Some Problems in Algebra and Topology

SOME GENERALIZATIONS OF FUZZY METRIZABILITY

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CERTIFICATE

This is to certify that the thesis entitled **"Some Generalizations of Fuzzy Metrizability"** is an authentic record of research carried out by **Sri. Sreekumar.R,** under our supervision and guidance in the Department of Mathematics, Cochin University of Science and Technology, Cochin-22, for the Ph.D. degree of the Cochin University of Science and Technology and no part of it has previously formed the basis for the award of any other degree or diploma in any other university.

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INTRODUCTION

There are two types of imprecision – vagueness and ambiguity. The difficulty of making sharp distinctions is vagueness and the situation of two or more alternatives not specified is ambiguity. An answer to capture the concept of imprecision in a way that would differentiate imprecision from uncertainty, L.A Zadeh [ZA] in 1965 introduced the concept of a fuzzy set.

In Mathematics a subset A of X can be equivalently represented by its characteristic function, a mapping χ_A from the universe X of discourse (region of consideration) containing A to the 2 - value set $\{0,1\}$. That is to say x belongs to A if and only if $\chi_A(x) = 1$. But the idea and concept of fuzzy set, introduced by Zadeh, used the unit interval I =[0,1] instead of $\{0,1\}$. That is in "fuzzy" case the "belonging to" relation $\chi_A(x)$ between x and A is no longer "either 0 or otherwise 1" but it has a membership degree, say A(x), belonging to [0,1].

The fuzzy set theory extended the basic mathematical concept of a set. Fuzzy Mathematics is just a kind of Mathematics developed in this frame work. In 1967 J.A Goguen [G] introduced the concept of L-fuzzy sets. Fuzzy set theory has become important with application in almost all areas of Mathematics, of which one is in the area of Topology.

Fuzzy topology is a kind of topology developed on fuzzy sets. It is a generalization of topology in classical mathematics, but it also has its own marked characteristics. For the first time in 1968, C.L.Chang [C] defined fuzzy topological spaces in the frame work of fuzzy sets. In 1976, R.Lowen [LO_1] has given another definition for a fuzzy topology by including all constant functions instead of just $\underline{0}$ and $\underline{1}$ (where $\underline{0}$ and $\underline{1}$ are fuzzy sets which takes every $x \in X$ to 0 and every $x \in X$ to 1 respectively) of Chang's definition. In this thesis we are following Chang's definition rather than Lowen's definition. Extensive work on the area of fuzzy topology was carried out by Goguen [G], Wong [WO], Lowen [LO], Hutton[HU] and others. More over this area of fuzzy Mathematics has applications in Science and Technology.

In 1984 Kaleva.O and Seikkala.S [K;S] introduced the concept of fuzzy metric. It provided a method for introducing fuzzy pseudo-metric topologies on sets. Earlier in 1982,Deng Zi-ke [DZ] introduced fuzzy pseudo metric spaces. In 1993 A. George and P. Veeramani [GV_1] modified the concept of fuzzy metric introduced by Kramosil and Michalek [K;M]. Also M.A Erceg [E], G. Artico and R. Moresco [A], B.Hutton and I. Reilly [HR] worked in this area. But not much work seems to have been done on the topological properties of fuzzy metrizable spaces.

The main purpose of our study is to extend the concept of the class of spaces called 'generalized metric spaces' to fuzzy context and investigate its properties.

Any class of spaces defined by a property possessed by all metric spaces could technically be called as a class of 'generalized metric spaces'. But the term is meant for classes, which are 'close' to metrizable spaces in some sense. They can be used to characterize the images or pre images of metric spaces under certain kinds of mappings.

The theory of generalized metric spaces is closely related to what is known as 'metrization theory'. These classes often appear in theorems, which characterize metrizability in terms weaker topological properties. The class of spaces like Morita's M- spaces, Borges's wΔ-spaces, Arhangelskii's p-spaces, Okuyama's σ-spaces have major roles in the theory of generalized metric spaces. They have appeared as a 'factor' in many metrization theorems. The first three are similar in some sense, being equivalent in the presence of paracompactness. In fact classes like 'p-spaces' generalize both metric spaces and compact spaces and various theorems which hold for both of these classes can often be generalized and hence unified by showing that they hold for p-spaces.

In this thesis we introduce fuzzy metrizable spaces, fuzzy submetrizable spaces and prove some characterizations of fuzzy submetrizable spaces. Also we introduce some fuzzy generalized metric spaces like fuzzy wΔ-spaces, fuzzy Moore spaces, fuzzy M-spaces, fuzzy k-spaces, fuzzy σ-spaces, study their properties, prove some equivalent conditions for fuzzy p- spaces. Also we prove some theorems to show that these classes of spaces are closely related to fuzzy metrizable spaces. The thesis is divided into six chapters, including the preliminary chapter.

In chapter 0, we collect the basic definitions, results and notations, which we require in the succeeding chapters. We use C.L.Chang's [C] definition of fuzzy topology. For a topological space (X, T) we denote by $\omega(T)$, the set of all lower semicontinuous maps f : X→ [0,1]. Then $\omega(T)$ is a fuzzy topology called the generated fuzzy topology. Also for a fuzzy topological space (X, F) the associated topology is denoted by $\iota(F)$ and is the weakest topology which makes every member of F lower semicontinuous. By x_α we mean a fuzzy point with support $x \in X$ and value $\alpha \in (0,1]$. All the fuzzy topological spaces (X, F) considered are assumed to be T_1 (That is every fuzzy point in the fuzzy topological space is a closed fuzzy set).

Metrizability is a very nice but restrictive property for topological spaces. The notion of submetrizability by Gary Gruenhage [GG] is less restrictive but retains much of this nicety. In chapter 1 we define fuzzy metrizable spaces, fuzzy submetrizable spaces and prove some characterizations of fuzzy submetrizable spaces. One of the property of a fuzzy metrizable space is that of having a G_δ -diagonal. We say that a fuzzy topological space (X, F) has a G_δ -diagonal if the diagonal Δ is a G_δ–set in (X^2, F_p), where F_p is the fuzzy product topology. We prove some equivalent conditions for a fuzzy topological space (X, F) to have a G_δ–diagonal. Also we study the relation between fuzzy paracompact spaces and fuzzy metrizable spaces.

The concepts like wΔ-spaces, developable spaces and Moore spaces were extensively discussed by various authors as a part of the study of generalized metric spaces.. In chapter 2 we introduce fuzzy wΔ-spaces, fuzzy Moore spaces, fuzzy submetacompact spaces, fuzzy subparacompact spaces and investigate some of their properties. We prove that every fuzzy subparacompact space is fuzzy submetacompact. Also we prove that every fuzzy Moore space is fuzzy subparacompact and a regular fuzzy topological space is a fuzzy Moore space if and only if it is a fuzzy submetacompact wΔ-spaces with a G_δ –diagonal.

The M- spaces, introduced by Morita, played a major role in the theory of generalized metric spaces. This class is very much related to metrizable spaces. In chapter 3 we introduce fuzzy M- spaces and study its relationship to fuzzy metrizable spaces. We prove that every fuzzy M- space is a fuzzy wΔ-space and is fuzzy submetrizable. Also we prove that an induced fuzzy topological space is fuzzy metrizable if it is a fuzzy M-space with a G_δ –diagonal.

The class 'p- spaces' generalizes both metrizable spaces and compact spaces. The concept of 'p-spaces' due to Arhangelskii is in terms of a sequence of open covers in some compactification of the space rather than the space itself. We refer[MH_1] for fuzzy Stone-$\check{C}$ech compactification. In chapter 4 we define fuzzy p-spaces, strict fuzzy p-spaces and prove some characterizations of both fuzzy p-spaces and strict fuzzy p-spaces. We also define fuzzy analogue of k-spaces and show that every regular fuzzy p-space is a fuzzy k- space.

The concept of a network is one of the most useful tools in the theory of generalized metric spaces. The σ-spaces is a class of generalized metric spaces having a network. In chapter 5 we introduce fuzzy σ-spaces and study its properties. We prove that every regular fuzzy σ-spaces is a fuzzy subparacompact space. A regular fuzzy topological space is a fuzzy Moore space if and only if it is a fuzzy σ-space and a fuzzy wΔ-space with a G_δ-diagonal.

The idea of fuzzy sets introduced by Zadeh using the unit interval, to describe and deal with the non-crisp phenomena was generalized by Goguen [G] using some lattice instead of [0,1]. Although in this thesis we use [0,1] fuzzy set up, most of the result could be extended to L-fuzzy setting.

CHAPTER 0

PRELIMINARIES AND BASIC CONCEPTS

The fuzzy set introduced by L.A.Zadeh, extended the basic Mathematical concept – set. In view of the fact that set theory is the corner stone of modern Mathematics, a new and more general frame work of Mathematics was established . Fuzzy topology is a kind of topology developed on fuzzy sets.

In this chapter we collect the basic definitions, results and notations, which we require in the succeeding chapters. Most of these are adapted from 'Fuzzy topology : Advances in Fuzzy Systems-Application and Theory' by Liu Ying-ming and Luo Mao kang [Y;M] by specializing L to [0,1]. Also we have used [ZA], [C], [MH_1] [MH_2][MH_3], [B], [B;W], [WL]'[P;Y_1] for some definitions and [GG] and [WI] for some results in generalized metric spaces.

0.1 Basic Operations on Fuzzy Sets

Definition 0.1.1

Let X be a set. A fuzzy set on X is a mapping $A : X \rightarrow [0,1]$. For a fuzzy set A, $\{x \in X : A(x) > 0\}$ is called the support of A and is denoted by supp A.

Definition 0.1.2

Let A and B be fuzzy sets on X. Then

(i) $A = B \Leftrightarrow A(x) = B(x)$ for all $x \in X$.

(ii) $A \leq B \Leftrightarrow A(x) \leq B(x)$ for all $x \in X$

(iii) $A \vee B$ is a fuzzy set on X defined by $(A \vee B)(x) = \max\{A(x), B(x)\}$ for all $x \in X$

(iv) $A \wedge B$ is a fuzzy set on X defined by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for all $x \in X$.

(v) For a fuzzy set A on X, its complement A' is a fuzzy set on X defined by $A'(x) = 1 - A(x)$ for all $x \in X$

Generally if $\{A_i\}_{i \in I}$ is any collection of fuzzy sets on X, then $\bigvee_{i \in I} A_i$ is a fuzzy set on X defined by $(\bigvee_{i \in I} A_i)(x) = \sup_{i \in I} \{ A_i(x) : x \in X\}$ and $\bigwedge_{i \in I} A_i$ is a fuzzy set on X defined by $(\bigwedge_{i \in I} A_i)(x) = \inf_{i \in I} \{ A_i(x) : x \in X\}$.

Definition 0.1.3

For $\alpha \in (0,1]$, $x \in X$, a fuzzy point x_α is defined to be the fuzzy set on X defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Notation

The set of all fuzzy sets on X is denote by I^X. The fuzzy set which takes every element in X to 0 is denoted by $\underline{0}$ and which takes every element in X to 1 is denoted by $\underline{1}$. N denotes the set of all natural numbers.

0.2 Fuzzy Topological spaces

Definition 0.2.1

A collection $F \subset I^X$ is called a fuzzy topology on X if it satisfies the following conditions

(i) $\underline{0}, \underline{1} \in F$

(ii) If A, B $\in$ F, then $A \wedge B \in F$

(iii) If $A_i \in F$ for each $i \in I$, then $\vee_{i \in I} A_i \in F$.

Then (X, F) is called a fuzzy topological spaces and the members of F are called fuzzy open sets. A fuzzy set is called fuzzy closed if its complement is fuzzy open.

Remark 0.2.2

For a fuzzy topological space (X, T), the set of all lower semi continuous functions from X to [0,1] generates a fuzzy topology on X, called the generated fuzzy topology and is denoted by ω(T).

Definition 0.2.3

The fuzzy set $A \subset I^X$ is called a crisp set on X if there exists an ordinary subset $U \subset X$ such that $A = \chi_U$.

Given a topological space (X,T), F $=\{ \chi_U : U \in T \}$ forms a fuzzy topology on X. Also given a fuzzy topological space (X, F), the set of supports of crisp members of F forms an ordinary topology on X, called the background space of (X, F) and is denoted by [F].

Definition 0.2.4

Let (X,F_1) and (Y,F_2) be two fuzzy topological spaces and let $f: X \to Y$ be a function. Then for a fuzzy set A on X, f(A) is a fuzzy set on Y defined by $f(A)(y) = \begin{cases} \vee \{A(x) : x \in X, f(x) = y\} & : \text{when } f^{-1}(y) \neq \phi \\ 0 & : \text{when } f^{-1}(y) = \phi \end{cases}$

and for a fuzzy set B of Y, $f^{-1}(B)$ is a fuzzy set on X defined by $f^{-1}(B)(x) = B(f(x))$ for all $x \in X$.

Definition 0.2.5

Let (X_1, F_1) and (X_2, F_2) be two fuzzy topological spaces. Let $X = X_1 \times X_2$ and $P_i : X \to X_i$ (i = 1,2) be the projections. Then the fuzzy product topology on X is the fuzzy topology F_p, generated by $\{P_i^{-1}(A_i) : A_i \in F_i, i = 1,2\}$. (X, F_p) is called the fuzzy product space.

Definition 0.2.6

For a fuzzy topological space (X, F), the diagonal Δ is the fuzzy set on $X \times X$ defined by $\Delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$

Definition 0.2.7

Let (X,F) be a fuzzy topological space. Then a fuzzy set A on X is called a G_δ -set if $A = \bigwedge_{i=1}^{\infty} A_i$, for $A_i \in F$

Definition 0.2.8

Let (X, F) be a fuzzy topological space and $A \in F$. Then A is said to be a quasi coincident neighbourhood of a fuzzy point x_α if $A'(x) < \alpha$. The set of all quasi coincident neighbourhood of x_α is denoted by $Q(x_\alpha)$.

We say that a fuzzy set A is quasi coincident with a fuzzy set B if $B'(x) < A(x)$ for some $x \in X$. We denote it by $A \hat{q} B$.

Definition 0.2.9

The fuzzy topological space (X, F) is said to be T_2 (Hausdorff) if for every two fuzzy points x_λ , y_γ with $x \neq y$, there exists $U \in Q(x_\lambda)$, $V \in Q(y_\gamma)$ such that $U \wedge V = \underline{0}$

Definition 0.2.10

The fuzzy topological space (X, F) is said to be strongly T_2 (or s-T_2) if for every two fuzzy points x_λ and y_γ with $x \neq y$, there exists $U \in Q(x_\lambda)$, $V \in Q(y_\gamma)$ such that $U_{(0)} \wedge V_{(0)} = \varphi$ where $U_{(0)} = \{x \in X : U(x) > \lambda\}$ and $V_{(0)} = \{x \in X : V(x) > \gamma\}$

Definition 0.2.11

The fuzzy topological space (X, F) is called α - T_2 for $\alpha \in (0,1]$, if for every distinct points $x,y \in X$ there exists $U \in Q(x_\alpha)$, $V \in Q(y_\alpha)$ such that $U \wedge V = \underline{0}$. (X, F) is called level-T_2 if (X, F) is α - T_2 for every $\alpha \in (0,1]$.

