STOCHASTIC PROCESSES AND THEIR APPLICATIONS

SEMI-MARKOV ANALYSIS OF SOME INVENTORY AND QUEUEING PROBLEMS

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

BY

B. LAKSHMY

DEPARTMENT OF MATHEMATICS AND STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
KOCHI - 682 022
INDIA

FEBRUARY 1991

CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Smt. B. Lakshmy under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

A. KRISHNAMOORTHY
Research Guide
Professor of Applied Mathematics
Department of Mathematics and Statistics
Cochin University of Science and Technology
KOCHI 682 022

February 18, 1991.

DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

Lakshmi

B. LAKSHMY

ACKNOWLEDGEMENTS

I wish to acknowledge first and foremost, the consistent help and invaluable criticism rendered by my guide Professor A.Krishnamoorthy. The assistance provided by Professor T.Thrivikraman, Head of the Department has been no less valuable. The help offered by Vinod Robert and Praseeth Lal of D.R.D.O., Cochin University of Science and Technology with regards to computational work is sincerely acknowledged.

A special note of thanks goes to the teachers, members of the office staff and the colleagues of the Department for their help and co-operation.

The excellent typing done by Mr.Joseph is thankfully acknowledged.

The financial support extended by the National Board for Higher Mathematics, TIFR, Bombay deserves an important mention.

Finally the strong faith, encouragement and the blessings of my family members, especially my late father Sri.V.Balakrishnan were with me throughout.

B. LAKSHMY

CONTENTS

Cont'd.

*****

Chapter-1

INTRODUCTION

This thesis analyses certain problems in Inventories and Queues. There are many situations in real-life where we encounter models as described in this thesis. It analyses in depth various models which can be applied to production, storage, telephone traffic, road traffic, economics, business administration, serving of customers, operations of particle counters and others. Certain models described here is not a complete representation of the true situation in all its complexity, but a simplified version amenable to analysis. While discussing the models, we show how a dependence structure can be suitably introduced in some problems of Inventories and Queues. Continuous review, single commodity inventory systems with Markov dependence structure introduced in the demand quantities, replenishment quantities and reordering levels are considered separately. Lead time is assumed to be zero in these models. An inventory model involving random lead time is also considered (Chapter-4). Further finite capacity single server queueing systems with single/bulk arrival, single/bulk services are also discussed. In some models the server is assumed to go on vacation (Chapters 7 and 8). In chapters 5 and 6 a sort of dependence is introduced in the service pattern in some queueing models.

This chapter reviews briefly some of the important developments in Inventories and Queues. It also explains the technical terms and notations used in this thesis. Further a brief outline of the work on which this thesis is based is also given towards the end of this chapter.

1.1. Historical background - Inventory Theory

The study of the quantitative analysis in inventory systems is considered to be originated with the work of Harris (1915) and he obtains a formula for the optimal production lot size given by the square root function of the fixed cost, holding cost and the demand. This formula referred to as the economic order quantity (EOQ) is popularised by Wilson. After World War II several authors have discussed the stochastic behaviour of the inventory in the case of scheduling the use of stored water to minimise the cost of supplying electric energy. Pierre Masse (1946), a French engineer is considered to be the first to achieve a satisfactory result regarding this problem. Arrow, Harris and Marschak(1951) have showed that the total expected cost incurred from use of an (s,S) policy satisfies a renewal equation. Further Dvoretzky, Kiefer and Wolfowitz (1952) have given some sufficient conditions for establishing that the optimal policy is an (s,S) policy for the single-stage inventory problem. A detailed account of the developments that have taken place till 1952 is given by Whitin (1953). Bellman, Glicksberg and Gross(1955) determine the optimal policy for the case in which the ordering and penalty cost are both linear. Gani (1957) studies some problems arising in the stochastic theory of storage systems.

