$

with the density function,

$$S(v+u)\left[\int_u^\infty S(t)\ dt\right]^{-1},\ \ v > 0$$

and hence the nth moment of the conditional distribution,

$$\begin{aligned} e_n(u) &= E\left(V_\tau^n | U_\tau > u\right)\ , \\ &= \left[\int_u^\infty S(t)\ dt\right]^{-1}\int_0^\infty t^n\ S(t+u)\ dt\ , \\ &= \left[\int_u^\infty S(t)\ dt\right]^{-1}\int_u^\infty (t-u)^n S(t)\ dt\ , \\ &= \left[(n+1)\int_u^\infty S(t)\ dt\right]^{-1}\int_u^\infty (t-u)^{n+1} f(t)\ dt\ , \\ &= \frac{E[(X-u)^{n+1}|X>u]}{(n+1)E[(X-u)|X>u]}\ , \end{aligned}$$

$$= \frac{r_{n+1}(u)}{(n+1)\ r_1(u)}. \tag{3.8}$$

Thus $m_n(x)$ and $e_n(x)$ are connected by

$$m_n(x) = \frac{r_n(x)}{n\ r_{n-1}(x)} = \frac{e_{n-1}(x)}{(n-1)\ e_{n-2}(x)}. \tag{3.9}$$

The quantities $r_1(x)$, $m_n(x)$ and $e_1(x) = e(x)$, represent mean residual life functions respectively of the original distribution, the n^{th} order equilibrium distribution and the residual life distribution of the equilibrium renewal process (RMRL). A weighted distribution argument can throw light on the differences between the three functions. While $r_1(x)$ is the mean residual life function of X, $e_1(x)$ is the corresponding quantity for the weighted density

$$f_w(x) = \frac{w(x)\ f(x)}{E(w(X))}\ ,$$

with $w(x) = [h(x)]^{-1}$. The density function of the n^{th} order equilibrium distribution is

$$f_n(x) = \frac{S_{n-1}(x)}{\mu_{n-1}}\ ,$$

which is the weighted form of $f_{n-1}(x)$ with $w_{n-1}(x) = [h_{n-1}(x)]^{-1}$ as the weight, and hence

$$f_n(x) = \frac{f_{n-1}(x)}{h_{n-1}(x)\ \mu_{n-1}}$$

or

$$h_{n-1}(x) = \frac{f_{n-1}(x)}{\mu_{n-1}\ f_n(x)}\ . \tag{3.10}$$

The impact of weighting the characteristics of the original distribution is seen from equations (3.7) through (3.10). Further, the failure rates of equilibrium distributions are easily computed from the density $f_n(x)$. Failure rates of lower orders are found from (3.5) and mean residual life functions from (2.38).

3.3 Characterizations of certain distributions

First we look at the characterizations of the generalized Pareto distribution (GPD) with survival function (Johnson *et al.* (1994, p. 614))

$$S(x) = \left(1 + \frac{ax}{b}\right)^{-\left(1+\frac{1}{a}\right)}, \; x > 0; \; a > -1, \; b > 0, \tag{3.11}$$

with mean

$$\begin{aligned} \mu &= \int_0^\infty \left(1 + \frac{au}{b}\right)^{-\left(1+\frac{1}{a}\right)} du, \\ &= b, \end{aligned}$$

in terms of the relationships between the reliability functions, failure rate functions and mean residual life functions of the original distribution (3.11) and its nth order equilibrium distribution. The importance of (3.11) is that it contains the exponential distribution with mean b as a tends to zero. Further setting $a = (\beta - 1)^{-1}$ and $b = a\alpha$, we get the Pareto II (Lomax) distribution

$$S(x) = \alpha^\beta \; (x + \alpha)^{-\beta}, \; x > 0; \; \alpha > 0, \; \beta > 0 \tag{3.12}$$

with linearly decreasing (increasing) failure rate (mean residual life), while, $a = -(1+d)^{-1}$ and $b = R\,(1 + d)^{-1}$ give the re-scaled beta model with

$$S(x) = \left(1 - \frac{x}{R}\right)^d, \; 0 < x < R, \; d, R > 0, \tag{3.13}$$

which has a linearly increasing (decreasing) failure rate (mean residual life). The following theorem extends the results (15), (16) and (17) of Gupta and Kirmani (1990) to the n^{th} order equilibrium distributions.

Theorem 3.3.1. *A non-negative random variable X with absolutely continuous survival function $S(x)$ and $E(X^n) < \infty$ for $n = 1, 2, \ldots$ is distributed as* GPD *in* (3.11) *if and only if one of the following properties hold for all $x > 0$ and two consecutive values of $n = i, i + 1$*

(i) $S_n(x) = (1 + cx)^n S(x)$, for some real c ,

(ii) $m_n(x) = C_n \; m(x)$,

(iii) $h_n(x) = K_n \; h(x)$,

(iv) $r_n(x) = A_n \; r_1^n(x)$,

where $C_n, \; K_n$ *and* A_n *are positive constants.*

Proof. (i): Assume that X has GPD (3.11). The survival function of X_n is obtained from (3.2) as

$$\begin{aligned} S_n(x) &= \left(1 + \frac{a}{b}x\right)^{-\frac{1}{a}+n-1}, \\ &= \left(1 + \frac{a}{b}x\right)^n S(x). \end{aligned} \tag{3.14}$$

Conversely, the condition in (i) means that

$$\frac{S_n(x)}{S_{n-1}(x)} = 1 + cx \tag{3.15}$$

or from (3.1),

$$\int_x^\infty S_{n-1}(u) \; du = (1 + cx) \; \mu_{n-1} S_{n-1}(x) \; .$$

Differentiating with respect to x,

$$-S_{n-1}(x) = \mu_{n-1} \left[c S_{n-1}(x) + (1 + cx) S'_{n-1}(x) \right],$$

and re-arranging terms we get,

$$\frac{S'_{n-1}(x)}{S_{n-1}(x)} = \frac{-(1 + c\mu_{n-1})}{(1 + cx) \; \mu_{n-1}} \; ,$$

where prime denotes the differentiation. Integrating with respect to x,

$$S_{n-1}(x) = K \; (1 + cx)^{-1-\frac{1}{c\mu_{n-1}}} \; . \tag{3.16}$$

As $x \to 0, \; K = 1$, showing that S_{n-1} is GPD with mean μ_{n-1}. This completes the proof for (i).

(ii): For the distribution (3.11),

$$\begin{aligned} m(x) &= \frac{1}{S(x)} \int_x^\infty S(u)\ du, \\ &= ax+b \end{aligned}$$

and

$$\begin{aligned} m_n(x) &= \frac{1}{S_n(x)} \int_x^\infty S_n(u)\ du, \\ &= \left[1+\frac{a}{b}x\right]^{\frac{1}{a}-n+1} \int_x^\infty \left(1+\frac{a}{b}u\right)^{-\frac{1}{a}+n-1} du, \quad -1 < a < \frac{1}{n}, \\ &= \frac{b+ax}{1-na}, \\ &= C_n\ m(x), \end{aligned} \tag{3.17}$$

from which the result in (ii) follows with $C_n = [1-na]^{-1}$ (>0). Conversely assuming the given condition we have from (3.4),

$$m_{n-1}(x) = \frac{m_n(x)}{1+m_n'(x)}$$

and

$$C_{n-1}m(x) = \frac{C_n\ m(x)}{1+C_n\ m'(x)},$$

$$m'(x) = \text{ a constant},$$

which on integration,

$$m(x) = ax+b,$$

for some constants a and b, which characterizes the GPD (3.11).

(iii): Once again, for the distribution (3.11),

$$\begin{aligned} h_n(x) &= \frac{f_n(x)}{S_n(x)}, \\ &= \frac{S_{n-1}(x)}{m_{n-1}(0)\ S_n(x)}, \text{ by (3.1)}, \\ &= \frac{1-(n-1)a}{b} \frac{\left(1+\frac{a}{b}x\right)^{n-1} S(x)}{\left(1+\frac{a}{b}x\right)^{n} S(x)}, \end{aligned}$$

$$\begin{aligned} &= \frac{1-(n-1)a}{b+ax}, \\ &= K_n\ h(x), \end{aligned}$$

from which (iii) follows with $K_n = 1-(n-1)a$. Conversely assuming (iii), from (3.6) we get

$$\frac{d}{dx}\log h_n(x) = h_{n-1}(x) - h_n(x),$$

$$\frac{d}{dx}\log h(x) = (K_{n-1} - K_n)h(x),$$

$$\frac{1}{h(x)}\frac{d}{dx}\log h(x) = \text{ a constant},$$

which on integration,

$$\frac{1}{h(x)} = ax + b,$$

for $a = K_{n-1} - K_n$ and some constant $b > 0$. By using the identity (2.3) this leads to

$$\begin{aligned} S(x) &= \exp\left[-\int_0^x h(u)\ du\right], \\ &= \exp\left[-\int_0^x \frac{1}{au+b}\ du\right], \\ &= \left(1+\frac{a}{b}x\right)^{\frac{-1}{a}}, \end{aligned}$$

which is of the form (3.11) and hence $S(x)$ is GPD.
(iv): Finally, for the distribution (3.11), the identity (3.9) becomes

$$\begin{aligned} r_n(x) &= nr_{n-1}(x)\ m_n(x), \\ &= n\ C_n\ r_1(x)r_{n-1}(x), \text{ by result (ii)}, \\ &= n!\ C_nC_{n-1}\cdots C_2r_1^n(x), \\ &= A_n\ r_1^n(x), \end{aligned}$$

where $A_n = n!\ C_nC_{n-1}\cdots C_2 > 0$ and hence the result (iv) follows. Conversely assuming $r_n(x) = A_n r_1^n(x)$ and the identity (3.9) that

$$\begin{aligned} m_n(x) &= \frac{r_n(x)}{n r_{n-1}(x)}, \\ &= \frac{A_n r_1^n(x)}{n A_{n-1} r_1^{n-1}(x)}, \\ &= \frac{A_n}{n A_{n-1}} r_1(x), \\ &= B_n\ m(x), \end{aligned}$$

satisfying the condition for the result (ii) with $B_n = A_n(nA_{n-1})^{-1}$. Hence $S(x)$ is GPD. ■

Remark 3.3.1. *(a) In the exponential case $S_n(x)$ and $S(x)$ are identical. Hence all the reliability characteristics of the exponential distribution remains unaltered for the n^{th} order equilibrium distribution.*

(b) All the three distributions, exponential, Pareto II *and beta keep the same form for $S_n(x)$, with failure rate (mean residual life) gradually decreasing (increasing) for* Pareto II *with increasing n. The beta model exhibits the opposite behaviour.*

Remark 3.3.2. *It is clear from* (3.17) *that $m(x) \leq m_1(x) \leq m_2(x) \leq \ldots \leq m_n(x)$ for the GPD with $a \in (-1, 1/n)$. By virtue of* (2.38)*, it is also clear that $h(x) \geq h_1(x) \geq h_2(x) \geq \ldots$.*

Lemma 3.3.1. *The distribution of X is a generalized mixture of exponentials specified by*

$$S(x) = \alpha e^{-\lambda_1 x} + (1-\alpha) e^{-\lambda_2 x},\ x > 0;\ \alpha \geq 0,\ 0 < \lambda_1 < \lambda_2, \tag{3.18}$$

if and only if $S_n(x)$ for $n = 1, 2, 3, \ldots$ is also a generalized mixture of exponential distributions, with different mixing constants.

Proof. Suppose X has the survival function (3.18), then the survival function of

X_1 is

$$\begin{aligned} S_1(x) &= \frac{\int_x^\infty S(u)\ du}{\int_0^\infty S(u)\ du}, \\ &= \frac{\frac{\alpha}{\lambda_1}e^{-\lambda_1 x} + \frac{1-\alpha}{\lambda_2}e^{-\lambda_2 x}}{\frac{\alpha}{\lambda_1} + \frac{1-\alpha}{\lambda_2}}, \\ &= \theta_1 e^{-\lambda_1 x} + (1-\theta_1)e^{-\lambda_2 x}, \end{aligned}$$

where

$$\theta_1 = \frac{\frac{\alpha}{\lambda_1}}{\frac{\alpha}{\lambda_1} + \frac{1-\alpha}{\lambda_2}}.$$

The survival function of X_2 is

$$\begin{aligned} S_2(x) &= \frac{\int_x^\infty S_1(u)\ du}{\int_0^\infty S_1(u)\ du}, \\ &= \frac{1}{\frac{\alpha}{\lambda_1^2} + \frac{1-\alpha}{\lambda_2^2}}\left[\frac{\alpha}{\lambda_1^2}e^{-\lambda_1 x} + \frac{1-\alpha}{\lambda_2^2}e^{-\lambda_2 x}\right] \end{aligned}$$

and in general,

$$S_n(x) = \frac{1}{\frac{\alpha}{\lambda_1^n} + \frac{1-\alpha}{\lambda_2^n}}\left[\frac{\alpha}{\lambda_1^n}e^{-\lambda_1 x} + \frac{1-\alpha}{\lambda_2^n}e^{-\lambda_2 x}\right], \tag{3.19}$$

is again a mixture of exponential distributions of the form

$$S_n(x) = \theta_n e^{-\lambda_1 x} + (1-\theta_n)e^{-\lambda_2 x}$$

with

$$\theta_n = \frac{\frac{\alpha}{\lambda_1^n}}{\frac{\alpha}{\lambda_1^n} + \frac{1-\alpha}{\lambda_2^n}}.$$

Conversely, assume that S_n is a mixture of exponential distributions of the form (3.18). Then by successive applications of Definition 2.4.1 on $S_n(x)$ for differentiation of order one we get

$$S(x) = \frac{\alpha\lambda_1^{n+1}e^{-\lambda_1 x} + (1-\alpha)\lambda_2^{n+1}e^{-\lambda_2 x}}{\alpha\lambda_1^{n+1} + (1-\alpha)\lambda_2^{n+1}},$$

which is again a mixture of exponential distributions. Hence the lemma is proved. ∎

The following theorem now follows from the Remark 2.3 of Navarro and Ruiz (2004) for the case $n = 1$ and from the Lemma 3.3.1.

Theorem 3.3.2. *A non-negative random variable X with absolutely continuous survival function $S(x)$ with $E(X^n) < \infty$ satisfies the relationship*

$$m_n(x) = \theta_1 + \theta_2 - \theta_1\theta_2 \; h_n(x) \;, \tag{3.20}$$

for all $x \geq 0$ and each $n = 1, 2, \ldots$ if and only if X has the generalized mixture of exponential distributions (3.18)*, where $\theta_i = \lambda_i^{-1}, i = 1, 2$.*

Proof. To prove the 'if' part, we note in the case of (3.19),

$$\begin{aligned} m_n(x) &= \frac{1}{S_n(x)} \int_x^\infty S_n(u)\, du, \\ &= \frac{\alpha\lambda_1^{-n-1}e^{-\lambda_1 x} + (1-\alpha)\lambda_2^{-n-1}e^{-\lambda_2 x}}{\alpha\lambda_1^{-n}e^{-\lambda_1 x} + (1-\alpha)\lambda_2^{-n}e^{-\lambda_2 x}}, \end{aligned}$$

$$\begin{aligned} h_n(x) &= \frac{1}{m_{n-1}(x)}, \\ &= \frac{\alpha\lambda_1^{1-n}e^{-\lambda_1 x} + (1-\alpha)\lambda_2^{1-n}e^{-\lambda_2 x}}{\alpha\lambda_1^{-n}e^{-\lambda_1 x} + (1-\alpha)\lambda_2^{-n}e^{-\lambda_2 x}} \end{aligned}$$

and

$$\theta_i = \frac{1}{\lambda_i}, \; i = 1, 2$$

verify the identity,

$$\begin{aligned} \theta_1 + \theta_2 - \theta_1\theta_2 h_n(x) &= \theta_1 + \theta_2 - \theta_1\theta_2 \frac{\alpha\theta_1^n\theta_2 e^{\frac{-x}{\theta_1}} + (1-\alpha)\theta_1\theta_2^n e^{\frac{-x}{\theta_2}}}{\alpha\theta_1^n e^{\frac{-x}{\theta_1}} + (1-\alpha)\theta_2^n e^{\frac{-x}{\theta_2}}\theta_1\theta_2}, \\ &= m_n(x). \end{aligned}$$

In establishing the only if part, we write (3.20) as

$$\int_x^\infty S_n(u)\ du = (\theta_1+\theta_2)\ S_n(x) + \theta_1\theta_2 \frac{dS_n(x)}{dx}. \tag{3.21}$$

Setting $\int_x^\infty S_n(u)\ du = y$, (3.21) becomes second order differential equation with constant coefficients

$$\theta_1\theta_2 \frac{d^2y}{dx^2} + (\theta_1+\theta_2)\frac{dy}{dx} + y = 0.$$

The corresponding auxiliary equation is

$$\theta_1\theta_2 m^2 + (\theta_1+\theta_2)\ m + 1 = 0,$$

which has roots $-\theta_1^{-1}$ and $-\theta_2^{-1}$ and hence the solution of (3.21) is of the form

$$S_n(x) = \frac{A_n}{\theta_1} e^{\frac{-x}{\theta_1}} + \frac{B_n}{\theta_2} e^{\frac{-x}{\theta_2}}.$$

This gives

$$S(x) = \frac{A_0}{\theta_1} e^{\frac{-x}{\theta_1}} + \frac{B_0}{\theta_2} e^{\frac{-x}{\theta_2}}.$$

As x tends to zero,

$$\frac{A_0}{\theta_1} + \frac{B_0}{\theta_2} = 1,$$

so that by taking $\theta_1^{-1}A_0 = \alpha$,

$$S(x) = \alpha e^{\frac{-x}{\theta_1}} + (1-\alpha) e^{\frac{-x}{\theta_2}},$$

a mixture of exponential with means θ_1 and θ_2 for the components . This completes the proof. ■

Remark 3.3.3. Nassar and Mahmood (1985) *have proved that, X is distributed as a mixture of exponentials of the form* (3.18) *with means λ_i and* $0 < \alpha < 1$, *if and only if*

$$m(x) = \lambda_1 + \lambda_2 - \lambda_1\lambda_2 h(x).$$

This result was generalized in Abraham and Nair (2001) *with the identity*

$$m(x) = (1-ax)\ (\lambda_1+\lambda_2+a\lambda_1\lambda_2) - \lambda_1\lambda_2\ (1+ax)^2 h(x)$$

characterizing the mixture of exponential $(a=0)$, *Pareto* $(a>0)$ *and rescaled beta* $(a<0)$. Navarro and Ruiz (2004) *modified the exponential case for general mixtures in which case* α *can also be negative.*

Our next theorem is to characterize the generalized Pareto law using a specific relationship between $e_n(x)$ and the mean residual life of X.

Theorem 3.3.3. *The distribution of* X *is* GPD *if for three consecutive integers* $n-2,\ n-1,\ n$ *and all* $x>0$

$$e_n(x) = C_n\ r_1^n(x) \tag{3.22}$$

where C_n *is some constant, independent of* X *and conversely if* (3.22) *holds for three consecutive values of* n, *then* X *has* GPD.

Proof. First we assume that X follows generalized Pareto distribution. Then

$$\begin{aligned} r_n(x) &= E\left[(X-x)^n | X>x\right]\ , \\ &= \left(1+\frac{a}{b}x\right)^{1+\frac{1}{a}} \frac{a}{b}\int_x^{\infty} (u-x)^n \left(1+\frac{a}{b}u\right)^{-(2+\frac{1}{a})} du\ , \\ &= \left(1+\frac{a}{b}x\right)^{1+\frac{1}{a}} \frac{a}{b}\int_0^{\infty} u^n \left(1+\frac{a}{b}\ (x+u)\right)^{-\left(2+\frac{1}{a}\right)} du\ , \\ &= \left(1+\frac{1}{a}\right)\left(1+\frac{a}{b}x\right)^n \left(\frac{b}{a}\right)^n B\left(n+1,\ 1+\frac{1}{a}-n\right)\ , \\ &= A_n\ r_1^n(x), \end{aligned}$$

since $r_1(x) = b+ax$. Hence from (3.8)

$$\begin{aligned} e_n(x) &= \frac{r_{n+1}(x)}{(n+1)\ r_1(x)}\ , \\ &= \frac{A_{n+1}\ r_1^{n+1}(x)}{(n+1)\ r_1(x)}\ , \end{aligned}$$

$$= \quad C_n \; r_1^n(x) \; .$$

Conversely if (3.22) is true for n, $n-1$ and $n-2$, we use (3.9) to find

$$m_n(x) = p_n \; r_1(x)$$

for some constant p_n and therefore, by Theorem 3.3.1, X has the distribution stated in (3.11). ■

3.4 Characteristic function approach

There are distributions specified by characteristic functions which do not have closed form expression for their distributions. In such cases, the computation of equilibrium distributions and verification of their properties becomes difficult. Further, sometimes it is easier to work with characteristic functions instead of distribution functions. We therefore investigate the relationship between the characteristic function of the n^{th} order equilibrium distribution and the parent distribution. If $\phi_n(t)$ is the characteristic function of $S_n(x)$,

$$\begin{aligned}
\phi_n(t) &= \int_0^\infty e^{itx} f_n(x) \; dx \; , \\
&= \mu_{n-1}^{-1} \int_0^\infty e^{itx} S_{n-1}(x) \; dx \; , \\
&= \frac{\phi_{n-1}(t) - 1}{it\mu_{n-1}} \; . \qquad (3.23)
\end{aligned}$$

Iterating for n,

$$\phi_n(t) = \frac{1}{(it)^n \mu\mu_1\mu_2 \cdots \mu_{n-1}} \left[\phi(t) - \sum_{r=0}^{n-1} (it)^r \mu\mu_1 \cdots \mu_r \right] \; , t \neq 0, \qquad (3.24)$$

where $\phi(t)$ is the characteristic function of X. Using the relationship (2.36) and the notation $\alpha_n = E(X^n)$ in (3.24),

$$\begin{aligned}
\phi_n(t) &= \frac{n!}{(it)^n\alpha_n}\left[\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\alpha_r - \sum_{r=0}^{n-1}\frac{(it)^r}{r!}\alpha_r\right], \\
&= \frac{n!}{(it)^n\alpha_n}\sum_{r=n}^{\infty}\frac{(it)^r}{r!}\alpha_r\ , \\
&= \frac{n!}{\alpha_n}\sum_{r=0}^{\infty}(it)^r\frac{\alpha_{n+r}}{(n+r)!}\ . \qquad (3.25)
\end{aligned}$$

Let X_n denote the random variable corresponding to the n^{th} order equilibrium distribution. Then the moments of X_n are related to the moments of X by

$$\beta_r = E(X_n^r) = \frac{n!\ r!}{(n+r)!}\frac{\alpha_{n+r}}{\alpha_n},\ r = 0, 1, 2, \dots\ . \qquad (3.26)$$

Specializing for $r = 1$ we get the results of Nanda *et al.* (1996b) relating the means of $S_n(x)$ and $S(x)$. Further if we work in the same manner with the residual life distribution of X, we have

$$E[(X_n - x)^r | X > x] = \frac{n!\ r!}{(n+r)!}\frac{E[(X-x)^{n+r}|X > x]}{E[(X-x)^n|X > x]}\ , \qquad (3.27)$$

which is a generalization of the formula for $m_n(x)$ given by Stein and Dattero (1999) stated in (3.7).

We now illustrate the use of characteristic function approach in the following theorems. Consider the gamma distribution ($\mathcal{G}(\theta, \lambda)$) having the density function

$$f(x) = \frac{\lambda^\theta}{\Gamma(\theta)}e^{-\lambda x}x^{\theta-1},\ x > 0;\ \lambda, \theta > 0, \qquad (3.28)$$

with parameters (θ, λ). Special cases of (3.28) are characterized in the following results.

Theorem 3.4.1. *A necessary and sufficient condition that X follows $\mathcal{G}(2,\ 2/\mu)$, where $0 < \mu = E(X)$, is that its equilibrium distribution of order n is a mixture of*

$\mathcal{G}(2,\ 2/\mu)$ and $\mathcal{G}(1,\ 2/\mu)$ and mixing constant

$$\theta_n = \frac{1}{n+1}$$

for $n = 1, 2, 3, \ldots$ and all $x > 0$.

Proof. First assume that X has $\mathcal{G}(2,\ 2/\mu)$ distribution with density function

$$f(x) = \frac{4}{\mu}\ x\ e^{-\frac{2x}{\mu}},\ x > 0$$

and characteristic function

$$\phi(t) = \left(1 - \frac{it\mu}{2}\right)^{-2}.$$

Then from (3.23),

$$\begin{aligned}
\phi_1(t) &= \frac{1}{it\mu}\left[\left(1 - \frac{it\mu}{2}\right)^{-2} - 1\right], \\
&= \frac{1}{it\mu}\left(1 - \frac{it\mu}{2}\right)^{-2}\left[it\mu - \left(\frac{it\mu}{2}\right)^{2}\right], \\
&= \left(1 - \frac{it\mu}{2}\right)^{-2}\left[\frac{1}{2} + \frac{1}{2}\left(1 - \frac{it\mu}{2}\right)\right], \\
&= \frac{1}{2}\left[\left(1 - \frac{it\mu}{2}\right)^{-1} + \left(1 - \frac{it\mu}{2}\right)^{-2}\right].
\end{aligned}$$

Now by induction using (3.23)

$$\phi_n(t) = \frac{n}{n+1}\left(1 - \frac{it\mu}{2}\right)^{-1} + \frac{1}{n+1}\left(1 - \frac{it\mu}{2}\right)^{-2} \tag{3.29}$$

which is the characteristic function of the mixture of distributions stated in the theorem. This proves the necessary part. To prove the sufficiency, we note that whenever S_n is the mixture of gamma distributions as stated in the theorem, (3.29) is true for $n = 1$. From (3.24) we have

$$\phi_1(t) = \frac{\phi(t) - 1}{it\mu}$$

or

$$\begin{aligned}
\phi(t) &= 1 + it\mu\ \phi_1(t)\ , \\
&= 1 + \frac{1}{2}it\mu \left[\left(1 - \frac{it\mu}{2}\right)^{-1} + \left(1 - \frac{it\mu}{2}\right)^{-2} \right]\ , \\
&= 1 + \frac{it\mu}{2} \left(1 - \frac{it\mu}{2}\right)^{-2} \left[1 - \frac{it\mu}{2} + 1\right], \\
&= 1 + \left(1 - \frac{it\mu}{2}\right)^{-2} \left[it\mu - \left(\frac{it\mu}{2}\right)^{2}\right], \\
&= 1 + \left(1 - \frac{it\mu}{2}\right)^{-2} \left[1 - \left(1 - \frac{it\mu}{2}\right)^{2}\right], \\
&= \left(1 - \frac{it\mu}{2}\right)^{-2}\ ,
\end{aligned}$$

as required. ■

A second application of the above result is in mutual characterizations of two probability distributions. Let X and X_n be random variables considered above and let Y be another random variable independent of X with characteristic function $\psi(t)$. Then we have the following theorem.

