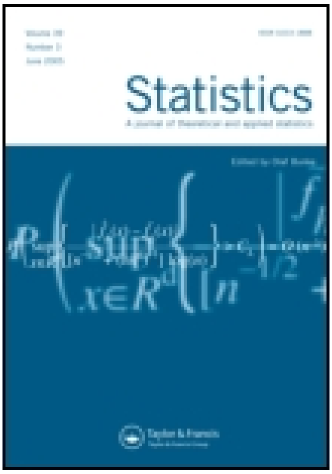

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Form-invariant bivariate weighted models

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FORM-INVARIANT BIVARIATE WEIGHTED MODELS

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In this paper the class of continuous bivariate distributions that has form-invariant weighted distribution with weight function $w(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ is identified. It is shown that the class includes some well known bivariate models. Bayesian inference on the parameters of the class is considered and it is shown that there exist natural conjugate priors for the parameters.

Keywords: Bivariate weighted distributions; Form-invariance; Bayesian inference; Lorenz surface

1 INTRODUCTION

The concept of weighted distributions has been introduced and formalized by Rao [13] by identifying various practical problems that can be modeled by such distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distributions. This may be due to non-observability of some events or damage caused to the original observation resulting in a reduced value, or adoption of a sampling procedure that gives unequal chances to the units in the original population. A detailed survey of literature on the application of weighted distributions in the analysis of family data, the problem of family size and alcoholism, study of albinism, human heredity, aerial survey and visibility bias, line transect sampling, renewal theory, cell cycle analysis and pulse labeling, efficacy of early screening for disease, etiological studies, statistical ecology and reliability modeling is available in Patil and Rao [9], Rao [14] and Gupta and Kirmani [4].

2 NOTATION AND TERMINOLOGY

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X\colon \Omega \to H$ be a random variable, where $H = (a, b)$ is a subset of the real line with $a > 0$ and $b > a$ being finite or infinite. When the distribution function $F(x)$ of X is absolutely continuous with density function $f(x)$ and

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$w(x)$, a non-negative weight function satisfying $E(w(X) = \mu) < \infty$, the random variable Y with density

$$g(x) = \frac{w(x)f(x)}{\mu}, \quad x > a \tag{2.1}$$

is said to have weighted distribution corresponding to X.

The bivariate extension of weighted distribution is discussed in Patil *et al.* [11]. For a pair of non-negative random variables (X_1, X_2) with joint density function $f(x_1, x_2)$ and a non-negative weight function $w(x_1, x_2)$ such that $Ew(X_1, X_2) < \infty$, the random vector (Y_1, Y_2) with density function

$$g(x_1, x_2) = \frac{w(x_1, x_2)f(x_1, x_2)}{Ew(X_1, X_2)} \tag{2.2}$$

is said to have weighted distribution corresponding to (X_1, X_2). For properties of (2.2) we refer to the above paper, Mahfoud and Patil [6], Arnold and Nagaraja [2], Patil *et al.* [10] and Jain and Nanda [5]. The multivariate aspects of the weighted distributions and some partial ordering and positive and negative dependence results related to weighted distributions are studied in Jain and Nanda [5]. For applications of bivariate weighted models in the context of reliability, contingency table analysis etc., we refer to Patil *et al.* [11] and Sunoj and Nair [16, 18].

3 FORM-INVARIANCE

If the distribution of the weighted random variable Y is of the same form as that of the original random variable X, we say that Y has form-invariant weighted distribution. Patil and Rao [9] by direct calculation have given several examples where the original and weighted distributions have identical form. This prompted Patil and Ord [8] to identify the general form of distributions that possesses this interesting property. They established that a necessary and sufficient condition for a distribution to be form-invariant under size biased sampling of order α (*i.e.*, $w(x) = x^{\alpha}$) is that the probability density function must belong to the log-exponential family specified by the density

$$f(x) = \frac{x^{\theta}a(x)}{m(\theta)} = \exp\{\theta \log x + A(x) - B(\theta)\}. \tag{3.1}$$

The model specification is an important component in any inference problem. The form-invariance property is very useful in inference problems concerning the parameters of the model, as the estimates of the parameters in the original model can also be used in the weighted version, with slight modifications. Some results regarding form-invariance of bivariate weighted distribution are discussed in Patil *et al.* [11]. Recently, Sunoj and Nair [18] derived the necessary and sufficient conditions for each type of the bivariate Pearson system under which the underlying system is form-invariant using the product weight function $w(x_1, x_2) = x_1x_2$. They have identified the members of the family possessing such a property by calculating the distributional form in each case. The interest in the concept of form-invariance has generated in the univariate case motivates its generalization to higher dimensions. However, the general form of bivariate densities that admits form-invariance under a more versatile weight function $w(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$ than considered in Sunoj and Nair [17] has not yet been discussed in the literature. We explore this problem and present a solution in Section 4 of this paper.