A systematic account of (s,S) inventory type is first provided by Arrow, Karlin and Scarf (1958). Their approach is based on renewal theory. It is natural to enquire how these models could be applied in practical situations. Hadley and Whittin (1963) provides an excellent account of the applications. A lucid survey of this field through 1962 is given by Scarf (1963). A complete computational approach for finding optimal (s,S) inventory policies is developed by Veinott and Wagner (1965). There is an excellent review by Veinott (1966) which summarizes the status of mathematical inventory theory. He focusses his attention on the determination of optimal policies of multi-item and/or multi-echelon inventory systems with certain and uncertain demands. Hurter and Kamisky (1967) find the limiting distribution of the number of units in the storage for a basic single commodity storage system by applying the theory of regenerative stochastic processes. The cost analysis of different inventory systems is given in Naddor (1966). Kaplan (1970) and Gross and Harris (1971) also make distinct contributions in these directions. Inventory systems with random lead time is discussed by

Ryshikov (1973) in his monograph.

Sivazlian (1974) considers the case of a continuous review inventory system with unit demand, zero lead time and arbitrary interarrival times of demands. He obtains the transient and steady state distribution for the position inventory and shows that the limiting distribution of the position inventory is uniform and is independent of the inter-arrival time distribution. Richards (1975) proves the same result for the case with random demand size. Srinivasan(1979) extends the result of Sahin (1974) to the case in which lead times are i.i.d random variables following a general distribution. This is further extended by Manoharan, Krishnamoorthy and Madhusoodanan (1987) to accommodate the case of non-identically distributed interarrival times.

An (s,S) inventory system with demand for items dependent on an external environment is studied by Feldman (1978). Constant lead time (S,s) inventory policy with demand quantities forming non-negative real valued random variables is anslysed by Sahin (1979). Ramaswami (1981) obtains algorithms for an (s,S) model where the demand is according to a versatile Markovian point process. Further, Sahin (1983) obtains the binomial moments of the time dependent and limiting distributions of the deficit in the case of a continuous review (s,S) policy with random lead time and demand process following a compound renewal process. Single product inventory systems relating to production process is seen in the works of De kok, Tijms and Van der Duynschouten (1984).

Thangaraj and Ramanarayanan (1983) discuss an inventory system with random lead time and having two reordering levels. Again Ramanarayanan and Jacob (1986) consider the same problem with varying reordering levels; but in their model passage to the limit is rather difficult. Also inventory system with varying reordering levels and random lead time is discussed by Krishnamoorthy and Manoharan (1991). They obtain the time dependent probability distribution of the inventory level and the correlation between the number of demands during a lead time and the length of the next inventory dry period.

A review of the work done in perishable inventory until 1982 can be had from Nahmias (1982). Kalpakam and Arivarignan (1985) consider the case of an inventory system with arbitrary interarrival time between demands in which one item is put into operation as an exhibiting item (they have assumed that an exhibiting item has exponentially distributed life time) and obtain the transient and steady state distributions for position inventory. Again the same system having one exhibiting item subject to random failures with failure times following exponential distribution and unit demand is dealt by the same authors (1985) and the expression for the limiting distribution of the position inventory is derived by applying the techniques of semi-regenerative process. Manoharan and Krishnamoorthy(1989) consider an inventory problem with all items subject to decay and derive the limiting probability distribution. In this the quantities demanded by arrivals are i.i.d.r.vs and interarrival times have a general distribution.

Ramanarayanan and Jacob (1987) analyses an inventory system with random lead time and bulk demands. They use the matrix of transition time densities and its convolutions to arrive at the expression for the probability distribution of the inventory level. Inventory system with random lead times and server going on vacations when the inventory becomes dry is introduced by Daniel and Ramanarayanan (1987, 1988). Jacob (1988) deals with bulk demand inventory models and server vacation. Further Krishnamoorthy and Manoharan (1990) investigate an inventory problem in which the quantities demanded by successive arrivals are assumed to follow distributions depending on the availability of the items. They obtain the limiting distribution of the inventory level. A stochastic inventory system with Poisson demand and exponentially distributed delivery time is discussed by Beckmann and Srinivasan (1987).