Theorem 3.4.2. *Let X_n be identically distributed as $X + Y$ for two independent random variables X and Y, then X is exponential if and only if Y degenerates at zero.*

Proof. Assume X_n and $X + Y$ have the same distribution, with X having exponential distribution. Then with the above notations,

$$\phi_n(t) = \phi(t)\ \psi(t)\ . \tag{3.30}$$

Since X is exponential with mean σ from (3.23)

$$\phi(t) = \phi_n(t) = \frac{1}{1 - it\sigma}$$

and hence (3.30) gives $\psi(t) \equiv 1$ or Y degenerates at 0. Converse is obtained by assuming $\psi(t) = 1$ and therefrom $\phi_n(t) = \phi(t)$ so that (3.23) gives

$$\phi(t) = \frac{1}{1 - it\sigma} \ ,$$

the characteristic function of the exponential distribution. ■

Remark 3.4.1. *The above Theorem extends some results for the case $n = 1$ studied in* Pakes (1996) *and* Pakes *et al.* (2003).

3.5 Characterizations of ageing classes

In this section, we deduce some results which give alternative definitions of certain ageing classes in common use, in terms of stochastic orders of equilibrium distributions and their residual life distributions. Most of the results, which discussed on baseline distribution or lower order equilibrium distributions, hold for their higher orders as well. Following theorem gives some interpretations of the IFR notion.

Theorem 3.5.1. *X, IFR is equivalent to $\dfrac{f(x)}{S(x+t)} \uparrow x$, for all $t > 0$.*

Proof.

$$\begin{aligned} X,\ \text{IFR} \ &\Leftrightarrow\ (X - u|X > u) \le_{hr} X, \text{by Definition 2.2.11}, \\ &\Leftrightarrow\ \frac{f(x+u)}{S(u)}\frac{S(u)}{S(u+x+y)} \ge \frac{f(x)}{S(x+y)}, \\ &\quad\ \ \text{for all } x, y, t, u > 0,\ \text{by Definition 2.2.3} \\ &\Leftrightarrow\ \frac{f(x)}{S(x+y)} \uparrow x \text{ for all } y > 0. \end{aligned}$$

■

Remark 3.5.1. *We can deduce similar interpretations for the DFR class by reversing the inequalities.*

Shaked and Shanthikumar (2007, p.243) have established that

$$(X-x|X>x) \leq_{Lt} \ (Y-x|Y>x) \Leftrightarrow \frac{\displaystyle\int_x^{\infty} e^{-s(u-x)}S(u)\ du}{\displaystyle\int_x^{\infty} e^{-s(u-x)}T(u)\ du} \downarrow x,$$

for all $s>0$. Following lemma is an extension of the result, which is used to prove some equivalence relations involving Laplace transforms.

Lemma 3.5.1. *Let X and Y be non negative continuous random variables with survival functions $S(x)$ and $T(x)$ respectively. Then for all $x,\ y>0$, and $s>0$,*

$$(X-(x+y)|X>(x+y)) \leq_{Lt} \ (Y-x|Y>x) \Leftrightarrow \frac{\displaystyle\int_{x+y}^{\infty} e^{-s(u-x-y)}S(u)\ du}{\displaystyle\int_x^{\infty} e^{-s(u-x)}T(u)\ du} \downarrow x.$$

Proof. For all $x,\ y>0$,

$$\frac{\displaystyle\int_{x+y}^{\infty} e^{-s(u-x-y)}S(u)\ du}{\displaystyle\int_x^{\infty} e^{-s(u-x)}T(u)\ du} \downarrow x$$

$$\Leftrightarrow \frac{\displaystyle\int_x^{\infty} e^{-s(u-x)}S(u+y)\ du}{\displaystyle\int_x^{\infty} e^{-s(u-x)}T(u)\ du} \downarrow x,$$

$$\Leftrightarrow \frac{d}{dx}\left[\frac{\displaystyle\int_x^{\infty} e^{-s(u-x)}S(u+y)\ du}{\displaystyle\int_x^{\infty} e^{-s(u-x)}T(u)\ du}\right] \leq 0,$$

$$\Leftrightarrow -S(x+y)\int_x^{\infty} e^{-s(u-x)}T(u)\ du + T(x)\int_x^{\infty} e^{-s(u-x)}S(u+y)\ du \leq 0,$$

$$\Leftrightarrow \frac{1}{S(x+y)}\int_x^{\infty} e^{-s(u-x)}S(u+y)\ du \leq \frac{1}{T(x)}\int_x^{\infty} e^{-s(u-x)}T(u)\ du,$$

$$\Leftrightarrow \frac{1}{S(x+y)} \int_0^\infty e^{-su} S(u+x+y)\ du \leq \frac{1}{T(x)} \int_0^\infty e^{-su} T(u+x)\ du,$$
$$\Leftrightarrow (X-(x+y)|X>(x+y)) \leq_{Lt}\ (Y-x|Y>x),\ \text{by Definition 2.2.2.}$$

■

The following theorem discusses the equivalence conditions for DMRL notion.

Theorem 3.5.2. *X, DMRL is equivalent to any of the following*

1. $h(x)\ m(x+y) \leq 1,\ x>0, y \geq 0,$
2. $\frac{1}{S(x)} \int_{x+y}^\infty S(u)\ du \downarrow x$, *for all* $y > 0$,
3. $(X_1 - y|X_1 > y) \leq_{hr} (X-x|X>x),\ x \leq y < \infty,$
4. $(X_1 - y|X_1 > y) \leq_{st} (X-x|X>x),\ x \leq y < \infty,$
5. $(X_1 - y|X_1 > y) \leq_{Lt} (X-x|X>x),\ x \leq y < \infty,$
6. $(X_1 - y|X_1 > y) \leq_{Lt} (X_1-x|X_1>x),\ x \leq y < \infty.$

Proof. 1. Assume that

$$h(x)\ m(x+y) \leq 1,\ x>0, y \geq 0.$$

Using the general relationship between $h(x)$ and $m(x)$ given in (2.8), the above condition is equivalent to

$$\left[1 + \frac{d}{dx} m(x)\right] \frac{m(x+y)}{m(x)} \leq 1.$$

The above inequality is true only when at least one of the terms of the product on the left is less than or equal to 1. But when one term is less than one, the other also satisfies the same condition so that X is DMRL. Conversely, if X is

DMRL, then $1 + m'(x) \leq 1$,

$$\frac{m(x+y)}{m(x)} \leq 1$$

and hence $h(x)\ m(x+y) \leq 1$, for all $x > 0$ and $y > 0$.

2.

$$\begin{aligned} X,\ \text{DMRL} \ &\Leftrightarrow\ (X - t|X > t) \leq_{mrl} X,\ \text{for all } t > 0,\ \text{by Definition 2.2.16}, \\ &\Leftrightarrow\ \frac{S(t)}{S(x+t)} \int_{x+y}^{\infty} \frac{S(u+t)}{S(t)}\ du \leq \frac{1}{S(x)} \int_{x+y}^{\infty} S(u)\ du, \\ &\quad\ \text{by Definition 2.2.9}, \\ &\Leftrightarrow\ \frac{1}{S(x+t)} \int_{x+y+t}^{\infty} S(u)\ du \leq \frac{1}{S(x)} \int_{x+y}^{\infty} S(u)\ du, \\ &\Leftrightarrow\ \frac{1}{S(x)} \int_{x+y}^{\infty} S(u)\ du\ \downarrow x,\ \text{for all } x,\ y > 0. \end{aligned}$$

3. For all $0 < x \leq y$,

$$\begin{aligned} (X_1 - y|X_1 > y) \leq_{hr} (X - x|X > x) \\ &\Leftrightarrow h_1(t+y) \geq h(t+x),\ \text{for all } t > 0, \\ &\Leftrightarrow h(t+x)\ m(t+y) \leq 1,\ \text{by using (2.38)}, \\ &\Leftrightarrow X \text{ is DMRL, by result 1 of this theorem.} \end{aligned}$$

4. For all $0 < x \leq y$,

$$\begin{aligned} (X_1 - y|X_1 > y) \leq_{st} (X - x|X > x) \\ &\Leftrightarrow \frac{S_1(t+y)}{S_1(y)} \leq \frac{S(t+x)}{S(x)}, \\ &\Leftrightarrow \frac{f_1(t+x)}{S_1(t+y)} \geq \frac{f_1(x)}{S_1(y)},\ \text{by (3.1)}, \\ &\Leftrightarrow X_1 \text{ is IFR by Theorem 3.5.1}, \\ &\Leftrightarrow X \text{ is DMRL.} \end{aligned}$$

5.

$$\begin{aligned} X, \text{ is DMRL } &\Rightarrow (X_1 - y|X_1 > y) \leq_{st} (X - x|X > x), \\ &\Rightarrow (X_1 - y|X_1 > y) \leq_{Lt} (X - x|X > x). \end{aligned}$$

Conversely for all $y \geq x > 0$ and $s > 0$,

$$\begin{aligned} &(X_1 - y|X_1 > y) \leq_{Lt} (X - x|X > x) \\ &\Leftrightarrow \frac{\int_{x+t}^{\infty} e^{-s(u-x-t)} S_1(u)du}{\int_{x}^{\infty} e^{-s(u-x)} S(u)du} \downarrow x, \ t = y - x, \\ &\quad \text{by Lemma 3.5.1 and which on integration by parts,} \\ &\Leftrightarrow \frac{S_1(x+t) - e^{s(x+t)} \int_{x+t}^{\infty} e^{-su} f_1(u)du}{S(x) - e^{sx} \int_{x}^{\infty} e^{-su} f(u)du} \downarrow x, \\ &\Rightarrow \frac{S_1(x+t)}{S(x)} \downarrow x, \ \text{as } s \to \infty, \\ &\Rightarrow \frac{f_1(x)}{S_1(x+t)} \uparrow x, \ \text{by (3.1)}, \\ &\Rightarrow X_1, \ \text{is IFR, by Theorem 3.5.1}, \\ &\Rightarrow X, \ \text{is DMRL}. \end{aligned}$$

6.

$$\begin{aligned} X, \text{ DMRL } &\Leftrightarrow X_1, \text{ IFR}, \\ &\Leftrightarrow (X_1 - y|X_1 > y) \leq_{Lt} (X_1 - x|X_1 > x), \text{ by Definition 2.2.11.} \end{aligned}$$

■

The behaviour of the IFR (DFR) classes of distributions in the case of equilibrium distribution of order n, is discussed in the following theorem. It is to be noted that

$$X,\ \text{IFR} \Rightarrow X,\ \text{IFR(2)} \Rightarrow X,\ \text{DMRL}\ (X_1,\ \text{IFR}) \Rightarrow X_1\ \text{IFR(2)} \Rightarrow X_1,\ \text{DMRL}\ \ldots.$$

Theorem 3.5.3. *If X is IFR (DFR) then*

1. $h(x) \leq (\geq) h_1(x) \ \leq (\geq)\ h_2(x) \ \leq (\geq) \ldots$ *and*
2. $m(x) \geq (\leq) m_1(x) \ \geq (\leq) m_2(x) \geq (\leq) \ldots.$

Proof. Suppose X is IFR, then $h_n'(x) \geq 0$ for all n and $x > 0$. Hence by (3.5), it is clear that

$$h(x) \leq h_1(x) \ \leq \ h_2(x) \ \leq \ldots.$$

For proving the second relation, we note that X is IFR $\Rightarrow$ X is DMRL and then $m_n'(x) \leq 0$ for all n and $x > 0$. Thus the result follows from the identity (3.4). We can prove the results for DFR by reversing the inequalities. ■

The following theorem discusses behaviour of equilibrium distributions having bath-tub shaped hazard rate. To prove the theorem, we use Theorem 4.2 of Lai and Xie (2006), which states that if X is BT then $m(x)$ is of UBT shape for $h(0)\mu > 1$ and $m(x)$ is decreasing for $h(0)\mu \leq 1$. It is also to be noted that $m_n(0) = \mu_n$ for all $n = 1, 2, 3, \ldots$.

Theorem 3.5.4. *Let $h(x)$ be differentiable. Then any one of the following hold.*

1. *X_n is BT $\Longrightarrow$ X_{n+1} is BT, if $\lim_{x\to 0} m_n'(x) = \lim_{x\to 0} h_n(x)\ m_n(x) - 1 > 0$; moreover if $h_{n+1}'(x_0) = 0$ for some $x_0 \geq 0$, then $h_n(x_0) = h_{n+1}(x_0)$.*
2. *X_n is BT $\Longrightarrow$ X_{n+1} is IFR, if $\lim_{x\to 0} m_n'(x) = \lim_{x\to 0} h_n(x)\ m_n(x) - 1 \leq 0$.*

Proof. Let us assume that $h_n(0)\ m_n(0) > 1$.

$$\begin{aligned} X_n \text{ is BT} \ &\Rightarrow \ m_n(x) = \frac{1}{h_{n+1}(x)} \text{ is UBT,} \\ &\Rightarrow \ X_{n+1} \text{ is BT.} \end{aligned}$$

Suppose x_0 is a change point of $h_{n+1}(x)$ then $h'_{n+1}(x_0) = 0$. Thus, using the identity (3.5), it is clear that $h_n(x_0) = h_{n+1}(x_0)$. The second part is also clear from the Theorem 4.2 of Lai and Xie (2006). ■

Remark 3.5.2. *Let $h(x)$ be differentiable. Then it is also clear that*

1. *X is UBT $\Longrightarrow$ X_1 is UBT, if $\lim_{x\to 0} h(x)\ m(x) < 1$.*
2. *X is UBT $\Longrightarrow$ X_1 is DFR, if $\lim_{x\to 0} h(x)\ m(x) \geq 1$.*

Now consider the UBA notion given in the Definition 2.2.21. It is clear that

$$X,\ \text{UBA} \Leftrightarrow (X - x|X > x) \geq_{st}\ X_E,$$

where X_E has the exponential distribution with mean $\mu_E = \lim_{x\to\infty} m(x)$. The following theorem discusses the relationship between the UBA and UBAE classes of equilibrium distributions.

Theorem 3.5.5. *Suppose $0 < m_{n-1}(\infty) < \infty$, then*

$$X_{n-1},\ \textit{UBAE (UWAE)}\ \Longrightarrow X_n,\ \textit{UBA (UWA)}.$$

Proof. From (2.3),

$$\begin{aligned}\frac{S_n(x+t)}{S_n(x)} &= \exp\left[-\int_0^{x+t} h_n(u)\ du + \int_0^x h_n(u)\right],\\ &= \exp\left[-\int_x^{x+t} h_n(u)du\right],\\ &= \exp\left[-\int_x^{x+t} \frac{1}{m_{n-1}(u)}du\right],\ \text{by (2.38)}.\end{aligned}$$

Form (2.9) we can see that $\lim_{x\to\infty} m'_n(x) = 0$ and also from (3.4),

$$m_{n-1}(x) = \frac{m_n(x)}{1 + m'_n(x)},$$

and

$$\lim_{x\to\infty} m_{n-1}(x) = \lim_{x\to\infty} m_n(x). \tag{3.31}$$

Now, if X_{n-1} is UBAE,

$$\begin{aligned}
m_{n-1}(x) \geq m_{n-1}(\infty) \ &\Rightarrow \ \int_x^{x+t} \frac{1}{m_{n-1}(u)}du \leq \int_x^{x+t} \frac{1}{m_{n-1}(\infty)}du, \\
&\Rightarrow \ \exp\left[-\int_x^{x+t} \frac{1}{m_{n-1}(u)}du\right] \geq \exp\left[-\int_x^{x+t} \frac{1}{m_{n-1}(\infty)}du\right] \\
&\Rightarrow \ \frac{S_n(x+t)}{S_n(x)} \geq \exp\left[\frac{-t}{m_{n-1}(\infty)}\right] = \exp\left[\frac{-t}{m_n(\infty)}\right], \text{ by (3.31)} \\
&\Rightarrow \ X_n \text{ is UBA.}
\end{aligned}$$

■

Remark 3.5.3. *If $0 < m_n(\infty) < \infty$, then for all n,*

$$X, \ UBA \Rightarrow X, \ UBAE \ \Rightarrow X_1, \ UBA \Rightarrow X_1, \ UBAE \ \Rightarrow \ldots.$$

In the following theorem, some characterizations of DVRL class are discussed using the equilibrium distributions and residual lives.

Theorem 3.5.6. *X, DVRL or X_1, DMRL is equivalent to any of the following*

1. $(X_1 - y|X_1 > y) \leq_{mrl} (X - x|X > x)$,
2. $(X_1 - y|X_1 > y) \leq_{icx} (X - x|X > x)$,
3. $(X_1 - y|X_1 > y) \leq_{hmrl} (X - x|X > x)$,

for all $y \geq x > 0$.

Proof. 1. For all $y \geq x > 0$,

$$\begin{aligned}(X_1 - y|X_1 > y) \leq_{mrl} (X - x|X > x) & \\ &\Leftrightarrow m_1(t+y) \leq m(t+x), \\ &\Leftrightarrow h_1(t+x)\; m_1(t+y) \leq 1, \text{ by (2.38)}, \\ &\Leftrightarrow X_1 \text{ is DMRL, by result 1 of Theorem 3.5.2}, \\ &\Leftrightarrow X \text{ is DVRL.}\end{aligned}$$

2. For all $y \geq x > 0$,

$$\begin{aligned}(X_1 - y|X_1 > y) \leq_{icx} (X - x|X > x) & \\ &\Leftrightarrow (X_1 - y|X_1 > y) \leq_{mrl} (X - x|X > x), \text{ by Definition 2.2.9}, \\ &\Leftrightarrow X_1 \text{ is DMRL}, \\ &\Leftrightarrow X \text{ is DVRL.}\end{aligned}$$

3. For all $y \geq x > 0$,

$$\begin{aligned}(X_1 - y|X_1 > y) \leq_{hmrl} (X - x|X > x) & \\ &\Leftrightarrow (X_1 - y|X_1 > y) \leq_{mrl} (X - x|X > x), \text{ by Definition 2.2.9}, \\ &\Leftrightarrow X_1 \text{ is DMRL}, \\ &\Leftrightarrow X \text{ is DVRL.}\end{aligned}$$

■

Theorem 3.5.7. *X_1, DVRL if and only if $(X_1 - y|X_1 > y) \leq_{vrl} (X - x|X > x)$, for $y \geq x > 0$.*

Proof.

$$(X_1 - y|X_1 > y) \leq_{vrl} (X - x|X > x)$$

$$\Leftrightarrow \frac{\frac{1}{S_1(y)}\int_t^\infty \int_u^\infty S_1(v+y)\ dv\ du}{\frac{1}{S(x)}\int_t^\infty \int_u^\infty S(v+x)\ dv\ du} \downarrow t, \text{ by Definition 2.2.6,}$$

$$\Leftrightarrow \frac{\int_t^\infty \int_{u+y}^\infty S_1(v)\ dv\ du}{\int_t^\infty \int_{u+x}^\infty S(v)\ dv\ du} \downarrow t,$$

$$\Leftrightarrow \frac{\int_t^\infty S_2(u+y)\ du}{\int_t^\infty S_1(u+x)\ du} \downarrow t, \text{ by (3.1),}$$

$$\Leftrightarrow \frac{S_3(t+y)\ dt}{S_2(t+x)\ dt} \downarrow t, \text{ by (3.1),}$$

$$\Leftrightarrow \frac{f_3(t+x)}{S_3(t+y)} \uparrow t,, \text{ again by (3.1)}$$

$\Leftrightarrow X_3$ is IFR, by result 1 of Theorem 3.5.2,

$\Leftrightarrow X_1$ is DVRL.

■

Now, let us consider the generalized increasing mean residual life (GIMRL) class discussed in the Definition 2.2.28. It follows from the definition that for any $x > 0$,

$$\begin{aligned} X, \text{ GIMRL} \ &\Leftrightarrow \ \frac{1}{S(x+t)}\int_t^\infty S(u)\ du \ \uparrow t, \\ &\Leftrightarrow \ \frac{S(t)\ m(t)}{S(x+t)} \uparrow t, \\ &\Leftrightarrow \ \frac{S(t)}{S(x+t)\ h_1(t)} \uparrow t, \\ &\Leftrightarrow \ \frac{f_1(t)}{f_1(x+t)}\frac{S_1(t)}{f_1(t)} \uparrow t, \\ &\Leftrightarrow \ \frac{S_1(t)}{f_1(x+t)} \uparrow t. \end{aligned} \tag{3.32}$$

It is also clear that GIMRL $\Rightarrow$ IMRL as $x \to 0$. The following theorem gives some interpretations of GIMRL in terms of the residual lives of equilibrium distributions.

Theorem 3.5.8. *X, GIMRL is equivalent to any of the following for all $y \geq x > 0$.*

1. $(X - y|X > y) \ \leq_{hr} \ (X_1 - x|X_1 > x)$,
2. $(X - y|X > y) \ \leq_{Lt} \ (X_1 - x|X_1 > x)$,
3. $(X_1 - y|X_1 > y) \ \leq_{lr} \ (X_2 - x|X_2 > x)$.

Proof. 1. For all $y \geq x > 0$,

$$\begin{aligned}
(X - y|X > y) \leq_{hr} \ & (X_1 - x|X_1 > x) \\
&\Leftrightarrow \frac{S_1(t+x)}{S_1(x)} \frac{S(y)}{S(t+y)} \uparrow t, \text{ by Definition 2.2.3,} \\
&\Leftrightarrow \frac{S_1(t+x)}{S(t+y)} \uparrow t, \\
&\Leftrightarrow \frac{S_1(t+x)}{f_1(t+y)} \uparrow t, \text{ by (3.1),} \\
&\Leftrightarrow X, \text{ GIMRL, by (3.32).}
\end{aligned}$$

2. For all $y \geq x > 0$,

$$\begin{aligned}
X, \text{ GIMRL} \ &\Rightarrow \ (X - y|X > y) \ \leq_{hr} \ (X_1 - x|X_1 > x), \\
&\Rightarrow \ (X - y|X > y) \ \leq_{Lt} \ (X_1 - x|X_1 > x).
\end{aligned}$$

Conversely by the Lemma 3.5.1,

$$\begin{aligned}
(X - y|X > y) \ \leq_{Lt} \ & (X_1 - x|X_1 > x) \\
&\Leftrightarrow \frac{\displaystyle\int_{x+t}^{\infty} e^{-s(u-x-t)} S(u) \, du}{\displaystyle\int_{x}^{\infty} e^{-s(u-x)} S_1(u) \, du} \downarrow x, \ t = y - x,
\end{aligned}$$

which on integration by parts,

$$\Leftrightarrow \frac{S(x+t) - \displaystyle\int_{x+t}^{\infty} e^{-s(u-x-t)} f(u) \, du}{S_1(x) - \displaystyle\int_{x}^{\infty} e^{-s(u-x)} f_1(u) \, du} \downarrow x,$$

$$\Rightarrow \frac{S(x+t)}{S_1(x)} \downarrow x \text{ as } s \to \infty,$$
$$\Rightarrow \frac{f_1(x+t)}{S_1(x)} \downarrow x, \text{ for all } t > 0,$$
$$\Rightarrow X, \text{ GIMRL, by (3.32).}$$

3. Finally, for all $y \geq x > 0$,

$$(X_1 - y|X_1 > y) \leq_{lr} (X_2 - x|X_2 > x)$$
$$\Leftrightarrow \frac{f_1(t+y)}{f_2(t+x)} \downarrow t, \text{ by Definition 2.2.4,}$$
$$\Leftrightarrow \frac{f_1(t+y)}{S_1(t+x)} \downarrow t, \text{ by (3.1),}$$
$$\Leftrightarrow X, \text{ GIMRL, by (3.32).}$$

■

The following theorem, as well as the above, characterizes the GIMRL of higher order equilibrium distributions.

Theorem 3.5.9. *X_n, GIMRL is equivalent to any of the following*

1. $(X_{n-1} - y|X_{n-1} > y) \leq_{mrl} (X_n - x|X_n > x)$,

2. $(X_{n-2} - y|X_{n-2} > y) \leq_{vrl} (X_{n-1} - x|X_{n-1} > x)$,

for all $y \geq x > 0$.

Proof. 1. For all $y \geq x > 0$,

$$
\begin{aligned}
(X_{n-1} - y | X_{n-1} > y) \ \leq_{mrl} \ (X_n - x | X_n > x) \\
\Leftrightarrow \frac{\frac{1}{S_{n-1}(y)} \displaystyle\int_t^{\infty} S_{n-1}(u+y)\ du}{\frac{1}{S_n(x)} \displaystyle\int_t^{\infty} S_n(u+x)\ du} \downarrow t, \\
\text{by Definition 2.2.9,} \\
\Leftrightarrow \frac{\displaystyle\int_{t+y}^{\infty} f_n(u)\ du}{\displaystyle\int_{t+x}^{\infty} f_{n+1}(u)\ du} \downarrow t, \ \text{by (3.1),} \\
\Leftrightarrow \frac{S_n(t+y)}{S_{n+1}(t+x)} \downarrow t, \ \text{by (3.1),} \\
\Leftrightarrow \frac{f_{n+1}(t+y)}{S_{n+1}(t+x)} \downarrow t, \ \text{again by (3.1),} \\
\Leftrightarrow X_n, \ \text{is GIMRL, by (3.32).}
\end{aligned}
$$

2. Again for all $y \geq x > 0$, by using the relation (3.1),

$$
\begin{aligned}
(X_{n-2} - y | X_{n-2} > y) \ \leq_{vrl} (X_{n-1} - x | X_{n-1} > x) \\
\Leftrightarrow \frac{\frac{1}{S_{n-2}(y)} \displaystyle\int_t^{\infty} \int_u^{\infty} S_{n-2}(v+y)\ dv\ du}{\frac{1}{S_{n-1}(x)} \displaystyle\int_t^{\infty} \int_u^{\infty} S_{n-1}(v+x)\ dv\ du} \downarrow t, \\
\text{by Definition 2.2.6,} \\
\Leftrightarrow \frac{\displaystyle\int_t^{\infty} \int_u^{\infty} f_{n-1}(v+y)\ dv\ du}{\displaystyle\int_t^{\infty} \int_u^{\infty} f_n(v+x)\ dv\ du} \downarrow t, \\
\Leftrightarrow \frac{\displaystyle\int_t^{\infty} S_{n-1}(u+y)\ du}{\displaystyle\int_t^{\infty} S_n(u+x)\ du} \downarrow t,
\end{aligned}
$$

$$\Leftrightarrow \frac{\displaystyle\int_{t+y}^{\infty} f_n(u)\ du}{\displaystyle\int_{t+x}^{\infty} f_{n+1}(u)\ du} \downarrow t,$$

$$\Leftrightarrow \frac{S_n(t+y)}{S_{n+1}(t+x)} \downarrow t,$$

$$\Leftrightarrow \frac{f_{n+1}(t+y)}{S_{n+1}(t+x)} \downarrow t,$$

$$\Leftrightarrow X_n,\ \text{is GIMRL, by (3.32).}$$

■

Chapter 4

Reliability Aspects of Discrete Equilibrium Distributions

4.1 Introduction

Discrete equilibrium distribution has originated as the asymptotic distribution of the residual waiting time for the first occurrence of an event in discrete renewal theory (Feller (1957)). The study of discrete equilibrium models was initiated by Gupta (1979), in which the properties of discrete equilibrium distribution in comparison with the basic distribution and characterizations of the geometric distribution were studied. This work was extended by Nair and Hitha (1989) by deriving the relationship between various reliability concepts in the original and equilibrium models that characterizes certain discrete models. The study was then followed up by Hitha and Nair (1989), Sen and Khattree (1996). Fagiuoli and Pellerey (1994) defined recursively the higher order discrete versions of their previous work in the continuous case. A recent contribution to the theory of discrete equilibrium distribution was made by Willmot et al. (2005), who derived the relationship between the baseline distribution and its kth order equilibrium version. They further discussed the equilibrium distributions of compound distributions, their reliability properties, and application to insurance claims modeling. A detailed discussion of some new properties of discrete equilibrium models with an emphasis on their applications to reliability analysis is aimed at in the present chapter.

