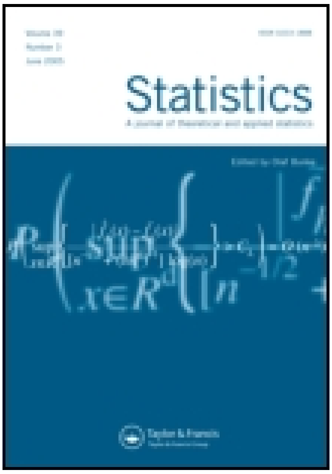

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Characterizations of distributions using log odds rate

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Characterizations of distributions using log odds rate

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In this paper, we examine the relationships between log odds rate and various reliability measures such as hazard rate and reversed hazard rate in the context of repairable systems. We also prove characterization theorems for some families of distributions *viz*. Burr, Pearson and log exponential models. We discuss the properties and applications of log odds rate in weighted models. Further we extend the concept to the bivariate set up and study its properties.

Keywords: Log odds rate; Hazard rate; Reversed hazard rate; Weighted models

1. Introduction

Let X be a random variable (rv) representing the lifetime of a component/system with reliability (survival) function $R_X(x)$, then the hazard rate is given by

$$h_X(x) = \frac{f_X(x)}{R_X(x)} = -\frac{\mathrm{d}}{\mathrm{d}x} \ln R_X(x) \tag{1}$$

for $x < b$, where $b = \sup\{x : R_X(x) > 0\}$ and $f_X(x)$ is the probability density function (pdf) of the random variable X. The hazard rate is one of the fundamental concepts of reliability analysis. In many practical situations it has been considered as a useful measure in modeling statistical data to derive the appropriate model. Based on the physical properties of the component, the monotone behaviour of the failure pattern is also an effective method to characterize the underlying model.

Recently, with the need for high reliability of the components, non-monotone hazard rates have also played an important role in the study of engineering reliability and biological survival analysis. The important distributions such as lognormal, Burr, inverse Gaussian and truncated normal are appropriate in such situations. It has been identified recently that the log odds rate (LOR) is a useful measure to model statistical data that shows non-monotone hazard rate (see [1]). A formal definition of the LOR is as follows. If $F_X(x)$ is the cumulative distribution

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function (cdf) and $R_X(x) = 1 - F_X(x)$ is the reliability function, the log odds function is

$$\mathrm{LO}_X(x) = \ln \frac{F_X(x)}{R_X(x)} = \ln F_X(x) - \ln R_X(x)$$

which gives

$$\frac{F_X(x)}{R_X(x)} = \exp(\mathrm{LO}_X(x))$$

or

$$F_X(x) = \frac{\exp(\mathrm{LO}_X(x))}{1 + \exp(\mathrm{LO}_X(x))}.$$

Then, LOR

$$H_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{LO}_X(x) = \frac{f_X(x)}{F_X(x)R_X(x)} \tag{2}$$

or

$$f_X(x) = \frac{H_X(x)\exp(\mathrm{LO}_X(x))}{(1 + \exp(\mathrm{LO}_X(x)))^2}.$$

In the present paper, we give an interpretation for LOR in the context of repairable systems. We also prove characterization theorems for some families of distributions *viz*. Burr, Pearson and log exponential densities. We discuss the properties and applications of log odds ratio in weighted models. We further extend the concept to the bivariate set up and study its properties.

2. Maintainability function and reliability function

Reliability and maintainability are important measures of the effectiveness of components. The major difference between these two measures is that reliability is the probability that a component has survived (or does not fail) in a particular time, whereas maintainability is the probability that any required maintenance would be successfully completed in a given time period.

Let Y denote the repair time of a component, then $F_Y(x) = P(Y \leq x)$ is known as the maintainability function (distribution function). It is used to predict the probability that a repair, beginning at time $x = 0$, will be accomplished in a time x. Then the reversed repair rate is defined as

$$\lambda_Y(x) = \frac{f_Y(x)}{F_Y(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln F_Y(x) \tag{3}$$

for $x > a$, where $a = \inf\{x : F_Y(x) > 0\}$. Equation (3) implies that the probability of its repair completed during the time $(x - \epsilon, x)$ is approximately equal to $\epsilon\lambda_Y(x)$. When Y denotes the failure time, $\lambda_Y(x)$ is then known as the reversed hazard rate (see [2]).