1.2. A brief account of the Inventory theory

An Inventory is a measured stock of some goods which is held or stored for the purpose of future sale or production. So it varies in quantity over time in response to a'demand' process which operates to diminish the stock and a 'replenishment' process which operates to increase it. The obvious applications to stocks of physical goods are light bulbs, raw materials to be used in some production process etc. whereas the number of engineers employed by a company, the number of students enrolled in a college or the amount of equity capital available for corporate growth are all regarded as inventory. When production is involved, the inventory problem might require, for example, determining how much wheat to plant per year or how much gasoline of certain variety to have blended. The amount of water to be released from a dam for electricity and irrigation purposes is also an inventory problem. Again inventory problems may involve scheduling, production, determining efficient distribution of commodities in certain markets, finding proper replacement policies for old equipment, determining proper prices for goods produced, or combinations of these elements.

Demand

Inventories are held for the ultimate purpose of satisfying demands. Usually the demand is not subject to control, but the timing and magnitude of the replenishments may be regulated.

Various models of natural attrition comprise what we call the demand process and the hiring or recruitment constitute the replenishment process. Inventory theory is concerned with the analysis of several types of decisions relating primarily to the problem of when to buy and how much to buy of a given item. The analysis involves consideration of when the item should be manufactured and problems of transportation and distribution of stock etc.

Motivation for Inventory

(a) Inventories are frequently held because of economies of scale in production or procurement. If the average cost of purchasing stock decreases when larger quantities are purchased, then it is economical to purchase in relatively large quantities. The result is the accumulation of stock prior to actual need.

(b) The requirements for items may vary substantially over time and this itself may serve as an incentive for holding stock. It is advantageous to procure the item before it is needed at a lower marginal cost, thus contributing to the formation of inventories. This motive for holding inventories will be reinforced if the cost function displays decreasing average cost.

(c) Another motive for holding stock is that the costs may themselves be a function of time.

(d) Uncertainty of future requirements is also a strong motive for holding inventories.

Inventory policies and objective function

In an inventory problem that lasts for some length of time, cost will generally be incurred at various moments of time. The main costs involved are: (i) the ordering cost which is composed of a cost proportional to the amount ordered plus a set up cost which is constant when the amount z ordered is positive and zero for z=0, (ii) storage cost or holding cost which is incurred by the actual maintenance of stocks or the rent of storage space or a measure of obsolescence or spoilage. The cost of repairing a defective item is also considered as a storage cost, (iii) penalty cost or shortage cost which arises when supply including both current output and accumulated stocks from the past, exceeds demand. If a demand occurs beyond the available inventory, it is met by a priority shipment or it is backlogged and satisfied when the commodity becomes available. These costs involved in the inventory are to be summarised to a single number so that alternative policies can be compared. An inventory policy is a set of rules that defines when and how much quantity to be ordered.

When any inventory model is investigated first we analyse the model to get the inventory equation which represents the inventory level at any instant of time. The purpose of obtaining the inventory equation is to determine the optimal policy. A policy is called optimal if it maximises the objective function when the objective function is a profit function or minimises the total expected cost per unit time if the objective function is a cost function. Several policies can be used to control an inventory system; but if it is known before hand that the policy has a particular form, then the time to compute optimal policies can be cut substantially. The most widely used policy is the (s,S) policy where the variables s and S are the two decision variables. The variable s is referred to as the reorder level while the variables s and S together stand for how much quantity to be ordered. Whenever the inventory position is equal to or less than s for the first time after a replenishment, a procurement or replenishment is made to bring the inventory to its maximum capacity S.

An inventory system can be either a continuous review or a periodic review system. In a continuous review policy the inventory position is monitored continuously over time whereas this is done at specified points of time in a periodic review system. We concentrate only on continuous review single commodity inventory systems.

