PROBABILITY THEORY, STOCHASTIC PROCESSES AND THEIR APPLICATIONS

PROBABILISTIC ANALYSIS OF SOME QUEUEING AND INVENTORY MODELS

THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By

JACOB M. J.

DEPARTMENT OF MATHEMATICS AND STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
COCHIN - 682 022
INDIA

1987

CERTIFICATE

Certified that the work reported in the present thesis is based on the bona-fide work done by Sri. Jacob M.J. under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

A. Krishnamoorthy
Research Guide
Professor in Applied Mathematics
Department of Mathematics and Statistics
Cochin University of Science and Technology
Cochin 682 022

DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

(JACOB M.J.)

ACKNOWLEDGEMENT

I wish to express my gratitude towards Professor A.Krishnamoorthy, my supervisor, Professor T.Thrivikraman, Head of the Department, Professor R. Ramanarayanan, Government Arts College, Krishnagiri and all my colleagues for their help and co-operation and Mr. Jose for his excellent typing.

CONTENTS

Cont'd.

Cont'd.

Chapter 1

INTRODUCTION

In this thesis we attempt to make a probabilistic analysis of some physically realizable, though complex, storage and queueing models. It is essentially a mathematical study of the stochastic processes underlying these models. Our aim is to have an improved understanding of the behaviour of such models, that may widen their applicability. Different inventory systems with random lead times, vacation to the server, bulk demands, varying ordering levels, etc. are considered. Also we study some finite and infinite capacity queueing systems with bulk service and vacation to the server and obtain the transient solution in certain cases. Each chapter in the thesis is provided with self introduction and some important references. This chapter gives a brief general introduction to the subject matter and related topics.

1.1 INVENTORY THEORY

An inventory is an amount of material stored for the purpose of sale or production. The inventory models are usually characterized by the demand pattern and the policy for replenishing the stock in the store. The two basic types of policy for replenishment are (i) the ordering cycle policy under which orders for replenishment are placed at regular intervals of time of length T, (ii) the (s,S) policy under which orders are placed as and when the stock in the store plus the quantity already on order falls to some fixed level s. The replenishments ordered under any of these policies are assumed to arrive after a time lag L, which may be fixed or a random variable. This time lag L is called 'lead time'. During a lead time the inventory level may fall to zero. The time duration for which the level of inventory continuously remains at zero is called a dry period.

A valuable review of the problems in the probability theory of storage systems is given by Gani [1957]. A systematic account of probabilistic treatment in the study of inventory systems using renewal theoretic arguments is given in Arrow, Karlin and Scarf [1958]. Hadley and Whitin [1963] deals with the applications of such models to practical situations. Tijms [1972] gives a detailed analysis of the inventory systems under (s,S) policy.

The cost analysis of different inventory systems is given in Naddor [1966]. A practical treatment of the (s,S) lost sales model can be found in the recent books by Silver and Peterson [1984] and Tijms [1986].

Veinott [1966] gives a detailed review of the work carried out in (s,S) inventory systems up to 1966. We refer to the monograph by Ryshikov [1973] for inventory systems with random lead times. Gross and Harris [1971] and Gross, Harris and Lechner [1971] deal with one for one ordering inventory policies with state dependent lead times. Sivazlian [1974] considers an (s,S) inventory model in which unit demands of items occur with arbitrary interarrival times between demands, but lead time is assumed to be zero. His results are extended by Srinivasan [1979] to the case in which lead times are independent and identically distributed random variables having a general distribution. Sahin[1979] considers an (s,S) inventory system in which demand quantities are random but lead time is a constant. Again in 1983 Sahin discussed an inventory system in which the interarrival times between consecutive demands, quantities demanded and lead times are all independent and generally distributed sequences of independent and identically distributed random variables. He obtained the binomial moments for the inventory deficit. Thangaraj and Ramanarayanan [1983] consider an inventory system with random lead times and having two ordering levels. Kalpakam and Ariviringnan [1985] deals with an inventory system having one exhibiting item subject to random failure. Daniel and Ramanarayanan [1987 a,b] consider inventory systems with vacation to the server during dry period.

1.2 QUEUEING THEORY

Queueing theory is a well developed branch of applied probability theory. Historically, the subject of queueing theory has been developed largely in the context of telephone traffic engineering. Over the past three decades, steady progress has been made towards solving increasingly difficult and realistic queueing models.

A queueing model is usually defined in terms of three characteristics-- the input process, the service mechanism and the queue discipline. The input process describes the sequence of requests for service. Often the input process is specified in terms of the distribution of the lengths of time between consecutive customer arrival instants. The service mechanism is the category that includes such characteristics as the number of servers and the lengths of time that customers hold the servers. The queue discipline deals with the rule by which customers are taken for service.

For the single server queue a busy period is the time interval during which the server is continuously busy.i.e. it is the length of time from the instant the (previously idle) server is seized until it next becomes idle. The time between the starting points of two consecutive busy periods is called a busy cycle. The actual waiting time in the queue of a customer is defined as the time between the moment of his arrival and the moment at which his service starts. The virtual waiting time at time t is the actual waiting time of a customer if he had arrived at time t.

For a complete reference on the earlier works of queueing theory we refer to the bibliographies given in the books by Syski [1960], Saaty [1961], Takacs [1962], Prabhu [1965], Cooper [1972], Gross and Harris [1974], Neuts [1981] and Medhi [1984].

M/G/1 queueing models where the server is not available over occasional intervals of time has been considered by many authors. The times, when the server is not available are called vacations (also referred to as rest). A queueing model in which the server goes for vacation whenever the system becomes empty is an 'exhaustive service system'. This model has been studied by Miller [1964], Cooper [1970], Levy and Yechiali [1975], Shantikumar [1980], Scholl and Kleinrock [1983], Lee [1984] and Fuhrmann [1984]. M/G/1 queueing systems without exhaustive service is studied by Neuts and Ramalhoto[1984], Ali and Neuts [1984] and Fuhrmann and Cooper [1985]. Daniel [1985] discusses several interesting models with vacation to the server. Doshi [1985] considers the G/G/1 exhaustive service system and proves that the 'decomposition property' holds. G/G/1 vacation system with Bernoulli schedules is considered by Keilson and Servi [1986]. For a complete survey of the queueing systems with vacations, we refer to Doshi [1986].

1.3 NOTATIONS

In this section we introduce the following notations, that may be frequently used in the thesis.

$*$ denotes the convolution operator.

$f^{*n}(x)$ is the n-fold convolution of $f(x)$ with itself.

For a distribution function $F(x)$; $\bar{F}(x) = 1-F(x)$

δ_{ij} is the Kronecker delta function given by $\delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$

$[x]$ denotes the integral part of x.

$\gamma_{\alpha,\beta}(\cdot)$ is the Gamma density function with parameters α and β.

$\Gamma_{\alpha,\beta}(\cdot)$ is the Gamma distribution function with parameters α and β.

$E(X)$ is the expectation of the random variable X.

We define the convolution of two matrices A and B as follows. If $A(t) = [a_{ij}(t)]$ is a matrix of order $m \times p$ and $B(t) = [b_{ij}(t)]$ is a matrix of order $p \times n$, then $A*B(t) = [c_{ij}(t)]$ is a matrix of order $m \times n$ whose elements are given by $c_{ij}(t) = \sum_{k=1}^{p} a_{ik}*b_{kj}(t)$.

1.4 <u>RENEWAL THEORY</u>

Let $\{X_n, n=1,2, \ldots\}$ be a sequence of nonnegative independent random variables with a common distribution function $F(x)$. Let $S_0 = 0$ and for $n \geq 1$, $S_n = \sum_{i=1}^{n} X_i$.

Define $N(t) = \text{Sup}\left\{n \mid S_n \leq t\right\}$

If $\mu = \int_0^\infty x \, dF(x)$, which we assume to exist, by the strong law of large numbers we have $\frac{S_n}{n} \longrightarrow \mu$ as $n \longrightarrow \infty$ with probability 1. Hence, for finite t, $S_n \leq t$ only finitely often and so $N(t) < \infty$ with probability 1. The process $\left\{N(t), \ t \geq o\right\}$ is a Renewal process.

It is easy to note that $N(t) \geq n \Leftrightarrow S_n \leq t$.
Using this one may obtain, $P\left\{N(t)=n\right\} = F^{*n}(t)-F^{*n+1}(t)$.
Let $M(t) = E(N(t))$; $M(t)$ is called the renewal function and it can be shown that $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$.
Let $m(t) = M'(t)$; $m(t)$ is called the renewal density function and $m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$ if the density function $f(x)=F'(x)$ exists.

Suppose $\left\{X_n, \ n = 1,2, \ldots\right\}$ is a sequence of independent nonnegative random variables with X_1 having distribution function $G(x)$ and X_n for $n > 1$ having distribution function $F(x)$.
Let $S_o = o$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$.

Define $N_D(t) = \operatorname{Sup}\left\{ n \mid S_n \leq t \right\}$, $N_D(t)$ is called a Delayed renewal process or a Modified renewal process.

Here we have, $P\left\{N_D(t)=n\right\} = G{*}F^{*n-1}(t) - G{*}F^{*n}(t)$.

The modified renewal function is $M_D(t) = E(N_D(t))$ and

$M_D(t) = \sum_{n=1}^{\infty} G{*}F^{*n-1}(t)$. The modified renewal density

function is $m_D(t) = M_D'(t)$ and it is given by

$m_D(t) = \sum_{n=1}^{\infty} g{*}f^{*n-1}(t)$, under the additional assumption

that the density function $g(x) = G'(x)$ and $f(x) = F'(x)$ exist.

For more details of the renewal theory we refer to Cox [1962].

1.5. SUMMARY OF THE WORK INCLUDED IN THIS THESIS

In the second chapter we consider three models on (s,S) inventory systems with finite backlog of demands and vacation to the server. In all the models the inter-arrival times of demands and lead times are independent sequences of independent and identically distributed random variables having general distributions. In the first two models, whenever the inventory becomes dry, the server goes for vacation. In the third model when the inventory becomes dry, a local purchase is made according to the availability and the server goes for vacation only if the local purchase is impossible. The vacation period is also random with a general distribution. If the server returns from vacation before the realization of the order, he permits a finite number of demands to wait. All the demands arriving during the vacation period of the server are lost. In the first and third model, order size is a constant and in the second model the order size can vary according to the inventory level. Using renewal theory, the inventory level and queue size probabilities are presented explicitly.

In chapter 3, we derive expressions to find the correlation between lead time and dry period for (s,S) inventory systems and finite capacity dam models. Also, assuming exponential distributions for interarrival times of demands and lead times, simple expressions for the joint moments are obtained.

Fourth chapter deals with an (s,S) policy inventory system under the assumption that intervals of time between successive demand points, quantities demanded at these points and lead times are independent sequences of independent and identically distributed random variables. Interarrival times of demands and lead times follow general distributions. The quantity demanded each time is a discrete random variable taking values between two integers a and b such that $s < a \leq b < S-s$. Backlogging of demands are not allowed. Exact expressions for the system size probabilities are derived.

In chapter 5, we consider an inventory system in which an ordering level is decided according to the number of demands during the previous lead time. Interarrival times of demands and lead times are generally distributed random variables and each demand is for one unit. All the demands that occur during the inventory dry period are lost. Using renewal theoretic arguments we derive the inventory level probabilities. Also we discuss the correlation between the number of demands in a lead time and the next dry period.

$G/M^{a,b}/1$ queueing system with vacation to the server is considered in chapter 6. The service time is exponentially distributed with parameter μ_i, if i is the size of the batch being served. The vacation periods are also exponentially distributed. Matrix-geometric method of Neuts is used to find the steady state probabilities of the system size. The structure of the matrix geometric equation is not simple and is not yielding to any easy algorithmic approach for solution in the general set up. Probability distribution of waiting time is given explicitly.

In chapter 7, we consider a finite capacity M/G/1 queueing system with server going for vacation whenever there is no unit in the system. The vacation periods are independent and identically distributed random variables having a general probability distribution function. The capacity of the waiting room is finite and all the demands that arrive when the waiting room is full are lost. Using renewal theory, we derive the transient system size probabilities at arbitrary time points. Also we derive expressions for the probability distribution of virtual waiting time in the queue at any time t.

In the last chapter we consider an $M/G^{a,b}/1$ queueing system with a waiting room that allows only a maximum of 'b' customers to wait at any time. A minimum of 'a' customers are required to start a service and the server goes for vacation whenever he finds less than 'a' customers in the waiting room after a service. If the server returns from vacation to find less than 'a' customers waiting, he begins another vacation immediately. Here also expressions for the time dependent system size probabilities at arbitrary time points are derived.

The expressions we derive are complicated and hence do not easily yield to give numerical solutions. Developing algorithms for these will be quite worthwhile work.

Chapter 2

INVENTORY SYSTEMS WITH FINITE BACKLOG OF DEMANDS AND VACATION TO THE SERVER*

2.1. INTRODUCTION

The probabilistic analysis of (s,S) inventory models using renewal theoretic arguments is considered by many authors. For instance, Arrow, Karlin and Scarf [1958] and Tijms [1972] contain detailed treatment of these models. Sivazlian [1974] deals with a continuous review (s,S) inventory system with general interarrival distributions between unit demands. Srinivasan [1979] considers the system with general demand arrival times, random lead times and unit demands. Thangaraj and Ramanarayanan [1983] consider an inventory system with two ordering levels. Daniel and Ramanarayanan [1987a,b] consider several models allowing vacation to the server during dry period.

In this chapter we consider three models of (s,S) policy inventory systems with finite backlog of demands and rest time for the server. In all the models, the interarrival times of demands and lead times are independent sequences of independent and identically

*To appear in Cahiers du C.E.R.O. Vol.29, 1987.

distributed random variables having general distributions. In the first two models, whenever the inventory becomes dry, the server goes for vacation. In the third model, when the inventory becomes dry, a local purchase is made according to the availability of the item and the server goes for vacation only if the local purchase is impossible. The vacation period is also random with a general distribution. If the server returns from vacation before the realization of the order, he permits a finite number of demands to wait. All the demands arriving during the vacation period of the server are lost. In the first and third model, order size is a constant and in the second model the order size can vary according to the inventory level.

In all these models, the intervals between placing successive orders are independent and identically distributed random variables. We calculate its probability density function and using renewal theory we derive the inventory level and queue size probabilities explicitly.

Now we introduce the following notations.

${}_s f_S(\cdot)$ Probability density function of the time between placing two successive orders.

$f_{s,i}(x)dx$ = Probability that the stock level drops to i in $(x, x+dx)$ due to the first demand served after the replenishment, given at time zero the order is placed.

$$k(x) = \sum_{n=0}^{\infty} f^{*n}(x)$$

$$q(x) = \sum_{n=0}^{\infty} {}_s f_s^{*n}(x)$$

For $1 \leq i \leq S$,

$\pi_i(t)$ = Probability that the stock level is i at time t, given at time zero the inventory size is S.

$\pi_o(t)$ = Probability that the inventory is dry and there is no waiting of demands at time t, given at time zero the inventory level is S.

$\pi_{-i}(t)$ = Probability that there are i demands waiting at time t, given the inventory level at time zero is S.

2.2. DESCRIPTION OF MODEL-1

In this section we give the details of the assumptions of this model. The maximum capacity of the store is S.

The interoccurrence times of demands are independent and identically distributed random variables with distribution function $F(.)$ and density function $f(.)$. Demands occur for one unit at a time. Whenever the inventory level falls to s, an order is placed for a quantity S-s. The lead time is a random variable with distribution function $G(.)$ and density function $g(.)$. When the inventory becomes dry (i.e. the inventory level falls to zero) the server goes for vacation for a random period with probability distribution function $H(.)$ and density function $h(.)$. All the demands that arrive during the rest time of the server are lost. During the inventory dry period, arriving demands are permitted to wait for service only after the rest time of the server, subject to a maximum of size S-2s-1. They are served when the order is realized. It may be noted that since the size of the order 'S-s' minus the maximum queue length S-2s-1 is s+1, we avoid placing a new order when an order is not realized. Finally we assume that, the interoccurrence times of demands, lead times and rest times are all independent.

In order to calculate the inventory level and queue size probabilities, we find the transition time probability density functions. It is easy to note that for $S-s \le i \le S-1$,

$$f_{s,i}(x) = \int_0^x f^{*S-i-1}(u)\ [G(x)-G(u)]\ f(x-u)\,du \qquad .. \quad (1)$$

Also,

$$f_{s,S-s-1}(x) = \int_0^x f^{*s}(u) \int_0^{x-u} k(v)\,[H(x-u)-H(v)]$$

$$[G(x)-G(u)]\ f(x-u-v)\,dv\ du \qquad .. \quad (2)$$

To write down equation (2) consider the inter-arrival (o,x). At u in this interval, the s^{th} demand occurs. During $(u,u+v)$ several demands are lost and at $u+v$ a demand is lost. The server who goes for rest at u returns only after $u+v$ but before x. The order placed at time zero is realized in (u,x) and a demand occurs at x.

We get for $s+1 \leq i \leq S-s-2$,

$$f_{s,i}(x) = \int_0^x f^{*s}(u) \int_0^{x-u} k(v) \int_0^{x-u-v} [H(w+v)-H(v)]\,f(w)$$

$$\int_0^{x-u-v-w} f^{*(S-s-i-2)}(y)\,[G(x)-G(u+v+w+y)]$$

$$f(x-u-v-w)\,dy\ dw\ dv\ du \qquad .. \quad (3)$$

To obtain equation (3) consider the interval (o,x). At u the s^{th} demand occurs. Demands are lost during $(u,u+v)$ and at $u+v$ a demand is lost. Server returns during $(u+v,\ u+v+w)$ after rest. $(S-s-i-1)$ demands arrive and wait for service before the order is realized and a demand occurs at x.