When X and Y are independent and identically distributed (i.i.d.) random variables, using (1) and (3), the LOR (2) becomes

$$H_X(x) = \lambda_X(x) + h_X(x) \quad \text{for } a < x < b. \tag{4}$$

Therefore, $H_X(x)$ reduces to the sum of the reversed repair rate and failure rate. One important property (4) posseses is that even if the survival data show a non-monotone hazard rate, the LOR would be monotone. For various properties of distributions with monotone $H_X(x)$ and examples, one could refer to [1].

3. Characterizations

Consider a rv X with support (a, b) and an absolutely continuous cdf $F_X(x)$, the system of distributions, introduced by Burr [3], given by

$$f_X(x) = F_X(x)(1 - F_X(x))g_X(x), \tag{5}$$

where $g_X(x)$ is some convenient function, which must be non-negative over the domain (a, b). Equations (2) and (5) together imply that

$$H_X(x) = g_X(x).$$

Hence, for Burr family of distributions, $g_X(x)$ directly gives the LOR and vice versa.

We now prove a characterization theorem for Pearson family of distributions.

THEOREM 1 *Let X be a rv having an absolutely continuous cdf $F_X(x)$ with support (a, b). Assume that $E(X) < \infty$, $m_X(x) = E(X|X > x)$ and $n_X(x) = E(X|X < x)$ denote the conditional expectations of X. Then the relationship*

$$m_X(x) = n_X(x) + (a_0 + a_1x + a_2x^2)H_X(x) \tag{6}$$

holds for all $x \in (a, b)$, if and only if the pdf of X satisfies the equation

$$\frac{\mathrm{d} \ln f_X(x)}{\mathrm{d}x} = \frac{-(x + d)}{b_0 + b_1x + b_2x^2} \tag{7}$$

with $a_i = (b_i/1 - 2b_2)$; $i = 0, 1, 2$.

Proof The family of distributions (7) is characterized by the identity

$$m_X(x) = \mu_X + (a_0 + a_1x + a_2x^2)h_X(x), \tag{8}$$

where $\mu_X = E(X)$ (see [4]). One can also establish that for the family (7), we have

$$n_X(x) = \mu_X - (a_0 + a_1x + a_2x^2)\lambda_X(x) \tag{9}$$

(see [5] and [6]).

From equations (8) and (9), we get

$$m_X(x) = n_X(x) + (a_0 + a_1x + a_2x^2)(h_X(x) + \lambda_X(x)),$$

which yields (6). Conversely, assume that (6) holds, multiplying (6) by $R_X(x)$ and $F_X(x)$ and on simplification we get,

$$F_X(x)\int_x^b tf_X(t)\mathrm{d}t = R_X(x)\int_a^x tf_X(t)\mathrm{d}t + (a_0 + a_1x + a_2x^2)f_X(x). \tag{10}$$

Differentiating equation (10) with respect to x, and simplifying we obtain the result (7). This completes the proof. ■

Examples Here we consider some of the important members of the Pearson family and their respective forms (6).

1. Normal: $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$,
   then $m_X(x) = n_X(x) + \sigma^2 H_X(x)$.
2. Beta: $f_X(x) = (1/B(m,n))x^{m-1}(1-x)^{n-1}$,
   then $m_X(x) = n_X(x) + (x(1-x)/(m+n))H_X(x)$.
3. Gamma: $f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$,
   then $m_X(x) = n_X(x) + \beta x H_X(x)$.

Next, we prove a characterization theorem using $H_X(x)$ for the one parameter log exponential family. The one parameter log exponential is defined by

$$f_X(x) = \frac{x^\theta C(x)}{A(\theta)}, \quad \theta > 0, \quad x \in (0, b), \tag{11}$$

where $0 < b \leq \infty$. $C(x)$ is a non-negative function of x, which is differentiable and $A(\theta)$ is a non-negative function of θ satisfying $A(\theta) = \int_0^b x^\theta C(x)\mathrm{d}x$.