4.2 Distribution theory

Let X be a discrete random variable taking values in $\mathcal{N} = \{0, 1, 2, \dots\}$ or its subset, with probability mass function $f(x)$, survival function $S(x) = P(X > x)$ and finite mean μ. Also the failure rate of X is $h(x)$ as in (2.14), mean residual life function is $m(x)$ as in (2.16), variance residual life function $V(x)$ as in (2.19) and the rth factorial stop-loss moment $\alpha_r(x)$ as in (2.22). With the above background materials we define equilibrium distributions and study their role in reliability.

Definition 4.2.1. *Let $E(X^n) < \infty$. Then the equilibrium distribution of order n of the random variable X is defined recursively for $n = 1, 2, 3, \dots$ by the probability mass function*

$$f_n(x) = \frac{1}{\mu_{n-1}} S_{n-1}(x),\ x = 0, 1, 2, \dots, \tag{4.1}$$

where

$$\mu_{n-1} = \sum_{x=0}^{\infty} S_{n-1}(x) < \infty,$$

$S_0(x) = S(x)$ and $\mu_0 = \mu$. In the sequel, we denote by X_n, the random variable with distribution (4.1), so that $X = X_0$.

Much of the discussions on (4.1) require a comparison of the characteristics of (4.1) with those of the baseline distribution of X. Therefore, we first derive a relationship between $S_n(x)$ and $S(x)$.

Theorem 4.2.1. *The survival functions of X_n and X are related by*

$$S_n(x) = \frac{1}{\mu_{(n)}} E\left((X - x - 1)^{(n)} | X > x + n\right) S(x + n), \tag{4.2}$$

where $\mu_{(n)} = E(X^{(n)})$ and $X^{(n)} = X(X-1)\cdots(X-n+1)$.

Proof. First, we note that with respect to the descending factorials $(t-x)^{(r)}$,

$$\begin{aligned}
\sum_{u=x+r+1}^{\infty}(u-x)^{(r+1)}f(u) &= \sum_{u=x+r+1}^{\infty}(u-x)^{(r+1)}[S(u-1)-S(u)],\\
&= \sum_{u=x+r+1}^{\infty}(u-x)^{(r+1)}S(u-1)-\sum_{u=x+r+1}^{\infty}(u-x)^{(r+1)}S(u),\\
&= \sum_{u=x+r}^{\infty}(u-x)^{(r)}[u+1-x-(u-x-r-1+1)]\ S(u),\\
&= (r+1)\sum_{u=x+r}^{\infty}(u-x)^{(r)}S(u). \qquad (4.3)
\end{aligned}$$

Hence from (4.1) and (4.3),

$$\begin{aligned}
\mu S_1(x) &= \sum_{u=x+1}^{\infty}S(u),\\
&= S(x+1)\ E(X-x-1|X>x+1)
\end{aligned}$$

and

$$\begin{aligned}
\mu_1 S_2(x) &= \sum_{u=x+1}^{\infty}S_1(u),\\
&= \sum_{u=x+2}^{\infty}(u-x-1)\ f_1(u),\\
&= \frac{1}{\mu}\sum_{u=x+2}^{\infty}(u-x-1)\ S(u),\\
&= \frac{1}{2\mu}\sum_{u=x+3}^{\infty}(u-x-1)^{(2)}\ f(u),
\end{aligned}$$

or

$$\begin{aligned}
2!\ \mu\mu_1 S_2(x) &= \sum_{u=x+3}^{\infty}(u-x-1)^{(2)}\ f(u),\\
&= E\left((X-x-1)^{(2)}|X>x+2\right)S(x+2).
\end{aligned}$$

Proceeding similarly, by induction we have for any n

$$n!\ \mu\mu_1\cdots\mu_{n-1}\ S_n(x) = E\left((X-x-1)^{(n)}|X>x+n\right)S(x+n) \tag{4.4}$$

Setting $x=-1$ in (4.4),

$$\begin{aligned} n!\ \mu\mu_1\cdots\mu_{n-1} &= E\left(X^{(n)}|X>n-1\right)S(n-1), \\ &= \sum_{u=n}^{\infty} u^{(n)} f(u), \\ &= \mu_{(n)}. \end{aligned} \tag{4.5}$$

Substituting (4.5) in (4.4) we have (4.2). ■

Remark 4.2.1. *The moments of* X_n *and* X *are connected through the relation* (4.5)*.*

Remark 4.2.2. *In terms of the stop loss moments defined earlier in* (2.22)*,*

$$S_n(x) = \frac{\alpha_n(x+1)}{\mu_{(n)}}. \tag{4.6}$$

One area in which the last result is useful is in the calculation of the stop loss premium and other actuarial applications (*see* Klugman *et al*. (1998)).

For many discrete laws it is more convenient to work with generating functions than with probability mass functions. Since factorial moments are predominant in our derivations, we look at the relationship between the factorial moment generating functions $W_n(t)$ and

$$\begin{aligned} W(t) &= E\left((1+t)^X\right), \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_{(r)} \end{aligned} \tag{4.7}$$

of X_n and X respectively.

We have

$$\begin{aligned} W(t) &= \sum_{u=0}^{\infty}(1+t)^u f(u), \\ &= \sum_{u=0}^{\infty}(1+t)^u \left[S(u-1) - S(u)\right], \\ &= 1 + t\sum_{u=0}^{\infty}(1+t)^u S(u) \end{aligned} \tag{4.8}$$

and hence from (4.8),

$$\begin{aligned} W_1(t) &= \frac{1}{\mu}\sum_{u=0}^{\infty}(1+t)^u S(u), \\ &= \frac{W(t)-1}{t\mu}, \end{aligned}$$

the relation between the factorial moment generating function of X and its equilibrium distribution. Using this for X_1 and X_2,

$$\begin{aligned} W_2(t) &= \frac{W_1(t)-1}{t\mu_1}, \\ &= \frac{W(t)-t\mu-1}{t^2\mu\mu_1}. \end{aligned}$$

By induction,

$$\begin{aligned} W_n(t) &= \frac{1}{t^n\mu\mu_1\cdots\mu_{n-1}}\left[W(t) - \sum_{r=1}^{n-1} t^r\mu\mu_1\cdots\mu_{r-1} - 1\right],\ n>1, \\ &= \frac{n!}{t^n\mu_{(n)}}\left[W(t) - \sum_{r=0}^{n-1}\frac{t^r\mu_{(r)}}{r!}\right],\ \text{by (4.5)}, \\ &= \frac{n!}{t^n\mu_{(n)}}\left[\sum_{r=0}^{\infty}\frac{t^r\mu_{(r)}}{r!} - \sum_{r=0}^{n-1}\frac{t^r\mu_{(r)}}{r!}\right], \\ &= \frac{n!}{t^n\mu_{(n)}}\sum_{r=n}^{\infty}\frac{t^r\mu_{(r)}}{r!}, \\ &= \frac{n!}{\mu_{(n)}}\sum_{r=0}^{\infty}\frac{t^r\mu_{(n+r)}}{(n+r)!}, \end{aligned}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\binom{n+r}{r}^{-1} \frac{\mu_{(n+r)}}{\mu_{(n)}} \right], \tag{4.9}$$

obtained on using (4.5) and (4.7). Thus the rth factorial moment of X_n becomes

$$\mu_{(r),n} = \binom{n+r}{r}^{-1} \frac{\mu_{(n+r)}}{\mu_{(n)}}. \tag{4.10}$$

When the moment generating function or moments are the starting points in the analysis, the distribution of X_n can be obtained from (4.10) as

$$f_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\mu_{(x+r),n}}{x!\ r!}.$$

The relationship between $\alpha_{r,n}(x)$, the r^{th} factorial stop loss moment of X_n and $\alpha_r(x)$ is

$$\alpha_{r+n}(x) = \binom{n+r}{r} \mu_{(n)} \alpha_{r,n}(x).$$

One can also write the relationship between the basic reliability characteristics of X_n and X_{n-1}. Denoting by $h_n(x)$ and $m_n(x)$ the failure rate and mean residual life functions of X_n, we have the identities,

$$\begin{aligned} h_n(x) &= \frac{f_n(x)}{S_n(x-1)}, \\ &= \frac{S_{n-1}(x)}{\sum_{u=x}^{\infty} S_{n-1}(u)}, \text{ by using (4.1)}, \\ &= \frac{1}{m_{n-1}(x)}, \text{ by (2.16)}, \end{aligned} \tag{4.11}$$

$$m_{n-1}(x+1) = \frac{m_n(x+1)}{1 + m_n(x+1) - m_n(x)}, \text{ by using (2.17)} \tag{4.12}$$

and

$$h_n(x+1) = 1 + h_{n+1}(x+1) \left[1 - \frac{1}{h_{n+1}(x)} \right], \text{ by (4.11) and (4.12)}, \tag{4.13}$$

connecting the reliability characteristics of equilibrium distributions of successive orders. Moreover, equations (4.3) and (4.11) through (4.13) are fundamental in studying the reliability aspects of X_n.

Discrete equilibrium distributions can be generated (interpreted) in different ways. In the continuous case, it is a weighted distribution of the original random variable with the reciprocal of the failure rate function as the weight function. The same interpretation does not hold in the discrete case.

However,

$$f_n(x) = \frac{\frac{1}{h_{n-1}(x)} - 1}{E\left(\frac{1}{h_{n-1}(X)} - 1\right)} f_{n-1}(x)$$

so that X_n is the weighted version of X_{n-1} with the weight function,

$$\begin{aligned} \frac{1}{h_{n-1}(x)} - 1 &= \frac{S_{n-1}(x-1)}{f_{n-1}(x)} - 1, \\ &= \frac{S_{n-1}(x-1) - f_{n-1}(x)}{f_{n-1}(x)}, \\ &= \frac{S_{n-1}(x)}{f_{n-1}(x)}. \end{aligned}$$

A second interpretation is through the bivariate distribution of a random vector (Y_1, Y_2) defined as

$$P(Y_1 = y_1, Y_2 = y_2) = \frac{p(y_2)}{\mu}, \; y_1 = 0, 1, \ldots, y_2 - 1; \; y_2 = x + 1, \ldots \tag{4.14}$$

where $p(\cdot)$ is the probability mass function of a random variable X defined on $\mathcal{N}$ with

$$\mu = E(X) < \infty.$$

The marginal distributions of (4.14) are

$$P(Y_1 = y_1) = \frac{1}{\mu} \sum_{u=y_1+1}^{\infty} p(u), \quad y_1 = 0, 1, 2, \ldots,$$

the equilibrium distribution of X and

$$\begin{aligned} P(Y_2 = y_2) &= \sum_{y_1=0}^{y_2-1} \frac{p(y_2)}{\mu}, \\ &= \frac{y_2}{\mu} p(y_2),\ y_2 = 1, 2, 3, \ldots, \end{aligned}$$

the length biased version of X. Further, the conditional distribution of Y_1 given $Y_2 = y_2$ becomes

$$P(Y_1 = y_1 | Y_2 = y_2) = \frac{1}{y_2}, \quad y_1 = 0, 1, 2, \ldots, y_2 - 1,$$

the uniform distribution. These results are discrete analogues of Lemma 2.1 of Brown (2006), but the representation $Y_1 = Y_2 U$ where U is uniform and independent of Y_2 in the continuous case does not appear to be true in the discrete case.

4.3 Characterizations

In this section we consider characterizations of some discrete distributions, by mutual relationships between characteristics of X and X_n, that are useful in reliability theory. The distributions in question specified by their probability mass functions are the geometric,

$$f_G(x) = q^x p,\ x = 0, 1, 2, \ldots, \tag{4.15}$$

where $q = 1 - p$ and $0 < p < 1$, the Waring,

$$f_W(x) = (a - b)\frac{(b)_x}{(a)_{x+1}},\ x = 0, 1, 2, \ldots;\ a > b,\ a, b \in \mathcal{N}, \tag{4.16}$$

where $(b)_x = b(b+1)\cdots(b+x-1)$ is the Pochhammer symbol and the negative hyper-geometric,

$$\begin{aligned} f_N(x) &= \frac{\binom{-1}{x}\binom{-k}{m-x}}{\binom{-1-k}{m}},\ x = 0, 1, 2, \ldots, m,\ k = 1, 2, \ldots, \qquad (4.17) \\ &= \frac{(-1)^x\binom{1+x-1}{x}\ (-1)^{m-x}\binom{k+m-x-1}{m-x}}{(-1)^m\binom{1+k+m-1}{m}}, \end{aligned}$$

$$= \frac{\binom{k+m-x-1}{m-x}}{\binom{k+m}{m}}.$$

These distributions are characterized in Nair and Hitha (1989) by a failure rate (mean residual life) function of the form

$$h(x) = \frac{1}{A + Bx}$$

or $m(x) = \alpha + \beta x$ with $B = 0$ ($\beta = 0$) for the geometric, $B > 0$ ($\beta > 0$) for the Waring and $B < 0$ ($\beta < 0$) for the negative hyper-geometric distributions. The three distributions will be abbreviated by $G(p)$, $W(a, b)$ and $NH(k, m)$.

Definition 4.3.1. *The zero-modified version of the random variable X is defined by the probability representation,*

$$P(X = 0) = \alpha + (1 - \alpha)\ f(0)$$

and

$$P(X = x) = (1 - \alpha)\ f(x),\ x = 1, 2, \ldots.$$

Note that the zero-modified distribution contains the original distribution when $\alpha = 0$.

Theorem 4.3.1. *If X is $G(p)$ $(W(a, b),\ NH(k, m))$, then X_n is $G(p)$ $(W(a, b + n),\ NH(k + n, m - n))$. Conversely if X_n has one of these distributions then X_{n-1} is distributed as its zero-modified version.*

Proof. Suppose X is $G(p)$ with survival function $S_G(x) = q^{x+1}$ and mean

$$\begin{aligned} \mu_G &= \sum_{u=0}^{\infty} S(u), \\ &= \frac{q}{p}. \end{aligned}$$

From (4.1), the survival function of X_1 is

$$\begin{aligned} S_{G,1}(x) &= \frac{1}{\mu_G} \sum_{u=x+1}^{\infty} S_G(u), \\ &= q^{x+1}, \end{aligned}$$

with mean

$$\mu_{G,1} = \frac{q}{p}.$$

In other words, X_1 is $G(p)$. Similarly, the distribution of X_2 is $G(p)$ with survival function

$$S_{G,2} = q^{x+1}$$

and mean

$$\mu_{G,2} = \frac{q}{p}.$$

Then by mathematical induction, the distribution of X_n is also $G(p)$.

Now suppose X is $W(a,b)$ with survival function

$$\begin{aligned} S_W(x) &= \sum_{u=x+1}^{\infty} f_W(u), \\ &= \sum_{u=x+1}^{\infty} (a-b)\ \frac{(b)_u}{(a)_{u+1}}, \\ &= \frac{(b)_{x+1}}{(a)_{x+2}} \sum_{u=0}^{\infty} (a-b)\ \frac{(b+x+1)_u}{(a+x+2)_u}, \\ &= (a+x+1)\ \frac{(b)_{x+1}}{(a)_{x+2}} \sum_{u=0}^{\infty} (a-b)\ \frac{(b+x+1)_u}{(a+x+1)_{u+1}}, \\ &= (a+x+1)\ \frac{(b)_{x+1}}{(a)_{x+2}}, \end{aligned}$$

and mean

$$\begin{aligned} \mu_W &= \sum_{u=0}^{\infty} S_W(u), \\ &= \sum_{u=0}^{\infty} (a+u+1)\ \frac{(b)_{u+1}}{(a)_{u+2}}, \end{aligned}$$

$$
\begin{aligned}
&= \sum_{u=0}^{\infty} \frac{b(b+1)\cdots(b+u)}{a(a+1)\cdots(a+u)}, \\
&= \frac{b}{a-b-1}, \text{ for } a > b+1.
\end{aligned}
$$

Again from (4.1) X_1 has the survival function

$$
\begin{aligned}
S_{W,1}(x) &= \sum_{u=x+1}^{\infty} \frac{S_W(u)}{\mu_W}, \\
&= \sum_{u=x+1}^{\infty} (a+u+1)\ \frac{b(b+1)\cdots(b+u)}{a(a+1)\cdots(a+u+1)} \frac{(a-b-1)}{b}, \\
&= \sum_{u=x+1}^{\infty} (a-b-1)\ \frac{(b+1)_u}{(a)_{u+1}}, \\
&= (a+x+1)\ \frac{(b+1)_{x+1}}{(a)_{x+2}},
\end{aligned}
$$

with mean

$$
\begin{aligned}
\mu_{W,1} &= \sum_{u=0}^{\infty} S_{W,1}(u), \\
&= \sum_{u=0}^{\infty} \frac{(b+1)(b+2)\cdots(b+u+2)}{a(a+1)\cdots(a+u)}, \\
&= \frac{b+1}{a-b-2}, \text{ for } a > b+2.
\end{aligned}
$$

This means that the distribution of X_1 is $W(a, b+1)$. Similarly, X_2 is $W(a, b+1)$ with survival function

$$
\begin{aligned}
S_{W,2}(x) &= \sum_{u=x+1}^{\infty} \frac{S_{W,1}(u)}{\mu_{W,1}}, \\
&= (a+x+1)\ \frac{(b+2)_{x+1}}{(a)_{x+2}}.
\end{aligned}
$$

In general, the distribution of X_n is $W(a, b+n)$.

Finally, suppose the distribution of X is $NH(k,m)$, having survival function

$$\begin{aligned}
S_N(x) &= \sum_{u=x+1}^{m} f_N(u), \\
&= \sum_{u=x+1}^{m} \frac{\binom{k+m-u-1}{m-u}}{\binom{k+m}{m}}, \\
&= \frac{1}{\binom{k+m}{m}} \sum_{u=x+1}^{m} \binom{k+m-u-1}{k-1}, \\
&= \frac{\binom{k+m-x-1}{k}}{\binom{k+m}{m}}
\end{aligned}$$

and mean

$$\begin{aligned}
\mu_N &= \sum_{u=0}^{m} S_N(u), \\
&= \sum_{u=0}^{m} \frac{\binom{k+m-u-1}{k}}{\binom{k+m}{m}}, \\
&= \frac{\binom{k+m}{k+1}}{\binom{k+m}{m}}, \\
&= \frac{m}{k+1}.
\end{aligned}$$

Once again from (4.1), X_1 is $NH(k+1, m-1)$ with survival function

$$\begin{aligned}
S_{N,1}(x) &= \sum_{u=x+1}^{m} \frac{S_N(u)}{\mu_N}, \\
&= \frac{k+1}{m} \sum_{u=x+1}^{m} \frac{\binom{k+m-u-1}{k}}{\binom{k+m}{m}}, \\
&= \frac{1}{\binom{k+m}{m-1}} \sum_{u=x+1}^{m} \binom{k+m-u-1}{k}, \\
&= \frac{\binom{k+m-x-1}{k+1}}{\binom{k+m}{m-1}}.
\end{aligned}$$

Similarly, X_2 is $NH(k+2, m-2)$ with survival function

$$S_{N,2}(x) = \frac{\binom{k+m-x-1}{k+2}}{\binom{k+m}{m-2}}.$$

Then by mathematical induction, X_n is $NH(k+n, m-n)$.

To prove the converse, we note that the survival function of the zero-modified version is $(1-\alpha)\ S(x)$ with mean $(1-\alpha)\ \mu$ and hence the result follows. ∎

Theorem 4.3.2. *The failure rate functions of X_n and X satisfy*

$$h_n(x) = (1+cn)\ h(x) \tag{4.18}$$

for all x and $n = 0, 1, 2, \ldots$ if and only if X is $G(p)$ $(W(a,b),\ NH(k,m))$ for $c = 0$ $(c < 0,\ c > 0)$.

Proof. Suppose X is $G(p)$ with hazard rate

$$\begin{aligned} h_G(x) &= \frac{f_G(x)}{S_G(x-1)}, \\ &= \frac{q^x p}{q^x}, \\ &= p. \end{aligned}$$

Then from Theorem 4.3.1, the hazard rate of X_n,

$$\begin{aligned} h_{G,n}(x) &= \frac{q^x p}{q^x}, \\ &= h_G(x), \end{aligned}$$

satisfies (4.18) for $c = 0$. Now if X is $W(a,b)$ with survival function and hazard rate

$$\begin{aligned} h_W(x) &= \frac{f_W(x)}{S_W(x-1)}, \\ &= \frac{a-b}{a+x}, \end{aligned}$$

then again from Theorem 4.3.1. the hazard rate of X_n

$$\begin{aligned} h_{W,n}(x) &= (a-b-n)\,\frac{(b+n)_x}{(a)_{x+1}}\,\frac{(a)_{x+1}}{(a+x)\,(b+n)_x}, \\ &= \frac{a-b-n}{a+x}, \\ &= \left(1-\frac{n}{a-b}\right) h_W(x), \end{aligned}$$

satisfy (4.18) with $c<0$. In the case of $NH(k,m)$ with the hazard rate,

$$\begin{aligned} h_N(x) &= \frac{f_N(x)}{S_N(x-1)}, \\ &= \frac{\binom{k+m-x-1}{k-1}}{\binom{k+m}{m}}\,\frac{\binom{k+m}{m}}{\binom{k+m-x}{k}}, \\ &= \frac{k}{k+m-x}, \end{aligned}$$

the hazard rate of X_n,

$$\begin{aligned} h_{N,n}(x) &= \frac{\binom{k+m-x-1}{k+n-1}}{\binom{k+m}{m-n}}\,\frac{\binom{k+m}{m-n}}{\binom{k+m-x}{k+n}}, \\ &= \frac{k+n}{k+m-x}, \\ &= \frac{k+n}{k}\;h_N(x), \\ &= \left(1+\frac{n}{k}\right)\;h_N(x), \end{aligned}$$

verifies (4.18) with $c>0$. This proves the if part.

Conversely assuming (4.18) we can write

$$h_1(x) = (1+c)\;h(x),$$

and hence using (4.13),

$$h(x+1) = h_1(x+1)\;\left[\frac{1}{h_1(x+1)} - \frac{1}{h_1(x)} + 1\right],$$

we arrive at the difference equation

$$\frac{1}{h_1(x+1)} = \frac{1}{h_1(x)} - \frac{c}{1+c},$$

with solution

$$h_1(x) = \frac{h_1(0)}{1 - \frac{c}{c+1}\, x}.$$

Since $h_1(x)$ is reciprocal linear, X_1 has the forms stated in the theorem for the designated values of c. Again using (4.18) for $n = 1$, we have the only if part proved. ■

Remark 4.3.1. *The mean residual life functions of X_n and X satisfy*

$$m_n(x) = \frac{m(x)}{1+Bn}, \tag{4.19}$$

for all x and $n = 0, 1, 2, \ldots$ if and only if X has $G(p)$ $(W(a, b);\ NH(k, m))$ according as $B = 0$ $(B > 0; B < 0)$. This follows from the identity (4.11) *and* Theorem 4.3.2.

Remark 4.3.2. Hitha and Nair (1989) *established that*

$$V(x) = C_1\ m(x)\ [m(x) - 1]$$

if and only if X is $G(p)$ $(W(a, b);\ NH(k, m))$ when $c_1 = 1$ $(> 1; < 1)$. Since X_n has the same distributional form as X by Theorem 4.3.1*, we can write variance residual life of X_n as*

$$V_n(x) = C_n\ m_n(x)\ [m_n(x) - 1].$$

Substituting (4.19) *we have a characteristic property for the three distributions in terms of $V_n(x)$ as a function of $m(x)$. In particular $V_n(x) = V(x)$ characterizes the geometric distribution.*

Characterization of life distributions by properties of moments of residual life is discussed by many authors. See Galambos and Kotz (1978) for the early literature on the subject and Gupta and Kirmani (2004) for recent results. We provide some similar results in the discrete case, using properties of equilibrium distributions.

Theorem 4.3.3. *Let X be a discrete random variable defined on $\mathcal{N}$ such that $E(X^n) < \infty$. Then X is geometric if and only if for all x and $n = 1, 2, 3, \ldots$*

$$E\left((X-x-1)^{(n)}|X > x+n\right) = c, \text{ a constant.} \tag{4.20}$$

Proof. Under the hypothesis of X geometric, we write from (4.2),

$$q^{x+1} = \frac{E\left((X-x-1)^{(n)}|X > x+n\right)}{E\left(X^{(n)}\right)} q^{x+n+1}$$

to verify (4.20).

Conversely under (4.20), (4.2) gives

$$S_n(x) = C_1 \; S(x+n)$$

or

$$S_1(x) = C_1 \; S(x+1),$$

showing that

$$\begin{aligned} m(x+1) &= \frac{1}{S(x+1)} \sum_{u=x+1}^{\infty} S(u), \\ &= \frac{\mu \; S_1(x)}{S(x+1)}, \\ &= c_2, \text{ a constant.} \end{aligned}$$

Hence X has geometric distribution. ■

Remark 4.3.3. *Similar results exist for the Waring and negative hyper-geometric laws. These are*

$$E\left((X-x-1)^{(n)}|X > x+n\right) = \frac{C_n}{(b)_n}(a+x-1)^{(n-1)}$$

and

$$E\left((X-x-1)^{(n)}|X > x+n\right) = \frac{C_n}{(m)_n}(k+m-x-1)^{(n)}.$$

The stop loss moments $\alpha_n(x)$ also determine the distribution of X_n uniquely for n. Because,

$$\begin{aligned}\frac{\alpha_n(x+1)}{\alpha_n(x)} &= \frac{S_n(x)}{S_n(x-1)}, \text{ by (4.6)}, \\ &= 1 - h_n(x), \text{ by (2.14)}\end{aligned}$$

or

$$h_n(x) = 1 - \frac{\alpha_n(x+1)}{\alpha_n(x)}. \tag{4.21}$$

Since expressions for $\alpha_n(x)$ are generally not of simple forms, their ratios can be employed for characterization. The following theorem makes an attempt in this direction.

Theorem 4.3.4.

$$\frac{\alpha_n(x+1)}{\alpha_n(x)} = \frac{(A-1)+Bx}{A+Bx}$$

if and only of X_n is geometric (Waring, negative hyper-geometric) for $B = 0$ ($B > 0, B < 0$) distribution.