An important element in the mechanism of inventory process is the lag in delivery of the commodity after an order is placed or decision is made to produce. This time lag is called the lead time. If replenishments take place instantaneously we say lead time is zero so that the possibility of penalty cost may not occur. In some cases lead time is fixed whereas in others it is a random variable with known distribution. The time interval for which the inventory is empty is termed as a dry period.

1.3. Queueing Theory

Queueing theory had its origin in the pioneering work done by Erlang (1909) on the application of probability theory to telephone traffic problems. It soon drew attention of many probabilists. We can have a queue of broken-down machines waiting for repair at a repair shop, a queue of customers at a store cash counter or a queue formed by planes circling above an air port waiting to land. These provide obvious examples of queues. Often we have cases where a physical queue is absent, such as the waiting list of passengers for a railway or air line ticket or of persons who register their names for the purchase of a car which is not readily available and is to be supplied from future production. So queueing is a mechanism that is used to handle congestion.

A system consisting of a service facility, a process of arrival of customers who wish to be served by the facility, and the process of service is called a queueing system. A queue or waiting line develops whenever the service facility cannot cope with the number of units requiring service. The units arriving for service are called customers in a generic sense. Thus a queueing system is regarded as an arrangement where the customers requiring service form the input, the serviced customers the output and the service rendered the transformation process.

Historical review

Since the work of Erlang (1909) with telephone engineering, applications have expanded into several areas. Interesting and fruitful interactions between theoretical structures and practical applicationshave led to the rapid development of the subject in areas like production planning, inventory control and maintenance problem. For about two decades various researchers and practitioners have looked at models either to solve particular problems at hand or to develop understanding of the stochastic processes that arise from them.

In any analysis of a queueing system one or more aspects like the queue length, the waiting time and the busy period are studied through their probability distributions, from which moments like mean, variance etc. can be obtained. For an ordinary M/M/1 queueing system, the system size probabilities are obtained by solving difference-differential equations. But for most of practical applications of the queueing model, a steady state or a state of statistical equilibrium solution is necessary. The time dependent or transient solution is first given by Bailey (1954 b) making use of generating function whereas Ledermann and Reuter (1956) obtain the solution with spectral theory. While Champernowne (1956) uses combinatorial method,Conolly (1958) uses difference equation techniques for the time dependent solution to an M/M/1 system. Pegden and Rosenshine (1982) also deal with the transient solution of M/M/1 queues. Parthasarathy (1987) provides a very simple and elegant approach to obtain the time-dependent solution to the M/M/1 queue. Again Parthasarathy and Sharafali (1989) extends this to the M/M/s queue. Syski (1988) shows that the result of Parathasarathy (1987) is equivalent to that obtained by Cohen (1982).

For an M/M/1 queueing system we do not have to take into account the time since the last arrival or the elapsed service time of the unit in service because the negative exponential distribution possesses the Markovian or the forgetfulness property and so the queue length process is Markov.

Several methods are available for the analysis of non-Markovian processes. These include:

(i) the use of regeneration points ie. of an embedded Markov chain. The behaviour is considered at a discrete set of time instants chosen in such a way that the resulting process is Markovian.

(ii) Erlang's method, in which life (service time) is divided into fictitious stages such that the time spent in each stage follows an exponential distribution

(iii) supplementary variable technique, whereby the inclusion of sufficient supplementary variables such as expended life-time, in the specification of the state of the system to make the whole process Markovian in continuous time.

The system size process, at arbitrary time points, in M/G/1 and GI/M/1 queueing systems are in general a non-Markovian processes. For an M/G/1 queue the successive departure instants constitute regeneration points whereas for a GI/M/1 queue the successive arrival epochs are the regeneration points. Thus a Markov chain is embedded at these regeneration points. Kendall (1951, 1953) makes use of this method. For an M/M/1 system, all time points are regeneration points so that the whole process in continuous time is Markovian. Cox (1955) uses the supplementary variable technique to analyse non-Markovian stochastic processes. The method of supplementary variable investigated by Cox (1955) is found in the thesis of L. Kosten in 1942. Lavenberg (1975) derives an expression for the Laplace-Stieltjes transform of steady-state distribution of the M/G/1 queueing systems.

