Studies in Topology and Related Areas

FUZZY ABSOLUTES AND RELATED TOPICS

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CERTIFICATE

This is to certify that the work reported in the thesis entitled **"Fuzzy Absolutes and Related Topics"** is a bona fide work done by **Mrs. Assia .N.V** under my guidance and supervision in the Department of Mathematics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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CHAPTER - 0

INTRODUCTION

0.1 Extensions and Absolutes

An extension of a topological space X is a space that contains X as a dense subspace. The different kinds of extensions especially compactification formed a major area of study in topology. As per one method of construction, the points of the extensions are ultrafilters on lattices such as lattices of open sets, lattices of zero sets, and lattices of clopen sets. From the introduction of Stone Čech compactification in the thirties this area caught the attention of researchers and many papers appeared there after in this area.

Related dual concepts of Absolutes due to Iliadis and Banaschewski also formed a major area of activity in Topology. Associated with each Hausdorff space always there exists an extremally disconnected zero dimensional Hausdorff space EX called the Iliadis absolute of X and a perfect irreducible Θ continuous surjection from EX on to X. In most of the constructions of absolutes the points of EX are ultrafilters on lattices associated with X. Thus extensions and absolutes although conceptually dual in nature are constructed using similar tools.

Perfect continuous mappings always preserve certain topological properties. Therefore, whenever dealing with a new topological property P and if X has P we have to look for a space which is a perfect continuous image of X or that can be mapped on to X by a perfect continuous surjection. If it is possible to find such a space we can study the properties of that space and from that we can infer the properties of X.

If X is regular, always there exists such a space called Iliadis absolute of X. It can be mapped on to X by a perfect continuous surjection K_X which is irreducible, i.e. if A is a proper closed subset of EX then $K_X(A)$ is a proper closed subset of X. The space EX is zero dimensional and also extremally disconnected. That is every point of EX has a clopen basis and closure of its open sets are always open.

Even though X is not regular, we can have an extremally disconnected space. This is called Banaschewski absolute of X denoted as PX. Here also there is a perfect irreducible surjection π_X: PX→X. Since X is not regular PX is not regular and so not zero dimensional. When X is regular PX and EX coincide and so does K_X and π_X .

Since perfect continuous mapping preserves some topological properties, PX and X have certain properties in common. More over there are certain properties which are preserved by the irreducibility of π_X. Examples of such properties are cellularity, π weight, density character and feeble compatness [P;W].

The absolute EX of X arises in the following situation . Suppose with each space X we can associate some algebraic object A (X). Let $A(\hat{X})$ be the algebraic completion of A (X). If X has certain nice properties, then $A(\hat{X})$ is isomorphic to A (EX).

The original motivation behind the study of absolutes was the problem of characterizing the projective objects in the category of compact spaces and continuous functions. In 1958 Gleason [GL] solved this problem. He showed that the projective objects of this category are precisely the compact extremally disconnected spaces. He constructed EX (when X is compact) as a part of the solution of this problem.

0.2 Fuzzy set theory

The concept of Fuzzy sets introduced by the American Cyberneticist L.A. Zadeh started a revolution in every branch of knowledge and in particular in every branch of mathematics. Zadeh introduced the fuzzy set theory [ZA] in 1965 inorder to study the control problem of complicated systems and dealings with fuzzy information. This theory described Fuzziness mathematically for the first time. Fuzziness is a kind of uncertainty and uncertainty of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol belongs.

Since the 16^{th} century probability theory has been studying a kind of uncertainty – randomness – i.e. the uncertainty of the occurrence of an event. But in this case the event itself is completely certain, the only uncertain thing is whether the event will occur or not. Fuzziness is another kind of theory of fuzzy topology. Using fuzzy sets introduced by Zadeh, C.L. Chang [CH] defined fuzzy topological space in 1968 for the first time. In 1976 Lowen $[LO]_1$ suggested a variant of this definition. Since then an extensive work on fuzzy topological space has been carried out by many researchers.

Many mathematicians while developing fuzzy topology have used different lattices for the membership sets like (1) completely distributive lattice with 0 and 1 by T.E. Gantner, R.C. Steinlage and R.H. Warren [G;S;W] (2) complete and completely distributive lattice equipped with order reversing involution by Bruce Hutton and Ivan Reilly [H;R](3) complete and completely distributive non atomic Boolean algebra by Mira Sarkar [SA] (4) complete chain by Robert Bernard [BE] and F. Conard [CO] (5) complete Brouwerian lattice with its dual also Brouwerian by Ulrich Hohle $[HO]_1$ (6) Complete and distributive lattice by S.E. Rodabaugh[RO] (7) complete Boolean algebra by Ulrich Hohle $[HO]_2$.

We take the definition of fuzzy topology in the line of Chang with membership set as the closed unit interval [0,1].

0.4 About this thesis

The theory of extensions has got a rich parallel theory in fuzzy topological spaces. Mathematicians like Cerutti, U [CE], Liu Ying Ming and Mao-Kang Luo [Y,M] have done a detailed study in this area. But not much work has been done regarding the theory of absolutes. In our work we are investigating this-fuzzy absolutes. In the development of a parallel theory based on fuzzy sets, here we can specifically notice many differences between the two theories.

As a prelude to construction of fuzzy absolutes we have done a detailed study of the proper analogue in the fuzzy context for the concepts like Stone spaces, regularly closed filters and open filters. The concepts of fuzzy filters introduced by A.K. Katsaras $[KA]_1$ and fuzzy regular closed sets by K.K. Azad $[AZ]_2$ have been used for this purpose.

0.5 Summary of the thesis

The thesis is divided into six chapters including this chapter 0.

The general preliminary results which are used in the succeeding chapters are given in the next section of this chapter. Due references are given wherever necessary. Some of the preliminary results which are related to each chapter are given at the beginning of the corresponding chapter itself.

The Stone duality theory was developed by Marshal H. Stone [ST] in 1937. Using the notion of ultrafilters he introduced the Stone space of a Boolean algebra and proved the Stone representation theorem.

In the first chapter we are doing the fuzzy analogue of this concept. Here we define a function $\Lambda: I^X \rightarrow P(\Omega(X))$ where $\Omega(X)$ is the set of all fuzzy ultrafilters on X and prove some of its properties. Using this we have introduced the Stone space of fuzzy sets denoted as f-S(X) and have proved the "Fuzzy Stone representation theorem". With suitable examples we point out the differences with the crisp situation.

In Chapter 2 we have introduced the concept of fuzzy regularly closed filters (FRC-filter) similar to the notion of fuzzy filters introduced by A.K. Katsaras $[KA]_2$. Then FRC ultrafilters are studied and an equivalent formulation for such filters have been given. M.A. De Prade Vicente in her paper [P;A] proved that every fuzzy ultrafilter is free. With suitable examples here we prove the existence of fixed FRC ultrafilters. Here we have also defined the s-fixed FRC ultrafilters in the same line as that by De Prade [P;A].

It is known that in the crisp situation the absolutes can be constructed using open ultrafilters also. So in the third chapter we introduce fuzzy open filters; and prove some of its properties. Also we define the fuzzy Hausdorff-closed spaces (f-H closed space) analogues to the concept of H-closed spaces. A characterization for f-H closed space has been given. Then we introduce an s-continuous mapping from a topological space to a fuzzy topological space and prove that the image of an H-closed space under an s-continuous mapping is f-H closed. Here we have also proved that the arbitrary product πf_i and the sum $\oplus f_i$ of the s -continuous maps f_i are also s-continuous.

Using the concepts in chapter 1 and chapter 2 in the fourth chapter we have introduced the fuzzy absolutes of a fuzzy topological space as a subspace of fuzzy Gleason space which is the Stone space of FRC(X). Thus associated with each fuzzy Hausdorff space there is a pair (f-EX, $K_{f\text{-}X}$) consisting of a B- extremally disconnected zero dimensional space f-EX and an s-continuous mapping $K_{f\text{-}X}$ which is compact (but not closed and hence not irreducible) from f-EX into X. Significant properties of the pair (f-EX, $K_{f\text{-}X}$) are given and have proved the uniqueness of the fuzzy absolutes. Fuzzy absolutes of sums and products of fuzzy topological spaces are also studied.

Fuzzy absolutes also can be constructed using the strong fixed (s-fixed) FRC ultrafilters. Properties of fuzzy absolutes in this situation is studied and identifies the differences occurring in the two cases.

In the fifth chapter we have given another construction for the fuzzy absolutes. Instead of using the fuzzy regularly closed subsets of X we can construct fuzzy absolutes by taking the pseudo-complemented lattice $\tau(X)$ of all fuzzy open subsets of X. It is denoted as f-E'X. Then the underlying set of f-E'X is the fixed f-open ultrafilters on X. Properties which are proved in chapter 4 are explicitly proved here for f-E'X. Suitable examples are given wherever necessary.

0.6 Basic Definitions:

The following definitions are adapted from [G;S],[CH],[LO]$_1$, [WO]$_1$, [G;K;M], [MA] , [P,W] [Y;M]$_2$.

0.6.1 Definition:- Let μ and γ be fuzzy subsets of a non empty set X. Then

$$\mu = \gamma \Leftrightarrow \mu(x) = \gamma(x) \ \ \forall\, x \in X.$$

$$\mu \leq \gamma \Leftrightarrow \mu(x) \leq \gamma(x) \ \forall\, x \in X.$$

$$\mu \vee \gamma = \eta \ \Leftrightarrow \ \eta(x) = \max(\mu(x), \gamma(x)) \ \ \forall\, x \in X.$$

$$\mu \wedge \gamma = \delta \ \Leftrightarrow \ \delta(x) = \min(\mu(x), \gamma(x)) \ \ \forall\, x \in X.$$

$$\mu' = \lambda \Leftrightarrow \lambda(x) = 1\text{-}\mu(x) \ \ \forall\, x \in X.$$

More generally, $\vee\mu_i$ and $(\wedge\mu_i)$ are defined as $(\vee\mu_i)(x) = \vee(\mu_i(x))$ and

$$(\wedge\mu_i)(x) = \wedge(\mu_i(x)), \ \forall\, x \in X.$$

The symbol 0 is used to denote the empty fuzzy subset defined by $\mu(x) = 0$, $\forall x \in X$ and 1 is used to denote the whole set X defined by $\mu(x) = 1$ $\forall x \in X$.

0.6.2 Definition:- If μ is a fuzzy subset of X, then $\{x \in X : \mu(x) > 0\}$ is called the support of μ and is denoted as supp μ or $\sigma_0(\mu)$

0.6.3 Definition:

A fuzzy point x_p in X is a fuzzy set in X defined by

$$x_p(y) = p \quad (p \in (0,1]) \text{ for } y = x$$
$$= 0 \qquad \text{for } y \neq x.$$

x and p are the support and value of x_p. A fuzzy point x_p belongs to a fuzzy set μ in X if and only if $p \leq \mu(x)$. In this case we use the notation $x_p \in \mu$. When $p = 1$, x_p is said to be fuzzy singleton.

0.6.4 Definition [CH]: A fuzzy topology on X is a subset $\delta \subset I^X$ such that

i) $0, 1 \in \delta$.

ii) If $\mu, \gamma \in \delta$ then $\mu \wedge \gamma \in \delta$

iii) If $\mu_i \in \delta$ for each i, then $\vee_i \mu_i \in \delta$.

δ is called a fuzzy topology on X and the pair (X, δ) is a fuzzy topological space or fts for short. Every member of δ is called an open fuzzy set. A fuzzy set is closed if and only if its complement is open.

0.6.5 Definition [LO]$_1$: $\delta \subset I^X$ is a fuzzy topology on X if and only if

i) for all constants α, $\alpha \in \delta$

ii) for all $\mu, \gamma \in \delta$, $\mu \wedge \gamma \in \delta$.

iii) If $\mu_i \in \delta$ for each i, then $\vee_i \mu_i \in \delta$.

0.6.6 Definition: Let (X, τ) be a fuzzy topological space. For $Y \subset X$ and $Y \neq \phi$, $\tau / Y = \{\mu / Y, \mu \in \tau\}$ is a fuzzy topology on Y. Then (Y, τ / Y) is called a subspace of (X, τ).

0.6.7 Definition: Let $\{X_\alpha\}$ $\alpha \in I$, be a family of fuzzy topological spaces with fuzzy topology τ_α. Let $X = \pi X_\alpha$ be the usual product space and P_α be the projection from X onto X_α. Then for $B \in \tau_\alpha$, $P_\alpha^{-1}(B)$ is a fuzzy set in X. Let $S = \{P_\alpha^{-1}(B) \,/\, B \in \tau_\alpha, \alpha \in I\}$. Let $\mathcal{B}$ be the family of all finite intersections of members of δ and τ be the family of all unions of members of $\mathcal{B}$. Then τ is a fuzzy topology for X with $\mathcal{B}$ as a base and S as a sub-base. Then (X, τ) is called the product fuzzy topological space.

0.6.8 Definition: Let $(X_i, \tau_i)_{i\in I}$ be a family of pair wise disjoint fuzzy topological spaces. Consider the set $X=\cup_\alpha X_\alpha$. Define the sum topology of $\{\tau_\alpha, \alpha\in I\}$ on I^X i.e. $\oplus\tau_\alpha$ as follows. For every $\mu\in I^X$, $\mu\in \oplus\,\tau_\alpha$ if and only if $\mu / X_\alpha \in\tau_\alpha$. Then (X, τ_α) is called sum fuzzy topological space or sum fts.

0.6.9 Definition: Let μ be a fuzzy set in a fuzzy topological space (X, τ). Then the largest open fuzzy set contained in μ is called the interior of μ and is denoted as "f-int μ" or μ^o.

i.e. $\mu^o = \vee\{\lambda:\lambda\in\tau,\ \lambda\leq\mu\}$.

The smallest closed fuzzy set containing μ is called the closure of μ denoted as 'f-cl μ' or $\bar{\mu}$.

i.e. $\bar{\mu} = \wedge\{\lambda' :\ \lambda\in\tau,\ \lambda'\geq\mu\}$.

0.6.10 Definition: Let (X, F) be a fuzzy topological space. A fuzzy subset μ on X is dense in (X, F) provided that $cl_F(\mu) = 1$. If (Y, H) is a fuzzy subspace of (X,F) then (Y,H) is dense in (X,F) provided μ_Y is dense in (X,F).

0.6.11 Definition: A fuzzy topological space (X,δ) is said to be fuzzy Hausdorff or fT_2 if for each distinct pair of points x and y in X, there exist open fuzzy sets μ and γ such that $\mu(x) = \gamma(y) = 1$ and $\mu\wedge\gamma = 0$.

0.6.12 Definition: Let (X, τ) be a fuzzy topological space. A family $\mathcal{A}$ of fuzzy sets is a cover of a fuzzy set μ if and only if $\mu \leq \vee \{ \lambda : \lambda \in \mathcal{A}\}$. It is an open cover if and only if each member of $\mathcal{A}$ is an open fuzzy set. A sub cover of $\mathcal{A}$ is a subfamily which is also a cover.

0.6.13 Definition: Let (X,δ) be a fuzzy topological space. Then the family $[\delta]$ of supports of all crisp subsets in δ is an ordinary topology on X. Then the topological space $(X, [\delta])$ is called the background space of (X,δ).

0.6.14 Definition: A function $f:X\rightarrow I$ is lower semi continuous if and only if for each $\alpha \in f^{-1}(\alpha,1]$ is open in X. The characteristic function $\chi_A:X\rightarrow[0,1]$ is lower semi continuous if and only if A is open in X.

0.6.15 Definition: A space X is zero dimensional if and only if each point of X has a neighbourhood base consisting of clopen sets.

For the elementary definitions and results in topology references may be made to [WI], [JO]. For the theory of Stone space and absolutes to [P;W],[WA].

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CHAPTER - 1

STONE SPACE OF FUZZY SETS[⊗]

1.0 Introduction

Using the notion of ultrafilters, Marshal H. Stone [ST] introduced in 1937 the concept of Stone space of a Boolean algebra and developed the Stone duality theory. The set of all clopen subsets of an arbitrary space forms a Boolean algebra with respect to the unions and intersections. He proved that any Boolean algebra is isomorphic to the Boolean algebra of clopen sets of a unique compact zero dimensional space. This is termed as Stone representation theorem. The absolute of a topological space X, constructed by Iliadis in 1963, mainly based on the Stone space of the set of all regular closed subsets on X. So as a prerequisite for the construction of fuzzy absolutes, in this chapter we are doing the fuzzy analogue of the concept "Stone space".

If X is any non empty set, the set of all fuzzy subsets of X i.e. I^X forms a pseudo Boolean algebra with pseudo complement defined as $\mu'(x) = 1 - \mu(x)$, $x \in X$, So the concept of Stone space can be extended to I^X. The notion of fuzzy filters introduced by A.K. Katsaras in $[KA]_1$ has been used for this purpose.

[⊗] An earlier version of this chapter has been published in the proceedings of the Annual Conference of Kerala Mathematical Association and National Seminar on Analysis and Applications, 26-29 March 1999.

In the first section of the chapter we give the necessary preliminary ideas. In section 1.2 we define a function Λ: $I^X \to P(\Omega(X))$ and prove some of its properties. Here $\Omega(X)$ denotes the set of all fuzzy ultrafilters on X. Using this we introduce the fuzzy analogue of the concept "Stone space"

In the third section we prove the "Fuzzy Stone representation theorem". Since we have only pseudo components in I^X and not complements, some results in crisp theory have no proper analogue in the fuzzy setting. With suitable examples we point out these differences.

1.1 Preliminaries

We use the notations μ, γ, δ etc to denote the fuzzy subsets on X, $\mathcal{F}$ to fuzzy filters and $\mathcal{U}$, $\mathcal{V}$ etc to fuzzy ultrafilters.

1.1.1 Definition [KA]$_1$: A fuzzy filter $\mathcal{F}$ on X is a non empty subset of I^X satisfying

i) $\lambda \in \mathcal{F}$, $\mu \geq \lambda$ imply $\mu \in \mathcal{F}$.

ii) $\lambda, \mu \in \mathcal{F}$ implies $\lambda \wedge \mu \in \mathcal{F}$.

iii) $0 \notin \mathcal{F}$.

1.1.2Definitions [P;A]: A non empty subset $\mathcal{B}$ of I^X is a base for some fuzzy filter if i) for $B_1, B_2 \in \mathcal{B}$, there exists $B_3 \in \mathcal{B}$ such that $B_3 \leq B_1 \wedge B_2$ and ii) $0 \notin \mathcal{B}$.

The collection $\mathcal{F} = \{F \in I^X / \exists\, B \in \mathcal{B} : F \geq B\}$ is a fuzzy filter. $\mathcal{F}$ is said to be generated by $\mathcal{B}$.

A subset $\mathcal{B}$ of $\mathcal{F}$ is a base for $\mathcal{F}$ if for each $F \in \mathcal{F}$, there is some $B \in \mathcal{B}$ such that $B \leq F$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are fuzzy filters on X, $\mathcal{F}_1$ is said to be finer than $\mathcal{F}_2$ if and only if $\mathcal{F}_1 \supset \mathcal{F}_2$.

A fuzzy filter $\mathcal{F}$ on X is a Fuzzy ultrafilter [f-ultrafilter for short] if there is no other fuzzy filter finer than $\mathcal{F}$. That is it is a maximal filter with respect to the inclusion relation.

1.1.3 Result [KA] $_2$:

i) Every fuzzy filter is contained in a fuzzy ultra filter

ii) If $\mathcal{U}$ is a fuzzy ultrafilter on X, then for every $\mu \in I^X$, either $\mu \in \mathcal{U}$ or $1\text{-}\mu \in \mathcal{U}$.

iii) A fuzzy filter $\mathcal{F}$ is a fuzzy ultra filter if and only if every $\lambda \in I^X$ with $\lambda \wedge \mu \neq 0$ for all $\mu \in \mathcal{F}$ belongs to $\mathcal{F}$.

1.1.4 Result [S; G]: If $\mathcal{U}$ is a fuzzy ultra filter on X, then $\lambda \vee \mu \in \mathcal{U}$ ($\lambda, \mu \in I^X$) if and only if $\lambda \in \mathcal{U}$ or $\mu \in \mathcal{U}$.

1.2 The Stone space of Fuzzy sets

1.2.1 Definition: Let $(L, \vee, \wedge)$ be a lattice with 0 and 1 as the smallest and largest element respectively. Then L is said to be pseudo complemented if for each $a \in L$, there is associated a unique a' such that

i) $(a')' = a$ for every $a \in L$

ii) $0' = 1$ and $1' = 0$

iii) $(a \vee b)' = (a' \wedge b')$ and $(a \wedge b)' = a' \vee b'$

Note: If further $a \vee a' = 1$ and $a \wedge a' = 0$ $\forall$ $a \in L$ then L is complemented.

1.2.2 Definition: A distributive lattice which is pseudo complemented is called a pseudo Boolean algebra (or p-Boolean algebra for short).

Note: I^X, the set of all fuzzy subsets of X forms a pseudo Boolean algebra.

1.2.3 Definition: If A and B are any two p-Boolean algebras, then a function $f: A \rightarrow B$ is said to be a p-Boolean homomorphism (or homomorphism for short) if

i) $f(a \vee b) = f(a) \vee f(b)$

ii) $f(a \wedge b) = f(a) \wedge f(b)$

iii) $(f(a))' \leq f(a')$

Let f be bijective. If both f and f^{-1} are homomorphisms, then f is called a p-Boolean isomorphism or just isomorphism.