Proof. By the Theorem 2.1 of Xekalaki (1983),

$$h_n(x) = \frac{1}{A+Bx}$$

if and only if X_n is geometric (Waring, negative hyper-geometric) for $B = 0$ ($B > 0, B < 0$) distribution. Then the theorem follows from the identity (4.21). ■

4.4 Mixtures of equilibrium distributions

Let Θ be a random variable (discrete or continuous) with distribution function $G(\theta)$. Let $X(\theta) = \{X|\theta \in \Theta\}$ be a non-negative discrete random variable with survival function

$$S(x|\theta) = P(X(\theta) > x),$$

probability mass function $f(x|\theta)$ and finite mean

$$\mu(\theta) = \sum_{u=0}^{\infty} S(u|\theta).$$

The survival function of the random variable X of $X(\theta)$ is given by

$$S(x) = \int_{-\infty}^{\infty} S(x|\theta)\; dG(\theta)$$

with finite mean

$$\mu = \sum_{u=0}^{\infty} S(u).$$

Let the equilibrium distribution of order n of $X(\theta)$ be with survival function,

$$S_n(x|\theta) = \frac{1}{\mu_{n-1}(\theta)} \sum_{u=x}^{\infty} S_{n-1}(u|\theta),$$

where

$$\mu_n(\theta) = \sum_{u=0}^{\infty} S_n(u|\theta),$$

the mean of $X_n(\theta)$, the random variable to the nth order equilibrium distribution of $X(\theta)$ and $\mu_n(\theta) < \infty$. Nanda et al. (1996a) has established that the equilibrium distribution of order n of X is given by the survival function

$$S_n(x) = \int_{-\infty}^{\infty} \lambda_{n-1}(\theta)\; S_n(x|\theta)\; dG(\theta), \tag{4.22}$$

where

$$\lambda_n(\theta) = \frac{\mu(\theta)\; \mu_1(\theta) \cdots \mu_n(\theta)}{\mu\; \mu_1 \cdots \mu_n}, \tag{4.23}$$

with $\lambda_{-1}(\theta) = 1$ and

$$\mu_n = \int_{-\infty}^{\infty} \lambda_{n-1}(\theta)\; \mu_n(\theta)\; dG(\theta),$$

the mean of the equilibrium distribution of order n of X. Now, using the relation (4.5) in (4.23) we get

$$\lambda_n(\theta) = \frac{E\left(X(\theta)^{(n+1)}\right)}{E\left(X^{(n+1)}\right)} \tag{4.24}$$

and hence (4.22) becomes

$$\begin{aligned} S_n(x) &= \int_{-\infty}^{\infty} \frac{E\left(X(\theta)^{(n)}\right)}{E\left(X^{(n)}\right)}\ S_n(x|\theta)\ dG(\theta), \\ &= \int_{-\infty}^{\infty} \frac{E\left((X(\theta)-x-1)^{(n)}|X(\theta)>x+n\right)}{E\left(X^{(n)}\right)} S(x+n|\theta)\ dG(\theta). \quad (4.25) \end{aligned}$$

Proceeding on similar lines we get the pmf of X_n (the random variable corresponding to the nth order equilibrium distribution of X) as

$$\begin{aligned} f_{n+1}(x) &= \int_{-\infty}^{\infty} \lambda_n(\theta)\ f_{n+1}(x|\theta)\ dG(\theta), \\ &= \int_{-\infty}^{\infty} \frac{E\left((X(\theta)-x-1)^{(n)}|X(\theta)>x+n\right)}{E\left(X^{(n+1)}\right)} S(x+n|\theta)\ dG(\theta), \quad (4.26) \end{aligned}$$

where $f_n(x|\theta)$ is the pmf of $X_n(\theta)$. Thus all the characteristics of X_n can be obtained from the base line distributions, by using the identities (4.24), (4.25) and (4.26).

The Waring and negative hyper-geometric equilibrium distributions enjoy a special property as mixture distributions as evidenced from the following examples.

Example 4.4.1. *Assume X to be negative binomial with probability mass function,*

$$f(x|q) = \binom{\alpha+x-1}{x} p^{\alpha} q^{x}, \quad x = 0,1,2,\ldots, \quad (4.27)$$

where q is distributed as beta $(1, \beta-\alpha)$ with density function

$$g(q) = (\beta-\alpha)\ (1-q)^{\beta-\alpha-1}, \ \ 0<q<1, \ \ \beta > \alpha+1. \quad (4.28)$$

It may be noted that this distribution can be obtained from (3.11) by setting $a = -(\beta-\alpha)^{-1}$ and $b = (\beta-\alpha)^{-1}$. Then the mixture formed with the pmf,

$$\begin{aligned} f(x) &= \int_0^1 f(x|q)\ g(q)\ dq, \\ &= \frac{\Gamma(\alpha+x)\ (\beta-\alpha)}{\Gamma(\alpha)\Gamma(x+1)} \int_0^1 q^x (1-q)^{\beta-1} dq, \\ &= (\beta-\alpha) \frac{\Gamma(\alpha+x)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta+x+1)}, \end{aligned}$$

$$= \quad (\beta-\alpha)\frac{(\alpha)_x}{(\beta)_{x+1}},$$

which has the Waring law ($W(\beta,\alpha)$), *defined in* (4.16). *The nth order equilibrium of* (4.28) *can be obtained from* (3.14) *by setting* $a=-(\beta-\alpha)^{-1}$ *and* $b=(\beta-\alpha)^{-1}$ *and that distribution is beta* $(1,\beta-\alpha+n)$. *Then the mixture of beta* $(1,\beta-\alpha+n)$ *with* (4.27) *is* $W(\beta,\alpha+n)$, *the nth order equilibrium distribution of* $W(\beta,\alpha)$ (*see* Theorem 4.3.1).

Example 4.4.2. *Similar is the case when binomial* (m,p) *is mixed with beta*$(k,1)$ *giving* $NH(k,m)$. *The nth order equilibrium distribution of beta*$(k,1)$ *mixed with the same binomial results in* $NH(k+n,m-n)$, *the nth order equilibrium distribution of* $NH(k,m)$.

When the mixing distribution is form-invariant under formation of equilibrium distributions, the resulting mixture also becomes form-invariant is obvious, but whether the procedure results in equilibrium distribution of the original model in all cases is an open question.

4.5 Ageing properties

From a reliability point of view it is of interest to examine how various ageing properties of X and X_n are related and also how many of the properties of X are preserved in X_n. All the concepts of ageing defined below are taken on the set $A=\{x|\ S(x)>0\}$.

Theorem 4.5.1. *If* X_n *is IFR* (DFR) *then* X_{n+1} *is IFR* (DFR) *for every* n. *But the converse need not be true.*

Proof. Since X_n is IFR, for all x,

$$h_n(x+1)\geq h_n(x).$$

Now,

$$\begin{aligned} S_n(x) &= \sum_{u=x+1}^{\infty} f_n(u), \\ &= \sum_{u=x+1}^{\infty} h_n(u)\ S_n(u-1) \\ &\geq h_n(x+1) \sum_{u=x}^{\infty} S_n(u). \end{aligned}$$

From (4.11),

$$h_{n+1}(x) \geq h_n(x+1).$$

Substituting in (4.13), the last inequality means

$$h_{n+1}(x+1) - h_{n+1}(x) \geq 0$$

or X_{n+1} is IFR. The proof for DFR is similar.

To prove the second part take the distribution of X as

$$f(x) = \begin{cases} \frac{2}{5}, & x = 0 \\ \frac{7}{20}, & x = 1 \\ \frac{11}{80}, & x = 2 \\ \frac{9}{80}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

Then the mean residual life function becomes

$$m(x) = \begin{cases} \frac{77}{48}, & x = 0 \\ \frac{29}{20}, & x = 1 \\ 1, & x = 2 \end{cases}$$

Thus $m(x)$ is decreasing for all x and therefore by (4.11) X_1 is IFR. On the other hand, $h(1) = 14/24$ and $h(2) = 11/20$ shows that X is not IFR. ■

Remark 4.5.1. *X, DMRL $\Rightarrow X_1$, IFR $\Rightarrow X_1$, DMRL $\Rightarrow \ldots \Rightarrow X_n$, DMRL.*

Theorem 4.5.2. *A necessary and sufficient condition that X_{n-1} is DMRL ($IMRL$) is that*

$$h_{n-1}(x) \leq (\geq) h_n(x) \text{ for all } x,\ n = 1, 2, \ldots.$$

Proof.

$$\begin{aligned} h_{n-1}(x) \leq h_n(x) \quad &\Leftrightarrow \quad h_{n-1}(x) - \frac{1}{m_{n-1}(x)} \leq 0, \\ &\Leftrightarrow \quad h_{n-1}(x)\ m_{n-1}(x) - 1 \leq 0, \\ &\Leftrightarrow \quad \left[\frac{m_{n-1}(x) - m_{n-1}(x-1) + 1}{m_{n-1}(x)}\right] m_{n-1}(x) - 1 \leq 0, \text{ by (2.17)}, \\ &\Leftrightarrow \quad m_{n-1}(x) - m_{n-1}(x-1) \leq 0, \\ &\Leftrightarrow \quad X_{n-1} \text{ is DMRL.} \end{aligned}$$

By reversing the inequalities the IMRL condition is obtained. ■

Theorem 4.5.3. *For every n,*

1. *X, IFR $\Rightarrow X_n$, IFR (2)*
2. *X_n, IFR (2) $\Rightarrow X_{n+1}$ IFR $\Leftrightarrow X_n$, DMRL.*

Proof. From the Definition 2.3.10 we see that

$$\begin{aligned} &a_t(x) - a_t(x+1) \\ &\quad = \frac{1}{S(x)} \sum_{u=x}^{x+t} S(u) - \frac{1}{S(x+1)} \sum_{u=x+1}^{x+t+1} S(u), \\ &\quad = \sum_{u=x+1}^{x+t} \prod_{v=x+1}^{u} (1 - h(v)) - \sum_{u=x+2}^{x+t+1} \prod_{v=x+2}^{u} (1 - h(v)), \text{ on using (2.15)}, \\ &\quad = (h(x+2) - h(x+1)) + \sum_{u=x+2}^{x+t+1} [h(u) - h(x+1)] \prod_{v=x+2}^{u} (1 - h(v)) \end{aligned}$$

When X is IFR, $h(k) - h(x) \geq 0$ for every $k \geq x$ and hence X is IFR (2). Thus we have X, IFR $\Rightarrow X_n$, IFR $\Rightarrow X_n$, IFR (2).

To prove (ii),

$$\begin{aligned} X_n, \text{IFR}(2) &\Rightarrow \frac{1}{S_n(x)} \sum_{u=x}^{x+t} S_n(u) \downarrow x \text{ for all } t \geq 0, \\ &\Rightarrow m_n(x) \downarrow x \text{ at } t \to \infty, \\ &\Rightarrow h_{n+1}(x) \uparrow x, \\ &\Rightarrow X_{n+1} \text{ IFR.} \end{aligned}$$

■

Remark 4.5.2. *Similar implications hold for the dual classes, DFR (2), IFR and IMRL.*

Theorem 4.5.4. *If $S(x)$ is strictly decreasing the following equivalence holds for every $n = 2, 3, \ldots$.*

$$X_n, \text{ IFR (DFR) } \Leftrightarrow X_{n-1}, \text{ DMRL (IMRL) } \Leftrightarrow X_{n-2}, \text{ DVRL (IVRL).}$$

Proof. The random variable X has decreasing variance residual life - DVRL (increasing variance residual life - IVRL) if $V(x)$ is decreasing (increasing) for all x. It is enough to prove the result for $n = 2$. The first implication is obvious. For the second, we write the equation (2.20) as

$$V(x) - V(x+1) = h(x+1)\ m(x+1)\ [m(x) - 1] - h(x+1)\ V(x+1)$$

and use (2.17) to verify

$$\begin{aligned} V(x) - V(x+1) &= m^2(x+1) - (m(x) - 1)^2 - h(x+1)\ [V(x+1) + m^2(x+1)], \\ &= m^2(x+1)\ h(x+1) \left[\frac{m(x) - 1}{m(x+1)} - \frac{V(x+1)}{m^2(x+1)} \right], \\ &= m^2(x+1)\ h(x+1) \left[m(x) - 1 - \frac{V(x+1)}{m(x+1)} \right], \end{aligned}$$

$$= \quad [m(x)-1]\ m(x+1)\ h(x+1)\left[1-\frac{V(x+1)}{m(x+1)}[m(x)-1]\right] \tag{4.29}$$

Also using (4.2),

$$\begin{aligned}
V(x)+m^2(x) &= m(x)+\frac{2\mu}{S(x)}\sum_{u=x+1}^{\infty} S(u),\\
&= m(x)+2m(x)\ m_1(x)\frac{S_1(x)}{S_1(x-1)},\\
&= m(x)+2m(x)\ m_1(x)\ [1-h_1(x)],\\
&= m(x)+2m_1(x)\ [m(x)-1].
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{V(x)}{m(x)\ [m(x)-1]} &= \frac{2m_1(x)}{m(x)}-1,\\
&= 2m_1(x)\ h_1(x)-1,\\
&= 2\ [m_1(x)-m_1(x-1)]+1.
\end{aligned} \tag{4.30}$$

Changing x to $x+1$ in (4.30) and simplifying,

$$\frac{V(x+1)}{m(x+1)\ [m(x)-1]}=\frac{[m(x+1)-1]\ [1+m_1(x+1)-m_1(x)]}{m(x)-1}. \tag{4.31}$$

Hence from (4.30) and (4.31)

$$\begin{aligned}
V(x)-V(x+1) &= m(x+1)\ h(x+1)\,[m(x)-1\\
&\quad -(m(x+1)-1)\ \ (1+m_1(x+1)-m_1(x))]\,,\\
&= \frac{m(x+1)\ h(x+1)}{1+m_1(x)-m_1(x-1)}\ [m_1(x-1)-m_1(x)],\\
&= m(x+1)\ h(x+1)\ [m(x)-m_1(x)],\\
&= m(x+1)\ m(x)\ h(x+1)\ \ [m_1(x-1)-m_1(x)]\,,
\end{aligned} \tag{4.32}$$

on utilizing the identity (4.12). Thus the sign of $V(x)-V(x+1)$ is the same as that of $m_1(x-1)-m_1(x)$, which proves the Theorem. ■

Remark 4.5.3. *Equation* (4.32) *means that* X *is DVRL (IVRL) if and only if* $m(x) \geq$

$(\leq) m_1(x)$.

Remark 4.5.4. *Since DMRL $\Rightarrow$ DVRL,*

$$X, \text{ DVRL } \Leftrightarrow X_1, \text{ DMRL } \Rightarrow X_1, \text{ DVRL } \Rightarrow \ldots \Rightarrow X_n, \text{ (DVRL)}.$$

Remark 4.5.5. *The expression $V(x)/m(x)(m(x)-1)$ takes the place of $V(x)/m^2(x)$, the square of the coefficient of variation of residual life in the continuous case, as regards many properties. For example exponential distribution is characterized by $V(x)/m^2(x) = 1$, where as the corresponding property for the geometric law is*

$$\frac{V(x)}{m(x)\ (m(x)-1)} = 1.$$

Definition 4.5.1. *The distribution of X for which $1 < m(\infty) < \infty$ is said to be UBA (UWA) if and only if*

$$S(x+1) \geq \ (\leq) \frac{m(\infty)-1}{m(\infty)}\ S(x),$$

for all x.

Definition 4.5.2. *X is said to UBAE (UWAE) if and only if $1 < m(\infty) < \infty$ and $m(x) \geq m(\infty)$ for all $x \geq 0$.*

Theorem 4.5.5. *If X is DMRL (IMRL) then X_n is used better (worse) than aged - UBA (used worse than aged - UWA) for each n.*

Proof. X is UBA when

$$1 - h(x+1) \geq 1 - h(\infty)$$

or when X is IFR. Whenever X is DMRL, X_{n+1} is also DMRL which is equivalent to X_n, IFR and hence UBA. ∎

Theorem 4.5.6. *A necessary and sufficient condition for X_n to be UBA (UWA) is that X_{n-1} is used better (worse) than aged in expectation - UBAE (UWAE).*

Proof. When X_{n-1} is UBAE,

$$\begin{aligned}\frac{S_n(x+1)}{S_n(x)} &= 1-h_n(x+1),\\ &= 1-\frac{1}{m_{n-1}(x+1)},\\ &\geq 1-\frac{1}{m_{n-1}(\infty)}=\frac{m_n(\infty)-1}{m_n(\infty)}, \qquad (4.33)\end{aligned}$$

that X_n is UBA.

Conversely when X is UBA, (4.33) holds and therefore,

$$1-h_n(x+1)\geq 1-h_n(\infty)$$

or

$$m_{n-1}(x+1)\geq m_{n-1}(\infty).$$

This proves the result. By reversing the inequality we get X_n, UWA $\Leftrightarrow X_{n-1}$, UWAE. ■

Remark 4.5.6. *When X is UBA*

$$\begin{aligned}m(x) &= \frac{1}{S(x)}\sum_{u=0}^{\infty}S(x+u),\\ &\geq \sum_{u=0}^{\infty}\left(\frac{m(\infty)-1}{m(\infty)}\right)^t=m(\infty).\end{aligned}$$

Hence UBA $\Rightarrow$ UBAE and X_n, UBA $\Rightarrow X_n$, UBAE $\Rightarrow X_{n+1}$ UBA for all n.

Remark 4.5.7. *X_n, IFR $\Rightarrow X_{n-1}$, DMRL $\Rightarrow X_{n-1}$, UBA $\Rightarrow X_{n-2}$, UBA.*

Theorem 4.5.7.

$$X,\ \textit{NBUE (NWUE)}\ \Leftrightarrow S_1(x-1)\leq(\geq)S(x).$$

Proof. X is said to be NBUE if

$$S(x)\sum_{u=0}^{\infty} S(u) \geq \sum_{u=x}^{\infty} S(u)$$

or $m(x) \leq \mu$ for all $x \geq 0$. Hence, we can write

$$\begin{aligned} S_1(x-1) &\Leftrightarrow \frac{m(x)\ s(x)}{\mu}, \\ &\Leftrightarrow S_1(x-1) \leq S(x), \\ &\Rightarrow m(x) \leq \mu, \\ &\Rightarrow X, \text{NBUE}. \end{aligned}$$

The second implication is obvious. ■

Theorem 4.5.8.

$$X_n,\ \textit{NBU (NWU)} \ \Rightarrow X_{n-1},\ \textit{NBUC (NWUC).}$$

Proof. X is said to be NBU (NWU) if for $x, y \geq 0$,

$$S(x+y) \leq (\geq) S(x)\ S(y)$$

and X is new better (worse) than used in convex ordering - NBUC (NWUC) if

$$S(x)\ S_1(y-1) \geq (\leq) S_1(x+y-1),\ x, y \geq 0. \tag{4.34}$$

We prove the result for NBU as the case for NWU follows similarly by reversing the inequalities.

$$\begin{aligned} X_n,\ \text{NBU} &\Rightarrow S_n(x+y) \leq S_n(x)\ S_n(y) \leq S_{n-1}(x+1)\ S_n(y), \\ &\Rightarrow X_{n-1},\ \text{NBUC}. \end{aligned}$$

■

Remark 4.5.8. *Setting $y = 0$ in (4.34), $S(x) \geq S_1(x-1)$ so that X is NBUE by* Theorem 4.5.6*. Thus NBUC $\Rightarrow$ NBUE. This means that by* Theorem 4.5.7*,*

$$X_n,\ NBU\ \Rightarrow X_{n-1},\quad NBUE.$$

Remark 4.5.9. *If X is NBUE, $m(x) \leq \mu$. Hence*

$$1 - h_1(x) \leq \frac{\mu - 1}{\mu},$$

giving

$$S_1(x) = \prod_{u=0}^{x} [1 - h_1(u)] \leq \left(\frac{\mu - 1}{\mu}\right)^{x+1}.$$

Further,

$$\begin{aligned} \mu_1 &= \frac{E\left(X(X-1)\right)}{2\mu}, \\ &= \sum_{u=0}^{\infty} S_1(u), \\ &\leq \mu - 1. \end{aligned}$$

Thus whenever X is NBUE,

$$\frac{V(x)}{\mu\ (\mu - 1)} \leq 1,$$

a necessary condition that is useful in data analysis. This is the equivalent of the result in the continuous case that the coefficient of variation of NBUE distributions (as well as its subclasses) is less than unity.

Theorem 4.5.9. *(i) X_{n-1}, NBUE (NWUE) $\Rightarrow X_n$, NBUFR (NWUFR)*
(ii) X_{n-1}, NBUC (NWUC) $\Rightarrow X_n$, NBUFR (NWUFR)

Proof. From Definition 2.3.13,

$$\begin{aligned} X_{n-1},\ \text{NBUE} &\Rightarrow m_{n-1}(x) \leq \mu_{n-1}, \\ &\Rightarrow m_{n-1}(x) \leq m_{n-1}(0) \leq m_{m-1}(0), \end{aligned}$$

$$\Rightarrow \quad h_n(x) \geq h_n(0),$$

proving (i). In order to establish (ii) we see that from Remark 4.5.8,

$$X_{n-1},\ \text{NBUC} \Rightarrow X_{n-1}, NBUE \Rightarrow X_n,\ \text{NBUFR}.$$

■

Remark 4.5.10. *X is new better (worse) than used in failure rate average if*

$$h(0) \leq \frac{1}{x}\left[h(0) + \ldots + h(x-1)\right].$$

It now follows that NBUFR $\Rightarrow$ NBUFRA so that the implication in Theorem 4.5.9 *extends to NBUFRA class as well.*

From our discussion of discrete equilibrium distributions, it is evident that all properties in the continuous counterpart are not shared in the discrete case. Further, the discrete models that satisfy characteristic properties analogous to those in the continuous case are identified.

Chapter 5

Stochastic Orders for Discrete Equilibrium Distributions

5.1 Introduction

This chapter is a continuation of the previous one. In comparison with the work in continuous case a major topic that does not appear to have been covered for discrete equilibrium distributions is stochastic ordering. Although discrete analogous of some stochastic orders in the continuous case have been presented in Shaked and Shanthikumar (2007), new definitions and results are required to complete the discussions relating to discrete equilibrium distributions. Also there is a need for redefining various discrete ageing concepts in terms of stochastic orders. Apart from providing alternative definitions of ageing concepts, they also give better insight into the ageing concepts and provide tools for establishing the behavior of various discrete reliability measures. The objective of present chapter is therefore to define various stochastic orders, re-interpret the discrete ageing concepts in terms of ordering and provide results that compares the properties of discrete equilibrium models.

5.2 Comparison of baseline and equilibrium distributions

In this section we derive several conditions in terms of stochastic orders connecting X and X_1 that define various ageing properties. For some of the ageing properties, continuous analogues have yet to be proved. First we look at conditions for increasing failure rate (IFR). Recall that X is IFR if and only if $h(x+1) \geq h(x)$ for all x. From equation (4.11), $h_1(x) = [m(x)]^{-1}$. Hence X_1, IFR is equivalent to X, DMRL. The following lemma as well as the Theorem 5.2.1 makes a discussion on an alternative definition of DMRL class.

Lemma 5.2.1. *For all $x,\ y \in \mathcal{N}$*

$$X,\ DMRL \Leftrightarrow \frac{1}{S(x)} \sum_{u=x+y}^{\infty} S(u) \downarrow x.$$

Proof. For all $x, y \in \mathcal{N}$,

$$\begin{aligned}
&\frac{1}{S(x)} \sum_{u=x+y}^{\infty} S(u) \downarrow x \\
&\qquad \Leftrightarrow \frac{1}{S(x)} \sum_{u=x+y}^{\infty} S(u) \geq \frac{1}{S(x+1)} \sum_{u=x+1+y}^{\infty} S(u) \\
&\qquad \Leftrightarrow (X - x | X > x) \geq_{mrl} (X - x - 1 | X > x+1), \text{ by Theorem 5.3.4} \\
&\qquad \Leftrightarrow m(x) \geq m(x+1) \\
&\qquad \Leftrightarrow X, \text{ DMRL.}
\end{aligned}$$

■

Theorem 5.2.1. *X_1 is IFR (X, DMRL) if and only if any one of the following holds for all $y \geq x$, and $x, y \in \mathcal{N}$.*

1. $X \geq_{hr} X_1$,
2. $(X - x | X > x) \geq_{hr} (X_1 - y | X_1 > y)$,

3. $(X_1 - x|X_1 > x) \geq_{hr} (X_1 - y|X_1 > y)$,
4. $X_1 \geq_{hr} (X_1 - x|X_1 > x)$,
5. $(X - x|X > x) \geq_{st} (X_1 - y|X_1 > y)$,
6. $(X_1 - x|X_1 > x) \geq_{st} (X_1 - y|X_1 > y)$,
7. $(X - x|X > x) \geq_{hmrl} (X - y|X > y)$,
8. $(X - x|X > x) \geq_{icx} (X - y|X > y)$,
9. $X \geq_{icx} (X - x|X > x)$,
10. $(X - x|X > x) \geq_{pgf} (X_1 - y|X_1 > y)$,
11. $(X - x|X > x) \geq_{mrl} (X - y|X > y)$,
12. $X \geq_{mrl} (X - x|X > x)$.

Proof. 1. The given condition is equivalent to

$$h(x) \leq h_1(x) = [m(x)]^{-1}$$

or $h(x)m(x) \leq 1$. Then (2.17) leads to $m(x) \leq m(x-1)$ or $h_1(x) \geq h_1(x-1)$ for all x and hence X_1 is IFR. The converse is obvious.

2. We note that

$$(X - x|X > x) \geq_{hr} (X_1 - y|X_1 > y) \Leftrightarrow h(t+x)\ m(t+y) \leq 1,$$

which is equivalent to

$$[m(t+x) - m(t+x-1) + 1]\ \frac{m(t+y)}{m(t+x)} \leq 1.$$

The above inequality is true only when at least one of the terms on the product on the left ≤ 1. But when one term is less than one, the other also satisfies the same condition. Then X is DMRL and hence X_1 is IFR.

3. $(X_1 - x|X_1 > x) \geq_{hr} (X_1 - y|X_1 > y) \Leftrightarrow h_1(t+x) \leq h_1(t+y)$, for all t. Hence X_1 is IFR.