Considering the fact that when the queue size during the lead time is $S-2s-1$, further arriving demands are lost, we get,

$$f_{s,s}(x) = \int_o^x f^{*s}(u) \int_o^{x-u} k(v) \int_o^{x-u-v} [H(w+v)-H(v)]f(w)$$

$$\int_o^{x-u-v-w} k*f^{*(S-2s-2)}(y)\,[G(x)-G(u+v+w+y)]$$

$$f(x-u-v-w-y)\,dy\ dw\ dv\ du \qquad \text{.. (4)}$$

Using (1), (2), (3) and (4) we find the probability density function of the time between successive orders as,

$$_sf_s(x) = \sum_{i=s}^{S-1} \int_o^x f_{s,i}(u)\ f^{*(i-s)}(x-u)\,du \qquad \text{.. (5)}$$

2.3. THE INVENTORY LEVEL AND QUEUE SIZE PROBABILITIES

It is easy to obtain,

$$\pi_S(t) = \bar{F}(t) + \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\ G(t-u-v)\bar{F}(t-u-v)\,dv\,du \quad .. (6)$$

Also for $s+1 \leq i \leq S-1$,

$$\pi_i(t) = [F^{*S-i}(t) - F^{*S-i+1}(t)] + \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)$$

$$\int_0^{t-u-v} \sum_{j=i}^{S-1} f_{s,j}(w)[F^{*j-i}(t-u-v-w)-F^{*j-i+1}(t-u-v-w)]$$

$$dw\,dv\,du$$

$$+ \delta_{S-s,i} \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v) \int_0^{t-u-v} f^{*s}(w)[G(t-u-v)-G(w)]$$

$$\bar{H}(t-u-v-w)\,dw\,dv\,du$$

$$+ \delta_{S-s,i} \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v) \int_0^{t-u-v} f^{*s}(w) \int_0^{t-u-v-w} k(y)$$

$$[H(t-u-v-w)-H(y)][G(t-u-v)-G(w)]\bar{F}(t-u-v-w-y)$$

$$dy\,dw\,dv\,du \quad .. (7)$$

The first term on the right side equation (7) is the probability that exactly S-i demands occur and the second term is the probability that the inventory level drops to s, several orders are placed and realized, a transition from level s to level j occurs and after which exactly j-i demands occur. The last two terms are written considering the realization of an order during the inventory dry period. For i = S-s, the third term is the probability that the inventory level is S-s at time t due to the realization of order before t but the server taking rest. For i=S-s, the fourth term is the probability that the inventory level becomes S-s due to the realization of an order, rest period of the server is over but no demand has occured after his return.

Now for $1 \leq i \leq s$, we get,

$$\pi_i(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\overline{G}(t-u-v)[F^{*s-i}(t-u-v) - F^{*s-i+1}(t-u-v)]dv\, du \qquad .. \quad (8)$$

Using the argument that during the inventory dry period (i) the server is absent or (ii) he is present but demands have not arrived after his rest time, we find,

$$\pi_o(t) = \int_o^t f^{*S-s}(u) \int_o^{t-u} q(v)\, \bar{G}(t-u-v) \int_o^{t-u-v} f^{*s}(w)\, \bar{H}(t-u-v-w)\, dw\, dv\, du$$

$$+ \int_o^t f^{*S-s}(u) \int_o^{t-u} q(v)\, \bar{G}(t-u-v) \int_o^{t-u-v} f^{*s}(w) \int_o^{t-u-v-w} k(y)\, [H(t-u-v-w)-H(y)]\bar{F}(t-u-v-w-y)\, dy\, dw\, dv\, du \qquad .. \quad (9)$$

Also for $1 \leq i \leq S-2s-2$,

$$\pi_{-i}(t) = \int_o^t f^{*S-s}(u) \int_o^{t-u} q(v)\, \bar{G}(t-u-v) \int_o^{t-u-v} f^{*s}(w) \int_o^{t-u-v-w} k(y) \int_o^{t-u-v-w-y} [H(y+z)-H(y)]\, f(z)\, [F^{*i-1}(t-u-v-w-y-z) - F^{*i}(t-u-v-w-y-z)]\, dz\, dy\, dw\, dv\, du \qquad .. \quad (10)$$

To obtain equation (10) we consider the interval (o,t). At u the $(S-s)^{th}$ demand occurs. During $(u,u+v)$ several orders are placed and realized. At $u+v$ an order is placed but not realized up to t. At $u+v+w$ inventory becomes dry. During $(u+v+w,\ u+v+w+y)$ several demands are lost and at $u+v+w+y$ a demand is lost. The next demand occurs at $u+v+w+y+z$ and the server returns during (u+v+w+y, u+v+w+y+z). Exactly i-1 demands occur in (u+v+w+y+z,x).

Considering the maximum size of the backlog is $S-2s-1$, we find,

$$\pi_{-(S-2s-1)}(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\, \bar{G}(t-u-v) \int_0^{t-u-v} f^{*s}(w) \int_0^{t-u-v-w} k(y) \int_0^{t-u-v-w-y} f(z)\, [H(y+z)-H(y)]\, F^{*S-2s-2}(t-u-v-w-y-z)\, dz\, dy\, dw\, dv\, du \quad .. \quad (11)$$

2.4 <u>DESCRIPTION OF MODEL-2</u>

In this model also we assume that the demands occur in accordance with a general renewal process and the lead time distribution for an order is general. Let $F(x)$ be the distribution function of the interoccurrence times of demands and let $f(x)$ be the corresponding probability density function. Demands are for one unit at a time. Maximum capacity of the store is S and an order is placed whenever the inventory level falls to s. The lead time distribution and density functions are respectively $G(x)$ and $g(x)$. After the lead time, an agent arrives and he can supply j units, $S-s \leq j \leq S$. If the inventory level is i, $o \leq i \leq s$, he supplies S-i units and so the inventory becomes full after each replenishment. When the inventory becomes dry, the server goes for rest for a random time whose distribution and density functions are $H(x)$ and $h(x)$ respectively. During the inventory dry period, arriving demands are permitted to wait for service only after the rest time of the server subject to a maximum of size S-s-1. They are served when the order is realized. Here we may note that since the maximum size of an order S minus the maximum queue length S-s-1 is s+1, we avoid placing a new order when an order is not realized. Also we assume that the interarrival times of demands, lead times and rest times are all independent.

Here we obtain the transition time density function as follows:

$$f_{s,S-1}(x) = \sum_{i=o}^{s-1} \int_{0}^{x} f^{*i}(u)[G(x)-G(u)]f(x-u)\,du + \int_{0}^{x} f^{*s}(u)$$

$$\int_{0}^{x-u} k(v)[G(x)-G(u)][H(x-u)-H(v)]f(x-u-v)\,dv\,du \qquad (12)$$

The first term corresponds to the case that the s to S-1 transition occurs before the inventory becomes dry. To write the second term we consider the interval (o,x). At u the inventory becomes dry. During $(u,u+v)$ several demands are lost and at $u+v$ a demand is lost. Next demand occurs at x. During (u,x) the order is realized and during $(u+v,x)$ the server returns.

For $s+1 \leq i \leq S-2$, we have,

$$f_{s,i}(x) = \int_0^x f^{*s}(u) \int_0^{x-u} k(v) \int_0^{x-u-v} [H(w+v)-H(v)]f(w)$$

$$\int_0^{x-u-v-w} f^{*S-i-2}(y)\,[G(x)-G(u+v+w+y)]\,\cdot$$

$$f(x-u-v-w)\,dy\,dw\,dv\,du \qquad .. (13)$$

To write down equation (13) consider the interval (o,x). At u the s^{th} demand occurs. Demands are lost during $(u,u+v)$ and at $u+v$ a demand is lost. Server returns during $(u+v, u+v+w)$ after rest. $S-i-1$ demands arrive and wait for service before the order is realized and a demand occurs at x.

Now,

$$f_{s,s}(x) = \int_0^x f^{*s}(u) \int_0^{x-u} k(v) \int_0^{x-u-v} [H(w+v)-H(v)]f(w)$$

$$\int_0^{x-u-v-w} k*f^{*S-s-2}(y)\,[G(x)-G(u+v+w+y)]$$

$$f(x-u-v-w-y)\,dy\,dw\,dv\,du \qquad .. (14)$$

Then we have,

$$ {}_{s}f_{s}(x) = \sum_{i=s}^{S-1} \int_{0}^{x} f_{s,i}(u)\, f^{*i-s}(x-u)\,du \qquad \text{.. (15)} $$

2.5. INVENTORY LEVEL AND QUEUE SIZE PROBABILITIES

Here we give the inventory level probabilities. It is easily seen that,

$$ \pi_{S}(t) = \bar{F}(t) + \int_{0}^{t} f^{*S-s}(u) \int_{0}^{t-u} q(v) \int_{0}^{t-u-v} \sum_{i=0}^{s-1} f^{*i}(w) $$

$$ [G(t-u-v)-G(w)]\bar{F}(t-u-v-w)\,dw\,dv\,du $$

$$ + \int_{0}^{t} f^{*S-s}(u) \int_{0}^{t-u} q(v) \int_{0}^{t-u-v} f^{*s}(w) \int_{0}^{t-u-v-w} k(y) $$

$$ [H(t-u-v-w)-H(y)]\ [G(t-u-v)-G(w)] $$

$$ \bar{F}(t-u-v-w-y)\,dy\,dw\,dv\,du $$

$$ + \int_{0}^{t} f^{*S-s}(u) \int_{0}^{t-u} q(v) \int_{0}^{t-u-v} f^{*s}(w)\,\bar{H}(t-u-v-w) $$

$$ [G(t-u-v)-G(w)]\,dw\,dv\,du \qquad \text{.. (16)} $$

The first term is the probability that no demand has occured during (o,t). The second term is the probability that during (o,t) several orders are realized, the last order placed is realized before the inventory becomes dry and no demand has occurred after its realization. The third term is the probability that the inventory level is S immediately after a dry period and the server is available. The fourth term is the same case when the server is taking rest.

For $s+1 \leq i \leq S-1$, we get,

$$\pi_i(t) = [F^{*(S-i)}(t) - F^{*(S-i+1)}(t)] + \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)$$

$$\int_0^{t-u-v} \sum_{j=i}^{S-1} f_{s,j}(w) \; [F^{*(j-i)}(t-u-v-w) -$$

$$F^{*(j-i+1)}(t-u-v-w)]dw \; dv \; du \qquad \text{.. (17)}$$

Now for $1 \leq i \leq s$,

$$\pi_i(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\bar{G}(t-u-v) \; [F^{*s-i}(t-u-v) -$$

$$F^{*s-i+1}(t-u-v)]dv \; du \qquad \text{.. (18)}$$

Also,

$$\pi_o(t) = \int_o^t f^{*S-s}(u) \int_o^{t-u} q(v)\, \bar{G}(t-u-v) \int_o^{t-u-v} f^{*s}(w)\, \bar{H}(t-u-v-w)\, dw\, dv\, du$$

$$+ \int_o^t f^{*S-s}(u) \int_o^{t-u} q(v)\, \bar{G}(t-u-v) \int_o^{t-u-v} f^{*s}(w) \int_o^{t-u-v-w} k(y)\, [H(t-u-v-w)-H(y)]\bar{F}(t-u-v-w-y)\, dy\, dw\, dv\, du$$

.. (19)

Now we find the queue size probabilities as follows.

For $1 \leq i \leq S-s-2$,

$$\pi_{-i}(t) = \int_o^t f^{*S-s}(u) \int_o^{t-u} q(v)\bar{G}(t-u-v) \int_o^{t-u-v} f^{*s}(w) \int_o^{t-u-v-w} k(y) \int_o^{t-u-v-w-y} [H(y+z)-H(y)]\, f(z)\, [F^{*i-1}(t-u-v-w-y-z) - F^{*i}(t-u-v-w-y-z)]\, dz\, dy\, dw\, dv\, du$$

.. (20)

To obtain equation (20) consider the interval (o,t).

At u the $(S-s)^{th}$ demand occurs. During (u,u+v) several orders are placed and realized. At u+v an order is placed but not realized up to t. At u+v+w inventory becomes dry. During (u+v+w, u+v+w+y) several demands are lost and at u+v+w+y a demand is lost. The next demand occurs at u+v+w+y+z and the server returns during (u+v+w+y, u+v+w+y+z). Exactly i-1 demands occur in (u+v+w+y+z,x).

Finally,

$$\pi_{-(S-s-1)}(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\bar{G}(t-u-v) \int_0^{t-u-v} f^{*s}(w)$$

$$\int_0^{t-u-v-w} k(y) \int_0^{t-u-v-w-y} f(z) \; [H(y+z)-H(y)]$$

$$F^{*S-s-2}(t-u-v-w-y-z)\,dz\,dy\,dw\,dv\,du \qquad \ldots (21)$$

2.6 DESCRIPTION OF MODEL-3

Here we consider an (s,S) inventory system with the following assumptions. The interarrival times of demands are independent and identically distributed random variables with distribution function F(x) and density function f(x). Demands are for one unit at a time. S is the maximum capacity of the store. When the inventory level falls to s, an order for S-s units is placed. Lead times are independent and identically distributed random variables with distribution function $G(x)$ and density function $g(x)$. Whenever the inventory level drops to zero, if the item is available, s units are brought to the store immediately at an additional cost. Irrespective of the time, let p be the probability that the item is available and let $q= 1-p$ be the probability that it is not available. When the inventory level drops to zero, if the item is not available, the store is closed for a random length of time, having distribution function $H(x)$ and density function $h(x)$. All the demands that arrive during this closed period is lost. If the store is opened before the realization of the order, no local purchase is made, but backlogging of demands is allowed to a maximum of S-2s-1 units. As the difference between the order size and the maximum queue length is s+1, we avoid placing a new order when an order has not realized. Interarrival times of demands, lead times and store closing periods are all assumed to be independent.

Here also using the notations introduced for transition probabilities, we obtain the following relations.

For $S-s \leq i \leq S-1$,

$$f_{s,i}(x) = \int_0^x \sum_{n=0}^{\infty} f^{*S+ns-i-1}(u)\ p^n [G(x)-G(u)] f(x-u)\,du \qquad (22)$$

To obtain (22) consider the interval (o,x). Whenever the level hit the zero level, local purchase was possible and all the demands are met in (o,u) and a demand is met at time u. The replenishment occurs in (u,x) and the first demand after u occurs at x.

For $i = S-s-1$, we easily get,

$$f_{s,S-s-1}(x) = \int_0^x \sum_{n=1}^{\infty} f^{*ns}(u)\,p^{n-1} q \int_0^{x-u} k(v)\,[H(x-u)-H(v)]$$

$$[G(x)-G(u)]\ f(x-u-v)\,dv\ du \qquad (23)$$

Also for $s+1 \leq i \leq S-s-2$,

$$f_{s,i}(x) = \int_0^x \sum_{n=1}^{\infty} f^{*ns}(u)\ p^{n-1} q \int_0^{x-u} k(v) \int_0^{x-u-v} [H(w+v)-H(v)] f(w)$$

$$\int_0^{x-u-v-w} f^{*S-s-i-2}(y)\ [G(x)-G(u+v+w+y)]$$

$$f(x-u-v-w-y)\,dy\ dw\ dv\ du \qquad .. (24)$$

Again, for $i = s$, we have,

$$f_{s,s}(x) = \int_0^x \sum_{n=1}^{\infty} f^{*ns}(u)\, p^{n-1}q \int_0^{x-u} k(v) \int_0^{x-u-v} [H(w+v)-H(v)]\, f(w)$$

$$\int_0^{x-u-v-w} f^{*S-2s-2}\, k(y)\, [G(x)-G(u+v+w+y)]$$

$$f(x-u-v-w-y)\, dy\, dw\, dv\, du \qquad \ldots (25)$$

Thus we get the probability density function of the time between placing two successive orders as,

$$_s f_s(x) = \sum_{i=s}^{S-1} \int_0^x f_{s,i}(u)\, f^{*i-s}(x-u)\, du \qquad \ldots (26)$$

2.7. <u>INVENTORY LEVEL AND QUEUE SIZE PROBABILITIES</u>

We get,

$$\pi_S(t) = \bar{F}(t) + \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v) \sum_{n=0}^{\infty} \int_0^{t-u-v} p^n\, f^{*ns}(w)$$

$$[G(t-u-v)-G(w)]\, \bar{F}(t-u-v-w)\, dw\, dv\, du \qquad \ldots (27)$$

Equation (27) is written considering the cases (i) No demand occurs upto time t, (ii) An order is placed at u, many orders are placed and realized and the last order is placed at u+v. Then several local purchase are made and at u+v+w, the level is s and the order is replenished in (w,x) but no demand occurs in (w,x).

For $s+1 \leq i \leq S-1$, it is easily seen that,

$$\pi_i(t) = [F^{*S-i}(t) - F^{*S-i+1}(t)] + \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)$$

$$\int_0^{t-u-v} \sum_{j=i}^{S-1} f_{s,j}(w) [F^{*(j-i)}(t-u-v-w) -$$

$$F^{*(j-i+1)}(t-u-v-w)] \, dw \, dv \, du$$

$$+ \delta_{S-s,i} \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v) \int_0^{t-u-v} \sum_{n=1}^{\infty} p^{n-1} q \, f^{*ns}(w)$$

$$[G(t-u-v) - G(w)] \, \bar{H}(t-u-v-w) \, dw \, dv \, du$$

$$+ \delta_{S-s,i} \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v) \int_0^{t-u-v} \sum_{n=1}^{\infty} p^{n-1} q \, f^{*ns}(w)$$

$$\int_0^{t-u-v-w} k(y) [H(t-u-v-w) - H(y)] \, [G(t-u-v) - G(w)]$$

$$\bar{F}(t-u-v-w-y) \, dy \, dw \, dv \, du \qquad .. \quad (28)$$

Now for $1 \leq i \leq s$,

$$\pi_i(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\, \bar{G}(t-u-v) \sum_{n=1}^{\infty} p^{n-1}$$

$$[F^{*(ns-i)}(t-u-v) - F^{*(ns-i+1)}(t-u-v)]\,dv\,du \qquad .. (29)$$

When the inventory is dry and when there is no demand being backlogged, we have two mutually exclusive cases.