Theorem 0.2.12

Let (X, F) be a fuzzy topological space. Then (X, F) is Hausdorff if and only if the diagonal of (X^2, F_p), where F_p is the fuzzy product topology, is a closed fuzzy set on (X^2, F_p).

Definition 0.2.13

A fuzzy topological space (X, F) is said to be regular if for each fuzzy point x_α and $U \in F$ with $x_\alpha \le U$, there exists $V \in F$ such that $x_\alpha \le V \le \overline{V} \le U$.

0.3 Stratified Spaces and Induced Spaces

Definition 0.3.1

A fuzzy topological space (X, F) is said to be induced if F is exactly the family of all lower semicontinuous mappings from $(X, [F]) \to [0,1]$, where [F] is the set of supports of crisp members of F.

(X, F) is said to be weakly induced if every $G \in F$ is a lower semicontinuous mapping from $(X, [F]) \to [0,1]$.

Definition 0.3.2

Let (X, F) be a fuzzy topological space and Y be a set. Let $f : X \to Y$ be a mapping. Then $F/f = \{V \in I^Y : f^{-1}(V) \in F\}$ is a fuzzy topology on Y and (Y, F/f) is called the fuzzy quotient space with respect to f.

Theorem 0.3.3

Let (X, F) be an induced fuzzy topological space and (Y, F/f) be the fuzzy quotient space with respect to the surjective mapping $f : X \to Y$. Then (Y, F/f) is also an induced fuzzy topological space.

Definition 0.3.4

Let (X, F) be a fuzzy topological space. Then the fuzzy topology F_1 generated by $F \cup \{\underline{a} \mid a \in [0,1]\}$ where $\underline{a} = a \wedge \chi_X$, is called the stratification of F and (X, F_1) is said to be stratified.

Theorem 0.3.5

Let (X, F) be a fuzzy topological space. Then (X, F) is stratified if and only if F contains all the lower semicotinuous functions from $(X, [F]) \to [0,1]$.

Theorem 0.3.6

Let (X, T) be an ordinary topological space and let $f : X \to [0,1]$ be a mapping. Then f is lower semicontinuous if and only if for every $a \in (0,1)$, $f^{[a]}$ is closed in (X, T), where $f^{[a]} = \{x \in X \mid f(x) \leq a\}$.

Theorem 0.3.7

Let (X, F) be a weakly induced fuzzy topological space. Then the following conditions are equivalent.

(i) (X, F) is s-T_2

(ii) (X, F) is T_2

(iii) (X, F) is level-T_2

(iv) There exists $\alpha \in (0,1]$ such that (X, F) is α - T_2.

(v) (X, [F]) is T_2.

0.4 Fuzzy Compact Spaces and Fuzzy Paracompact Spaces

Definition 0.4.1

Let (X, F) be a fuzzy topological space. For a fuzzy set G on X, a family $\mathcal{A}$ of fuzzy sets on X is called a cover of G if $\vee\mathcal{A} \geq G$. $\mathcal{A}$ is called a cover of (X, F) if it is a cover of $\underline{1}$ and is an open cover if $\mathcal{A} \subset F$.

Definition 0.4.2

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of fuzzy sets on (X, F). We say that $\mathcal{A}$ is a refinement of $\mathcal{B}$ if for each G $\in \mathcal{A}$ there is an H $\in \mathcal{B}$ such that G $\leq$ H.

Definition 0.4.3

A fuzzy set A in a fuzzy topological space (X, F) is said to be fuzzy compact if for every $\mathcal{A} \subset F$ with $\vee\mathcal{A} \geq$ A and every $\varepsilon > 0$, there exists a finite subfamily $\mathcal{A}' \subset \mathcal{A}$ such that $\vee\mathcal{A}' \geq A - \varepsilon$. In particular X is fuzzy compact if $\underline{1}$ is fuzzy compact.

Definition 0.4.4

A family $\{A_t : t \in \mathcal{T}\}$ of fuzzy sets on (X, F) is said to be locally finite at x_λ, if there exists $U \in Q(x_\alpha)$ such that $A_t \hat{q}\, U$ holds except for finitely many $t \in \mathcal{T}$.

For a fuzzy set $A \in I^X$, $\mathcal{A} = \{A_t : t \in \mathcal{T}\}$ of fuzzy sets is called locally finite in A, if $\mathcal{A}$ is locally finite at every x_λ, where λ is such that $\lambda \leq A(x)$ for some x $\in$X.

Definition 0.4.5

Let (X, F) be a fuzzy topological space and $\alpha \in (0,1]$. A collection $\mathcal{B}$ of fuzzy sets is called an α - Q cover of a fuzzy set A, if for each fuzzy point x_α

with $x_\alpha \leq A$, there exists $B \in \mathcal{B}$ with $B'(x) < \alpha$ (That is x_α quasi coincident with B)

If $\mathcal{B} \subset F$, then $\mathcal{B}$ is called an open α - Q cover of A.

Definition 0.4.6

Let (X, F) be a fuzzy topological space and let $A \in I^X$ and $\alpha \in [0, 1]$. Then A is said to be α - fuzzy paracompact if for every open α - Q cover of $\mathcal{A}$ of A, there exists an open refinement $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is a locally finite in A and is an α - cover of A.

A is called fuzzy paracompact if A is α- fuzzy paracompact for every $\alpha \in [0,1]$. (X, F) is said to fuzzy paracompact if $\underline{1}$ is fuzzy paracompact.

Definition 0.4.7

Let (X, F) be a fuzzy topological space and let A be a fuzzy set. Then $\mathcal{A} = \{A_t : t \in \mathcal{T}\}$ of fuzzy sets on X is called *- locally finite in A if for every x_λ, where λ is such that $\lambda \leq A(x)$ for some $x \in X$, there exists $U \in Q(x_\lambda)$ and a finite subset T_0 of T such that $t \in T - T_0 \Rightarrow A_t \wedge U = \underline{0}$.

Definition 0.4.8

For $\alpha \in [0, 1]$, a fuzzy set A on (X, F) is said to be α^*- fuzzy paracompact if for every α - Q cover of $\mathcal{A}$ of A, there exists an open refinement $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is * - locally finite in A and is an α - Q cover of A.

A is * - fuzzy paracompact if A is α^* - fuzzy paracompact for every $\alpha \in [0, 1]$. (X, F) is * - fuzzy paracompact if $\underline{1}$ is * - fuzzy paracompact.

Theorem 0.4.10

Let (X, F) be a weakly induced fuzzy topological space. Then the following conditions are equivalent.

(i) (X, F) is fuzzy paracompact

(ii) There exists $\alpha \in (0,1)$ such that (X, F) is α - fuzzy paracompact.

(iii) (X, [F]) is paracompact.

Theorem 0.4.11

Let (X, F) be a fuzzy topological space and $A \in I^X$, $\alpha \in [0,1)$. Then

(i) A is α^* - fuzzy paracompact $\Rightarrow$ A is α - fuzzy paracompact.

(ii) A is * - fuzzy paracompact $\Rightarrow$ A is fuzzy paracompact.

Theorem 0.4.12

Every T_2 and fuzzy compact fuzzy topological space is * - fuzzy paracompact.

Remark 0.4.13

The concept of "good extensions" in fuzzy topological spaces was introduced by Lowen [LO_1]. For a property P of ordinary topological spaces, a property P* of fuzzy topological spaces is called a good extension of P, if for every ordinary topological space (X, T), (X, T) has property P if and only if (X, T) has property P*.

Fuzzy compactness is a " good extension" of compactness in ordinary topological spaces.

0.5 Some Results from Generalized Metric Spaces

Definition 0.5.1

Let X be a topological space . For $x \in X$ and a collection $\mathcal{U}$ of subsets of X , $st(x, \mathcal{U}) = \cup \{ U \in \mathcal{U} : x \in U \}$. For $A \subset X$, $st(A, \mathcal{U}) = \cup \{U \in \mathcal{U} : U \cap A \neq \phi\}$.

A cover ψ of a space X is called a star refinement of a cover $\mathcal{U}$ if $\{st(x, \psi) : x \in X\}$ is a refinement of $\mathcal{U}$.

Theorem 0.5.2

A T_0- space X is metrizable if and only if X has a development $(\mathcal{G}_n)$ such that whenever $G, G' \in \mathcal{G}_{n+1}$ and $G \cap G' \neq \varphi$, then $G \cup G'$ is contained in some member of $\mathcal{G}_n$.

Theorem 0.5.3

A T_0- space X is metrizable if and only if X has a development $(\mathcal{G}_n)$ such that for each n, $\mathcal{G}_{n+1}$ is a star- refinement of $\mathcal{G}_n$.

Lemma 0.5.4

Let $(\mathcal{G}_n)$ be a sequence of open covers of X such that $(\mathcal{G}_{n+1})$ is a regular refinement of $\mathcal{G}_n$ for each n. Then there is a pseudo metric ρ on X such that

(i) $\rho(x, y) = 0$ if $y \in \cap \operatorname{st}(x, \mathcal{G}_n)$

(ii) U is open in the topology generated by ρ if and only if for each $x \in U$ there exists $n \in N$ such that $\operatorname{st}(x, \mathcal{G}_n) \subset U$.

Lemma 0.5.5

Let Y be a submetacompact subspace of a topological space X. For each n, let $\mathcal{U}_n$ be a collection of open subsets of X covering Y. Then there exists a sequence (ψ_n) of open collections covering Y such that, for each $y \in Y$

$$\bigcap_n \overline{\operatorname{st}(y, \psi_n)} = \bigcap_n \operatorname{st}(y, \psi_n) \subset \bigcap_n \operatorname{st}(y, \mathcal{U}_n)$$

Theorem 0.5.6

A countably compact space with a G_δ- diagonal is compact, hence metrizable.

Theorem 0.5.7

A topological space X is a Moore space (respectively a metrizable space) if and only if X is submetacompact (respectively a paracompact) wΔ-space with a G_δ- diagonal.

Definition 0.5.8

A map f from a topological space X onto a topological Y is perfect if it is continuous, closed and for each $y \in Y$, $f^{-1}\{y\}$ is compact.

A map $f : X \rightarrow Y$ is said to be quasi perfect if it is continuous , closed and $f^{-1}\{y\}$ is countably compact for each $y \in Y$

Theorem 0.5.9

A topological space X is an M-space if and only if there exists a metric space Y and a quasi perfect map from X onto Y.

Theorem 0.5.10

A T_1-space is paracompact if and only if every open cover of the space has an open star refinement.

CHAPTER 1

FUZZY SUBMETRIZABILITY

1.1 Introduction

Metrizability is a very nice but restrictive property for topological spaces. The notion of submetrizability (for details refer [G G]) is less restrictive but retains some of the nice properties of metrizability. A topological space (X, T) is submetrizable if their exists a topology $T' \subset T$ with (X, T') metrizable. In this chapter we define analogously the concept of fuzzy metrizable spaces, fuzzy submetrizable spaces and obtain some characterizations of fuzzy submetrizable spaces. Also we study the relation between fuzzy paracompact spaces and fuzzy metrizable spaces .

1.2 Fuzzy Submetrzable Spaces

Definition 1.2.1

Let (X, F) be a fuzzy topological space. . Let $\iota(F)$ be the weakest topology on X which makes all the functions in F are lower semicontinuous. Then the fuzzy topological space (X, F) is said to be fuzzy metrizable if $(X, \iota(F))$ is metrizable.

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* We have included some results of this chapter in the paper titled *On Fuzzy Submetrizablity* in **The Journal of Fuzzy Mathematics**, Vol. 10 No. 2 (2002).

** Some Results mentioned in this chapter are published in the paper titled "*Fuzzy Metrizablity and Fuzzy Compactness*" in the proceedings of **The International Workshop and seminar on Transform Techniques and Their Applications** held at St. Joseph's College, Irinjalakkuda, Kerala (2001)

Definition 1.2.2

The fuzzy topological space (X , F) is said to be fuzzy submetrizable if there exists $F' \subset F$ such that (X, F') is fuzzy metrizable.

Example 1.2.3 - Fuzzy metrizable spaces

1. Consider X= R, the real line. Let F be the fuzzy topology generated by the set $\{\chi_U \mid U$ open in the usual topology on $R\}$. Then (X, F) is fuzzy metrizable.

2. Let (X, T) be a metrizable topological space . Then $\omega(T) = \{ f \mid f: (X,T) \to [0,1]$ is lower semicontinuous $\}$ is a fuzzy topology on X ,called the generated fuzzy topology and the fuzzy topological space (X, ω(T)) is fuzzy metrizable .

Example 1.2.4 - Fuzzy Submetrizable Spaces

1. Consider X= R. For intervals of the type [a,b) define $f_{[a,b)}: X \to [0,1]$ by

$$f_{[a,b)}(x) = \begin{cases} 1 & \text{if } x \in (a,b) \\ \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x \notin [a,b) \end{cases}$$

Let F be the fuzzy topology generated by $\{ f_{[a,b)}, \chi_{(a,b)} \mid a, b \in R\}$. Now the weakest topology $\iota(F)$, which makes all elements of F lower semicontinuous, is the lower limit topology , and (X , $\iota(F)$) is not metrizable. Therefore (X, F) is not fuzzy metrizable.

If F' is the fuzzy topology generated by $\{ \chi_{(a,b)} \mid a,b \in R \}$, then (X, F') is fuzzy metrizable, since $(X, \iota(F'))$, where $\iota(F')$ is the usual topology, is metrizable. Now $F' \subset F$. Therefore (X, F) is fuzzy submetrizable.

2. Consider $X = R$. For G, H subsets of R, G open with respect to the usual topology on R and H any subset of irrationals, define $f_{G,H} : X \to [0,1]$ by

$$f_{G,H}(x) = \begin{cases} 1 & \text{if } x \in G \\ \frac{1}{2} & \text{if } x \notin G \text{ and } x \in H \\ 0 & \text{otherwise} \end{cases}$$

Let T be the topology on R with basic open sets as $\{ G \cup H \mid G$ open with respect to usual topology, H subset of irrationals$\}$. Consider $F = \{f_{G,H} \mid G, H \subset R$, G open in the usual topology, H any subset of irrationals$\} \cup \{\underline{0}, \underline{1}\}$. Then the fuzzy topological space (X, F) is not fuzzy metrizable, since (X, T) is not metrizable and $T = \iota(F)$. Let $F' = \{ f_{G,\phi} \mid G$ open in the usual topology$\} \cup \{\underline{0}, \underline{1}\}$. Then the weakest topology $\iota(F')$, which makes every member of F' lower semicontinuous, is the usual topology on R. Now $(X, \iota(F'))$ is metrizable. Therefore (X, F') is fuzzy metrizable. Also $F' \subset F$. Hence (X, F) is fuzzy submetrizable.

Remark 1.2.5

The concept of fuzzy metrizability and fuzzy submetrizability that we have introduced above are 'good extensions' of the crisp metrizability and submetrizability in the sense of R.Lowen[LO_1].