Apart from providing an extension of the result of Patil and Ord [8] to the bivariate case, Section 5 demonstrates its usefulness in inference problems. We also give an example where form invariance becomes handy in the computation of bivariate Lorenz surfaces. It is often encountered in agriculture to estimate the average yield produced in unit area. The weighted model corresponding to $\alpha_1 = \alpha_2 = 1$ corresponds to the situation that the probability of choosing a specified sample is proportional to its area. Form-invariance in such situations implies that the model remains the same whether or not we choose random samples in which the field areas are the same.

4 CLASS OF DISTRIBUTIONS ADMITTING FORM-INVARIANCE

Following Patil and Ord [8], the distribution of (Y_1, Y_2) will be of the same form as that of (X_1, X_2) if there exist parametric functions $\eta_1(\theta_1)$ and $\eta_2(\theta_2)$ such that

$$g(x_1, x_2) = f(x_1, x_2; \eta_1, \eta_2) = \frac{x_1^{\alpha_1} x_2^{\alpha_2} f(x_1, x_2; \theta_1, \theta_2)}{\mu_{\alpha_1,\alpha_2}(\theta_1, \theta_2)} \tag{4.1}$$

provided $\mu_{\alpha_1,\alpha_2}(\theta_1, \theta_2) = E(X_1^{\alpha_1} X_2^{\alpha_2})$ is finite.

With this notion of form-invariance we prove

THEOREM 4.1 *If a non-negative random vector (X_1, X_2) is such that*

(a) $E(\log X_i)$ *is finite,*
(b) $\lim_{\alpha_i \to 0} ((\eta_i - \theta_i)/\alpha_i)$ *and* $\lim_{\eta_i \to \theta_i} (\alpha_i/(\eta_i - \theta_i))$ *are non-negative and finite and*
(c) $\int_0^\infty x_i^{\alpha_i} f(x_i)\,\mathrm{d}x_i$ *and* $\int_0^\infty (\mathrm{d}/\mathrm{d}\alpha_i)[x_i^{\alpha_i} f(x_i)]\mathrm{d}x_i$ *converge uniformly in* $[0, \infty)$,

then the distribution of (Y_1, Y_2) is of the same form as that of (X_1, X_2) if, and only if, the joint density of (X_1, X_2) has the form

$$f(x_1, x_2) = x_1^{\theta_1} x_2^{\theta_2} A(x_1, x_2) B_1(\theta_1) B_2(\theta_2) C(\theta_1, \theta_2). \tag{4.2}$$

Proof Assume that the distribution is of the form (4.2). Then $g(x_1, x_2)$ is of the same form as (4.2) with

$$\lim_{\alpha_i \to 0}\left(\frac{\eta_i - \theta_i}{\alpha_i}\right) = \lim_{\eta_i \to \theta_i}\left(\frac{\alpha_i}{\eta_i - \theta_i}\right) = 1, \quad i = 1, 2$$

and

$$E(\log X_i) = -\frac{\partial}{\partial \theta_i} \log(B_i(\theta_i) C(\theta_1, \theta_2)) < \infty, \quad i = 1, 2.$$

To prove the 'only if' part, we note that

$$\frac{\log f(x_1, x_2; \eta_1, \eta_2) - \log f(x_1, x_2; \theta_1, \theta_2)}{\eta_1 - \theta_1} = \left(\frac{\alpha_1}{\eta_1 - \theta_1}\right)\log x_1 + \left(\frac{\alpha_2}{\eta_1 - \theta_1}\right)\log x_2 - \frac{\log \mu_{\alpha_1,\alpha_2}(\theta_1, \theta_2)}{\eta_1 - \theta_1}. \tag{4.3}$$