Queueing systems with server vacation arise in many computer, communication, production and other stochastic systems. Welsch (1964) characterises the transient and asymptotic distributions of the queue size, waiting time and waiting-plus service time of an M/G/1 queue in which he assumes that the first customer arriving when the server is idle has a distribution different from that when the server is busy. Miller (1964), Avi-Itzhak, Maxwell and Miller (1965), Cooper (1970, 1981), Levy and Yechiali (1975), Heymann (1977), Shantikumar (1980, 1982), Scholl and Kleinrock (1983), Ali and Neuts (1984), Doshi (1985) all deal with vacation models. An extensive survey of the queueing system with vacation to the server is given by Doshi (1986). Daniel (1985) studies some queueing models with vacation to the server where the server takes rest either after serving a certain fixed number of customers or whenever the system becomes empty, whichever occurs first. A finite capacity M/G/1 queue with server vacation is considered by Lee (1984) where the vacation is initiated if either the queue is empty or M customers have been served during a busy period. Manoharan and Krishnamoorthy (1989) also consider a model similar to Lee (1984) and obtain the time dependent queue size distribution and virtual waiting time distribution. Ramachandran Nair (1988) analyses extensively queues with vacation to the server after serving a random number of units. Jacob (1988) and Madhusoodanan (1989) deal extensively with several queueing models with server vacation and derive their time dependent behaviour.

Over the past two decades steady progress has been made towards solving increasingly difficult and realistic queueing models. Lack of results suited for ready practical implementation is observed in several areas in queueing theory. One such class of models is distinguished by the presence of a specified feature, namely, that customers arrive in groups of random size and are served in groups that are themselves of random size. Queueing models belonging to the above category are termed "bulk queues" in literature.

Bailey (1954a) is the first to carry out the mathematical investigation of queues involving batch service. He studies the stationary behaviour of the system in terms of probability generating function. This is followed by a series of papers with group arrival and/or batch service. Gaver (1959) seems to be the first to handle queues involving group arrivals. He is followed by Jaiswal (1960, 1962). Saaty (1961) provides an excellent account of some of these works. Miller (1959) is the first to examine a queueing system in which customers arrive in groups and are served in groups. He obtains the stationary distribution of the number of units in the system making use of embedded Markov chain method. Bhat (1964) studies the equilibrium behaviour of the $M^X/G^Y/1$ and the $GI^X/M^Y/1$ systems using fluctuation theory. Again bulk service queue with infinite waiting room is investigated by Bhat (1967) to obtain the busy period and the busy cycle distribution of the queue length process. Further Teghum, Loris-Teghum and Lambotte(1969) also deal with bulk arrival, bulk service queueing model. Chaudhary and Templeton (1981) obtain the limiting behaviour of an $M/G^B/1$ queueing system. The books by Chaudhary and Templeton (1984) and Medhi (1984) give a detailed account of the work done in bulk queues. Jacob (1988) and Madhusoodanan (1989) also deal extensively with several bulk service queueing models. Morse (1955), Takacs (1961, 1962), Cohen (1969), Prabhu (1965, 1980), Gnedenko and Kovalenko(1968), Cooper (1972), Gross and Harris (1974, 1984), Bagchi and Templeton (1972) and Asmussen (1987) analyse in depth several queueing problems.

Liu, Kashyap and Templeton (1987) deal with an infinite server queueing system providing both individual service and batch service and obtain the transient results for the first two moments of the system size distribution. Waiting time distribution and steady state results are also computed by them.