4. By proceeding along the similar lines as in 3., we have the result.

5. For all $y \geq x$,

$$\begin{aligned}
(X - x|X > x) \geq_{st} (X_1 - y|X_1 > y) & \\
&\Leftrightarrow \frac{S(t+x)}{S(x)} \geq \frac{S_1(t+y)}{S_1(y)}, \\
&\Leftrightarrow \frac{1}{S(x)} \sum_{u=y+1}^{\infty} S(u) \geq \frac{1}{S(t+x)} \sum_{u=t+y+1}^{\infty} S(u), \text{ by using (4.1)}, \\
&\Leftrightarrow X \text{ is DMRL}, \\
&\Leftrightarrow X_1 \text{ is IFR}.
\end{aligned}$$

6. For all $y \geq x$,

$$\begin{aligned}
(X_1 - x|X_1 > x) \geq_{st} (X_1 - y|X_1 > y) & \\
&\Leftrightarrow \frac{S_1(t+x)}{S_1(x)} \geq \frac{S_1(t+y)}{S_1(y)}, \\
&\Leftrightarrow \prod_{u=x+1}^{t+x} (1 - h_1(u)) \geq \prod_{u=y+1}^{t+y} (1 - h_1(u)), \\
&\text{for all } t \geq 1, \text{ by using (2.15)}, \\
&\Leftrightarrow 1 - h_1(x+1) \geq 1 - h_1(y+1), \\
&\Leftrightarrow h_1(x) \leq h_1(y), \\
&\Leftrightarrow X_1 \text{ is IFR}.
\end{aligned}$$

7. For all $y \geq x$,

$$\begin{aligned}
(X - x|X > x) \geq_{hmrl} (X - y|X > y) & \\
&\Leftrightarrow \frac{\sum_{u=t}^{\infty} S(u+x)}{S(x)\ m(x)} \geq \frac{\sum_{u=t}^{\infty} S(u+y)}{S(y)\ m(y)}, \text{ by Definition 2.3.5},
\end{aligned}$$

$$\Leftrightarrow \frac{\sum_{u=t+x}^{\infty} S(u)}{\sum_{u=x}^{\infty} S(u)} \geq \frac{\sum_{u=t+y}^{\infty} S(u)}{\sum_{u=y}^{\infty} S(u)}, \text{ by using (2.16)},$$

$$\Leftrightarrow \frac{\sum_{u=t+x}^{\infty} S(u)}{\sum_{u=x}^{\infty} S(u)} \geq \frac{\sum_{u=t+x+z}^{\infty} S(u)}{\sum_{u=x+z}^{\infty} S(u)}, \ z = y - x,$$

$$\Leftrightarrow \frac{\sum_{u=t+x}^{\infty} S(u)}{\sum_{u=t+x}^{\infty} S(u+z)} \geq \frac{\sum_{u=x}^{\infty} S(u)}{\sum_{u=x}^{\infty} S(u+z)},$$

$$\Leftrightarrow \frac{\sum_{u=x}^{\infty} S(u)}{\sum_{u=x}^{\infty} S(u+z)} \uparrow \text{ in } x,$$

$$\Leftrightarrow m(x) \geq m(y), \text{ by Definition 2.3.4},$$

$$\Leftrightarrow X \text{ is DMRL},$$

$$\Leftrightarrow X_1 \text{ is IFR}.$$

8. For all $y \geq x$,

$$(X - x|X > x) \geq_{icx} (X - y|X > y)$$

$$\Leftrightarrow \frac{\sum_{u=t+1}^{\infty} S(u+x)}{S(x)} \geq \frac{\sum_{u=t+1}^{\infty} S(u+y)}{S(y)}, \text{ by Definition 2.3.6},$$

$$\Leftrightarrow \frac{1}{S(x)} \sum_{u=t+x+1}^{\infty} S(u) \geq \frac{1}{S(y)} \sum_{u=t+y+1}^{\infty} S(u),$$

$$\Leftrightarrow X \text{ is DMRL, by Lemma 5.2.1},$$

$$\Leftrightarrow X_1 \text{ is IFR..}$$

9. By setting $x = -1$ in 8., we have the result.

10. For all $y \geq x,\ x,\ y \in \mathcal{N}$,

$$\begin{aligned}
X_1,\ \text{IFR} \Leftrightarrow & (X - x|X > x) \geq_{st} (X_1 - y|X_1 > y) \\
\Leftrightarrow & \frac{S(u+x)}{S(x)} \geq \frac{S_1(u+y)}{S_1(y)},\ u \in \mathcal{N},\ \text{by Definition 2.3.1}, \\
\Rightarrow & \sum_{u=0}^{\infty} s^u \frac{S(u+x)}{S(x)} \geq \sum_{u=0}^{\infty} s^u \frac{S_1(u+y)}{S_1(y)},\ \text{for all } s \in (0,1), \\
\Rightarrow & (X - x|X > x) \geq_{pgf} (X_1 - y|X_1 > y).
\end{aligned}$$

Conversely for all $y \geq x$,

$$\begin{aligned}
(X - x|X > x) & \geq_{pgf} (X_1 - y|X_1 > y) \\
\Leftrightarrow & \frac{\sum_{u=x+t}^{\infty} s^{u-x-t} S_1(u)}{\sum_{u=x}^{\infty} s^{u-x} S(u)} \downarrow x,\ t = y - x, \\
\Leftrightarrow & \frac{S_1(x+t) + \sum_{u=x+t+1}^{\infty} s^{u-x-t} S_1(u)}{S(x) + \sum_{u=x+1}^{\infty} s^{u-x} S(u)} \downarrow x, \\
\Rightarrow & \frac{S_1(x+t)}{S(x)} \downarrow x,\ \text{as } s \to 0, \\
\Rightarrow & \frac{1}{S(x)} \sum_{u=x+t+1}^{\infty} S(u) \downarrow x,\ \text{by (4.1)}, \\
\Rightarrow & X,\ \text{DMRL, by Lemma 5.2.1}, \\
\Rightarrow & X_1,\ \text{IFR}.
\end{aligned}$$

Since *11.* and *12.* are the same as the condition for DMRL, the proof is trivial.

■

Theorem 5.2.2. *The random variable X is IFR if and only if*

1. $X \geq_{lr} X_1$,
2. $X \geq_{lr} (X_1 - x|X_1 > x)$,
3. $(X - x|X > x) \geq_{lr} (X_1 - y|X_1 > y)$,
4. $X_1 \geq_{lr} (X_1 - x|X_1 > x)$,
5. $(X - x|X > x) \geq_{pgf} (X - y|X > y)$,

for all $x, y \in \mathcal{N}$, and $x \leq y$.

Proof. 1.

$$X \geq_{lr} X_1 \Leftrightarrow \frac{\mu f(x)}{S(x)} \uparrow x \Leftrightarrow h(x) \uparrow x \Leftrightarrow X \text{ is IFR.}$$

2.

$$\begin{aligned} X \geq_{lr} (X_1 - x|X_1 > x) &\Leftrightarrow \frac{S(t+x)}{\mu S_1(x)} \frac{1}{S(t-1) - S(t)} \downarrow t, \\ &\Leftrightarrow \frac{S(t+x)}{S(t-1) - S(t)} \downarrow t, \\ &\Leftrightarrow \frac{1}{h(t)} \prod_{z=t}^{t+x} (1 - h(z)) \downarrow t, \\ &\Leftrightarrow h(t) \uparrow t, \\ &\Leftrightarrow X \text{ is IFR.} \end{aligned}$$

3.

$$\begin{aligned} (X - x|X > x) \geq_{lr} (X_1 - y|X_1 > y) &\Leftrightarrow \frac{f_1(t+y)}{S_1(y)} \frac{S(x)}{f(t+x)} \downarrow t, \\ &\Leftrightarrow \frac{f_1(t+y)}{f(t+x)} \downarrow t, \\ &\Leftrightarrow \frac{S(t+y)}{f(t+x)} \downarrow t, \end{aligned}$$

which is equivalent to

$$\left[\frac{1}{h(t+x)}-1\right]\prod_{z=t+x}^{t+y}[1-h(z)]\downarrow t,$$

on using (2.15). At least one of the terms is decreasing in t, which means that all terms are decreasing in t and $h(x)$ is increasing or X is IFR.

4.

$$\begin{aligned} X_1 \geq_{lr} (X_1-x|X_1>x) &\Leftrightarrow \frac{f_1(t+x)}{f_1(t)\ S_1(x)}\downarrow t,\\ &\Leftrightarrow \frac{S(x+t)}{S(t)\ S_1(x)}\downarrow t,\\ &\Leftrightarrow \frac{S(x+t)}{S(t)}\downarrow t,\\ &\Leftrightarrow X \text{ is IFR.}\end{aligned}$$

5. For all $y \geq x,\ x,\ y \in \mathcal{N}$,

$$\begin{aligned} X,\ \text{IFR} &\Leftrightarrow \frac{S(x+1)}{S(x)}\downarrow x\in\mathcal{N},\ \text{by Definition 2.3.9},\\ &\Leftrightarrow \frac{S(u+x)}{S(x)} \geq \frac{S(u+y)}{S(y)},\ u\in\mathcal{N},\\ &\Rightarrow \sum_{u=0}^{\infty} s^u\frac{S(u+x)}{S(x)} \geq \sum_{u=0}^{\infty} s^u\frac{S(u+y)}{S(y)},\\ &\Rightarrow (X-x|X>x) \geq_{pgf} (X-y|X>y),\ y\geq x,\ \text{by Definition 2.3.7.}\end{aligned}$$

Conversely for all $y \geq x$,

$$\begin{aligned} &(X-x|X>x) \geq_{pgf} (X-y|X>y)\\ &\qquad\Leftrightarrow \frac{\sum_{u=x+t}^{\infty} s^{u-x-t}S(u)}{\sum_{u=x}^{\infty} s^{u-x}S(u)}\downarrow x,\ t=y-x,\ \text{by Theorem 5.3.15,}\end{aligned}$$

$$\Leftrightarrow \frac{S(x+t) + \sum_{u=x+t+1}^{\infty} s^{u-x-t} S(u)}{S(x) + \sum_{u=x+1}^{\infty} s^{u-x} S(u)} \downarrow x,$$
$$\Rightarrow \frac{S(x+t)}{S(x)} \downarrow x, \text{ as } s \to 0,$$
$$\Rightarrow X, \text{ IFR, by Definition 2.3.9.}$$

■

We now prove some results that concerns the nature of mean residual life function.

Theorem 5.2.3. *The random variable X_1 is DMRL (X is DVRL) if and only if*

1. $X \geq_{mrl} X_1$
2. $X \geq_{mrl} (X_1 - x|X_1 > x)$
3. $(X - x|X > x) \geq_{mrl} (X_1 - y|X_1 > y),\ y \geq x,$
4. $(X - x|X > x) \geq_{hmrl} (X_1 - y|X_1 > y),\ y \geq x,$

Proof. The proof of all the above cases depends on the following result

$$h(x+t)\ m(x+z) \leq 1 \Leftrightarrow X \text{ is DMRL for } z \geq t. \tag{5.1}$$

The left side can be seen to be equivalent to

$$[m(x+t) - m(x+t-1) + 1]\ \frac{m(x+z)}{m(x+t)} \leq 1,$$

which holds if and only if X is DMRL. Now,

$$\begin{aligned} X \geq_{mrl} X_1 &\Leftrightarrow m(x) \geq m_1(x) \\ &\Leftrightarrow h_1(x)\ m_1(x) \leq 1 \\ &\Leftrightarrow X \text{ is DMRL by (5.1).} \end{aligned}$$

By the same argument in *2.* and *3.*, we have the inequalities $h_1(t)\ m_1(t+x) \leq 1$ and $h_1(t+x)\ m_1(t+y) \leq 1$, each of which implies that X is DMRL. Finally, the proof of 4 is as follows. For all $y \geq x$,

$$
\begin{aligned}
(X-x|X>x) \geq_{hmrl} (X_1-y|X_1>y) &\\
&\Leftrightarrow \frac{\sum\limits_{u=t}^{\infty} S(u+x)}{S(x)\ m(x)} \geq \frac{\sum\limits_{u=t}^{\infty} S_1(u+y)}{S_1(y)\ m_1(y)}, \text{ by Definition 2.3.5,}\\
&\Leftrightarrow \frac{\sum\limits_{u=t+x}^{\infty} S(u)}{\sum\limits_{u=x}^{\infty} S(u)} \geq \frac{\sum\limits_{u=t+y}^{\infty} S_1(u)}{\sum\limits_{u=y}^{\infty} S_1(u)}, \text{ by using (2.16),}\\
&\Leftrightarrow \frac{\sum\limits_{u=t+x}^{\infty} S(u)}{\sum\limits_{u=x}^{\infty} S(u)} \geq \frac{\sum\limits_{u=t+x+z}^{\infty} S_1(u)}{\sum\limits_{u=x+z}^{\infty} S_1(u)},\ z=y-x,\\
&\Leftrightarrow \frac{\sum\limits_{u=t+x}^{\infty} S(u)}{\sum\limits_{u=t+x}^{\infty} S_1(u+z)} \geq \frac{\sum\limits_{u=x}^{\infty} S(u)}{\sum\limits_{u=x}^{\infty} S_1(u+z)},\\
&\Leftrightarrow \frac{\sum\limits_{u=x}^{\infty} S(u)}{\sum\limits_{u=x}^{\infty} S_1(u+z)} \uparrow \text{ in } x,\\
&\Leftrightarrow m(x) \geq m_1(y), \text{ by Definition 2.3.4,}\\
&\Leftrightarrow X_1 \text{ is DMRL.}
\end{aligned}
$$

■

Two other basic ageing concepts are NBU and NBUE. We say that X is NBU iff for all $x,\ t$

$$S(t+x) \leq S(t)\ S(x)$$

and X is NBUE if and only if $m(x) \leq \mu$. We have the following implications connecting these concepts.

Theorem 5.2.4. *1. X is NBU $\Leftrightarrow X \geq_{st} (X - x|X > x)$.*

2. X is NBUE $\Rightarrow X_1 \leq_{st} X$, but the converse is not true.

3. X is NBUE $\Leftrightarrow X_1 \leq_{st} X_G$, where X_G is geometric with parameter μ^{-1}.

Proof. 1. follows from the definition of NBU. To prove *2.*,

$$\begin{aligned} m(x) \leq \mu \quad &\Leftrightarrow \quad \frac{1}{S(x)} \sum_{t=x}^{\infty} S(t) \leq \sum_{t=0}^{\infty} S(t) \\ &\Leftrightarrow \quad S(x) \geq \frac{1}{\sum_{t=0}^{\infty} S(t)} \sum_{t=x}^{\infty} S(t) \\ &\Leftrightarrow \quad S(x) \geq S_1(x-1) \\ &\Rightarrow \quad S_1(x-1) \leq S(x) \leq S(x-1) \text{ for all } x \\ &\Rightarrow \quad X_1 \leq_{st} X. \end{aligned}$$

In order to establish the second part choose

$$S(x) = \begin{cases} \frac{2}{3}, & x = 0 \\ \frac{1}{3}, & x = 1 \\ 0, & x > 1. \end{cases}$$

Clearly $X_1 \leq_{st} X$. But $m(0) = 1.5 > \mu = 1$, showing that X is not NBUE. Finally, to prove *3.*,

$$\begin{aligned} X \text{ is NBUE} \quad &\Leftrightarrow \quad m(x) \leq \mu \\ &\Leftrightarrow \quad h_1(x) \geq \frac{1}{\mu} \\ &\Leftrightarrow \quad \prod_{t=0}^{x} [1 - h_1(t)] \leq \left(1 - \frac{1}{\mu}\right)^x \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow \quad S_1(x) \le S_G(x) \\ &\Leftrightarrow \quad X_1 \le_{st} X_G. \end{aligned} \tag{5.2}$$

■

Note that $\mu > 1$ as in the discrete case the failure rate is less than 1. Inequality (5.2) defines the basic ageing concept NBUE in terms of stochastic order.

We now look at some other ageing criteria which are less frequently used than those discussed above. We say that X is NBUC or NBRU (new better than used in convex order or new better than renewal used) if and only if $S(x) > S_1(x+y)/S_1(x)$ for all $x,\ y$. This means that a new unit has better reliability than the residual life $(X_1 - x|X_1 > x)$ (See Cao and Wang (1991) for the definition in the continuous case.). Obviously X is NBUC or NBRU $\Leftrightarrow$ $X \ge_{st} (X_1 - x|X_1 > x)$. Another equivalent criterion is given in the following theorem.

Theorem 5.2.5. *$X \ge_{icx} (X - x|X > x) \Leftrightarrow X$ is NBUC or NBRU.*

Proof.

$$\begin{aligned} X \ge_{icx} (X - x|X > x) \quad &\Leftrightarrow \quad \sum_{z=t+1}^{\infty} S(z) \ge \sum_{z=t+1}^{\infty} \frac{S(z+x)}{S(x)}, \\ &\Leftrightarrow \quad S(x)\ S_1(t) \ge S_1(x+t), \\ &\Leftrightarrow \quad X \text{ is NBUC or NBRU.} \end{aligned}$$

■

A relationship involving the means of X and the residual life of X_1 is new better than renewal used in expectation (NBRUE), introduced in the continuous case by Abouammoh and Qamber (2003). X is said to be NBRUE iff

$$E(X) \ge E(X_1 - x|X_1 > x).$$

Adopting the same definition in the discrete case also we have the following results connecting NBRUE with some other stochastic orders.

Theorem 5.2.6. *1. X is NBRUE $\Rightarrow X \geq_{icx} X_1$ and the converse need not be true,*

2. $X \geq_{pgf} (X_1 - x|X_1 > x) \Rightarrow X$ is NBRUE.

Proof. 1.

$$\begin{aligned} X \text{ is NBRUE } &\Rightarrow \mu \geq m_1(x) \\ &\Rightarrow \mu\, S_1(x) \geq \sum_{t=x}^{\infty} S_1(t) \\ &\Rightarrow \sum_{t=x+1}^{\infty} S(t) \geq \sum_{t=x}^{\infty} S_1(t) \\ &\Rightarrow X \geq_{icx} X_1. \end{aligned}$$

For the converse part consider the survival function

$$S(x) = \begin{cases} \frac{3}{4}, & x = 0, \\ \frac{2}{4}, & x = 1, \\ \frac{1}{4}, & x = 2, \\ 0, & x > 2. \end{cases}$$

It is clear that $X \geq_{icx} X_1$. However, $m_1(0) = 3/4 \; > \mu = 1$, showing that X is not NBRUE.

2.

$$\begin{aligned} &X \geq_{pgf} (X_1 - x|X_1 > x) \\ &\Rightarrow \sum_{t=0}^{\infty} s^t S(t) \geq \frac{1}{S_1(x)} \sum_{t=0}^{\infty} s^t S_1(t+x) \\ &\Rightarrow \sum_{t=0}^{\infty} S(t) \geq \frac{1}{S_1(x)} \sum_{t=0}^{\infty} S_1(t+x), \text{ as } s \to 1 \end{aligned}$$

$$\Rightarrow \mu \geq m_1(x)$$
$$\Rightarrow X \text{ is NBRUE.}$$

■

5.3 Comparison of equilibrium distributions

In this section we compare the equilibrium distributions of two random variables X and Y in terms of various stochastic orders that have implications in reliability applications. Note that all the reliability characteristics of the random variable Y are as defined in the Section 2.3.1.

Theorem 5.3.1.

$$X \geq_{hr} Y \Leftrightarrow X_1 \geq_{lr} Y_1.$$

Proof.

$$\begin{aligned} X \geq_{hr} Y \quad &\Leftrightarrow \quad \frac{S(x)}{T(x)} \uparrow \text{ in } x \in \mathcal{N} \\ &\Leftrightarrow \quad \frac{f_1(x)}{g_1(x)} \uparrow \text{ in } x \\ &\Leftrightarrow \quad X_1 \geq_{lr} Y_1. \end{aligned}$$

■

Remark 5.3.1. *Generally,* $X \geq_{lr} Y \Rightarrow X \geq_{hr} Y \Rightarrow X \geq_{st} Y$*. Thus from* Theorem 5.3.1,

$$X \geq_{hr} Y \Rightarrow X_1 \geq_{lr} Y_1 \Rightarrow X_1 \geq_{hr} Y_1,$$

preserving the hazard rate order in the formation of equilibrium distributions. It is not true that $X_1 \geq_{hr} Y_1 \Rightarrow X \geq_{hr} Y$*, but from* Theorem 1.C.4 *of* Shaked and

Shanthikumar (2007) *we get*

$$X_1 \geq_{hr} Y_1 \text{ and } \frac{h_1(x)}{k_1(x)} \uparrow \text{ in } x \Rightarrow X \geq_{lr} Y$$
$$\Rightarrow X \geq_{hr} Y.$$

Since $h_1(x) = (m(x))^{-1}$ *and* $k_1(x) = (r(x))^{-1}$, *the above result can be re-stated as*

$$X_1 \geq_{hr} Y_1 \text{ and } \frac{r(x)}{m(x)} \uparrow \text{ in } x \Rightarrow X \geq_{hr} Y.$$

Remark 5.3.2. *From* Theorem 5.3.1 *we get*

$$X \geq_{lr} Y \Rightarrow X \geq_{hr} Y \Rightarrow X_1 \geq_{lr} Y_1,$$

preserving the likelihood order in the formation of equilibrium distributions. It is not true that $X_1 \geq_{lr} Y_1 \Rightarrow X \geq_{lr} Y$, *by the* Remark 1.C.2 *of* Shaked and Shanthikumar (2007). *Also by the* Remark 5.3.1 *we get the following result;*

$$X_1 \geq_{lr} Y_1 \text{ and } \frac{h_1(x)}{k_1(x)} \uparrow \text{ in } x \Rightarrow X \geq_{lr} Y,$$

or

$$X_1 \geq_{lr} Y_1 \text{ and } \frac{r(x)}{m(x)} \uparrow \text{ in } x \Rightarrow X \geq_{lr} Y.$$

The results in the following theorem are straight-forward from the definitions.

Theorem 5.3.2. *1.* $X \geq_{hr} Y \Leftrightarrow (X_1 - x|X_1 > x) \geq_{lr} (Y_1 - x|Y_1 > x)$.

2. $X_1 \geq_{lr} Y_1 \Leftrightarrow (X - x|X > x) \geq_{hr} (Y - x|Y > x)$.

Remark 5.3.3. *From* Theorems 5.3.1 and 5.3.2 *we have*

$$X \geq_{hr} Y \Leftrightarrow (X - x|X > x) \geq_{hr} (Y - x|Y > x)$$

and

$$X \geq_{lr} Y \Leftrightarrow (X - x|X > x) \geq_{lr} (Y - x|Y > x).$$

Theorem 5.3.3. *Let X and Y be independent, DMRL random variables, then*

$$\min(X_1, Y_1) \leq_{lr} Z,$$

where Z is the random variable representing the equilibrium distribution of $\min(X, Y)$.

Proof. The random variable $W = \min(X_1, Y_1)$ has survival function $S_1(x)T_1(x)$ and the $pmf.$,

$$f_W(x) = S_1(x-1)\ T_1(x-1) - S_1(x)\ T_1(x).$$

Also the $pmf.$ of Z is

$$f_Z(x) = \frac{S_Z(x)}{E(Z)} = \frac{S(x)\ T(x)}{E(Z)}.$$

Hence the ratio

$$\begin{aligned} \frac{f_W(x)}{f_Z(x)} &= \frac{S_1(x-1)\ T_1(x-1) - S_1(x)\ T_1(x)}{S(x)\ T(x)} E(Z) \\ &= \frac{E(Z)}{S(x)T(x)} \left[\frac{S(x)m(x)}{\mu} \frac{T(x)r(x)}{\lambda} - \frac{S(x+1)m(x+1)}{\mu} \frac{T(x+1)r(x+1)}{\lambda} \right] \\ &= \frac{E(Z)}{\mu\lambda} \left[m(x)\ r(x) - (1 - h(x+1))\ m(x+1)\ (1 - k(x+1))\ r(x+1) \right], \\ &\quad \text{obtained on using (2.15)} \\ &= \frac{E(Z)}{\mu\lambda} \left[m(x)\ r(x) - (1 - m(x))\ (1 - r(x)) \right]. \end{aligned}$$

Since X and Y are DMRL, the expression in the square braces decreases with respect to x. Hence $W \leq_{lr} Z$.

■

The following theorem discusses a refinement on the definition of mean residual life order.

Theorem 5.3.4.

$$X \leq_{mrl} Y \Leftrightarrow \frac{1}{S(x)} \sum_{u=y}^{\infty} S(u) \leq \frac{1}{T(x)} \sum_{u=y}^{\infty} T(u),$$

for all $y \geq x$.

Proof. By the Definition 2.3.4,

$$
\begin{aligned}
X \leq_{mrl} Y \quad &\Leftrightarrow \quad \frac{\sum_{u=x}^{\infty} T(u)}{\sum_{u=x}^{\infty} S(u)} \uparrow \text{ in } x \\
&\Leftrightarrow \quad \frac{\sum_{u=y}^{\infty} T(u)}{\sum_{u=y}^{\infty} S(u)} \geq \frac{\sum_{u=x}^{\infty} T(u)}{\sum_{u=x}^{\infty} S(u)}, \text{ for all } y \geq x \\
&\Leftrightarrow \quad \sum_{u=x}^{\infty} T(u) \sum_{u=y}^{\infty} S(u) \leq \sum_{u=y}^{\infty} T(u) \sum_{u=x}^{\infty} S(u), \text{ for all } y \geq x. \qquad (5.3)
\end{aligned}
$$

Also,

$$
\begin{aligned}
X \leq_{mrl} Y \quad &\Leftrightarrow \quad \frac{1}{S(x)} \sum_{u=x}^{\infty} S(u) \leq \frac{1}{T(x)} \sum_{u=x}^{\infty} T(u) \\
&\Leftrightarrow \quad \frac{\sum_{u=x}^{\infty} S(u) \sum_{u=y}^{\infty} S(u)}{S(x) \sum_{u=y}^{\infty} S(u)} \leq \frac{\sum_{u=x}^{\infty} T(u) \sum_{u=y}^{\infty} S(u)}{T(x) \sum_{u=y}^{\infty} S(u)} \\
&\Leftrightarrow \quad \frac{\sum_{u=x}^{\infty} S(u) \sum_{u=y}^{\infty} S(u)}{S(x) \sum_{u=y}^{\infty} S(u)} \leq \frac{\sum_{u=y}^{\infty} T(u) \sum_{u=x}^{\infty} S(u)}{T(x) \sum_{u=y}^{\infty} S(u)}, \text{ by using (5.3)} \\
&\Leftrightarrow \quad \frac{1}{S(x)} \sum_{u=y}^{\infty} S(u) \leq \frac{1}{T(x)} \sum_{u=y}^{\infty} T(u).
\end{aligned}
$$

■

The following theorem immediately follows from the relation,

$$h_1(x) = \frac{1}{m(x)}.$$

Theorem 5.3.5.

$$X \leq_{mrl} Y \Leftrightarrow X_1 \leq_{hr} Y_1.$$

Theorem 5.3.6.

$$X \leq_{mrl} Y \Rightarrow X_1 \leq_{hr} Y_1 \Rightarrow X_1 \leq_{mrl} Y_1,$$

preserving the mean residual order in the formation of equilibrium distributions; but the converse need not be true.

Proof. First, we note that

$$\begin{aligned}
X \leq_{hr} Y &\Rightarrow h(x) \geq k(x), \text{ for all } x \in \mathcal{N} \\
&\Rightarrow 1 - h(x) \leq 1 - k(x) \\
&\Rightarrow \prod_{u=x+1}^{y} (1 - h(u)) \leq \prod_{u=x+1}^{y} (1 - k(u)), \ y > x \\
&\Rightarrow \frac{\prod_{u=0}^{y}(1 - h(u))}{\prod_{u=0}^{x}(1 - h(u))} \leq \frac{\prod_{u=0}^{y}(1 - k(u))}{\prod_{u=0}^{x}(1 - k(u))} \\
&\Rightarrow \frac{S(y)}{S(x)} \leq \frac{T(y)}{T(x)}, \text{ by } (2.15) \\
&\Rightarrow \frac{1}{S(x)} \sum_{u=x}^{y} S(u) \leq \frac{1}{T(x)} \sum_{u=x}^{y} T(u) \\
&\Rightarrow m(x) \leq r(x), \text{ by taking the limit as } y \to \infty \\
&\Rightarrow X \leq_{mrl} Y.
\end{aligned}$$

Hence from Theorem 5.3.5 the conclusion follows. For the converse part, let the survival functions of X and Y be respectively

$$S(x) = \begin{cases} \frac{4-x}{5}. & x = 0, 1, 2, 3 \\ 1, & x < 0 \\ 0, & x > 3 \end{cases}$$

and

$$T(x) = \begin{cases} \left(\frac{1}{2}\right)^{x+1}, \; x = 0, 1, 2, \ldots \\ 1, \; x < 0. \end{cases}$$

Clearly $m_1(x) < r_1(x)$ and therefore $X_1 \leq_{mrl} Y_1$; but $m(0) = 5/2 > r(0) = 2$, showing that $X \leq_{mrl} Y$ does not hold. ■

Then we have the following result.