When $\alpha_2 \to 0$, in (4.3)

$$\begin{aligned}\frac{\log f(x_1, x_2; \eta_1, \theta_2) - \log f(x_1, x_2; \theta_1, \theta_2)}{\eta_1 - \theta_1} &= \left(\frac{\alpha_1}{\eta_1 - \theta_1}\right)\log x_1 - \frac{\log \mu_{\alpha_1}(\theta_1, \theta_2)}{\eta_1 - \theta_1} \\ &= \left(\frac{\alpha_1}{\eta_1 - \theta_1}\right)\log x_1 - \left(\frac{\alpha_1}{\eta_1 - \theta_1}\right)\frac{\log \mu_{\alpha_1}(\theta_1, \theta_2)}{\alpha_1}.\end{aligned} \tag{4.4}$$

Now as $\alpha_2 \to 0$, using L'Hospital's rule

$$\begin{aligned}\lim_{\alpha_1 \to 0} \frac{\log \mu_{\alpha_1}(\theta_1, \theta_2)}{\alpha_1} &= \lim_{\alpha_1 \to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha_1} \int_0^\infty \int_0^\infty x_1^{\alpha_1} f(x_1, x_2; \theta_1, \theta_2)\, \mathrm{d}x_1 \mathrm{d}x_2, \\ &= \lim_{\alpha_1 \to 0} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\alpha_1} x_1^{\alpha_1} f(x_1)\, \mathrm{d}x_1, \\ &= \lim_{\alpha_1 \to 0} \int_0^\infty \log x_1 \cdot x_1^{\alpha_1} f(x_1)\, \mathrm{d}x_1, \\ &= \int_0^\infty \log x_1 \cdot f(x_1)\, \mathrm{d}x_1 = E(\log X_1) = C_1(\theta_1, \theta_2), \text{ say}\end{aligned}$$

so that (4.4) becomes

$$\frac{\partial \log f}{\partial \theta_1} = \gamma_1(\theta_1)\log x_1 - \gamma_1(\theta_1) C_1(\theta_1, \theta_2)$$

where

$$\gamma_1(\theta_1) = \lim_{\eta_1 \to \theta_1} \frac{\alpha_1}{\eta_1 - \theta_1}.$$

Integrating with respect to θ_1 yield,

$$\log f = b_1(\theta_1)\log x_1 + \log q_1(\theta_1, \theta_2) + \log p_1(x_1, x_2; \theta_2)$$

where

$$b_1(\theta_1) = \int \gamma_1(\theta_1)\, \mathrm{d}\theta_1, \qquad \log q_1(\theta_1, \theta_2) = -\int \gamma_1(\theta_1) C_1(\theta_1, \theta_2)\, \mathrm{d}\theta_1$$

and $p_1(x_1, x_2; \theta_2)$ is the constant of integration. Thus,

$$f(x_1, x_2; \theta_1, \theta_2) = x_1^{b_1(\theta_1)} p_1(x_1, x_2; \theta_2) q_1(\theta_1, \theta_2). \tag{4.5}$$

Similarly, working with $\eta_2 \to \theta_2$,

$$f(x_1, x_2; \theta_1, \theta_2) = x_2^{b_2(\theta_2)} p_2(x_1, x_2; \theta_1) q_2(\theta_1, \theta_2). \tag{4.6}$$

From (4.5) and (4.6),

$$\frac{\partial^2 \log q_1(\theta_1, \theta_2)}{\partial\theta_1 \partial\theta_2} = \frac{\partial^2 \log q_2(\theta_1, \theta_2)}{\partial\theta_1 \partial\theta_2}$$

which should mean that for some $P_1(\theta_1)$ and $Q_1(\theta_2)$,

$$q_1(\theta_1, \theta_2) = P_1(\theta_1) Q_1(\theta_2) C(\theta_1, \theta_2). \tag{4.7}$$

Substituting (4.7) in (4.5) and (4.6) and equating the resulting expressions

$$\frac{x_1^{b_1(\theta_1)} P_1(\theta_1)}{P_2(\theta_1) P_2(x_1, x_2; \theta_1)} = \frac{x_2^{b_2(\theta_2)} Q_2(\theta_2)}{Q_1(\theta_2) P_1(x_1, x_2; \theta_2)}. \tag{4.8}$$

In order that (4.8) holds for all θ_1, θ_2, either side must be of the form $B(x_1, x_2)$, independent of θ_1, θ_2 which gives

$$P_2(x_1, x_2; \theta_1) = \frac{A(x_1, x_2) x_1^{b_1(\theta_1)} P_1(\theta_1)}{P_2(\theta_1)} \tag{4.9}$$

where $A(x_1, x_2) = (B(x_1, x_2))^{-1}$.