Another important feature of a bulk queue is that the system follows a general bulk service rule with range (a,b) and with or without vacation. In 1942 Kosten discussed a deterministic service time system with capacity range (a,∞). Further Neuts (1967), Borthakur (1971 a,b), Medhi (1975, 1984), Holman et. al. (1981), Kambo and Chaudhary (1982), Easton and Chaudhary (1982), Chaudhary and Templeton (1981) all consider bulk service queueing system with range (a,b). Fabens (1961, 1963) studies the transient state of the system by identifying the underlying semi-Markov process. Most of these works require the application of Rouche's theorem. Neuts (1979) develops an algorithmic method for the solution of $M/G^{a,b}/1$ system. His approach involves only real arithmetic and avoids the calculation of the complex roots based on Rouche's theorem. Cohen(1982) seems to be the only author to have developed waiting time results for bulk arrival and bulk service queue where the server becomes idle when the system is empty. His results are given in terms of integrals. Most of the above mentioned works concentrate on the steady state behaviour of the system. Jacob, Krishnamoorthy and Madhusoodanan (1988) obtain the time dependent solution to $M/G^{a,b}/1$ queue with finite capacity and the same model with server vacation is analysed by Jacob and Madhusoodanan (1987). Manoharan (1990) extends their result to $E_k/G^{a,b}/1$ queue with server vacation. Transient solution and virtual waiting time distribution are discussed by him. He also considers a queueing situation where the service is carried out either singly or in batches depending upon the number of customers waiting for service in the waiting room. Steady state behaviour of the system is examined by him.

Another notable feature in the queueing system is the state dependence of the service characteristics. Hiller et.al. (1964), Gupta (1967) and Rosenshine (1967) examine queueing systems in which the service rates are an instantaneous functions of the system state. Harris (1967) considers the standard M/G/1 system in which the service time parameter is a random variable dependent upon the state of the system at the moment the customer's service is begun. Murari (1969) and Harris (1970) discuss bulk arrival queue with state dependent service rate. Ponser (1973) investigates a queueing model in which the service time of a customer depends upon his waiting time in the queue and at the same time independent of all other parameters associated with the system size. Shantikumar (1979) discusses a class of queueing models in which the service time of a customer at a single server facility is dependent on the queue size at the onset of its service. He extends Harris's two state, state dependent service to M/G/1 queue.

1.4. Relation between Queues and Inventories

Applications of and fruitful connections between queueing theory and inventory theory occur numerously. Steady progress has been made to solve problems which are difficult but realistic in inventory and queues. Similarities between the mathematical formalisms of both models have been observed from early times.

The amount of goods or material held in stock for future purpose can be identified as a group of customers waiting for some sort of service at a service facility. The arrival of an order or a demand for an item is likened to a service completion since such an arrival or demand results in the departure of a customer in the queue which corresponds to the depletion of the inventory level. The demand for an item to an inventory arrives singly or in batch of fixed or variable size. The bulk demand corresponds to the bulk arrival in queueing theory and single demand that of single arrival. The interarrival times of demands are regarded as the service time.

A better correspondence between an inventory system and a queueing system is seen by regarding the demands occuring to the inventory system as the arrival of customers to the queue because both of these are more or less uncontrollable. The inventory replenishment time or leadtime can be compared to the service time of the queueing system and both of these are, in general controllable by the management of the system.

1.5. Renewal process

Let $\{X_n, n=1,2,\ldots\}$ be a sequence of non-negative independent and identically distributed random variables with $X_1, X_2, X_3, \ldots$ representing the times between successive occurrences of a fixed phenomenon. Then $S_o=0$; $S_{n+1}=S_n+X_{n+1}$, $n=0,1,2,\ldots$ define the times of occurrence of 1st, 2nd,... events, assuming that the time origin is taken to be an instant of such an occurrence. Then S_n's are called renewal times.

Let F(.) denote the distribution of the interrenewal times. Assume that $\Pr\{X_o=0\} < 1$. Since X_n's are non-negative $E(X_n)$ exists.