Definition 1.2.6

The fuzzy topological space (X , F) is said to have a G_δ -diagonal if the diagonal Δ is a G_δ -set in (X^2 , F_p) where F_p is the fuzzy product topology.

Definition 1.2.7

Let $\mathcal{A}$ be a cover of (X , F) . For $\alpha \in (0,1]$ and a fuzzy point x_α , $st(x_\alpha, \mathcal{A}_n) = \vee\{B: B\in \mathcal{A} \text{ and } B(x) \geq \alpha\}$ and for a fuzzy set G, $st(G , \mathcal{A}) = \vee\{B : B\in \mathcal{A} \text{ and } B_n \wedge G \neq \underline{0}\}$.

Theorem 1.2.8

A fuzzy topological space (X , F) has a G_δ -diagonal if and only if there exists a sequence $(\mathcal{A}_n)$ of open covers of (X , F) such that for x , y $\in$ X with $x \neq y$, $\alpha, \beta \in (0,1]$ there exists $n\in N$ with $y_\beta \nleq st(x_\alpha, \mathcal{A}_n)$.

Proof

First suppose that (X , F) has a G_δ -diagonal. Then $\Delta = \wedge_n G_n$ where each G_n is a fuzzy open set on (X^2 , F_p). For each $n \in N$, $x\in X$ and for each $\alpha \in (0,1]$ we have $x_\alpha \times x_\alpha \leq \Delta \leq G_n$ (here $x_\alpha \times x_\alpha \leq \Delta$ maens that $\Delta(x,x) \geq \alpha$). Since $G_n\in F_p$, there exists $H_{n,x}^{\alpha} \in F$ such that $x_\alpha \leq H_{n,x}^{\alpha}$

and $H_{n,x}^{\alpha} \times H_{n,x}^{\alpha} \leq G_n$. Then $\mathcal{A}_n = \{ H_{n,x}^{\alpha} \mid x \in X \ , \alpha \in (0,1] \}$ forms an open cover of (X,F) .

For x, y $\in$ X with $x \neq y$, $\alpha, \beta \in (0,1]$, we claim that there exists $n \in N$ with $y_\beta \nleq st(x_\alpha, \mathcal{A}_n)$. For otherwise suppose that $y_\beta \leq st(x_\alpha, \mathcal{A}_n)$ for all n . Then $y_\beta \leq H_{n,x}^{\gamma}$ for some $\gamma \in (0, 1]$ and $H_{n,x}^{\gamma} \leq st(x_\alpha, \mathcal{A}_n)$. Also $x_\alpha \leq H_{n,x}^{\gamma}$. Therefore $x_\alpha \times y_\beta \leq H_{n,x}^{\gamma} \times H_{n,x}^{\gamma} \leq G_n$ for each n . That is $x_\alpha \times y_\beta \leq \wedge_n G_n = \Delta$, which is a contradiction. Thus $(\mathcal{A}_n)$ satisfies the conclusion of the theorem .

Conversely suppose that $(\mathcal{A}_n)$ be a sequence of open covers which satisfies the conditions mentioned in the theorem . Let $G_n = \vee \{ A \times A \mid A \in \mathcal{A}_n \}$. Then for $\alpha \in (0, 1]$, $x_\alpha \leq A$ for some $A \in \mathcal{A}_n$. Therefore $x_\alpha \times x_\alpha \leq A \times A$. Hence $\Delta = \vee(x_\alpha \times x_\alpha) \leq \vee \{ A \times A \mid A \in \mathcal{A}_n \} = G_n$ for each n. Therefore $\Delta \leq \wedge_n G_n$. If for each α ,$\beta \in (0,1]$ and $x \neq y$ with $x_\alpha \times y_\beta \leq \wedge_n G_n$, we have $x_\alpha \times y_\beta \leq G_n$ for each n . Hence there exists $A_n \in \mathcal{A}_n$ with $x_\alpha \times y_\beta \leq A_n \times A_n$. That is $y_\beta \leq st(x_\alpha, \mathcal{A}_n)$, which is a contradiction. Therefore $\wedge_n G_n = \vee \{(x_\alpha \times x_\beta) \mid x \in X$ and $\alpha, \beta \in (0, 1]\} \leq \Delta$. That is (X, F) has a G_δ-diagonal.

Remark

We write $\not\leq$ and not >, although they are the same here , keeping in mind the fact that the same concept of the theorem can be extended to L- fuzzy topological spaces.

Remark 1.2.9

If $(\mathcal{A}_n)$ is a sequence of fyzzy open covers with the property in theorem 1.2.8, then for $x \in X$, $\alpha \in (0, 1]$, $\wedge_n st(x_\alpha, \mathcal{A}_n) = x_\alpha$.

Proof

Let $y \in$ support of $\wedge_n st(x_\alpha, \mathcal{A}_n)$. Let $\wedge_n st(x_\alpha, \mathcal{A}_n)(y) = \beta$. Therefore $y_\beta \leq \wedge_n st(x_\alpha, \mathcal{A}_n)$. Therefore $y_\beta \leq \wedge_n st(x_\alpha, \mathcal{A}_n)$ for all n. Hence by the above theorem $y = x$. Therefore support of $\wedge_n st(x_\alpha, \mathcal{A}_n) = \{x\}$. Let $\gamma = \wedge_n st(x_\alpha, \mathcal{A}_n)(x)$ (note that $\gamma \geq \alpha > 0$) .If $\gamma > \alpha$ then by passing onto refinements we can form a sequence of fuzzy open covers $(\mathcal{A}_n^*)$ such that $\wedge_n st(x_\alpha, \mathcal{A}_n)(x) = \alpha$. Hence $\wedge_n st(x_\alpha, \mathcal{A}_n) = x_\alpha$.

Example 1.2.10

Consider X=R. For G, H subsets of R, G open with respect to the usual topology on R and H any subset of irrationals, define $f_{G,H} : X \to [0,1]$ by

$$f_{G,H}(x) = \begin{cases} 1 & \text{if } x \in G \\ \frac{1}{2} & \text{if } x \notin G \text{ and } x \in H \\ 0 & \text{otherwise} \end{cases}$$

Let $F = \{f_{G,H} \mid G, H \subset R$, G open in the usual topology, H any subset of irrationals$\} \cup \{\underline{0}, \underline{1}\}$. Consider (X^2, F_p), where F_p is the fuzzy product topology. Basic fuzzy open sets in (X^2, F_p) can be written as $f_{G_1,H_1} \times f_{G_2,H_2}$ where $(f_{G_1,H_1} \times f_{G_2,H_2})(x,y) = \min\{f_{G_1,H_1}(x), f_{G_2,H_2}(y)\}$. For each n, take $G_n = f_{\left(\frac{-1}{n},\frac{1}{n}\right),\phi} \times f_{\left(\frac{-1}{n},\frac{1}{n}\right),\phi}$. Then

$$G_n(x,y) = \begin{cases} 1 & \text{if } x,y \in \left(\frac{-1}{n}, \frac{-1}{n}\right) \\ 0 & \text{otherwise} \end{cases}$$

Then the diagonal $\Delta = \wedge_n G_n$. That is Δ is a G_δ-set . Therefore (X, F) is having a G_δ-diagonal

Definition 1.2.11

A sequence $(\mathcal{A}_n)$ of fuzzy open covers of (X,F) is called a G_δ -diagonal sequence , if for each x , y$\in$ X with $x \neq y$, $\alpha, \beta \in (0,1]$, there exists

$n \in N$ with $y_\beta \nleq st(x_\alpha\ \mathcal{A}_n)$. That is $\underset{n}{\wedge} st\left(x_\alpha, \mathcal{A}_n\right) = x_\alpha$. A space (X, F) has a G_δ -diagonal if there exists a G_δ -diagonal sequence.

Definition 1.2.12

A cover $\mathcal{A}$ of (X, F) is called a star refinement of the cover $\mathcal{B}$ if $\{st(G, \mathcal{A}) : G \in \mathcal{A}\}$ refines $\mathcal{B}$. That is for each $G \in \mathcal{A}$, there exists $H \in \mathcal{B}$ such that $st(G, \mathcal{A}) \leq H$.

Definition 1.2.13

Let (X, F) be a fuzzy topological space. A fuzzy covering $\mathcal{A}_{n+1}$ is a regular refinement of the covering $\mathcal{A}_n$ if for $G, H \in \mathcal{A}_{n+1}$ with $G \wedge H \neq \underline{0}$, $G \vee H \leq B_n$ for some $B_n \in \mathcal{A}_n$.

Theorem 1.2.14

The following are equivalent for an induced fuzzy topological space (X , F) .

(a) (X, F) is fuzzy submetrizable

(b) (X, F) has a G_δ -diagonal sequence $(\mathcal{A}_n)$ such that $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$ for each n

(c) (X, F) has a G_δ -diagonal sequence $(\mathcal{A}_n)$ such that $\mathcal{A}_{n+1}$ is a regular refinement of $\mathcal{A}_n$ for each n.

Proof

(a)$\Rightarrow$ (b)

Assume that (X , F) is fuzzy submetrizable. Then there exists $F' \subset F$ such that (X, F') is fuzzy metrizable. Therefore (X, $\iota(F')$) is metrizable. Hence (X,$\iota(F')$) is paracompact. Therefore each open cover of (X, $\iota(F')$) has an open star refinement (see [WI]). By metrizability we can form a G_δ-diagonal sequence ($\mathcal{G}_n$) for (X ,$\iota(F')$). For each $G_n \in \mathcal{G}_n$, let A_{G_n} be the characteristic function of G_n. Each A_{G_n} is lower semicontinuous and hence belongs to F'. We claim that ($\mathcal{A}_{\mathcal{G}_n}$) where $\mathcal{A}_{\mathcal{G}_n} = \{ A_{G_n} \mid G_n \in \mathcal{G}_n \}$ forms a G_δ-diagonal sequence for (X, F').

Consider x, y$\in$ X with x $\neq$ y, $\alpha, \beta \in$ (0, 1]. Since ($\mathcal{G}_n$) forms a G_δ-diagonal sequence for (X,$\iota(F')$), there exists n $\in$N such that y $\notin$ st (x, $\mathcal{G}_n$). Now $x_\alpha \neq y_\beta$ and st(x_α, $\mathcal{A}_{\mathcal{G}_n}$) $=\vee\{ B_{G_n} \mid B_{G_n} \in \mathcal{A}_{\mathcal{G}_n}$ and $B_{G_n}(x) \geq \alpha \}$. Therefore st($x_\alpha$, $\mathcal{A}_{\mathcal{G}_n}$) (y)= 0 where as $y_\beta(y) = \beta$. Therefore $y_\beta \not\leq$ st(x_α, $\mathcal{A}_{\mathcal{G}_n}$). That is ($\mathcal{A}_{\mathcal{G}_n}$) forms a G_δ-diagonal sequence for (X , F'). Since $\mathcal{G}_{n+1}$ star refines $\mathcal{G}_n$, it follows that $\mathcal{A}_{\mathcal{G}_{n+1}}$ star refines $\mathcal{A}_{\mathcal{G}_n}$.

(b) $\Rightarrow$ (c)

Let $(\mathcal{A}_n)$ be a G_δ-diagonal sequence for (X , F) such that $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$ for each n . Let G , H $\in \mathcal{A}_{n+1}$ be such that $G \wedge H \neq \underline{0}$. Since $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$,there exists $A_n \in \mathcal{A}_n$ such that $G \leq st(G, \mathcal{A}_{n+1}) \leq A_n$. Since $G \wedge H \neq \underline{0}$, $H \leq st(G, \mathcal{A}_{n+1})$. Therefore $G \vee H \leq st(G, \mathcal{A}_{n+1}) \leq A_n \in \mathcal{A}_n$. Hence $\mathcal{A}_{n+1}$ is a regular refinement of $\mathcal{A}_n$ for each n.

(c) $\Rightarrow$ (a)

Let (X , F) has a G_δ-diagonal sequence $(\mathcal{A}_n)$ such that $\mathcal{A}_{n+1}$ is a regular refinement of $\mathcal{A}_n$ for each n . For each $A_n \in \mathcal{A}_n$,let $G_n = \bigcup_{\alpha \in (0,1]} A_n^{-1}((\alpha,1])$

Then $\mathcal{G}_n = \{ G_n \}$ forms G_δ-diagonal sequence for (X , ι(F)) . Since F is an induced fuzzy topological space $[F] = \iota(F)$. Take G_{n+1} , $G'_{n+1} \in \mathcal{G}_{n+1}$ with $G'_{n+1} \cap G_{n+1} \neq \phi$. Then there exists A_{n+1} , $A'_{n+1} \in \mathcal{A}_{n+1}$ with $A_{n+1} \wedge A'_{n+1} \neq \underline{0}$ and $G_{n+1} = \bigcup_{\alpha \in (0,1]} A_{n+1}^{-1}((\alpha,1])$, $G'_{n+1} = \bigcup_{\alpha \in (0,1]} A'^{-1}_{n+1}((\alpha,1])$.

By assumption $A_{n+1} \vee A'_{n+1} \leq A_n$ for some $A_n \in \mathcal{A}_n$. Therefore

$\bigcup_{\alpha\in(0,1]} \{A^{-1}_{n+1}(\alpha,1] \cup A'^{-1}_{n+1}(\alpha,1]\} \subset \bigcup_{\alpha\in(0,1]} A^{-1}_{n}(\alpha,1])$. That is $G_{n+1} \cup G'_{n+1} \subset G_n$.

Thus the sequence $(\mathcal{G}_n)$ of open covers are such that $\mathcal{G}_{n+1}$ is a regular refinement of $\mathcal{G}_n$. Therefore by the Lemma 0.5.4, there exists a pseudometric ρ on X such that U is open in the topology generated by ρ if and only if for each $x \in U$,there exists $n \in N$ such that st(x, $\mathcal{G}_n$) $\subset$ U . Now $\{x\}$ = $\bigcap_n$ st(x, $\mathcal{G}_n$) .Therefore ρ is metric on X [by part(i) of Lemma 0.5.4]. Also by part (ii) of the Lemma 0.5.4, the topology generated by ρ, say T', is contained in the topology $\iota(F)$. Therefore (X , $\iota(F)$) is submetrizable . Let $\omega(T')$ be the collection of all lower semicontinuous mappings from (X, T') $\rightarrow$ [0,1]. Now $T' \subset \iota(F)$ and since F is induced $\omega(T') \subset F$. Therefore (X, $\omega(T')$) is fuzzy metrizable. Hence(X , F) is fuzzy submetrizable .

1.3 Fuzzy Compactness, Fuzzy Paracompactness and Fuzzy Metrizability

In this section we connect fuzzy paracompact spaces with fuzzy submetrizable spaces. Also we prove some relationship between fuzzy metrizable spaces and fuzzy compact spaces

Lemma 1.3.1

If (X, F) is an induced fuzzy paracompact space, then the generated topological space $(X, \iota(F))$ is paracompact.

Proof

Since (X, F) is induced, $[F] = \iota(F)$. Also as (X, F) is a fuzzy paracompact space, (X, [F]) is paracompact [see theorem 0.4.10]. That is $(X, \iota(F))$ is paracompact.

