On the generalized obnoxious center problem: antimedian subsets*

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Abstract

The set of vertices that maximize (minimize) the remoteness is the antimedian (median) set of the profile. It is proved that for an arbitrary graph G and $S \subseteq V(G)$ it can be decided in polynomial time whether S is the antimedian set of some profile. Graphs in which every antimedian set is connected are also considered.

Keywords: graph distance, remoteness, antimedian set, computational complexity, linear programming

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1 Introduction

A *profile* $\pi = (x_1, \ldots, x_k)$ on a graph G is a finite sequence of vertices of G. For a profile π on G and a vertex u of G the sum

$$D(u,\pi) = \sum_{x \in \pi} d_G(u,x)$$

is called the *remoteness* of u, see [2, 6]. (As usual, $d_G(u,x)$ is the length of a shortest u,x-path in G.) The vertex u is called an *(anti)median vertex for* π if $D(u,\pi)$ is minimum (maximum). The *(anti)median set* $(AM(\pi,G))$ $M(\pi,G)$ of π in G is the set of all (anti)median vertices for π.

Median sets have been intensively studied by now, see for instance [2, 3, 7]. In this note we concentrate on antimedian sets but nevertheless, the methods used also yield new results for median sets. The special case in which profiles consist of all vertices is known as the *obnoxious facility location problem* and has been previously studied; see, for instance, [4, 9, 10]. Another special case, in which the profiles consist of exactly three vertices, is the central topic of [1]. We also add that very recently Rao and Vijayakumar studied (anti)medians in cographs [8].

In the next section we ask what is the computational complexity of the following problem: given a graph G and a subset X of its vertices, does there exist a profile on G such that $X = AM(\pi,G)$? There is an exponential number of profiles with different vertices and, moreover, since it is allowed that in a profile a vertex is repeated several times, a profile can be arbitrarily large. However, using a linear programming approach we prove that the problem is polynomial. The same approach also works for median sets.

A related question is: which are the graphs in which all (anti)median sets are connected. The problem was solved for median sets by Bandelt and Chepoi in [3]. They also proved that it can be decided in polynomial time whether all median sets of a given graph are connected. In Section 3 we consider graphs in which all antimedian sets are connected. We do not give a characterization of such graphs but prove this property for a certain class of graphs that in particular includes cocktail-party graphs. Another such class is formed by Hamming graphs which follows from the fact that the Cartesian product preserves the property.

2 Polynomial recognition of (anti)median subsets

Note that there are $2^n - 1$ different profiles without repetition for a graph on n vertices. Hence even in this special case (when repetitions are not allowed) the direct approach to the problem whether a given subset of $V(G)$ is the antimedian set of some profile is exponential. However, we have:

Theorem 2.1 *Let G be a connected graph and $S \subseteq V(G)$. Then it can be determined in polynomial time whether S is the antimedian set of some profile on G.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and assume without loss of generality that $S = \{v_1, \ldots, v_k\}$ for some $1 \leq k \leq n$. For $i = 1, \ldots, n$ set

$$f_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} d_G(v_i, v_j) x_j \,.$$

Note that if x_i, $1 \leq i \leq n$, is interpreted as the number of times the vertex v_i appears in a given profile π, then $f_i(x_1, \ldots, x_n) = D(v_i, \pi)$. Consider the following linear program LP(f):

$$\begin{aligned}
&\min\{x_1 + \cdots + x_n\};\\
&x_i \geq 0,\ i = 1, \ldots, n;\\
&f_1(x_1, \ldots, x_n) = f_j(x_1, \ldots, x_n);\ j = 2, \ldots, k;\\
&f_1(x_1, \ldots, x_n) \geq f_j(x_1, \ldots, x_n) + 1;\ j = k+1, \ldots, n.
\end{aligned}$$

Suppose first that the linear program LP(f) has a solution. Since the coefficients are all integers, the solution x_i^*, $1 \leq i \leq n$, must be rational. Let r be an integer such that rx_i^* are integers. Then

$$f_i(rx_1, \ldots, rx_n) = \sum_{j=1}^{n} d_G(v_i, v_j) r x_j = r \sum_{j=1}^{n} d_G(v_i, v_j) x_j = r f_i(x_1, \ldots, x_n) \,.$$

It follows that $(rx_1, \ldots, rx_n)$ is a profile for which S is the antimedian set.

Suppose next that LP(f) has no solution. In other words, the constraint conditions have no feasible solution. Therefore, by the above interpretation of LP(f) we conclude that S is not an antimedian set for any profile on G.

In conclusion, S is an antimedian set if and only if LP(f) has a solution. Now, the number of (in)equalities of the LP is n and the size of the coefficients is bounded by $n-1$ - the largest possible distance in a connected graph on n vertices. Therefore LP can be solved in polynomial time and the theorem is proved. $\square$

In the proof of Theorem 2.1 the objective function of the LP(f) can be taken as any function which guarantees a finite optimum, we have just selected an instance of such a function.

Observe that the same approach (just reversing the inequalities in the constraints of the LP) also works for median sets. We therefore also have:

Theorem 2.2 *Let G be a connected graph and $S \subseteq V(G)$. Then it can be determined in polynomial time whether S is the median set of some profile on G.*

3 Graphs with connected antimedian sets

In this section we consider the question for which graphs G every antimedian set of G is connected. Note first that this is clearly the case for complete graphs. Another such class of graphs is described in the next result.