Define N(t) as $\text{Sup}\{n \mid S_n \leqslant t\}$. Then the process $\{N(t), t \geqslant 0\}$ is called a renewal process or a counting process. Obviously the state space of the renewal process consists of a single element. The random variable $N(t)$ gives the number of renewals in the interval $(o,t]$. The distribution of S_n is given by $Pr\{S_n \leqslant x\} = F_n(x)$, where $F_n(x) = F^{*n}(x)$, (since X_i's are i.i.d random variables) and $F^{*n}(.)$ denotes the n-fold convolution of $F(.)$ with itself. $(F^{*o}(.) \equiv 1)$.

It is easily verified that

$$N(t) \geqslant n \Longleftrightarrow S_n \leqslant t$$

so that the distribution of $N(t)$ is

$$Pr\{N(t) = n\} = F^{*n}(t) - F^{*(n+1)}(t).$$

Using this distribution, the expected number of renewals in $(o,t]$ denoted by $M(t)$ is given by

$$M(t) = E[N(t)] = \sum_{n=1}^{\infty} F^{*n}(t)$$

$M(t)$ is called the renewal function.

Consider a stochastic process $Z = \{Z(t), t \geqslant 0\}$ with state space E. Assume that every time a certain event occurs, the future of the process Z after that time is a probabilistic replica of the future after time 0. Such times are called regeneration times of Z and the process Z is said to be a regenerative process. If $T_1, T_2, T_3, \ldots$ constitute a sequence of regeneration points, then $\{T_n, n=1,2,\ldots\}$ forms a renewal process and the time between successive renewal points is called a cycle of the process. Cox and Smith (1961), Cox (1962), Feller (1965), Ross (1975), Cinlar (1975 b), Bhat (1984) give a detailed account of renewal theory.

1.6. Semi-Markov and Markov renewal process

Consider a stochastic process which moves from one state to another of a countable number of states in such a way that the successive states visited forms a Markov chain. Assume that the process remains in a given state for a random length of time whose distribution depends upon the state being visited and the one to be visited next. Such a process is defined as a semi-Markov process since it is a Markov chain with the time scale being randomly selected. Thus a semi-Markov process identifies or gives the state of the process at each time point. For the same stochastic process, let $N_i(t)$ denotes the number of transitions or renewals into the state i (E be the state space of the Markov chain) which occur in $(o,t]$. Set

$$N(t) = ((N_1(t), N_2(t), \ldots)$$

Then the stochastic process $\{N(t), t \geqslant 0\}$ is a Markov renewal process. Thus a Markov renewal process is a counting process which records at each time point t the number of times each of the possible states have been visited. Such a process becomes a Markov process if the sojourn times are all exponentially distributed independent of the next state to be visited; it reduces to a Markov chain if sojourn times are all equal to one, and becomes a renewal process if there is only one state. This means that a stochastic process $\{(X,T)\}=\{(X_n,T_n),\ n\in N\}$ defined over a finite set E is a Markov renewal process if

$$\Pr\{(X_{n+1}=j;\ T_{n+1}-T_n\leqslant t \mid X_o,X_1,\ldots,X_n;T_o,T_1,\ldots,T_n\}$$

$$=\Pr\{X_{n+1}=j;\ T_{n+1}-T_n\leqslant t \mid X_n\} \text{ for all } n\in N \text{ and } i,j\in E \text{ and } t\geqslant 0 \qquad (1)$$

Denote the R.H.S. of (1) by $Q(i,j,t)$, if $X_n=i$.

Clearly

$$Q(i,j,t)\geqslant 0;\quad i,j\in E;\ t\geqslant 0$$

$$\sum_{j\in E} Q(i,j,\infty) = 1$$

The family of probabilities

$\mathbb{Q}=\{Q(i,j,t),\ i,j\in E;\ t\geqslant 0\}$ is called a semi-Markov kernel.