Theorem 2 *Let*

$$m_{X,C}(x) = E\left(\frac{XC'(X)}{C(X)}|X > x\right),$$
$$n_{X,C}(x) = E\left(\frac{XC'(X)}{C(X)}|X < x\right) \quad \textit{and} \quad E\left(\frac{XC'(X)}{C(X)}\right) < \infty$$

and assume that $\lim_{x\to b} C(x)x^{\theta+1} = 0$. *Then the distribution of X belongs to the one parameter log exponential family, if and only if*

$$m_{X,C}(x) = n_{X,C}(x) - xH_X(x). \tag{12}$$

Proof For the model (11), we have

$$R_X(x) = \frac{-C(x)x^{\theta+1}}{A(\theta)(\theta+1)} - \frac{1}{A(\theta)(\theta+1)} \int_x^b C'(t)t^{\theta+1}\mathrm{d}t$$

or

$$m_{X,C}(x) = -xh_X(x) - (\theta+1). \tag{13}$$

Similarly, one can obtain

$$F_X(x) = \frac{C(x)x^{\theta+1}}{A(\theta)(\theta+1)} - \frac{1}{A(\theta)(\theta+1)} \int_x^b C'(t)t^{\theta+1}\mathrm{d}t$$

or

$$n_{X,C}(x) = x\lambda_X(x) - (\theta+1). \tag{14}$$

Combining equations (13) and (14), we obtain the required form (12). The converse part is straightforward. ■

4. Weighted models

The concept of weighted distributions was introduced by Rao [7], explaining how such distributions arise in practice. Motivated from this, several research works were carried out in theoretical and practical setting. These include analyses of family data, the problem of family size and alcoholism, study of albinism, human heredity, aerial survey and visibility bias, line transect sampling, renewal theory, cell cycle analysis and pulse labeling, efficacy of early screening for disease, etiological studies, statistical ecology and reliability modeling. An exhaustive account of research in this area and the latest survey of literature we refer to [8], [9], [10] and [11].

Let $f_X(x)$ be the pdf of a non-negative rv X denoting the life length of a component having cdf $F_X(x)$ with $F_X(0) = 0$. Then a rv Z with density

$$f_Z(x) = \frac{w(x)}{\mu_w} f_X(x) \tag{15}$$

such that $\mu_w = E(w(X))$ is said to have weighted distribution corresponding to X and w. Denoting $F_Z(x) = P(Z \leq x)$ and $R_Z(x) = P(Z > x)$, the cdf and survival function, respectively of the rv Z, then the log odds function denoted by $\mathrm{LO}_Z(x)$ is given by

$$\mathrm{LO}_Z(x) = \ln\left(\frac{F_Z(x)}{R_Z(x)}\right) = \ln F_Z(x) - \ln R_Z(x). \tag{16}$$

But it can be obtained directly, the quantities

$$F_Z(x) = \frac{n_X^w(x)}{\mu_w} F_X(x) \tag{17}$$

and

$$R_Z(x) = \frac{m_X^w(x)}{\mu_w} R_X(x), \tag{18}$$

where $m_X^w(x) = E(w(X)|X > x)$ and $n_X^w(x) = E(w(X)|X < x)$ are the conditional means of X (see [12]). From expressions (16), (17) and (18) the log odds function becomes

$$\mathrm{LO}_Z(x) = \mathrm{LO}_X(x) + \ln\left(\frac{n_X^w(x)}{m_X^w(x)}\right).$$

Using equation (2), we obtain

$$H_Z(x) = \frac{f_Z(x)}{F_Z(x)R_Z(x)} = \frac{w(x)\mu_w}{n_X^w(x)m_X^w(x)} H_X(x).$$

From a direct computation with certain specified weight functions, it can be observed that the weighted version often retains the same form as their parent distribution. This property is usually termed as form-invariance. Initially Rao [7] identified this property in some discrete models, but a comprehensive study of the topic was initiated by Patil and Ord [13]. They proved that a necessary and sufficient condition for X to be a form-invariant under size-bias of order α (i.e., when $w(x) = x^\alpha$, $\alpha > 0$) is that its pdf is of the form (11). Motivated by the relevance of form-invariance in characterizing families of distributions and usefulness of the same in modeling random phenomena by employing various families of distributions, Sankaran and Nair [14] derived the conditions under which the Pearson & Ord families are form-invariant with respect to the length biased samplings (i.e., when $w(x) = x$). In view of the form-invariant property for families (7) and (11), the analogous statements for Theorems 1 and 2 in the context of weighted models are immediate, which is stated as follows.