Let,

$\pi_0^c(t)$ = Probability that the store is closed and the inventory level is zero, given the level at time zero is S.

Also let,

$\pi_0^o(t)$ = Probability that the inventory is dry, the store is open and there is no waiting of demands at time t, given at time zero inventory level is S.

Then we have,

$$\pi_0^c(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\bar{G}(t-u-v) \int_0^{t-u-v} \sum_{n=1}^{\infty} p^{n-1} q\, f^{*ns}(w)$$

$$\bar{H}(t-u-v-w)\,dw\,dv\,du \qquad .. (30)$$

$$\pi_o^{o}(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\, \bar{G}(t-u-v)$$

$$\int_0^{t-u-v} \sum_{n=1}^{\infty} p^{n-1} q\, f^{*ns}(w) \int_0^{t-u-v-w} k(y)$$

$$[H(t-u-v-w)-H(y)]\, \bar{F}(t-u-v-w-y)\,dy\, dw\, dv\, du \quad .. (31)$$

The queue size probabilities are obtained as follows.

For $1 \leq i \leq S-2s-2$,

$$\pi_{-i}(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\, \bar{G}(t-u-v) \int_0^{t-u-v} \sum_{n=1}^{\infty} p^{n-1} q\, f^{*ns}(w)$$

$$\int_0^{t-u-v-w} k(y) \int_0^{t-u-v-w-y} [H(y+z)-H(y)] f(z)$$

$$[F^{*(i-1)}(t-u-v-w-y-z)-F^{*i}(t-u-v-w-y-z)]\,dz\, dy\, dw\, dv\, du$$

$$.. (32)$$

Finally we have,

$$\pi_{-(S-2s-1)}(t) = \int_0^t f^{*S-s}(u) \int_0^{t-u} q(v)\, \bar{G}(t-u-v) \int_0^{t-u-v} f^{*s}(w)$$

$$\int_0^{t-u-v-w} k(y) \int_0^{t-u-v-w-y} f(z)\, [H(y+z)-H(y)]$$

$$F^{*S-2s-2}(t-u-v-w-y-z)\,dz\, dy\, dw\, dv\, du \quad .. (33)$$

Remarks

The probabilities calculated in the above models may be used for finding the expected cost during (o,t). We assume that the inventory carrying cost per unit is 'a' units per unit time and queue maintaining cost per demand is 'b' units per unit time. Then we find the expected cost during (o,t) as,

$$E(C(t)) = \sum_{n=1}^{S} na \int_{o}^{t} \pi_n(u)du + \sum_{m=1}^{d} mb \int_{o}^{t} \pi_{-m}(u)du$$

where C(t) is the total cost during (o,t). For models 1 and 3, d = S-2s-1 and for model 2, d = S-s-1.

Chapter 3

CORRELATION BETWEEN LEAD TIME AND DRY PERIOD FOR INVENTORIES AND DAMS

3.1 INTRODUCTION

A detailed review of inventory systems was given by Gani [1957] and applications of such models to practical situations, are provided by Hadley and Whittin [1963]. Moran [1959] gave the probability theory of a dam and later it was further extended by several authors. In many of the models developed, under the assumptions of general distributions for interarrival times of demands and lead times (time between seasons), the system size probabilities are obtained. For instance one may refer Sahin[1979], Srinivasan [1979], Thangaraj and Ramanarayanan [1983] and Roes [1970]. But the relationsobtained are too involved to yield for any further analysis.

So far, the correlation between lead times (time between seasons) and storage dry periods have not been studied at any depth. In this chapter we develop some simple results and use it to find the correlation between lead time and dry period for (s,S) policy inventory systems and finite capacity, continuous demand dam models. Also in some particular cases simple expressions for the joint moments are obtained.

3.2. SOME GENERAL RESULTS

Let X and Y be two positive independent random variables with probability distribution functions $F(x)$ and $G(x)$ and probability density functions $f(x)$ and $g(x)$ respectively. Define a random variable Z as

$$Z = \begin{cases} X-Y & \text{if } X > Y \\ 0 & \text{otherwise} \end{cases}$$

The joint density functions of X and Z is given by

$$f(x,z) = \begin{cases} f(x)\bar{G}(x) & \text{for } z=0 \text{ and } x > 0 \\ f(x)g(x-z) & \text{for } z > 0 \text{ and } z \leq x \\ 0 & \text{otherwise} \end{cases} \qquad \text{.. (1)}$$

[Here $f(x,z)$ is not a proper probability density function, for, non zero probability is attached with a set of Lebesgue measure zero. But $f(x,z)$ can be used for our computations.]

The Double Laplace Stieltjes Transform (DLST) of X and Z is given by

$$E(e^{-\xi X-\eta Z}) = \int_0^\infty e^{-\xi x} f(x)\,\bar{G}(x)dx + \int_0^\infty \int_0^x e^{-\xi x-\eta z} f(x)\ g(x-z)dz\ dx$$

After some simplifications we get,

$$E(e^{-\xi X-\eta Z}) = \int_0^\infty e^{-\xi x} f(x)dx - \eta \int_0^\infty e^{-\xi x-\eta x} f(x) \int_0^x e^{\eta y}\ G(y)dy\ \ dx \qquad ..\ (2)$$

Now differentiating (2) partially with respect to η and putting ξ = o and η = o, we obtain after changing the sign,

$$E(Z) = \int_0^\infty f(x) \int_0^x G(y)dy\ dx \qquad ..\ (3)$$

Similarly taking the second partial derivative with respect to η and putting ξ = o, η = o, we get

$$E(Z^2) = 2 \int_0^\infty f(x) \int_0^x \int_0^y G(z)dz\ dy\ dx \qquad ..\ (4)$$

Now differentiating equation (2) partially with respect to η and then with respect to ξ and putting $\eta=o$, $\xi=o$, we obtain,

$$E(XZ) = \int_0^\infty x\, f(x) \int_0^x G(y)dy\, dx \qquad .. (5)$$

Using (3) and (5) the covariance of X and Z can be found. From (3) and (4) the variance of Z is calculated and so the correlation between X and Z can be obtained.

Similarly we can compute the correlation between Y and Z as follows. The joint density functions of Y and Z (as in the earlier case, here also it is not a proper density function) can be easily written as,

$$f(y,z) = \begin{cases} F(y)\ g(y) & \text{for } z = o,\ y > o \\ g(y)\ f(y+z) & \text{for } z > o,\ y > o \\ 0 & \text{otherwise} \end{cases} \qquad .. (6)$$

Hence DLST of Y and Z is,

$$E(e^{-\xi Y-\eta Z}) = \int_0^\infty e^{-\xi y} g(y)\ F(y)dy + \int_0^\infty \int_0^\infty e^{-\xi y-\eta z} g(y)\ f(y+z)dz\ dy$$

This can be simplified into the form,

$$E(e^{-\xi Y-\eta Z}) = \int_0^{\infty} e^{-\xi y} g(y)\ F(y)dy +$$

$$\int_0^{\infty} e^{-\xi y+\eta y} g(y) \int_0^{\infty} e^{-\eta x} f(x)dx\ dy \qquad .. (7)$$

Then we find that,

$$E(YZ) = \int_0^{\infty} y\ g(y) \int_y^{\infty} \bar{F}(x)dx\ dy \qquad .. (8)$$

Then as in the earlier case, the correlation between Y and Z can be calculated.

3.3 ON (s,S) POLICY INVENTORY SYSTEMS

Consider an (s,S) policy inventory system under the assumption that the interarrival times of demands and lead times are independent sequences of independent and identically distributed random variables with general distributions and the demands are for one unit at a time. Let H(x) be the cumulative distribution function (c.d.f) of interarrival times of demands and let h(x) be the corresponding probability density function (p.d.f). Also let $F(x)$ be the c.d.f of lead times and let $f(x)$ be its p.d.f. The order is placed whenever the inventory level falls to s. Hence the time for the inventory to become dry is the time needed for the occurrence of s demands which is the sum of s independent and identically distributed random variables having c.d.f $H(x)$. Therefore $H^{*s}(x)$, the s-fold convolution of $H(x)$ with itself, is the c.d.f of the time to dry.

In section 3.2, if we take X as the lead time and Y as the time to dry, then Z will be the dry period. X is having c.d.f $F(x)$ and Y is with c.d.f $H^{*s}(x)$. Substituting these distribution functions in the earlier equations, the correlation between lead time and dry period and time to dry and dry period can be obtained.

Now we study a special case in which both the interarrival times of demands and lead times are exponentially distributed.

Let

$$H(x) = 1-e^{-\mu x} \text{ and } F(x) = 1 - e^{-\lambda x}$$

Using the equations in section 3.2 we get,

(i) The expected dry period, $E(Z) = \frac{1}{\lambda}\left(\frac{\mu}{\mu+\lambda}\right)^{s}$

(ii) $E(Z^2) = \frac{2}{\mu^2}\left(\frac{\mu}{\mu+\lambda}\right)^{s}$

(iii) $E(XZ) = \frac{1}{\lambda^2}\left(\frac{\mu}{\mu+\lambda}\right)^{s}\left(\frac{2\mu+2\lambda+\lambda s}{\mu+\lambda}\right)$

(iv) $Cov(X,Z) = \frac{1}{\lambda^2}\left(\frac{\mu}{\mu+\lambda}\right)^{s}\left(\frac{\mu+\lambda+\lambda s}{\mu+\lambda}\right)$

(v) $E(YZ) = \frac{1}{\lambda}\left(\frac{\mu}{\mu+\lambda}\right)^{s}\left(\frac{s}{\mu+\lambda}\right)$

(vi) $Cov(Y,Z) = -s\left(\frac{\mu}{\mu+\lambda}\right)^{s}\frac{1}{\mu(\mu+\lambda)}$

Then the correlation between X and Z is given by

$$\rho(X,Z) = \frac{\mu(\mu+\lambda+\lambda s)}{(\mu+\lambda)\sqrt{2\lambda^2\left(\frac{\mu+\lambda}{\mu}\right)^{s} - \mu^2}}$$

and the correlation between Y and Z is given by

$$\rho(Y,Z) = \frac{-\lambda\mu\sqrt{s}}{(\mu+\lambda)\sqrt{2\lambda^2\left(\frac{\mu+\lambda}{\mu}\right)^{s} - \mu^2}}$$

<u>Remark</u>:

The expected length of dry period can be increased or decreased using (i) by decreasing or increasing the value of s. A pre-planned dry period will be useful for doing activities like clearing the accounts, cleaning the store etc. Also note that covariance of X and Z is positive and the covariance of Y and Z is negative.

3.4. ON DAM MODELS WITH CONTINUOUS DEMANDS

We consider a dam with a finite capacity C. Time zero is a season epoch and the dam gets water of random amount N having c.d.f $N(x)$ and p.d.f. $n(x)$. Hence the water contained in the dam initially is N if $N \leq C$ and it is C if $N > C$. The next season occurs after a random time having c.d.f. $F(x)$. Demands for water occur with interarrival times having c.d.f $M(x)$ and density function $m(x)$. The quantity demanded each time is random with c.d.f. $K(x)$.

Let $P_k = \text{Prob}\left\{\text{the dam survives } k \text{ demands}\right\}$

$$= \int_0^C K^{*k}(x) n(x)\,dx + K^{*k}(C)\,\bar{N}(C)$$

Let p_i = Prob $\{$the dam becomes empty due to the i^{th} demand$\}$

for $i = 1, 2, \ldots$

Then $p_i = P_{i-1} - P_i$

Then the time to dry the dam is a random variable having c.d.f

$$G(x) = \int_0^x \sum_{i=1}^{\infty} p_i \, m^{*i}(s)ds \qquad .. (9)$$

Here also if we take X as the time between seasons and Y as the time to dry, Z will be the dry period. So results of Section 3.2 can be used to find the correlation between the time between seasons and dam dry periods.

Consider a special case in which the time between seasons having c.d.f $F(x) = 1-e^{-\lambda x}$ and time between demands having c.d.f $M(x) = 1-e^{-\mu x}$.

Let $\varphi(s) = \sum_{i=1}^{\infty} p_i s^i$ and let $r = \frac{\mu}{\mu+\lambda}$

Then the following relations can be obtained (using the equations in 3.2).

(i) The expected dry period,

$$E(Z) = \frac{1}{\lambda} \varphi(r)$$

(ii) $$E(Z^2) = \frac{2}{\lambda^2} \varphi(r)$$

(iii) $$E(XZ) = \frac{\mu}{\lambda(\lambda+\mu)^2} \varphi'(r) + \frac{2}{\lambda^2} \varphi(r)$$

(iv) $$\text{Cov}(X,Z) = \frac{\mu}{\lambda(\lambda+\mu)^2} \varphi'(r) + \frac{1}{\lambda^2} \varphi(r)$$

(v) $$E(YZ) = \frac{\mu}{\lambda(\lambda+\mu)^2} \varphi'(r)$$

(vi) $$\text{Cov}(Y,Z) = \frac{\mu}{\lambda(\lambda+\mu)^2} \varphi'(r) - \frac{\varphi'(1)}{\lambda\mu} \varphi(r)$$

Then the correlation between X and Z is obtained as

$$\rho(X,Z) = \frac{\frac{\lambda\mu}{(\lambda+\mu)^2} \varphi'(r) + \varphi(r)}{\sqrt{2\varphi(r) - \varphi^2(r)}}$$

and the correlation between Y and Z is obtained as

$$\rho(Y,Z) = \frac{\frac{\mu^2}{(\lambda+\mu)^2} \varphi'(r) - \varphi'(1)\, \varphi(r)}{\sqrt{\varphi''(1) + 2\varphi'(1) - [\varphi'(1)]^2}\sqrt{2\varphi(r)-\varphi^2(r)}}$$

Chapter 4

AN INVENTORY SYSTEM WITH RANDOM LEAD TIMES AND BULK DEMANDS

4.1 INTRODUCTION

In this chapter we consider an (s,S) policy inventory system under the assumption that intervals of time between successive demand points, quantities demanded at these points and lead times are independent sequences of independent and identically distributed random variables. All the demands that occur during an inventory dry period are lost. We derive expressions for the inventory level probabilities explicitly.

Gross, Harris and Lechner [1971] considered (S-1,S) inventory models with bulk demand and state dependent lead times. They have assumed that inter-arrival times of demands and lead times are exponentially distributed random variables and obtained the expected inventory cost in order to obtain an optimal value of S. Srinivasan [1979] considers an (s,S) policy inventory system with general demand arrival times, random lead times and unit demands. In this paper, he has given the explicit expression of the probability mass function of the stock level at any time t as well as other statistical characteristics governing the actual sales and shortages. Thangaraj and Ramanarayanan [1983] considered an inventory system with two ordering levels. Sahin [1979] considers (s,S) inventory systems in which the quantity demanded is random but the lead time is a constant and full backlogging is allowed. He derives time dependent and stationary distribution of inventory position and on hand inventory and discusses some results for the characterization of the optimal policies. Also Sahin [1983] considered an (s,S) inventory model with random lead times and bulk demand and obtained the binomial moments of the time dependent and limiting distributions of inventory deficit.

In section 4.2 we give details of the assumptions and notations used in this chapter. The transition time probabilities are given in 4.3 and in 4.4, the exact expressions for the inventory level probabilities are written.

4.2 THE MODEL AND PRELIMINARIES

S is the maximum capacity of the ware house and s is the ordering level. The interarrival times of demands are independent and identically distributed random variables with c.d.f. $F(x)$ and p.d.f. $f(x)$. The quantity demanded each time is a discrete random variable taking the value i with probability p_i. The minimum quantity demanded is 'a' and the maximum quantity that can be demanded is 'b'; where a and b are two integers such that $s < a \leq b < S-s$. Then $\sum_{i=a}^{b} p_i = 1$. Whenever the inventory level falls to i such that $0 \leq i \leq s$, an order is placed for S-i units. Lead times are i.i.d random variables having probability distribution function $G(x)$ and density function $g(x)$. When the inventory level is i, if a demand occurs for more than i units, all the i units are given. No demand is allowed to wait during the inventory dry period. The interarrival times of demands, quantities demanded and lead times are all independent. Finally we assume that at time zero the inventory is full and the demand process starts.

Let

$$\pi_i(t) = \text{Prob}\{\text{there are } i \text{ units in the system at time } t / \text{ at time zero the level is } S \text{ and demand process starts}\}$$

$$\varphi(r) = \sum_{i=a}^{b} p_i \, r^i$$

$$p_k^{(n)} = \text{the coefficient of } r^k \text{ in } [\ \varphi(r)\ ]^n$$

For $i > j > o$,

$$\text{let } h_{i,j}(x) = \sum_{n=1}^{\infty} f^{*n}(x)\, P_{i-j}^{(n)}$$

For $i > o$

$$\text{let } h_{i,o}(x) = \sum_{n=1}^{\infty} f^{*n}(x) \sum_{k=s+1}^{b} P_{i-k}^{(n-1)} \sum_{j=k}^{b} p_j$$

Consider the time points at which the first demand after each order realization occurs and look at the inventory level at these points. S is the level at time zero and if ξ $(S-s-b \leq \xi \leq S-a)$ is the level after the first transition (i.e. due to the first demand occurring after the first order realization, the inventory level becomes ξ) Let $f_{S,\xi}(x)$ denote the probability density function of the transition time. Similarly if ξ is the inventory level at one such time point and if η is the level at the next such time point (ie. the time point at which the first demand after the next order realization occurs), then $f_{\xi,\eta}(x)$ denotes the transition time probability density function $(S-s-b \leq \xi, \eta \leq S-a)$. These transitions can occur with a dry period during lead time or without a dry period during lead time. Let ${}_1f_{i,j}(x)$ denote the transition time probability density function with a dry period and let

${}_2f_{i,j}(x)$ denote the transition time probability density function without a dry period.