Lemma 1.3.2

Let (X, F) be an induced fuzzy topological space. If (X, F) is fuzzy paracompact with a G_δ-diagonal then it is fuzzy submetrizable.

Proof

Let $(\mathcal{A}_n)$ be a G_δ-diagonal sequence for the induced fuzzy paracompact space (X, F). Then by lemma 1.3.1 $(X, \iota(F))$ is paracompact. Therefore every open cover of $(X, \iota(F))$ has an open star refinement [see Theorem 0.5.10].

Consider $\mathcal{G}_n = \{ G_n \mid G_n = \bigcup_{\alpha \in (0,1]} A_n^{-1}(\alpha, 1], A_n \in \mathcal{A}_n \}$. Then $(\mathcal{G}_n)$ forms a G_δ-diagonal sequence for $(X, \iota(F))$. For if $x, y \in X, x \neq y$ and there exists no $n \in N$ with $y \notin st(x, \mathcal{G}_n)$, then we have $y \in st(x, \mathcal{G}_n)$ for all n.

For each n, $y \in st(x,\mathcal{G}_n) \Rightarrow x, y \in G_n$, for some $G_n \in \mathcal{G}_n$.

$$\Rightarrow y \in A_n^{-1}(\beta,1],\ x \in A_n^{-1}(\alpha,1] \text{ for some } \alpha,\beta \in (0,1]$$

$$\Rightarrow A_n(y) > \beta,\ A_n(x) > \alpha$$

$$\Rightarrow y_\beta < A_n \text{ and } x_\alpha < A_n \text{ for some } A_n \in \mathcal{A}_n.$$

$$\Rightarrow y_\beta < st(x_\alpha, \mathcal{A}_n).$$

This is a contradiction as $(\mathcal{A}_n)$ is a G_δ-diagonal sequence for (X, F).

As (X, T) is paracompact, there exists a G_δ-diagonal sequence say $(\mathcal{G}_n')$ such that, $\mathcal{G}_{n+1}'$ star refines $\mathcal{G}_n'$ (see[GG]). Corresponding to each $\mathcal{G}_n'$, form $\mathcal{A}_n'$ where $\mathcal{A}_n' = \{\chi_{G_n'},\ G_n' \in \mathcal{G}_n'\}$. Then $(\mathcal{A}_n')$ forms a G_δ-diagonal sequence for (X,F) such that $\mathcal{A}_{n+1}'$ star refines $\mathcal{A}_n'$. Therefore by theorem 1.2.14(b), (X, F) is fuzzy submetrizable.

Theorem1.3.3

Let (X,F) be an induced Hausdorff fuzzy topological space. If (X, F) is a fuzzy compact space with a G_δ-diagonal, then it is fuzzy metrizable.

Proof

Let (X, F) be an induced Hausdorff fuzzy compact space with a G_δ-diagonal. Then (X, F) is a fuzzy paracompact space [see Theorem 0.4.11and Theorem 0.4.12]. Since (X, F) is induced, by theorem 1.3.2 ,it follows that (X, F) is fuzzy submetrizable. Therefore there exists $F' \subset F$ such that (X, F') is fuzzy metrizable. Now, as (X, F) is induced, $\iota(F) = [F]$. Then $(X, \iota(F))$ is a compact Hausdorff space [see Theorem 0.3.7 and Remark 0.4.13]. Also$(X, \iota(F')) \subset (X, \iota(F))$ and $(X, \iota(F'))$ is metrizable. But a topology which is strictly weaker than a compact Hausdorff topology cannot be Hausdorff. Therefore $\iota(F') = \iota(F)$ so that $F' = F$. Hence (X, F) is fuzzy metrizable.

CHAPTER 2

FUZZY WΔ- SPACES AND FUZZY MOORE SPACES

2.1 Introduction

The notion of generalized metric spaces is closely related to metrization theory . The concepts like wΔ-spaces, developable spaces and Moore spaces were extensively studied by various authors, as a part of the study of generalized metric spaces. In this chapter we introduce fuzzy submetacompact spaces, fuzzy subparacompact spaces, fuzzy wΔ-spaces, , fuzzy Moore spaces and investigate some of their properties.

2.2 Fuzzy sub metacompact spaces.

A topological space X is submetacompact if for each open cover $\mathcal{U}$ of X there is a sequence (ψ_n) of open refinements of $\mathcal{U}$ such that for each $x \in X$, there exists $n \in N$ such that x is in only finitely many elements of ψ_n. In this section we define fuzzy submetacompact spaces and study some of its properties.

We have included some results of this chapter in the paper titled '*On Fuzzy wΔ-Spaces and fuzzy Moore Spaces*' **Journal of Thripura Mathematical Society** Vol.4 (2002) 47-52.

Definition 2.2.1

A sequence $(\mathcal{A}_n)$ of fuzzy open covers of a fuzzy topological space (X, F) is called a G_δ^* - diagonal sequence if for each $x \in X$, $\alpha \in (0, 1]$ $\wedge_n st(x_\alpha, \mathcal{A}_n) = \wedge_n \overline{st(x_\alpha, \mathcal{A}_n)} = x_\alpha$.

Definition 2.2.2

A fuzzy topological space (X, F) is said to be fuzzy submetacompact, if for each fuzzy open cover $\mathcal{A}$ of X, there exists a sequence (ψ_n) of fuzzy open refinements of $\mathcal{A}$ such that, for each fuzzy point x_α, $x \in X$, $\alpha \in (0, 1]$, there exists $n \in N$ such that $x_\alpha \leq v_n \in \psi_n$ holds for finitely many elements of ψ_n.

Theorem 2.2.3

A regular fuzzy submetacompact space with a G_δ - diagonal has a G_δ^* - diagonal.

Proof

Let (X, F) be a regular fuzzy submetacompact space with a G_δ -diagonal. Then there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of (X, F) such that for a fuzzy point x_α, $\wedge_n st(x_\alpha, \mathcal{A}_n) = x_\alpha$. Consider $\mathcal{A}_1$. Then by sub metacompactness of X, $\mathcal{A}_1$ has a sequence of open refinements say $(\mathcal{U}_{1n})_{n \in N}$, such that for each fuzzy point x_α there exists $n \in N$ such that x_α is only in finitely many elements of $\mathcal{U}_{1n}$ .Let $\mathcal{U}_{11}$ be one such open refinement corresponding to the fuzzy point x_α . By regularity of X, for each $\mathcal{U}_{1n}$, we can find an open refinement, say $\psi_{1,n}$ such that, for $x_\alpha \le U_{1n} \in \mathcal{U}_{1n}$, there exists $V_{1n} \in \psi_{1,n}$ with $x_\alpha \le V_{1n} \le \overline{V_{1n}} \le U_{1n}$.

Similarly for $\mathcal{A}_2$, by submetacompactness, there exist $(\mathcal{B}_{2n})_{n\in N}$ and by regularly each $\mathcal{B}_{2n}$ has an open refinement $\mathcal{U}_{2n}$. Then take $(\psi_{2,n})_{n\in N}$ as follows.

$\psi_{2,n} = \mathcal{U}_{2n} \wedge \psi_{1,1} = \{ U \wedge V \mid U \in \mathcal{U}_{2n}, V \in \psi_{1,1} \}$ .For $\mathcal{A}_3$, by sub metacompactness, there exists a sequence of open refinements $(\mathcal{B}_{3n})_{n\in N}$ and by regularity each $\mathcal{B}_{3n}$ has an open refinement $\mathcal{U}_{3n}$..Take $(\psi_{3,n})_{n\in N}$ as follows .

$$\psi_{3,n} = \mathcal{U}_{3n} \wedge \psi_{1,1} \wedge \psi_{1,2} \wedge \psi_{2,1} \wedge \psi_{2,2}$$

Repeating this process for each m, we have a sequence $(\psi_{m,n})_{n\in N}$ of open covers of (X, F) such that

(i) $(\psi_{m,n})_{n\in N}$ is a refinement of each $\psi_{i,j}$ such that $i < m$, $j < m$ and for each fuzzy point x_α there exists $n \in N$ such that x_α is in only finitely many members of $\psi_{m,n}$.

(ii) If $V \in \psi_{m,n}$ and $i,j < m$ there exists $w \in \psi_{i,j}$ such that, $\overline{V} \leq W$ and for $k \leq m$ there exists $A \in \mathcal{A}_K$ such that $\overline{V} \leq A$.

Let $y_\alpha \leq \overline{\wedge_{i,j} st\left(x_\alpha , \psi_{i,j}\right)}$, for the fuzzy points x_α, y_α. Fix i and j and let $m > \max\{i, j\}$. Now the fuzzy point x_α is in only finitely many members of $\psi_{m,n}$ for some $n \in N$.

$$\text{Therefore } y_\alpha \leq \overline{st\left(x_\alpha , \psi_{m,n}\right)} = \vee \{\overline{V} : x_\alpha \leq V \in \psi_{m,n}\}$$

$$\leq \vee \{w : x_\alpha \leq V \leq \overline{V} < w \in \psi_{i,j}), \text{ by (ii)}$$

$$= st(x_\alpha, \psi_{i,j})$$

Therefore, for each $i,j \in N$, $y_\alpha \leq \overline{\wedge_{i,j} st\left(x_\alpha , \psi_{i,j}\right)} \Rightarrow y_\alpha \leq st(x_\alpha, \psi_{i,j})$.

Therefore by using (i) $\overline{\wedge_{i,j} st\left(x_\alpha , \psi_{i,j}\right)} = \wedge_{i,j} st\left(x_\alpha , \psi_{i,j}\right) \leq \wedge_n st\left(x_\alpha, \mathcal{A}_n\right) = x_\alpha$.

Therefore $(\psi_{i,j})_{i,j \in N}$ is a G_δ*- diagonal sequence for (X, F). Hence (X, F) has a G_δ*- diagonal.

2.3 Fuzzy wΔ-Spaces and Fuzzy Developable Spaces.

In this section we define fuzzy developable spaces, fuzzy wΔ-spaces and find some relationship between them.

Definition 2.3.1

Let (X, F) be a fuzzy topological space. A fuzzy point x_α, $\alpha \in (0,1]$ is said to be a cluster point of the set $\{(x_n)_\alpha : n \in N\}$, where $(x_n)_\alpha$ is a fuzzy set with support x_n and value α, if for each fuzzy set $G \in F$ such that $x_\alpha \leq G$, there exists $n_0 \in N$ with $x_{n0} \neq x$ and $(x_{n0})_\alpha \leq G$.

Definition 2.3.2

A fuzzy topological space (X, F) is called a fuzzy wΔ-space if there exists sequence($\mathcal{A}_n$)of fuzzy open covers of X such that for each $n \in N$, fuzzy points$(x_n)_\alpha$ with support $x_n \in X$ and value α and $(x_n)_\alpha \leq st(x_\alpha, \mathcal{A}_n)$, set $\{(x_n)_\alpha : n \in N\}$ has a cluster point.

Definition 2.3.3

A sequence ($\mathcal{A}_n$) of fuzzy open covers of (X, F) is a fuzzy development for X, if for $\alpha \in (0, 1]$, a fuzzy point x_α, the set $\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ is a base at x_α. A fuzzy topological space (X, F) is fuzzy developable if it has a fuzzy development.

Example 2.3.4

Consider X = R with usual topology T. Let F be the topology generated by the set $\{\chi_U \mid U$ open in the usual topology on R$\}$.For each $x \in X$, consider $\chi_{\left(x-\frac{1}{n}, x+\frac{1}{n}\right)}$ and form

$\mathcal{A}_n = \{ \chi_{\left(x-\frac{1}{n}, x+\frac{1}{n}\right)} \mid x \in X \}$ .Then $(\mathcal{A}_n)$ forms a sequence of fuzzy open covers of (X, F). For $\alpha \in (0, 1]$, a fuzzy point x_α, st $(x_\alpha, \mathcal{A}_n) = \vee \{ A_n \in \mathcal{A}_n \mid A_n(x) \geq \alpha \}$

$$= \vee \{ \chi_{\left(x-\frac{1}{n}, x+\frac{1}{n}\right)} \mid \chi_{\left(x-\frac{1}{n}, x+\frac{1}{n}\right)}(x) \geq \alpha \}$$

$$= \chi_{\left(x-\frac{2}{n}, x+\frac{2}{n}\right)} .$$

If G is an open fuzzy set in (X, F) and $x_\alpha \leq G$, then there exists $U \in T$ such that $x_\alpha \leq \chi_U \leq G$ .Now $x \in U$ so that there exists $n \in N$ such that $\left(x - \frac{2}{n}, x + \frac{2}{n}\right) \subset U$. Therefore $x_\alpha \leq \chi_{\left(x-\frac{2}{n}, x+\frac{2}{n}\right)} \leq \chi_U \leq G$. Hence the set $\{st (x_\alpha, \mathcal{A}_n) : n \in N\}$ is a base at x_α .Therefore (X,F) is a fuzzy developable space .

For $\alpha \in (0, 1]$, if we choose fuzzy points$(x_n)_\alpha$ with $(x_n)_\alpha \leq st(x_\alpha, \mathcal{A}_n)$, then x_α is a cluster point of the set $\{(x_n)_\alpha : n \in N\}$. Therefore (X,F) is a fuzzy wΔ-space .

Lemma 2.3.5

Let (X, F) be a fuzzy topological space. Suppose $\{U_n\}$ is a decreasing sequence of fuzzy open sets such that $\wedge_n U_n = \wedge_n \overline{U_n}$ and for $\alpha \in (0,1]$, fuzzy points $(x_n)_\alpha$ with $(x_n)_\alpha \leq U_n$ implies ,the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point.

Then $\{U_n\}$ is a base for the fuzzy set $\wedge_n U_n$. (That is for every open fuzzy set V with $\wedge_n U_n \leq V$, there exists some U_n such that $U_n \leq V$) .

Proof

Suppose that $\{U_n\}$ satisfies the hypothesis of the lemma, but not the conclusion. Then we can find a fuzzy open set V such that $\wedge_n U_n \leq V$ and for each n],there exists fuzzy points$(x_n)_\alpha$ with $(x_n)_\alpha \leq U_n$, but $(x_n)_\alpha \nleq V$. Since $\{U_n\}$ is decreasing, any cluster point of $\{(x_n)_\alpha : n \in N\}$ must be in $\wedge_n \overline{U_n}$ and $\wedge_n U_n = \wedge_n \overline{U_n}$. But as $\wedge_n U_n \leq V$, this implies that the set $\{ (x_n)_\alpha : n \in N\}$ has no cluster point, which is a contradiction. Thus for every open fuzzy set V with $\wedge_n U_n \leq V$, there exists U_n such that $U_n \leq V$.

Theorem 2.3.6

A regular fuzzy topological space (X, F) is Fuzzy developable if and only if it is a fuzzy wΔ -space with a G_δ * - diagonal.