THEOREM 3 *Let Z be the weighted rv associated to X and $w(x) = x^{\alpha}$, $\alpha > 0$. Then X is a member of the Pearson system of distributions equation* (7) *with $b_0 = 0$ and $\lim_{x\to a}(b_1 x + b_2 x^2) f_X(x) = 0$, if and only if*

$$m_Z(x) = n_Z(x) + (c_1 x + c_2 x^2) H_Z(x)$$

where $m_Z(x) = E(Z|Z > x)$ and $n_Z(x) = E(Z|Z < x)$.

THEOREM 4 *Let Z be the weighted rv associated to X and assume that $\lim_{x\to b} c(x)\, x^{\theta+1} = 0$, with $w(x) = x^{\alpha}$, $\alpha > 0$ and $g(x) = xC'(x)/C(x)$, then the relationship*

$$m_Z^g(x) = n_Z^g(x) - xH_Z(x)$$

holds if and only if Z belongs to the one parameter exponential family (12), *where $m_Z^g(x) = E(g(Z)|Z > x)$ and $n_Z^g(x) = E(g(Z)|Z < x)$.*

In the following, we present a characterization theorem for the one parameter exponential family defined by

$$f_X(x) = \frac{\alpha(x)\theta^x}{b(\theta)}; \quad x \in (a, b), \quad \theta > 0, \tag{19}$$

where $\alpha(x)$ is a non-negative function and $b(\theta) = \int_a^b \alpha(x)\theta^x \, \mathrm{d}x$.

THEOREM 5 *Let Z be the weighted rv associated with X and $w(x) = x^{\alpha}$, $\alpha > 0$, then the relationship*

$$H_Z(x) = n_Z^d(x) - m_Z^d(x) \tag{20}$$

holds, if and only if X belongs to the one parameter exponential family (19) *with $\alpha(x) = (w(x)/d(x))$ provided*

$$d(b)\theta^b - d(a)\theta^a = 0, \quad C(\theta) = \int_a^b \mathrm{d}(x)\theta^x \, \mathrm{d}x, n_Z^d(x) = E\left(\frac{d'(Z)}{d(Z)}|Z < x\right) \quad \textit{and}$$

$$m_Z^d(x) = E\left(\frac{d'(Z)}{d(Z)}|Z > x\right).$$

5. Bivariate Case

In this section, we extend the concept of log odds function and LOR to higher dimensions. We confine our study to the bivariate setup. The extensions to higher dimensions are direct. Let $X = (X_1, X_2)$ be a bivariate random vector with an absolutely continuous distribution function $F(x_1, x_2)$ and survival function $R(x_1, x_2)$ and pdf $f(x_1, x_2)$. Let $F_i(x_i)$ and $R_i(x_i)$,

$i = 1, 2$ denote the marginal distribution function and survival function of X_i. Let $f_i(x_i)$ be the density function of X_i. Then we propose the bivariate log odds function by

$$L = \mathrm{LO}_{x_1,x_2}(x_1, x_2) = \ln \frac{F(x_1, x_2)}{R(x_1, x_2)} = \ln F(x_1, x_2) - \ln R(x_1, x_2), \tag{21}$$

which gives

$$\frac{F(x_1, x_2)}{R(x_1, x_2)} = \exp(\mathrm{LO}_{x_1,x_2}(x_1, x_2)). \tag{22}$$

The corresponding LOR is defined as a vector

$$H(x_1, x_2) = (H_1(x_1, x_2), H_2(x_1, x_2)), \tag{23}$$

where

$$H_i(x_1, x_2) = \frac{\partial \mathrm{LO}_{x_1,x_2}(x_1, x_2)}{\partial x_i}; \quad i = 1, 2. \tag{24}$$

Using the bivariate vector failure rate due to Johnson and Kotz [15] and bivariate reversed hazard rate due to Roy [16], equation (24) becomes

$$H_i(x_1, x_2) = \lambda_i(x_1, x_2) + h_i(x_1, x_2), \tag{25}$$

where $\lambda_i(x_1, x_2) = \partial/\partial x_i \ln F(x_1, x_2)$ and $h_i(x_1, x_2) = -(\partial/\partial x_i) \ln R(x_1, x_2)$, $i = 1, 2$, are the ith components of the reversed hazard rates and failure rates, respectively.
Here we consider some bivariate densities having simple vector valued LOR.