Theorem 5.3.7. *If* $\frac{m_1(x)}{r_1(x)}$ *is increasing in* x*, then*

$$X \leq_{mrl} Y \Leftrightarrow X_1 \leq_{mrl} Y_1.$$

Proof. From 2.17 and $h_1(x) = \frac{1}{m(x)}$,

$$\begin{aligned} \frac{1}{m(x)} &= \frac{m_1(x) - m_1(x-1) + 1}{m_1(x)} \\ &= 1 + \frac{1}{m_1(x)} - \frac{m_1(x-1)}{m_1(x)}. \end{aligned}$$

Also,

$$\begin{aligned} \frac{m_1(x)}{r_1(x)} \uparrow \text{ in } x \quad &\Leftrightarrow \quad \frac{m_1(x)}{r_1(x)} \geq \frac{m_1(x-1)}{r_1(x-1)} \\ &\Leftrightarrow \quad \frac{r_1(x-1)}{r_1(x)} \geq \frac{m_1(x-1)}{m_1(x)}. \end{aligned}$$

Thus if $\frac{m_1(x)}{r_1(x)} \uparrow$ in x,

$$\begin{aligned} X \leq_{mrl} Y &\Leftrightarrow 1 + \frac{1}{m_1(x)} - \frac{m_1(x-1)}{m_1(x)} \geq 1 + \frac{1}{r_1(x)} - \frac{r_1(x-1)}{r_1(x)} \\ &\Leftrightarrow X_1 \leq_{mrl} Y_1. \end{aligned}$$

■

We now discuss the relationships of harmonic mean residual life order with other partial orders.

Theorem 5.3.8. *If X and Y have finite means, then*

$$X_1 \leq_{st} Y_1 \Leftrightarrow X \leq_{hmrl} Y.$$

Proof.

$$\begin{aligned} X_1 \leq_{st} Y_1 &\Leftrightarrow S_1(x) \leq T_1(x) \\ &\Leftrightarrow \frac{1}{\mu} \sum_{u=x+1}^{\infty} S(u) \leq \frac{1}{\lambda} \sum_{u=x+1}^{\infty} T(u) \\ &\Leftrightarrow X \leq_{hmrl} Y. \end{aligned}$$

■

Theorem 5.3.9.

$$X \leq_{mrl} Y \Rightarrow X \leq_{hmrl} Y.$$

Proof.

$$\begin{aligned} X \leq_{mrl} Y &\Rightarrow m(x) \leq r(x), \text{ for all } x \\ &\Rightarrow 1 - \frac{1}{m(x)} \leq 1 - \frac{1}{r(x)} \\ &\Rightarrow \prod_{u=0}^{x} \left(1 - \frac{1}{m(u)}\right) \leq \prod_{u=0}^{x} \left(1 - \frac{1}{r(u)}\right) \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \prod_{u=0}^{x}(1-h_1(u)) \leq \prod_{u=0}^{x}(1-k_1(u)) \\
&\Rightarrow S_1(x) \leq T_1(x) \\
&\Rightarrow X_1 \leq_{st} Y_1 \\
&\Rightarrow X \leq_{hmrl} Y.
\end{aligned}
$$

■

Remark 5.3.4.

$$X \leq_{mrl} Y \Rightarrow X_1 \leq_{st} Y_1.$$

Remark 5.3.5.

$$X \leq_{hmrl} X_1 \Leftrightarrow X_1 \leq_{st} X_2.$$

The following results discuss the relationships of increasing convex order with other partial orders.

Theorem 5.3.10.

$$X \leq_{mrl} Y \Leftrightarrow (X-x|X>x) \leq_{icx} (Y-x|Y>x).$$

Proof.

$$
\begin{aligned}
&(X-x|X>x) \leq_{icx} (Y-x|Y>x) \\
&\qquad \Leftrightarrow \frac{1}{S(x)}\sum_{u=t+1}^{\infty} S(u+x) \leq \frac{1}{T(x)}\sum_{u=t+1}^{\infty} T(u+x),\ t \in \mathcal{N} \\
&\qquad \Leftrightarrow \frac{1}{S(x)}\sum_{u=t+x+1}^{\infty} S(u) \leq \frac{1}{T(x)}\sum_{u=t+x+1}^{\infty} T(u) \\
&\qquad \Leftrightarrow X \leq_{mrl} Y.
\end{aligned}
$$

■

Remark 5.3.6. *In the above theorem, when $x=-1$,*

$$X \leq_{mrl} Y \Rightarrow X \leq_{icx} Y.$$

Theorem 5.3.11. *If X and Y have finite means,*

$$X \leq_{hmrl} Y \Rightarrow X \leq_{icx} Y.$$

Proof. From the definition,

$$X \leq_{hmrl} Y \Rightarrow \frac{1}{\mu} \sum_{u=x+1}^{\infty} S(u) \leq \frac{1}{\lambda} \sum_{u=x+1}^{\infty} T(u).$$

Also

$$X \leq_{hmrl} Y \Rightarrow \mu \leq \lambda$$

and hence $X \leq_{icx} Y$. ■

Theorem 5.3.12.

$$X \leq_{st} Y \Rightarrow X \leq_{icx} Y.$$

Proof.

$$\begin{aligned} X \leq_{st} Y &\Rightarrow S(x) \leq T(x) \\ &\Rightarrow \sum_{u=x+1}^{\infty} S(u) \leq \sum_{u=x+1}^{\infty} T(u) \\ &\Rightarrow X \leq_{icx} Y. \end{aligned}$$

■

Theorem 5.3.13. *If $\mu = \lambda$, then*

$$X \leq_{icx} Y \Rightarrow X_1 \leq_{icx} Y_1.$$

Proof.

$$\begin{aligned} X \leq_{icx} Y &\Rightarrow \sum_{u=x+1}^{\infty} S(u) \leq \sum_{u=x+1}^{\infty} T(u) \\ &\Rightarrow \mu S_1(x) \leq \lambda T_1(x) \\ &\Rightarrow S_1(x) \leq T_1(x) \\ &\Rightarrow \sum_{u=x+1}^{\infty} S_1(u) \leq \sum_{u=x+1}^{\infty} T_1(u) \\ &\Rightarrow X_1 \leq_{icx} Y_1. \end{aligned}$$

∎

Remark 5.3.7. *If $\mu = \lambda$, then*

$$X \leq_{icx} Y \Rightarrow S_1(x) \leq T_1(x) \Rightarrow X_1 \leq_{st} Y_1.$$

The following results discuss some properties of the probability generating order and its relationship with other partial orders.

Theorem 5.3.14. *If $\mu = \lambda$, then*

$$X \leq_{pgf} Y \Leftrightarrow X_1 \leq_{pgf} Y_1.$$

Proof.

$$\begin{aligned} X \leq_{pgf} Y &\Leftrightarrow \sum_{u=0}^{\infty} s^u f(u) \geq \sum_{u=0}^{\infty} s^u g(u) \\ &\Leftrightarrow \sum_{u=0}^{\infty} s^u S(u) \leq \sum_{u=0}^{\infty} s^u T(u) \\ &\Leftrightarrow \sum_{u=0}^{\infty} s^u f_1(u) \leq \sum_{u=0}^{\infty} s^u g_1(u), \text{ since } \mu = \lambda \\ &\Leftrightarrow X_1 \leq_{pgf} Y_1. \end{aligned}$$

∎

We now prove the following result, which gives an alternative definition for the probability generating function order.

Theorem 5.3.15. *For all $x,\ y \in \mathcal{N}$ and $0 < s < 1$,*

$$(X - x - y | X > x + y) \le_{pgf} (Y - x | Y > x) \Leftrightarrow \frac{\sum_{u=x+y}^{\infty} s^{u-y} S(u)}{\sum_{u=x}^{\infty} s^{u-x} T(u)} \downarrow x \in \mathcal{N}.$$

Proof.

$$\frac{\sum_{u=x+y}^{\infty} s^{u-x-y} S(u)}{\sum_{u=x}^{\infty} s^{u-x} T(u)} \downarrow x \in \mathcal{N}$$

$$\Leftrightarrow \frac{\sum_{u=x+y}^{\infty} s^{u-x-y} S(u)}{\sum_{u=x}^{\infty} s^{u-x} T(u)} - \frac{\sum_{u=x+y+1}^{\infty} s^{u-x-y-1} S(u)}{\sum_{u=x+1}^{\infty} s^{u-x-1} T(u)} \ge 0$$

$$\Leftrightarrow \left[\sum_{u=x+y}^{\infty} s^{u} S(u)\right] \left[\sum_{u=x+1}^{\infty} s^{u} T(u)\right] \ge \left[\sum_{u=x}^{\infty} s^{u} T(u)\right] \left[\sum_{u=x+y+1}^{\infty} s^{u} S(u)\right]$$

$$\Leftrightarrow s^{x+y} S(x+y) \sum_{u=x}^{\infty} s^{u} T(u) \ge s^{x} T(x) \sum_{u=x+y}^{\infty} s^{u} S(u)$$

$$\Leftrightarrow \frac{1}{S(x+y)} \sum_{u=x+y}^{\infty} s^{u-x-y} S(u) \le \frac{1}{T(x)} \sum_{u=x}^{\infty} s^{u-x} T(u)$$

$$\Leftrightarrow (X - x - y | X > x + y) \le_{pgf} (Y - x | Y > x), \text{ by Definition 2.3.7.}$$

■

Then we have the following results.

Theorem 5.3.16. *For all $x \in \mathcal{N}$,*

$$X \leq_{hr} Y \Leftrightarrow (X - x|X > x) \leq_{pgf} (Y - x|Y > x).$$

Proof.

$$\begin{aligned}
X \leq_{hr} Y \Leftrightarrow & \frac{S(x)}{T(x)} \downarrow x \in \mathcal{N}, \text{ by Definition 2.3.2} \\
\Leftrightarrow & \frac{S(u+x)}{T(u+x)} \leq \frac{S(x)}{T(x)}, \text{ for all } x, u \in \mathcal{N} \\
\Leftrightarrow & \frac{S(u+x)}{S(x)} \leq \frac{T(u+x)}{T(x)}, \text{ for all } x, u \in \mathcal{N} \\
\Rightarrow & \frac{1}{S(x)} \sum_{u=0}^{\infty} s^u S(u+x) \leq \frac{1}{T(x)} \sum_{u=0}^{\infty} s^u T(u+x), \text{ for some } 0 < s < 1 \\
\Rightarrow & (X - x|X > x) \leq_{pgf} (Y - x|Y > x), \text{ by Definition 2.3.7.}
\end{aligned}$$

Conversely,

$$\begin{aligned}
(X - x|X > x) & \leq_{pgf} (Y - x|Y > x) \\
& \Leftrightarrow \frac{\sum_{u=x}^{\infty} s^{u-x} S(u)}{\sum_{u=x}^{\infty} s^{u-x} T(u)} \downarrow x \in \mathcal{N}, \text{ by Theorem 5.3.15} \\
& \Leftrightarrow \frac{S(x) + \sum_{u=x+1}^{\infty} s^{u-x} S(u)}{T(x) + \sum_{u=x+1}^{\infty} s^{u-x} T(u)} \downarrow x \in \mathcal{N} \\
& \Rightarrow \frac{S(x)}{T(x)} \downarrow x \in \mathcal{N}, \text{ as } s \to 0 \\
& \Rightarrow X \leq_{hr} Y \text{ by Definition 2.3.2.}
\end{aligned}$$

■

Chapter 6

Multivariate Equilibrium Distributions of Order n

6.1 Introduction

In view of the importance of equilibrium distribution in various fields of applications and theoretical work, there have been some attempts to generalize it to higher dimensions. We present a brief review of some of the important developments in this context.

Gupta and Sankaran (1998) considered a bivariate extension of the equilibrium distribution by requiring that the marginal and conditional distributions are of the equilibrium form. Given a noon-negative random vector (X_1, X_2) with survival function

$$S(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

they defined a vector (Y_1, Y_2) with conditional distributions satisfying the properties

$$g_1(y_1|Y_2 > y_2) = \frac{S(y_1, y_2)}{r_1(0, y_2)\ S(0, y_2)}$$

and

$$g_2(y_2|Y_1 > y_1) = \frac{S(y_1, y_2)}{r_2(y_1, 0)\ S(y_1, 0)},$$

where

$$r_i(x_1, x_2) = E(X_i - x_i|X_1 > x_1, X_2 > x_2),\ i = 1, 2 \tag{6.1}$$

are the components of the bivariate mean residual life $(r_1(x_1, x_2), r_2(x_1, x_2))$. They characterized the Gumbel's bivariate exponential distribution

$$S_E(x_1, x_2) = \exp[-\theta_1 x_1 - \theta_2 x_2 - \theta x_1 x_2],$$

where $x_1, x_2 > 0,\ \theta_1, \theta_2 > 0,\ 0 \leq \theta \leq \theta_1\theta_2$, the bivariate Pareto distribution (Sankaran and Nair (1993))

$$S_P(x_1, x_2) = (1 + a_1 x_1 + a_2 x_2 + b x_1 x_2)^{-\alpha},$$

where $x_1, x_2 > 0,\ a_1, a_2 > 0,\ 0 \leq b \leq (\alpha + 1)a_1 a_2$ and the bivariate beta

$$S_B(x_1, x_2) = (1 - p_1 x_1 - p_2 x_2 + q x_1 x_2)^d,$$

where $0 < x_1 < \frac{1}{p_1},\ 0 < x_2 < \frac{1 - p_1 x_1}{p_2 - q x_1},\ 1 - d \leq \frac{q}{p_1 p_2} \leq 1,\ p_1, p_2, d > 0$, by certain relationships between hazard rates and mean residual function of (X_1, X_2) and (Y_1, Y_2). Gupta and Sankaran (1998) did not consider

- the form of the joint distribution of (Y_1, Y_2) arising from the conditional densities g_1 and g_2
- and the condition under which g_1 and g_2 generate a bivariate distribution.

Of the two, the second problem was resolved by Navarro and Sarabia (2010). Navarro and Sarabia (2010) proposed

$$f_E(x_1, x_2) = \frac{S(x_1, x_2)}{E(X_1, X_2)}$$

as the joint density function of the equilibrium random vector (Y_1, Y_2). A new version of the bivariate equilibrium distribution based on conditional distributions were presented in Navarro and Sarabia (2010). It is defined as the joint distribution of (Y_1, Y_2) determined by

$$a_1(y_1|Y_2 = y_2) = \frac{P(Y_1 > y_1|Y_2 = y_2)}{E(Y_1|Y_2 = y_2)}$$

and

$$a_2(y_2|Y_1 = y_1) = \frac{P(Y_2 > y_2|Y_1 = y_1)}{E(Y_2|Y_1 = y_1)}.$$

They also derived reliability properties of the new version. In the present chapter, we discuss two different approaches to define MVED of order n and to derive the properties of the models obtained therefrom. A missing aspect in the earlier papers in Multivariate equilibrium distributions (MVED) is the expression for the joint distribution of order n which is essential in comparing the nth order distribution with the original model and this is rectified in our work.

6.2 Equilibrium distribution based on joint survival functions

Let $\boldsymbol{X} = (X_1,\ X_2,\ \ldots,\ X_p)$ be a random vector defined on the non-negative orthant $\Re_p^+$ of the p-dimensional Euclidean space, with absolutely continuous survival function $S(\boldsymbol{x}) = P(\boldsymbol{X} > \boldsymbol{x})$ and density function $f(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1,\ x_2,\ \ldots,\ x_p)$ and the ordering in $\boldsymbol{X} > \boldsymbol{x}$ is component-wise, $X_1 > x_1,\ \ldots, X_p > x_p$. Then the reliability concepts relating to $\boldsymbol{X}$ of interest in the sequel are the scalar failure rate (Basu (1971))

$$k(\boldsymbol{x}) = \frac{f(\boldsymbol{x})}{S(\boldsymbol{x})}, \tag{6.2}$$

the vector valued failure rate of Johnson and Kotz (1975)

$$\boldsymbol{h}(\boldsymbol{x}) = -\nabla \log S(\boldsymbol{x}),$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)$ is the p-dimensional gradient operator, so that if $h_i(\boldsymbol{x})$ is the i^{th} component of $\boldsymbol{h}(\boldsymbol{x})$

$$h_i(\boldsymbol{x}) = -\frac{\partial \log S(\boldsymbol{x})}{\partial x_i}, \; i = 1, 2, \ldots, p. \tag{6.3}$$

Further, the mean residual life function of $\boldsymbol{X}$ is (Zahedi (1985))

$$\boldsymbol{m}(\boldsymbol{x}) = E\left[\boldsymbol{X} - \boldsymbol{x} | \boldsymbol{X} > \boldsymbol{x}\right]$$

with i^{th} component

$$\begin{aligned} m_i(\boldsymbol{x}) &= E[X_i - x_i | \boldsymbol{X} > \boldsymbol{x}], \\ &= \frac{1}{S(\boldsymbol{x})} \int_{x_i}^{\infty} S(\boldsymbol{x}_{(i)}, t_i)\; dt_i, \end{aligned} \tag{6.4}$$

where $(\boldsymbol{x}_{(i)}, t_i)$ stands for the vector $\boldsymbol{x}$ in which the i^{th} element x_i is replaced by t_i. Differentiating (6.4) partially with respect to x_i,

$$\begin{aligned} \frac{\partial m_i(\boldsymbol{x})}{\partial x_i} &= \frac{1}{S^2(\boldsymbol{x})}\left[-S^2(\boldsymbol{x}) - \frac{\partial S(\boldsymbol{x})}{\partial x_i}\int_{x_i}^{\infty} S(\boldsymbol{x}_{(i)}, t_i)\; dt_i\right], \\ &= -1 + h_i(\boldsymbol{x})\; m_i(\boldsymbol{x}), \end{aligned}$$

we get the relationship between (6.3) and (6.4) as

$$h_i(\boldsymbol{x}) = \frac{1}{m_i(\boldsymbol{x})}\left(1 + \frac{\partial m_i(\boldsymbol{x})}{\partial x_i}\right), \; i = 1, 2, \ldots, p. \tag{6.5}$$

We also need the concept of product moment of residual life defined as (Nair et al. (2004))

$$\begin{aligned} P_0(\boldsymbol{x}) &= E\left[\prod_{i=1}^{p}(X_i - x_i)\middle| \boldsymbol{X} > \boldsymbol{x}\right], \\ &= \frac{1}{S(\boldsymbol{x})}\int_{(\boldsymbol{x},\; \boldsymbol{\infty})} S(\boldsymbol{t})\; d\boldsymbol{t}, \end{aligned} \tag{6.6}$$

where the single integral symbol over $(\boldsymbol{x}, \boldsymbol{\infty})$ in (6.6) means the p-tuple integral in which the i^{th} integral has the range (x_i, ∞) and $\boldsymbol{t} = (t_1, t_2, \ldots, t_p)$. We now introduce a definition of MVED of order n which corresponds to the recursive application of the MVED (of order one) defined in Navarro et al. (2006).

Definition 6.2.1. *The multivariate equilibrium distribution of order n based on $S(\boldsymbol{x})$ is defined recursively through the relations*

$$S_n(\boldsymbol{x}) = \frac{\displaystyle\int_{(\boldsymbol{x},\ \boldsymbol{\infty})} S_{n-1}(\boldsymbol{t})\ d\boldsymbol{t}}{\displaystyle\int_{(\boldsymbol{0},\ \boldsymbol{\infty})} S_{n-1}(\boldsymbol{t})\ d\boldsymbol{t}},\ n = 1, 2, \ldots \tag{6.7}$$

with $S_0(\boldsymbol{x}) = S(\boldsymbol{x})$ and

$$\mu_n = \int_{(\boldsymbol{0},\ \boldsymbol{\infty})} S_n(\boldsymbol{t})\ d\boldsymbol{t}.$$

Some interpretations offered to the univariate equilibrium distributions can be extended to the MVED's as well. Firstly, $S_n(\boldsymbol{x})$ is the weighted distribution of $S_{n-1}(\boldsymbol{x})$ with weight function as the reciprocal of the multivariate failure rate (This property was noted by Navarro et al. (2006) for S_1). To see this, we note that the weighted density function corresponding to $f_{n-1}(\boldsymbol{x})$, the density of $S_{n-1}(\boldsymbol{x})$, with weight $[k_{n-1}(\boldsymbol{x})]^{-1}$, becomes

$$\begin{aligned} f_w(\boldsymbol{x}) &= \frac{[k_{n-1}(\boldsymbol{x})]^{-1}\ f_{n-1}(\boldsymbol{x})}{E\left[(k_{n-1}(\boldsymbol{X}))^{-1}\right]}, \\ &= S_{n-1}(\boldsymbol{x}) \left[\int_{(\boldsymbol{0},\ \boldsymbol{\infty})} \frac{S_{n-1}(\boldsymbol{x})}{f_{n-1}(\boldsymbol{x})}\ f_{n-1}(\boldsymbol{x})\ d\boldsymbol{x}\right]^{-1}, \\ &= S_{n-1}(\boldsymbol{x}) \left[\int_{(\boldsymbol{0},\ \boldsymbol{\infty})} S_{n-1}(\boldsymbol{x})\ d\boldsymbol{x}\right]^{-1}. \end{aligned}$$

The last expression is the density function corresponding to $S_n(\boldsymbol{x})$. Secondly, in the univariate case, if X is a non-negative random variable with density function $f(x)$, then the distribution of WZ is the equilibrium distribution, where W is the length biased random variable of X specified by the density function

$$f_W(x) = \frac{x\ f(x)}{E(X)}$$

and Z is uniform over $(0,\ 1)$ independently of W.

We give an interpretation of $S_n(\boldsymbol{x})$ in terms of uniform random variables and size-biased distributions in the next theorem.

Theorem 6.2.1. *The MVED of order n corresponding to* $\boldsymbol{X}$ *is the joint distribution of* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_p)$ *where* $Y_i = W_i Z_i$, $\boldsymbol{W} = (W_1, W_2, \ldots, W_p)$ *has multivariate size-biased distribution of order n with density function*

$$f_{\boldsymbol{W}}(\boldsymbol{x}) = \frac{x_1^n x_2^n \cdots x_p^n}{E(X_1^n X_2^n \cdots X_p^n)} f_{\boldsymbol{X}}(\boldsymbol{x}),$$

$f_{\boldsymbol{X}}(\boldsymbol{x})$ *being the density function of* $\boldsymbol{X}$ *and* Z_i *are independent random variables each being the minimum of n independent and identically distributed uniform random variables over* $(0,1)$*. Further,* W_i*'s and* Z_i*'s are independent.*

Proof. The density function of Z_i is

$$f_{Z_i}(u_i) = n\ (1-u_i)^{n-1},\ 0 < u_i < 1.$$

Using the independence of $\boldsymbol{W}$ and $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_p)$, the joint density of $(\boldsymbol{W}, \boldsymbol{Z})$ becomes

$$f_{(\boldsymbol{W},\boldsymbol{Z})}(\boldsymbol{x}, \boldsymbol{u}) = \frac{x_1^n x_2^n \cdots x_p^n}{E(X_1^n X_2^n \cdots X_p^n)} f_{\boldsymbol{X}}(\boldsymbol{x})\ n^p \prod_{i=1}^{p} (1-u_i)^{n-1}.$$

The Jacobian of the transformation $\boldsymbol{Y} = (W_1 Z_1, W_2 Z_2, \ldots, W_p Z_p)$ is

$$\left| \left(\frac{\partial W_i}{\partial Y_j} \right)_{p \times p} \right| \left| \left(\frac{\partial Z_i}{\partial Z_j} \right)_{p \times p} \right| = \prod_{i=1}^{p} \frac{1}{Z_i}$$

and the joint distribution of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ has density

$$f_{(\boldsymbol{Y},\boldsymbol{Z})}(\boldsymbol{y}, \boldsymbol{u}) = \frac{n^p\ f_{\boldsymbol{X}}\left(\frac{y_1}{u_1}, \ldots, \frac{y_p}{u_p} \right)}{E(X_1^n X_2^n \cdots X_p^n)} \prod_{i=1}^{p} \left[\left(\frac{y_i}{u_i} \right)^n (1-u_i)^{n-1} u_i^{-1} \right]$$

The last expression gives the marginal density of $\boldsymbol{Y}$ as

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) \ = \ \int_{(\mathbf{0},\mathbf{1})} f_{(\boldsymbol{Y},\boldsymbol{Z})}(\boldsymbol{y}, \boldsymbol{u})\ d\boldsymbol{u},$$

$$
\begin{aligned}
&= (-1)^p \int_{(\boldsymbol{y},\boldsymbol{\infty})} \frac{n^p}{E(X_1^n X_2^n \cdots X_p^n)} f_{\boldsymbol{X}}(\boldsymbol{x}) \left[\prod_{i=1}^{p} \frac{-y_i}{x_i^2} \frac{x_i^{n+1}}{y_i} \left(1 - \frac{y_i}{x_i}\right)^{n-1}\right] d\boldsymbol{x}, \\
&= \frac{n^p}{E(X_1^n X_2^n \cdots X_p^n)} \int_{(\boldsymbol{y},\boldsymbol{\infty})} \left[\prod_{i=1}^{p} (t_i - y_i)^{n-1}\right] f_{\boldsymbol{X}}(\boldsymbol{t})\ d\boldsymbol{t}
\end{aligned}
$$

and also its survival function as

$$
\begin{aligned}
S_{\boldsymbol{Y}}(\boldsymbol{y}) &= \int_{(\boldsymbol{y},\boldsymbol{\infty})} f_{\boldsymbol{Y}}(\boldsymbol{t})\ d\boldsymbol{t}, \\
&= \frac{n^p}{E(X_1^n X_2^n \cdots X_p^n)} \frac{1}{n^p} \int_{(\boldsymbol{y},\boldsymbol{\infty})} \left[\prod_{i=1}^{p} (t_i - y_i)^{n}\right] f_{\boldsymbol{X}}(\boldsymbol{t})\ d\boldsymbol{t}, \text{ by Remark 6.2.4,} \\
&= S_{\boldsymbol{X}}(\boldsymbol{y}) \frac{E\left[\prod_{i=1}^{p} (X_i - y_i)^n | \boldsymbol{X} > \boldsymbol{y}\right]}{E(X_1^n X_2^n \cdots X_p^n)},
\end{aligned}
$$

which coincides survival function of $\boldsymbol{X}_n$ obtained in Theorem 6.2.4. ■

Let $\boldsymbol{X}_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,p})$ be the random vector having survival function $S_n(\boldsymbol{x})$ with

$$
\begin{aligned}
\mu_{n,p} &= E(X_{n,1} X_{n,2} \cdots X_{n,p}), \\
&= \int_{(\boldsymbol{0},\ \boldsymbol{\infty})} S_n(\boldsymbol{t})\ d\boldsymbol{t} < \infty.
\end{aligned}
$$

Then we have the following theorems as a consequence of equations (13) and (14) in Navarro et al. (2006).