1.2.4 Definition: A p-Boolean algebra is complete if it is complete as a lattice.

1.2.5 Definition: Let X be any non empty set and $\Omega(X)$, the set of all fuzzy ultrafilters on X. For $\mu \in I^X$, define $\Lambda(\mu) = \{\mathcal{U} \in \Omega(X): \mu \in \mathcal{U}\}$.

Then Λ is a function from I^X into $P(\Omega(X))$

1.2.6 Result: For any $\mu, \gamma \in I^X$.

i) $\Lambda(0) = \phi$ and $\Lambda(1) = \Omega(X)$

ii) $\Lambda(\mu \vee \gamma) = \Lambda(\mu) \cup \Lambda(\gamma)$

iii) $\Lambda(\mu \wedge \gamma) = \Lambda(\mu) \cap \Lambda(\gamma)$

iv) $(\Lambda(\mu))' \subseteq \Lambda(\mu')$

Proof :

i) $\Lambda(0) = \{\mathcal{U} \in \Omega(X): 0 \in \mathcal{U}\}$

Since there is no ultrafilter containing zero, $\Lambda(0) = \phi$.

When $0 \notin \mathcal{U}$, by 1.1.3 (ii), $1 \in \mathcal{U}$. Therefore every ultrafilter contains 1. i.e. $\Lambda(1) = \Omega(X)$.

ii) $\mathcal{U} \in \Lambda(\mu \vee \gamma) \Rightarrow \mu \vee \gamma \in \mathcal{U}$.

$$\Rightarrow \mu \in \mathcal{U} \text{ or } \gamma \in \mathcal{U} \text{ (by 1.1.4)}$$

$$\Rightarrow \mathcal{U} \in \Lambda(\mu) \text{ or } \mathcal{U} \in \Lambda(\gamma)$$

$$\Rightarrow \mathcal{U} \in \Lambda(\mu) \cup \Lambda(\gamma)$$

$\mathcal{V} \in \Lambda(\mu) \cup \Lambda(\gamma) \quad \Rightarrow \mathcal{V} \in \Lambda(\mu)$ or $\mathcal{V} \in \Lambda(\gamma)$

$\Rightarrow \mu \in \mathcal{V}$ or $\gamma \in \mathcal{V}$

$\Rightarrow \mu \vee \gamma \in \mathcal{V}$ (by 1.1.1.(i))

$\Rightarrow \mathcal{V} \in \Lambda(\mu \vee \gamma)$

There fore $\Lambda(\mu \vee \gamma) = \Lambda(\mu) \cup \Lambda(\gamma)$

iii) $\mathcal{U} \in \Lambda(\mu \wedge \gamma)$ $\Rightarrow \mu \wedge \gamma \in \mathcal{U}$

$\Rightarrow \mu \in \mathcal{U}$ and $\gamma \in \mathcal{U}$ (by 1.1.1)(i)

$\Rightarrow \mathcal{U} \in \Lambda(\mu)$ and $\mathcal{U} \in \Lambda(\gamma)$

$\Rightarrow \mathcal{U} \in \Lambda(\mu) \cap \Lambda(\gamma)$.

$\mathcal{V} \in \Lambda(\mu) \cap \Lambda(\gamma)$ $\Rightarrow \mathcal{V} \in \Lambda(\mu)$ and $\mathcal{V} \in \Lambda(\gamma)$

$\Rightarrow \mu \in \mathcal{V}$ and $\gamma \in \mathcal{V}$

$\Rightarrow \mu \wedge \gamma \in \mathcal{V}$ (by 1.1.1.(ii))

$\Rightarrow \mathcal{V} \in \Lambda(\mu \wedge \gamma)$

Therefore $\Lambda(\mu \wedge \gamma) = \Lambda(\mu) \cap \Lambda(\gamma)$

(iv) Let $\mathcal{U} \in (\Lambda(\mu))'$. Then $\mathcal{U} \notin \Lambda(\mu)$

i.e $\mu \notin \mathcal{U}$. Therefore by 1.1.3 (ii) $\mu' \in \mathcal{U}$. i.e. $\mathcal{U} \in \Lambda(\mu')$.

Therefore $(\Lambda(\mu))' \subset \Lambda(\mu')$.

1.2.10 Note: For any set Y, B(Y) represents the set of all clopen sets in Y.

Therefore, $\{\Lambda\mu): \mu \in I^X\} \subset B(f\text{-}S(X))$.

1.3 Stone representation theorem for fuzzy sets

1.3.1 Definition [P;W]: A space X is zero dimensional if B(X) is an open base for X.

1.3.2 Theorem:- Let X be any set and f- $S(X)$ be as defined in 1.2.8. Then

a. f- $S(X)$ is a compact, T_2, zero dimensional space.

b. $\{\Lambda(\mu): \mu \in I^X\} = B(f\text{-}S(X))$

c. Λ is a homomorphism from I^X onto $B(f\text{-}S(X))$

Proof:

a. Let $\mathcal{F}$ be a filter of closed sets in f-$S(X)$ and

$G = \{\mu \in I^X : \Lambda(\mu) \supseteq F \text{ for some } F \in \mathcal{F}\}$

$\{\Lambda(\mu) : \mu \in I^X\}$ is a base for closed sets in f-$S(X)$

$\therefore \cap\, \mathcal{F} = \cap\, \{\Lambda(\gamma) : \gamma \in G\}$

Let $\mu_1, \mu_2 \in G$. Then there exist F_1 and F_2 in $\mathcal{F}$ such that $\Lambda(\mu_1) \supseteq F_1$ and $\Lambda(\mu_2) \supseteq F_2$.

$$\Lambda(\mu_1 \wedge \mu_2) = \Lambda(\mu_1) \cap \Lambda(\mu_2)$$

$$\supseteq F_1 \cap F_2, \text{ where } F_1 \cap F_2 \in \mathcal{F}.$$

$\therefore \mu_1 \wedge \mu_2 \in G$.

Let $\mu \in G$ and $\mu_1 \geq \mu$. Since $\mu \in G$, there exists $F_1 \in \mathcal{F}$ such that $F_1 \subseteq \Lambda(\mu) \subseteq \Lambda(\mu_1)$.

$\therefore \mu_1 \in G$ and hence G is a fuzzy filter.

So by 1.1.3(i) there exists an f-ultrafilter $\mathcal{U} \in$ f-$\mathcal{S}$(X) such that $G \subset \mathcal{U}$.

Therefore $\mu \in \mathcal{U}$ and so $\mathcal{U} \in \Lambda(\mu)$; $\mu \in G$.

Therefore, $\mathcal{U} \in \cap \{\Lambda(\mu): \mu \in G\}$. i.e. $\mathcal{U} \in \cap F$.

Therefore $\cap F \neq \phi$. Hence f-$\mathcal{S}$(X) is compact.

Let $\mathcal{U}$ and $\mathcal{V} \in$ f-$\mathcal{S}$(X) such that $\mathcal{U} \neq \mathcal{V}$

Therefore there exists $\mu \in I^X$ such that $\mu \in \mathcal{U}$ but not belongs to $\mathcal{V}$.

i.e $\mathcal{U} \in \Lambda(\mu)$ and $\mu \notin \mathcal{V}$.

Since $\mu \notin \mathcal{V}$, by 1.1.3(iii), there exists $\gamma \in \mathcal{V}$ such that $\mu \wedge \gamma = O$.

Therefore $\Lambda(\mu \wedge \gamma) = \phi$

i.e. $\Lambda(\mu) \cap \Lambda(\gamma) = \phi$ where $\mathcal{U} \in \Lambda(\mu)$ and $\mathcal{V} \in \Lambda(\gamma)$. Also $\Lambda(\mu)$ and $\Lambda(\gamma)$ are open in f-$\mathcal{S}$(X).

Therefore f-$\mathcal{S}$(X) is Hausdorff.

By result 1.2.9, $\Lambda(\mu)$ is clopen in f-$\mathcal{S}$(X) and $\{\Lambda(\mu): \mu \in I^X\}$ forms a base for f-$\mathcal{S}$(X). Therefore by definition 1.3.1 f-$\mathcal{S}$(X) is zero dimensional. That is f-$\mathcal{S}$(X) is a compact, T_2, zero dimensional space.

b) By note 1.2.10 we have $\{ \Lambda(\mu) : \mu \in I^X \} \subseteq B(f\text{-}\mathcal{S}(X))$ -(1)

Now, let $C \in B(f\text{-}\mathcal{S}(X))$

Since C is open, there exists a non empty subset D of I^X such that

$C = \cup \{\Lambda(\mu) : \mu \in D\}$

C is also closed and so compact. Therefore there exists a finite subset H of D such that

$$\begin{aligned} C &= \cup\{ \Lambda(\gamma) : \gamma \in H\} \\ &= \Lambda(\vee \gamma : \gamma \in H) \text{ by result 1.2.6 (ii)} \\ &= \Lambda(\eta) \quad \text{where } \eta = \vee \{ \gamma : \gamma \in H\} \in I^X \end{aligned}$$

i.e, $C \in \{\Lambda(\mu) : \mu \in I^X\}$

$\therefore B(f\text{-}\mathcal{S}(X)) \subset \{ \Lambda(\mu) : \mu \in I^X \}$ -(2)

Hence $B(f\text{-}\mathcal{S}(X)) = \{\Lambda(\mu) : \mu \in I^X \}$

(c) By part b) we have $B(f\text{-}\mathcal{S}(X)) = \{ \Lambda(\mu) : \mu \in I^X \}$ where

$\Lambda(\mu) = \{ \mathcal{U} \in \Omega(X) : \mu \in \mathcal{U} \}$

Therefore, Λ is a function from I^X onto $B(f\text{-}\mathcal{S}(X))$

Also by the result 1.2.6 Λ is a homomorphism. Therefore Λ is a homomorphism, from I^X onto $B(f\text{-}\mathcal{S}(X))$.

1.3.3 Remark: The function Λ from I^X onto $B(f\text{-}\mathcal{S}(X)$ is not one to one in general.

Example:- Let X be any non empty set .

Define μ and γ from X to $[0,1]$ as $\mu(x) = \gamma(x)$ for every $x \neq x_0$ and $\mu(x_0) > \gamma(x_0)$.

Then $\mu \neq \gamma$.

Let $\mathcal{F} \in \Lambda(\mu)$. i.e. $\mu \in \mathcal{F}$.

Therefore $\mu \wedge F \neq 0$ for every $F \in \mathcal{F}$.

i.e. $F(x_0) \neq 0$ for every $F \in \mathcal{F}$.

There fore $(\gamma \wedge F)(x_0) \neq 0$, $\forall F \in \mathcal{F}$.

i.e. $\gamma \wedge F \neq 0$, $\forall F \in \mathcal{F}$.

There fore by 1.1.3 (iii) $\gamma \in \mathcal{F}$ and so $\mathcal{F} \in \Lambda(\gamma)$.

$\therefore \Lambda(\mu) \subseteq \Lambda(\gamma)$

Similarly the reverse inclusion also holds.

So $\Lambda(\mu) = \Lambda(\gamma)$, but $\mu \neq \gamma$.

1.4 B- extremally disconnected space

1.4.1 Definition: A topological space X with a clopen basis $\mathcal{B}$ is said to be B- extremally disconnected if and only if the closure of every open set in X is contained in a basic open set.

1.4.2 Result: Any dense subset of a B-extremally disconnected space is also B- extremally disconnected.

Proof:

Let X be a B-extremally disconnected space and H be any dense subset of X.

Let V be any open set in H. Then $V = H \cap W_1$ where W_1 is open in X.

Therefore $cl_x V = cl(H \cap W_1)$

$= cl(H) \cap cl W_1$

$= X \cap cl(W_1) = cl(W_1)$

Since W_1 is open in X and X is B-extremally disconnected, there exists a basic open set W_2 in X such that $cl(W_1) \subset W_2$.

Therefore $cl_X V = cl_X(W_1) \subset W_2$.

$H \cap cl_X V \subseteq H \cap W_2$,where $H \cap W_2$ is a basic open set in H.

That is $cl_H V \subset H \cap W_2$.

Therefore H is B- extremally disconnected.

1.4.3 Theorem: Let X be any non empty set. Then f-$\mathcal{S}$(X) , the fuzzy Stone space of X is B-extremally disconnected.

Proof: Let U be any arbitrary open set in f-$\mathcal{S}$(X). Since $\{\Lambda(\mu):\mu \in I^X\}$ is an open basis, there is a non empty subset D of I^X such that $U = \cup\{\Lambda(\mu_i) : \mu_i \in D\}$.

I^X is complete. Therefore $\vee \mu_i$ exist.

Let $\vee \{\mu_i : \mu_i \in D\} = \gamma$. Then $\mu_i \leq \gamma$ for every i.

$\therefore \Lambda(\mu_i) \subseteq \Lambda(\gamma)$

So $\cup \Lambda(\mu_i) \subseteq \Lambda(\gamma)$, $\Lambda(\gamma)$ is clopen

i.e. $U \subseteq \Lambda(\gamma)$

$\therefore cl(U) \subseteq \Lambda(\gamma)$ -(i)

Hence f-$\mathcal{S}$(X) is B- extremally disconnected

1.4.4 Note: The reverse inclusion of (i) does not hold in the case of non crisp sets.

Example:

Let X be any non empty set with at least two distinct points x_0 and x_1.

Define $\mu_1: X \to [0,1]$ as $\mu_1(x_0) = 1/3$ and $\mu_1(x) = 0$, $\forall x \neq x_0$.

Now $U[x] = \{\mu \in I^X : \mu(x) > 0\}$ is a fuzzy ultrafilter on X.

Let f-$\mathcal{S}$(X) be the Stone space of X. Then $\{\Lambda\{\mu) : \mu \in I^X\}$ is a basis for open sets in X.

Consider the open set $U = \Lambda(\mu_1)$.

This open set will contain the ultrafilter $U[x_0]$

Define $\mu_2 : X \to [0,1]$ as $\mu_2(x) = 1/3$ for every $x \in X$

Then $\mu_1 \leq \mu_2$. Therefore $\Lambda(\mu_1) \subset \Lambda(\mu_2)$.

That is $U \subset \Lambda(\mu_2)$, where $U = \Lambda(\mu_1)$

Therefore cl (U) $\subset \Lambda(\mu_2)$ (1)

Now let $\mathcal{V} = \{ \mu \in I^X : \mu(x_1) > 0, \text{ for } x_1 \neq 0\}$. Then $\mathcal{V}$ is a fuzzy ultra filter on X.

Then $\mathcal{V}$ contains μ_2 but not μ_1 .

$\therefore$ $\mathcal{V} \in \Lambda(\mu_2)$ and $\mathcal{V} \notin \Lambda(\mu_1) = U$

Therefore $\Lambda(\mu_2) \not\subset U$.

1.4.5 Conclusion

Let X be any non empty set and A be any (crisp) subset of X. If F is an ultrafilter on X and A $\in F$, then $A^c \notin F$ since $A \cap A^c = \phi$. But in the case of fuzzy ultrafilter $\mathcal{U}$ on X, $\mu \in \mathcal{U}$ does not imply that $\mu' \notin \mathcal{U}$ as $\mu \wedge \mu'$ need not be equal to zero. Due to this inherited draw back in fuzzy filters, some results in the crisp set theory may not be true in the fuzzy context.

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CHAPTER - 2

FUZZY REGULARLY CLOSED FILTERS[⊗]

2.0 Introduction

A subset A of a topological space X is said to be regularly closed if A = cl (int A) and regularly open if A = int (clA). The analogous concept "Fuzzy regularly closed sets" was introduced by K.K. Azad in [AZ] $_2$. He proved that the closure of every fuzzy open set is fuzzy regularly closed and interior of every fuzzy closed set is fuzzy regularly open.

Corresponding to a given a topological space (X,T), always there is a generated fuzzy topology $\omega(T)$ consisting of all lower semi continuous functions from [X,T] to [0,1]. Similarly corresponding to each fuzzy topological space (X,F) there is an associated topology $l(F)$ which is the weakest one such that every element of F is lower semi continuous . If $A \subset X$ is regularly closed in X, then χ_A is fuzzy regularly closed in the corresponding generated fuzzy topology and conversely if χ_B is f-regularly closed in the fuzzy topological space (X,F), then B is regularly closed in the corresponding associated topology.

FRC(X)- the set of all fuzzy regularly closed subsets of X is a partially ordered set with respect to the usual '$\leq$' relation. Also it becomes a lattice with '$\wedge$' and '$\vee$' defined by $\mu \vee \gamma = \mu \cup \gamma$ and $\mu \wedge \gamma$=f-cl(f-int $(\mu \cap \gamma)$).For $\mu \in$FRC(X),

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[⊗] Some results of this chapter were communicated to fuzzy sets and system.

taking the complement as μ^c = f-cl $(1-\mu)$ = $1-$ f-int μ, in the first part of section 2.2. we prove that (FRC(X), $\wedge,\vee$) is pseudo complemented and complete.

In [BI] it was proved that the set of all fuzzy regularly closed subsets of a B-fuzzy topological space form a fuzzy Boolean algebra.

In the next section we define the fuzzy regularly closed filters (FRC-filter for short) similar to the notion of fuzzy filters introduced by A.K. Katsaras. The FRC ultrafilters are defined and an equivalent formulation has been given for such filters. Properties of fuzzy filters given by 1.1.3 and 1.1.4 hold good for FRC filters also.

M.A. De Prade Vicente in her paper [P;A] proved that every fuzzy ultrafilter is free. In the third section of this chapter with suitable examples we prove the existence of fixed FRC-ultrafilters and we obtain a characterization for such ultrafilters.

2.1 Preliminaries

Throughout this chapter (X,τ) or simply X represents a fuzzy topological space in Chang's sense [CH]. Also we use the notation '$\vee$' and '$\wedge$' for lattice union and intersection, and '$\cup$' & '$\cap$' for fuzzy union and intersection.

2.1.1 Definition[AZ]$_2$: A fuzzy subset μ of (X, τ) is said to be fuzzy regularly closed if $\mu = \overline{\mu^o}$ and fuzzy regularly open if $\mu = \overline{\mu}^o$ where $\mu^o = \vee\{\lambda : \lambda \in \tau, \lambda \leq \mu\}$ and $\overline{\mu} = \wedge\{\gamma : \gamma' \in \tau, \gamma \geq \mu\}$.

2.1.2 Result:

a. Given a topological space (X,T) and B regularly closed in (X,T), then χ_B is fuzzy regularly closed in the corresponding generated fuzzy topology (X, F_T).

Conversely, given a topological space (X,T) and χ_B fuzzy regularly closed in (X, F_T), then B is regularly closed in (X,T).

b. Given a fuzzy topological space (X,F) and B regularly closed in the associated topology l(F), then χ_B is fuzzy regularly closed in (X,F).

Conversely given (X,F) and χ_B fuzzy regularly closed in (X,F) then B is regularly closed in l(F).

2.1.3 Definition [P;A]: A fuzzy ultrafilter $\mathcal{U}$ on X is said to be free if $\wedge\ \mathcal{U} = 0$ and is fixed if $\wedge\ \mathcal{U} \neq 0$

2.1.4 Proposition [P;A]: Every fuzzy ultrafilter is free

2.1.5 Definition [P;A]: A fuzzy ultrafilter $\mathcal{F}$ is strong free (or s-free) if and only if $\cap\{\text{supp } F \,/\, F \in \mathcal{F}\} = \phi$ and s-fixed if $\cap\{\text{supp } F \,/\, F \in \mathcal{F}\} \neq \phi$.

If $\cap\{\text{supp } F \,/\, F \in \mathcal{F}\} \neq \phi$ and $\mathcal{F}$ is a fuzzy ultrafilter, then the intersection must be reduced to a single point. The only s-fixed fuzzy ultrafilters are $\mathfrak{I}_x = \{ F \in I^X / F(x) > 0\}$ called tivial f-ultrafilters.

2.2 Fuzzy regularly closed filters

2.2.1 Definition: Let FRC (X) denote the set of all fuzzy regularly closed subsets of a fuzzy topological space X. Then for $\mu, \gamma \in$ FRC (X) define $\mu \leq \gamma$

if and only if $\mu(x)\leq \gamma(x)$ for every $x \in X$. Then '$\leq$' is a partial order on FRC(X) and hence FRC(X) is a poset.

If μ and $\gamma \in$ FRC (X), $\mu \cup \gamma$ is fuzzy regularly closed. But $\mu \cap \gamma$ need not be fuzzy regularly closed. So define $\mu \vee \gamma = \mu \cup \gamma$ and $\mu \wedge \gamma =$ f-cl (f-int $(\mu \cap \gamma)$). Then (FRC(X), $\leq$) is a lattice.