Using (4.9) and (4.7) in (4.6), the joint density takes the form

$$f(x_1, x_2) = x_1^{b_1(\theta_1)} x_2^{b_2(\theta_2)} A(x_1, x_2) B_1(\theta_1) B_2(\theta_2) C(\theta_1, \theta_2)$$

from which the required result follows.

COROLLARY 4.1 *When the weight function is of the form $w(x_1, x_2) = x_1^{\alpha_1}$, the distribution of (Y_1, Y_2) is form-invariant if, and only if, the joint density is of the form*

$$f(x_1, x_2) = A(x_1, x_2) C(\theta_1) x_1^{\theta_1}. \tag{4.10}$$

Some members of the family (4.2) are presented in Table I. For an application of the member of the family (4.2) in engineering studies, we refer to Schneider and Holst [15]. Furthermore, if X_1 and X_2 represent the warning time and failure rate respectively, then the joint distribution is bivariate Beta-Stacy density given in Table I (see Mihiram and Hultquist [7]). Figures 4.1 and 4.2 exhibit the shape of the Dirchlet density for the original and weighted random variables, which shows how the concept of form-invariance proves in a graphical sense.

TABLE I Distributions Admitting Form-invariance.

Distribution	*Density*	*(θ_1, θ_2)*
Dirichlet	$\frac{\Gamma(n_1+n_2+n_3)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)}x_1^{n_1-1}x_2^{n_2-1}(1-x_1-x_2)^{n_3-1}$	(n_1-1, n_2-1)
Inverted beta	$\frac{\Gamma(-n_3+1)}{\Gamma(n_1)\Gamma(n_2)\Gamma(-n_1-n_2-n_3+1)}x_1^{n_1-1}x_2^{n_2-1}(1+x_1+x_2)^{n_3-1}$	(n_1-1, n_2-1)
Beta-Stacy	$\frac{cx_1^{\alpha c-\theta_1-\theta_2}x_2^{\theta_1-1}(x_1-x_2)^{\theta_2-1}}{B(\theta_1,\theta_2)\Gamma(\alpha)a^{c\alpha}}\exp\left\{-\left(\frac{x_1}{a}\right)^c\right\}$	$(c\alpha-\theta_1-1, \theta_1-1)$
Gamma	$\frac{x_1^{n_1-1}x_2^{n_2-1}}{\Gamma(n_1)\Gamma(-n_1-n_2)}\exp\left\{-\frac{(x_1+1)}{x_2}\right\}$	(n_1-1, n_2-1)
Lognormal	$\frac{1}{2\pi x_1x_2\sigma_1\sigma_2\sqrt{(1-\rho^2)}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\log x_1-\mu_1}{\sigma_1}\right)^2 -2\rho\left(\frac{\log x_1-\mu_1}{\sigma_1}\right)\left(\frac{\log x_2-\mu_2}{\sigma_2}\right)+\left(\frac{\log x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$	$\left(\frac{\mu_1}{\sigma_1^2(1-\rho^2)}-1, \frac{\mu_2}{\sigma_2^2(1-\rho^2)}-1\right)$
Generalised beta	$\frac{\sqrt{\pi}\Gamma((n_1+n_2)/2)\Gamma((n_1+n_2-1)/2)(x_2-x_1)}{\Gamma(n_1/2)\Gamma(n_2/2)\Gamma((n_1-1)/2)\Gamma((n_2-1)/2)} \times x_1^{(n_1-3)/2}x_2^{(n_1-3)/2}(1-x_1)^{(n_2-3)/2}(1-x_2)^{(n_2-3)/2}$	$\left(\frac{n_1-3}{2}, \frac{n_1-3}{2}\right)$
Type II	$\frac{\Gamma(-n_1+1)}{\Gamma(n_2)\Gamma(n_3)\Gamma(-n_1-n_2-n_3+1)}x_1^{n_1-1}x_2^{n_2-1}(-1+x_1-x_2)^{n_3-1}$	(n_1-1, n_2-1)