Proof

First suppose that (X, F) is a fuzzy developable space. Let $(\mathcal{A}_n)$ be a fuzzy development. Then for $\alpha \in (0,1]$, a fuzzy point x_α with $(x_n)_\alpha \leq st\ (x_\alpha, \mathcal{A}_n)$, the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point 'x_α' . This is because the set

$\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ forma a base at x_α. Therefore if $x_\alpha \leq G$ with $G \in F$ there exists $n \in N$ such that $x_\alpha \leq st(x_\alpha, \mathcal{A}_n) \leq G$. Hence $(x_n)_\alpha \leq G$. Therefore (X, F) is a fuzzy wΔ - space. Also for $x \neq y$, $\alpha \in (0,1]$, $\{y_\alpha\}'$ is an open fuzzy set and $x_\alpha \leq \{y_\alpha\}'$. Therefore $x_\alpha \leq st(x_\alpha, \mathcal{A}_n) \leq \{y_\alpha\}'$ for some n. Hence $y_\alpha \nleq st(x_\alpha, \mathcal{A}_n)$ for some n. Since (X, F) is regular it follows that $\bigwedge_n \overline{st(x_\alpha, \mathcal{A}_n)} = x_\alpha$. Therefore (X, F) has a G_δ* - diagonal.

Conversely assume that (X, F) is a fuzzy wΔ-space with a G_δ * - diagonal. Let $(\mathcal{A}_n)$ be a sequence of fuzzy open covers of X such that, for $\alpha \in (0,1]$, a fuzzy point x_α with $(x_n)_\alpha \leq st(x_\alpha, \mathcal{A}_n)$, the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point and $\bigwedge_n st(x_\alpha, \mathcal{A}_n) = x_\alpha$. In lemma 2.3.5, take $U_n = st(x_\alpha, \mathcal{A}_n)$. By passing onto refinement, one can make $\{U_n\}$ decreasing. Therefore by lemma 2.3.5 $\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ is a base at x_α. Thus $(\mathcal{A}_n)$ is a fuzzy development for (X, F). Hence (X, F) is fuzzy developable.

2.4 Fuzzy Moore Spaces

A topological space X is said to be subparacompact if for each open cover $\mathcal{U}$ of X there exists a sequence (ψ_n) of open covers such that, for each $x \in X$, there exists $n \in N$ such that $st(x, \psi_n)$ is contained in some members of $\mathcal{U}$.

In this section we define fuzzy subparacompact spaces, fuzzy Moore spaces and study their properties.

Definition 2.4.1

A fuzzy topological space (X,F) is said to be fuzzy subparacompact if for every fuzzy open cover $\mathcal{U}$ of X, there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of X such that for $\alpha \in (0,1]$, a fuzzy point of x_α, there exists $n \in N$ such that $st(x_\alpha, \mathcal{A}_n) \leq U_n$ for some $U_n \in \mathcal{U}$.

Remark 2.4.2

Every fuzzy subparacompact space is a fuzzy submetacompact space.

Proof

Let $\mathcal{U}$ be any fuzzy open cover of the fuzzy subparacompact space (X, F). Then by the definition there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of X such that for $\alpha \in (0,1]$, a fuzzy point of x_α, there exists $n \in N$ such that $st(x_\alpha, \mathcal{A}_n) \leq U_n$ for some $U_n \in \mathcal{U}$ .Take $\psi_n = \{ st(x_\alpha, \mathcal{A}_n) \mid x \in X, \alpha \in (0,1] \}$. Then (ψ_n) forms a sequence of open refinements of $\mathcal{U}$ such that for each fuzzy point of x_α, there exists only one $st(x_\alpha, \mathcal{A}_n)$ such that $x_\alpha \leq st(x_\alpha, \mathcal{A}_n) \in \psi_n$. Therefore (X,F) is a fuzzy submetacompact space .

Definition 2.4.3

A fuzzy topological space (X, F) is said to be a fuzzy Moore space, if it is regular and fuzzy developable.

Remark

By Theorem 2.3.6 it follows that (X, F) is a Moore space if and only if it is a fuzzy wΔ -space with a G_δ * - diagonal.

Example 2.4.4

In Example 2.3.4 {st (x_α, $\mathcal{A}_n$) : n $\in$ N} forms a base at x_α . Let $\mathcal{U}$ be any open cover of X. Then for fuzzy point x_α there exists U$\in\mathcal{U}$ such that $x_\alpha \leq$ U. Then there exists n$\in$ N such that st (x_α, $\mathcal{A}_n$) $\leq$ U. Therefore (X, F) is fuzzy subparacompact. Also (X, F) is regular and fuzzy developable . Therefore it follows that (X, F) is a fuzzy Moore space.

Remark 2.4.5

Every fuzzy Moore space is a fuzzy subparacompact space.

Proof

Let (X, F) be a fuzzy Moore space. Let ($\mathcal{A}_n$) be a development for X and let $\mathcal{U}$ be any open cover of (X,F). For $\alpha \in$ (0,1], a fuzzy point of x_α, {st (x_α, $\mathcal{A}_n$) : n $\in$ N} forms a base at x_α . Therefore if $x_\alpha \leq$ U with U $\in \mathcal{U}$

there exists $n \in N$ with st $(x_\alpha, \mathcal{A}_n) \leq U$. Hence (X, F) is a fuzzy subparacompact space.

Theorem 2.4.6

A regular fuzzy topological space (X, F) is a fuzzy Moore space if and only if it is a fuzzy submetacompact, fuzzy wΔ- space with a G_δ -diagonal.

Proof

First assume that (X, F) is a fuzzy Moore space. Then (X, F) is fuzzy subparacompact [by remark 2.4.5]. Therefore (X, F) is fuzzy submetacompact [by Remark 2.4.2]. Since (X, F) is fuzzy developable, by Theorem 2.3.6, it is a fuzzy wΔ - space with a G_δ* - diagonal and hence a fuzzy wΔ - space with a G_δ - diagonal .

Conversely assume that (X, F) is a fuzzy submetacompact, fuzzy wΔ-space with a G_δ - diagonal. By Theorem 2.3.3, (X, F) has a G_δ* -diagonal. Therefore it follows that (X, F) is a fuzzy wΔ - space with a G_δ*- diagonal and hence fuzzy developable, by Theorem 2.3.6. Hence as (X, F) is regular, it is a fuzzy Moore space.

CHAPTER 3

FUZZY M-SPACES AND FUZZY METRIZABILITY

3.1 Introduction

The M-spaces, introduced by Morita, is one among those vital in the theory of generalized metric spaces. This class is very much related to that of metrizable spaces. In this chapter we introduce the fuzzy M- spaces, fuzzy quasi perfect maps and study their relationships to fuzzy wΔ-spaces and fuzzy metrizable spaces. We prove that an induced fuzzy topological space is fuzzy metrizable if it is a fuzzy M- spaces with a G_δ -diagonal.

3.2 Fuzzy M- spaces and Fuzzy quasi perfect maps

In this section we define fuzzy M-spaces, fuzzy quasi perfect maps and study their relationships to fuzzy wΔ-spaces and fuzzy metrizable spaces.

Definition 3.2.1

A fuzzy topological space (X, F) is said to be weakly countably fuzzy compact if each countable infinite set of fuzzy points clusters at some fuzzy point.

Definition 3.2.2

Let (X, F_1) and (Y, F_2) be two fuzzy topological spaces. A mapping $f: X \to Y$ is called a fuzzy quasi perfect map if

(i) f is fuzzy continuous

(ii) f is fuzzy closed

(iii) for each fuzzy point y_α of Y, $f^{-1}\{y_\alpha\}$ is weakly countably fuzzy compact.

Definition 3.2.3

A fuzzy topological space (X,F) is called a fuzzy M- space if there exists a sequence$(\mathcal{A}_n)$ of fuzzy open covers of X such that

(i) for $\alpha \in (0,1]$, if $(x_n)_\alpha$ are fuzzy points with support x_n and value α and $(x_n)_\alpha \leq st(x_\alpha, \mathcal{A}_n)$ for each $n \in N$, then the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point

(ii) each $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$.

Remark 3.2.4

Every fuzzy M- space is a fuzzy wΔ-space.

Theorem 3.2.5

Fuzzy paracompact fuzzy wΔ-spaces are fuzzy M- spaces

Proof

Let (X, F) be a fuzzy paracompact fuzzy wΔ-space .Then by the definition of fuzzy wΔ-space there exists a sequence ($\mathcal{A}_n$) be a sequence of fuzzy open covers of X which satisfies condition (i) for a fuzzy M- space. Then by the fuzzy paracompactness of (X, F) , we can modify ($\mathcal{A}_n$) to satisfy (ii) also [see proof of Theorem1.2.14, part (a)] . Thus (X, F) is a fuzzy M- space.

Theorem 3.2.6

Let (X, F_1) and (Y, F_2) be two fuzzy topological spaces with (Y, F_2) fuzzy metrizable. If f from (X, F_1) onto (Y,F_2) is a fuzzy quasi perfect map, then (X, F_1) is a fuzzy M- space.

Proof

Since (Y, F_2) is fuzzy metrizable, (Y, $\iota(F_2)$) where $\iota(F_2)$ is the weakest topology which makes every members of F_2 lower semicontinuous is metrizable. Therefore Y has a development ($\mathcal{U}_n$) such that, for each n , $\mathcal{U}_{n+1}$star refines $\mathcal{U}_n$ [see Theorem 0.5.3]. For each U∈ $\mathcal{U}_n$, define A_U : Y →[0,1] by

$$A_U(y)=\begin{cases}\frac{1}{n} & \text{if } y\in U\\ 1 & \text{if } y\notin U\end{cases}$$

Then $\mathcal{A}_n$={ A_U : U ∈ $\mathcal{U}_n$ } forms a fuzzy cover of (Y,F_2) such that $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$. Now ($\mathcal{B}_n$) where $\mathcal{B}_n$={ $f^{-1}(A_U)$ | A_U ∈ $\mathcal{A}_n$}, forms a sequence of fuzzy covers of (X, F_1) such that $\mathcal{B}_{n+1}$ star refines $\mathcal{B}_n$. For $\alpha \in (0,1]$, fuzzy point x_α and fuzzy points $(x_n)_\alpha$ with $(x_n)_\alpha \le st(x_\alpha, \mathcal{B}_n)$ for each n, we show that the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point. If there are infinitely many $(x_n)_\alpha$ with $(x_n)_\alpha \le f^{-1}(y_\alpha)$ where $y_\alpha = f(x_\alpha)$, by weakly countably fuzzy compactness of $f^{-1}\{y_\alpha\}$, the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point. Otherwise for each n, choose $(x_n)_\alpha$ with $(x_n)_\alpha \nleq f^{-1}(y_\alpha)$. Then $f((x_n)_\alpha) \le st(f(x_\alpha), \mathcal{A}_n) = st(y_\alpha, \mathcal{A}_n)$, so that $f((x_n)_\alpha) \to f(x_\alpha)$. Since f is fuzzy closed, it follows that the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point in $f^{-1}\{y_\alpha\}$. Therefore (X, F_1) is a fuzzy M-space.

Theorem 3.2.7

If (X, F) is a stratified fuzzy M-space, then there exists a fuzzy metrizable space Y and a fuzzy quasi perfect map from X onto Y.

Proof

Let (X, F) be a stratified fuzzy M- space. Let $(\mathcal{A}_n)$ be a sequence of fuzzy open covers of X satisfying (i) and (ii) in the definition of fuzzy M- space. Let $T_1 = [F]$, be the set of supports of crisp members of F. Consider $\mathcal{G}_n = \{suppA; A \in \mathcal{A}_n\}$. Then $(\mathcal{G}_n)$ forms a sequence of open covers of (X, T_1) such that $\mathcal{G}_{n+1}$ star refines $\mathcal{G}_n$ and if $x_n \in st(x, \mathcal{G}_n)$ for each $n \in N$, then (x_n) has a cluster point. Therefore (X, T_1) is an M- space. Hence there exists a metrizable space Y and a quasi perfect map f from X onto Y [see Theorem 0.5.9]. Let T be the metric topology on Y and let ω(T) be the generated fuzzy topology on Y. Then (Y, ω(T)) is fuzzy metrizable. We show that f :(X, F) $\rightarrow$ (Y, ω(T)) is a fuzzy quasi perfect map, which will complete the proof of the theorem.

(i) f is fuzzy continuous

Take B $\in \omega$(T). Then $f^{-1}(B)(x) = B(f(x))$. Therefore $f^{-1}(B) = B \circ f$, is lower semicontinuous as f is continuous and B is lower semicontinuous. Therefore $f^{-1}(B) \in F$, since (X, F) is a stratified fuzzy topological space. Hence f is fuzzy continuous.

(ii) f is fuzzy closed

Take A $\in$ F and put B = f(A). Then for a $\in$ (0,1), as f is onto,

$$B^{[a]} = \{y \in Y \mid B(y) \leq a\} = \{ y \in Y \mid \vee\{A(x) : x \in X, f(x)=y \} \leq a \}$$

$$= \{f(x) \in Y \mid A(x) \leq a \}$$

$$= \{f(x) \in Y \mid x \in A^{[a]} \}$$

$$= f(A^{[a]}).$$

Since A is lower semicontinuous, by Theorem 0.3.6, $A^{[a]}$ is closed in (X, T_l). Since f is closed $B^{[a]} = f(A^{[a]})$ is closed in Y. Therefore B $\in \omega$(T) again by Theorem 0.3.6. Since f is onto, f (A') = f(A)' = B' is closed in (Y, ω(T)). Therefore f is fuzzy closed.

(iii) $f^{-1}\{y_\alpha\}$ is weakly countably fuzzy compact for each fuzzy point y_α in Y.

Let $(x_n)_\alpha \leq f^{-1}\{y_\alpha$ for each n $\in$ N

$$(x_n)_\alpha \leq f^{-1}(\{y_\alpha\}) \Rightarrow (x_n)_\alpha(x_n) \leq f^{-1}(\{y_\alpha\})(x_n) = y_\alpha(f(x_n))$$

$$\Rightarrow \; y = f(x_n) \quad \Rightarrow x_n \in f^{-1}(y).$$

Since f is quasi perfect, $f^{-1}\{y\}$ is countably compact. Therefore (x_n) has a cluster point, say x . Then the fuzzy point x_α is a cluster point of the set $\{(x_n)_\alpha : n\in N\}$. Hence the theorem.

Remark 3.2.8

From Theorem 3.2.6 and Theorem 3.2.7 it follows that a stratified fuzzy topological space (X, F) is a fuzzy M-space if and only if there exists a fuzzy metrizable space Y and a fuzzy quasi perfect map from X onto Y.

3.3 Fuzzy M-spaces and Fuzzy metrizability

In this section we prove some connection between fuzzy M-spaces and fuzzy metrizable spaces.

Theorem 3.3.1

An induced fuzzy topological space is fuzzy metrizable if it is a fuzzy M- space with a G_δ-diagonal

Proof

Let (X, F) be an induced fuzzy M-space with a G_δ-diagonal. Then by Theorem 3.2.7, there exists a fuzzy metrizable space Y and a fuzzy quasi perfect map f from X onto Y. Then if F_Y is the fuzzy topology on Y, the fuzzy quotient space with respect to f, $(Y, F/f) = (Y, F_Y)$. Since (X, F) is induced by

Theorem 0.3.3, (Y, F_Y) is also induced. Therefore (Y, T_Y), where $T_Y = [F_Y]$, is a metrizable space. Take $[F] = T_X$. We prove that f from (X,T_X) onto (Y,T_Y) is a quasi perfect map.