2.2.2 Result: For a fuzzy topological space X, (FRC(X),$\leq$) is pseudo complemented .

Proof: For $\mu \in$ FRC(X), take $\mu^c =$ f-cl$(1-\mu)$ $= 1-($ f-int $(\mu))$

(a) $(\mu^c)^c$ $=$ f-cl$(1-\mu^c)$ = f-cl $(1-$f-cl$(1-\mu))$

$=$ f-cl $(1-(1-$f-int $\mu))$

$=$ f-cl (f-int $\mu) = \mu$

(b) $(\mu \wedge \gamma)^c$ $=$ f-cl $(1-(\mu \wedge \gamma))$

$=$ f-cl$(1-$f-cl (f-int $(\mu \cap \gamma)))$

$=$ f-cl (f-int $(1-$f-int $(\mu \cap \gamma)))$

$=$ f-cl (f-int $((1-$ f-int $\mu) \cup (1-$ f-int $\gamma)))$

$=$ f-cl (f-int $(\mu^c \cup \gamma^c))$

$= (\mu^c \cup \gamma^c) = \mu^c \vee \gamma^c$

(c) $(\mu \vee \gamma)^c$ $=$ f-cl $(1-(\mu \vee \gamma) =$ f-cl$(1-(\mu \cup \gamma))$

$=$ f-cl$(1-$ f-cl(f-int $(\mu \cup \gamma)))$

$=$ f-cl $(1-[$f-cl (f-int $\mu) \cup$ f-cl (f-int $\gamma)]$)

$=$ f-cl $([1-$ f-cl (f-int $\mu)] \cap [1-$ f-cl (f-int $\gamma)]$)

$=$ f-cl [f-int $(1-$f-int $\mu) \cap$ f-int $(1-$ f-int $\gamma)]$

$= \quad$ f-cl (f-int [$(1-\text{f-int } \mu) \cap (1-\text{f-int } \gamma)$])

$= \quad$ f-cl (f-int $(\mu^c \cap \gamma^c)) = \mu^c \wedge \gamma^c$

$\therefore$(FRC (X), $\leq$) is pseudo complemented.

2.2.3 Result: For a fuzzy topological space X, (FRC(X), $\leq$) is a complete lattice .

Proof: Let λ_α be members of FRC(X). Then we show that,

a) $\wedge_\alpha \lambda_\alpha$ = f-cl (f-int $(\cap_\alpha \lambda_\alpha)$) b) $\vee_\alpha \lambda_\alpha$=f-cl (f-int $(\cup_\alpha \lambda_\alpha)$).

a) Put η= f-cl (f-int $(\cap_\alpha \lambda_\alpha)$). Then $\eta \in$ FRC(X).

$\cap \lambda_\alpha \leq \lambda_\alpha$ for every α.

$\therefore$ f-cl (f-int $(\cap \lambda_\alpha)$) $\leq \lambda_\alpha$. i.e. $\eta \leq \lambda_\alpha$.

Therefore η is a lower bound of λ_α.

Let $\delta \in$ FRC(X) such that $\delta \leq \lambda_\alpha$ for every α.

Therefore $\delta \leq \cap \lambda_\alpha$.

Therefore f-cl (f-int δ) $\leq$ f-cl (f-int $(\cap \lambda_\alpha)$).

i.e. $\delta \leq$ f-cl(f-int $(\cap \lambda_\alpha)$).

i.e. $\delta \leq \eta$.

$\therefore \eta$ is the greatest lower bound.

Hence $\wedge_\alpha \lambda_\alpha$= f-cl(f-int($\cap_\alpha \lambda_\alpha$)).

b) Now put μ =f-cl (f-int $(\cup_\alpha \lambda_\alpha)$). Since μ is the closure of a fuzzy open set, μ is fuzzy regularly closed. ie. $\mu \in$ FRC(X).

$\cup_\alpha \lambda_\alpha \geq \lambda_\alpha$ for every α.

i.e. f-cl (f-int(λ_α)) $\leq$ f-cl (f-int ($\cup_\alpha\lambda_\alpha$)) . i.e. $\lambda_\alpha \leq \mu$.

$\therefore$ μ is an upper bound of λ_α .

Let $\gamma \in$FRC (X) such that $\lambda_\alpha \leq \gamma$ for every α.

Then $(\cup_\alpha\lambda_\alpha) \leq \gamma$.

$\therefore$f-cl (f-int ($\cup_\alpha\lambda_\alpha$)) $\leq$ f-cl(f-int γ) = γ. i.e. $\mu \leq \gamma$.

Therefore μ is the least upper bound

$\therefore$ $\vee_\alpha\lambda_\alpha$=f-cl (f-int ($\cup_\alpha\lambda_\alpha$)).

Therefore (FRC(X), $\leq$) is a complete lattice.

2.2.4 Definition: A fuzzy regularly closed filter (FRC –filter for short) $\mathcal{F}$ on X is a non empty subset of FRC(X) satisfying,

i) $\lambda \in \mathcal{F}$ and $\mu \in$ FRC(X) such that $\mu > \lambda$ imply $\mu \in \mathcal{F}$.

ii) $\lambda, \mu \in \mathcal{F}$ implies $\lambda \wedge \mu \in \mathcal{F}$.

iii) $0 \notin \mathcal{F}$.

The following definitions are analogous to those by M.A. De Prade [P;A]

2.2.5 Definition: A subset $\mathcal{B}$ of FRC(X) is a base for some FRC filter if and only if $\mathcal{B} \neq \phi$ and (i) if $\mu_1, \mu_2 \in \mathcal{B}$ then $\mu_3 \leq \mu_1 \wedge \mu_2$ for some $\mu_3 \in \mathcal{B}$. (ii) $0 \notin \mathcal{B}$.

The collection $\mathcal{F} = \{\mu \in \text{FRC (X)}: \exists\, \gamma \in \mathcal{B} \text{ and } \mu \geq \gamma\}$ is an FRC filter. Then $\mathcal{F}$ is said to be generated by $\mathcal{B}$.

A subset $\mathcal{B}$ of FRC(X) is a base for $\mathcal{F}$ if and only if for each $\mu \in \mathcal{F}$, there is a $\gamma \in \mathcal{B}$ and $\gamma \leq \mu$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are FRC filters on X, $\mathcal{F}_1$ is said to be finer than $\mathcal{F}_2$ if and only if $\mathcal{F}_1 \supset \mathcal{F}_2$. An FRC-Filter $\mathcal{F}$ on X is an FRC ultra filter if there is no other FRC filter finer than $\mathcal{F}$.

2.2.6 Definition: Let $\mathcal{F}$ be an FRC filter on X. Then a (crisp) regularly closed subset Y of X is said to be included in $\mathcal{F}$ if and only if every fuzzy regularly closed subsets of X with support Y is an element of $\mathcal{F}$.

2.2.7 Theorem: If $\mathcal{F}$ is an FRC-Filter on X, then the following are equivalent.

i) $\mathcal{F}$ is an FRC-ultrafilter

ii) Let $\mu \in$ FRC(X). If $\mu \notin \mathcal{F}$ then there is some $\gamma \in \mathcal{F}$ such that $\mu \wedge \gamma = 0$.

iii) Let Y be a (crisp) regularly closed subset of X. Then either Y or Y^c is included in $\mathcal{F}$ where $Y^c = cl\,(X/Y)$.

Proof:- i) $\Rightarrow$ii) Let $\mu \notin \mathcal{F}$. If $\mu \wedge \gamma \neq 0$ for every $\gamma \in \mathcal{F}$ the collection $B = \{\mu \wedge \gamma : \gamma \in \mathcal{F}\}$ is a base for an FRC filter which is finer than $\mathcal{F}$, contradicting i).

Therefore there exists at least one $\gamma \in \mathcal{F}$ such that $\mu \wedge \gamma = 0$.

ii) $\Rightarrow$ iii) Let Y be a regularly closed subset of X. Suppose both Y and Y^c were not included in $\mathcal{F}$. Therefore there exist $\mu, \gamma \in$ FRC (X) with supports

Y and Y^c respectively such that both do not belong to $\mathcal{F}$. By part ii) if $\mu \notin \mathcal{F}$, there exists a $\delta \in \mathcal{F}$ such that $\mu \wedge \delta = 0$. Similarly there is an $\eta \in \mathcal{F}$ such that $\gamma \wedge \eta = 0$.

Supp $\mu = Y$. Therefore $\mu(x) \neq 0$ for every $x \in Y$.

So $\delta(x) = 0$ for every $x \in Y$.

Supp $\gamma = Y^c$ $\therefore$ $\gamma(x) \neq 0$ for every $x \in Y^c$ and so $\eta(x) = 0$ for every $x \in Y^c$.

$\therefore (\delta \wedge \eta)(x) = 0$ for every $x \in X$.

i.e. $\delta \wedge \eta = 0$ which is not possible since $\delta, \eta \in \mathcal{F}$. Therefore either Y or Y^c is included in $\mathcal{F}$.

iii) $\Rightarrow$ (i) If $\mathcal{F}$ is not an FRC –ultrafilter, let $G \supset \mathcal{F}$. Let $\gamma \in G$ such that $\gamma \notin \mathcal{F}$ and supp $\gamma = Y$. Therefore Y is not included in $\mathcal{F}$. Hence by (iii) Y^c is included in $\mathcal{F}$. That is any fuzzy regularly closed set δ with support Y^c belongs to $\mathcal{F}$ and hence in G. Therefore $\gamma \wedge \delta \in G$ which is not possible since $\gamma \wedge \delta = 0$. Therefore $\mathcal{F}$ is an FRC ultrafilter.

2.2.8 Proposition: Every FRC filter is contained in some FRC–ultrafilter.
Proof:- Follows from Zorn's Lemma.

2.2.9 Proposition: If $\mathcal{U}$ is an FRC ultrafilter on X then for every $\mu \in FRC(X)$ either $\mu \in \mathcal{U}$ or $\mu^c \in \mathcal{U}$ where $\mu^c =$ f-cl$(1-\mu)$. (Proof is similar to that for fuzzy filters by Katsaras [KA]$_2$).

Proof:- Suppose both μ and μ^c do not belong to $\mathcal{U}$. Therefore by theorem 2.2.7 (ii), there exist γ and $\delta \in \mathcal{U}$ such that $\mu \wedge \gamma = 0$ and $\mu^c \wedge \delta = 0$.

$\therefore \mu \wedge \gamma \wedge \delta = 0$ and $\mu^c \wedge \gamma \wedge \delta = 0$

i.e. $\mu \wedge (\gamma \wedge \delta) = 0$ and $\mu^c \wedge (\gamma \wedge \delta) = 0$

$\gamma \wedge \delta \neq 0$ since γ and $\delta \in \mathcal{U}$. Therefore there exists at least one $x_0 \in X$ such that $(\gamma \wedge \delta)(x_0) \neq 0$.

$\therefore \mu(x_0) = 0 = \mu^c(x_0)$ which is not possible since $\mu^c(x_0) = \text{f-cl}\,(1-\mu(x_0)) \neq 0$.

Therefore either μ or μ^c belongs to $\mathcal{U}$.

2.2.10 Remark: In general topology if $\mathcal{F}$ is an ultrafilter on X, and if A is a regularly closed subset of X such that $A \in \mathcal{F}$, then $A^c \notin \mathcal{F}$ since $A \cap A^c = \phi$.

Here in the case of fuzzy regularly closed filters. If $\mathcal{U}$ is an FRC-ultrafilter, $\mu \in \mathcal{U}$ does not imply that $\mu^c \notin \mathcal{U}$. But if A is a (crisp) regularly closed subset of X such that A is included (in the sense of definition 2.2.6) in $\mathcal{U}$ then A^c is not included in $\mathcal{U}$.

2.3 Fixed FRC-ultrafilters

2.3.1 Definition: An FRC-ultrafilter $\mathcal{U}$ on X is said to be free if $\wedge\, \mathcal{U} = 0$ and is fixed if $\wedge\, \mathcal{U} \neq 0$ Where $\wedge\, \mathcal{U} = \wedge \{\mu : \mu \in \mathcal{U}\}$

2.3.2 Note: Proposition 2.1.4 states that every fuzzy ultrafilter is free. But if we consider FRC ultrafilters on X such a result is not true. The following examples illustrate the existence of fixed FRC -ultrafilters.

2.3.3 Example: Let $\boldsymbol{R}$ be the set of real numbers with usual topology. Define μ_1 & μ_2 from $\boldsymbol{R}$ to $[0,1]$ as

$\mu_1(0) = 1/4$, $\mu_1(x) = 0$ for every $x \neq 0$.

$\mu_2(0) = 4/5$, $\mu_2(x) = 0$ for every $x \neq 0$.

Let F be the fuzzy topology generated by usual crisp topology $\boldsymbol{R}$, μ_1 and μ_2. Now consider the associated topology l(F) of F. i.e. l (F) is the smallest topology on X which makes every element of F lower semi continuous . μ_1 and μ_2 are lower semi continuous only if $\{0\}$ is open in l (F). Therefore l (F) is the topology generated by usual open intervals in R and $\{0\}$. Then the closed intervals in R and $\{0\}^c$ are regularly closed in l(F). Let $\mathcal{U}$ be the set of all regularly closed sets in l (F) containing zero. Then $\mathcal{U}$ is a fixed ultrafilter in l (F) and $\cap \mathcal{U} = \{0\}$.

Let $\mathcal{F} = \{ \chi_A : A \in \mathcal{U} \}$.

Then $\mathcal{F}$ is an FRC-filter. Therefore by proposition 2.2.8 there exists an FRC ultrafilter $\mathcal{U}$ containing $\mathcal{F}$. Then $\mathcal{U}$ contains elements of F and fuzzy regularly closed sets formed by μ_1' and μ_2'. Since $\chi_{\{0\}} \in \mathcal{U}$, $\wedge \mathcal{U}$ is always a fuzzy point with support zero. Also $\wedge \mathcal{U}(0) = 1/5$. i.e. $\wedge \mathcal{U} \neq 0$.

Therefore $\mathcal{U}$ is fixed.

2.3.4 Example: Consider $\boldsymbol{R}$ with usual topology .Define μ: $\boldsymbol{R} \to [0,1]$ as $\mu(0) = 1/4$ and $\mu(x) = 0$ for every $x \neq 0$. Let $\boldsymbol{R}_F$ be the fuzzy topology generated by usual crisp topology $\boldsymbol{R}$ and μ. Then f-closed sets in $\boldsymbol{R}_F$ are of the form χ_K, μ', $\chi_K \wedge \mu'$ where K is any closed set in R

Let $K_0 = [a,b]$ be any closed interval in $\boldsymbol{R}$ where $a \neq 0$ or $b \neq 0$.

f-int $\chi_{K0} =$ f-int $\chi_{[a,b]} = \chi_{(a,b)}$

f-cl (f-int $\chi_{[a,b]}) = \chi_{[a.b]} = \chi_{ko}$

$\therefore$ χ_{ko} is fuzzy-regularly closed in $\boldsymbol{R}_F$.

If $b = 0$, f-int $\chi_{[a,0]} = \chi_{(a,0)} \vee \mu$.

Therefore f-cl(f-int $\chi_{[a,0]}$) $= \chi_{(a,0)} \wedge \mu'$.

That is $\chi_{[a,0]}$ is not f-regularly closed. Similarly $\chi_{[0,b]}$ is also not f-regularly closed.

f-int $\mu' = \chi_{\mathbf{R}\setminus\{0\}} \vee \mu$

f-cl (f-int μ') $= \mu'$

$\therefore \mu'$ is f-regularly closed.

In a similar manner we can prove that $\chi_{\{0\}}$, $\chi_K \wedge \mu'$, 1,0 are f-regularly closed in $\boldsymbol{R}_F$. Let FRC($\boldsymbol{R}_F$) be the set of all fuzzy regularly closed subsets of $\boldsymbol{R}_F$.

Define $\mathcal{F} = \{ \mu \in \text{FRC}(\boldsymbol{R}_F): \mu(0) > 3/4\}$.

Then $\mathcal{F}$ is an FRC -filter. Let $\mathcal{U}$ be the fuzzy ultrafilter containing $\mathcal{F}$. Then $\wedge\, \mathcal{U}(0) = 3/4$.

$\therefore \wedge\, \mathcal{U} \neq 0$. Therefore, $\mathcal{U}$ is a fixed FRC-ultrafilter.

2.3.5 Result:

$\sigma_0(\wedge\mathcal{U}) \subseteq \cap\{\sigma_0(\mu):\mu\in\mathcal{U}\}$ where $\wedge\mathcal{U} = \wedge\{\mu : \mu \in \mathcal{U}\}$ and $\sigma_0(\mu) = \{x \in X: \mu(x) > 0\}$.

Proof: Let $x \in \sigma_0(\wedge\mathcal{U})$. Then $(\wedge\mathcal{U})(x) \neq 0$.

That is $\mu(x) \neq 0$ for every $\mu \in \mathcal{U}$.

i.e. $x \in \sigma_0(\mu) \quad \forall \mu \in \mathcal{U}$.

i.e. $x \in \cap\{\sigma_0(\mu) : \mu \in \mathcal{U}\}$

$\therefore \sigma_0(\wedge\mathcal{U}) \subseteq \cap\{\sigma_0(\mu) : \mu\in\mathcal{U}\}$

2.3.6 Theorem :- If $\mathcal{U}$ is a fixed FRC-ultrafilter on X, then

i) $\cap\{\sigma_0(\mu) : \mu \in \mathcal{U}\}$ contains exactly one point

ii) $\wedge\mathcal{U}$ is a unique fuzzy point.

Proof:

i) By the result 2.3.5 $\sigma_0(\wedge\mathcal{U}) \subseteq \cap\{\sigma_0(\mu) : \mu \in \mathcal{U}\}$. Since $\mathcal{U}$ is fixed, $\wedge\mathcal{U} \neq 0$. Therefore there exists atleast one $x\in X$ such that $\wedge\mathcal{U}(x)\neq 0$. So $\sigma_0(\wedge\mathcal{U}) \neq \phi$. Hence $\cap\{(\sigma_0(\mu):\mu \in \mathcal{U}\} \neq \phi$.

Suppose $\cap\{\sigma_0(\mu):\mu\in\mathcal{U}\}$ contains two points x and y and $x\neq y$. Let A be a regularly closed subset of X containing x but not y. Since $\mathcal{U}$ is an FRC-ultrafilter by 2.2.7 (iii) either A or A^c is included in $\mathcal{U}$. If A is included in $\mathcal{U}$, then the set of all fuzzy regularly closed subsets of X with support A belongs to $\mathcal{U}$. Therefore no fuzzy regularly closed subsets of X with A^c as support can belong to $\mathcal{U}$ .So y cannot belong to $\cap(\sigma_0(\mu):\mu\in\mathcal{U}\}$.Therefore $\cap\{\sigma_0(\mu):\mu\in\mathcal{U}\}$ contains exactly one point.

ii) By part i) and result 2.3.5 $\sigma_0(\wedge\mathcal{U})$ is a singleton. $\therefore$ $\wedge\mathcal{U}$ is a fuzzy point.

Let $\sigma_0(\wedge\mathcal{U}) = x_0$ and p, q be two distinct fuzzy points with support x_0.

Choose μ ($\neq 0$ and 1) belonging to FRC(X) such that $p \in$ f-int μ and $q \notin \mu$.

i.e. $p(x_0) \leq$ f-int $\mu(x_0)$ and $q(x_0) > \mu(x_0)$.

Let $p \in \wedge\mathcal{U}$.

Then $p(x_0) \leq (\wedge\mathcal{U})(x_0)$.

Therefore, $p(x_0) \leq \gamma(x_0)$, $\forall$ $\gamma\in \mathcal{U}$.

i.e. $p(x_0) \leq$ (f-int $(\mu\wedge\gamma)$) (x_0)

$\therefore$ f-int $(\mu \wedge \gamma) \neq 0$.

Hence $\mu \wedge \gamma \neq 0$ $\forall$ $\gamma\in \mathcal{U}$. So by theorem 2.2.7, $\mu\in\mathcal{U}$. But $q\notin\mu$. Therefore $q \notin \wedge\mathcal{U}$.

Hence $\wedge\mathcal{U}$ is a unique fuzzy point.

2.3.7 Definition:- An FRC-ultrafilter $\mathcal{U}$ on X is said to be strong fixed or s-fixed if $\wedge\mathcal{U} = 0$ and $\{ \sigma_0(\mu):\mu\in\mathcal{U} \} \neq\phi$.

2.3.8 Remark:- In this case also we can prove that $\{\sigma_0(\mu):\mu\in\mathcal{U}\}$ is singleton. Then such ultrafilters are only of the form $\mathcal{U}[x] = \{\mu\in FRC(X):\mu(x)>0\}$ which are called trivial FRC-ultrafilters.

CHAPTER - 3

F-OPEN FILTERS AND F-H CLOSED SPACES[⊗]

3.0 Introduction

It is known that in the crisp situation the absolute of a topological space can be constructed using the open ultrafilters also [P;W]. The set $\tau(X)$ of all fuzzy open subsets of X forms a pseudo complemented lattice with $\mu^c = 1 - \text{f-cl}(\mu) = \text{f-int}\,(1 - \mu)$. Therefore as in the case of topological spaces fuzzy absolutes also can be constructed using f-open ultrafilters. So in the first section of this chapter we introduce fuzzy open filters. Fuzzy open ultrafilters are defined and an equivalent formulation for such ultrafilters has been given. Using the concept of adherence of an f-open filter we define fixed f-open ultrafilters and they are used to characterize the convergence property of a filter. Here also some results in the crisp topology are not true in the fuzzy context. Counter examples are given where strict analogous results are not possible.

A space X is said to be H-closed if it is closed in every Hausdorff space containing X as a subspace. In section 3.2 we give the fuzzy analogue of this concept "Fuzzy Hausdorff closed" or (f-H closed) spaces. The characterization for f-H closed spaces given here establishes the relationship between the f-H closed spaces and their f-open filters.