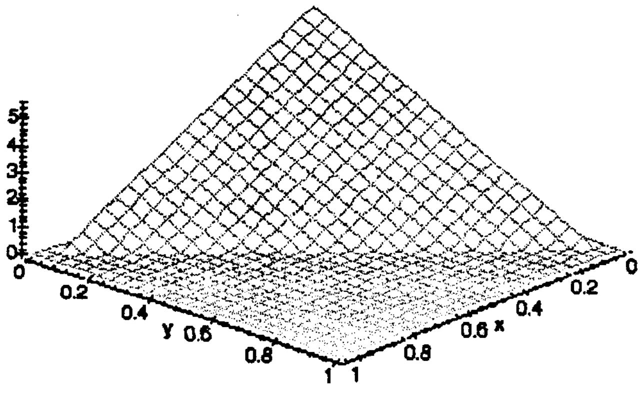

FIGURE 4.1 $f(x_1, x_2)$ for Dirichlet density (for $n_1 = n_2 = 1$, $n_3 = 2$).

5 AN EXAMPLE

In this section we point out some areas of applications where the concept of form-invariance with respect to the simple product weight function $w(x_1, x_2) = x_1x_2$ becomes useful. The Lorenz surface of a bivariate continuous random variable (X_1, X_2), in the support of the first octant in the two-dimensional space with joint density $f(x_1, x_2)$ and marginal densities $f_1(x_1)$ and $f_2(x_2)$, is defined by Arnold [1] as

$$L(u, v) = \int_0^x \int_0^y \frac{x_1x_2 f(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2}{E(X_1X_2)} \tag{5.1}$$

where

$$u = \int_0^x f_1(x_1)\, \mathrm{d}x_1 \quad \text{and} \quad v = \int_0^y f_2(x_2)\, \mathrm{d}x_2. \tag{5.2}$$

This is a direct extension of the Lorenz curve of a univariate continuous r.v. X in the support of $(0, \infty)$ defined by

$$L(p) = \frac{1}{\mu} \int_0^{F^{-1}(p)} x\, \mathrm{d}F(x), \quad 0 \le p \le 1. \tag{5.3}$$

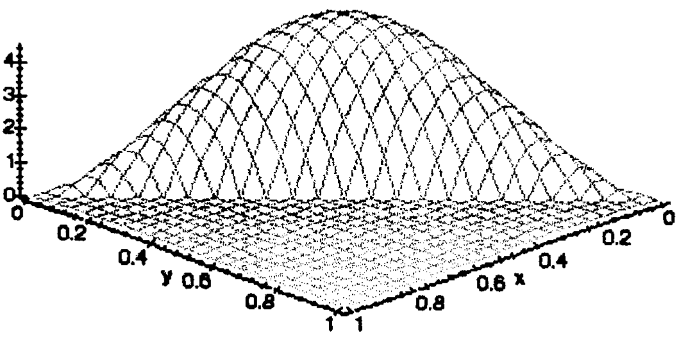

FIGURE 4.2 $g(x_1, x_2)$ for Dirichlet density (for $\alpha_1 = 1$, $\alpha_2 = 2$, $n_1 = n_2 = 1$, $n_3 = 2$).

Obviously, $L(u, v)$ can be expressed in terms of the distribution function $G(x_1, x_2) = P(Y_1 \leq x_1, Y_2 \leq x_2)$ of the weighted variables with $x_1 = F_1^{-1}(u)$ and $x_2 = F_2^{-1}(v)$. The corresponding weighted Lorenz surface has the form

$$L^*(u,v) = \int_0^x \int_0^y \frac{x_1 x_2 g(x_1, x_2)\,\mathrm{d}x_1 \mathrm{d}x_2}{E(X_1 X_2)} \tag{5.4}$$

with the same u and v.

Consider the bivariate beta density

$$f(x_1, x_2) = \frac{\Gamma(n_1 + n_2 + 1)}{\Gamma(n_1)\Gamma(n_2)} x_1^{n_1-1} x_2^{n_2-1}; \quad n_1, n_2 > 0, \quad x_1, x_2 > 0, \ x_1 + x_2 \leq 1. \tag{5.5}$$