(i) f is continuous

Let $V \in T_Y$, then $\chi_V \in F_Y$. Since f is fuzzy continuous, $f^{-1}(\chi_V) \in F$. Now $f^{-1}(\chi_V) = \chi_V(f)$. Therefore $x \in Supp(f^{-1}(\chi_V)) \Leftrightarrow \chi_V(f(x)) > 0$

$$\Leftrightarrow f(x) \in V$$

$$\Leftrightarrow x \in f^{-1}(V)$$

Hence $f^{-1}(V) \in T_X$ so that f is continuous.

(ii) f is closed

Let A be a closed set in (X, T_X). Then $\chi_A' \in F$ where χ_A' denotes the compliment of χ_A. Therefore χ_A is a closed fuzzy set in (X, F). Since f is fuzzy closed, $f(\chi_A)$ is a closed fuzzy set in F_Y. Therefore supp $f(\chi_A)$ is a closed set in (Y,T_Y).

$$y \in \text{supp } f(\chi_A) \Leftrightarrow f(\chi_A)(y) \geq 0$$

$$\Leftrightarrow \vee \{\chi_A(x) \mid x \in X, f(x) = y\} \geq 0, \text{ (since f is onto)}$$

$$\Leftrightarrow x \in A \text{ and } f(x) = y$$

$$\Leftrightarrow y \in f(A)$$

Hence $f(A) = \text{supp } f(\chi_A)$ is closed in (Y, T_Y). Therefore f is a closed map.

(iii) $f^{-1}\{y\}$ is countably compact for each $y \in Y$.

Let $y \in Y$ and $x_n \in f^{-1}(y)$ for $n \in N$.

For each $n \in N$, $x_n \in f^{-1}(y) \Rightarrow y = f(x_n)$

$$\Rightarrow (x_n)_\alpha(x_n) \leq y_\alpha(f(x_n)) \text{ for } \alpha \in (0,1].$$

$$\Rightarrow (x_n)_\alpha \leq y_\alpha(f) \Rightarrow (x_n)_\alpha \leq f^{-1}\{y_\alpha\}.$$

Since $f^{-1}\{y_\alpha\}$ is weakly countably fuzzy compact, the set $\{(x_n)_\alpha : n \in N\}$ has a cluster point, say x_α and $x_\alpha \leq f^{-1}\{y_\alpha\}$. Therefore (x_n) has a cluster point at x. Thus f from (X, T_X) onto (Y, T_Y) is a quasi perfect map with (Y, T_Y) metrizable.

Let $(\mathcal{A}_n)$ be a G_δ-diagonal sequence for (X, F) Then for each $A_n \in \mathcal{A}_n$ define G_n by $G_n = \bigcup_{\alpha \in (0,1]} A_n^{-1}(\alpha, 1]$. Then $(\mathcal{G}_n)$ where $G_n \in \mathcal{G}_n$, forms a G_δ-diagonal sequence for (X, T_X). Therefore (X, T_X) is compact [see Theorem 0.5.6]. Hence f is a perfect mapping. Therefore (X, T_X) is paracompact and hence metrizable [seeTheorem0.5.7] Therefore (X, F) is a fuzzy metrizable space.

CHAPTER 4

FUZZY P- SPACES

4.1 Introduction

The class of 'p- spaces' generalizes both metrizable spaces and compact spaces. Various theorems which hold for both metrizable spaces and compact spaces can often be generalized and hence unified by showing that they hold for p- spaces . The concept of 'p- spaces' due to Arhangelskii is in terms of a sequence of open covers in some compactification of the space rather than the space itself (for details, cf[GG]) . In this chapter we define fuzzy p-spaces, strict fuzzy p-spaces and prove some characterizations of both of these spaces. We refer [MH_1] for fuzzy Stone-$\check{C}$ech compactification. We also define fuzzy k-spaces and establish some relation between fuzzy p- spaces and fuzzy k-spaces . We say that a fuzzy topological space(X, F) is completely regular if the topological space $(X, \iota(F))$ is completely regular, where $\iota(F)$ is the weakest topology on X which makes every member of F lower semicontinuous function from$(X, \iota(F)) \to [0,1]$. All the fuzzy topological spaces considered in this chapter are assumed to be completely regular .

4.2 Fuzzy P- Spaces

Definition 4.2.1(Fuzzy compactification) [MH_1]

Let (X, F) be a fuzzy topological space. Let (βX, T) be any compact topological space which contains $(X, \iota(F))$ as a dense subspace. Then F_T, the set of all lower semicontinuous mappings g : (βX, T) → [0, 1] such that $g\chi_X \in F$, is a fuzzy topology on βX and (βX, F_T) is fuzzy compact.

If (βX, T) is the Stone-$\check{C}$ech compactification of (X, ι(F)) then (βX, F_T) is the fuzzy Stone-$\check{C}$ech compactification of (X, F).

Definition 4.2.2

A completely regular fuzzy topological space (X, F) is called a fuzzy p- space if there exists a sequence ($\mathcal{B}_n$) of families of fuzzy open sets on (βX, F_T) such that

(i) for each n, $(\bigvee_{B \in \mathcal{B}_n} B) \geq \chi_X$.

(ii) for each $x \in X$, $\alpha \in (0,1]$, $\bigwedge_n st(x_\alpha, \mathcal{B}_n) \leq \chi_X$. If we also have

(iii) for $x \in X$, $\alpha \in (0, 1]$, $\bigwedge_n st(x_\alpha, \mathcal{B}_n) = \bigwedge_n \overline{st(x_\alpha, \mathcal{B}_n)}$, then (X, F) is said to be a strict fuzzy p- space.

Theorem 4.2.3

A fuzzy topological space (X, F) is a strict fuzzy p-space if and only if there exists a sequence ($\mathcal{A}_n$) of fuzzy open covers of (X, F) such that for each $x \in X$, $\alpha \in (0,1]$

(a) $Cx_\alpha = \bigwedge_n st(x_\alpha, \mathcal{A}_n)$ is fuzzy compact.

(b) $\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ forms a base for Cx_α .(That is if $Cx_\alpha \le G \in F$, there exists some n_0 such that $st(x_\alpha, \mathcal{A}_{n_0}) \le G$).

Proof

(Necessity)

First suppose that (X, F) is a strict fuzzy p- space. Then there exists a sequence $(\mathcal{B}_n)$ of families of fuzzy open sets on $(\beta X, F_T)$ which satisfies (i), (ii) and (iii) in the definition of a strict fuzzy p-space. We can assume that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$. For if $(\mathcal{B}_n)$'s are not so, then $(\mathcal{B}_n')$ where $\mathcal{B}_n' = \{ \bigwedge_{i \le n} B_i \mid B_i \in \mathcal{B}_i \}$ will do so.

For each $B_n \in \mathcal{B}_n$ choose $A_n \in F$ such that $A_n = B_n \chi_X$ and denote such collection of A_n by $\mathcal{A}_n$ .By (i) $(\mathcal{A}_n)$ forms a sequence of fuzzy open covers of (X, F). By (iii) $\bigwedge_n st(x_\alpha, \mathcal{B}_n) = \bigwedge_n \overline{st(x_\alpha, \mathcal{B}_n)}$ is closed in $(\beta X, F_T)$ and hence fuzzy compact. Therefore $Cx_\alpha = \bigwedge_n st(x_\alpha, \mathcal{A}_n) = \bigwedge_n \overline{st(x_\alpha, \mathcal{A}_n)}$ is fuzzy compact in (X, F), since $(\beta X, F_T)$ is the fuzzy Stone-$\check{C}$ech compactification of (X, F). This proves (a). Also as $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$, we have $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$. Take $U_n = st(x_\alpha, \mathcal{A}_n)$. Now $\bigwedge_n U_n = \bigwedge_n \overline{U_n}$ and as $Cx_\alpha = \bigwedge_n U_n$ is fuzzy compact, $\{(x_n)_\alpha : n \in N\}$ has a cluster point, where $(x_n)_\alpha$ are the fuzzy points with support x_n, value α and $(x_n)_\alpha \le U_n$. Therefore by lemma 2.3.5, $\{U_n\}$ is a base for the fuzzy set $\wedge_n U_n$. That is $\{st(x_\alpha, \mathcal{A}_n): n \in N\}$ is a base for Cx_α. This proves (b).

(Sufficiency)

Conversely assume that $(\mathcal{A}_n)$ is a sequence of open covers of (X, F) satisfying (a) and (b). For each $A_n \in \mathcal{A}_n$, we can find a $B_{An} \in F_T$ and can form $(\mathcal{B}_n)$ in the following way.

For each $\alpha \in (0,1]$, take $V_\alpha = A_n^{-1}(\alpha, 1] \in \iota(F)$. Let $\overset{*}{V}_\alpha = \beta X - \underset{\beta X}{cl}(X - V_\alpha)$. Then $\overset{*}{V}_\alpha \in T$ and $V_\alpha = \overset{*}{V}_\alpha \cap X$. Define $B_{An} : \beta X \to [0, 1]$ by $B_{An}(x) = \sup\{\alpha : x \in \overset{*}{V}_\alpha\}$. Then for $x \in X$, $B_{An}(x) = \sup\{\alpha : x \in V_\alpha\} = A_n(x)$. Therefore $B_{An}\chi_X = A_n$. Now we show that B_{An}'s are lower semicontinuous. Take $b \in [0,1]$. Let $G = B_{An}^{-1}(b,1]$.

Then $x \in G \Rightarrow B_{An}(x) > b$

$\Rightarrow B_{An}(x) > \alpha > b$ for some $\alpha \in (0,1)$

$\Rightarrow x \in \overset{*}{V}_\alpha \subset B_{An}^{-1}(b,1] = G$.

Therefore $G \in T$. Hence B_{An} is lower semicontinuous. That is $B_{An} \in F_T$. Take $\mathcal{B}_n = \{ B_{An} \mid A_n \in \mathcal{A}_n\}$. Then $(\mathcal{B}_n)$ forms a sequence of families of open fuzzy sets on $(\beta X, F_T)$ which satisfies condition (i) of a fuzzy p-space.

For each $A_n \in \mathcal{A}_n$, define $\overset{*}{A}_n$ as $\overset{*}{A}_n(x) = \begin{cases} A_n(x) & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases}$

$$\text{Denote } \underset{n}{\wedge} st\left(x_\alpha , \overset{*}{\mathcal{A}}_n\right) \text{ as } \overset{*}{C} x_\alpha \text{ where } \overset{*}{\mathcal{A}}_n = \{ \overset{*}{A}_n \mid A_n \in \mathcal{A}_n \}.$$

We show that $\overset{*}{C} x_\alpha = \underset{n}{\wedge} st\left(x_\alpha, \mathcal{B}_n\right) = \underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)}$. This gives conditions (ii) and (iii) for strict fuzzy p- space.

$$\text{Now } \overset{*}{C} x_\alpha = \underset{n}{\wedge} st\left(x_\alpha , \overset{*}{\mathcal{A}}_n\right) \le \underset{n}{\wedge} st\left(x_\alpha, \mathcal{B}_n\right) \le \underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)} \rightarrow \text{(i)}$$

since each $B \in \mathcal{B}_n$ is such that $B\chi_X = A \in \mathcal{A}_n$. Suppose that $\underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)} \nleq \overset{*}{C} x_\alpha$. Then there exists some $y \in \beta X$ with $\left(\underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)}\right)(y) > \overset{*}{C} x_\alpha(y)$. Let $\gamma = \left(\underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)}\right)(y)$. Therefore $y_\gamma \le \underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)}$ but $y_\gamma \nleq \overset{*}{C} x_\alpha$. Since Cx_α is fuzzy compact $\overset{*}{C} x_\alpha$ is also fuzzy compact. Therefore there exists some open fuzzy set W on $(\beta X, F_T)$ such that $y_\gamma \le W$ and $\overline{W} \wedge \overset{*}{C} x_\alpha = \underline{0}$. Also as the $\{st(x_\alpha , \mathcal{A}_n) : n \in N \}$ forms a base for Cx_α, $\{st\left(x_\alpha , \overset{*}{\mathcal{A}}_n\right) : n \in N \}$ forms a base for $\overset{*}{C}x_\alpha$. Therefore there exists some m ∈ N such that $st\left(x_\alpha, \overset{*}{\mathcal{A}}_m\right) \wedge \overline{W} = \underline{0}$ Hence

$\overline{st\left(x_\alpha, \mathcal{A}^*_m\right)} \wedge W = \underline{0}$. In particular $\overline{st\left(x_\alpha, \mathcal{B}_m\right)} \wedge W = \underline{0}$. This is a contradiction as

$y_\gamma \leq \underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)}$. Thus $\overset{*}{C}x_\alpha \geq \underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)} \rightarrow (2)$.

From (1) and (2) $\overset{*}{C}x_\alpha = \underset{n}{\wedge} st\left(x_\alpha, \overset{*}{\mathcal{A}}_n\right) = \underset{n}{\wedge} st\left(x_\alpha, \mathcal{B}_n\right) = \underset{n}{\wedge} \overline{st\left(x_\alpha, \mathcal{B}_n\right)}$. Therefore (X, F) is a strict fuzzy p- space.

Corollary 4.2.4

Every fuzzy Moore space is a strict fuzzy p- space.

Proof

Let (X, F) be a fuzzy Moore space. Let $(\mathcal{A}_n)$ be a fuzzy development for (X, F). Therefore for $\alpha \in (0, 1]$, a fuzzy point x_α, $\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ forms a base for x_α. Let $\{B_i \mid i \in I\}$, where I is an index set, be an open cover for $Cx_\alpha = \underset{n}{\wedge} st\left(x_\alpha, \mathcal{A}_n\right)$. Then there exists some B_{i0} such that $x_\alpha \leq B_{i0}$. Since $\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ is a base at x_α, there exists m∈N such that $x_\alpha \leq st(x_\alpha, \mathcal{A}_m) \leq B_{i0}$. Hence $Cx_\alpha \leq B_{i0}$, so that Cx_α is fuzzy compact.

For any fuzzy open set G with $Cx_\alpha \leq G$, we can also have $x_\alpha \leq G$. Therefore there exists n_0 such that $x_\alpha \leq st(x_\alpha, \mathcal{A}_{n0}) \leq G$. Hence

$\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ is a base for Cx_α. Thus (X, F) satisfies conditions (a) and (b) of theorem 4.2.3. Thus (X, F) is a strict fuzzy p- space.

Corollary 4.2.5

Every strict fuzzy p- space is a fuzzy wΔ-space.

Proof

Assume that (X, F) is a strict fuzzy p- space. Then by theorem4.2.3, there exists a $(\mathcal{A}_n)$ a fuzzy open covers of X such that $\{st(x_\alpha, \mathcal{A}_n) : n \in N\}$ forms a base for $Cx_\alpha = \wedge_n st(x_\alpha, \mathcal{A}_n)$ and Cx_α is fuzzy compact. Therefore if we choose fuzzy points $(x_n)_\alpha$ with $(x_n)_\alpha \leq st(x_\alpha, \mathcal{A}_n)$, $\{(x_n)_\alpha : n \in N\}$ has a cluster point in Cx_α . Thus (X, F) is a fuzzy wΔ-space.