[⊗] Some results of this chapter were presented in the Annual Conference of Kerala Mathematical Association at Kottayam, December 2001.

In the third section we introduce an s-continuous mapping from a topological space to a fuzzy topological space and prove that the image of an H-closed space under an s-continuous map is f-H closed. Here we have also proved that the arbitrary product πf_i and the sum $\oplus f_i$ of the s -continuous maps f_i are also s-continuous.

3.1 f-open filters:

3.1.1 Definition: Let (X,δ) be a fuzzy topological space. Then $\mathcal{F} \subset \delta$ is said to be a fuzzy open filter or f-open filter if it satisfies the following conditions.

i) $\lambda \in \mathcal{F}$ and $\mu \in \delta$ such that $\mu > \lambda$ imply $\mu \in \mathcal{F}$

ii) $\lambda, \mu \in \mathcal{F}$ implies $\lambda \wedge \mu \in \mathcal{F}$.

iii) $0 \notin \mathcal{F}$.

3.1.2 Note: The f-open filter basis and f-open ultrafilter can be defined as in 2.2.5.

3.1.3 Definition: Let $\mathcal{F}$ be an f-open filter on X. Then a crisp open subset Y of X is said to be included in $\mathcal{F}$ if and only if every f-open subset of X with support Y is an element of $\mathcal{F}$.

3.1.4 Theorem: Let $\mathcal{F}$ be an f-open filter on $(X.\delta)$. Then the following are equivalent.

i) $\mathcal{F}$ is an f-open ultrafilter.

ii) Let $\mu \in \delta$. If $\mu \notin \mathcal{F}$, then there is some $\gamma \in \mathcal{F}$ such that $\mu \wedge \gamma = 0$

iii) Let Y be a crisp open subset of X. Then either Y or Y^c is included in $\mathcal{F}$ where $Y^c = X \setminus cl(Y)$

Proof: Similar to that of theorem 2.2.7.

3.1.5 Note: Propositions 2.2.8 and 2.2.9 also hold good for f-open ultrafilters.

3.1.6 Definitions:

a. An f-open ultrafilter $\mathcal{U}$ on X is said to be fixed if $a(\mathcal{U}) = \wedge\{\bar{\mu} : \mu \in \mathcal{U}\} \neq 0$ and free if $a(\mathcal{U}) = 0$. $a(\mathcal{U})$ is called the adherence of f-open ultrafilter $\mathcal{U}$.

b. If x_p is a fuzzy point in (X, δ) with support x, then $\mathcal{N}(x_p) = \{\mu \in \delta : p \leq \mu(x)\}$ is called the set of neighbourhoods of x_p.

3.1.7 Definition: Let x_p be an f- point in X. Then an f- open filter $\mathcal{F}$ on X is said to converge to x_p if $\mathcal{N}(x_p) \subset \mathcal{F}$ and it is denoted as $\mathcal{F} \rightarrow x_p$. The point x_p is said to be a cluster point of $\mathcal{F}$ if $\mu \wedge F \neq 0$ for all $F \in \mathcal{F}$ and $\mu \in \mathcal{N}(x_p)$.

3.1.8 Lemma:

Let X be a fuzzy topological space and $\mathcal{U}$ be a fixed f-open ultra filter on X. Let x_p be a fuzzy singleton in X. Then $x_p \in a(\mathcal{U}) \Rightarrow \mathcal{N}(x_p) \subset \mathcal{U}$.

Proof:

Suppose $x_p \in a(\mathcal{U})$ where $a(\mathcal{U}) = \wedge\{\bar{\mu} : \mu \in \mathcal{U}\}$

Then $a(\mathcal{U})(x) = 1$.

That is $\wedge\{\bar{\mu} : \mu \in \mathcal{U}\}(x) = 1$

Therefore, $\bar{\mu}(x) = 1$ for every $\mu \in \mathcal{U}$.

Let $\delta \in N(x_p)$ and γ be any arbitrary element of $\mathcal{U}$. Then $\delta(x)=1$ and $\bar{\gamma}(x) = 1$

Therefore $(\delta \wedge \bar{\gamma})(x) = 1$ and so $\delta \wedge \bar{\gamma} \neq 0$.

If $\delta \wedge \gamma = 0$, then $\gamma \leq \delta^c$. i.e $\bar{\gamma} \leq \delta^c$.

$\bar{\gamma}(x) \leq \delta^c(x)$, for every $x \in X$ which is not possible.

There fore $\delta \wedge \gamma \neq 0$

i.e. $\delta \wedge \gamma \neq 0$ for every $\gamma \in \mathcal{U}$.

Therefore $\delta \in \mathcal{U}$ (by the theorem 3.1.4(ii))

Therefore $N(x_p) \subset \mathcal{U}$.

3.1.9 Remark: In the case of non crisp sets the converse of 3.1.8 need not be true as is seen from the following example.

3.1.10 Example: Let $X = \{a, b, c\}$.

Define μ_1, μ_2 and μ_3 from X to [0,1] as

$\mu_1(a) = \frac{1}{2}$, $\mu_1(b) = \mu_1(c) = 0$

$\mu_2(b) = \frac{1}{2}$, $\mu_2(a) = \mu_2(c) = 0$

$\mu_3(c) = \frac{1}{2}$, $\mu_3(a) = \mu_3(b) = 0$

Then $\tau = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_1 \vee \mu_2, \mu_1 \vee \mu_3, \mu_2 \vee \mu_3, \mu_1 \vee \mu_2 \vee \mu_3\}$ is a fuzzy topology on X. Then the closed sets are $\{0, 1, \mu_1', \mu_2' \mu_3', (\mu_1 \vee \mu_2)', (\mu_2 \vee \mu_3)', (\mu_1 \vee \mu_3)', (\mu_1 \vee \mu_2 \vee \mu_3)\}$, Let $\mathcal{U} = \{1, \mu_1, \mu_1\vee\mu_2, \mu_1\vee\mu_3, \mu_1\vee\mu_2\vee\mu_3\}$. Then $\mathcal{U}$ is an f-open ultrafilter.

$a(\mathcal{U}) = \wedge \{ \bar{\mu} : \mu \in \mathcal{U} \}$

$= \wedge \{1, \mu_1 \vee \mu_2 \vee \mu_3 \} = \mu_1\vee\mu_2\vee\mu_3.$

Now, consider the fuzzy singleton a_1. Then

$N(a_1) = \{ \mu \in \tau(X) : \mu(a) = 1 \} = \{1\}$

$\therefore N(a_1) \subseteq \mathcal{U}$.

But $a(\mathcal{U})(a) = (\mu_1\vee\mu_2\vee\mu_3)(a) = \frac{1}{2} \neq 1$

$\therefore a_1 \notin a(\mathcal{U})$.

3.1.11 Result: Let $\mathcal{U}$ be an f-open ultrafilter and $x_p \in a(\mathcal{U})$. Then x_p is a cluster point of $\mathcal{U}$.

Proof:

If $\mathcal{U}$ is an f-open ultrafilter and $x_p \in a(\mathcal{U})$ then by lemma 3.1.8 $N(x_p) \subset \mathcal{U}$.

Therefore $\mu \wedge F \neq 0$ for every $\mu \in N(x_p)$ and $F \in \mathcal{U}$.

That is x_p is a cluster point of $\mathcal{U}$.

3.1.12 Lemma: Let $C(\mathcal{U})$ denote the set of all cluster points of $\mathcal{U}$. If $\mathcal{U}$ is a fixed f-open ultrafilter then $C(\mathcal{U})$ contains exactly one point.

Proof: Since $\mathcal{U}$ is fixed $a(\mathcal{U}) \neq 0$ and so $C(\mathcal{U}) \neq \phi$.

Let x_1 and y_1 be two distinct fuzzy singletons such that both belong to C ($\mathcal{U}$). Since X is fT_2, we can have $\mu \in N$ (x_1) and $\gamma \in N$ (y_1) such that $\mu \wedge \gamma = 0$. If x_1 and y_1 are cluster points, then $\mu \wedge U \neq 0$ and $\gamma \wedge U \neq 0$ for every $U \in \mathcal{U}$. Therefore $\mu \in \mathcal{U}$ and $\gamma \in \mathcal{U}$ which is not possible.

Hence $C(\mathcal{U})$ contains exactly one point.

3.2 f-H closed spaces: -

3.2.1. Definition: A fuzzy topological space (X,δ) is said to be closed in (Y,τ) if $Cl_Y X = X$. (Closure is with respect to the fuzzy topology of Y).

X is said to be fuzzy Hausdorff closed or (f-H closed) if X is closed in every fuzzy Hausdorff space containing X as a subspace.

3.2.2. Theorem:- For a fuzzy topological space X , the following are equivalent.

i) X is f-H closed

ii) Every f-open filter on X has a non empty adherence .

iii) For every f- open cover of X , there is a finite subfamily whose union is dense in X

Proof:

i)⇒ii) Let (X, δ) be f-H closed and $\mathcal{F}$ be an f-open filter on X such that a ($\mathcal{F}$) = 0 where a ($\mathcal{F}$) = $\wedge\{ \bar{\mu} : \mu \in \mathcal{F} \}$.

Let $Y = X \cup \{ \mathcal{F} \}$.

Define μ to be f-open in Y and write $\mu \in \tau$ if $\mu \wedge X$ is open in X and $\mathcal{F} \in \mu$ implies $\mu \wedge X \in \mathcal{F}$.

Then μ_1 and $\mu_2 \in \tau \Rightarrow \mu_1 \wedge X \in \delta$ and $\mu_2 \wedge X \in \delta$.

$\therefore (\mu_1 \wedge \mu_2) \wedge X \in \delta$.

$\mathcal{F} \in \mu_1 \wedge \mu_2 \Rightarrow \mathcal{F} \in \mu_1$ and $\mathcal{F} \in \mu_2$

$\Rightarrow (\mu_1 \wedge X) \in \mathcal{F}$ and $(\mu_2 \wedge X) \in \mathcal{F}$.

$\Rightarrow (\mu_1 \wedge \mu_2) \wedge X \in \mathcal{F}$.

$\therefore \mu_1 \wedge \mu_2 \in \tau$.

$\mu_i \in \tau \Rightarrow \mu_i \wedge X \in \delta$ and $\mathcal{F} \in \mu_i$ implies $\mu_i \wedge X \in \mathcal{F}$.

Then $(\vee \mu_i) \wedge X = \vee (\mu_i \wedge X) \in \delta$ and $(\vee \mu_i \wedge X) \in \mathcal{F}$.

Therefore $\vee \mu_i \in \tau$.

That is (Y,τ) is a fuzzy topological space and X is not closed in Y.

Since X is fuzzy Hausdorff, for every distinct fuzzy points in X we can have disjoint f-open sets in Y.

Let x_p be a fuzzy singleton in X with support x. Since $a(\mathcal{F}) = 0$, $a(\mathcal{F})(x) = 0$ for every $x \in X$. i.e. there exists some $\gamma \in \mathcal{F}$ such that $\overline{\gamma}(x) = 0$, i.e. $\gamma(x) = 0$. Then there exists an open neighbourhood η of x_p such that $\gamma \wedge \eta = 0$. Therefore, $\gamma \cup \{\mathcal{F}\}$ and η are disjoint open neighbourhoods of $\mathcal{F}$ and x_p in Y.

Therefore Y is Fuzzy Hausdorff. But X is not closed in Y. Thus X is not f-H closed.

Therefore, X is f-H closed $\Rightarrow$ a $(\mathcal{F}) \neq 0$.

ii)$\Rightarrow$i) Suppose every f-open filter on X has a non empty adherence. To show that X is f-H closed .

Suppose X is not f-H closed. i.e., there exists a fuzzy Hausdorff space Y such that X is a subspace and $cl_Y(X) \neq X$. Let x_p be a fuzzy singleton in $cl_Y(X) \setminus X$ with support x.

Let $\mathcal{F} = \{\mu \wedge X : \mu \in \tau, \mu(x) = 1\}$. Then $0 \notin \mathcal{F}$.

$$\gamma_1 \text{ and } \gamma_2 \in \mathcal{F} \Rightarrow \gamma_1 = \mu_1 \wedge X, \quad \mu_1(x) = 1$$

$$\gamma_2 = \mu_2 \wedge X, \quad \mu_2(x) = 1$$

$$\gamma_1 \wedge \gamma_2 = (\mu_1 \wedge \mu_2) \wedge X, \qquad (\mu_1 \wedge \mu_2)(x) = 1$$

Therefore $\gamma_1 \wedge \gamma_2 \in \mathcal{F}$.

If $\delta > \gamma$ and $\gamma \in \mathcal{F}$, then $\delta \in \mathcal{F}$. Therefore $\mathcal{F}$ is an f-open filter on X.

$a(\mathcal{F}) = \wedge \{ \bar{\mu} : \mu \in \mathcal{F} \}$.

Y is fuzzy Hausdorff. Therefore for every distinct fuzzy singleton in Y, there exist f-open sets γ and δ such that $\gamma \wedge \delta = 0$

Therefore a $(\mathcal{F}) = 0$, which is a contradiction.

Hence X is f-H closed.

ii)$\Rightarrow$iii) Suppose every f- open filter on X has a non empty adherence. Let $C = \{ \mu_i : i \in I\}$ be an open cover for X such that for any finite subset A of C, $cl(\vee \gamma_i : \gamma_i \in A) \neq 1$.

Let $\mathcal{F} = \{\delta : \delta$ open and $\delta \geq 1 - cl(\vee\gamma_i)$, $\gamma_i \in A\}$

Then $0 \notin \mathcal{F}$.

$\mu_1, \mu_2 \in \mathcal{F} \Rightarrow \mu_1 \geq 1 - cl\ (\vee \gamma_i)$ and $\mu_2 \geq 1 - cl\ (\vee \gamma_i)$

$\therefore \mu_1 \wedge \mu_2 \geq 1 - cl\ (\vee \gamma_i)$

$\therefore \mu_1 \wedge \mu_2 \in \mathcal{F}$.

Let η be an f-open set such that $\eta \geq \mu$ and $\mu \in \mathcal{F}$.

Then $\eta \in \mathcal{F}$. Therefore $\mathcal{F}$ is an f-open filter on X.

$a_x(\mathcal{F}) = \wedge\{\overline{\delta} : \delta \in \mathcal{F}\}$

$\delta \geq 1 - cl(\vee\mu_i)$. Therefore, $\wedge\overline{\delta} \leq \wedge \{cl\ (1 - cl(\vee\mu_i)\}$

$a_x(\mathcal{F}) = \wedge\{\delta : \delta \in \mathcal{F}\}$

$\leq \wedge\{cl\ (1 - cl\ (\vee\gamma_i) : \gamma_i \in A\}$.

$\leq \wedge\{cl\ (1 - cl(\vee\mu_i) :\ \mu_i \in C\}$.

$= \wedge\{1 - int\ cl\ (\vee\mu_i) :\ \mu_i \in C\}$

$\leq \wedge\{1 - \vee\mu_i) : \mu_i \in C\}$

$= 0$ Since $\vee\mu_i = 1$ for $\mu_i \in C$

Therefore $a_x(\mathcal{F}) \neq 0 \Rightarrow C$ has a finite sub family whose union is dense in X

iii)$\Rightarrow$ii) Let $\mathcal{F}$ be a fuzzy-open filter on X such that $a(\mathcal{F}) = 0$. That is $\wedge\{\overline{\mu} : \mu \in \mathcal{F}\} = 0$. So $\{1 - \overline{\mu} : \mu \in \mathcal{F}\}$ is an f-open cover for X. Then for a finite subset $\mathcal{A}$ of $\mathcal{F}$, $\{1 - \overline{\gamma} : \gamma \in \mathcal{A}\}$ is a finite sub family of this open cover.

$f\text{-}cl\ \{\vee\ \{1 - \overline{\gamma} : \gamma \in \mathcal{A}\} = \vee\ (f\text{-}cl(1 - \overline{\gamma}) : \gamma \in \mathcal{A}\}$.

$= \vee \{1 - \text{f-int}(\text{f-cl}\gamma) : \gamma \in \mathcal{A}\}$

$\leq 1 - (\wedge\gamma : \gamma \in \mathcal{A}) \neq 1$ (since $\wedge\{\gamma : \gamma \in \mathcal{A}\} \neq 0$)

which is not true by iii)

Therefore $a_x(\mathcal{F}) \neq 0$

3.3 s-continuous mapping:-

3.3.1 Definitions: Let X be a fuzzy topological space and $\{p_i\}$ be a set of fuzzy points in X. Then for any fuzzy set γ in X, $\{p_i\}$ is said to be subordinate to γ denoted as $\{p_i\} \subset \gamma$ if and only if $p_i \leq \gamma$ for every i.

3.3.2 Definition: Let X be a topological space and Y be a fuzzy topological space. A mapping f from X to the set of fuzzy points in Y is said to be s-continuous at $x_o \in X$ if for every f-open set γ such that $f(x_o) \in \gamma$ there is an open neighbourhood U of x_o such that $f(cl(U)) \subset cl(\gamma)$. If f is s-continuous at every $x_0 \in X$, then f is s-continuous on X.

3.3.3 Example: Let X = ***R*** be the set of real numbers with usual topology and Y be the fuzzy topology generated by usual crisp topology in ***R*** and μ where $\mu : \boldsymbol{R} \rightarrow [0,1]$ is defined as $\mu(0) = \frac{1}{4}$ and $\mu(x)=0 \quad \forall\ x \neq 0$.

Define f from X to the set of fuzzy points in Y as $f(x) = (x^2)_{½}$ (fuzzy point with support x^2 and value ½). Let $x \in \boldsymbol{R}$ and (a,b) be an open interval such that $x^2 \in (a,b)$. Therefore $\chi_{(a,b)}$ is an f-open set in Y containing x^2.

Correspondingly there is an open set $U=(\sqrt{a}, \sqrt{b})$ in X such that $f([\sqrt{a}, \sqrt{b}])$ is subordinate to $\chi_{[a,b]}$. i.e. $f(cl(U)) \subset cl\chi_{(a,b)}$.

3.3.4 Definition(s-regular): A fuzzy topological space X is said to be strong regular (or s-regular) at $x \in X$ if for each f-open set γ such that $\gamma(x) = 1$ there exists a crisp open set U containing x in X such that $U \subset cl\ U \subset \gamma$.

3.3.5 Result: Let S(X,Y) denote the set of all s-continuous mappings from the topological space X to the fuzzy topological space Y. If Y is s-regular then $S(X,Y) = C(X,Y')$ where $C(X,Y')$ is the set of all continuous functions from X to Y' - the back ground space of the fuzzy topological space Y.

Proof: Let $f \in S(X,Y)$. Let $x_0 \in X$ and V be any open set in Y' such that $f(x_0) \in V$.(Then V is an f-open set). Since f is s-continuous there exists an open set U containing x_0 such that $f(cl(U)) \subset cl(V)$.i.e. $f(U) \subset V$. Therefore, f is s-continuous.

$\therefore S(X,Y) \subset C(X,Y')$ (1)

Now let $f \in C(X,Y')$ and γ be an f-open set in Y' such that $f(x_0) \in \gamma$. Since Y' is s-regular there exists an open set V in γ such that $f(x_0) \in V \subset \bar{V} \subset \gamma$. f is continuous. Therefore there exists an open set U in X such that $x_0 \in U$ and $f(U) \subset V$. Therefore $f(cl(U)) \subset \bar{V} \subset \gamma \subset \bar{\gamma}$.

i.e. f is s-continuous.

Therefore $C(X,Y') \subset S(X,Y)$ (2)

Hence $S(X,Y) = C(X,Y')$

3.3.6 Remark: The result holds also when Y' has the associated topology instead of the background topology.

3.3.7 Result: Let f be an s-continuous mapping from the topological space X to the fuzzy topological space Y. If X is H-closed, then Y is f-H closed.

Proof:- Let $\mathcal{C}$ be an open cover for Y. For each $x \in X$, there is an open set $\mu \in \mathcal{C}$ such that $f(x) \in \mu$. Since f is s-continuous, there exists an open set say U (x) in X such that $f(\mathrm{cl}U(x)) \subseteq \mathrm{cl}(\mu)$.

X is H-closed. Therefore for every open cover of X, there is a finite family whose union is dense in X. Therefore there is a finite subset F such that

$$X = \mathrm{cl}\,(\cup \{U(x) : x \in F\})$$

$$= \cup\, \mathrm{cl}\{U(x) : x \in F\}$$

$$\text{Therefore } f(X) = \cup\,(f\,\{\mathrm{cl}U(x) : x \in F\})$$

$$\leq \vee\,(\mathrm{cl}\mu_i), \quad i = 1,2\ldots.n, \quad n = |F|$$

That is $Y \leq \vee\, \mathrm{cl}(\mu_i)$, $\mu_i \in \mathcal{C}$

$\therefore$ The open cover $\mathcal{C}$ of Y has a finite sub cover whose union is dense in Y. Therefore Y is f-H closed.

3.3.8 Theorem: Let $\{X_i, i \in I\}$ be a set of topological spaces and $\{Y_i, i \in I\}$ be a set of fuzzy topological spaces. Also for each $i \in I$, let f_i be s- continuous from X_i onto the set of fuzzy singletons in Y_i. Then

a) $\pi f_i: \pi X_i \rightarrow \pi Y_i$ and

b) $\oplus f_i: \oplus X_i \rightarrow \oplus Y_i$ are both s-continuous .