Some simple calculations yield the Lorenz surface for the original variables as

$$L(u, v) = \frac{\Gamma(n_1 + n_2 + 3)}{\Gamma(n_1 + 2)\Gamma(n_2 + 2)} x^{n_1+1} y^{n_2+1} \tag{5.6}$$

where

$$u = \frac{B_x(n_1, n_2 + 1)}{B(n_1, n_2 + 1)} \quad \text{and} \quad v = \frac{B_y(n_2, n_1 + 1)}{B(n_2, n_1 + 1)},$$

and $B_x(p, q)$ is the incomplete beta function

$$B_x(p, q) = \int_0^x t^{p-1}(1 - t)^{q-1}\,\mathrm{d}t.$$

The Lorenz surface of the weighted random variable is

$$L^*(u, v) = \frac{\Gamma(n_1 + n_2 + 5)}{\Gamma(n_1 + 3)\Gamma(n_2 + 3)} x^{n_1+2} y^{n_2+2} \tag{5.7}$$

with the same u and v. From (5.6) and (5.7), the convenience in computing the Lorenz surfaces and ease of interpretation of the income inequality for the original and weighted variables is evident for form-invariance weighted distributions.

6 BAYESIAN INFERENCE

Apart from possessing the form-invariance property, the family of distributions presented in Theorem 4.1 is important in its own right. It includes several bivariate distributions that are well known for their applications. In addition, (4.2) provides some interesting features in the context of Bayesian inference and decision.

Let $(\underline{X}_{1i}, \underline{X}_{2i})$, $i = 1, 2, \ldots, n$ be a random sample of n observations taken from the joint density (4.2). Then the likelihood function based on this sample can be written as

$$L(\underline{x}_1, \underline{x}_2 \mid \theta_1, \theta_2) = (B_1(\theta_1))^n (B_2(\theta_2))^n (C(\theta_1, \theta_2))^n P(\underline{x}_1, \underline{x}_2) \exp(n(\theta_1 \log g_1 + \theta_2 \log g_2)) \tag{6.1}$$

where

$$\underline{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n}), \quad \underline{x}_2 = (x_{21}, x_{22}, \ldots, x_{2n})$$

and

$$g_i(\underline{x}_i) = \left(\prod_{j=1}^{n} x_{ij}\right)^{1/n}, \quad i = 1, 2$$

is the geometric mean of the observations in $\underline{x}_i$.

The kernel of the likelihood function given the sample values is

$$K(\theta_1, \theta_2 \mid \underline{x}_1, \underline{x}_2) = (B_1(\theta_1))^n (B_2(\theta_2))^n (C(\theta_1, \theta_2))^n \exp(n(\theta_1 \log g_1 + \theta_2 \log g_2)). \tag{6.2}$$

Hence (g_1, g_2) or equivalently $(\log g_1, \log g_2)$ is a sufficient statistic for (θ_1, θ_2). Notice that other parameters, if any, in the model are taken in $P(\underline{x}_1, \underline{x}_2)$ and will be treated as nuisance parameters or known. Thus the inference framework discussed here relates only to the vector (θ_1, θ_2). Hence a prior distribution for (θ_1, θ_2) can be prescribed by normalizing the kernel (6.2).

$$f'(\theta_1, \theta_2) \propto (B_1(\theta_1))^{n'} (B_2(\theta_2))^{n'} (C(\theta_1, \theta_2))^{n'} \exp(n'(\theta_1 \log a_1 + \theta_2 \log a_2)) \tag{6.3}$$

with a_1, a_2, n' as the hyper parameters in the prior with $a_1, a_2, n' > 0$. Combining (6.2) and (6.3), we arrive at the density of the posterior distribution of (θ_1, θ_2) as

$$f''(\theta_1, \theta_2) \propto (B_1(\theta_1))^{n''} (B_2(\theta_2))^{n''} (C(\theta_1, \theta_2))^{n''} \exp(n''(\theta_1 \log b_1 + \theta_2 \log b_2))$$

or

$$f''(\theta_1, \theta_2) = A(b_1, b_2; n'')(B_1(\theta_1))^{n''} (B_2(\theta_2))^{n''} (C(\theta_1, \theta_2))^{n''} \exp(n''(\theta_1 \log b_1 + \theta_2 \log b_2)) \tag{6.4}$$

which is of the same form as the prior. Thus (6.3) is a natural conjugate prior density in the sense of Raiffa and Schlaifer [12]. With the aid of the posterior density, inference can be made on (θ_1, θ_2) by employing the established techniques. In the form-invariant case, the original estimates of θ_1 and θ_2 can be used in inferring the characteristics of the weighted model.