Theorem 6.2.2.

$$
k_n(\boldsymbol{x}) = \frac{1}{P_{n-1}(\boldsymbol{x})}, \ n = 1, 2, \ldots \tag{6.8}
$$

where $k_n(\boldsymbol{x})$ is the scalar failure rate of $\boldsymbol{X}_n$ and

$$
P_n(\boldsymbol{x}) = \frac{1}{S_n(\boldsymbol{x})} \int_{(\boldsymbol{x},\ \boldsymbol{\infty})} S_n(\boldsymbol{t})\ d\boldsymbol{t}.
$$

Proof. From (6.7), the density function of $\boldsymbol{X}_n$ is

$$f_n(\boldsymbol{x}) = \frac{S_{n-1}(\boldsymbol{x})}{\mu_{n-1,p}}$$

and hence by (6.2), the scalar failure rate of $\boldsymbol{X}_n$ is

$$\begin{aligned} k_n(\boldsymbol{x}) &= \frac{f_n(\boldsymbol{x})}{S_n(\boldsymbol{x})}, \\ &= \frac{S_{n-1}(\boldsymbol{x})}{\int_{(\boldsymbol{x},\infty)} S_{n-1}(\boldsymbol{t})\, d\boldsymbol{t}}, \text{ by (6.7)}, \\ &= \frac{1}{P_{n-1}(\boldsymbol{x})}. \end{aligned}$$

■

Theorem 6.2.3. *The vector-valued failure rate $\boldsymbol{h}_n(\boldsymbol{x})$ of $\boldsymbol{X}_n$ is related to $P_n(\boldsymbol{x})$ by*

$$\boldsymbol{h}_n(\boldsymbol{x}) = \boldsymbol{h}_{n-1}(\boldsymbol{x}) - \nabla \log P_{n-1}(\boldsymbol{x}),\ n = 1, 2, \ldots. \tag{6.9}$$

Proof. By the definition in (6.3), the ith component of $\boldsymbol{h}_n(\boldsymbol{x})$ is

$$\begin{aligned} h_{n,i}(\boldsymbol{x}) &= \frac{-\partial S_n(\boldsymbol{x})}{\partial x_i}, \\ &= \frac{-1}{\mu_{n-1,p}} \frac{\partial}{\partial x_i} \left(S_{n-1}(\boldsymbol{x}) P_{n-1}(\boldsymbol{x}) \right), \\ &= \frac{-\partial S_{n-1}(\boldsymbol{x})}{\partial x_i} - \frac{\partial P_{n-1}(\boldsymbol{x})}{\partial x_i}, \\ &= h_{n-1,i}(\boldsymbol{x}) - \frac{\partial P_{n-1}(\boldsymbol{x})}{\partial x_i},\ i = 1, 2, \ldots, p. \end{aligned} \tag{6.10}$$

Applying (6.10) into vector form, (6.9) follows. ■

Remark 6.2.1. *We get a specific relation between vector and scalar failure rates by using* (6.8) *in* (6.9) *as*

$$\boldsymbol{h}_n(\boldsymbol{x}) = \boldsymbol{h}_{n-1}(\boldsymbol{x}) + \nabla \log k_n(\boldsymbol{x})$$

.

Remark 6.2.2. *The main objective of studying equilibrium distributions is to compare its properties with those of the baseline distribution* $S(\boldsymbol{x})$*. Equations* (6.8) *and* (6.9) *can be used for this purpose in the context of reliability analysis. Further from*

$$S_n(\boldsymbol{x}) = \frac{S_{n-1}(\boldsymbol{x})\ P_{n-1}(\boldsymbol{x})}{\mu_{n-1,p}}$$

we get the following relationship between $S_n(\boldsymbol{x})$ *and* $S(\boldsymbol{x})$

$$S_n(\boldsymbol{x}) = \frac{P_0(\boldsymbol{x})P_1(\boldsymbol{x})\cdots P_{n-1}(\boldsymbol{x})}{\mu_{0,p}\ \mu_{1,p}\cdots\mu_{n-1,p}}S(\boldsymbol{x}),\ \ n=1,2,\ldots \tag{6.11}$$

where $\mu_{0,p} = E(X_1X_2\cdots X_p)$*.*

A more explicit relationship is stated in the next theorem.

Theorem 6.2.4.

$$S_n(\boldsymbol{x}) = \frac{E\left[(X_1-x_1)^n\cdots(X_p-x_p)^n|\boldsymbol{X}>\boldsymbol{x}\right]}{E(X_1^nX_2^n\cdots X_p^n)}S(\boldsymbol{x}),\ \ n=1,2,\ldots. \tag{6.12}$$

Proof. From the definition (6.7), specializing for $n=1$,

$$\begin{aligned}\mu_{0,p}\ S_1(\boldsymbol{x}) &= \int_{(\boldsymbol{x},\ \boldsymbol{\infty})} S(\boldsymbol{t})\ d\boldsymbol{t},\\ &= \int_{(\boldsymbol{x},\ \boldsymbol{\infty})}\left[\prod_{i=1}^{p}(t_i-x_i)\right]f(\boldsymbol{t})\ d\boldsymbol{t},\\ &= E\left[\prod_{i=1}^{p}(X_i-x_i)\middle|\boldsymbol{X}>\boldsymbol{x}\right]\ S(\boldsymbol{x}). \end{aligned}\tag{6.13}$$

Further, using (6.13),

$$\begin{aligned}\mu_{0,p}\int_{(\boldsymbol{x},\boldsymbol{\infty})} S_1(\boldsymbol{t})\ d\boldsymbol{t} &= \int_{(\boldsymbol{x},\boldsymbol{\infty})}\int_{(\boldsymbol{t},\ \boldsymbol{\infty})} S(\boldsymbol{y})\ d\boldsymbol{y}\ d\boldsymbol{t},\\ &= \int_{(\boldsymbol{x},\boldsymbol{\infty})}\left[\prod_{i=1}^{p}(t_i-x_i)\right]S(\boldsymbol{t})\ d\boldsymbol{t},\end{aligned}$$

on successive partial integration of each of the integrals from (x_i, ∞). Again by partial integration on the right side,

$$\begin{aligned}
\mu_{0,p} \int_{(\boldsymbol{x},\boldsymbol{\infty})} S_1(\boldsymbol{t})\, d\boldsymbol{t} &= \int_{(\boldsymbol{x},\boldsymbol{\infty})} \frac{1}{(2!)^p} \left[\prod_{i=1}^{p} (t_i - x_i)^2\right] f(\boldsymbol{t})\, d\boldsymbol{t}, \\
&= \frac{1}{(2!)^p}\, E\left[\prod_{i=1}^{p} (X_i - x_i)^2 \middle| \boldsymbol{X} > \boldsymbol{x}\right] S(\boldsymbol{x}).
\end{aligned}$$

Once again, from (6.7) and then on successive partial integration discussed above, we obtain

$$\begin{aligned}
\mu_{0,p}\mu_{1,p} \int_{(\boldsymbol{x},\boldsymbol{\infty})} S_2(\boldsymbol{t})\, d\boldsymbol{t} &= \mu_{0,p} \int_{(\boldsymbol{x},\boldsymbol{\infty})} \int_{(\boldsymbol{t},\boldsymbol{\infty})} S_1(\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{t}, \\
&= \int_{(\boldsymbol{x},\boldsymbol{\infty})} \int_{(\boldsymbol{t},\boldsymbol{\infty})} \int_{(\boldsymbol{y},\boldsymbol{\infty})} S(\boldsymbol{z})\, d\boldsymbol{z}\, d\boldsymbol{y}\, d\boldsymbol{t}, \\
&= \int_{(\boldsymbol{x},\boldsymbol{\infty})} \left[\prod_{i=1}^{p} (t_i - x_i)\right] \int_{(\boldsymbol{t},\boldsymbol{\infty})} S(\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{t}, \\
&= \int_{(\boldsymbol{x},\boldsymbol{\infty})} \frac{1}{(2!)^p} \left[\prod_{i=1}^{p} (t_i - x_i)^2\right] S(\boldsymbol{t})\, d\boldsymbol{t}, \\
&= \int_{(\boldsymbol{x},\boldsymbol{\infty})} \frac{1}{(3!)^p} \left[\prod_{i=1}^{p} (t_i - x_i)^3\right] f(\boldsymbol{t})\, d\boldsymbol{t}, \\
&= \frac{1}{(3!)^p}\, E\left[\prod_{i=1}^{p} (X_i - x_i)^3 \middle| \boldsymbol{X} > \boldsymbol{x}\right] S(\boldsymbol{x}).
\end{aligned}$$

Proceeding along the same lines and using (6.7) repeatedly for $n = 2, 3, \ldots$ we find that for any positive integer m

$$\mu_{0,p}\mu_{1,p} \cdots \mu_{m-1,p} \int_{(\boldsymbol{x},\ \boldsymbol{\infty})} S_m(\boldsymbol{t})\, d\boldsymbol{t} = \frac{E\left[\prod_{i=1}^{p} (X_i - x_i)^{m+1} \middle| \boldsymbol{X} > \boldsymbol{x}\right]}{[(m+1)!]^p} S(\boldsymbol{x}) \qquad (6.14)$$

and from (6.7)

$$\mu_{0,p}\mu_{1,p} \cdots \mu_{m,p} S_{m+1}(\boldsymbol{x}) = \frac{E\left[\prod_{i=1}^{p} (X_i - x_i)^{m+1} \middle| \boldsymbol{X} > \boldsymbol{x}\right]}{[(m+1)!]^p} S(\boldsymbol{x}). \qquad (6.15)$$

As $\boldsymbol{x} \to \mathbf{0}_p$ in (6.15), where $\mathbf{0}_p$ is a p-vector of zeroes,

$$\mu_{0,p}\mu_{1,p}\cdots\mu_{m,p} = \frac{1}{[(m+1)!]^p}\ E[X_1^{m+1}\cdots X_p^{m+1}] \tag{6.16}$$

and hence, using (6.16) in (6.14), we recover the form (6.12). ■

Remark 6.2.3. *From equation* (6.16) *we find that*

$$\mu_{n-1,p} = \frac{E(X_1^n X_2^n \cdots X_p^n)}{n^p E(X_1^{n-1} X_2^{n-1}\cdots X_p^{n-1})}.$$

Further the covariance between $X_{i,n-1}$ and $X_{j,n-1}$ in terms of the original variable, becomes

$$Cov(X_{i,n-1}, X_{j,n-1}) = n^{-2}\left[\frac{E(X_i^n X_j^n)}{E(X_i^{n-1}X_j^{n-1})} - \frac{E(X_i^n)E(X_j^n)}{E(X_i^{n-1})E(X_j^{n-1})}\right],\ n = 1, 2, \ldots.$$

The last relation can be employed to compare the dependence structure of the baseline distribution and its n^{th} order MVED.

Remark 6.2.4. *We can see that*

$$\begin{aligned}
&\int_{(\boldsymbol{x},\boldsymbol{\infty})}\int_{(\boldsymbol{t},\boldsymbol{\infty})}\left[\prod_{i=1}^{p}(y_i - t_i)^n\right] f(\boldsymbol{y})\ d\boldsymbol{y}\ d\boldsymbol{t} \\
&\quad = (n!)^p\mu_{0,p}\mu_{1,p}\cdots\mu_{n-1,p}\int_{(\boldsymbol{x},\boldsymbol{\infty})} S_n(\boldsymbol{t})\ d\boldsymbol{t},\ \ \textit{by}\ (6.15), \\
&\quad = (n!)^p\mu_{0,p}\mu_{1,p}\cdots\mu_{n,p}S_{n+1}(\boldsymbol{x}),\ \ \textit{by}\ (6.7), \\
&\quad = \frac{(n!)^p}{[(n+1)!]^p}E\left[\prod_{i=1}^{p}(X_i - x_i)^{n+1}\middle|\ \boldsymbol{X} > \boldsymbol{x}\right] S(\boldsymbol{x}),\ \ \textit{again by}\ (6.15), \\
&\quad = \frac{1}{(n+1)^p}\int_{(\boldsymbol{x},\boldsymbol{\infty})}\left[\prod_{i=1}^{p}(t_i - x_i)^{n+1}\right] f(\boldsymbol{t})\ d\boldsymbol{t}.
\end{aligned}$$

Remark 6.2.5. *From* (6.14) *and* (6.15), *we get*

$$P_n(\boldsymbol{x}) \ = \ \frac{1}{S_n(\boldsymbol{x})}\int_{(\boldsymbol{x},\ \boldsymbol{\infty})} S_n(\boldsymbol{t})\ d\boldsymbol{t},$$

$$= \frac{E[(X_1 - x_1)^{n+1} \cdots (X_p - x_p)^{n+1} | \boldsymbol{X} > \boldsymbol{x}]}{(n+1)^p \; E[(X_1 - x_1)^n \cdots (X_p - x_p)^n | \boldsymbol{X} > \boldsymbol{x}]}$$

and

$$P_0(\boldsymbol{x}) \cdots P_{n-1}(\boldsymbol{x}) = \frac{1}{(n!)^p} \; E[(X_1 - x_1)^n \cdots (X_p - x_p)^n | \boldsymbol{X} > \boldsymbol{x}]. \tag{6.17}$$

Remark 6.2.6. *The marginal distributions of $\boldsymbol{X}_n$ are the n^{th} order equilibrium distributions of the corresponding marginal distributions of $\boldsymbol{X}$.*

6.3 Equilibrium distributions based on conditional distributions

Chatterjee and Mukherjee (2000, p. 125) introduced higher order equilibrium distributions in the multivariate set up by generating a sequence of equilibrium distributions corresponding to a survival function $R(\boldsymbol{x})$ for all positive integers n through the definition

$$R_{i,n+1}(\boldsymbol{x}) = \frac{\displaystyle\int_{x_i}^{\infty} R_{i,n}(\boldsymbol{x}_{(i)}, t_i) \; dt_i}{\displaystyle\int_{0}^{\infty} R_{i,n}(\boldsymbol{x}_{(i)}, t_i) \; dt_i} \tag{6.18}$$

and state that $R_{i,n+1}(\boldsymbol{x})$ represents a proper multivariate reliability function in the usual sense. We note that the ratio on the right hand side when $n = 0$ depends on i and is in-fact

$$\frac{\displaystyle\int_{x_i}^{\infty} P(X_i > t_i | \boldsymbol{X}_{(i)} > \boldsymbol{x}_{(i)}) \; dt_i}{\displaystyle\int_{0}^{\infty} P(X_i > t_i | \boldsymbol{X}_{(i)} > \boldsymbol{x}_{(i)}) \; dt_i}$$

representing the equilibrium distribution of the univariate conditional distribution of X_i given $\boldsymbol{X}_{(i)} > \boldsymbol{x}_{(i)}$ where $\boldsymbol{x}_{(i)} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p)$. Further these conditional densities may not be compatible as pointed out in Navarro et al. (2006). For any random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ and $i = 1, 2, \ldots, p$ a necessary and sufficient condition for compatibility of conditional distributions $P(X_i | \boldsymbol{X}_{(i)} > \boldsymbol{x}_{(i)})$ is that for each $i \neq j$

$$\frac{P(X_i > x_i|\boldsymbol{X}_{(i)} > \boldsymbol{x}_{(i)})}{P(X_j > x_j|\boldsymbol{X}_{(j)} > \boldsymbol{x}_{(j)})} = \frac{A_j(\boldsymbol{x}_{(j)})}{A_i(\boldsymbol{x}_{(i)})}$$

where $A_j(\boldsymbol{x}_{(j)})$ are survival functions (Arnold (1994)). In view of the above facts, we modify the definition of Chatterjee and Mukherjee (2000) of the MVED corresponding to a survival function $S(\boldsymbol{x})$ in $\Re_p^+$ in the following manner.

Definition 6.3.1. *The MVED of $\boldsymbol{X}$ in $\Re_p^+$ with survival function $S(\boldsymbol{x})$ is defined as the joint survival function of $\boldsymbol{Y}_1 = (Y_{1,1}, Y_{1,2}, \ldots, Y_{1,p})$ determined by the conditional survival functions*

$$\begin{aligned} R_{1,i}(\boldsymbol{x}) &= P(Y_{1,i} > x_i|\boldsymbol{Y}_{1,(i)} > \boldsymbol{x}_{(i)}), \\ &= \frac{\displaystyle\int_{x_i}^{\infty} S(\boldsymbol{x}_{(i)}, t_i)\ dt_i}{\displaystyle\int_{0}^{\infty} S(\boldsymbol{x}_{(i)}, t_i)\ dt_i}, \\ &= \frac{\displaystyle\int_{x_i}^{\infty} S(t_i|\boldsymbol{x}_{(i)})\ dt_i}{\displaystyle\int_{0}^{\infty} S(t_i|\boldsymbol{x}_{(i)})\ dt_i},\ \ i = 1, 2, \ldots \end{aligned} \tag{6.19}$$

where $S(t_i|\boldsymbol{x}_{(i)}) = P(X_i > t_i|\boldsymbol{X}_{(i)} > \boldsymbol{x}_{(i)})$ are subject to the compatibility condition mentioned above. It is easy to see that

$$R_{1,i}(\boldsymbol{x}) = \frac{m_i(\boldsymbol{x})\ S(x_i|\boldsymbol{x}_{(i)})}{E[X_i|\boldsymbol{X}_{(i)} > \boldsymbol{x}_{(i)}]}.$$

In the next theorem, we find the joint survival function $R_1(\boldsymbol{x})$ of $\boldsymbol{Y}_1$.

Theorem 6.3.1. *Equation* (6.19) *characterizes the distribution of $\boldsymbol{Y}_1$ through the survival function*

$$R_1(\boldsymbol{x}) = S(\boldsymbol{x}) \prod_{i=1}^{p} \frac{E[(X_i - x_i)|X_1 > x_1, X_2 > x_2, \ldots, X_i > x_i]}{E[X_i|X_1 > x_1, X_2 > x_2, \ldots, X_{i-1} > x_{i-1}]}. \tag{6.20}$$

Proof. Allowing $\boldsymbol{x}_{(1)} \to \mathbf{0}_{p-1}$ in (6.19) where $\mathbf{0}_p$ is a p-vector of zeroes,

$$P(Y_{1,1} > x_1) = \frac{m_1(x_1, \mathbf{0}_{p-1})\ S(x_1, \mathbf{0}_{p-1})}{m_1(\mathbf{0}_p)}.$$

Again when $x_i \to 0$, for $i = 3, 4, \ldots, p$,

$$P(Y_{1,2} > x_2 | Y_{1,1} > x_1) = \frac{m_2(x_1, x_2, \mathbf{0}_{p-2})\ S(x_1, x_2, \mathbf{0}_{p-2})}{m_2(x_1, \mathbf{0}_{p-1})\ S(x_1, \mathbf{0}_{p-1})}$$

so that from the last two equations

$$P(Y_{1,1} > x_1, Y_{1,2} > x_2) = \frac{m_1(x_1, \mathbf{0}_{p-1})\ m_2(x_1, x_2, \mathbf{0}_{p-2})}{m_1(\mathbf{0}_p)\ m_2(x_1, \mathbf{0}_{p-1})} S(x_1, x_2, \mathbf{0}_{p-2}).$$

Proceeding like this, by induction

$$R_1(\boldsymbol{x}) = S(\boldsymbol{x}) \prod_{i=1}^{p} \frac{m_i(x_1, x_2, \ldots, x_i, \mathbf{0}_{p-i})}{m_i(x_1, x_2, \ldots, x_{i-1}, \mathbf{0}_{p-i+1})} \tag{6.21}$$

which is the same as (6.20). Hence (6.19) implies (6.20). The converse is obtained directly from (6.21). ■

Remark 6.3.1. *There are other* $(p-1)$ *equivalent representations for* $R_1(\boldsymbol{x})$*, by starting with any of the* $Y_{1,i}$ *and adopting a proof similar to that of* Theorem 6.3.1.

Remark 6.3.2. *From* (6.21),

$$P(Y_{1,i} > x_i) = \frac{m_i(0, 0, \ldots, x_i, 0, \ldots, 0)}{E(X_i)} P(X_i > x_i)$$

showing that all univariate marginals are equilibrium distributions of the corresponding univariate distribution of $\boldsymbol{X}$.

With the above definition of MVED, our new definition of MVED of order n is as follows.

Definition 6.3.2. *Let* $\mathbf{Y}_n = (Y_{n,1}, Y_{n,2}, \ldots, Y_{n,p})$ *be a random vector in* $\Re_p^+$ *with sur-*

vival function $R_n(\boldsymbol{x})$ determined by

$$\begin{aligned} R_{n,i}(\boldsymbol{x}) &= P(Y_{n,i} > x_i | \boldsymbol{Y}_{n,i} > \boldsymbol{x}_{(i)}), \\ &= \frac{\displaystyle\int_{x_i}^{\infty} R_{n-1}(\boldsymbol{x}_i, t_i)\ dt_i}{\displaystyle\int_0^{\infty} R_{n-1}(\boldsymbol{x}_i, t_i)\ dt_i},\ n = 1, 2, 3, \ldots \end{aligned} \tag{6.22}$$

with $R_0(\boldsymbol{x}) = S(\boldsymbol{x})$. Then $R_n(\boldsymbol{x})$ is called the n^{th} order MVED corresponding to $S(\boldsymbol{x})$.

The joint distribution of $\boldsymbol{Y}_n$ is derived in the next theorem.

Theorem 6.3.2.

$$R_n(\boldsymbol{x}) = S(\boldsymbol{x}) \prod_{i=1}^{p} \frac{E[(X_i - x_i)^n | X_1 > x_1, X_2 > x_2, \ldots, X_i > x_i]}{E[X_i^n | X_1 > x_1, X_2 > x_2, \ldots, X_{i-1} > x_{i-1}]}. \tag{6.23}$$

Proof. Equation (6.20) provides the relationship between a survival function and its equilibrium distribution. Applying it to the random vector $\boldsymbol{Y}_1$,

$$R_2(\boldsymbol{x}) = R_1(\boldsymbol{x}) \prod_{i=1}^{p} \frac{E[Y_{1,i} - x_i | Y_{1,1} > x_1, Y_{1,2} > x_2, \ldots, Y_{1,i} > x_i]}{E[Y_{1,i} | Y_{1,1} > x_1, Y_{1,2} > x_2, \ldots, Y_{1,i-1} > x_{i-1}]}.$$

Now,

$$\begin{aligned} E[Y_{1,1} - x_1 | Y_{1,1} > x_1] &= \frac{1}{R_1(x_1, \mathbf{0}_{p-1})} \int_{x_1}^{\infty} R_1(t, \mathbf{0}_{p-1})\ dt, \\ &= \frac{1}{R_1(x_1, \mathbf{0}_{p-1})} \int_{x_1}^{\infty} \left[\int_t^{\infty} \frac{S(y, \mathbf{0}_{p-1})}{E(X_1)}\ dy \right] dt, \text{ by (6.22)}, \\ &= \frac{1}{R_1(x_1, \mathbf{0}_{p-1})} \int_{x_1}^{\infty} \frac{(t - x_1)}{E(X_1)}\ S(t, \mathbf{0}_{p-1})\ dt, \\ &= \frac{1}{R_1(x_1, \mathbf{0}_{p-1})} \int_{x_1}^{\infty} \frac{(t - x_1)^2}{2E(X_1)}\ f(t, \mathbf{0}_{p-1})\ dt, \\ &= \frac{E[(X_1 - x_1)^2 | X_1 > x_1]\ S(x_1, \mathbf{0}_{p-1})}{2E(X_1)\ R_1(x_1, \mathbf{0}_{p-1})}, \end{aligned}$$

on using integration by parts in the integral on the right side. As $x_1 \to 0$,

$$E(Y_{1,1}) = \frac{E(X_1^2)}{2E(X_1)}.$$

Then,

$$\begin{aligned}
\frac{E[Y_{1,1} - x_1 | Y_{1,1} > x_1]}{E(Y_{1,1})} &= \frac{S(x_1, \mathbf{0}_{p-1})\ E[(X_1 - x_1)^2 | X_1 > x_1]}{E(X_1^2)\ R_1(x_1, \mathbf{0}_{p-1})}, \\
&= \frac{E[(X_1 - x_1)^2 | X_1 > x_1]}{E(X_1^2)} \frac{E(X_1)\ S(x_1, \mathbf{0}_{p-1})}{\int_{x_1}^{\infty} S(t, \mathbf{0}_{p-1})\ dt}, \text{ by (6.22)}, \\
&= \frac{E[(X_1 - x_1)^2 | X_1 > x_1]}{E(X_1^2)} \frac{E(X_1)}{E[(X_1 - x_1) | X_1 > x_1]}.
\end{aligned}$$

By the same type of calculation for any $i = 1, 2, \ldots, p$,

$$\begin{aligned}
\frac{E[Y_{1,i} - x_i | Y_{1,1} > x_1, \ldots Y_{1,i} > x_i]}{E[Y_{1,i} | Y_{1,1} > x_1, \ldots Y_{1,i-1} > x_{i-1}]} &= \Bigg[\frac{E[X_i | X_1 > x_1, \ldots, X_{i-1} > x_{i-1}]}{E[(X_i - x_i) | X_1 > x_1, \ldots, X_i > x_i]} \times \\
&\quad \frac{E[(X_i - x_i)^2 | X_1 > x_1, \ldots, X_i > x_i]}{E[X_i^2 | X_1 > x_1, \ldots, X_{i-1} > x_{i-1}]} \Bigg].
\end{aligned}$$

Hence

$$R_2(\boldsymbol{x}) = S(\boldsymbol{x}) \prod_{i=1}^{p} \frac{E[(X_i - x_i)^2 | X_1 > x_1, X_2 > x_2, \ldots, X_i > x_i]}{E[X_i^2 | X_1 > x_1, X_2 > x_2, \ldots, X_{i-1} > x_{i-1}]}$$

on using the expression for $R_1(\boldsymbol{x})$ in (6.20). Finally (6.23) is arrived at by induction on n. ■

Remark 6.3.3. *All univariate marginals of $\boldsymbol{Y}_n$ are equilibrium distributions of the corresponding marginals of $\boldsymbol{Y}_{n-1}$ by* Remark 6.3.2. *The same property need not hold for higher dimensional marginals.*

Equation (6.23) enables to establish some identities connecting the reliability characteristics of $\boldsymbol{Y}_n$ and $\boldsymbol{X}$. If

$$\boldsymbol{\Lambda}_n(\boldsymbol{x}) = (\Lambda_{n,1}(\boldsymbol{x}), \Lambda_{n,2}(\boldsymbol{x}), \ldots, \Lambda_{n,p}(\boldsymbol{x}))$$

is the vector failure rate of $\boldsymbol{Y}_n$, so that

$$\Lambda_{n,i}(\boldsymbol{x}) = -\frac{\partial}{\partial x_i} \log R_n(\boldsymbol{x}),$$

we have by logarithmic differentiation of (6.23) with respect to x_p,

$$\Lambda_{n,p}(\boldsymbol{x}) = h_p(\boldsymbol{x}) - \frac{\frac{\partial}{\partial x_p} E[(X_p - x_p)^n | \boldsymbol{X} > \boldsymbol{x}]}{E[(X_p - x_p)^n | \boldsymbol{X} > \boldsymbol{x}]}$$

or, in general

$$\Lambda_{n,i}(\boldsymbol{x}) = h_i(\boldsymbol{x}) - \frac{\frac{\partial}{\partial x_i} E[(X_i - x_i)^n | \boldsymbol{X} > \boldsymbol{x}]}{E[(X_i - x_i)^n | \boldsymbol{X} > \boldsymbol{x}]}, \ i = 1, 2, \ldots, p. \tag{6.24}$$

Chatterjee and Mukherjee (2000) proved that

$$\Lambda_{n,i}(\boldsymbol{x}) = \frac{1}{M_{n-1,i}(\boldsymbol{x})}, \tag{6.25}$$

where $\boldsymbol{M}_n(\boldsymbol{x}) = (M_{n,1}(\boldsymbol{x}), M_{n,2}(\boldsymbol{x}), \ldots, M_{n,p}(\boldsymbol{x}))$ is the mean residual life of $\boldsymbol{Y}_n$, so that from (6.4) and (6.5),

$$\begin{aligned} M_{n,i}(\boldsymbol{x}) &= \frac{1}{R_n(\boldsymbol{x})} \int_{x_i}^{\infty} R_n(\boldsymbol{x}_{(i)}, t_i)\, dt_i, \\ &= \frac{1}{R_n(x_i | \boldsymbol{x}_{\boldsymbol{(i)}})} \int_{x_i}^{\infty} R_n(t_i | \boldsymbol{x}_{\boldsymbol{(i)}})\, dt_i, \end{aligned}$$

$$\Lambda_{n,i}(\boldsymbol{x}) = \frac{1 + \frac{\partial}{\partial x_i} M_{n,i}(\boldsymbol{x})}{M_{n,i}(\boldsymbol{x})}.$$

Hence

$$M_{n-1,i}(\boldsymbol{x}) = \frac{M_{n,i}(\boldsymbol{x})}{1 + \frac{\partial}{\partial x_i} M_{n,i}(\boldsymbol{x})} \tag{6.26}$$

and

$$\Lambda_{n-1,i}(\boldsymbol{x}) = \Lambda_{n,i}(\boldsymbol{x}) - \frac{\frac{\partial}{\partial x_i} \Lambda_{n,i}(\boldsymbol{x})}{H_{n,i}(\boldsymbol{x})} \tag{6.27}$$

are recurrence relations connecting the failure rates and mean residual life functions of the MVED's of order n and $n-1$. The expression $M_{n,i}(\boldsymbol{x})$ in terms of the random vector $\boldsymbol{X}$ is

$$M_{n,i}(\boldsymbol{x}) = \frac{E[(X_i - x_i)^n | X_i > x_i]}{nE[(X_i - x_i)^{n-1} | X_i > x_i]}.$$

6.4 Characterizations

In this section, we derive characterizations of some multivariate distributions by properties of their higher order equilibrium distributions. In the univariate case it is well-known that the only distribution for which the original and equilibrium distributions are identical is the exponential. We now look at multivariate analogues of this result.