Proof:

a) For each i, fi's are s-continuous from X_i to Y_i. Therefore if $x_i \in X_i$ and γ_i's are open sets in Y_i such that $f_i(x_i) \in \gamma_i$, then there exist open sets U_i containing x_i such that $f_i\,(\overline{U}_i)$ is subordinate to $\overline{\gamma_i}$.

i.e. $f_i(\overline{U}_i) \subset \overline{\gamma}_i$.

Since γ_i's are open in y_i, $\gamma = \bigwedge_{i=1}^{n} \pi_i^{-1}(\gamma_i)$ is open in $Y = \pi Y_i$

Let $x \in X = \pi X_i$

$\pi f_i(x) = f(x) = (p_1, p_2, p_3 \ldots)$ where $p_i = f_i(x_i)$, $p_i \leq \gamma$ for every i.

Therefore $f(x) \in \gamma$ and γ is open.

Let $U = U_1 \times U_2 \times \ldots \times U_n \times X \times X \times \ldots$

Then U is an open set containing x.

$$f(\overline{U}) = f(\overline{U}_1 \times \overline{U}_2 \times \ldots\ldots\ldots \overline{U}_n \times X \times X \ldots)$$
$$= (f_1(\overline{U}_1), f_2(\overline{U}_2), \ldots)$$

Each $f_1(\overline{U}_1)$, $f_2(\overline{U}_2)$ etc are subordinate to,.$\overline{\gamma}_1$, $\overline{\gamma}_2$ ....

Therefore $f(\overline{U})$ is subordinate to $\overline{\gamma}$.

Hence $\pi f_i = f$ is s-continuous.

b) Now $f = \oplus f_i$ defined from $\oplus X_i$ to $\oplus Y_i$ as $f(x) = \begin{cases} f_i(x) & \text{if } x \in X_i \\ \phi, & \text{otherwise,} \end{cases}$

is s-continuous since each f_i is s-continuous .

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CHAPTER - 4

FUZZY ABSOLUTES[⊗□]

4.0 Introduction: -

The set R(X) of all regularly closed subsets of a topological space X form a Boolean algebra, with complement defined as $A^c = cl\ (X \setminus A)$ for $A \in R(X)$. The absolute of a topological space X introduced by Iliadis in 1963 involves the Stone space of R(X) which is based on its ultrafilters.

I^X -the set of all fuzzy subsets of X forms a pseudo Boolean algebra with pseudo complement as $\mu' = 1-\mu$. In chapter 1 we have constructed the Stone space of fuzzy sets. In chapter 2 we proved that the set FRC(X) of all fuzzy regularly closed subsets of X is a pseudo complemented lattice which is also complete. Assuming the distributivity, we can say that FRC(X) is a pseudo Boolean algebra and hence the concept of absolutes can be extended to fuzzy context. But in the case of fuzzy sets, FRC(X) becomes distributive in fuzzy topological spaces satisfying the following property.

" If μ is fuzzy regularly open and γ is any fuzzy open set in X such that f-cl μ = f-cl γ, then for any fuzzy open set δ in X, f-cl $(\mu \wedge \delta)$ = f-cl $(\gamma \wedge \delta)$."

[At the moment we do not know whether this is a necessary condition for distributivity].

[⊗] Some of the results in this chapter were communicated to Indian Journal of Mathematics, Allahabad Mathematical Society.

[□] Some results of this chapter were published in the proceedings of the U.G.C. Sponsored National Seminar on Fuzzy Mathematics and Applications at U.C.College , Aluva March 1999.

A fuzzy topological space having this property is called B-fuzzy topological space [BI]. In [BI] it was proved that FRC(X) is a fuzzy Boolean algebra by considering Lowen's fuzzy topological space $[LO]_1$ and by taking the complement of $\mu \in I^x$ as the unique element μ' such that $\mu \wedge \mu' \leq \frac{1}{2}$ and $\mu \vee \mu' \geq \frac{1}{2}$

Throughout our work we are using Chang's fuzzy topological space. So in the first section of this chapter we are giving examples to B-fuzzy topological space in Chang's sense. Using this we prove distributivity in FRC(X). Therefore considering the B-fuzzy topological space, the set FRC(X) becomes a pseudo Boolean algebra.

In the second section of the chapter we introduce fuzzy absolutes as the Stone space of FRC(X). For constructions we are making use of the fixed FRC ultrafilters obtained in chapter 2.

The third section proves the uniqueness and some properties of the fuzzy absolute. In section 4 we are constructing fuzzy absolutes by using s-fixed FRC ultra filters instead of fixed FRC ultra filters. Its properties are proved with suitable examples.

In the last section we introduce the fuzzy absolutes of sums and products of fuzzy topological spaces.

4.1 Pseudo Boolean algebra of fuzzy regularly closed sets

4.1.1 Definition : A fuzzy topological space X (in Chang's sense)satisfying the following condition (A) is called B-fuzzy topological space.

Condition A:- If μ is fuzzy regularly open and γ is fuzzy open in X such that f-cl μ = f-cl γ then for any fuzzy open set δ in X , f-cl ($\mu \wedge \delta$) = f-cl($\gamma \wedge \delta$).

4.1.2 Example:

Let $\boldsymbol{R}$ be the set of real numbers with usual topology. Define μ_1 and μ_2 from $\boldsymbol{R}$ to [0,1] as $\mu_1(0) = \frac{1}{4}$, $\mu_1(x) = 0$ $\forall$ $x \neq 0$ and

$$\mu_2(0) = \frac{2}{5}, \quad \mu_2(x) = 0 \quad \forall\ x \neq 0.$$

Let F be the fuzzy topology generated by usual crisp topology in $\boldsymbol{R}$ and μ_1 and μ_2.

Then f-cl (μ_1) $= \chi_{\{0\}} \wedge \mu'_2$

f-int (f-cl (μ_1)) = μ_2.

f-cl (μ_2) $= \chi_{\{0\}} \wedge \mu_2'$ and f-int (f-cl (μ_2))= μ_2.

$\therefore \mu_2$ is fuzzy regularly open and μ_1 is open in F such that f-cl (μ_1) = f-cl (μ_2).

If $0 \in (a,b)$, then $\chi_{\{a,b\}} \wedge \mu_1 = \mu_1$ and

$$\chi_{\{a,b\}} \wedge \mu_2 = \mu_2$$

$$\therefore \text{f-cl}(\chi_{\{a,b\}} \wedge \mu_1) = \text{f-cl}\,(\chi_{\{a,b\}} \wedge \mu_2)$$

Also , $\chi_{\{a,b\}} \vee \mu_1 = \chi_{\{a,b\}}$

$\therefore$ f-cl($(\chi_{\{a,b\}} \vee \mu_1) \wedge \mu_1)$ = f-cl $((\chi_{\{a,b\}} \vee \mu_1) \wedge \mu_2)$

`If $0 \notin (a,b)$, $\chi_{\{a,b\}} \wedge \mu_1 = \chi_{\{a,b\}} \wedge \mu_2 = 0$.

$\therefore$ f-cl$(\chi_{\{a,b\}} \wedge \mu_1)$ = f-cl $(\chi_{\{a,b\}} \wedge \mu_2)$

$\chi_{\{a,b\}} \vee \mu_1$ and $\chi_{\{a,b\}} \vee \mu_2$ are open sets .

Then f-cl $(\chi_{\{a,b\}} \vee \mu_1) \wedge \mu_1)$ = f-clμ_1 and f-cl $(\chi_{\{a,b\}} \vee \mu_1) \wedge \mu_2)$ = f-clμ_1.

$\therefore$ f-cl $(\chi_{\{a,b\}} \vee \mu_1) \wedge \mu_1)$ = f-cl $(\chi_{\{a,b\}} \vee \mu_1) \wedge \mu_2)$

Similarly f-cl $(\chi_{\{a,b\}} \vee \mu_2) \wedge \mu_1)$ = f-clμ_1 and f-cl $(\chi_{\{a,b\}} \vee \mu_2) \wedge \mu_2)$ = f-clμ_2.

Therefore, f-cl $(\chi_{\{a,b\}} \vee \mu_2) \wedge \mu_1)$ = f-cl $(\chi_{\{a,b\}} \vee \mu_2) \wedge \mu_2)$ since f-clμ_1 = f-clμ_2.

All the fuzzy regularly open sets and f-open sets are of these types and so the condition is satisfied for all f-open sets and f-regular open sets Therefore F is a B- fuzzy topological space.

4.1.3 Example:

Let X = {a,b,c}. Define μ_1, μ_2 and μ_3 from X to [0,1] as

$$\mu_1(a) = \frac{1}{2}, \quad \mu_1(b) = \mu_1(c) = 0$$

$$\mu_2(a) = 0, \quad \mu_2(b) = 1, \quad \mu_2(c) = 0$$

$$\mu_3(a) = \frac{1}{2}, \quad \mu_3(b) = 1, \quad \mu_3(c) = 0$$

Then $\tau = \{0, 1, \mu_1, \mu_2, \mu_3\}$ is a fuzzy topology on X.

f-cl$(\mu_1) = \mu_3'$, f-cl$(\mu_2) = \mu_1'$, and f-cl$(\mu_3) = \mu_1'$.

f-int (f-cl(μ_3) = f-int $\mu_1' = \mu_3$

Therefore μ_3 is fuzzy regularly open and μ_2 is f open such that f-cl(μ_2) = f-cl(μ_3).

$\mu_1 \wedge \mu_2 = 0$. Therefore, f-cl $(\mu_1 \wedge \mu_2) = 0$

$\mu_1 \wedge \mu_3 = \mu_1$. Therefore, f-cl$(\mu_1 \wedge \mu_3)$ =f-cl $\mu_1 = \mu_3'$

Hence f-cl $(\mu_1 \wedge \mu_2) \neq$ f-cl $(\mu_1 \wedge \mu_3)$.

Therefore (X,τ) is not a B-fuzzy topological space.

4.1.4 Example.

Let X={a, b}. Define μ_1 and μ_2 from X to [0,1] as

$$\mu_1(a) = \frac{1}{4}, \quad \mu_1(b) = \frac{3}{4}$$

$$\mu_2(a) = \frac{2}{5}, \quad \mu_2(b) = \frac{1}{2}$$

Then $\tau = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2\}$ is a fuzzy topology on X.

f-cl $(\mu_1) = 1$, f-cl $(\mu_2) = \mu_2'$, f-cl$(\mu_1 \wedge \mu_2) = \mu_2'$

f-int (f-cl μ_2)= f-int $\mu_2' = \mu_2$.

$\therefore \mu_2$ is fuzzy regularly open and $\mu_1 \wedge \mu_2$ is f-open such that f-cl(μ_2) = f-cl($\mu_1 \wedge \mu_2$).

Also f-cl$(\mu_1 \wedge \mu_2)$ = f-cl$(\mu_1 \wedge (\mu_1 \wedge \mu_2))$ and

f-cl$((\mu_1 \vee \mu_2) \wedge \mu_2)$ = f-cl$((\mu_1 \vee \mu_2) \wedge (\mu_1 \wedge \mu_2))$.

Here μ_2 is the only f- regularly open set and so this exhausts the various possibilities. Therefore (X,τ) is a B-fuzzy topological space.

4.1.4 Theorem: Let X be a B-fuzzy topological space. Then the set FRC(X) of all fuzzy regularly closed subsets of X is a complete pseudo Boolean algebra.

Proof:

By proposition 2.2.2 and 2.2.3 we have FRC(X) is a pseudo complemented lattice which is complete. Therefore it is enough to prove that the distributive law holds in FRC(X).

For λ, $\mu \in$ FRC(X), $\lambda \vee \mu = \lambda \cup \mu$ and $\lambda \wedge \mu =$ f-cl (f-int $(\lambda \cap \mu)$).

$\therefore$ f-cl (f-int $(\lambda \vee \mu)$) $= \lambda \vee \mu =$ f-cl(f-int λ) $\cup$ f-cl(f-int μ)

$=$ f-cl [f-int $\lambda \cup$ f-int μ]

Since X is a B-fuzzy topological space, for any open set δ in X,

f-cl ($\delta \wedge$ f-int $(\lambda \vee \mu)$)= f-cl[$\delta \wedge$ (f-int $\lambda \cup$ f-int μ)]

Now, $\lambda \wedge (\mu \vee \gamma)$ $=$ f-cl[f-int $(\lambda \cap (\mu \vee \gamma)$]

$=$ f-cl [f-int $\lambda \cap$ f-int $(\mu \vee \gamma)$]

$=$ f-cl [f-int $\lambda \cap$ (f-int $\mu \cup$ f-int γ)

$=$ f-cl [(f-int $\lambda \cap$ f-int μ) $\cup$(f-int $\lambda \cap$ f-int γ)]

$=$ f-cl (f-int $(\lambda \cap \mu)$) $\cup$ f-cl (f-int $(\lambda \cap \gamma)$)

$= (\lambda \wedge \mu) \cup (\lambda \wedge \gamma)$

$= (\lambda \wedge \mu) \vee (\lambda \wedge \gamma)$.

Similarly $\lambda \vee (\mu \wedge \gamma) = (\lambda \vee \mu) \wedge (\lambda \vee \gamma)$

$\therefore$ FRC(X) is a pseudo Boolean algebra.

4. 2 Fuzzy Absolute:

4.2.1: Definition: Let X be a B-fuzzy topological space and $\Omega_r(X)$ be the set of all FRC- ultrafilters on X. For $\mu \in FRC(X)$, let $\Lambda_r(\mu) = \{\mathcal{U} \in \Omega_r(X) : \mu \in \mathcal{U}\}$ Then Λ_r is a function from FRC(X) onto $P(\Omega_r(X))$.

4.2.2 Definition: The fuzzy Stone space of FRC(X) is called the fuzzy Gleason space of X (f- Gleason space in short) and is denoted as f-θX. Therefore the elements of f-θX are FRC-ultrafilters on X and a basis for open sets of f-θX is $\{\Lambda_r(\mu): \mu \in FRC(X)\}$

4.2.3 Theorem: f-θX is a compact, T_2, zero dimensional space.

Proof: f-θX is the Stone space of FRC(X). There for the theorem follows from the theorem 1.3.2.

4.2.4 Theorem: The fuzzy Gleason space f-θX is B-extremally disconnected.

Proof: Since f-θX is a Stone space the theorem follows from 1.4.3.

4.2.5 Remark: In crisp topology the Gleason space is extremally disconnected. But here, as in example 1.4.4 the reverse inclusion does not hold. Therefore the space is only B- extremally disconnected.

4. 2.6 Definition: A fuzzy topological space X is said to be weak regular if for each $x_0 \in X$ and an f- closed set γ not containing x_0 in X, there exists a fuzzy regularly closed set λ in X such that $\lambda(x_0) = 1$ and $\gamma \wedge \lambda = 0$.

4.2.7 Definition: Let X be a B-fuzzy topological space which is weak regular. Then the space $\{\mathcal{U} \in$ f-θX : $\wedge \mathcal{U} \neq 0\}$ equipped with a subspace topology inherited from f-θX, is called the fuzzy absolute of X and is denoted by f-EX. Therefore elements of f-EX are fixed ultra filters on X and a basis for open sets is $\{\boldsymbol{\Lambda}_{\mathbf{r}}(\mu) \cap$ f- EX : $\mu \in$ FRC(X)$\}$.

4.2.8 Definition: Let X be an fT_2 space and p be a fuzzy singleton in X with support x. Then by F(p) we denote the family of all fuzzy regularly closed neighbourhoods of p. i.e., F(p) = $\{\mu \in$ FRC(X): f-int $\mu(x)=1\}$.

4.2.9 Result: F(p) is an FRC-filter.

Proof : Let μ_1 and $\mu_2 \in$ F(p).

Then f-int $\mu_1(x) = 1$ and f-int $\mu_2(x) = 1$. $\therefore$ f-int $(\mu_1 \wedge \mu_2)$ (x) = 1

i.e. $\mu_1 \wedge \mu_2 \in$ F(p).

Let $\mu_1 \in$ F(p) and $\mu_2 \geq \mu_1$. Then f-int $\mu_2(x)=1$.

$\therefore \mu_2 \in$ F(p). i.e. F(p) is an FRC-filter.

4.2.10. Definition: A fuzzy singleton which is regularly closed is called a fuzzy regularly closed point or frc- point.

4.2.11: Lemma: Let p be an frc-point in X with support x_o. Then there exists a $\mathcal{U} \in$ f-EX such that $\wedge\ \mathcal{U}$ = p.

Proof: Let F = $\{\mu \in$ FRC(X): $\mu(x_o) = 1\}$. Then F is an FRC –filter. Since every FRC-filter is contained in some FRC-ultrafilter, there exists a $\mathcal{U} \in$ f-θX such that F$\subseteq \mathcal{U}$.

Therefore $\wedge \mathcal{U} \leq \wedge F$.

Thus $(\wedge \mathcal{U})(x_0) \leq (\wedge F)(x_0)$.

i.e. $(\wedge \mathcal{U})(x_0) \leq 1$.

If $(\wedge \mathcal{U})(x_0) < 1$, there exists $\gamma \in \mathcal{U}$ such that $\gamma(x_0) < 1$. Therefore γ is a fuzzy regularly closed set not containing p. Therefore by weak regularity of X there exists $\lambda \in FRC(X)$ such that $\lambda(x_0) = 1$ and $\lambda \wedge \gamma = 0$ .Since $\lambda(x_0) = 1$, $\lambda \in F \subset \mathcal{U}$.

Therefore $\lambda \wedge \gamma \in \mathcal{U}$ which is not possible.

$\therefore (\wedge \mathcal{U})(x_0) = 1$. i.e. $(\wedge \mathcal{U}) \neq 0$

Therefore, $\mathcal{U} \in$ f-EX. By theorem 2.3.6, $\wedge \mathcal{U}$ is a unique fuzzy point.

Therefore $\wedge \mathcal{U} = p$.

4.2.12 Note: Corresponding to every $\mathcal{U} \in$ f-EX, there is a unique fuzzy singleton 'p' in X with support as that of $\wedge \mathcal{U}$. This unique fuzzy singleton is denoted as $K_{f\text{-}X}(\mathcal{U})$. This defines a mapping $K_{f\text{-}X}$ from f-EX into the set of fuzzy singletons in X. Therefore we use the pair (f-EX, $K_{f\text{-}X}$) to denote fuzzy absolute of X.

Now, by the above lemma if p is an frc-point in X, then correspondingly there exists a $\mathcal{U} \in$ f-EX such that $\wedge \mathcal{U} = p$. Therefore $K_{f\text{-}X}$ becomes a surjection from f-EX to the set of frc -points on X.

This shows the difference from the crisp case .In the crisp case, corresponding to every $x \in X$, there exists a $\mathcal{U} \in$ f-EX such that $\cap\, \mathcal{U} = \{x\}$.

4.2.13 Theorem: f-EX is a dense, zero dimensional sub space of f-θX.

Proof: f-θX is a zero dimensional space. Being a subspace of f-θX, f-EX is also zero dimensional.

To show that f-EX is dense in f-θX, it is enough to show that for each $\mu \in FRC(X)$ such that $\mu \neq 0$, f-EX $\cap\, \Lambda_r(\mu) \neq \phi$.

Choose an frc- point p with support x_o such that $\mu(x_o)=1$.

Let $F = \{\gamma \in FRC(X): \gamma(x_o) = 1\}$. Then F is an FRC-filter which is contained in some FRC-ultrafilter (say) $\mathcal{U}$. Then $\mu \in \mathcal{U}$. Therefore $\mathcal{U} \in \Lambda_r(\mu)$. Also $\wedge\, \mathcal{U} \neq 0$. $\therefore$ $\mathcal{U} \in$ f-EX. $\mathcal{U} \in$ f-EX $\cap\, \Lambda_r(\mu)$.

i.e. f-EX $\cap\, \Lambda_r(\mu) \neq \phi$. Hence f-EX is dense in f-θX.

4.2.14. Note: f-EX is a dense subspace of f-θX and f-θX is compact. Therefore f-θX is a compactification of f-EX.

4.2.15. Theorem: f-EX is B- extremally disconnected

Proof: By 1.4.2 any dense subspace of a B-extremally disconnected space is B- extremally disconnected. f-EX is a dense subspace of f-θX which is B -extremally disconnected. Therefore f-EX is B -extremally disconnected.

4.3 Properties of the pair (f-EX, $K_{f\text{-}X}$)

4.3.1 Proposition: Let $\mathcal{U} \in$ f-θX and p be a fuzzy singleton in X with support x_o. If $\mathcal{U} \in$ f-EX and $K_{f\text{-}X}(\mathcal{U}) = p$, then $F(p) \subset \mathcal{U}$.

Proof: $F(p) = \{\mu \in FRC(X) : \text{f-int } \mu(x)=1\}$

Suppose $F(p) \not\subset \mathcal{U}$.. Then there exists $\gamma \in F(p)$ such that $\gamma \notin \mathcal{U}$.

$\gamma \in F(p) \Rightarrow$ f-int $\gamma(x_o)=1$.

When $\gamma \notin \mathcal{U}$, by theorem 2.2.7(ii) there exists a $\delta \in \mathcal{U}$ such that $\gamma \wedge \delta = 0$.

$\gamma(x_o) = 1$. Therefore $\delta(x_o) = 0$.