Theorem 4.2.6

Every regular fuzzy submetacompact, fuzzy p-space is a strict fuzzy p- space.

Proof

Let (X, F) be a regular fuzzy submetacompact space which is also a fuzzy p-space. Let $(\mathcal{B}_n)$ be a sequence of families of fuzzy open subsets on $(\beta X, F_T)$ which satisfies conditions (i) and (ii) for the fuzzy p-space. For each $B_n \in \mathcal{B}_n$, choose $A_n \in F$ such that $A_n = B_n \chi_X$ and form $\mathcal{A}_n$. Then $(\mathcal{A}_n)$ forms a sequence of fuzzy open covers of X. Consider $\mathcal{A}_1$. By the fuzzy sub metacompactness of X , $\mathcal{A}_1$ has a sequence of open refinements, say $(\mathcal{U}_{1n})_{n\in N}$, such that for a fuzzy point x_α , there exists $n \in N$ such that $x_\alpha \leq U_{1n} \in \mathcal{U}_{1n}$ holds only for finitely many elements of $\mathcal{U}_{1n}$. Let $\mathcal{U}_{11}$ be one such open refinement corresponding to the fuzzy point x_α. By regularity of X , for each $\mathcal{U}_{1n}$, we can find an open refinement, say $\mathcal{A}_{1n}$, such that, for $x_\alpha \leq U_{1n} \in \mathcal{U}_{1n}$ there exists $A_{1n} \in \mathcal{A}_{1n}$ with $x_\alpha \leq A_{1n} \leq \overline{A_{1n}} \leq U_{1n}$. Similarly for $\mathcal{A}_2$, by fuzzy submetacompactness, there exists $(\mathcal{U}_{2n})_{n\in N}$ and by regularly each $\mathcal{U}_{2n}$ has a refinement $\mathcal{B}_{2n}$. Then take $(\mathcal{A}_{2n})_{n\in N}$ as follows

$\mathcal{A}_{2n} = \mathcal{B}_{2n} \wedge \mathcal{A}_{11} = \{B \wedge U / B \in \mathcal{B}_{2n}, U \in \mathcal{A}_{11}\}$. For $\mathcal{A}_3$, by fuzzy submetacompactness , there exists a sequence of open refinements $(\mathcal{U}_{3n})_{n\in N}$ and by regularly each $\mathcal{U}_{3n}$ has an open refinement $\mathcal{B}_{3n}$. Take $(\mathcal{A}_{3n})_{n\in N}$ as follows

$\mathcal{A}_{3n} = \mathcal{B}_{3n} \wedge \mathcal{A}_{11} \wedge \mathcal{A}_{12} \wedge \mathcal{A}_{21} \wedge \mathcal{A}_{22}$.

Repeating this process for each m, we have a sequence $(\mathcal{A}_{m,n})_{n\in N}$ of fuzzy open covers of X such that

(a) $(\mathcal{A}_{m,n})_{n\in N}$ is a refinement of each $\mathcal{A}_{ij}$ such that $i < m$, $j < m$ and for every fuzzy point x_α , there exists $n \in N$ such that x_α is in only finitely many members of $\mathcal{A}_{m,n}$.

(b) If $V \in \mathcal{A}_{m,n}$ and $i, j < m$,there exists $w \in \mathcal{A}_{ij}$ such that $\overline{V} \leq W$ and for $k \leq m$ there exists $A \in \mathcal{A}_K$ such that $\overline{V} \leq A$.

Let $y_\alpha \leq \overline{\bigwedge_{i,j} st\left(x_\alpha, \mathcal{A}_{i,j}\right)}$ .Fix i and j and let $m > \max\{i, j\}$. Then there exists $n \in N$ such that x_α is in only finitely many members of $\mathcal{A}_{m,n}$

$$\text{That is } y_\alpha \leq \overline{st\left(x_\alpha, \mathcal{A}_{m,n}\right)} = \vee \{\overline{V} : x_\alpha \leq V \in \mathcal{A}_{m,n}\}$$

$$\leq \vee \{w : x_\alpha \leq V \leq \overline{V} < w \in \mathcal{A}_{ij}\}, \text{ by (b)}$$

$$= st\,(x_\alpha, \mathcal{A}_{ij})$$

Therefore for each $i,j \in N$ $\overline{\bigwedge_{i,j} st\left(x_\alpha, \mathcal{A}_{i,j}\right)} = \bigwedge_{i,j} st\left(x_\alpha, \mathcal{A}_{i,j}\right) \leq \bigwedge_n st\left(x_\alpha, \mathcal{A}_n\right)$.

Corresponding to each $A_{ij} \in \mathcal{A}_{ij}$, we can find a $B_{Aij} \in F_T$ such that $A_{ij} = B_{Aij}\, \chi_X$ and can form $\mathcal{B}_{ij}$. That is $\mathcal{B}_{ij} = \{ B_{Aij} \mid A_{ij} \in \mathcal{A}_{ij} \}$. Now

$$\bigwedge_n \overline{st\,(x_\alpha, \mathcal{B}_n)} = \bigwedge_{ij} \overline{st\,\left(x_\alpha, \mathcal{B}_{i,j}\right)} = \bigwedge_{ij} st\left(x_\alpha\,,\, \overset{*}{\mathcal{A}}_{i,j}\right)$$

$\leq \quad \wedge_n st\left(x_\alpha , \overset{*}{\mathcal{A}}_n \right) \leq \wedge_n st \left(x_\alpha , \mathcal{B}_n \right)$ where $\overset{*}{A}_n(x) = \begin{cases} A_n(x) & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases}$

and $\overset{*}{\mathcal{A}}_n = \{ \overset{*}{A}_n \mid A_n \in \mathcal{A}_n \}$.Therefore $\wedge_n st \left(x_\alpha , \mathcal{B}_n \right) = \wedge_n \overline{st \left(x_\alpha , \mathcal{B}_n \right)}$. That is (X, F) is a strict fuzzy p- space .

Theorem 4.2.7

Every regular fuzzy paracompact, fuzzy p-space is a fuzzy M-space.

Proof

Let (X, F) be a regular fuzzy paracompact space, which is also a fuzzy p- space. Since every fuzzy paracompact space is a fuzzy subparacompact space and hence submetacompact space, it follows from theorem 4.2.6 that (X, F) is a strict fuzzy p- space. Hence by corollary 4.2.5, (X, F) is a fuzzy w Δ- space. Also fuzzy paracompact, fuzzy wΔ- spaces are fuzzy M-spaces, since using fuzzy paracompactness one can modify ($\mathcal{A}_n$) such that $\mathcal{A}_{n+1}$ star refines $\mathcal{A}_n$. Thus (X, F) is a fuzzy M-space.

4.3 Fuzzy P-Spaces and Fuzzy K-Spaces.

In this section we prove some characterizations for fuzzy p-spaces and prove some relationship between fuzzy p-spaces and fuzzy k-spaces.

Theorem 4.3.1

A fuzzy topological space (X, F) is a fuzzy p-space if and only if there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of (X, F) satisfying the following conditions. If for each n, fuzzy point x_α with support $x \in X$, value $\alpha \in (0,1]$ and $x_\alpha \leq A_n \leq \mathcal{A}_n$

(a) $\bigwedge_n \bar{A}_n$ is fuzzy compact

(b) $\{\bigwedge_{i \leq n} \bar{A}_i \mid n \in N\}$ is an outer network for the fuzzy set $\bigwedge_n \bar{A}_n$ That is for every fuzzy open set G with $\bigwedge_n \bar{A}_n \leq G$, there exists some $\bigwedge_{i \leq n} \bar{A}_i$ with $\bigwedge_{i \leq n} \bar{A}_i \leq G$.)

Proof

(Necessity)

Assume that (X, F) is a fuzzy p-space. Let $(\mathcal{B}_n)$ be a sequence of fuzzy open sets in $(\beta X, F_T)$ which satisfies (i) and(ii) in the definition of fuzzy p-space. Then we can choose a sequence $(\mathcal{A}_n)$ of fuzzy open covers of (X,F) such that

$\{\overline{A_n^*} \mid A_n \in \mathcal{A}_n\}$ refines $\mathcal{B}_n$ where $A_n^*(x) = \begin{cases} A_n(x) & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases}$

Let $\{G_i\}$ be a fuzzy open cover of $\wedge_n A_n$ .For each G_i , choose $H_i \in F_T$ such that $G_i = H_i \chi_X$. Now these H_i's forms a fuzzy open cover of $\wedge_n \overline{A_n^*}$ and hence possesses a finite subfamily which cover $\wedge_n \overline{A_n^*}$ .The corresponding G_i's then form a subfamily of $\{G_i\}$which cover $\wedge_n \bar{A}_n$. Therefore $\wedge_n \bar{A}_n$ is fuzzy compact , which proves (a) .

Let G be any fuzzy open set in (X, F) with $\wedge_n \bar{A}_n \leq G$. Suppose that for each n, $\bigwedge_{i \leq n} \bar{A}_i \not\leq G$. Take $K_n = \bigwedge_{i \leq n} \bar{A}_i$. For each n, choose $x_n \in X$ such that $K_n(x_n) > G(x_n)$. Let $\alpha_n = K_n(x_n)$. Then the fuzzy points $(x_n)_{\alpha n}$, where $(x_n)_{\alpha n}$ is a fuzzy point with support x_n and value α_n , are such that $(x_n)_{\alpha n} \leq K_n$ and $(x_n)_{\alpha n} \not\leq G$. Since sequence (K_n) is decreasing, the set of fuzzy points $\{(x_n)_{\alpha n} : n \in N\}$ has a cluster point, say x_α. Now $\wedge_n K_n = \wedge_n \bar{A}_n$ which is fuzzy compact, so that $x_\alpha \leq \wedge_n K_n$. That is $x_\alpha \leq \wedge_n \bar{A}_n \leq G$, which is a contradiction to the choice of $(x_n)_{\alpha n}$.Thus for some n, $\bigwedge_{i \leq n} \bar{A}_i \leq G$, which proves (b).

(Sufficiency)

Assume that there exists a sequence $(\mathcal{A}_n)$ of fuzzy open covers of (X, F) with satisfies (a) and (b) . Let $A_n \in \mathcal{A}_n$. Then for each $\alpha \in [0 , 1]$,

$A_n^{-1}(\alpha,1] \in \iota(F)$,. Take $U_{(\alpha)} = A_n^{-1}(\alpha,1]$. Then $U_{(\alpha)} \subset X$ and let $U^*_{(\alpha)} = \beta X - \underset{\beta X}{cl}(X-U_{(\alpha)})$. Now $U^*_{(\alpha)} \in T$, where $(\beta X, T)$ is the Stone-$\check{C}$ech compactification of $(X, \iota(F))$ and $U_{(\alpha)}=U^*_{(\alpha)} \cap X$. Define $B_{An} : \beta X \to [0, 1]$ by $B_{An}(x) = \sup\{\alpha : x \in U^*_{(\alpha)}\}$. Then for $x \in X$, $B_{An}(x) = \sup\{\alpha : x \in U_{(\alpha)}\} = A_n(x)$. Therefore $B_{An}\,\chi_X = A_n$. Now B_{An}'s are lower semicontinuous [see sufficiency part of theorem 4.2.3] .That is $B_{An} \in F_T$. Take $\mathcal{B}_n = \{ B_{An} \mid A_n \in \mathcal{A}_n\}$. Then $(\mathcal{B}_n)$ forms a sequence of families of fuzzy open sets on $(\beta X, F_T)$ which satisfies condition (i) for a fuzzy p- space.

Now we show that $\underset{n}{\wedge}\, st\,(x_\alpha, \mathcal{B}_n) \leq \chi_X$ for every fuzzy point x_α with support $x \in X$ and value $\alpha \in (0,1]$ .For $A_n \in \mathcal{A}_n$ we have $\overline{A_n^*} \leq B_{An}$ where

$$A_n^*(x) = \begin{cases} A_n(x) & \text{for } x \in X \\ 0 & \text{for } x \in \beta X - X \end{cases}$$

Let $x_\alpha \leq A_n \leq \mathcal{A}_n$. Then $\underset{n}{\wedge}\, \overline{A_n^*} \leq \underset{n}{\wedge}\, st\,(x_\alpha, \mathcal{B}_n) \to (1)$

Now suppose that $\underset{n}{\wedge}\, st\,(x_\alpha, \mathcal{B}_n) \not\leq \underset{n}{\wedge}\, \overline{A_n^*}$ .Then there exists some $y \in \beta X$ such that $\underset{n}{\wedge}\, st\,(x_\alpha, \mathcal{B}_n)(y) > \underset{n}{\wedge}\, \overline{A_n^*}(y)$. Let $\underset{n}{\wedge}\, st\,(x_\alpha, \mathcal{B}_n) = \gamma$. Therefore $y_\gamma \leq \underset{n}{\wedge}\, st\,(x_\alpha, \mathcal{B}_n)$, but $y_\gamma \not\leq \underset{n}{\wedge}\, \overline{A_n^*}$ .Since $\underset{n}{\wedge}\, \overline{A_n^*}$ is fuzzy compact ,there exists a fuzzy open set W in βX with $y_\gamma \leq W$ and $\overline{W} \wedge (\wedge_n \overline{A_n^*}) = \underline{0}$. Therefore $\overline{W} \wedge (\wedge_{i \leq n} \overline{A}_i^*) = \underline{0} \rightarrow (2)$ by (b). We claim that $\overline{W} \wedge (\wedge_{i \leq n} st(x_\alpha, \bar{\mathcal{A}}_i^*)) = \underline{0}$ where $\bar{\mathcal{A}}_i^* = \{\overline{A}_i^* \mid A_i \in \mathcal{A}_i\}$. For if $\{\overline{W} \wedge (\wedge_{i \leq n} st(x_\alpha, \bar{\mathcal{A}}_i^*))\}(z) > 0$, then $\overline{W}(z) > 0$ and $(\wedge_{i \leq n} st(x_\alpha, \bar{\mathcal{A}}_i^*))(z) > 0$. Hence $st(x_\alpha, \bar{\mathcal{A}}_i^*)(z) > 0$ for all $i = 1,2, \ldots\ldots n$. Therefore we can choose $\overline{A}_i^* \in \bar{\mathcal{A}}_i^*$ with $\overline{A}_i^*(z) > 0$ for all $i = 1,2, \ldots\ldots n$, which is a contradiction to equation (2). Thus $\overline{W} \wedge (\wedge_{i \leq n} st(x_\alpha, \bar{\mathcal{A}}_i^*)) = \underline{0}$. In particular $\overline{W} \wedge (\wedge_n st(x_\alpha, \bar{\mathcal{A}}_n^*)) = \underline{0}$. Therefore $W \wedge (\wedge_n st(x_\alpha, \mathcal{B}_n)) = \underline{0}$. Therefore $y_\gamma \not\leq \wedge_n st(x_\alpha, \mathcal{B}_n)$, which is a contradiction.