As an illustration we consider the inference problem relating to the parameters (θ_1, θ_2) in the Dirichlet density specified by

$$f(x_1, x_2) = \frac{\Gamma(\theta_1 + \theta_2 + \theta_3 + 3)}{\Gamma(\theta_1 + 1)\Gamma(\theta_2 + 1)\Gamma(\theta_3 + 1)} x_1^{\theta_1} x_2^{\theta_2} (1 - x_1 - x_2)^{\theta_3} \tag{6.5}$$
$$x_1, x_2 > 0; \; x_1 + x_2 \leq 1; \; \theta_1, \theta_2, \theta_3 > -1$$

with θ_3 treated as known. Comparison with the bivariate family (4.2) shows that

$$B_1(\theta_1) = \frac{1}{\Gamma(\theta_1 + 1)}, \; B_2(\theta_2) = \frac{1}{\Gamma(\theta_2 + 1)} \text{ and } C(\theta_1, \theta_2) = \Gamma(\theta_1 + \theta_2 + \theta_3 + 3).$$

Thus the posterior density of (θ_1, θ_2) is

$$f''(\theta_1, \theta_2) = \frac{A(b_1, b_2; n'')(\Gamma(\theta_1 + \theta_2 + \theta_3 + 3))^{n''}}{(\Gamma(\theta_1 + 1))^{n''}(\Gamma(\theta_2 + 1))^{n''}} b_1^{n''\theta_1} b_2^{n''\theta_2} \tag{6.6}$$

where

$$[A(b_1, b_2; n'')]^{-1} = \int_{\theta_1} \int_{\theta_2} \frac{(\Gamma(\theta_1 + \theta_2 + \theta_3 + 3))^{n''}}{(\Gamma(\theta_1 + 1))^{n''}(\Gamma(\theta_2 + 1))^{n''}} b_1^{n''\theta_1} b_2^{n''\theta_2} \, d\theta_1 d\theta_2.$$

Since the density is continuous, the estimators of θ_1 and θ_2 can be prescribed as those values of the parameters that render $f''(\theta_1, \theta_2)$ a maximum (see DeGroot [3], p. 236). Thus the estimates of (θ_1, θ_2) are solutions of

$$\frac{\partial \log f''(\theta_1, \theta_1)}{\partial \theta_i} = 0, \quad i = 1, 2 \tag{6.7}$$

and

$$\frac{\partial^2 \log f''(\theta_1, \theta_2)}{\partial \theta_1^2} \frac{\partial^2 \log f''(\theta_1, \theta_2)}{\partial \theta_2^2} - \left[\frac{\partial^2 \log f''(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2}\right]^2 > 0. \tag{6.8}$$

Equation (6.7) reduces to

$$n'' \frac{\partial}{\partial \theta_1} \log \Gamma(\theta_1 + \theta_2 + \theta_3 + 3) - n'' \frac{\partial}{\partial \theta_1} \log \Gamma(\theta_1 + 1) = -n'' \log b_1 \tag{6.9}$$

and

$$n'' \frac{\partial}{\partial \theta_2} \log \Gamma(\theta_1 + \theta_2 + \theta_3 + 3) - n'' \frac{\partial}{\partial \theta_2} \log \Gamma(\theta_2 + 1) = -n'' \log b_2. \tag{6.10}$$

Equations (6.9) and (6.10) involve digamma functions and therefore numerical methods are required to solve them. One can use the Stirling's approximation and write (6.9) as (6.10) as

$$\frac{1}{2\theta_1} + \log \theta_1 \cong \frac{1}{2(\theta_1 + \theta_2 + \theta_3 + 2)} + \log(\theta_1 + \theta_2 + \theta_3 + 2) + \log b_1 \tag{6.11}$$

and

$$\frac{1}{2\theta_2} + \log \theta_2 \cong \frac{1}{2(\theta_1 + \theta_2 + \theta_3 + 2)} + \log(\theta_1 + \theta_2 + \theta_3 + 2) + \log b_2. \tag{6.12}$$

From (6.11) and (6.12), approximate solutions for θ_1 and θ_2 can be obtained. These can be substituted to verify the second order conditions. Values satisfying the two conditions can be used as initial approximations to solve (6.9) and (6.10). In case more than one maxima occurs, the (θ_1, θ_2) value for which the posterior density is the largest is the estimator.

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