Let

$$\underline{f}_S(x) = (\, f_{S,S-s-b}(x),\ f_{S,S-s-b+1}(x), \ldots, f_{S,S-a}(x))$$

(it is a vector of order b+s-a+1)

Now we introduce a square matrix of order b+s-a+1 given by,

$$\mathbb{F}(x) = \begin{bmatrix} f_{S-s-b,S-s-b}(x) & \cdots & f_{S-s-b,S-a}(x) \\ \vdots & & \\ f_{S-a,S-s-b}(x) & \cdots & f_{S-a,S-a}(x) \end{bmatrix}$$

Let $\mathbb{F}^{*0}(x)$ be the identity matrix of order b+s-a+1 and for $n \geqslant 1$ let $\mathbb{F}^{*n}(x)$ be the n-fold convolution of the matrix $\mathbb{F}(x)$ with itself.

Then $(\underline{f}_S * \sum_{n=0}^{\infty} \mathbb{F}^{*n})(x)$ is a vector of order b+s-a+1.

Let

$$F_\eta(x) = (\underline{f}_S * \sum_{n=0}^{\infty} \mathbb{F}^{*n})_\eta (x) \text{ be the } (\eta-S+s+b+1)^{th} \quad .. (1)$$

coordinate of this vector, where η = S-s-b, ..., S-a.

4.3. TRANSITION TIME PROBABILITIES

The following relations for the transition time probabilities can be obtained easily.

For $S-s-b \leq \xi \leq S-a$

$$_1f_{S,\xi}(x) = \int_0^x \sum_{i=0}^{s} h_{S,i}(u) \int_0^{x-u} \sum_{n=1}^{\infty} f^{*n}(v)$$

$$[G(x-u)-G(v)]p_{S-i-\xi}\, f(x-u-v)dv\; du \quad .. (2)$$

For $S-b \leq \xi \leq S-a$

$$_2f_{S,\xi}(x) = \int_0^x \sum_{i=0}^{s} h_{S,i}(u)\; G(x-u)p_{S-\xi}\, f(x-u)du \quad .. (3)$$

Then for ξ satisfying $S-b \leq \xi \leq S-a$, we have

$$f_{S,\xi}(x) = {_1f_{S,\xi}(x)} + {_2f_{S,\xi}(x)} \quad .. (4)$$

and for $S-s-b \leq \xi \leq S-b-1$ we have

$$f_{S,\xi}(x) \quad = \quad {}_1f_{S,\xi}(x) \qquad .. (5)$$

For $S-s-b \leq \xi$, $\eta \leq S-a$,

$${}_1f_{\xi,\eta}(x) = \int_0^x \sum_{i=0}^{s} h_{\xi,i}(u) \int_0^{x-u} \sum_{n=1}^{\infty} f^{*n}(v)$$

$$[G(x-u)-G(v)]\, p_{S-i-\eta} f(x-u-v)\, dv\, du \qquad .. (6)$$

Also for $S-b \leq \eta \leq S-a$ and $S-s-b \leq \xi \leq S-a$

$${}_2f_{\xi,\eta}(x) = \int_0^x \sum_{i=0}^{s} h_{\xi,i}(u)\; G(x-u)\; p_{S-\eta} f(x-u)\, du \qquad .. (7)$$

Then we have

$$f_{\xi,\eta}(x) = {}_1f_{\xi,\eta}(x) + {}_2f_{\xi,\eta}(x) \text{ for } S-b \leq \eta \leq S-a$$

$$.. (8)$$

and $\qquad f_{\xi,\eta}(x) = {}_1f_{\xi,\eta}(x) \quad \text{for } S-s-b \leq \eta \leq S-b-1$

4.4. INVENTORY LEVEL PROBABILITIES

Now we give the relations for system size probabilities.

$$\pi_S(t) = \bar{F}(t) + \sum_{\xi=1}^{s} \int_0^t h_{S,\xi}(x)\, \bar{F}(t-x)\, G(t-x)\,dx$$

$$+ \int_0^t h_{S,o}(x) \int_0^{t-x} \sum_{n=o}^{\infty} f^{*n}(v)\,[G(t-x)-G(v)]\,\bar{F}(t-v)\,dv\,dx$$

$$+ \sum_{\eta=S-s-b}^{S-a} \sum_{i=1}^{s} \int_0^t F_\eta * h_{\eta,i}(x)\, \bar{F}(t-x)\, G(t-x)\,dx \qquad \cdots (9)$$

$$+ \sum_{\eta=S-s-b}^{S-a} \int_0^t F_\eta * h_{\eta,o}(x) \int_0^{t-x} \sum_{n=o}^{\infty} f^{*n}(v)$$

$$[G(t-x)-G(v)]\,\bar{F}(t-v)\,dv\,dx$$

The above expression is written considering the following mutually exclusive and exhaustive cases: (i) no demand during (o,t), (ii) first order is placed at level $\xi \neq o$, no demand occurs thereafter and order is realized, (iii) first order is made at level zero, several demands are lost, order is realized but no demand after the realization of order, (iv) several orders are realized, an order is made at level $i \neq o$, but no demand occurs and the order is realized, (v) several orders are realized, an order is made at level zero, several demands are lost, order is realized but no demand occurs.

Now for $1 \leq i \leq s$,

$$\pi_{S-i}(t) = \int_0^t h_{S,i}(x) \int_0^{t-x} \sum_{n=1}^{\infty} f^{*n}(v)[G(t-x)-G(v)] \bar{F}(t-x-v)dv\ dx$$

$$+ \sum_{\eta=S-s-b}^{S-a} \int_0^t F_\eta * h_{\eta,i}(x) \int_0^{t-x} \sum_{n=1}^{\infty} f^{*n}(v) [G(t-x)-G(v)]\ \bar{F}(t-x-v)dv\ dx \qquad ..\ (10)$$

To get (10) we have to consider two cases: (i) first order is made at level i for S-i units, inventory becomes dry due to a demand, order is realized and then no demand occurs, (ii) several orders are realized, an order is made at level i for S-i units, inventory becomes dry, order is realized and no demand occurs.

For $S-s-1 \leq i \leq S-a+1$,

$$\pi_i(t) = 0 \qquad ..\ (11)$$

For $S-s-b \leq i \leq S-a$

$$\pi_i(t) = \int_0^t h_{S,i}(u)\ \bar{F}(t-u)du \sum_{\eta=i+a}^{S-a} \int_0^t F_\eta * h_{\eta,i}(u)\bar{F}(t-u)du + \int_0^t F_i(u)\bar{F}(t-u)du \qquad (12)$$

The equation (12) is written considering the cases: (i) the level drops to i from S and remains in it, (ii) several orders are realized and level becomes j due to a demand after the last order realization, and then it becomes i due to further demands, (iii) several orders are realized and the level becomes i due to a demand after the last order realization and no demand occurs.

For $s+1 \leq i \leq S-s-b-1$, let $k = \max\{i+a, S-s-b\}$, then

$$\pi_i(t) = \int_0^t h_{S,i}(u)\, \bar{F}(t-u)\,du + \int_0^t \sum_{j=k}^{S-a} (F_j * h_{j,i})(u)\, \bar{F}(t-u)\,du \qquad \text{.. (13)}$$

In deriving (13) we considered the cases: (i) From S the level drops to i and remains there, (ii) several orders are realized and j is the level due to a demand after the last order realization and level drops to i due to further demands.

Next for $1 \leq i \leq s$,

$$\pi_i(t) = \int_0^t h_{S,i}(x)\, \bar{F}(t-x)\, \bar{G}(t-x)\,dx + \sum_{j=S-s-b}^{S-a} \int_0^t (F_j * h_{j,i})(x)\, \bar{F}(t-x)\, \bar{G}(t-x)\,dx \qquad \text{.. (14)}$$

To arrive at equation (14) consider the exclusive cases: (i) first order is made at i and no demand occurs and order is not realized, (ii) several orders are realized and the level becomes j due to the first demand after the last order realization and the next order is placed at level i; no demand occurs and order not realized.

Finally,

$$\pi_o(t) = \int_o^t h_{S,o}(u)\, \bar{G}(t-u)du + \sum_{i=1}^{s} \int_o^t h_{S,i}(x)$$

$$\bar{G}(t-x)\; F(t-x)dx + \sum_{j=S-s-b}^{S-a} (F_j * h_{j,o})(u)\bar{G}(t-u)du$$

$$+ \sum_{j=S-s-b}^{S-a} \sum_{i=1}^{s} \int_o^t (F_j * h_{j,i})(u)\bar{G}(t-u)F(t-u)du \qquad (15)$$

Equation (15) is written considering the cases: (i) first order is made at level zero and it is not realized, (ii) first order is made at level $i \neq o$ and then a demand occurs and order is not realized, (iii) several orders are realized, and then an order is made at level zero and it is not realized upto time t, (iv) several orders are realized, an order is made at level $i \neq o$ which is not realized upto time t and a demand occurs.

Chapter-5

AN INVENTORY SYSTEM WITH RANDOM LEAD TIMES AND VARYING ORDERING LEVELS

5.1 INTRODUCTION

In the study of inventory problems, usually two basic types of policy for replenishing the stock of an item in a store are considered. (i) The ordering cycle policy, under which orders for replenishments are placed at regular intervals of time of length T, (ii) The (s,S) policy, under which orders are placed as and when the stock in the store, plus any quantity already on order, falls to some fixed level s. In both the cases the quantity to be ordered is calculated so as to bring the amount in stock plus the amount on order, upto some fixed level S. The replenishments ordered under any of these policies are assumed to arrive after a time lag, which may be either fixed or a random variable. If a demand arises at a time when there is no stock in the store, there is said to be a shortage. Then in some models the customer has to wait until the next replenishment takes place and in some models the customer will leave the system unsatisfied.

In this chapter we consider a continuous review inventory system in which the capacity of the store is a fixed number S, but the ordering level in one cycle is decided according to the number of demands during the previous lead time. The interarrival times of demands and lead times are independent sequences of independent and identically distributed random variables. The demands occur for one unit at a time and no backlogging of demands is allowed. We derive expressions for the stock level probabilities and give some relations to find the correlation between the number of demands during a lead time and the next inventory dry period.

5.2 ASSUMPTIONS OF THE MODEL

The maximum capacity of the store is S and we assume that the inventory is full at time zero. The demands occur for one unit at a time and the time intervals between the arrivals of two consecutive demands constitute a family of independent and identically distributed random variables having the common probability distribution function $F(x)$ and density function $f(x)$. The ordering policy for replenishment of the item is as follows. We fix a number c such that $S-c > c$ as the highest ordering level. The first order is placed at fixed level s $(o \leq s \leq c)$ and the remaining orders are placed at levels decided according to the number of demands during the previous lead time. An order is placed at a level i if there were i demands during the previous lead time such that $o \leq i \leq c$. If the number of demands during a lead time is more than c, we make the next order at c only. Each time order is placed to fill the inventory. The lead times are independent and identically distributed random variables with probability distribution function $G(x)$ and density function $g(x)$. Backlogging of demands is not allowed. Also we assume that interarrival times of demands and lead times are independent sequences of random variables.

5.3 NOTATIONS

Let

$\pi_i(t)$ = Prob {the inventory level is i at time t/ the inventory level at time zero is S}

For $o \leq \xi, \eta \leq c$, let

$f_{\xi,\eta}(x)dx$ = Probability that the ordering level η is reached in $(x, x+dx)$ given the previous order was placed at time zero when the ordering level was ξ.

i.e. $f_{\xi,\eta}(x)$ is the probability density function of the transition time between two consecutive ordering points given the ordering levels at these points.

Let

$\underline{f}_s(x) = (f_{s,o}(x),\ f_{s,1}(x),\ \ldots,\ f_{s,c}(x))$, it is a vector of order c+1.

We define a square matrix of order c+1 given by,

$$\mathbb{F}(x) = \begin{bmatrix} f_{o,o}(x) & \cdots & f_{o,c}(x) \\ \vdots & & \\ f_{c,o}(x) & \cdots & f_{c,c}(x) \end{bmatrix}$$

Let $\mathbb{F}^{*o}(x)$ be the identity matrix of order c+1 and for $n \geqslant 1$, let $\mathbb{F}^{*n}(x)$ be the n-fold convolution of $\mathbb{F}(x)$ with itself.

$(f^{*S-s} * \underline{f}_s)(x)$ is the vector obtained by convoluting each element of the vector $\underline{f}_s(x)$ by the function $f^{*S-s}(x)$.

Then $(f^{*S-s} * \underline{f}_s * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ is a vector of order c+1.

Let $K_i(x)$ be the $(i+1)^{th}$ coordinate of the vector $(f^{*S-s} * \underline{f}_s * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$, where $i = o, 1, \ldots, c$.

5.4 THE TRANSITION TIME PROBABILITIES

In this section we give the relations for the transition time probability density functions. Here ξ and η are such that $o \leq \xi, \eta \leq c$.

For $\xi < \eta < c$,

$$f_{\xi,\eta}(x) = \int_o^x \int_o^u f^{*\eta}(v) \ [G(u)-G(v)] \ f(u-v) \ f^{*(S-\xi-\eta-1)}(x-u)\,dv\,du \qquad (1)$$

$$f_{\xi,c}(x) = \int_o^x \int_o^u \sum_{n=o}^{\infty} f^{*n+c}(v) \ [G(u)-G(v)] f(u-v) \ f^{*S-\xi-c-1}(x-u)\,dv\,du \qquad (2)$$

Also,

$$f_{\xi,\xi}(x) = \int_o^x \int_o^u f^{*\xi}(v) \ [G(u)-G(v)] \ f(u-v) \ f^{\,S-2\xi-1}(x-u)\,dv\,du \qquad (3)$$

For $o < \eta < \xi$,

$$f_{\xi,\eta}(x) = \int_o^x \int_o^u f^{*\eta}(v) \ [G(u)-G(v)] \ f(u-v) \ f^{*S-2\eta-1}(x-u)\,dv\,du \qquad (4)$$

Also,

$$f_{\xi,o}(x) = \int_o^x G(u)\ f(u)\ f^{*S-1}(x-u)\,du \tag{5}$$

5.5 INVENTORY LEVEL PROBABILITIES

Now we compute the exact expressions for the system size probabilities at any time t.

For $o \leq j < s$,

$$\pi_{S-j}(t) = [F^{*j}(t) - F^{*j+1}(t)] + \int_o^t f^{*S-s}(u)\,[F^{*j}(t-u)-F^{*j+1}(t-u)]G(t-u)\,du$$

$$+ \sum_{i=j+1}^{c} \int_o^t K_i(u)[F^{*j}(t-u)-F^{*j+1}(t-u)]G(t-u)\,du$$

$$+ \sum_{i=o}^{j-1} \int_o^t K_i(u) \int_o^{t-u} \sum_{n=o}^{\infty} f^{*n+i}(v) \int_o^{t-u-v} [G(v+y)-G(v)]\, f(y)[F^{*j-i-1}(t-u-v-y)-F^{*j-i}(t-u-v-y)]\,dy\,dv\,d$$

$$+ \sum_{i=1}^{j} \int_o^t K_i(u) \int_o^{t-u} \sum_{k=1}^{i} f^{*k}(v)\ G(v)\,[F^{*i-k}(t-u-v)-F^{*i-k+1}(t-u-v)]\,dv\,du$$

$$+ \int_o^t K_j(u) \int_o^{t-u} \sum_{n=o}^{\infty} f^{*n+j}(v)\ \bar{F}(t-u-v)[G(t-u)-G(t-u-v)]\,d \tag{6}$$

The above equation is obtained by considering the cases (i) there are exactly j demands up to time t, (ii) the first order is placed at time u, then exactly j demands occur in (u,t) and order is replenished before time t. (iii) Several orders are placed and the last order is placed at level i and at time u; where $i > j$. Then exactly j demands occur in (u,t) and order is replenished before t. (iv) Several orders are placed and the last order is placed at level i ($< j$) and at time u. Then at or prior to a demand occuring at v the inventory becomes dry, the next demand occurs at y and the order is replenished in (v,y) and then exactly j-i-1 demands occurs before time t. (v) Many orders are realized and the last order before time t is placed at u when the level is i ($\leq j$). The number of demands during lead time is k($\leq i$) and then exactly i-k demands occur. (vi) the last order before time t is placed at u when the level is j and the inventory becomes dry before replenishment, no demand occurs after replenishment.