Hence $\wedge_n \overline{A_n^*} > \wedge_n st(x_\alpha, \mathcal{B}_n) \rightarrow (3)$. Thus from (1) and (3) it follows that $\wedge_n \overline{A_n^*} = \wedge_n st(x_\alpha, \mathcal{B}_n)$. For $x \in X$, $A_n^*(x) = A_n(x)$ and $\wedge_n \bar{A}_n$ is fuzzy compact. Therefore it follows that $\wedge_n st(x_\alpha, \mathcal{B}_n) \leq \chi_X$, which is condition (ii) for a fuzzy p- space. Hence (X, F) is a fuzzy p- space.

Definition 4.3.2

A fuzzy topological space (X, F) is called a fuzzy k- space if the fuzzy set A is closed in (X, F) whenever $(A \wedge K)/Y : Y \rightarrow [0,1]$ is a closed fuzzy set in (Y, F_Y), where $Y = \text{supp } K$ and $F_Y = \{A/Y \mid A \in F\}$, for each compact fuzzy set K.

Theorem 3.3

Every regular fuzzy p- space is a fuzzy k- space .

Proof

Let (X, F) be a fuzzy p-space. Let A be not a closed fuzzy set on (X, F). We find a compact fuzzy set K in X such that $(A \wedge K)/Y$ is not a closed fuzzy set on (Y, F_Y) . By theorem 4.3.1 there exists a $(\mathcal{A}_n)$ of fuzzy open covers of (X, F) such that for each n, fuzzy point x_α with support $x \in X$ value $\alpha \in (0,1]$ and $x_\alpha \leq A_n \leq \mathcal{A}_n$, (a) $\wedge_n \bar{A}_n$ is fuzzy compact. (b) $\{ \wedge_{i \leq n} \bar{A}_i \mid n \in N\}$ is an outer network for the fuzzy set $\wedge_n \bar{A}_n$.

Let $y \in$ support of $(\bar{A})$ – support of (A) and $\alpha \in (0,1]$. For each n, choose a fuzzy point y_α and a fuzzy open set G_n with $y_\alpha \leq G_n$ and $A_n \in \mathcal{A}_n$ such that $y_\alpha \leq G_{n+1} \leq \bar{G}_{n+1} \leq G_n \leq \wedge_{i \leq n} A_i \leq \wedge_{i \leq n} \bar{A}_i$. Then $\wedge_n G_n = \wedge_n \bar{G}_n \leq \wedge_n \bar{A}_n$. By (a) $\wedge_n \bar{G}_n$ is fuzzy compact and hence $\wedge_n G_n$ is fuzzy compact . If $(x_n)_\alpha$ denote a fuzzy point with support x_n and value α and $(x_n)_\alpha \leq G_n$,then by (b), $\{(x_n)_\alpha : n \in N \}$ has a cluster point . This is because for every fuzzy open set W with $\wedge_n \bar{A}_n \leq W$ there exists some $n \in N$ with $\wedge_{i \leq n} \bar{A}_i \leq W$. Therefore all but finitely many fuzzy points of $(x_n)_\alpha$ are such that $(x_n)_\alpha \leq W$. Hence some point of $\wedge_n \bar{A}_n$ must be a cluster point.

Hence by lemma 2.3.5 , $\{ G_n \mid n \in N \}$ is a base for the fuzzy compact set $K = \bigwedge_n G_n$. If $(A \wedge K)/Y$ is not a closed fuzzy set on (Y, F_Y), then the proof is over.

Suppose that $(A \wedge K)/Y$ is a closed fuzzy set on (Y, F_Y) .Then there exists a closed fuzzy set B on (X,F) such that $(A \wedge K)/Y = B/Y$. Denote B as $\overline{(A \wedge K)}$ Let (H_n) be a sequence of fuzzy open sets such that $y_\alpha \leq H_n$ for all n and $\bar{H}_{n+1} \leq H_n$ and $H_0 \wedge (A \wedge K) = \underline{0}$. Then as above we can show that $\{ G_n \wedge H_n \mid n \in N \}$ is a base for the fuzzy compact set $K' = \bigwedge_n (G_n \wedge H_n)$. Also $K' \wedge A = \underline{0}$. Choose fuzzy point $(x_n)_\alpha$ with $(x_n)_\alpha \leq G_n \wedge H_n \wedge A$. Let $K'' = K' \vee \{(x_n)_\alpha \mid n \in N\}$. Then we show that K'' is fuzzy compact and $K'' \wedge A$ is not a closed fuzzy set on (X,F), so that $(K'' \wedge A)/Y$ is not a closed fuzzy set on (Y, F_Y) , which will complete the proof.

Now $G_n \wedge H_n \wedge A$ is decreasing and since $K' = \bigwedge_n (G_n \wedge H_n)$ is fuzzy compact , $\{(x_n)_\alpha \mid n \in N\}$ has a cluster point x_α such that $x_\alpha \leq K'$. Therefore any fuzzy open cover of K'' contains a finite sub cover of the compact fuzzy set K' and at least one member in the sub cover, say G , is such that $x_\alpha \leq G$ .Therefore all but finitely many $(x_n)_\alpha$ are such that $(x_n)_\alpha \leq G$. Hence the finite sub cover of K' together with fuzzy open set corresponding to those finitely many $(x_n)_\alpha \nleq G$, forms a finite sub cover of the original cover of K'' .Hence K'' is fuzzy compact. But $K'' \wedge A$ is not a closed fuzzy set on (X, F), since the cluster point x_α of $\{(x_n)_\alpha \mid n \in N\}$ is such that $x_\alpha \leq K'$ and $K' \wedge A = \underline{0}$. Hence the theorem.

CHAPTER 5

FUZZY σ- SPACES AND FUZZY METRIZABILITY

5.1 Introduction

The concept of a network is one of the most useful tools in the theory of generalized metric space. The σ-spaces is a class of generalized metric space having a network. Okuyama[referGG]defined these class of spaces as spaces having a σ-discrete network. In this chapter we introduce the fuzzy analogue of σ-spaces and investigate some of its properties. Also we study the relationship between fuzzyσ-spaces, fuzzy Moore spaces ,fuzzy p- spaces, fuzzy M-spaces and fuzzy metrizable spaces.

5.2 Fuzzy σ-spaces

A net work for a topological space is like a base , but its elements are not required to be open. In this section we define fuzzy network , fuzzyσ-spaces and study the properties of fuzzyσ-spaces .

Some results mentioned in this chapter are published in the paper titled '*Fuzzy σ-Spaces and Fuzzy Moore Spaces'*, in the proceedings of the **Annual Conference of the Kerala Mathematical Association' and 'the National Seminar on Mathematical Modelling'**,at Beseluis College, Kottayam (2002)

Definition 5.2.1

A fuzzy network for a fuzzy topological space (X, F) is a collection $\mathcal{C}$ of fuzzy sets on X such that whenever x_α is a fuzzy point and G is a fuzzy open set with $x_\alpha \le G$, there exists an element $C \in \mathcal{C}$ with $x_\alpha \le C \le G$.

Remark 5.2.2

For a regular fuzzy topological space, the set of closures of the elements of a fuzzy network is also a fuzzy network.

Example 5.2.3

Consider X = R with usual topology T. Let F be the set of all lower semicontinuos mappings from (X, T) $\to$ [0,1].For each $x \in X$, define

$f_{x,n} : X \to [0,1]$ by

$$f_{x,n}(y) = \begin{cases} \frac{1}{n} & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Let $\mathcal{C}_n = \{ f_{x,n} : x \in X \}$ .Then for each n, $\mathcal{C}_n$ forms a fuzzy network for (X, F).

Definition 5.2.4

A collection $\mathcal{A}$ of fuzzy sets on a fuzzy topological space (X, F) is said to be discrete if for every fuzzy point x_α in X, there is a $G \in F$ with $x_\alpha \le G$

and $G \wedge A \neq \underline{0}$ holds for at most one element A of $\mathcal{A}$. If $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where each $\mathcal{A}_n$ is discrete, then $\mathcal{A}$ is said to be a σ-discrete collection.

Definition 5.2.5

A fuzzy topological space (X, F) is a fuzzy σ-space if X has a σ-discrete fuzzy network.

Theorem 5.2.6

A regular fuzzy σ-space is a fuzzy subparacompact space.

Proof

Let (X, F) be a regular fuzzy σ-space. Let $\mathcal{C}$ be a closed σ-discrete fuzzy network for (X, F). Therefore $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ where $\mathcal{C}_n$ is discrete. Let $\mathcal{U}$ be a fuzzy open cover of (X, F) and let $E_n = \vee \{c_i \mid c_i \in \mathcal{C}_n\}$. Since (X, F) is regular and $\mathcal{C}_n$ is a closed discrete network, for each $c_n \in \mathcal{C}_n$ we can choose $U_{c_n} \in \mathcal{U}$ such that $c_n \leq U_{c_n}$. Take $A_{c_n} = \bigwedge_{i \neq n} \{ U_{c_n} \wedge c_i' \mid c_i' \in \mathcal{C}_n\}$ where c_i' denote the compliment of c_i and $\mathcal{A}_n = \{ A_{c_n} \mid c_n \in \mathcal{C}_n \} \bigcup \{U \wedge E_n' \mid U \in \mathcal{U}\}$ where E_n' denote the compliment of E_n. Then $(\mathcal{A}_n)$ forms a sequence of fuzzy open covers of (X,F). For a fuzzy point x_α, there exists at most one c_n such that $x_\alpha \leq c_n \leq U$ for some $U \in \mathcal{U}$.

Therefore for each n, st $(x_\alpha, \mathcal{A}_n) \leq U$ for some $U \in \mathcal{U}$. Hence (X, F) is a fuzzy subparacompact space.

Theorem 5.2.7

Every closed fuzzy set in a fuzzy σ-space is a G_δ-set.

Proof

Let A be a closed fuzzy set on a fuzzy σ-space (X, F). Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ be a σ-discrete closed fuzzy network of (X, F) [see Remark 5.2.2]. For $c \in \mathcal{C}_n$, let c' denotes its complement. Let $G_n = \wedge \{c' : c \in \mathcal{C}_n \text{ and } c \wedge A = \underline{0}\}$. Then $A = \bigwedge_n G_n$. Therefore A is a G_δ-set.

Theorem 5.2.8

Let (X, F) be a Hausdorff fuzzy topological space. If (X, F) is a regular fuzzy σ-space, then it has a G_δ*-diagonal.

Proof

Let (X, F) be a regular fuzzy σ-space. Let $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ be a σ- discrete closed fuzzy network of (X, F). If F_p denote the fuzzy product topology, then for (X^2, F_p), $\mathcal{C}_p = \bigcup_{n=1}^{\infty} \mathcal{C}_{np}$ where $\mathcal{C}_{np} = \{ c_i \times c_j : c_i, c_j \in \mathcal{C}_n\}$ and

$(C_i \times C_j)(x,y) = \min\{ C_i(x), C_j(y) \}$, forms a σ-discrete closed fuzzy network. Therefore (X^2,F_p) is also a fuzzy σ-space.

Now by Theorem 0.2.12, the diagonal Δ is a closed fuzzy set on (X^2,F_p). Therefore by, Theorem 5.2.7, Δ is a G_δ-set. Thus (X, F) has a G_δ-diagonal. Also by Theorem 5.2.6, (X, F) is a fuzzy subparacompact space. Therefore (X, F) is a fuzzy submetacompact space [see Remark 2.4.2]. Since (X, F) is regular it follows from Theorem 2.2.3 that (X, F) has a G_δ*-diagonal.

5.3 Fuzzy σ-spaces, Fuzzy Moore spaces and Fuzzy Metrizability.

In this section we study the relation between fuzzy σ-space, fuzzy wΔ- spaces, fuzzy p- spaces ,fuzzy M- saces and fuzzy metrizable spaces.

Theorem 5.3.1

A regular fuzzy topological space is a fuzzy Moore space if and only if it is a fuzzy σ-space and a fuzzy wΔ- space with a G_δ-diagonal.

Proof

Let (X, F) be a fuzzy Moore space. Then (X, F) is a fuzzy subparacompact space [see Remark 2.4.5]. Let $\mathcal{U}$ be a fuzzy open cover of (X, F) and let $(\mathcal{A}_n)$ be a sequence of fuzzy open covers of (X, F) such that for $\alpha \in (0,1]$, a fuzzy point of x_α, there exists $n \in N$ such that $st(x_\alpha,\mathcal{A}_n) \leq U_n$

for some $U_n \in \mathcal{U}$. Let $\mathcal{C}_n = \{ st(x_\alpha, \mathcal{A}_n) \mid x \in X, \alpha \in (0,1] \}$. Then each $\mathcal{C}_n$ is discrete and $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a σ-discrete refinement of $\mathcal{U}$. Hence each cover of (X, F) has a σ-discrete refinement. Let $(\mathcal{B}_n)$ be a fuzzy development of (X, F). Now each $\mathcal{B}_n$ has a σ-discrete refinement, say $\mathcal{D}_n$. Then $\mathcal{D} = \bigcup_n \mathcal{D}_n$ is a σ-discrete fuzzy network for (X, F).That is (X, F) is a fuzzy σ-space. Also by Theorem 2.4.6 (X, F) is a fuzzy wΔ- space, with a G_δ-diagonal.

Conversely assume that (X, F) is a fuzzy σ-space and a fuzzy wΔ- space with a G_δ-diagonal. Since (X, F) is a regular fuzzy σ-space it is a fuzzy sub paracompact space[by Theorem 5.2.6] and hence a fuzzy sub metacompact space [by Remark2.4.2]. Hence by theorem 2.4.6, (X, F) is a fuzzy Moore space.

Theorem 5.3.2

Let (X, F) be a Hausdorff, regular fuzzy topological space. Then (X, F) is a fuzzy Moore space if and only if (X, F) is a fuzzy σ- space and a fuzzy p-space.

Proof

Let (X, F) be a fuzzy Moore space. Then by Corollary 4.2.4, it is a strict fuzzy p- space and hence a fuzzy p-space. Also by Theorem 5.3.1 (X, F) is a fuzzy σ-space.

Conversely assume that (X, F) be a fuzzy σ-space and a fuzzy p-space. Since (X, F) is a fuzzy σ-space, by Theorem 5.2.6, (X, F) is a fuzzy subparacompact space and hence a fuzzy submetacompact space. By Theorem 5.2.8, (X, F) has a G_δ -diagonal. In a regular fuzzy submetacompact space, every fuzzy p- space is a strict fuzzy p- space and hence a fuzzy wΔ-space [see Corollary 4.2.5]. Thus (X, F) is a fuzzy submetacompact , fuzzy wΔ- space with a G_δ -diagonal and hence a fuzzy Moore space by Theorem 2.4.6.

Theorem 5.3.3

Let (X, F) be an induced regular fuzzy topological space. If (X, F) is a Hausdorff fuzzy σ-space and a fuzzy M-space, then it is fuzzy metrizable.

Proof

Let (X, F) be a Hausdorff fuzzy σ-space and a fuzzy M-space. Then by Theorem 5.2.8, (X, F) has a G_δ *-diagonal and hence a G_δ -diagonal. Since (X, F) is a fuzzy M-space, it follows from Theorem 3.3.1, that (X, F) is fuzzy metrizable. Hence the theorem.

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