Therefore $\delta \in \mathcal{U}$ and $\delta(x_o)=0$. Therefore $\wedge \mathcal{U} \neq p$. i.e. $K_{f\text{-}X}(\mathcal{U}) \neq p$.

Hence if $\mathcal{U} \in$ f-EX and $K_{f\text{-}X}(\mathcal{U}) = p$, then $F(p) \subset \mathcal{U}$.

4.3.2 Proposition. If $\mu \in FRC(X)$, then $K_{f\text{-}X}(\text{f-EX} \cap \Lambda_r(\mu)) = \{p_i\}$ where $\{p_i\}$ is subordinate to μ.

Proof: By definition 3.3.1 , $\{p_i\}$ is subordinate to μ if $p_i \le \mu$ for every i.

Let $\mathcal{U} \in \text{f-EX} \cap \Lambda_r(\mu)$ and $K_{f\text{-}X}(\mathcal{U}) = p$ where $\sigma_0(p) = \{x\}$

Then $\mathcal{U} \in$ f-EX and $\mathcal{U} \in \Lambda_r(\mu)$.

That is $\wedge \mathcal{U} \neq 0$ and $\mu \in \mathcal{U}$.

Since $K_{f\text{-}X}(\mathcal{U}) = p$, $\wedge \mathcal{U} = p$.

Therefore, $\mu \in \mathcal{U} \Rightarrow \mu(x)=1$.

That is $p \le \mu$.

Therefore, $K_{f\text{-}X}(\text{f-EX} \cap \Lambda_r(\mu)) = \{p_i\}$ where $p_i \le \mu$ for every i.

That is $\{p_i\}$ is subordinate to μ.

4.3.3 Proposition: Let p be an frc-point in X with support x and $\mu \in FRC(X)$. Then $\mu \in F(p) \Rightarrow \overleftarrow{K}_{f\text{-}X}(p) \subset \Lambda_r(\mu)$.

Proof: Let $\mathcal{U} \in \overleftarrow{K}_{f\text{-}X}(p)$. Then $\mathcal{U} \in$ f-EX and $K_{f\text{-}X}(\mathcal{U}) = p$.

Therefore by proposition 4.3.1 $F(p) \subset \mathcal{U}$.

Therefore, if $\mu \in F(p)$ then $\mu \in \mathcal{U}$.

That is $\mathcal{U} \in \Lambda_r(\mu)$.

$\therefore K^{\leftarrow}_{f-x}(p) \subset \Lambda_r(\mu)$.

The converse of this result is not true as it is seen from the following example.

4.3.4 Example: Let $\boldsymbol{R}$ be the set of real numbers with usual topology. Define μ_1, μ_2 and μ_3 from $\boldsymbol{R}$ to $[0,1]$ as

$$\mu_1(0) = \frac{3}{4} \text{ and } \mu_1(x) = 0 \;\; \forall\, x \neq 0.$$

$$\mu_2(0) = \frac{3}{5} \text{ and } \mu_2(x) = 0 \;\; \forall\, x \neq 0.$$

$$\mu_3(0) = \frac{1}{2} \text{ and } \mu_3(x) = 0 \;\; \forall\, x \neq 0.$$

Let $\boldsymbol{R}_F$ be the fuzzy topology generated by usual crisp topology in $\boldsymbol{R}$ and μ_1, μ_2 and μ_3. Then as in example 4.1.2 $\boldsymbol{R}_F$ is a B- fuzzy topology on $\boldsymbol{R}$.

Let K be any closed set in $\boldsymbol{R}$. Then closed sets in $\boldsymbol{R}_F$ are $\{K\}$, μ_1', μ_2', $\{K\} \wedge \mu_1'$, $\{K\} \wedge \mu_2'$, 1,0 . Let K_o be any closed interval in $\boldsymbol{R}$ with zero not an end point. Then as in example 2.3.4 $\chi_{\{K_o\}}$ is fuzzy regularly closed in $\boldsymbol{R}_F$.

$$\text{f-int } \mu_2' = \chi_{\boldsymbol{R} \setminus \{0\}}$$

$$\text{f-cl(f-int } \mu_2') = \mu_2'$$

$\therefore$ μ_2' is fuzzy regularly closed

Similarly we can show that $\chi_{Ko} \wedge \mu_1'$, $\chi_{Ko} \wedge \mu_2'$ are fuzzy regularly closed.

$\therefore$FRC($\boldsymbol{R}_F$) $= \{ \chi_{Ko}, \mu_1', \mu_2', \chi_{Ko}\wedge\mu_1', \chi_{Ko} \wedge \mu_2', \chi_{\{0\}}, 1, 0\}$

Let f-θX $= \{ \mathcal{U} : \mathcal{U}$ is an FRC ultrafilter on $\boldsymbol{R}_F\}$

and f-EX $= \{ \mathcal{U} \in$ f-θX $: \wedge \mathcal{U} \neq 0\}$.

Let $K_{f\text{-}X}$ be the mapping from f-EX onto the set of frc-points on $\boldsymbol{R}_F$.

Take $p = \chi_{\{0\}}$.

Then $K^{\leftarrow}_{f-X}(p) = \{ \mathcal{U} \in$ f-EX $: \wedge \mathcal{U} = p\}$.

Let $\mathcal{U} \in K^{\leftarrow}_{f-X}(p)$. Then $\wedge \mathcal{U} = p$.

That is $\mu(0) = 1$, for every $\mu \in \mathcal{U}$.

If $\mathcal{U} \notin \Lambda_r(\mu_3')$, then $\mu_3' \notin \mathcal{U}$.

That is $\mu_3 \in \mathcal{U}$ (by proposition 2.2.9).

i.e. $\mu_3(0) = 1$ which is not true since $\mu_3(0) = \frac{1}{2}$.

Therefore $\mathcal{U} \in \Lambda_r(\mu_3')$

$\therefore$ $K^{\leftarrow}_{f-X}(p) \subseteq \Lambda_r(\mu_3')$

But, f-int $\mu_3' = \chi_{R\setminus\{0\}} \vee \mu_3$.

$\therefore$ f-int $\mu_3'(0) = (\chi_{R\setminus\{0\}} \vee \mu_3)(0) = \frac{1}{2} \neq 1$.

Therefore $\mu_3' \notin F(p)$.

4.3.5 Remark: In the case of non crisp sets, the following converse holds.

If p is an frc-point with support x and if $K^{\leftarrow}_{f\text{-}x}(p) \subseteq (\Lambda_r(\mu^c))^c$, then $\mu \in F(p)$.

Proof:- Suppose $\mu \notin F(p)$ where $F(p) = \{\mu \in FRC(X) : \text{f-int}\mu(x)=1\}$.

If $\mu \notin F(p)$, then f-int $\mu(x) \neq 1$.

Let $\mathcal{U} \in K^{\leftarrow}_{f\text{-}x}(p)$. Then $K_{f\text{-}x}(\mathcal{U}) = p$. i.e. $\wedge \mathcal{U} = p$.

Suppose $\mathcal{U} \notin \Lambda_r(\mu^c)$. Then $\mu^c \notin \mathcal{U}$. Therefore by the theorem 2.2.7 (ii) there exists $\eta \in \mathcal{U}$ such that $\mu^c \wedge \eta = 0$.

$\eta \in \mathcal{U} \Rightarrow \eta(x)=1$

$\therefore \mu^c(x) = 0$. But $\mu^c(x) = 1 - \text{f-int}(x) \neq 0$ since f-int $\mu(x) \neq 1$.

$\therefore \mathcal{U} \in \Lambda_r(\mu^c)$

i.e. $K^{\leftarrow}_{f\text{-}x}(p) \subset \Lambda_r(\mu^c)$

$\therefore$ If $\mu \notin F(p)$, then $K^{\leftarrow}_{f\text{-}x}(p) \subset \Lambda_r(\mu^c)$

That is if $K^{\leftarrow}_{f\text{-}x}(p) \not\subset \Lambda_r(\mu^c)$ then $\mu \in F(p)$. i.e. if there is an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \in K^{\leftarrow}_{f\text{-}x}(p)$ and $\mathcal{U} \notin \Lambda_r(\mu^c)$, then $\mu \in F(p)$.

i.e. if $\mathcal{U} \in K^{\leftarrow}_{f\text{-}x}(p)$ and $\mathcal{U} \in (\Lambda_r(\mu^c))^c$,then $\mu \in F(p)$.

There fore if $K^{\leftarrow}_{f\text{-}x}(p) \subseteq (\Lambda_r(\mu^c))^c$, then $\mu \in F(p)$.

4.3.6 Note: In the case of crisp sets, $(\Lambda_r(\mu^c))^c = \Lambda_r(\mu)$. Therefore the above results becomes $\mu \in F(p) \Leftrightarrow K^{\leftarrow}_{f\text{-}x}(p) \subseteq \Lambda_r(\mu)$.

4.3.7 Theorem: The mapping $K_{f\text{-}X}$ from f-EX into the set of f-singleton in X is s- continuous and compact.

Proof: To prove the s- continuity of $K_{f\text{-}X}$.

Let $\mathcal{U} \in$ f-EX and γ be an f-open set in X such that $K_{f\text{-}X}(\mathcal{U}) \in \gamma$.

Let $K_{f\text{-}X}(\mathcal{U}) = p$. Then $\wedge\ \mathcal{U} = p$ and $p \in \gamma$.

Let $\sigma_0(p)=\{x\}$.

Put δ = f-cl(γ). Then f-int $\delta = \gamma$.

$\therefore p \in \gamma \Rightarrow p \in$ f-int $\delta \Rightarrow$ f-int $\delta\,(x)=1$

$\therefore \delta \in F(p)$.

Hence by proposition 4.3.3 $\overleftarrow{K}_{f\text{-}X}(p) \subseteq \Lambda_r(\delta)$.

$\therefore\ \mathcal{U} \in \Lambda_r(\delta)$ .i.e. $\delta \in \mathcal{U}$.

$\therefore\ \mathcal{U} \in$ f-EX $\cap \Lambda_r(\delta)$, where f-EX $\cap \Lambda_r(\delta)$ is an open set.

f-EX$\cap \Lambda_r(\delta)$ is also closed.

By proposition 4.3.2 $K_{f\text{-}X}$(f-EX $\cap \Lambda_r(\delta)) = \{p_i\}$, where$\{p_i\}$ is subordinate to δ.

i.e. $K_{f\text{-}X}$ (f-EX $\cap \Lambda_r(\delta)$) is subordinate to δ = f-cl(γ).

Therefore by definition 3.3.2 $K_{f\text{-}X}$ is s-continuous.

To show that $K_{f\text{-}X}$ is compact it is enough to show that $\overleftarrow{K}_{f\text{-}X}(p)$ is a closed subset of f-θX.

Let $\mathcal{U} \in (\overleftarrow{K}_{f\text{-}X}(p))^c$. Then $\mathcal{U} \notin \overleftarrow{K}_{f\text{-}X}(p)$ and so $K_{f\text{-}X}(\ \mathcal{U}) \neq p$. i.e. there exists a $\mu \in\ \mathcal{U}$ such that $\mu(x) \neq 1$. Since $\mu \in\ \mathcal{U}$, $\mathcal{U}\ \in \Lambda_r(\mu)$ and $\Lambda_r(\mu)$ is open. Now we can show that $\Lambda_r(\mu)\ \subseteq (\overleftarrow{K}_{f\text{-}X}(p))^c$.

If not, there exists an ultrafilter say $\mathcal{F}$ in $\Lambda_r(\mu)$ and $\mathcal{F} \notin (K^{\leftarrow}_{f\text{-}X}(p))^c$.

That is $\mathcal{F} \in \Lambda_r(\mu)$ and $\mathcal{F} \in K^{\leftarrow}_{f\text{-}X}(p)$.

i.e. $\mu \in \mathcal{F}$ and $K_{f\text{-}X}(\mathcal{F}) = p$. i.e. $\mu \in \mathcal{F}$ and $\wedge \mathcal{F} = 1$

$\therefore (\wedge \mathcal{F})(x) = 1$

i.e. $\gamma(x)=1$ for every $\gamma \in \mathcal{F}$. Therefore $\mu(x)=1$ which is not true.

Therefore $\Lambda_r(\mu) \subseteq (K^{\leftarrow}_{f\text{-}X}(p))^c$

Hence $(K^{\leftarrow}_{f\text{-}X}(p))^c$ is an open set and so $K^{\leftarrow}_{f\text{-}X}(p)$ is closed.

$\therefore$ The mapping $K_{f\text{-}X}$ is compact .

Note: Unlike crisp situation, the mapping $K_{f\text{-}X}$ is not closed.

4.3.8 Example: Let $\boldsymbol{R}$ be the set of real numbers with usual topology. Define μ_1 and μ_2 from $\boldsymbol{R}$ to $[0,1]$ as $\mu_1(0) = \frac{3}{4}$ and $\mu_1(x) = 0 \ \forall \ x \neq 0$,

$$\mu_2(0) = \frac{3}{5} \text{ and } \mu_2(x) = 0 \ \forall \ x \neq 0 .$$

Let $\boldsymbol{R}_F$ be the fuzzy topology generated by $\boldsymbol{R}$, μ_1 and μ_2. Now as in example 4.3.4, $\boldsymbol{R}_F$ is a B-fuzzy topological space with $FRC(\boldsymbol{R}_F) = \{\chi_{K_o}, \mu_1', \mu_2', \chi_{K_o} \wedge \mu_1', \chi_{K_o} \wedge \mu_2', \chi_{\{0\}}, 1, 0\}$

Let $f\text{-}\theta X = \{ \mathcal{U} : \mathcal{U}$ is an FRC ultrafilter on $\boldsymbol{R}_F\}$ and

$f\text{-}EX = \{ \mathcal{U} \in f\text{-}\theta X : \wedge \mathcal{U} \neq 0\}$.

Let $H = \{ \mathcal{U} \in f\text{-}EX : \mu_1' \in \mathcal{U}\}$

$= f\text{-}EX \cap \Lambda_r(\mu_1')$

$\therefore H$ is closed.

Now let $\mathcal{U} \in H$. Then $\mathcal{U} \in$ f-EX and $\mathcal{U} \in \Lambda_r(\mu_1')$

$\therefore \wedge\, \mathcal{U} \neq 0$ and $\mu_1' \in \mathcal{U}$.

Therefore $K_{f\text{-}X}(\mathcal{U})$ is a fuzzy singleton with support other than zero.

$\therefore K_{f\text{-}X}(H) \subset \{ p_i : p_i(0) = 0\}$ (1)

That is $K_{f\text{-}X}(H)$ is a sub set of the set of fuzzy singletons with a non zero support.

Let q be any fuzzy singleton with support $x_o \neq 0$.

We have, f-int $\mu_1' = \chi_{R \setminus \{0\}}$. Therefore f-int $\mu_1'(x_o) = 1$, for $x_o \neq 0$

$\therefore \mu_1' \in F(q)$.

Hence by proposition 4.3.3 $K^{\leftarrow}_{f\text{-}X}(q) \subseteq \Lambda_r(\mu_1')$

$\therefore K^{\leftarrow}_{f\text{-}X}(q) \subset H$.

i.e. $q \in K_{f\text{-}X}(H)$.

Therefore $\{p_i : p_i(0) = 0\} \subseteq K_{f\text{-}X}(H)$ (2)

Hence $K_{f\text{-}X}(H) = \{p_i : p_i(0) = 0\}$

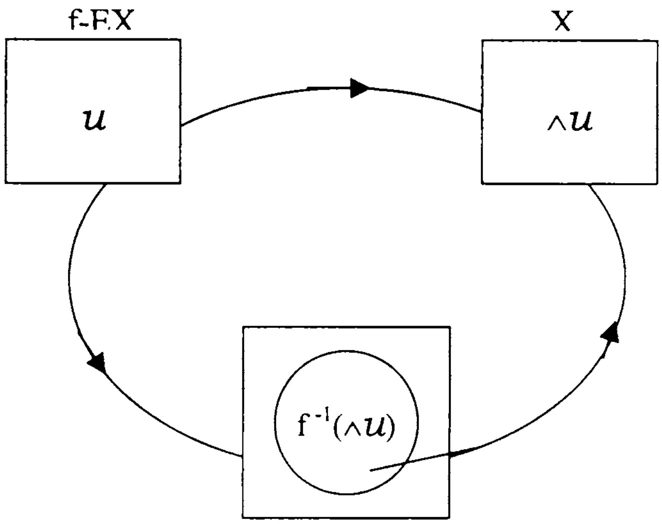
$= R\setminus\{0\}$ which is not closed.

Therefore $K_{f\text{-}X}(H)$ is not a closed map.

4.3.9 Uniqueness of Fuzzy absolutes

Theorem: Let X be a B-fuzzy topological space and $\mathcal{Y}$ be a B-extremally disconnected zero dimensional space of fuzzy sets defined on some set Y in which every FRC filter is convergent. Let ' f ' be an s- continuous surjection from $\mathcal{Y}$ onto the set of frc points on X. Then there is a continuous mapping h from f-EX onto the set of fuzzy singletons in $\mathcal{Y}$ such that $K_{f-X} = f \circ h$.

Proof:



$\mathcal{Y}$ is a B- extremally disconnected zero dimensional space of fuzzy sets in which every FRC-filter is convergent.

Let $B \in B(\mathcal{Y})$ where $B(\mathcal{Y})$ is a clopen base in $\mathcal{Y}$. Then B is both open and closed. Let $y \in B$. Since f is s-continuous for every fuzzy open set μ in X such that $f(y) \in \mu$, there is an open set B containing y in $\mathcal{Y}$ such that f(cl B) is subordinate to $cl(\mu) = \gamma$. i.e. $f(cl(B) \subseteq \gamma$. i.e. $f(B) \subseteq \gamma$. Then this γ is f-regularly closed and $\gamma \in \mathcal{U}$, for some $\mathcal{U} \in$ f-EX.

Define $F = \{\eta : \eta \in B, B \in B(\mathcal{Y}), f(B) \subseteq \gamma\ ,\ \gamma \in \mathcal{U}\}$.Then F will form a filter base which will generate an FRC filter (say) $\mathcal{F}$. Let $\mathcal{U}'$ be an FRC ultrafilter containing $\mathcal{F}$. Then $\wedge\mathcal{U}'$ is a unique fuzzy point in $\mathcal{Y}$. Let $\wedge\mathcal{U}' = x_q$.

Define h from f-EX to $\mathcal{Y}$ as $h(\mathcal{U}) = \wedge \mathcal{U}' = x_q$.

Then $(f \circ h)\, \mathcal{U} = f(h(\mathcal{U})) = f(\wedge \mathcal{U}')$

$$\leq f(\wedge\{\eta : \eta \in B;\ f(B) \subseteq \gamma,\ \gamma \in \mathcal{U}\})$$

$$= \wedge\{f(\eta) : \eta \in B,\ f(B) \subseteq \gamma,\ \gamma \in \mathcal{U}\}$$

$$\leq \wedge\, \mathcal{U} = K_{f\text{-}X}(\mathcal{U}).$$

$$(f \circ h)(\mathcal{U}) \leq \wedge \mathcal{U} = K_{f\text{-}X}(\mathcal{U}).$$

But $K_{f\text{-}X}(\mathcal{U})$ is a fuzzy singleton.

Therefore $f \circ h = K_{f\text{-}X}$.

To show that h is closed.

Let F be a closed set in f-EX. Then there exists a family $\{\mu_i,\ i \in I\} \subset FRC(X)$ such that $F = \cap\{\Lambda_r(\mu_i) \cap \text{f-EX}\}$.

To each i, there is a $B \in B(\mathcal{Y})$ such that $f(B_i) \leq \mu_i$, $\mu_i \in \mathcal{U}$, $\mathcal{U} \in$ f-EX

Then $h(F) = h(\cap\{\Lambda_r(\mu_i) \cap \text{f-EX}\})$

$$= \cap\{h(\Lambda_r(\mu_i) \cap \text{f-EX})\}$$

$$= \cap\{B_i : f(B_i) \subseteq \mu_i\}$$ which is closed in $\mathcal{Y}$.

Therefore, h is a closed map.

To show that h is continuous.

Suppose $\mathcal{U} \in$ f-EX and $B \in B(\mathcal{Y})$ where $h(\mathcal{U}) \in B$.

When $h(\mathcal{U}) \in B$, $f(B) \leq \eta$, $\eta \in \mathcal{U}$ and so $\mathcal{U} \in \Lambda_r(\mu)$

Let $W \in \Lambda_r(\eta)$. Therefore $\eta \in W$.

Then $h(W) \in B$, $f(B) \leq \mu_i$, $\mu_i \in W$.

Let $W' = \{\gamma : \gamma \in B, f(B) \leq \eta, \eta \in W\}$

Then $h(W) = \cap W' = \cap\{\gamma : \gamma \in B, f(B) \leq \mu_i, \mu_i \in W\} \in B$.

i.e. $W \in \Lambda_r(\eta) \Rightarrow h(W) \in B$, open

$\therefore h(\Lambda_r(\eta)) \subset B$

$\therefore$ h is continuous.

Now to show that h is onto.

Let p be any fuzzy singleton in $\mathcal{Y}$ with support y and

$\mathcal{U}[y] = \{\mu_i : f(B_i) \leq \mu_i, y \in B_i\}$, $B_i \in B(\mathcal{Y})$

Then $\mathcal{U}[y]$ is a filter base which generate a filter say $\mathcal{U}'[y]$. Let V(y) be the ultrafilter containing $\mathcal{U}'[y]$. Then $\wedge\mathcal{U}'[y] = \wedge\{\mu_i : f(B_i) \leq \mu_i\}$

$= f(B_k)$, where $f(B_k) = \min f(B_i)$.