Now,

$$\pi_{S-s}(t) = [F^{*s}(t) - F^{*s+1}(t)] + \int_0^t f^{*S-s}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+s}(v)\, \bar{F}(t-u-v)$$

$$[G(t-u) - G(t-u-v)]\, dv\, du$$

$$+ \int_{o}^{t} f^{*S-s}(u) \int_{o}^{t-u} \sum_{k=1}^{s} f^{*k}(v)\ G(v)$$

$$[F^{*s-k}(t-u-v) - F^{*s-k+1}(t-u-v)]dv\ du$$

$$+ \sum_{i=s+1}^{c} \int_{o}^{t} K_i(u)\ [F^{*s}(t-u) - F^{*s+1}(t-u)]\ G(t-u)du$$

$$+ \int_{o}^{t} K_s(u) \int_{o}^{t-u} \sum_{n=o}^{\infty} f^{*n+s}(v)\ \bar{F}(t-u-v)$$

$$[G(t-u) - G(t-u-v)]dv\ du$$

$$+ \int_{o}^{t} K_s(u) \int_{o}^{t-u} \sum_{k=1}^{s} f^{*k}(v)\ G(v)$$

$$[F^{*s-k}(t-u-v) - F^{*s-k+1}(t-u-v)]dv\ du$$

$$+ \sum_{j=o}^{s} \int_{o}^{t} K_j(u) \int_{o}^{t-u} \sum_{n=o}^{\infty} f^{*n+j}(v) \int_{o}^{t-u-v} [G(v+y) - G(v)]f(y)$$

$$[F^{*s-j-1}(t-u-v-y) - F^{*s-j}(t-u-v-y)]dy\ dv\ du$$

$$+ \sum_{j=1}^{s-1} \int_{o}^{t} K_j(u) \int_{o}^{t-u} \sum_{k=1}^{j} f^{*k}(v) G(v) [F^{*s-k}(t-u-v) - F^{*s-k+1}(t-u-v)]dv\ du \qquad \dots (7)$$

Equation (7) is written considering the following cases. (i) Exactly s demands up to time t. (ii) the first order is placed at u, the inventory becomes dry during the lead time and no demand occurs after the replenishment (iii) the first order is placed at u, then there are k ($\leq$ s) demands during lead time and exactly s-k demands after the replenishment (iv) the last order before time t is placed at u when the level is i ($>$ s) and exactly s demands occurs in (u,t) and replenishment is done before t. (v) the last order is placed at level s, the inventory becomes dry during lead time and no demand occurs after replenishment (vi) the last order is placed at level s and there are k ($<$ s) demands before replenishment and s-k demands after replenishment (vii) the last order is placed at u and the level is j; the inventory becomes dry and exactly s-j demands occur after the replenishment (viii) the last order is placed at u when the level is j and the k^{th} demand after time u occurs at v and the replenishment occur in (u,u+v); then there are exactly s-k demands in the interval (u+v,t).

Next, for $s < j < c$

$$\pi_{S-j}(t) = [F^{*j}(t)-F^{*j+1}(t)] + \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+s}(v) \int_0^{t-u-v} [G(v+y)-G(v)]\, f(y)\,[F^{*j-s-1}(t-u-v-y) - F^{*j-s}(t-u-v-y)]\,dy\,dv\,du$$

$$+ \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{k=1}^{s} f^{*k}(v)\, G(v)\, [F^{*j-k}(t-u-v) -$$

$$F^{*j-k+1}(t-u-v)]\ dv\ du$$

$$+ \sum_{i=j+1}^{c} \int_0^t K_i(u)\ [F^{*j}(t-u) - F^{*j+1}(t-u)]\ G(t-u)\, du$$

$$+ \int_0^t K_j(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+j}(v)\, \overline{F}(t-u-v)\, [G(t-u)-G(t-u-v)]\, dv\ du$$

$$+ \int_0^t K_j(u) \int_0^{t-u} \sum_{k=1}^{j} f^{*k}(v)\, G(v)\, [F^{*j-k}(t-u-v)-F^{*j-k+1}(t-u-v)]\, dv\ du$$

$$+ \sum_{i=0}^{j-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+i}(v) \int_0^{t-u-v} [G(v+y)-G(v)]\ f(y)$$

$$[F^{*j-i-1}(t-u-v-y) - F^{*j-i}(t-u-v-y)]\, dy\ dv\ du$$

$$+ \sum_{i=1}^{j-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=1}^{i} f^{*k}(v)\, G(v)\, [F^{*j-k}(t-u-v)-F^{*j-k+1}(t-u-v)]$$

$$dv\ du$$

(8)

To derive equation (8) consider the following cases (i) Exactly j demands up to time t (ii) the first order is placed at u, inventory becomes dry and exactly j-s demands occur after replenishment (iii) the first order is placed at u, then the k^{th} demand occurs only after replenishment and exactly j-k more demands occur after that (iv) the last order is placed at u when the level is i ($>j$) and exactly j demands occur in (u,t) and replenishment occurs before t. (v) After realizing many orders, the last order is placed at level j, the inventory becomes dry before replenishment and there are no demands after replenishment (vi) the last order is placed at level j and at time u, then the k^{th} demand occurs at u+v and order is realized before u+v and in the interval (u+v,t) exactly j-k demands occur (vii) the last order is placed at a level i($<j$), the inventory becomes dry and exactly j-i demands occur after the replenishment (viii) the last order is placed at a level i ($<j$) and at time u, then the k^{th} demand occur at u+v and order is realized in (u, u+v); exactly j-k demands occur in (u+v, t).

Now,

$$\pi_{S-c}(t) = [F^{*c}(t) - F^{*c+1}(t)] + \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+s}(v)$$

$$\int_0^{t-u-v} [G(v+y)-G(v)] f(y) [F^{*c-s-1}(t-u-v-y) - F^{*c-s}(t-u-v-y)]\, dy\, dv\, du$$

$$+ \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{k=1}^{s} f^{*k}(v)\ G(v)\ [F^{*c-k}(t-u-v) -$$

$$F^{*c-k+1}(t-u-v)]\ dv\ du$$

$$+ \int_0^t K_c(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+c}(v)\ \bar{F}(t-u-v)[G(t-u)-G(t-u-v)]dv\ du$$

$$+ \sum_{i=1}^{c} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=1}^{i} f^{*k}(v)\ G(v)[F^{*c-k}(t-u-v) - \tag{9}$$

$$F^{*c-k+1}(t-u-v)]dv\ du$$

$$+ \sum_{i=0}^{c-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+i}(v) \int_0^{t-u-v} [G(v+y)-G(v)]f(y)$$

$$[F^{*c-i-1}(t-u-v-y) - F^{*c-i}(t-u-v-y)]dy\ dv\ du$$

The above equation is written considering the cases (i) there are exactly c demands upto time t (ii) the first order is placed at u, then the inventory becomes dry and exactly c-s demands after replenishment occurs. (iii) the first demand is placed at u, the k^{th} demand after that occurs at u+v and the replenishment happens in (u,u+v) and in (u+v,t) exactly c-k demands occur. (iv) After realizing many orders, the last order is placed at u when the level is c and in (u, u+v) there are more than c demands. The replenishment occurs in (u+v,t) and no demand occurs.

(v) the last order is placed at level i at time u; the k^{th} demand occurs at u+v and in (u,u+v) the replenishment happens and in (u+v,t) there are exactly c-k demands.

(vi) the last order is placed at u when the level is i, the inventory becomes dry and then there are exactly c-i demands after the replenishment.

For $S-c > j > c$,

$$\pi_j(t) = [F^{*S-j}(t) - F^{*S-j+1}(t)] + \int_0^t f^{*S-s}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+s}(v) \int_0^{t-u-v} [G(v+y)-G(v)]$$

$$f(y)\ [F^{*S-s-j-1}(t-u-v-y)-F^{*S-s-j}(t-u-v-y)]dy\ dv\ du$$

$$+ \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{k=1}^{s} f^{*k}(v)\ G(v)\ [F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]\ dv\ du \tag{10}$$

$$+ \sum_{i=0}^{c} \int_0^t K_i(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{n+i}(v) \int_0^{t-u-v} [G(v+y)-G(v)] f(y)$$

$$[F^{*S-i-j-1}(t-u-v-y) - F^{*S-i-j}(t-u-v-y)]\,dy\,dv\,du$$

$$+ \sum_{i=1}^{c} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=1}^{i} f^{*k}(v)\; G(v)\; [F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]\,dv\,du$$

Equation (10) is obtained by considering the cases (i) exactly S-j demands up to time t (ii) the first order is placed at u and then the inventory becomes dry and there are exactly S-s-j demands after the replenishment (iii) first order is placed at u and the k^{th} demand occurs at u+v and replenishment occurs in (u,u+v) and there are exactly S-k-j demands in (u+v,t) (iv) After realizing many orders, the last order is placed at u when the level is i, inventory becomes dry and there are exactly S-i-j demands after replenishment (v) the last order is placed at level i at time u, the k^{th} demand occur at u+v and replenishment occur in (u,u+v) and there are S-k-j demands in the interval (u+v,t).

Now,

$$\pi_c(t) = [F^{*S-c}(t) - F^{*S-c+1}(t)] + \int_0^t f^{*S-s}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+s}(v) \int_0^{t-u-v} [G(v+y)-G(v)]f(y)$$

$$[F^{*S-s-c-1}(t-u-v-y)-F^{*S-s-c}(t-u-v-y)]dy\ dv\ du$$

$$+ \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{k=1}^{s} f^{*k}(v)\ G(v)$$

$$[F^{*S-k-c}(t-u-v) - F^{*S-k-c+1}(t-u-v)]dv\ du$$

$$+ \int_0^t K_c(u)\ \bar{F}(t-u)\ \bar{G}(t-u)du \tag{11}$$

$$+ \sum_{i=0}^{c-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{n=0}^{\infty} f^{*n+i}(v) \int_0^{t-u-v} [G(v+y)-G(v)]f(y)$$

$$[F^{*S-i-c-1}(t-u-v-y) - F^{*S-i-c}(t-u-v-y)]dy\ dv\ du$$

$$+ \sum_{i=0}^{c-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=1}^{i} f^{*k}(v)\ G(v)$$

$$[F^{*S-k-c}(t-u-v) - F^{*S-k-c+1}(t-u-v)]dv\ du$$

Equation (11) is written by considering the cases (i) exactly S-c demands up to time t (ii) first order is placed at u, then the inventory becomes dry before replenishment and exactly S-s-c demands occur after the replenishment (iii) first order is placed at u, then the k^{th} demand occur at u+v and the order is replenished in (u,u+v), then exactly S-k-c demands occur in (u+v,t). (iv) After realizing many orders, the last order is placed at u when the level is c, then no demand occurs and the order is not materialized (v) the last order before time t is placed at u when the level is i and the inventory becomes dry before the replenishment and there are exactly S-i-c demands after the replenishment (vi) the last order is placed at level i and at time u and the k^{th} demand occurs at u+v and order is replenished in (u,u+v), there are exactly S-k-c demands after the replenishment.

For $c > j > s$, we have

$$\pi_j(t) = [F^{*S-j}(t) - F^{*S-j+1}(t)] + \int_0^t f^{*S-s}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{j-s-1} f^{*n+s}(v) \int_0^{t-u-v} [G(v+y)-G(v)]\ f(y)$$

$$[F^{*S-s-j-1}(t-u-v-y) - F^{*S-s-j}(t-u-v-y)]dy\ dv\ du$$

$$+ \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{k=1}^{s} f^{*k}(v)\ G(v)\ [F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]dv\ du$$

$$+ \sum_{i=j}^{c} \int_0^t K_i(u)\ [F^{*i-j}(t-u) - F^{*i-j+1}(t-u)]\bar{G}(t-u)du \qquad (12)$$

$$+ \sum_{i=j}^{c} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=o}^{j} f^{*k}(v)\ G(v)\ [F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]dv\ du$$

$$+ \sum_{i=o}^{j-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{n=o}^{j-i-1} f^{*n+i}(v) \int_0^{t-u-v} [G(v+y)-G(v)]f(y)$$

$$[F^{*S-i-j-1}(t-u-v-y)-F^{*S-i-j}(t-u-v-y)]dy\ dv\ du$$

$$+ \sum_{i=1}^{j-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=1}^{i} f^{*k}(v)\ G(v)\ [F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]dv\ du$$

To obtain equation (12) we consider the cases (i) exactly S-j demands up to time t (ii) the first order is placed at u and the inventory becomes dry and exactly S-s-j demands occur after the replenishment (iii) the first order is placed at u, then the k^{th} demand occurs at u+v and order is replenished in (u,u+v) and exactly S-k-j demands after replenishment (iv) After realizing many orders, the last order is placed at u when the level is i and it is not realized but exactly i-j demands occur. (v) The last order is placed at level i at time u, the k^{th} demand occurs at u+v and order is realized in (u,u+v) and exactly S-k-j demands in (u+v,t). (vi) the last order is placed at u when the level is i and then the inventory becomes dry (but total number of demands during lead time is $< j$) and then S-i-j demands occur after replenishment. (vii) the last order is placed at level i, then the k^{th} ($k \leq i$) demand occurs at u+v and replenishment occurs in (u,u+v), then there are exactly S-k-j demands in the interval (u+v, t).

Now for $1 \leq j \leq s$,

$$\pi_j(t) = \int_0^t f^{*S-s}(u)\ \bar{G}(t-u)\ [F^{*s-j}(t-u)-F^{*s-j+1}(t-u)]du$$

$$+ \int_0^t f^{*S-s}(u) \int_0^{t-u} \sum_{k=1}^{j-1} f^{*k}(v)\ G(v)\ [F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]dv\ du$$

$$+ \sum_{i=o}^{j-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{n=o}^{j-i-1} f^{*n+i}(v) \int_0^{t-u-v} [G(v+y)-G(v)]f(y)$$

$$[F^{*S-i-j-1}(t-u-v-y) - F^{*S-i-j}(t-u-v-y)]dy\ dv\ du$$

$$+ \sum_{i=1}^{j-1} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=1}^{i} f^{*k}(v)\ G(v)[F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]\ dv\ du$$

$$+ \sum_{i=j}^{c} \int_0^t K_i(u)\ \bar{G}(t-u)\ [F^{*i-j}(t-u) - F^{*i-j+1}(t-u)]du$$

$$+ \sum_{i=j}^{c} \int_0^t K_i(u) \int_0^{t-u} \sum_{k=o}^{j} f^{*k}(v)\ G(v)\ [F^{*S-k-j}(t-u-v) -$$

$$F^{*S-k-j+1}(t-u-v)]\ dv\ du \qquad (13)$$

To arrive at (13) we have to consider the cases (i) the first order is placed at u and in (u,t) exactly

s-j demands occur and order is not realized, (ii) first demand is placed at u, the k^{th} demand (for $k < j$) occurs at u+v and replenishment happens in (u,u+v), then there are exactly S-k-j demands in (u+v,t) (iii) After realizing many orders, the last order before time t is placed at u when the level is i and the inventory becomes dry and there are exactly S-i-j demands after replenishment (iv) the last order is placed at level i and at time u, then the k^{th} demand occurs at u+v and order is realized in (u,u+v) and there are S-k-j demands in (u+v,t).
(v) the last order is placed at level i ($\geq j$), order is not realized, but there are i-j demands. (vi) the last order before time t is placed at u when the level is i ($\geq j$), the k^{th} demand (for $k < j$) occurs at u+v and order is realized in (u,u+v), then there are S-k-j demands in the interval (u+v,t).

Finally we have,

$$\pi_0(t) = \int_0^t f^{*S-s}(u)\, \bar{G}(t-u)\, F^{*s}(t-u)\,du$$

$$+ \sum_{i=0}^{c} \int_0^t K_i(u)\, \bar{G}(t-u)\, F^{*i}(t-u)\,du \qquad (14)$$

The above equation is written considering the cases (i) the first order is placed at u and there are at least s demands in (u,t), but order is not realized up to t (ii) After realizing many orders, the last order is placed at u when the level is i, that order is not realized upto t, but there are at least i demands in the interval (u,t).

5.6 ON THE CORRELATION BETWEEN NUMBER OF DEMANDS DURING A LEAD TIME AND THE NEXT INVENTORY DRY PERIOD.

Depending upon the number of demands during one lead time, define the random variable J as follows. The random variable J takes the value j if there are j demands during the lead time, provided j is less than c. If there are more than c-1 demands during that lead time, J takes the value c. Thus J is the ordering level that we fix for the next order. Also let Z be the duration of the dry period during the next lead time. Applying the method used in Chapter 3 we derive expressions to find the correlation between J and Z.

For $o \leq j \leq c$, let $p_j = \text{Prob}\left\{ J = j \right\}$

then

$$p_j = \int_0^\infty g(y) \; [F^{*j}(y) - F^{*j+1}(y)]dy \text{ if } j = 0,1,\ldots,c-1$$

and $$p_c = \int_0^\infty g(y)\, F^{*c}(y)\,dy$$

Here p_j is the probability that the second ordering level is j and so the inventory will become dry if there are at least j demands during that lead time. Here we easily obtain,

$$P\{J=j,\ Z=o\} = p_j \int_0^\infty g(y)\, \bar{F}^{*j}(y)\,dy,\quad o \le j \le c$$

$$P\{J=j,\ z < Z \le z+dz\} = p_j \left(\int_z^\infty g(y) f^{*j}(y-z)\,dy \right) dz$$

$$\text{for } z > o \text{ and } o \le j \le c$$

Then the Double Laplace Stieltjes transform of J and Z will be given by,

$$E(e^{-\eta Z} r^J) = \sum_{j=o}^{c} p_j\, r^j \int_0^\infty g(y)\, \bar{F}^{*j}(y)\,dy$$

$$+ \sum_{j=o}^{c} p_j r^j \int_0^\infty e^{-\eta z} \int_z^\infty g(y) f^{*j}(y-z)\,dy\,dz$$

After some simplifications we get

$$E(e^{-\eta Z} r^J) = \sum_{j=o}^{c} p_j r^j - \eta \sum_{j=o}^{c} p_j r^j \int_0^\infty e^{-\eta y} g(y) \int_0^y e^{\eta x} F^{*j}(x)\,dx\,dy \qquad (15)$$

Now the expected dry period is given by

$$E(Z) = - \frac{\partial}{\partial \eta} E\left(e^{-\eta Z} r^{J}\right) \Big|_{\eta=o,\ r=1}$$

$$= \sum_{j=o}^{c} p_j \int_{o}^{\infty} g(y) \int_{o}^{y} F^{*j}(x)\,dx\,dy \tag{16}$$

$$E(Z^2) = \frac{\partial^2}{\partial \eta^2} E\left(e^{-\eta Z} r^{J}\right) \Big|_{\eta=o,\ r=1}$$

$$= 2 \sum_{j=o}^{c} p_j \int_{o}^{\infty} g(y) \int_{o}^{y} \int_{o}^{x} F^{*j}(u)\,du\,dx\,dy \tag{17}$$

Also

$$E(JZ) = \frac{\partial^2}{\partial r \partial \eta} E\left(e^{-\eta Z} r^{J}\right) \Big|_{\eta=o,\ r=1}$$

$$= \sum_{j=o}^{c} j\, p_j \int_{o}^{\infty} g(y) \int_{o}^{y} F^{*j}(x)\,dx\,dy \tag{18}$$

Then as in chapter 3, the variance of J and Z can be computed and hence the correlation between J and Z can be obtained.