Example 1 Bivariate normal:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{x_1^2}{\sigma_1^2} - \frac{2\rho x_1 x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right]\right\};$$
$$-\infty < x_1, x_2 < \infty, \sigma_1, \sigma_2 > 0, |\rho| < 1. \tag{26}$$

Taking logarithm on both sides of (26) and differentiating with respect to x_i and then integrating twice between the limits x_i to b_i and x_j to b_j, $i = 1, 2$ and $j = 3 - i$, we get

$$(1-\rho^2)\sigma_i^2\sigma_j h_i(x_1, x_2) = \sigma_j m_i(x_1, x_2) - \rho\sigma_i m_j(x_1, x_2), \tag{27}$$

where $m_i(x_1, x_2) = E(X_i | X_i > x_i, X_j > x_j)$, $i = 1, 2$ and $j = 3 - i$ (see [17] and [18]). Similarly, integrating twice between the limits a_i to x_i and a_j to x_j, $i = 1, 2$ and $j = 3 - i$, we obtain

$$(1-\rho^2)\sigma_i^2\sigma_j \lambda_i(x_1, x_2) = \rho\sigma_i n_j(x_1, x_2) - \sigma_j n_i(x_1, x_2), \quad i = 1, 2 \quad \text{and} \quad j = 3 - i, \tag{28}$$

where $n_i(x_1, x_2) = E(X_i | X_i < x_i, X_j < x_j)$, $i = 1, 2$ and $j = 3 - i$. Now adding (27) and (28), we get

$$(1-\rho^2)\sigma_i^2\sigma_j H_i(x_1, x_2) = \sigma_j(m_i(x_1, x_2) - n_i(x_1, x_2)) - \rho\sigma_i(n_j(x_1, x_2) - m_j(x_1, x_2)),$$
$$i = 1, 2 \quad \text{and} \quad j = 3 - i.$$

Example 2 Bivariate exponential:
The joint density function of the exponential conditional due to Arnold & Strauss [19] is

$$f(x_1, x_2) = C\exp(-\alpha_1 x_1 - \alpha_2 x_2 - \beta x_1 x_2), x_1, x_2, \alpha_1, \alpha_2 > 0, \beta \geq 0, \tag{29}$$

where

$$C = -\beta\exp\left[\frac{-\alpha_1\alpha_2}{\beta}\right]\left(E_i\left(\frac{-\alpha_1\alpha_2}{\beta}\right)\right)^{-1}.$$

Now proceeding in the similar manner as above, the identity connecting the vector valued LOR and the conditional moments for the density (29) becomes

$$H_i(x_1, x_2) = \beta(m_j(x_1, x_2) - n_j(x_1, x_2)), \quad i = 1, 2 \quad \text{and} \quad j = 3 - i.$$

THEOREM 6 *The relationship*

$$LO_{X_1,X_2}(x_1, x_2) = \mathrm{LO}_{X_1}(x_1) + \mathrm{LO}_{X_2}(x_2) \tag{30}$$

holds for all x_1, x_2, if and only if X_1 and X_2 are independent.

Proof Suppose equation (30) holds, then

$$\frac{F(x_1, x_2)}{R(x_1, x_2)} = \frac{F_1(x_1)}{R_1(x_1)}\frac{F_2(x_2)}{R_2(x_2)},$$

which is equivalent to

$$F(x_1, x_2)(1 - F_1(x_1))(1 - F_2(x_2)) = F_1(x_1)F_2(x_2)(1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2)).$$

On simplification, we obtain

$$F(x_1, x_2) = F_1(x_1)F_2(x_2),$$

which proves the result. The converse part is straightforward. ■

Remark Theorem 6 can be useful to test the independence among the variables. This might be helpful in reliability analysis to study the dependence structure between the components of a system.

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