Therefore $\wedge V(y) = f(B_k)$

Hence $\wedge V(y) \neq 0$.

That is $V(y) \in$ f-EX.

$\therefore$ corresponding to every $p \in \mathcal{Y}$ there is a $V(y) \in$ f-EX such that $h(V(y)) = p$.

Therefore the mapping h is onto.

4.3.10 Note: The pair($\mathcal{Y}$,f)constructed in 4.3.9 is said to be equivalent to (f-EX, $K_{f\text{-}X}$) and is denoted by (f-EX, $K_{f\text{-}X}$) $\sim$ ($\mathcal{Y}$,f) . Then ($\mathcal{Y}$,f) is the fuzzy absolute of X up to equivalence.

4.4 Fuzzy absolutes using s-fixed FRC ultrafilters

Instead of taking the fixed FRC ultrafilters, the fuzzy absolutes also can be constructed using the strong fixed (s-fixed) FRC ultrafilters defined in (2.3.7). Then fuzzy Gleason space f-θX is defined as in definition 4.2.2. But the fuzzy absolute is defined as follows.

4.4.1 Definition: Let X be a B-fuzzy topological space which is fT_2. Then the set $\{\mathcal{U} \in$ f-$\theta X : \cap\{\sigma_o(\mu) : \mu \in \mathcal{U}\} \neq \phi\} = \{\mathcal{U}[x] \in$ f-$\theta X, x \in X\}$ equipped with a subspace topology inherited from f-θX is called the fuzzy absolute of X and is denoted by f- E'X.

Therefore here f-E'X consists of fuzzy principal ultrafilters on FRC(X) viewed as subspace of f-θX.

4.4.2 Lemma: Let X be B-fuzzy topological space. Then

1) If $\mathcal{U} \in$ f- E'X, then $\cap\{\sigma_0(\mu):\mu\in\mathcal{U}\}$ contains exactly one point.

2) if $x \in X$, there exist $\mathcal{U} \in$ f- E'X such that $\cap\{\sigma_0(\mu):\mu\in\mathcal{U}\} = \{x\}$.

Proof:

1) If $\mathcal{U} \in$ f- E'X, then it is a principal FRC-ultrafilter of the form

$\mathcal{U}[x] = \{\mu\in \text{FRC}(X):\mu(x) > 0\}$. Therefore, $\cap\{\sigma_0(\mu):\mu\in\mathcal{U}\} = \{x\}$.

2) Let $x \in X$. Then $F_{(x)}=\{\mu\in\text{FRC}(X)$: f-int $\mu(x)=1\}$ is an FRC-filter. Therefore always there exists a $\mathcal{U}[x]$ in f-θX such that $F(x) \subset \mathcal{U}[x]$ and this $\mathcal{U}[x]\in$ f- E'X. Then $\mu(x)>0$, $\forall$ $\mu\in\mathcal{U}[x]$.

Therefore $\cap\{\sigma_0(\mu) : \mu\in\mathcal{U}\} = \{x\}$.

$\therefore$ If $x\in X$, there exists $\mathcal{U}\in$ f- E'X such that $\cap\{\sigma_0(\mu):\mu\in\mathcal{U}\} = \{x\}$.

4.4.3 Note: If $\mathcal{U}\in$ f-E'X, we denote the unique point of X belonging to $\cap\{\sigma_0(\mu):\mu\in\mathcal{U}\}$ by $K_{f\text{-}x'}(\mathcal{U})$. Therefore by lemma 4.4.2. $K_{f\text{-}x'}$ is a well defined surjection from f-E'X onto the set of fuzzy singletons in X. So we use the pair (f-E'X , $K_{f\text{-}x'}$) to denote the fuzzy absolute .

4.4.4 Remark : In the case of absolutes using fixed FRC ultrafilter, we have $K_{f\text{-}X}$ is a mapping from f-EX into X. In the lemma 4.2.8 it was proved that $K_{f\text{-}X}$ is a surjection from f-EX onto the set of frc-points in X. But here $K_{f\text{-}x'}$ is a surjection from f-EX onto the set of crisp points in X.

As in the case of fixed ultrafilters, here also we can prove the properties of f-E'X. Some of them are explicitly proved here, to show the differences occurring in the two cases.

4.4.5 Theorem: Let X be a B- fuzzy topological space. Let $\mathcal{U}\in$ f-θX and $x\in X$. If $\mathcal{U}\in$ f-E'X , and $K_{f\text{-}x'}(\mathcal{U}) = \{x\}$, then $F(x) \subseteq\mathcal{U}$.

Proof: F (x) = { $\mu \in$ FRC(X) : f-int $\mu(x)=1$ }

Suppose $F(x)\not\subset \mathcal{U}$ .Then there exists a $\mu\in F(x)$ such that $\mu\notin\mathcal{U}$. Therefore there exists a $\gamma\in \mathcal{U}$ such that $\mu \wedge \gamma = 0$. $\therefore(\mu\wedge\gamma)(x) = 0$ for every x.

But $\mu(x)\neq0$. $\therefore\gamma(x)=0$.

i.e. $\gamma\in\mathcal{U}$ and $\gamma(x)=0$. i.e. $x \notin\sigma_0(\gamma)$.

Therefore $\cap\{\sigma_0(\mu): \mu\in\mathcal{U}\} \neq \{x\}$.

$\therefore$ if $K_{f\text{-}x'}(\mathcal{U}) = \{x\}$, then $F(x) \subseteq \mathcal{U}$.

Converse of this result is not true in the case of fuzzy sets.

4.4.6 Example: Let X, {a,b,c}. Define μ_1, μ_2, μ_3 from X to [0,1] as

$$\mu_1(a) = 0 \quad \mu_1(b) = 0 \quad \mu_1(c) = \frac{1}{2}$$

$$\mu_1(a) = \frac{1}{2} \quad \mu_2(b) = \frac{1}{2} \quad \mu_1(c) = 0$$

$$\mu_3(a) = 1 \quad \mu_3(b) = 1 \quad \mu_3(c) = \frac{1}{2}$$

Then $\tau = \{0, 1, \mu_1, \mu_2, \mu_3, \mu_1 \vee \mu_2)$ is a fuzzy topology on X. Also as in example 4.1.4 (X, τ) is a B-fuzzy topological space.

Here the closed sets are $\{0, 1, \mu_1, \mu_3, \mu_2', \mu_1' \wedge \mu_2'\}$

f-int $(\mu_1' \wedge \mu_2') = \mu_2$ and f-cl (f-int $(\mu_1' \wedge \mu_2')) =$ f-cl $(\mu_2) = \mu_1' \wedge \mu_2'$

$\therefore \mu_1' \wedge \mu_2'$ is fuzzy regularly closed.

Similarly we can show that f-cl(f-int $(\mu_1) \neq \mu_1$. Therefore μ_1 is not f-regularly closed. Also μ_3 and μ_2' are not fuzzy regularly closed.

$\therefore$ FRC(X) $= \{0, 1, \mu_1' \wedge \mu_2'\}$

$F(a) = \{\mu \in$ FRC(X):f- int $\mu(a) = 1\} = \{1\}$

$\mathcal{U} = \{1, \mu_1' \wedge \mu_2'\}$

Therefore, $F(a) \subset \mathcal{U}$.

$\mu(a) > 0$ for every $\mu \in \mathcal{U}$. $\therefore$ $\mathcal{U} \in$ f-E'X .

But $K_{f\text{-}x'}(\mathcal{U}) = \cap \{\sigma_0(\mu) : \mu \in \mathcal{U}\} = \{a, b, c\} \neq \{a\}$.

Therefore even if $F(x) \subset \mathcal{U}$, $K_{f\text{-}x'}(\mathcal{U}) \neq \{x\}$

4.4.7 Remark : Converse of 4.4.5 is true only in the case of crisp sets.

Proof: Let $\mathcal{U} \in$ f-θX and $F(X) \subseteq \mathcal{U}$.

Let $\gamma \in FRC(X)$ such that $\gamma(x)=0$

Then $(1-\gamma)(x) = 1$

i.e. f-int (f-cl$(1-\gamma)$) $(x) = 1$

i.e. f-int $(\gamma^c)(x) = 1$

Therefore, $\gamma^c \in F(x) \subset \mathcal{U}$. Therefore $\gamma^c \in \mathcal{U}$.

If γ is not crisp, then $\gamma \wedge \gamma^c$ is not equal to zero. $\therefore$ $\gamma^c \in \mathcal{U}$ does not imply $\gamma \notin \mathcal{U}$.

But if γ is crisp , $\gamma^c \in \mathcal{U} \Rightarrow \gamma \notin \mathcal{U}$.

$\therefore$ if $\gamma(x)=0$, $\gamma \notin \mathcal{U}$.

Hence for every $\gamma \in \mathcal{U}$, $\gamma(x)>0$.

$\therefore \mathcal{U} \in$ f-E'X and $\cap \{\sigma_0(\mu):\mu \in \mathcal{U}\} = \{x\}$.

i.e. $K_{f\text{-}x'}(\mathcal{U}) = \{x\}$

4.4.8 Theorem:- If $\mu \in FRC(X)$, then $K_{f\text{-}x'}$ (f-E'X $\cap \Lambda_r(\mu)) = \sigma_0(\mu)$.

Proof: Let $\mu \in$ f-E'X and $\mathcal{U} \in$ (f-E'X $\cap$ $\Lambda_r(\mu)$). Then $\mathcal{U} \in$ f-E'X and $\mathcal{U} \in \Lambda_r(\mu)$.

i.e. $\gamma(x)> 0$ for every $\gamma \in \mathcal{U}$ and $\mu \in \mathcal{U}$.

Therefore $\mu(x)> 0$. That is $x \in \sigma_0(\mu)$.

i.e $K_{f\text{-}x'}(\mathcal{U}) = \{x\}$ and $x \in \sigma_0(\mu)$.

Therefore $K_{f\text{-}x'}$ (f-E'X $\cap$ $\Lambda_r(\mu)) = \sigma_0(\mu)$.

Conversely, let $x \in \sigma_0(\mu)$. Then $\mu(x) > 0$.

Therefore , there exists $\mathcal{U}[x] \in$ f-E'X such that $\mu \in \mathcal{U}[x]$.

That is $\mathcal{U}[x] \in$(f-E'X $\cap \Lambda_r(\mu)$) and $K_{f\text{-}x'}(\mathcal{U}[x]) = \{x\}$.

$\therefore x \in K_{f\text{-}x'}$ (f-E'X $\cap \Lambda_r(\mu)$) . $\therefore K_{f\text{-}x'}$ (f-E'X $\cap \Lambda_r(\mu)) = \sigma_0(\mu)$.

4.4.9 Remark: Here corresponding to each $x \in X$, there is a unique $\mathcal{U}[x] \in$ f-E'X such that $K_{f\text{-}x'}(\mathcal{U}) = \{x\}$. Therefore unlike in crisp topology this mapping $K_{f\text{-}x'}$ is one to one in nature. But as in the section 4.3 $K_{f\text{-}x'}$ is s-continuous and compact but not closed.

If we consider the s-fixed FRC ultraflters, the mapping h defined in 4.3.9 becomes one to one. Therefore in this case the theorem 4.3.9 can be stated as follows.

"Let X be a B-fuzzy topological space and $\mathcal{Y}$ be a B-extremally disconnected zero dimensional space of fuzzy sets defined on some set Y in which every FRC filter is convergent. Let f be an s-continuous surjection from $\mathcal{Y}$ onto the set of fuzzy singletons in X. Then there is a homeomorphism h from f-E'X onto the set of fuzzy singletons in $\mathcal{Y}$ such that $K_{f\text{-}x'} = f \circ h$.

So the absolute f-E'X is unique and we say (f-E'X $K_{f\text{-}x'}$) is equivalent to ($\mathcal{Y}$,f). i.e. (f-EX, $k_{f\text{-}x}$) ~ ($\mathcal{Y}$,f).

4.5 Fuzzy absolutes of products and sums:

4.5.1 Theorem: If $\{X_i,\ i \in I\}$ is a set of fuzzy topological spaces, then

(a) (f-EX, $K_{f\text{-}X}$) ~ (f-E(πf-E X_i), $\pi\, K_{f\text{-}Xi} \circ K_{f\text{-}\pi(f\text{-}E\ Xi)}$)where $X = \pi X_i$.

(b) (f-E ($\oplus X_i$),$K_{f(\oplus Xi)}$) ~ ($\oplus$ f- EXi, $\oplus K_{f\text{-}Xi}$)

Proof: $K_{f\text{-}Xi}$ is an s-continuous surjection from f-EX_i onto X_i and $K_{f\text{-}x}$ is an s-continuous surjection from f-EX onto X where $X=\pi X_i$ .Since each $K_{f\text{-}Xi}$ is s-continuous, by theorem 3.3.8 $\pi K_{f\text{-}Xi}$ is s-continuous from π(f-EX_i) onto πX_i.

f-EX_i's are B-extremally disconnected and Hausdorff . Therefore π(f-EX_i) is also Hausdorff and so f-E(π(f-EX_i)) is B- extremally disconnected. $K_{f\text{-}(\pi(f\text{-}EXi)}$ is s-continuous from f-E(π(f-EX_i)) onto π(f-EX_i) .Therefore, $\pi K_{f\text{-}Xi} \circ K_{f\text{-}\pi(f\text{-}EXi)}$ is an s- continuous surjection from f-E(π(f-EX_i)) onto πX_i. Also f-E(π(f-EX_i)) is zero dimensional. Therefore, f-E(π(f-EX_i)) is a fuzzy absolute of πX_i = X up to equivalence .

Therefore (f-EX , $K_{f\text{-}X}$) ~ (f-E(π(f-EX_i), $\pi K_{f\text{-}xi} \circ K_{f\text{-}\pi(f\text{-}EXi)}$).

a) $K_{f\text{-}Xi}$'s are s-continuous from f-EX_i onto X_i . Therefore by theorem 3.3.8 $\oplus$ $K_{f\text{-}Xi}$ is s-continuous from $\oplus$f-EX_i onto $\oplus X_i$. F-EX_i's are B-extremally disconnected and zero dimensional. Therefore $\oplus$(f-EX_i) is also B-extremally disconnected and zero dimensional. $\therefore$ (f-E($\oplus X_i$), $K_{f\text{-}\oplus Xi}$)~ ($\oplus$f-EXi,$\oplus K_{f\text{-}Xi}$).

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CHAPTER -5

FUZZY ABSOLUTE AS A SET OF F- OPEN ULTRA FILTERS[⊗]

5.0. Introduction

If X is a fuzzy topological space, then $\tau(X)$ - the set of all fuzzy open subsets of X forms a pseudo Boolean algebra with pseudo complement defined as, for $\mu \in \tau(X)$, $\mu^c = 1 - \text{f-cl}(\mu) = \text{f-int}(1-\mu)$. In the third chapter we have already introduced fuzzy open filters (f-open filters for short), f-open ultrafilters and fixed f-open ultrafilters.

It is known that in crisp topology, absolutes can be constructed using open ultrafilters on X. So in this chapter we are constructing fuzzy absolutes by taking $\tau(X)$, instead of the fuzzy regularly closed subsets of X. The absolute so constructed is denoted by f-E'X. Then the underlying set of f- E'X is the fixed f-open ultra filters on X. The second section of this chapter gives some properties of f-E'X. Though they are similar to that of f-EX in chapter 4, we are explicitly proving this to note the differences in the two cases.

5.1. Fuzzy absolute using f-open ultrafilters

5.1.1. Definition: Let X be a fuzzy topological space. Then the set of all convergent f-open ultra filters on X will be denoted by f-E'X. If x_p is a fuzzy point in X with support x then $\mathcal{N}(x_p) = \{\mu \in \tau(X) : p \leq \mu(x)\}$.

[⊗] Some results of this chapter were communicated to Ganitha Sandesh, Rajasthan Ganitha Parishad.

5.1.2 Lemma:

Let X be a fuzzy topological space and x_p be a fuzzy singleton in X. Let $\mathcal{U} \in$ f-E'X. Then ,

i) if $x_p \in a(\mathcal{U})$, $\mathcal{N}(X_p) \subset \mathcal{U}$.

ii) there is exactly one $x \in X$ such that $a(\mathcal{U})(x)=1$.

Proof:

(i) Since $\mathcal{U} \in$ f-E'X it is a fixed f-open ultrafilter. Therefore the result follows from 3.1.8

ii) By the result 3.1.11, if $x_p \in a(\mathcal{U})$, then x_p is a cluster point. But by lemma 3.1.12 $C(\mathcal{U})$, the set of all cluster points of $\mathcal{U}$ contains exactly one point. Therefore, there exists exactly one $x \in X$ such that $x_p \in a(\mathcal{U})$. i.e. there exists exactly one $x \in X$ such that $a(\mathcal{U})(x) = 1$.

5.1.3 Remark: From the example 3.1.10 we can see that the converse of 5.1.2(i) need not be true in the case of non crisp sets. In the case of crisp sets corresponding to every $x \in X$, there exists a $\mathcal{U}(x) \in EX$ such that $a(\mathcal{U}(x)) = \{x\}$. But because of the example 3.1.10 this is not true in the case of non crisp sets.

5.1.4 Definition: Let X be a fuzzy topological space and let f-θ'X denote the set of all f- open ultrafilters on X. If $\mu \in \tau(X)$, let $O(\mu) = \{ \mathcal{U} \in \text{f-}\theta'X : \mu \in \mathcal{U} \}$. Then f-E'X= $\{ \mathcal{U} \in \text{f-}\theta'X : a(\mathcal{U}) \neq 0\}$.

5.1.5. Lemma: Let $\mu, \gamma \in \tau(X)$. Then,

(i) $O(\mu) = \phi$ if and only if $\mu = 0$.

(ii) $O(\mu \wedge \gamma) = O(\mu) \cap O(\gamma)$

(iii) $(O(\mu))^c \subset O(\mu^c)$.

(iv) $O(\mu) = f\text{-}\theta'X$ if μ is dense in X and in particular $O(1) = f\text{-}\theta'X$

(v) $\{O(\mu) : \mu \in \tau(X)\}$ is a base for a Hausdorff topology on $f\text{-}\theta'X$

Proof:

i) Let $\mu = 0$. Since there is no ultrafilter contains zero, $O(\mu) = \phi$.

Conversely suppose $\mu \neq 0$. Then there exists $x \in X$ such that $\mu(x) \neq 0$. Let $\mathcal{F} = \{\gamma \in \tau(X) : \gamma\ (x) > 0\}$. Then $\mathcal{F}$ is an f-open ultrafilter containing μ.

i.e. $\mathcal{F} \in O(\mu)$ $\therefore O(\mu) \neq \phi$.

i.e. $O(\mu) = \phi$ if and only if $\mu = 0$.

ii). Let $\mathcal{U} \in O(\mu \wedge \gamma)$

$\mathcal{U} \in O(\mu \wedge \gamma) \Leftrightarrow \mu \wedge \gamma \in \mathcal{U}$

$\Leftrightarrow \mu \in \mathcal{U}$ and $\gamma \in \mathcal{U}$ by definition of f-open ultrafilter.

$\Leftrightarrow \mathcal{U} \in O(\mu)$ and $\mathcal{U} \in O(\gamma)$

$\Leftrightarrow \mathcal{U} \in O(\mu) \cap O(\gamma)$

$\therefore O(\mu \wedge \gamma) = O(\mu) \cap O(\gamma)$

iii) Let $\mathcal{U} \in (O(\mu))^c$

Then $\mathcal{U} \notin O(\mu)$. ie. $\mu \notin \mathcal{U}$.

Therefore by note 3.1.5 $\mu^c \in \mathcal{U}$ and so $\mathcal{U} \in O(\mu^c)$

$\therefore (O(\mu))^c \subset O(\mu^c)$.

The reverse inclusion does not hold here. The example 1.2.7 will prove this since the set of all constant functions from X to [0,1] forms a fuzzy topology on X.

iv) If $\mu \in \tau(X)$, then $1-\bar{\mu} = 0$ if and only if μ is dense in X.

Therefore, $O(\mu^c) = \phi$ if and only if μ is dense in X.

But $(O(\mu))^c \subset O(\mu^c)$. There fore$(O(\mu))^c = \phi$ if μ is dense in X. .

That is $O(\mu) = f\text{-}\theta'X$ if μ is dense in X. .

So in particular $O(1) = f\text{-}\theta'X$

v) By i), ii) and iv), the set $\{O(\mu) : \mu \in \tau(X)\}$ forms a basis for $f\text{-}\theta'X$.

Let $\mathcal{U}$ and $\mathcal{V}$ be two distinct f- open ultrafilters. Then $\mathcal{U} \not\subset \mathcal{V}$.

Therefore there exists $\mu \in \mathcal{U}$ such that $\mu \notin \mathcal{V}$.

Since $\mu \notin \mathcal{V}$, by theorem 3.1.4 there exists $\gamma \in \mathcal{V}$, such that $\mu \wedge \gamma = 0$.

Therefore $\mathcal{U} \in O(\mu)$ and $\mathcal{V} \in O(\gamma)$ such that $O(\mu) \cap O(\gamma) = \phi$.