Chapter 6

ON A GENERAL ARRIVAL, BULK SERVICE QUEUE WITH VACATIONS TO THE SERVER

6.1. INTRODUCTION

Single arrival and bulk service queueing systems have been considered by several authors. Bailey [1954] and Downton [1955] considered single server queues having Poisson input, intermittently available server and service in batches of a fixed maximum size. The usual bulk service rule in which the service is done in batches of a fixed maximum size and the server may wait if he finds none in the queue on a completion of service is considered by Jaiswal [1960] and Chaudhry and Templeton [1972]. A general bulk service rule was introduced by Neuts [1967] where a minimum of 'a' units are needed to start a service and a maximum of 'b' units can be served at a time and he obtained the transition probabilities of the number of customers in the system for the model $M/G^{a,b}/1$. Then Borthakur [1971] obtained the steady state probabilities of the queue size for the model $M/M^{a,b}/1$. Ghare [1968] has studied a multichannel queueing system with bulk service. Also Curry and Feldman [1985] considers an $M/M^{a,b}/1$ system with state dependent service parameters.

In all the queueing models discussed above, an idle server remains alert awaiting a new arrival and will commence service immediately upon the customer's arrival. The effect of rest periods in queueing models is studied by many authors. Scholl and Kleinrock [1983] discusses M/G/1 queues with vacations to the server. Assuming the steady state exists, Fuhrmann and Cooper [1985] shows that for a class of M/G/1 queueing systems with generalized vacations, the 'decomposition property' holds.

$M/M^{a,b}/1$ queueing systems with vacations to the server is considered by Nadarajan and Subramanian [1984]. A more detailed and explicit expressions for $M/M^{a,b}/1$ queues with vacation to the server and state dependent service rates is given by Daniel [1985]. In this chapter we extend these to a queue with general arrival distributions. Matrix-geometric approach of Neuts [1981] is utilized. As will be seen, the structure of the matrix-geometric equation is not simple and will not be yielding to any easy algorithmic approach for solutions in the general set up.

In section 6.2 we give details of the model and in section 6.3 we obtain the transition probability matrix.

Matrix-geometric solution of the system is written in section 6.4 and finally in section 6.5 we give the waiting time distribution explicitly.

6.2. THE MODEL

We consider a queueing system in which interarrival times are independent and identically distributed random variables with a general probability distribution function $G(x)$ and let $g(x)$ be the corresponding probability density function. The units are served in batches according to a bulk service rule that, a minimum of 'a' units are needed to start the service and a maximum of 'b' units can be served at a time. Service time is an exponentially distributed random variable with parameter μ_i if i is the batch size being served. At any time immediately after a service if the server finds less than 'a' units in the system, he goes for rest for a random period which is exponentially distributed with parameter α. If after the rest completion the server finds again less than 'a' units in the queue, he remains idle and starts service when queue size becomes 'a'.

Observing the system just prior to the arrival points, we can obtain an imbedded Markov chain with the following state space.

$$S = \Big\{(i,V) \text{ for } i \geqslant o;\ (i,I) \text{ for } o \leq i \leq a-1;\ (i,j) \text{ for } i \geqslant a,\ a \leq j \leq \min\{i,b\}\Big\}$$

Here the state (i,V) for $i \geqslant o$ denotes that there are i units in the system and server is under vacation. The state (i,I) for $o \leq i \leq a-1$ denotes that there are i units in the system and server is idle. The state (i,j) for $i \geqslant a$, $a \leq j \leq \min\{i,b\}$ denote that there are i units in the system and a batch of j units is being served. We shall denote the level i by $\underline{i}$, which is the ordered vector of all possible states having i as the first component.

i.e., for $o \leq i \leq a-1$, $\underline{i} = ((i,V),(i,I))$ is a 2-vector

for $a \leq i \leq b$, $\underline{i} = ((i,V),(i,a),\ldots,(i,i))$ is a $i-a+2$ vector

for $i > b$, $\underline{i} = ((i,V),(i,a),\ldots,(i,b))$ is a $b-a+2$ vector

6.3. TRANSITION PROBABILITY MATRIX

The pattern of the transition probability matrix P will depend on the values of a and b. We shall write the matrix P in the block partitioned form with the assumption that $(b-a-1) < a$. In the case of $(b-a-1) \geqslant a$ also P can be written similarly.

Let $P = [A_{ij}]_{\substack{i \geq o \\ j \geq o}}$ be the block partitioned form of P.

We shall note that if i is a level with m states and j is a level with n states, then A_{ij} is a matrix of order m x n. The matrices A_{ii} for all $i \geq o$ are zero matrices and A_{ij} for all $i \geq o$ and $j > i+1$ are also zero matrices. The non zero A_{ij}'s are given in the following:

For i = o, 1, ..., a-2,

$$A_{i,i+1} = \begin{bmatrix} a_o & \hat{a}_o \\ 0 & 1 \end{bmatrix}$$

where
$$a_o = \int_o^\infty g(x)\ e^{-\alpha x}\ dx$$

$$\hat{a}_o = \int_o^\infty g(x)\ (1-e^{-\alpha x})dx$$

For i = a-1,

$$A_{a-1,a} = \begin{bmatrix} a_o & a_1 \\ 0 & a_2 \end{bmatrix}$$

where
$$a_o = \int_o^\infty g(x)\ e^{-\alpha x}\ dx$$

$$a_1 = \int_o^\infty g(x) \int_o^x \alpha e^{-\alpha u}\ e^{-\mu_a (x-u)} du\ dx$$

$$a_2 = \int_o^\infty g(x)\ e^{-\mu_a x}\ dx$$

For $i = a, \ldots, b-1$,

$$A_{i,i+1} = [a_{rs}]$$

where

$$a_{11} = a_o$$

$$a_{rr} = \int_0^\infty g(x) e^{-\mu_{a+r-2}x}\, dx, \quad r > 1$$

$$a_{1,i-a+3} = \int_0^\infty g(x) \int_0^x \alpha\, e^{-\alpha u}\, e^{-\mu_{i+1}(x-u)} du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

Also,

$$A_{a-1,o} = [a_{rs}]$$

where

$$a_{11} = \int_0^\infty g(x) \int_0^x \gamma_{\alpha,1} * \gamma_{\mu_a,1}(u)\, e^{-\alpha(x-u)} du\, dx$$

$$a_{12} = \int_0^\infty g(x) \int_0^x \gamma_{\alpha,2}(u) \mu_a\, e^{-\mu_a(x-u)} du\, dx$$

$$a_{21} = \int_0^\infty g(x) \int_0^x \gamma_{\mu_a,1}(u)\, e^{-\alpha(x-u)} du\, dx$$

$$a_{22} = \int_0^\infty g(x) \int_0^x \alpha\, e^{-\alpha u} \mu_a\, e^{-\mu_a(x-u)} du\, dx$$

For $i = a, a+1, \ldots, b-1$,

$$A_{i,o} = [a_{rs}]$$

where
$$a_{11} = \int_0^\infty g(x) \int_0^x \gamma_{\alpha,1} * \gamma_{\mu_{i+1},1}(u) e^{-\alpha(x-u)} du\, dx$$

$$a_{12} = \int_0^\infty g(x) \int_0^x \gamma_{\alpha,2}(u)\ \mu_{i+1}\ e^{-\mu_{i+1}(x-u)} du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

For $i = a, a+1, \ldots, b-1$

$$A_{i,1} = [a_{rs}]$$

where
$$a_{i-a+2,1} = \int_0^\infty g(x) \int_0^x \mu_i\ e^{-\mu_i u}\ e^{-\alpha(x-u)} du\, dx$$

$$a_{i-a+2,2} = \int_0^\infty g(x) \int_0^x \mu_i\ e^{-\mu_i u} \alpha e^{-\alpha(x-u)} du\, dx$$

$$a_{rs} = 0 \text{ otherwise}$$

Similarly, $A_{i,\ell}$ for $i = a+\ell-1, \ldots, b-1$ will have the ℓ^{th} row from the last as nonzero and all other rows will be zeros. Expressions are similar.

Now for $i = b, b+1, \ldots,$

$$A_{i,i+1} = [a_{rs}] \text{ is a } (b-a+2)x(b-a+2) \text{ matrix}$$

where
$$a_{11} = \int_0^\infty g(x)\, e^{-\alpha x}\, dx$$

$$a_{rr} = \int_0^\infty g(x)\, e^{-\mu_{a+r-2}x}\, dx, \quad r > 1$$

$$a_{1,b-a+2} = \int_0^\infty g(x) \int_0^x \alpha\, e^{-\alpha u}\, e^{-\mu_b(x-u)}\, du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

Again $A_{b+a-1,b} = A_{b+a,b+1} = A_{b+a+1,b+2} = \ldots = [a_{rs}]$ (say)

where
$$a_{2,b-a+2} = \int_0^\infty g(x) \int_0^x \mu_a\, e^{-\mu_a u}\, e^{-\mu_b(x-u)}\, du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

Also $A_{b+a,b} = A_{b+a+1,b+1} = A_{b+a+2,b+2} = \ldots = [a_{rs}]$

where
$$a_{3,b-a+2} = \int_0^\infty g(x) \int_0^x \mu_{a+1}\, e^{-\mu_{a+1}u}\, e^{-\mu_b(x-u)}\, du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

Similarly,

$$A_{b+a+2,b} = A_{b+a+3,b+1} = \ldots = [a_{rs}]$$

where $$a_{4,b-a+2} = \int_0^\infty g(x) \int_0^x \mu_{a+2}\, e^{-\mu_{a+2}u}\, e^{-\mu_b(x-u)}\, du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

Proceeding like this, we have,

$$A_{2b-1,b} = A_{2b,b+1} = \ldots = [a_{rs}]$$

where $$a_{1,b-a+2} = \int_0^\infty g(x) \int_0^x \gamma_{\alpha,1} * \gamma_{\mu_b,1}(u)\, e^{-\mu_b(x-u)}\, du\, dx$$

$$a_{b-a+2,b-a+2} = \int_0^\infty g(x) \int_0^x \mu_b\, e^{-\mu_b u}\, e^{-\mu_b(x-u)}\, du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

Proceeding further,

$$A_{2b+a-1,b} = A_{2b+a,b+1} = \ldots = [\, a_{rs}\,]$$

where $$a_{2,b-a+2} = \int_0^\infty g(x) \int_0^x \gamma_{\mu_a,1} * \gamma_{\mu_b,1}(u)\, e^{-\mu_b(x-u)}\, du\, dx$$

$$a_{rs} = 0 \quad \text{otherwise}$$

Similarly, $A_{2b+a,b} = A_{2b+a+1,b+1} = \ldots$ and so on.

.4 MATRIX-GEOMETRIC SOLUTION

Deleting all the rows and columns upto $\underline{b-1}$ from the atrix P and using the notations,

$$M_0 = A_{b,b+1}, \quad M_1^0 = A_{b+a-1,b}, \quad M_2^0 = A_{b+a,b}, \quad \ldots$$

$$M_1^1 = A_{2b+a-1,b}, \quad M_2^1 = A_{2b+a,b}, \quad \ldots \text{ etc.}$$

e get,

$$\begin{array}{rl}
 & \begin{array}{cccccccccccc} \underline{b} & \underline{b+1} & \underline{b+2} & \cdots & \underline{b+a-1} & \underline{b+a} & \underline{b+a+1} & \cdots & \underline{2b-1} & \underline{2b} & \underline{2b+1} & \cdot \end{array} \\
\begin{array}{c} \underline{b} \\ \underline{b+1} \\ \vdots \\ \underline{b+a-1} \\ \underline{b+a} \\ \vdots \\ \underline{2b-1} \\ \underline{2b} \\ \vdots \\ \underline{2b+a-1} \\ \underline{2b+a} \\ \vdots \end{array} & \left[\begin{array}{cccccccccccc}
0 & M_0 & 0 & & 0 & 0 & 0 & & 0 & 0 & 0 & \\
0 & 0 & M_0 & & 0 & 0 & 0 & & 0 & 0 & 0 & \\
 & & & & & & & & & & & \\
M_1^0 & 0 & 0 & & 0 & M_0 & 0 & & 0 & 0 & 0 & \\
M_2^0 & M_1^0 & 0 & & 0 & 0 & M_0 & & 0 & 0 & 0 & \\
 & & & & & & & & & & & \\
M_{b-a+1}^0 & M_{b-a}^0 & M_{b-a-1}^0 & & \cdots & & & & 0 & M_0 & 0 & \\
0 & M_{b-a+1}^0 & M_{b-a}^0 & & \cdots & & & & 0 & 0 & M_0 & \\
 & & & & & & & & & & & \\
M_1^1 & 0 & 0 & & \cdots & & & & & & & \\
M_2^1 & M_1^1 & 0 & & \cdots & & & & & & & \\
 & & & & & & & & & & &
\end{array} \right.
\end{array}$$

Let $\underline{x}$ be the stationary probability vector satisfying $\underline{x}\, P = \underline{x}$ and $\underline{x}\, \underline{e} = 1$, where $\underline{e} = (1, 1, \ldots)^T$, then $\underline{x}$ has a modified matrix geometric form (Neuts [1981]).
We can partition $\underline{x}$ in the form,

$$\underline{x} = (\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{a-1}, \underline{x}_a, \ldots, \underline{x}_{b-1}, \underline{x}_b, \underline{x}_{b+1}, \ldots)$$

where $\underline{x}_i$ is a vector having its order equal to the order of i. For $i > b$, we can look for solution of the form $\underline{x}_i = \underline{x}_b R^{i-b}$ where R is a square matrix of order b-a+2, which is called the rate matrix and is explained in Neuts [1981].

Using $\underline{x}\, P = \underline{x}$, we get

$$\underline{x}_b M_o + \underline{x}_{b+a} M_1^o + \underline{x}_{b+a+1} M_2^o + \ldots + \underline{x}_{2b} M_{b-a+1}^o +$$
$$\underline{x}_{2b+a} M_1^1 + \underline{x}_{2b+a+1} M_2^1 + \ldots + \underline{x}_{3b} M_{b-a+1}^1 +$$
$$\ldots\ldots\ldots\ldots\ldots\ldots = \underline{x}_{b+1}$$

Using $\underline{x}_i = \underline{x}_b R^{i-b}$ for $i > b$

$$\underline{x}_b M_o + \underline{x}_b R^a M_1^o + \underline{x}_b R^{a+1} M_2^o + \ldots + \underline{x}_b R^b M_{b-a+1}^o +$$
$$\underline{x}_b R^{b+a} M_1^1 + \underline{x}_b R^{b+a+1} M_2^1 + \ldots + \underline{x}_b R^{2b} M_{b-a+1}^1 +$$
$$\ldots\ldots\ldots\ldots\ldots\ldots = \underline{x}_b R$$

Using $\underline{x}_b > \underline{0}$ we get,

$$M_o + \sum_{i=1}^{b-a+1} R^{a+i-1} M_i^o + \sum_{i=1}^{b-a+1} R^{b+a+i-1} M_i^1 + \ldots = R$$

i.e. $$M_o + \sum_{k=o}^{\infty} \sum_{i=1}^{b+a-1} R^{kb+a+i-1} M_i^k = R \quad \text{———} \quad (A)$$

The minimal solution ($\geqslant 0$) of the equation (A) will give the rate matrix R and thus for $i > b$, the matrix-geometric solution is given by $\underline{x}_i = \underline{x}_{i-1} R$.

The vector $(\underline{x}_o, \underline{x}_1, \ldots, \underline{x}_b)$ can be obtained by solving the system,

$$[\underline{x}_o, \underline{x}_1, \ldots \underline{x}_b] \begin{bmatrix} 0 & A_o & 0 & \ldots & 0 \\ 0 & 0 & A_{12} & \ldots & 0 \\ \vdots & & & & \\ A_{a-1,o} & 0 & 0 & \ldots & \\ A_{a,o} & A_{a,1} & 0 & \ldots & \\ \vdots & & & & \\ A_{b-1,o} & A_{b-1,1} & & \ldots & A_{b-1,b} \\ Y_1 & Y_2 & & & Y_b \end{bmatrix} = [\underline{x}_o, \underline{x}_1, \ldots, \underline{x}_b]$$

and $\underline{x}_o \, \underline{e} + \underline{x}_1 \, \underline{e} + \ldots + \underline{x}_b \, (I-R)^{-1} \, \underline{e} = 1$

where $Y_1 = \sum_{k=o}^{\infty} \sum_{i=o}^{b-a} R^{kb+a+i-1} A_{(k+1)b+a-i-1,o}$

$$Y_2 = A_{b,1} + \sum_{k=o}^{\infty} \sum_{i=o}^{b-a} R^{kb+a+i} A_{(k+1)b+a-i,1}$$

$$\vdots$$

$$Y_b = \sum_{k=o}^{\infty} \sum_{i=1}^{b+a-1} R^{kb+a+i-2} M_i^k$$

6.5 WAITING TIME DISTRIBUTION

The waiting time of a new arrival depends on the state of the server at the arrival point. An arriving unit may find the server in one of the following states. (a) Server is idle, (b) Server is working, (c) Server is under vacation.

(a) Server is idle.

Let (i,I) for $1 \leq i \leq a-1$ be the state of the system when the arrival occurs. Let W be the waiting time of the arriving unit in the queue. The unit has to wait for the arrival of next $a-i-1$ units.