Theorem 6.4.1. *The only absolutely continuous multivariate distribution for which* $S_n(\boldsymbol{x})$ *is always equal to* $S(\boldsymbol{x})$ *for two consecutive values of* n *and all* $\boldsymbol{x} > \mathbf{0}$ *is the mixture of exponentials with density function*

$$f(\boldsymbol{x}) = \frac{1}{\mu_p} \int_{(\mathbf{0},\ \infty)} \exp(-\boldsymbol{x}\boldsymbol{\lambda}^T)\ \nu(d\boldsymbol{\lambda}),\ \ \boldsymbol{x} > \mathbf{0},\ \ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p) > \mathbf{0} \tag{6.28}$$

where ν *is a probability measure on the event that*

$$\lambda_1 \lambda_2 \cdots \lambda_p = \frac{1}{\mu_p}.$$

Proof. Let $S_n(\boldsymbol{x})$ and $S(\boldsymbol{x})$ be identical for $n = m, m+1$. Then from (6.11),

$$P_m(\boldsymbol{x}) = \mu_{m,p}.$$

Further,

$$\begin{aligned} \mu_{m,p} &= \int_{(\mathbf{0},\infty)} S_m(\boldsymbol{x})\ d\boldsymbol{x}, \\ &= \int_{(\mathbf{0},\infty)} S(\boldsymbol{x})\ d\boldsymbol{x} \end{aligned}$$

is independent of m. Hence $P_m(\boldsymbol{x}) = \mu_p$, say, which is independent of $\boldsymbol{x}$ as well. Thus

$P_m(\boldsymbol{x})$ is a constant and hence from (6.8),

$$k_{m+1}(\boldsymbol{x}) = \frac{1}{\mu_p}.$$

From Puri and Rubin (1974), for a given $\mu_p > 0$, the only absolutely continuous distribution with constant failure rate is (6.28). Conversely, when $\boldsymbol{X}$ has the distribution specified by (6.28), direct calculations yield $S_n(\boldsymbol{x}) = S(\boldsymbol{x})$. ■

Remark 6.4.1. *The above theorem also follows from* Puri and Rubin (1974) *and* Navarro et al. (2006, p. 58).

Theorem 6.4.2. *For the random vectors* $\boldsymbol{X}$ *and* $\mathbf{Y}_n$ *defined on* Section 6.3, *the survival function* $S(\boldsymbol{x})$ *is always equal to* $R_n(\boldsymbol{x})$ *for all* $\boldsymbol{x} > \mathbf{0}$ *if and only if* $\boldsymbol{X}$ *follows Gumbel's multivariate exponential distribution with survival function*

$$S(\boldsymbol{x}) = \exp\left[-\sum_{i=1}^{p} \lambda_i \; x_i - \sum_{i<j,\; i,j=1}^{p} \lambda_{ij} \; x_i x_j - \ldots - \lambda_{12\ldots p} \; x_1 x_2 \cdots x_p\right]. \qquad (6.29)$$

Proof. Since the vector failure rate determines the corresponding multivariate distribution uniquely (Galambos and Kotz (1978, p. 129)), when $S(\boldsymbol{x})$ and $R_n(\boldsymbol{x})$ are identical, their failure rates must be the same, which means that $\Lambda_{n,i}(\boldsymbol{x}) = h_i(\boldsymbol{x})$. Hence from (6.24)

$$\frac{\partial}{\partial x_i} E[(X_i - x_i)^n | \boldsymbol{X} > \boldsymbol{x}] = 0.$$

This implies

$$E[(X_i - x_i)^n | \boldsymbol{X} > \boldsymbol{x}] = c_i(\boldsymbol{x}_{(i)}), \; i = 1, 2, \ldots, p$$

a function independent of x_i, or

$$n \int_{x_i}^{\infty} (t_i - x_i)^{n-1} S(t_i | \boldsymbol{x}_{(i)}) \; dt_i = c_i(\boldsymbol{x}_{(i)}) \; S(x_i | \boldsymbol{x}_{(i)}).$$

Differentiating the last equation successively with respect to x_i, leads to the n^{th} order partial differential equation

$$S(x_i | \boldsymbol{x}_{(i)}) = c_i(\boldsymbol{x}_{(i)}) \; \frac{(-1)^n}{n!} \; \frac{\partial^n}{\partial x_i^n} S(x_i | \boldsymbol{x}_{(i)}).$$

Adopting the method of proof in solving a similar differential equation in Theorem 2.3.2 in Galambos and Kotz (1978, p. 33) we find

$$S(x_i|\boldsymbol{x}_{(i)}) = \exp[-b_i(\boldsymbol{x}_{(i)})\ x_i].$$

This gives the i^{th} component of the vector valued failure rate as

$$\begin{aligned} h_i(\boldsymbol{x}) &= -\frac{\partial}{\partial x_i} \log S(x_i|\boldsymbol{x}_{(i)}), \\ &= b_i(\boldsymbol{x}_{(i)}), \end{aligned}$$

which is independent of x_i. Now by Theorem 5.4.11 in Galambos and Kotz (1978, p. 129) $\boldsymbol{X}$ follows Gumbel's multivariate exponential distribution. Conversely,

$$\begin{aligned} E\left[(X_i - x_i)^n|\boldsymbol{X} > \boldsymbol{x}\right] &= \frac{1}{S(\boldsymbol{x})} \int_{(\boldsymbol{x},\ \boldsymbol{\infty})} (t - x_i)^n \frac{\partial^p S(\boldsymbol{x})}{\partial x_1 \cdots \partial x_p} dx_1 \cdots dx_p, \\ &= \frac{1}{S(\boldsymbol{x})} \int_{x_i}^{\infty} (t - x_i)^n \frac{\partial S(\boldsymbol{x})}{\partial x_i} dx_i. \end{aligned}$$

Substituting (6.29) for $S(\boldsymbol{x})$ and integrating by parts successively, we find

$$\begin{aligned} E[(X_i - x_i)^n|\boldsymbol{X} > \boldsymbol{x}] =& n! \left[\lambda_i + \sum_{j\neq i,\ j=1}^{p} \lambda_{ij}\ x_j + \sum_{j,k\neq i,\ j<k=1}^{p} \lambda_{ijk}\ x_j x_k + \ldots + \right. \\ & \left. \lambda_{12\ldots p} \prod_{j\neq i,\ j=1}^{p} x_j \right]^{-n},\ i = 1, 2, \ldots, p. \end{aligned} \tag{6.30}$$

Since (6.30) is independent of x_i, from (6.24)

$$\Lambda_{n,i}(\boldsymbol{x}) = h_i(\boldsymbol{x}),\ i = 1, 2, \ldots, p.$$

Thus the failure rates of $R_n(\boldsymbol{x})$ and $S(\boldsymbol{x})$ are the same and hence $R_n(\boldsymbol{x})$ is identical with $S(\boldsymbol{x})$. This completes the proof. ■

Chapter 7

Conclusions and Future Work

The present thesis has considered various aspects of equilibrium distributions of order n in discrete, continuous and multivariate cases. Emphasis is given to establish results related to relationships among reliability characteristics, characterization of distributions, ageing criteria and stochastic orders of equilibrium distributions.

Chapter 1 was intended to make a brief introduction on equilibrium distributions and their applications, which explain the motivation and objectives of the present study.

In the beginning of Chapter 2, we presented some basic concepts in reliability that help to explain the existing results as well as the results obtained in the subsequent chapters. After that, a brief discussion on the existing results was made, which included both continuous and discrete cases. The review included the origin and interpretations, characterizations, stochastic orders and ageing of equilibrium distributions and their higher orders.

In Chapter 3, we considered equilibrium distributions of non-negative continuous random variables representing lifetimes of components or devices. First, we derived some basic identities that have importance in reliability theory. The survival function of equilibrium distribution of order n and that of the baseline (original) distribution were linked through the moments of the residual lives of the original distribution. Further, the hazard rate and the mean residual life function of the higher order equilibrium distribution were expressed in terms of their respective lower orders. We also derived the identity connecting the mean residual life functions of the original distribution, the equilibrium distribution of order n and the residual life distribution of the equilibrium renewal process. These identities were then used to establish some characterization results involving generalized Pareto distribution and generalized mixture of exponential distributions. An approach using the characteristic function was also used to deduce a characterization result involving mixture of gamma distributions and exponential distribution. In the final section of the chapter, we established some alternative definitions for the ageing notions - IFR, DMRL, DVRL and GIMRL in terms of partial orders of equilibrium distributions and their residual lives.

Chapter 4 dealt with higher order equilibrium distributions in discrete time and their various aspects in reliability. We derived some basic identities that have significance in reliability theory. The survival function of n^{th} order equilibrium distribution was expressed in terms of the survival function and the residual life moments of the original distribution. It was further noted that the former can be expressed in terms of the stop - loss moments of the same order. The factorial moment generating function was linked to the factorial moments of the original distribution. From this identity the relation between the factorial moments of the higher order equilibrium distribution and that of the baseline distribution were obtained. It was then explained that the equilibrium distributions can be viewed as a weighted distribution in the discrete. The above discussed identities were employed to establish some characterization results of geometric, Waring and negative hyper-geometric distributions. The characterizations involve survival functions, hazard rates, mean residual life functions and stop - loss moments. After this a discussion on the higher order equilibrium distributions of mixture of distributions was made. These were illustrated with negative binomial distribution mixed with finite range beta and binomial distribution mixed with beta. Finally, some ageing notions of higher order equilibrium distributions were discussed. The ageing notions involve IFR, DMRL, IFR(2), DVRL, UBA, UBAE, NBU, NBUE, NBUC, NBUFR and NBUFRA. Various implications of the above notions of lower order and higher order equilibrium distributions were also established.

Chapter 5 also dealt with higher order equilibrium distributions in discrete time and their various aspects in reliability. Here we compared the baseline distribution with their higher order equilibrium distributions as well as the equilibrium distributions of two different variables in terms of stochastic orders. By comparing the baseline and the n^{th} order equilibrium distributions, we have established several interpretations of the ageing concepts - DMRL, IFR, DVRL, NBU, NBUE, NBUC, NBRU and NBRUE. Later, we have examined various implications among the partial orders such as hazard rate order, likelihood ratio order, mean residual life order, stochastic order, harmonic mean residual life order, increasing convex order and probability generating function order of equilibrium distributions of two different random variables. It is being revealed that some of the reversed implications do not hold and have presented examples to support the claim.

In the Chapter 6, we discussed the equilibrium distribution of higher orders in higher dimension. At first, two different approaches were made to define the multivariate equilibrium distributions of order n. They are distribution based on joint survival functions and the distribution based on conditional distributions. We have deduced that multivariate equilibrium distribution of order n can be viewed as the distribution of a random vector, whose components are the product of two independent random variables one has the marginal distribution of a multivariate size biased distribution of order n and the other is the minimum of n independent and identically distributed uniform random variables over $(0, 1)$. We have extended several results of univariate equilibrium distributions to multivariate cases. Using these results, we have established some characterizations involving mixture of exponentials and Gumbel's multivariate exponential distribution.

In the previous chapters we have seen that more results and findings are required to make more advanced study in connection with equilibrium distributions of higher orders. Some of the areas that need more emphasis are ageing criteria and stochastic orders based on higher order discrete equilibrium distributions, ageing properties of higher order multivariate equilibrium distributions and their applications to systems. Further research in these directions is in progress.

List of Published Works

1. Nair, N. U., Preeth, M. (2008). Multivariate equilibrium distributions of order n, *Statistics and Probability Letters*, **78**, 3312 - 3320.
2. Nair, N. U., Preeth, M. (2009). On some properties of equilibrium distributions of order n, *Statistical Methods and Applications*, **18**, 453 - 464.
3. Nair, N. U., Sankaran, P. G., Preeth, M. (2012). Reliability aspects of discrete equilibrium distributions, *Communications in Statistics - Theory and Methods*, **41:**3, 500 - 515.

Bibliography

[1] Abouammoh, A. M., Ahmad, A. N., Barry, A. M. (1993). Shock models and testing for the renewal mean remaining life, *Microelectronics Reliability*, **33**, 729 - 740.

[2] Abouammoh, A. M., Ahmad, R., Khalique, A. (2000). On new renewal better than used classes of life distributions, *Statistics and Probability Letters*, **48**, 189 - 194.

[3] Abouammoh, A. M., Kanjo, A., Khalique, A. (1990) On aspects of variance remaining life distributions, *Microelectron Reliability*, 30 (4), 751 - 760.

[4] Abouammoh, A. M., Qamber, I. S. (2003). New better than renewal used class of life distributions, *IEEE Transactions on Reliability*, **52**, 150 - 153.

[5] Abraham, B., Nair, N. U. (2001) On characterizing mixtures of some life distributions, *Statistical Papers*, **42**, 387 - 393.

[6] Arnold, B. C. (1994). In: Balakrishnan, N. (Ed.), Conditional Survival Models in Recent Advances in Life Testing and Reliability A Volume in Honour of Alonzo Clifford Cohen Jr. CRC Press, New York, pp. 589 - 601.

[7] Averous, J. and Meste, M. (1989). Tailweight and life distributions, *Statistics and Probability Letters*, **8**, 381 - 387.

[8] Barlow, R. E., Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing*, Silver Spring.

[9] Basu, A. P. (1971). Bivariate failure rate, *Journal of the American Statistical Association*, **66**, 103 - 104.

[10] Bhattacharjee, M. C., Abouammoh, A. M., Ahmed, A. N., Berry, R. M. (2000). Preservation results for life distributions based on comparisons with asymmetric remaining life under replacements, *Journal of Applied Probability*, **37**, 999 - 1009.

[11] Bluementhal, S. (1967). Proportional sampling in life length studies, *Technometrics*, **9**, 205 - 2018.

[12] Bon, J., Illayk, A. (2002). A note on some new renewal ageing notions, *Statistics and Probability Letters*, **57**, 151 - 155.

[13] Brown, M. (2006). Exploiting the waiting time paradox, *Probability in the Engineering and Informational Sciences*, **20**, 195 - 230.

[14] Bryson, M. C., Siddiqui, M.M. (1969). Some criteria for ageing, *Journal of the American Statistical Association*, **64**, 1472 - 1483.

[15] Calabria, R., Pulcini, G. (1987). On the asymptotic behaviour of the mean residual life function, *Reliability Engineering*, **19**, 165 - 170.

[16] Cao, J., Wang, Y. (1991). The NBUC and NWUC classes of life distributions, *Journal of Applied Probability*, **28**, 473 - 479.

[17] Chatterjee, A., Mukherjee, S. P. (2000). Equilibrium distribution - Its role in reliability theory, In:N. Balakrishnan and C.R. Rao (Eds.) *Handbook of Statistics*,Elsevier, **20**, 105 - 137.

[18] Cox, D. R. (1962). *Renewal Theory*, Methuen and Co, London.

[19] Cox, D. R., Lewis, P.A.W. (1966). *The Statistical Analysis of Series of Events*, New York: Wiley.

[20] Denuit, M., Lefevre, C., Shaked, M. (1998). The s-convex orders among real random variables, *Mathematical Inequalities and Applications,* **1**, 585 - 613.

[21] Deshpande, J. V., Kochar, S. C., Singh, H. (1986). Aspects of Positive Ageing, *Journal of Applied Probability*, **23**, 748 - 758.

[22] Ebrahimi, N. (1989). How to define system improvement and deterioration for a repairable system, *IEEE Transactions on Reliability*, **38**, 214 - 217.

[23] Fagiuoli, E., Pellerey, F. (1993). New partial orderings and applications, *Naval Research Logistics*, **40**, 829 - 842.

[24] Fagiuoli, E., Pellerey, F. (1994). Prevention of certain classes of life distributions under Poisson shock models, *Journal of Applied Probability*, **31**, 458 - 465.

[25] Feller, W. (1957). *An introduction to Probability Theory and its Applications*, 2nd ed. New York: Wiley.

[26] Galambos, J., Kotz, S. (1978). *Characterization of Probability Distributions*, Lecture Notes in Mathematics, Vol. 675. New York: Academic Press.

[27] Gupta, R. C. (1976). Some characterizations of distributions by properties of their forward and backward recurrence times in a renewal process, *Scandinavain Journal of Statistics*, **3**, 215 - 216.

[28] Gupta, R. C. (1979). On the characterization of survival distributions in reliability by properties of their renewal densities, *Communications in Statistics - Theory and Methods*, **8**:7, 685 - 697.

[29] Gupta, R. C. (1984). Some characterizations of renewal densities with emphasis in reliability, *Mathematische Operations-forschung and Statistics*, **15**, 571 - 579.

[30] Gupta, R. C. (2007). Role of equilibrium distribution in reliability studies, *Probability in the Engineering and Informational Sciences*, **21**, 315 - 334.

[31] Gupta, R. C., Kirmani, S. N. U. A. (1990). The role of weighted distributions in stochastic modeling, *Communications in Statistics - Theory and Methods*, **19**: 9, 3147 3162.

[32] Gupta, R. C., Kirmani, S. N. U. A. (2000) Residual coefficient of variation and some characterization results, *Journal of Statistical Planning and Inference*, **91**, 23 - 31.

[33] Gupta, R. C., Kirmani, S. N. U. A. (2004). Some characterizations of distributions by functions of failure rate and mean residual life, *Communications Statistics - Theory and Methods*, **33**, 3115 - 3131.

[34] Gupta, R. C., Kirmani, S. N. U. A., Launer, R. L. (1987). On life distributions having monotone residual variance, *Probability in the Engineering and Informational Sciences*, **1**, 299 - 307.

[35] Gupta, R. P., Sankaran, P. G. (1998). Bivariate equilibrium distribution and its applications to reliability, *Communications in Statistics - Theory and Methods*, **27**, 385 - 394.

[36] Harkness, W. L., Shantaram, R. (1969). Convergence of a sequence of transformations of distribution functions, *Pacific Journal of Mathematics*, **31**: 2, 403 - 415.

[37] Hitha, N., Nair, N. U. (1989). Characterization of some discrete models by properties of residual life function, *Calcutta Statistical Association Bulletin*, **38**, 151 - 154.

[38] Hu, T., Kundu, A., Nanda, A. K. (2001). On generalized orderings and ageing properties with their implications. In: Hayakawa, Y., Irony T., Xie, M. (ed) System and Bayesian Reliability. World Scientific, Singapore, 199 - 228.

[39] Huang, J. S., Lin, G. D. (1995). Characterization of distributions using equilibrium transformations, *Sankhya: The Indian Journal of Statistics*, **57**:2, 179 - 185.

[40] Jain, K, Singh, H., Bagai, I. (1989). Relations for reliability measures of weighted distributions, *Communications in Statistics - Theory and Methods*, **18**: 12, 4393 - 4412.

[41] Jean-Louis, B., Abbas, I. (2005). Ageing properties and series system, *Journal of Applied Probability*, **42**, 279 - 286.

[42] Johnson, N. L., Kotz, S. (1975). A vector valued multivariate hazard rate, *Journal of Multivariate Analysis*, **5**, 53 - 66.

[43] Johnson, N. L., Kotz, S., Balakrishnan, N. (1994). *Continuous Univariate Distributions*, Vol 1, 2nd edn. Wiley, New York.

[44] Kaas, R., Van Heerwaarden, A., Goovaerts, M. (1994). *Ordering of Actuarial Risks*, Xrussels: CAIRE.

[45] Kleiber, C., Kotz, S. (2003). *Statistical Size Distributions in Economics and Actuarial Sciences*, Hoboken, NJ: Wiley Interscience.

[46] Klugman, S., Panjer, H., Willmot, G. (1998). *Loss Models - From Data to Decisions*, New York: Wiley.

[47] Lai, C. D., Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability*, Springer, New York.

[48] Launer, R. L. (1984). Inequalities for NBUE and NWUE classes of life distributions, *Operations Research*, **32**, 660 - 667.

[49] Li, X., Xu, M. (2007). Reversed hazard rate order of equilibrium distributions and a related ageing notion, *Statistical Papers*, **49**, 749 - 767.

[50] Loh, W. Y. (1984). A new generalisation of NBU distributions, *IEEE Transactions on Reliability*, **33**, 419 - 422.

[51] Mi, J. (1998). Some comparison results of system availability, *Naval Research Logistics*, **45**, 205 - 218.

[52] Mugdadi, A. R., Ahmad, I. A. (2005). Moments inequalities derived from comparing life with its equilibrium form, *Journal of Statistical Planning and Inference*, **134**, 303 - 317.

[53] Mukherjee, S. P., Chatterjee, A. (1992). Stochastic dominance of higher orders and its implications, *Communications in Statistics - Theory and Methods*, **21**: 7, 1977 - 1986.

[54] Muth, E. J. (1977). Reliability models with positive memory derived from the mean residual life function, In: *The Theory and Applications of Reliability*, Vol. II, C.P. Tsokos and I.N. Shimi (eds.). New York: Academic Press, 401 - 435.

[55] Nair, N. U. (1989). Characterization of failure time models by properties of residual life. In *Proceedings of the Symposium on Reliability and Renewal Theory*, Ed. A. M. Mathai , pp. 1 - 9.

[56] Nair, N. U., Asha, G., Sankaran, P.G. (2004). On the covariance of residual lives, *Statistica*, **44**, 573 - 585.

[57] Nair, N. U., Geetha, K. G., Priya, P. (2000). On partial moments of discrete distributions and their applications, *Far East Journal of Theoretical Statistics*, **4**, 153 - 164.

[58] Nair, N. U., Hitha, N. (1989). *Characterization of discrete models by distribution based on their partial sums,* Statistics and Probability Letters, **8**, 335 - 337.

[59] Nair, N. U., Preeth, M. (2008). Multivariate equilibrium distributions of order n, *Statistics and Probability Letters*, **78**, 3312 - 3320.

[60] Nair, N. U., Preeth, M. (2009). On some properties of equilibrium distributions of order n, *Statistical Methods and Applications*, **18**, 453 - 464.

[61] Nair, N. U., Sankaran, P. G. (2010). Properties of a mean residual life function arising from renewal theory, *Naval Research Logistics*, **57**, 373 - 379.

[62] Nair, N. U., Sankaran, P. G., Preeth, M. (2012). Reliability aspects of discrete equilibrium distributions, *Communications in Statistics - Theory and Methods*, **41:**3, 500 - 515.

[63] Nanda, A. K., Jain, K., Singh, H. (1996a). On closure of some partial orderings under mixtures, *Journal of Applied Probability*, **33**: 3, 698 - 706.

[64] Nanda, A. K., Jain, K., Singh, H. (1996b). Properties for moments for s-order equilibrium distributions, *Journal of Applied Probability*, **33**: 4, 1108 - 1111.

[65] Nassar, M. M., Mahmood, M. R. (1985). On characterization of a mixture of exponential distributions, *IEEE Transactions on Reliability*, **34**, 484 - 488.

[66] Navarro, J., Hernandez, P. J. (2004). How to obtain bathtub-shaped failure rate models from normal mixtures, *Probability in the Engineering and Informational Sciences*, **18**, 511 - 531.

[67] Navarro, J., Ruiz, J. M. (2004). Characterizations from relationships between failure rate functions and conditional moments, *Communications in Statistics - Theory and Methods*, **33**, 3159 - 3171

[68] Navarro, J, Ruiz, J. M., Aguila, Y.D. (2006). Multivariate weighted distributions: A review and some extensions, *Statistics*, **40**, 51 - 64.

[69] Navarro, J., Sarabia, J. M. (2010). Alternative definitions of bivariate equilibrium distributions, *Journal of Statistical Planning and Inference*, **140**, 2046 - 2056.

[70] Pakes, A. G. (1996). Length biasing and laws equivalent to the log-normal, *Journal of Mathematical Analysis and Applications*, **197**, 825 - 854.

[71] Pakes, A. G., Navarro, J. (2007). Distributional characterizations through scaling relations, *Australian and New-Zealand Journal of Statistics*,**49**: 2, 115 - 135.

[72] Pakes, A. G., Navarro, J., Ruiz, J.M., Aguila, Y. (2003). Characterizations using weighted distributions, *Journal of Statistical Planning and Inference*, **116**, 389 - 420.

[73] Puri, P. S., Robin, H. (1974). On a characterization of the family of distributions with constant multivariate failure rate, *Annals of Probability*, **2**, 738 - 740.

[74] Rao, C. R. (1965). On discrete distributions arising out of methods of ascertainments. In *Classical and Contagious Discrete distributions*. G. P. Patil, Ed., Permagon Press and Statistical Publishing Society, Calcutta, 320 - 332.

[75] Sankaran, P. G., Nair, N. U. (1993). A bivariate Pareto model and its applications to reliability, *Naval Research Logistics*, **40**, 1013 - 1020.

[76] Scheaffer, P. L. (1972). Size biased sampling, *Technometrics*, **14**, 635 - 644.

[77] Sen, A., Khattree, R. (1996). Length biased distribution, equilibrium distribution and characterization of probability laws, *Journal of Applied Statistical Science*, **3**: 4, 239 - 252.

[78] Shaked, M., Shanthikumar, J. G. (2007). *Stochastic Orders*, Springer, New York.

[79] Stein, W. E., Dattero, R. (1999). Bondesson's functions in reliability theory, *Applied Stochastic Models in Business and Industry*, **15**, 103 - 109.

[80] Whitt, W. (1985). The Renewal-Process Stationary-Excess operator, *Journal of Applied Probability*, **22**:1, 156 - 167.

[81] Willmot, G. E., Drekic, S., Cai, J. (2005). Equilibrium component distributions and stop-loss moments, *Scandinavian Actuarian Journal*, **1**, 6 - 24.

[82] Xekalaki, E. (1983). Hazard functions and life distributions in discrete time, *Communications in Statistics - Theory and Methods*, **12**(21), 2503 - 2509.

[83] Zahedi, H. (1985). Some new classes of multivariate survival distribution functions, *Journal of Statistical Planning and Inference*, **11**, 171 - 178.

[84] Zelen, M. (1974). Problems in cell kinetics and early detection of disease, *Reliability and Biometry*, SIAM, Philadelphia, 701 - 726.

[85] Zelen, M., Feinleib, M. (1969). On the theory of screening for chronic diseases, *Biometrika*, **56**, 601 - 613.