$\therefore$ $f\text{-}\theta'X$ is Hausdorff .

Note : By the above lemma, $f\text{-}\theta'X$ can be regarded as a topological space with $\{O(\mu) : \mu \in \tau(X)\}$ is a basis for open sets.

5.1.6. Definition:

Let X be a fuzzy topological space.

The set f-E'X is topologised by giving the subspace topology inherited from the space f-θ'X. That is $\{O(\mu) \cap \text{f-E'X}: \mu \in \tau(X)\}$ is an open base for the topology of f- E'X. This topological space f-E'X is called the fuzzy absolute of X.

Note: To each $\mathcal{U} \in$ f-E'X, there is a unique point x in X such that $a(\mathcal{U})(x) = 1$. This defines a function from f-E'X into X, which will be denoted by $K_{f\text{-}X'}$. i.e. $K_{f\text{-}X'}(\mathcal{U}) = x$.

5.1.7 Result: For $\mu \in \tau(X)$, $O(\mu)$ is clopen in f-θ'X

Proof:

$$O(\mu) = \{ \mathcal{U} \in \text{f-}\theta'X : \mu \in \mathcal{U} \}$$

$$(O(\mu))^c = \{ \mathcal{V} \in \text{f-}\theta'X : \mu \notin \mathcal{V} \}$$

When $\mu \notin \mathcal{V}$ by theorem 3.1.4 there exists a $\gamma \in \mathcal{V}$ such that $\mu \wedge \gamma = 0$. For every $\mu \notin \mathcal{V}$ there is some γ satisfying this condition.

Therefore $(O(\mu))^c = \cup_i(O(\gamma_i))$, where $\mu \wedge \gamma_i = 0$.

$O(\gamma_i)$ is open for every i and so is $\cup O(\gamma_i)$.

$\therefore$ $O(\mu)$ is closed.

i.e. $O(\mu)$ is clopen in f-θ'X .

Therefore $\{O(\mu) : \mu \in \tau(X)\} \subset B(\text{f-}\theta'X)$ where $B(\text{f-}\theta'X)$ is the set of all clopen sets in f-θ'X .

5.1.8. Theorem: Let X be a fuzzy topological space. Then f-θ'X is a B-extremally disconnected, zero dimensional, compact space.

Proof: Let U be any open set in f-θ'X. Then there exists a finite subset D of $\tau(X)$ such that $U = \cup\{O(\mu_i) : \mu_i \in D\}$

Since $\tau(X)$ is complete $\vee\mu_i$ exist.

Let $\vee\mu_i = \gamma$. Then $\gamma \in D$

$\mu_i \leq \gamma$

$\therefore\ O(\mu_i) \subseteq O(\gamma)$

Therefore $\cup\, O(\mu_i) \subseteq O(\gamma)$

$O(\gamma)$ is clopen by 5.1.7.

$\therefore$ cl $\cup(O(\mu_i)) \subseteq O(\gamma)$.

That is $cl(U) \subseteq O(\gamma)$. (1)

Therefore f-θ'X is B- extremally disconnected.

By 5.1.7, we have $O(\mu)$ is clopen in f-θ'X . Also $\{O(\mu):\mu\in\tau(X)\}$ forms an open base for f-θ'X. Therefore by definition 1.3.1, f-θ'X is zero dimensional.

To show that f-θ'X is compact.

Let $\mathcal{F}$ be a filter of closed sets in f-θ'X and

$G = \{\mu \in \tau(X) : O(\mu) \subset F \text{ for some } F \in \mathcal{F}\}$.

$\{O(\mu):\mu\in \tau(X)\}$ is a base for closed sets in f-θ'X.

Therefore $\cap F= \cap\{(O(\mu):\mu\in G\}$.

Then G is an f-open filter. So there exists an f-open ultrafilter say $\mathcal{U}$ such that $G\subset \mathcal{U}$.

Therefore $\mu\in \mathcal{U}$ for every $\mu\in G$

That is, $\mathcal{U} \in O(\mu)$ for all $\mu\in G$.

Therefore, $\mathcal{U} \in \cap \{O(\mu):\mu\in G\}$.

$\therefore$ $\mathcal{U} \in \cap F$. Hence $\cap F \neq \phi$.

$\therefore$ f-θ'X is compact.

5.1.9. Note: As in the case of f-EX, the reverse inclusion of (1) in the above theorem does not hold. As in example 1.4.4 we can prove this.

Example: Let X be any non empty set.

Define $\mu_1:X\rightarrow[0,1]$ as $\mu_1(0) =\frac{1}{3}$ and $\mu_1(x) = 0$, $\forall$ $x \neq 0$.

Consider the fuzzy topology τ on X generated by the set of all constant functions and μ_1.

Let f-θ'X be the collection of all f-open ultrafilters on X. Then the base for open sets in f-θ'X is $\{O(\gamma) : \gamma \in\tau(X)\}$.

Consider the open set $U= O(\mu_1) = \{\mathcal{U} \in$ f-θ'X $: \mu_1\in \mathcal{U}\}$.

Then the f-open ultrafilter $\mathcal{U}\,[0]= \{\mu\in\tau(X):\mu(0)>0\}$ will contain μ_1.

Let μ_2: $X\rightarrow[0,1]$ be defined as $\mu_2(x) = \frac{1}{3}$, $\forall\, x \in R$

Then $\mu_1\leq\mu_2$.

Therefore $O(\mu_1) \subseteq O(\mu_2)$

That is $U \subseteq O(\mu_2)$

i.e. $cl(U) \subseteq O(\mu_2)$

Let $\mathcal{V}[x] =\{\mu \in\tau(X):\mu(x)>0, x\neq0\}$.

Then $\mathcal{V}$ [x] is an f- open ultrafilter containing μ_2 but not μ_1.

i.e. $\mathcal{V}[x] \in O(\mu_2)$ and $\mathcal{V}[x] \notin O(\mu_1)$.

$\therefore O(\mu_2) \not\subset cl(U)$.

5.1.10. Result:

$B(f\text{-}\theta'X) = \{O(\mu) : \mu\in\tau(X)\}$

Proof: By the result 5.1.7, we have $\{O(\mu):\mu\in\tau(X)\} \subset B(f\text{-}\theta'X)$ (1)

Now let $C\in B(f\text{-}\theta'X)$. Then C is both open and closed. Since C is open there exists a subset D of $\tau(X)$ such that $C = \cup\{O(\mu):\mu \in D\}$. C is also closed. Therefore C is compact. Therefore there exists a finite subset H of D such that

$C = \cup\{O(\mu): \mu\in H\}$

$\subseteq \{ O(\vee\mu_i) :\mu \in H \}$

$\in \{O(\mu_i\}:\mu_i\in\tau(X)\}$.

Therefore, $C \in \{O(\mu_i): \mu_i \in \tau(X)\}$.

$$B(f\text{-}\theta'(X)) \subset \{O(\mu):\mu \in \tau(X)\} \qquad (2)$$

Hence $B(f\text{-}\theta'X) = \{O(\mu):\mu\in \tau(X)\}$.

5.2. Properties of the pair (f- E'X, $K_{f\text{-}x'}$)

5.2.1 Definition: A fuzzy singleton in X which is also f- open is called an f- open point in X

5.2.2 Theorem: f-E'X is a dense B-extremally disconnected zero dimensional subspace of f-θ'X.

Proof: Being a subspace of f-θ'X which is zero dimensional f-E'X is also zero dimensional.

To show that f-E'X is dense in f-θ'X .

Let $\mu\in\tau(X)$ such that $\mu\neq 0$. Choose an f-open point p in X such that $p(x)=1$

Let $\mathcal{F} = \{\gamma\in\tau(X) :\gamma(x)=1\}$.

Then $\mathcal{F}$ is an f-openfilter and so there is an f-open ultrafilter say $\mathcal{U}$ such that $\mathcal{F}\subset \mathcal{U}$.

$\mu\in \mathcal{F}$, therefore $\mu \in \mathcal{U}$. i.e. $\mathcal{U} \in O(\mu)$.

If $a(\mathcal{U})=0$, then there is $\eta\in \mathcal{U}$ such that $\bar{\eta}(x)=0$. i.e.$\eta(x)=0$

$\therefore p\wedge\eta=0$, which is not true since $p\in \mathcal{U}$.

$\therefore$ $a(\mathcal{U}) \neq 0$. i.e. $\mathcal{U} \in$ f-E'X.

Therefore, $\mathcal{U} \in O(\mu) \cap$ f-E'X.

i.e. for $\mu \in \tau(X)$, f-E'X $\cap O(\mu) \neq \phi$. Therefore f-E'X is dense in f-θ'X.

By result 1.4.2 ,any dense subset of a B-extremally disconnected space is B-extremally disconnected. f-E'X is a dense subspace of f-θ'X which is B-extremally disconnected. Therefore f-E'X is B-extremally disconnected.

5.2.3 Theorem: For $\mu \in \tau(X)$, $k_{f\text{-}X'}(O(\mu) \cap$ f-E'X$) = \{p_i\}$ where $\{p_i\}$ is subordinate to $\bar{\mu}$.

Proof: Let $\mathcal{U} \in (O(\mu) \cap$ f-E'X$)$ and $K_{f\text{-}X'}(\mathcal{U}) = p$ where $p(x_o)=1$ and $p(x)=0$ for every $x \neq x_o$.

Then , $\mathcal{U} \in O(\mu)$ and $\mathcal{U} \in$ f-E'X

i.e. $\mu \in \mathcal{U}$ and $a(\mathcal{U}) \neq 0$.

Since $K_{f\text{-}X'}(\mathcal{U}) = p$, $a(\mathcal{U})(x_0)=1$

i.e. $\bar{\gamma}(x_0) = 1$ for every $\gamma \in \mathcal{U}$.

$\therefore \bar{\mu}(x_0) = 1$, since $\mu \in \mathcal{U}$.

i.e. $p \leq \bar{\mu}$.

Therefore, $K_{f\text{-}X'}(O(\mu) \cap$ f-E'X$) = \{p_i\}$ where $p_i \leq \bar{\mu}$ for every i. Therefore by definition 3.3.1 $\{p_i\}$ is subordinate to $\bar{\mu}$.

5.2.4 Theorem: Let p be an f-open point in X with support x and $\mu\in\tau(X)$. If $p\in$ f-int (f-cl(μ)), then $K^{\leftarrow}_{f\text{-}x'}(p) \subseteq O(\mu)$.

Proof: Let $p\in$ f-int ($\bar{\mu}$). i.e. f-int $\bar{\mu}(x)=1$. Therefore , $\bar{\mu}(x)=1$.

Let $\mathcal{U} \in K^{\leftarrow}_{f\text{-}x'}(p)$. Then $K_{f\text{-}x'}(\mathcal{U}) = p$

Therefore, a $(\mathcal{U})$ $(x)= 1$.

i.e. $\bar{\gamma}(x)=1$ for every $\gamma\in \mathcal{U}$.

$\bar{\mu}$ $(x)=1$. Therefore, $1-\bar{\mu}(x)\neq 1$.

i.e. $1-\bar{\mu}\notin \mathcal{U}$. That is $\mu^c\notin \mathcal{U}$. Therefore, $\mu\in \mathcal{U}$.

That is $\mathcal{U} \in O(\mu)$.

$\therefore K^{\leftarrow}_{f\text{-}x'}(p) \subset O(\mu)$.

Note: The converse of this theorem is not true in the case of non crisp sets. By an example similar to that of 4.3.4 , we can prove this.

5.2.5 Theorem: The function $K_{f\text{-}x'}$ from f-E'X into X is s- continuous and compact.

Proof: Similar to that of 4.3.7.

5.2.6 Note: The mapping $K_{f\text{-}x'}$ is not a closed map. An example similar to that of 4.3.8 will serve the purpose .

5.2.7 Definition: An f-open ultrafilter $\mathcal{U}$ on X is said to be strong fixed (s-fixed) if a $(\mathcal{U}) = 0$ and $\cap\ \{\sigma_0(\bar{\mu}) : \mu\in \mathcal{U}\} \neq \phi$.

5.2.8 Remark:

As in the case of s-fixed ultrafilters, if $\cap$ $\{\sigma_0(\bar{\mu}) : \mu\in \mathcal{U}\} \neq\phi$, then it reduces to be a singleton. Therefore the only s-fixed f-open ultrafilters are of the form $\mathcal{F}[x] = \{\mu\in\tau(X) : 0 \quad \mu(x)>0\}$. They are called principal f-open ultrafilters. So as in section 4.4 we can construct the absolutes using the principal f-open ultrafilters. The results proved in section 4.4 hold in this case also.

In such construction corresponding to every $x\in X$, we can have $\mathcal{U}[x]$ belonging to f-E'X such that $\cap$ $\{\sigma_0(\bar{\mu}):\mu\in\mathcal{U}\} =\{x\}$. Therefore the mapping $K_{f\text{-}x'}$ becomes a surjection from f-E'X onto X.

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BIBLIOGRAPHY

[AZ]$_1$ Azad, K.K. *Fuzzy Grills and A Characterization of Fuzzy Proximity,* J.Math. Anal.Appl 79(1981) 13-17.

[AZ]$_2$ Azad, K.K. *On Fuzzy Semi Continuity, Fuzzy Almost Continuity and Fuzzy Weakly Continuity,* J.Math Anal.Appl.82 (1981) 14- 32.

[BE] Bernard, R. *Fuzzy Topological Spaces and Their Representation,* J. Math. Anal. Appl 81 (1981) 378-390.

[BI] Bin Shahna, A.S. *Fuzzy Boolean algebra of Fuzzy Regular Closed Sets,* Fuzzy Sets and Systems 106 (1999) 285-287.

[BU] Bul bul, A. *A Note on L0- Fuzzy Nearly Compact Spaces,* Fuzzy Sets and System 93 (1998) 235-239.

[B;M;G]$_1$ Burton, M.H; Muraleetharan, M; Gutierrez, G. *Generalised Filters 1,* Fuzzy Sets and System 106 (1999) 275-284.

[B;M;G]$_2$ Burton,M.H; Muraleetharan, M; Gutierrez, G. *Generalised Filters 2,* Fuzzy Sets and System 106 (1999) 393-400.

[CE] Cerutti,U. *The Stone Čech Compactification in the Category of Fuzzy Topological Spaces.* Fuzzy Sets and Systems *(1981) 197-204.*

[CH] Chang, C. L. *Fuzzy Topological Spaces,* J.Math.Anal.Appl.24 (1968) 182-189.

[CO] Conard, F. *Fuzzy Topological Concepts.* J.Math.Anal.App (1980) 432-440 .

[D; P] Dubois, D; Prade, H. *Fuzzy Sets and Systems Theory and Applications*, Academic Press Newyork (1980).

[E ; G]$_1$ Eklund, P; Gahler, W. *Basic Notions for Fuzzy Topology I,* Fuzzy Sets and Systems 26 (1988) 333-356.

[E ; G]$_2$ Eklund, P; Gahler, W. *Basic Notions for Fuzzy Topology II* , Fuzzy Sets and Systems 27, 333-356 (1988).

[G ; S]$_1$ Ganguly, S; Saha, S. *On Separation Axiom and T_I Fuzzy Continuity*, Fuzzy sets and systems 16, 265-275 (1985).

[G ; S]$_2$ Ganguly, S; Saha,S. *A Note on Semi Open Sets in Fuzzy Topological Spaces,* Fuzzy Sets and System 18 (1985) 83-96.

[GH] Ghosh, B. *Fuzzy Extremally Disconnected Spaces*, Fuzzy Sets and Systems 46 (1992) 245-250.

[G ; K; M] Ghanim, M.H; Kerre, E.E; Mashhour, A.S. *Separation Axioms, Sub Spaces and Sums in Fuzzy Topology*, J. Math. Anal. Appl 102 (1984) 189-202.

[GL] Gleason, A.M. *Projective Topological Spaces III,* J. Math.2 (1958) 482- 489.

[GO] Goguen, J. A. *L. Fuzzy Sets*, J. Math. Anal. Appl. 18 (1967) 145-174.

[G ; S ; W] Gantner, T.E; Steinlage, R.C; Warren, R.H.*Compactness in Fuzzy Topological Spaces*, J. Math.Anal. Appl. 62 (1978) 547-562.

[HO]$_1$ Hohle, U. *L. Fuzzy Topologies on Algebraic Structure,* J. Math. Anal. Appl. 108 (1985) 113-150.

[HO]$_2$ Hohle, U. *Fuzzy Topologies and Topological Spaces Objects in a TOPOS*, Fuzzy Sets and Systems 19 (1986) 299-304.

[HU] Hutton, B. *Normality in Fuzzy Topological Spaces*, J. Math. Anal. Appl 50 (1975).

[H ; R] Hutton, B; Reilly, I. *Separation Axioms in Fuzzy Topological Spaces*, Fuzzy Sets and System 3 (1980) 93-104.

[JO] Joshy, K.D. *Introduction to General Topology*, Wiley Eastern (1983).

[KA]$_1$ Katsaras, A.K. *Fuzzy Proximity Spaces*, J. Math. Anal. Appl. 68 (1979) 100-110.

[KA]$_2$ Katsaras, A.K. *On Fuzzy Proximity Spaces*, J. Math. Anal. Appl. 75 (1980) 571-583.

[KAU] Kaufmann, A. *Introduction to the Theory of Fuzzy Subsets*, Academic Press. Inc New York (1975).

[K ;F] Klir, G.J; Folger, T.A. *Fuzzy Sets Uncertainty and Information,* Prentice Hall of India (1993).

[LO]$_1$ Lowen, R. *Fuzzy Topological Spaces and Fuzzy Compactness*, J. Math.Anal. Appl. 56 (1976) 621-633.

[LO]$_2$ Lowen, R. *Initial and Final Fuzzy Topologies and The Fuzzy Tychonoff Theorem,* J. Math. Anal. Appl. 58, (1977) 11-21.

[LO]$_3$ Lowen, R. F*uzzy Neighbourhood Spaces*, Fuzzy Sets and Systems 7 (1982) 165-189.

[MA] Martin, H.W. *A Stone Čech Ultra Fuzzy Compactifications*, J. Math.Anal. Appl. 73 (1980) 453-456.

[M ; B] Malghan, S.R; Benchalli, S.S. *On Fuzzy Topological Spaces*, Glasnik Mathematick Vol. 16 (36), (1981) 313-325.

[M ;G ; F] Mashhour, A.S; Ghanim, M.H; Fath Allah, M.A. *Fuzzy Ultrafilters and Fuzzy Grills*, Simon Stevin 60 (1986) 3-14.

[P ; A] De Prade Vicente, M.A; Aranguren, M.S. *Fuzzy Filters*, J.Math Anal. Appl.129 (1988) 560- 568.

[P ; W] Porter, J.R. ; Woods, R.G. *Extensions and Absolutes of Hausdorff Spaces*, Springer Verlag New York (1988).

[RO] Rodabaugh, S.E. *A Categorical Accommodation of Various Notions of Fuzzy Topology*, Fuzzy Sets and Systems 9 (1983) 241-265.

[SA] Sarkar, M. *On Fuzzy Topological Spaces*, J. Math.Anal. Appl. 79 (1981) 384-394.

[S ; G] Srivastava, P; Gupta, R.L. *Fuzzy Proximity Structures and Fuzzy Ultrafilters*, J.Math.Anal.Appl.92 (1983) 297-311.

[S ; K]$_1$ Srivastava, P; Khare, M. *Fuzzy Grills, Fuzzy Ultrafilters and Counter Example*. Simon Stevin, Vol. 67 (1993) Supplement December 1993.

[S ; K]$_2$ Srivastava, P; Khare, M. *Examples of Classical and Fuzzy Riesz Proximities*, J. Math. Solvaca 45 (1995). No. 4, 519-522.

[S ;K]$_3$ Srivastava, P; Khare, M. *Fuzzy f- Proximities*, Fuzzy Sets and Systems 59 (1993) 197-203.

[ST] Stone, M.H. *Application of Theory of Boolean Rings to General Topology*, Trans.Amer. Math. Soc 41 (1937) 375-481.

[WA] Walker, R.C. *The Stone Čech Compactification*, Springer Verlay, Berlin Heidelberg, New York (1974).

[WE] Weiss, M.D. *Fixed Points Separation and Induced Topologies for Fuzzy Sets*, J.Math.Anal. Appl. 50 (1975) 142-150.

[WI] Willard, S. *General Topology*, Addison Wesley pub. Company (1970).

[WO]$_1$ Wong, C.K. *Fuzzy Topology in Fuzzy Sets and Their Applications to Cognitive and Decision Process*, Academic Press, Newyork (1975).

[WO]$_2$ Wong, C.K. *Fuzzy Topology, Product and Quotient Theorem*, J. Math.Anal. Appl. 45 (1974) 512-521.

[Y ;M]$_1$ Ying-Ming, Liu; Mao- Kang, Luo. *Induced Spaces and Fuzzy Stone Čech Compactification*, Sientia Sinica, Ser.A 30 (1987) 1034-1044.

[Y ; M]$_2$, Ying-Ming, Liu; Mao- Kang, Luo. *Fuzzy Topology*, World Scientific, Singapore (1997).

[ZA] Zadeh, L.A. *Fuzzy Sets,* Information Control 8 (1965) 388-353.

[ZI] Zimmermann, H.J. *Fuzzy Sets Theory and its Application*, Kluwer Academic (1996).