Therefore $\quad P\{W \leq t\} = G^{*(a-i-1)}(t)$

(b) Server is working

Let (i,j) be the state of the system just prior to the arrival of the unit, where $i \geq a$, $a \leq j \leq \min\{i,b\}$. Let $K = i-j - [\frac{i-j}{b}]b$. Then the waiting time of that unit in the queue is as follows.

$$P\{W \leq t\} = \Gamma_{\mu_j,1} * \Gamma_{\mu_b,[\frac{i-j}{b}]}(t), \quad \text{if } K \geq a-1$$

Now for $K < a-1$

$$P\{W \leq t\} = \left\{\Gamma_{\mu_j,1} * \Gamma_{\mu_b,[\frac{i-j}{b}]}(t)\right\} \sum_{n=a-k-1}^{\infty} G^{*n}(t)$$

$$+ \sum_{l=0}^{a-k-2} \int_0^t \Gamma_{\mu_j,1} * \Gamma_{\mu_b,[\frac{i-j}{b}]} * \Gamma_{\alpha,1}(u)$$

$$(G^{*l}(u) - G^{*l+1}(u))\, d\, G^{*(a-k-l-1)}(t-u)$$

(c) Server is under vacation

Let (i,V) be the state of the system just prior to the arrival of the unit.

Let $K = i - [\frac{i}{b}]b$.

If $K \geqslant a-1$,

$$P\left\{W \leq t\right\} = \Gamma_{\alpha,1} * \Gamma_{\mu_b,[\frac{i}{b}]}(t)$$

and if $K < a-1$,

$$P\left\{W \leq t\right\} = \left\{\Gamma_{\alpha,1} * \Gamma_{\mu_b,[\frac{i}{b}]}(t)\right\} \sum_{n=a-k-1}^{\infty} G^{*n}(t)$$

$$+ \sum_{\ell=0}^{a-k-2} \int_0^t \Gamma_{\alpha,2} * \Gamma_{\mu_b,[\frac{i}{b}]}(u)$$

$$(G^{*\ell}(u) - G^{*\ell+1}(u)) \, dG^{*(a-k-\ell-1)}(t-u)$$

Chapter 7

A FINITE CAPACITY M/G/1 QUEUEING SYSTEM WITH VACATIONS TO THE SERVER

7.1 INTRODUCTION

In this chapter we consider a single server queueing system with the server going for vacation whenever there is no unit in the system. The rest times are independent and identically distributed random variables having a general probability distribution function. The arrival process is Poisson. We assume that the capacity of the waiting room is a fixed positive integer and all the arrivals taking place when the waiting room is full are lost. The service times of units are independent and identically distributed random variables with a general probability distribution function. Using renewal theoretic arguments we derive the transient solution for the system size probabilities at arbitrary time points. Also we obtain expressions for the probability distribution of the virtual waiting time in the queue at any time t.

Several authors have considered M/G/1 queues with vacations to the server. See for example Miller [1964] and

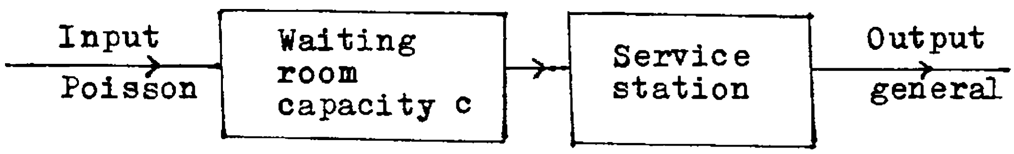
Scholl and Kleinrock [1983]. They have all treated the situations where the steady state exists and investigated the system size probabilities at the departure points. Assuming the steady state exists, Fuhrmann and Cooper[1985] shows that for a class of M/G/1 queueing systems with generalized vacations, the 'decomposition property' holds. That is, the (stationary) number of customers present in the system at a random time point is distributed as the sum of two or more independent random variables, one of which is the (stationary) number of customers present in the corresponding standard M/G/1 queue (ie. the server is always available) at a random point in time. For a complete reference on vacation models one may refer to Doshi [1986].

Physical and economic considerations reveal that many realistic waiting line systems have only finite capacity. Under such conditions the arrivals are not accepted to the system if the waiting room is full. Cohen [1976] gives a detailed account of M/G/K loss systems. Lee [1984] investigates the stationary behaviour of a finite capacity M/G/1 queue with vacation time and exhaustive service. In the literature, very few results are available giving the time dependent system size probabilities of Non-Poisson queues.

In section 7.2, we give the details of the assumptions and notations used in this chapter; together with the different probability density functions used in section 7.3. In section 7.3 the transient system size probabilities are derived. Section 7.4 deals with the virtual waiting time in the queue at any time t.

7.2 DESCRIPTION OF THE MODEL

The arrival of customers to the system is according to a homogeneous Poisson process of rate μ. Service times are independent and identically distributed random variables with distribution function G(.) and density function g(.). The waiting room is of finite capacity c(> o). All the arrivals taking place when the waiting room is full are lost.



At time zero the system starts with 'a' (> o) units in the waiting room. The server takes all the 'a' units to the service station and serves them one by one in the order of their arrival. When all the 'a' units are served, the server goes back to the waiting room. If there is at least one unit waiting, he takes all of them to the service station and serves them one by one. If there is no unit in the waiting room, the server goes for vacation for a random duration having distribution function H(.) and density function h(.). If the server returns from a vacation to find no customer waiting, he begins another vacation immediately, independent of the previous one and having the same distribution function H(.). This process is continued until there is atleast one unit in the waiting room.

For j = o,1, ..., c-1, let $\mu_j(x)$ be the probability that there are exactly j arrivals during an interval of length x and let $\mu_c(x)$ be the probability that there are atleast c arrivals during an interval of length x.

Then

$$\mu_j(x) = \frac{e^{-\mu x}(\mu x)^j}{j!}, \quad j = o,1, \ldots, c-1$$

and $$\mu_c(x) = \sum_{j=c}^{\infty} \frac{e^{-\mu x}(\mu x)^j}{j!}$$

For $o \le i, j \le c$, let

$f_{ij}(x)\,dx$ = Probability that starting at time zero, the service of i units is over in the interval $(x, x+dx)$ and there are j arrivals accepted to the system in $(o, x]$.

Then

$$f_{ij}(x) = \mu_j(x)\, g^{*i}(x) \tag{1}$$

For $i = 1, 2, \ldots, c$, let

$$\underline{f}_i(x) = (f_{i1}(x), \ldots, f_{ic}(x))$$

Also let

$$\underline{f}_o(x) = (f_{1o}(x), \ldots, f_{co}(x))^T$$

(It is a column vector of order c)

Now define a square matrix of order c, given by

$$\mathbb{F}(x) = \begin{bmatrix} f_{11}(x) & \cdots & f_{1c}(x) \\ \vdots & & \\ f_{c1}(x) & \cdots & f_{cc}(x) \end{bmatrix}$$

Let $\mathbb{F}^{*o}(x)$ be the identity matrix of order c and for $n \geq 1$, $\mathbb{F}^{*n}(x)$ be the n-fold convolution of the matrix $\mathbb{F}(x)$ with itself.

Then, for i=1,2,...,c, $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ is a vector of order c. Let $K_\ell^i(x)$ be the ℓ^{th} coordinate of the vector $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$; where $1 \leq i, \ell \leq c$.

Also let,

$$K^i(x) = (\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \underline{f}_o)(x) \qquad (2)$$

Let $F_i(x)$ be the probability density function of a busy period initiated by i customers.ie, the probability that the busy period initiated by i units is over between x and x+dx. Then we have,

$$F_i(x) = f_{io}(x) + K^i(x) \qquad (3)$$

The renewal points of the process are those time points at which the server goes for rest after a busy period. Let Z be the time between two such successive renewal points, then the probability density function of Z is obtained as,

$$k(z) = P\left\{z < Z \le z+dz\right\}$$

$$= \int_{o}^{z} \mu_{o}(u) \sum_{m=o}^{\infty} h^{*m}(u) \int_{u}^{z} h(v-u) \sum_{i=1}^{c} \mu_{i}(v-u)$$

$$F_{i}(z-v)\,dv\,du \tag{4}$$

Then the renewal density function of the delayed renewal process is given by

$$M(u) = \sum_{n=o}^{\infty} (F_{a} * k^{*n})(u) \tag{5}$$

Finally, the state space of the system is

$S = \left\{(i,j) \middle| \; o \le i,\ j \le c\right\}$ where, for $i = o,1,\ldots c$, (i,o) denotes the state that there are i units in the waiting room and server is under vacation. For $1 \le j \le c$ and $o \le i \le c$, (i,j) denotes the state that there are i units in the waiting room and there are j units in the service station including the one being served.

7.3 THE SYSTEM SIZE PROBABILITIES

Let $P_{ij}(t)$ = The probability that the system is in state (i,j) at time t given the system starts with 'a' units at time zero.

considering all the mutually exclusive and exhaustive cases, the following relations can be written.

For $i = o,1,\ldots,c$,

$$P_{io}(t) = \int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} \mu_o(v-u)h^{*m}(v-u)$$

$$\bar{H}(t-v)\ \mu_i(t-v)dv\ du \tag{6}$$

The above equation is obtained as follows. Consider the interval (o,t). Many busy cycles are over prior to time u and the last busy period is over at u and the server goes for vacation. In (u,v), many vacation periods are over but no unit arrives in the waiting room. Finally the vacation started at v is not over up to time t and there are i arrivals during (v,t).

For $o < j \leq a$ and $o \leq i \leq c$,

$$P_{ij}(t) = [G^{*(a-j)}(t) - G^{*(a-j+1)}(t)]\ \mu_i(t) +$$

$$\int_o^t \sum_{\ell=j}^{c} K_\ell^a(u)\ [G^{*(\ell-j)}(t-u) - G^{*(\ell-j+1)}(t-u)]\mu_i(t-u)du$$

$$+ \int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u)\ \mu_o(v-u) \int_v^t h(w-v)$$

$$\sum_{\ell=j}^{c} \mu_\ell(w-v) \; [G^{*(\ell-j)}(t-w) - G^{*(\ell-j+1)}(t-w)]\mu_i(t-w)dw \; dv \; du$$

$$+ \int_0^t M(u) \int_u^t \sum_{m=0}^{\infty} h^{*m}(v-u) \; \mu_0(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{c} \mu_\ell(w-v) \int_w^t \tag{7}$$

$$\sum_{k=j}^{c} K_k^\ell(y-w) \; [G^{*(k-j)}(t-y)-G^{*(k-j+1)}(t-y)]\mu_i(t-y)dy \; dw \; dv \; du$$

To obtain equation (7) the following mutually exclusive cases are to be considered (i) starting the service process at time zero, exactly a-j services are over upto time t during which time there are i arrivals (ii) The busy period is not over, but after many services, at time u, the server takes a batch of size ℓ to the service station and exactly ℓ-j services are over and there are i arrivals during the interval (u,t). (iii) After many busy cycles, the last busy period is over at u and after the vacations the server starts service with ℓ units and w and then exactly ℓ-j services are over during which time i arrivals occur. (iv) As in the last case after many busy cycles, a busy period is started with ℓ units and then after many service cycles, at time y the server takes a batch of k units to the service station at time y of which exactly k-j units are served out and i arrivals occur during the interval (y,t).

Now for $a < j \leq c$ and $o \leq i \leq c$

$$P_{ij}(t) = \int_o^t \sum_{\ell=j}^{c} K_\ell^a(u) \, [G^{*(\ell-j)}(t-u) - G^{*(\ell-j+1)}(t-u)] \mu_i(t-u) du$$

$$+ \int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u) \, \mu_o(v-u) \int_v^t h(w-v)$$

$$\sum_{\ell=j}^{c} \mu_\ell(w-v) \, [G^{*(\ell-j)}(t-w) - G^{*(\ell-j+1)}(t-w)]$$

$$\mu_i(t-w) dw \, dv \, du + \int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u)$$

$$\mu_o(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{c} \mu_\ell(w-v) \int_w^t \sum_{k=j}^{c} K_k^\ell(y-w)$$

$$[G^{*(k-j)}(t-y) - G^{*(k-j+1)}(t-y)] \mu_i(t-y) dy \, dw \, dv \, du \qquad (8)$$

To arrive at equation (8) we need just consider the cases of (ii), (iii) and (iv) of equation (7).

7.4 VIRTUAL WAITING TIME IN THE QUEUE

The virtual waiting time in the queue at time t is defined as the waiting time of a unit in the queue if it were to arrive at time t. (See Takacs [1962]). In our model, a queue in the waiting room is shifted as such to the service station. So an arrival has to wait in a queue in the waiting room and then in a queue in the service station until he is taken for service. Let W_t be the virtual waiting time at time t. Assuming that the virtual customer arriving at time t is accepted even if the waiting room is full, we compute the probability distribution of W_t, conditional to the state of the system at time t and it is enough because the system size probabilities are already given. We consider the following cases separately.

case (i): The state is (i,j) for $o \le i \le c$, $o < j \le c$, so that the server is working.

Then

$$P\left\{W_t \le x\right\} = \int_o^t \sum_{\ell=j}^{c} K_\ell^a(u)\ G^{*\ell+i}(t+x-u)\,du$$

$$+\int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u)\ \mu_o(v-u)$$

$$\int_u^t h(w-v) \sum_{\ell=1}^{c} \mu_\ell(w-v) \int_w^t \sum_{k=j}^{c} K_k^\ell(y-w)$$

$$G^{*k+i}(t+x-y)\,dy\ dw\ dv\ du \qquad (9)$$

Case (ii): The server is taking rest and the state is (i,o) for $1 \leq i \leq c$. In this case we can easily obtain,

$$P\left\{W_t \leq x\right\} = \int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u)\ \mu_o(v-u)$$

$$\int_t^{t+x} h(w-v)\ G^{*i}(t+x-w)\,dw\ dv\ du \tag{10}$$

Case (iii): The server is taking rest and state is (o,o)

Then

$$P\left\{W_t \leq x\right\} = \int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u)\ \mu_o(v-u)$$

$$[H(t+x-v)-H(t-v)]\ dv\ du \tag{11}$$

Chapter 8

ON A FINITE CAPACITY $M/G^{a,b}/1$ QUEUEING SYSTEM WITH VACATIONS TO THE SERVER

8.1 INTRODUCTION

Bailey [1954] carried out the first mathematical investigation of queues involving batch service. He studied the stationary behaviour of a single server queue having simple Poisson input, intermittently available server and service in batches of a fixed maximum size. This study was followed by a series of papers involving the treatment of queueing processes with group arrival or batch service. Neuts [1967], Borthakur [1971] and Medhi [1975] deal with queueing systems with Poisson input and bulk service. More complex waiting line systems in which customers arrive in groups of random size and are served in groups of random size, are also considered by many authors. Keilson [1962] discusses the $M^X/G^Y/1$ queue using supplementary variable technique. Bhat[1964] analyses the imbedded Markov chains of $M^X/G^Y/1$ and $G^X/M^Y/1$ queueing systems using fluctuation theory.

In this chapter we consider a single server queueing system with Poisson input, general batch service and a waiting room that allows only a maximum of 'b' customers to wait at any time. A minimum of 'a' customers are required to start a service and the server goes for a vacation whenever he finds less than 'a' customers in the waiting room after a service. If the server returns from a vacation to find less than 'a' customers waiting, he begins another vacation immediately. Here also we derive expressions for the time dependent system size probabilities at arbitrary time points.

Section 8.2 contains the details of the assumptions and notations used in this chapter. The different probability density functions used in section 8.3 are also derived in this section. In section 8.3 expressions for the system size probabilities at arbitrary time points are given.

8.2 DESCRIPTION OF THE MODEL

The arrival of customers into the system is according to a homogeneous Poisson process of rate μ. The units are served in batches according to a bulk service rule, namely, a minimum of 'a' units are needed to start a service and a maximum of 'b' units can be served at a time. Service times are independent random variables having distribution function $G_i(.)$, if i is the size of the batch being served and let $g_1(.)$ be the corresponding probability density function. The waiting room is of capacity 'b', so that each service is started with all the units that are waiting for service at that time. All the arrivals taking place when the waiting room is full are lost.

At time zero, the system starts with 'r' units; $a \leq r \leq b$. At any time immediately after a service, if the server finds at least 'a' customers waiting, all of them are taken for service. If he finds less than 'a' customers in the system, he goes for vacation for a random period having probability distribution function $H(.)$ and probability density function $h(.)$. If the server returns from a vacation to find less than 'a' units waiting, he begins another vacation immediately, independent of the previous one and having the same distribution function $H(.)$. This process is continued until there are at least 'a' units in the waiting room.

For $j = o,1, \ldots, b-1$, let $\mu_j(x)$ be the probability that there are exactly j arrivals during an interval of length x and let $\mu_b(x)$ be the probability that there are at least b arrivals during an interval of length x.

Then

$$\mu_j(x) = \frac{e^{-\mu x}(\mu x)^j}{j!}, \; j = o,1, \ldots, b-1$$

$$\mu_b(x) = \sum_{j=b}^{\infty} \frac{e^{-\mu x}(\mu x)^j}{j!}$$

For $a \leq i \leq b$ and $o \leq j \leq b$, let

$f_{ij}(x)dx$ = Probability that starting at time zero, the service of a batch of size i units is over in the interval $(x, x+dx)$ and there are j accepted arrivals during the interval $(o,x]$.

Then $f_{ij}(x) = g_i(x) \; \mu_j(x)$ (1)

For $a \leq i \leq b$, let $\underline{f}_i(x) = (f_{ia}(x), f_{i(a+1)}(x), \ldots, f_{ib}(x))$;

it is a vector of order b-a+1.