Proposition 3.1 *Let G be the graph obtained by removing k, $2 \leq k \leq n/2$, independent edges from the complete graph K_n. Then any antimedian set of G is connected.*

Proof. We use the LP from the proof of Theorem 2.1. Let $V(G) = \{v_1, \ldots, v_n\}$ and assume without loss of generality that $M = \{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\}$ is the set of non-edges of G.

First observe that if $S \subset V(G)$ induces a disconnected subgraph, then $S = \{v_p, v_{p+1}\}$ for some fixed odd p, $1 \leq p \leq 2k-1$. Suppose S is the antimedian set of some profile π on G. Then the following LP

$$\begin{array}{l}
\min\{x_1 + \cdots + x_n\};\\
x_i \geq 0,\ i = 1, \ldots, n;\\
f_p(x_1, \ldots, x_n) = f_{p+1}(x_1, \ldots, x_n);\\
f_j(x_1, \ldots, x_n) + 1 \leq f_p(x_1, \ldots, x_n);\ j \in \{1, \ldots, n\}, j \neq p, p+1\,,
\end{array}$$

has a solution, where

$$f_j(x_1, \ldots, x_n) = 2x_{j+1} + \sum_{\ell \notin \{j, j+1\}} x_\ell\ ; \quad j = 1, 3, \ldots, 2k-1, \tag{1}$$

$$f_j(x_1, \ldots, x_n) = 2x_{j-1} + \sum_{\ell \notin \{j-1, j\}} x_\ell\ ; \quad j = 2, 4, \ldots, 2k, \tag{2}$$

and

$$f_j(x_1, \ldots, x_n) = \sum_{\substack{\ell=1\\ \ell \neq j}}^{n} x_\ell\ ; \quad j = 2k+1, 2k+2, \ldots, n\,. \tag{3}$$

Since

$$f_p(x_1, \ldots, x_n) = 2x_{p+1} + \sum_{\ell \notin \{p, p+1\}} x_\ell$$

and

$$f_{p+1}(x_1, \ldots, x_n) = 2x_p + \sum_{\ell \notin \{p, p+1\}} x_\ell$$

we infer that $x_p = x_{p+1}$.

Moreover, because $f_q(x_1,\ldots,x_n) = 2x_{q+1} + \sum_{\ell\notin\{q,q+1\}} x_\ell$, $x_p = x_{p+1}$, and $f_q(x_1,\ldots,x_n) < f_p(x_1,\ldots,x_n)$, we also find that

$$2x_{q+1} + \sum_{\ell\notin\{q,q+1\}} x_\ell < x_p + x_{p+1} + \sum_{\ell\notin\{p,p+1\}} x_\ell = \sum_{\ell} x_\ell\,.$$

This in turn implies that $x_{q+1} > x_q$. Analogously, because $f_{q+1}(x_1,\ldots,x_n) = 2x_q + \sum_{\ell\notin\{q,q+1\}} x_\ell$, $x_p = x_{p+1}$, and $f_{q+1}(x_1,\ldots,x_n) < f_p(x_1,\ldots,x_n)$ we obtain that $x_q > x_{q+1}$ which is not possible. Hence the LP has no solution and therefore S cannot be an antimedian set. $\square$

Note that cocktail-party graphs (complete graphs K_{2n} minus a perfect matching) are special instances of graphs with connected antimedian sets.

To obtain more graphs with connected antimedian sets it is useful to consider Cartesian products of graphs. Recall that the *Cartesian product* $G\,\Box\,H$ of graphs G and H is the graph with the vertex set $V(G)\times V(H)$ where vertices (g,h) and (g',h') are adjacent if $gg'\in E(G)$ and $h=h'$, or $g=g'$ and $hh'\in E(H)$. *Hamming graphs* are defined as Cartesian products of complete graphs. For connected graphs G and H, $d_{G\,\Box\,H} = d_G + d_H$, an utmost useful fact known as *the Distance Lemma*, see [5].

Let π be a profile on a Cartesian product $G\,\Box\,H$. Then by the projection of π on G, $\mathrm{proj}_G\pi$, we mean the projection of π on G taking into account the multiplicities of projected vertices.

Proposition 3.2 *Let π be a profile on $G\,\Box\,H$. Let $\pi_G = \mathrm{proj}_G\pi$ and $\pi_H = \mathrm{proj}_H\pi$. Then*

$$AM(\pi, G\,\Box\,H) = AM(\pi_G, G) \times AM(\pi_H, H)\,.$$

Proof. Let (g,h) be an arbitrary vertex of $G\,\Box\,H$. Then, using the Distance Lemma, we compute the remoteness as follows:

$$\begin{aligned}
D_{G\,\Box\,H}((g,h),\pi) &= \sum_{(g',h')\in\pi} d_{G\,\Box\,H}((g,h),(g',h')) \\
&= \sum_{(g',h')\in\pi} (d_G(g,g') + d_H(h,h')) \\
&= \sum_{g'\in\pi_G} d_G(g,g') + \sum_{h'\in\pi_H} d_H(g,g') \\
&= D_G(g,\pi_G) + D_H(h,\pi_H)\,.
\end{aligned}$$

We conclude that $(g,h)\in AM(\pi, G\,\Box\,H)$ if and only if $g\in AM(\pi_G,G)$ and $h\in AM(\pi_H,H)$. $\square$

Corollary 3.3 *Let G and H be connected graphs with connected antimedian sets. Then $G \,\Box\, H$ has connected antimedian sets. In particular, Hamming graphs have connected antimedian sets.*

We conclude by asking for a characterization of graphs with connected antimedian sets.

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