Now define a square matrix of order b-a+1, given by

$$\mathbb{F}(x) = \begin{bmatrix} f_{aa}(x) & \cdots\cdots & f_{ab}(x) \\ \vdots & & \\ f_{ba}(x) & \cdots\cdots & f_{bb}(x) \end{bmatrix}$$

Also we define a matrix $\mathbb{G}(x)$ of order $(b-a+1) \times a$ given by

$$\mathbb{G}(x) = \begin{bmatrix} f_{ao}(x) & \cdots & f_{a(a-1)}(x) \\ \vdots & & \\ f_{bo}(x) & \cdots & f_{b(a-1)}(x) \end{bmatrix}$$

Let $\mathbb{F}^{*o}(x)$ be the identity matrix of order $(b-a+1)$ and for $n \geq 1$, $\mathbb{F}^{*n}(x)$ be the n-fold convolution of $\mathbb{F}(x)$ with itself. Then for $a \leq i \leq b$, $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ will be a vector of order $b-a+1$ and $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{G})(x)$ will be a vector of order a.

For $\eta = a, a+1, \ldots, b$, let $M_i^{\eta}(x)$ be the $(\eta-a+1)^{th}$ coordinate of $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ and for $\eta = o, 1, \ldots, a-1$, let $K_i^{\eta}(x)$ be the $(\eta+1)^{th}$ coordinate of $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{G})(x)$.

Thus we obtain the probability density function of a busy period, starting with i units and ending with η units left over, as

$$F_i^{\eta}(x) = f_{i\eta}(x) + K_i^{\eta}(x) \text{ for } a \le i \le b \text{ and } 0 \le \eta \le a-1 \tag{2}$$

As described earlier, at any time immediately after a service if the server finds less than 'a' customers in the system, he goes for a vacation and the vacations are repeated until he finds at least 'a' customers in the system. Vacation periods are assumed to be independent and identically distributed random variables with distribution function H(.) and density function h(.). For $0 \le j \le a-1$ and $a \le k \le b$, let $h_{jk}(x)dx$ be the probability that after a busy period the server goes for a vacation at time zero, when there were j units waiting and after many vacations, the next busy period starts in $(x, x+dx)$ when there are k units in the system.

Then

$$h_{jk}(x) = \int_0^x \sum_{m=0}^{\infty} h^{*m}(u) \sum_{\ell=0}^{a-j-1} \mu_\ell(u) h(x-u) \mu_{k-\ell-j}(x-u)\, du$$

$$\text{for } 0 \le j \le a-1 \text{ and } a \le k < b \tag{3}$$

and

$$h_{jb}(x) = \int_0^x \sum_{m=0}^{\infty} h^{*m}(u) \sum_{\ell=0}^{a-j-1} \mu_\ell(u) h(x-u) \sum_{i=b-\ell-j} \mu_i(x-u)\, du$$

Now we look at the time points at which the busy periods start and obtain the probability density function of the time between two such consecutive points.

For $a \leq i,\ k \leq b$, let

$b_{ik}(x)dx$ = Probability that a busy period is started at time zero with i units in the system and the next busy period begins in the time interval $(x, x+dx)$ with k units in the system.

Then $$b_{ik}(x) = \int_0^x \sum_{\eta=0}^{a-1} F_i^{\eta}(u)\ h_{\eta k}(x-u)du \qquad (4)$$

Now we define a vector of order (b-a+1) by

$$\underline{b}_r(x) = (b_{ra}(x),\ b_{r(a+1)}(x)\ ,\ldots,\ b_{rb}(x))$$

Also define a square matrix of order (b-a+1) by,

$$\mathbb{B}(x) = \begin{bmatrix} b_{aa}(x) & \cdots & b_{ab}(x) \\ \vdots & & \\ b_{ba}(x) & \cdots & b_{bb}(x) \end{bmatrix}$$

Let $\mathbb{B}^{*o}(x)$ denote the identity matrix of order $(b-a+1)$ and for $n \geqslant 1$, let $\mathbb{B}^{*n}(x)$ be the n-fold convolution of the matrix $\mathbb{B}(x)$ with itself. Then $(\underline{b}_r * \sum_{n=o}^{\infty} \mathbb{B}^{*n})(x)$ is a vector of order $(b-a+1)$. For $a \leq j \leq b$, let $B_j(x)$ be the $(j-a+1)^{th}$ coordinate of $(\underline{b}_r * \sum_{n=o}^{\infty} \mathbb{B}^{*n})(x)$.

Finally the state space of the system is given by

$$S = \left\{(i,j) \,\middle|\, a \leq i \leq b,\ o \leq j \leq b\right\} \cup \left\{(o,j) \,\middle|\, o \leq j \leq b\right\}$$

For $a \leq i \leq b$, the state (i,j) denotes that a batch of i units is being served and there are j units waiting at that time. Also (o,j) denotes the state that server is under vacation and there are j units waiting in the system.

8.3. THE SYSTEM SIZE PROBABILITIES

For $i = o, a, a+1, \ldots, b$ and $j = o, 1, \ldots, b$,

Let $P_{ij}(t)$ = The probability that the state of the system is (i,j) at time t, given the system starts with r units at time zero.

Considering all the mutually exclusive and exhaustive cases, the following relations can be obtained.

For $o \le j \le b$,

$$P_{rj}(t) = \bar{G}_r(t)\ \mu_j(t) + \int_o^t M_r^r(u)\ \bar{G}_r(t-u)\mu_j(t-u)du$$

$$+ \sum_{\eta=a}^{b} \int_o^t B_\eta(u) \int_u^t M_\eta^r(v-u)\ \bar{G}_r(t-v)\mu_j(t-v)dv\ du \qquad (5)$$

$$+ \int_o^t B_r(u)\ \bar{G}_r(t-u)\ \mu_j(t-u)du$$

To obtain equation (5) consider the cases (i) service of the first batch is not over up to time t and there are j arrivals during this time (ii) the first busy period is not over, a batch of size r is being served and j arrivals has taken place (iii) Many busy cycles are over, the last busy period starts with η units, then after many service completions a batch of size r is being served and there are j arrivals (iv) Many busy cycles are over, the last busy period starts with r units, then no service completion but j arrivals occur.

For $j = o,1, \ldots, b$ and $i = a, \ldots, r-1, r+1, \ldots, b$,

$$P_{ij}(t) = \int_o^t M_r^i(u)\, \bar{G}_i(t-u)\, \mu_j(t-u)du$$

$$+ \sum_{\eta=a}^{b} \int_o^t B_\eta(u) \int_u^t M_\eta^i(v-u)\, \bar{G}_i(t-v)\, \mu_j(t-v)dv\, du \qquad (6)$$

$$+ \int_o^t B_i(u)\, \bar{G}_i(t-u)\quad \mu_j(t-u)du$$

Equation (6) is derived considering the cases (i) the first busy period is not over and a batch of size i is being served during which time there are j arrivals (ii) Many busy cycles are over and the last busy period is started with η units. Then after many service completion, a batch of size i is being served and there are j arrivals (iii) Many busy cycles are over by time u, the last busy period starts with i units and their service is not over up to time t and there are j arrivals.

Now for $j = o,1, \ldots, a-1$, we get

$$P_{oj}(t) = \int_o^t \sum_{\eta=a}^{b} B_\eta(u) \int_u^t \sum_{k=o}^{j} F_\eta^k(v-u)\mu_{j-k}(t-v)dv\ du$$

$$+ \int_o^t \sum_{k=o}^{j} F_r^k(u)\ \mu_{j-k}(t-u)du \qquad (7)$$

To write down equation (7), we consider the cases (i) Many busy cycles are over and last busy period is started with η units and ends with k units left over at time v and then j-k arrivals in (u,t] (ii) the first busy period is over with k units left over at time u and then j-k arrivals.

For j = a,a+1, ..., b-1,

$$P_{oj}(t) = \int_o^t \sum_{\eta=a}^{b} B_\eta(u) \int_u^t \sum_{k=o}^{a-1} F_\eta^k(v-u) \int_v^t \sum_{m=o}^{\infty} h^{*m}(v-u)$$

$$\sum_{\ell=o}^{a-k-1} \mu_\ell(w-v)\ \overline{H}(t-w)\ \mu_{j-k-\ell}(t-w)dw\ dv\ du$$

$$+ \int_o^t \sum_{k=o}^{a-1} F_r^k(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{\ell=o}^{a-k-1} \mu_\ell(v-u)$$

$$\overline{H}(t-v)\quad \mu_{j-k-\ell}(t-v)dv\ du \tag{9}$$

To obtain equation (8) we consider the cases (i) Many busy cycles are over and the last busy period starts with η units and ends with k units left over and then many vacation periods are over and only ℓ arrivals during this period and the next vacation period not over during which there are j-k-ℓ arrivals (ii) the first busy period is over with k units left over in the system at time u, then many vacation periods are over during which only ℓ arrivals take place and the next vacation is not over up to time t and there are $j-k-\ell$ arrivals.

Finally,

$$P_{ob}(t) = \int_0^t \sum_{\eta=a}^{b} B_\eta(u) \int_u^t \sum_{k=o}^{a-1} F_\eta^k(v-u) \int_v^t \sum_{m=o}^{\infty} h^{*m}(w-v)$$

$$\sum_{\ell=o}^{a-k-1} \mu_\ell(w-v)\, \bar{H}(t-w) \sum_{i=b-k-\ell}^{b} \mu_i(t-w)\,dw\,dv\,du$$

$$+ \int_o^t \sum_{k=o}^{a-1} F_r^k(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{\ell=o}^{a-k-1} \mu_\ell(v-u)$$

$$\bar{H}(t-v) \sum_{i=b-k-\ell}^{b} \mu_i(t-v)\,dv\,du \tag{9}$$

Equation (9) is written considering the cases (i) many busy cycles are over and the last busy period started with η units is over with k units left over in the system and then many vacation periods are over during which time ℓ arrivals take place; the next vacation is not over up to time t and there are at least $b-k-\ell$ arrivals

(ii) the first busy period is over with k units left in the system, then many vacation periods are over during which ℓ arrivals take place; the next vacation is not over up to time t and there are at least $b-k-\ell$ arrivals.

Remark:

As in the previous chapter, the probability distribution of the virtual waiting time in the queue at any time t can be written conditional to the state of the system at time t.

REFERENCES

1. O.M.E. Ali and M.F. Neuts [1984], " A service system with two stages of waiting and feedback of customers" , J.Appl. Prob. 21, 404-413.

2. Arrow K.J., S.Karlin and H. Scarf [1985], " Studies in the mathematical theory of Inventory and Production", Stanford University Press.

3. Bailey N.T.J. [1954], " On queueing processes with bulk service", J.Roy.Statist,Soc.Ser.B, 16, 80-87.

4. Bhat U.N. [1964], " Imbedded Markov chain analysis of single server bulk queues", J.Austral.Math.Soc.4, 244-263.

5. Borthakur A. [1971], " Poisson queue with a general bulk service rule", J.Assam Sciences Soc. XIV, 162-167.

6. Chaudhry M.L. and J.G.C.Templeton [1972], " The theory of bulk arrival bulk service queues" , Opsearch 9, 103-121.

7. Cohen J.W. [1976], " On regenerative processes in queueing theory" , Lecture notes in Eco. and Math. systems, 121, Springer-Verlag, New York.

8. Cooper R.B. [1970], " Queues served in cyclic order: waiting times". Bell System Technical Journal, 49, 399-413.

9. Cooper R.B. [1972], " Introduction to queueing theory" . The Macmillan Company, New York.

10. Cox D.R. [1962], "Renewal theory" . Methuen, London.

11. Curry G.L. and Feldman R.M. [1985], " An M/M/1 queue with a general bulk service rule ", Nav. Res. Logist. Quart., 32, 595-603.

12. Daniel J.K. [1985], " Queueing and inventory models with rest periods", Ph.D. thesis (un-published), Cochin University of Science and Technology.

13. Daniel J.K. and Ramanarayanan R. [1987a], " An (s,S) inventory system with rest to the server" , Nav.Res.Logist. Quart. (to appear).

14. Daniel J.K. and Ramanarayanan R. [1987b], " An (s,S) inventory system with two servers and rest periods" , Cahiers du C.E.R. Operationnelle (to appear)

15. Doshi B.T. [1985], " A note on stochastic decomposition in a GI/G/1 queue with vacations or set-up times" , J.Appl.Prob. 22, 419-428.

16. Doshi. B.T. [1986], " Queueing systems with vacations- A Survey", Queueing Systems, 1, 29-66.

17. Downton F. [1955], " Waiting time in bulk service queues", J.Roy. Statist. Soc.B.17, 256-261.

18. Fuhrmann S.W. [1984], " A note on the M/G/1 queue with server vacations", Oper. Res. 32, 1368-1373.

19. Fuhrmann S.W. and Cooper R.B. [1985], " Stochastic decomposition in the M/G/1 queue with generalized vacations", Oper. Res. 33, 1117-1129.

20. Gani J. [1957], " Problems in the probability theory of storage systems", J.Roy.Statist.Soc.B, 19, 181-206.

21. Ghare P.M. [1968], " Multichannel queueing system with bulk service" , Oper. Res. 16, 189-192.

22. Gross D and Harris C.M. [1971], " On one for one ordering inventory policies with state dependent lead times", Oper. Res. 19, 735-760.

23. Gross D and Harris C.M. [1974], " Fundamentals of Queueing Theory" , Wiley, New York.

24. Gross D, Harris C.M. and Lechner J.A. [1971], "Stochastic inventory models with bulk demand and state dependent lead times", J.Appl.Prob. 8, 521-534.

25. Hadley G and Whitin T.M. [1963], " Analysis of Inventory Systems", Prentice Hall Inc., New Jersey.

26. Jaiswal N.K. [1960], " Bulk service queueing problem" , Oper.Res. 8, 139-143.

27. Kalpakam S and Arivarignan G. [1985], " Analysis of an exhibiting inventory system", Stoch. Anal. Appl., 3, 447-465.

28. Keilson J. [1962], " The general bulk queue as a Hilbert problem" , J.Roy.Statist.Soc. B, 24, 344-358.

29. Keilson J and Servi L.D. [1986], " Oscillating random walk models for GI/G/1 vacation systems with Bernoulli schedules",J.Appl.Prob. 23, 790-802.

30. Lee T.T. [1984], " M/G/1/N queue with vacation time and exhaustive service discipline" , Oper.Res.32,774-784.

31. Levy Y and Yechiali U. [1975], " Utilization of idle time in an M/G/1 queueing system", Manag.Sci.22, 202-211.

32. Medhi J. [1975], " Waiting time distribution in a Poisson queue with general bulk service rule" , Manag. Sci. 21, 777-782.

33. Medhi J. [1984], " Recent developments in bulk queueing models" , Wiley Eastern Ltd., New Delhi.

34. Miller L.W. [1964], " Alternating priorities in multi-class queues" , Ph.D. dissertation, Cornell University, Ithaca, NY.

35. Moran P.A.P. [1959], " The theory of storage" , Methuen, London.

36. Nadarajan R and Subramanian A. [1984], " A general bulk service queue with server's vacation", Operations Research in Managerial system, Academic Publications, India.

37. Naddor E. [1966], " Inventory Systems" , John Wiley and Sons, New York.

38. Neuts M.F. [1967], " A general class of bulk queues with Poisson input" , Ann.Math.Stat. 38, 759-770.

39. Neuts M.F. [1981], " Matrix-geometric solutions in stochastic models: An algorithmic approach', The John Hopkins University Press, Baltimore.

40. Neuts M.F and Ramalhoto M.F. [1984], " A service model in which the server is required to search for customers" , J. Appl. Prob.21, 157-166.

41. Prabhu N.U. [1965], " Queues and Inventories" , Wiley, New York.

42. Roes P.B.M. [1970], " The finite dam " , J. Appl. Prob.7, 316-326.

43. Ross S.M. [1970], " Applied Probability models with optimization" , Holden-Day, San Francisco.

44. Ryshikov J.I. [1973], " Lagerhalting" , Akademic Verlag, Berlin.

45. Saaty T.L. [1961], " Elements of Queueing theory with applications" Mc-Graw Hill, New York.

46. Sahin I. [1979], " On the stationary analysis of continuous review (s,S) inventory systems with constant lead times" , Oper.Res. 27, 717-729.

47. Sahin I. [1983], " On the continuous review (s,S) inventory model under compound renewal demand and random lead times" , J.Appl.Prob. 20, 213-219.

48. Scholl M and Kleinrock L [1983], " On the M/G/1 queue with rest periods and certain service independent queue disciplines" , Oper.Res.31, 705-719.

49. Shantikumar J.G. [1980], " Some analysis on the control of queues using level crossing of regenerative processes", J.Appl. Prob. 17, 814-821.

50. Silver E and Peterson R. [1984], " Decision systems for Inventory Management and Production Planning" , Wiley, New York.

51. Sivazlian B.D. [1974], " A continuous review (s,S) inventory system with arbitrary interarrival distribution between unit demands", Oper.Res.22, 65-71.

52. Srinivasan S.K. [1979], " General analysis of s-S inventory systems with random lead time and unit demands" , J.Math.Phy.Sci. 13, 107-129.

53. Syski R. [1960], " Introduction to congestion theory in telephone systems", Oliver and Boyd, London.

54. Takacs L. [1962], " Introduction to the theory of Queues" , Oxford University Press, London.

55. Thangaraj V and Ramanarayanan R. [1983], " An operating policy in Inventory systems with random lead times and unit demands" , Math. Operationforch. U. Statist. Ser. Optimization, 14, 111-124.

56. Tijms H. [1972], " Analysis of (s,S) inventory models", Mathematical Centre Tracts 40, Amsterdam.

57. Tijms H. [1986], " Stochastic modelling and analysis" , Wiley, New York.

58. Veinott A.F. [1966], " The status of Mathematical Inventory theory" , Manag. Sci. 12, 745-777.