For this Markov renewal process, the expected number of returns to state j in an amount of time t given that the system has started from state i is the Markov renewal function $R(i,j,t)$

which is given by

$$R(i,j,t) = \sum_{n=o}^{\infty} Q^{*n}(i,j,t) \quad \text{where}$$

$$Q^{*(n+1)}(i,j,t) = \sum_{k \in E} \int_{o}^{t} Q(i,k,du)\ Q^{*n}(k,j,t-u) \quad \text{for } n \geqslant 0$$

and

$$Q^{o}(i,j,t) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Define the process $Y = \{Y(t),\ t \geqslant 0\}$ with state space E by $Y(t) = X_n$ for $T_n \leqslant t < T_{n+1}$. Then the process $\{Y(t),\ t \geqslant 0\}$ is called the semi-Markov process defined over the state space E with the semi-Markov transition kernel $\mathbb{Q} = \{Q(i,j,t)\}$. Thus the semi-Markov process Y provides a picture which is convenient in describing the Markov renewal process underlying it.

<u>Markov renewal equations</u>

Let (X,T) be a Markov renewal process defined over a finite state space E with the semi-Markov kernel Q(i,j,t) and Markov renewal function R(i,j,t), $i,j \in E$, $t \geqslant 0$. Let R_+ and R denote the set of non-negative real numbers and real numbers respectively. Assume that f is a function defined by $f: E \times R_+ \longrightarrow R$ such that for every $i \in E$, the mapping $t \longrightarrow f(i,t)$ is Borel measurable and bounded over finite intervals. Let $\mathcal{F}$ be the class of functions f. Then a function $f \in \mathcal{F}$ is said to satisfy the Markov renewal equation if f(i,t) can be written as

$$f(i,t) = g(i,t) + \sum_{j \in E} \int_0^t Q(i,j,du)\ f(j,t-u),$$
$$i \in E,\ t \in R_+ \qquad (2)$$

for some function $g \in \mathcal{F}$. Here $Q(i,j,t)$ and $g(i,t)$ are known and so the problem is to solve for $f(i,t)$. Further the Markov renewal function (2) has one and only one solution given by

$$f(i,t) = \sum_{j \in E} \int_0^t R(i,j,du)\ g(j,t-u),\ i \in E,\ t \in R_+$$

Levy (1954) and Smith (1955) independently introduced semi-Markov processes. A detailed description of the Markov renewal process is given in Pyke (1961 a,b). Cinlar (1969, '75 a,b) provide a detailed account of Markov renewal and semi-Markov processes. Inventory and queueing models based on the theory of semi-Markov process is studied by Fabens (1961, '63). Further Schal (1971) analyses M/G/1 and G/M/1 queues and obtains their assymptotic behaviour and rates of convergence. His approach is also based on the theory of semi-Markov process.

1.7 A brief account of the results in this thesis

The aim of the thesis is to study the time-dependent and steady state behaviour of certain problems in Inventories and Queues. This is achieved by identifying the underlying semi-Markov processe and the embedded Markov renewal process of the basic process. It is assumed that the inventory (assumed to be single commodity) is continuously monitored over an infinite horizon period. In the case of some of the problems discussed we have analysed certain control problems associated with them. All queueing problems investigated deal with finite capacity.

Chapter 2 deals with an (s,S) inventory policy where each arrival demands a random number of items, the maximum size being a with $a \leqslant s$. We assume that the successive quantities demanded form a Markov chain. Replenishment is instantaneous and the quantity replenished is such that the inventory is brought back to its maximum capacity S. The probability distribution of the stock level at arbitrary time points and also the steady state inventory level distribution are obtained. The optimal value of the pair (s,S) is computed.

In chapter 3, the dependence structure is introduced in the (s,S) inventory problems in two different ways. Model I discusses a bulk demand inventory policy with the successive quantities replenished forming a Markov chain. Model II studies a unit demand (s,S) policy with the successive reorder levels varying according to a Markov chain. In Model II, the replenishment quantity is always equal to M=S-s. In both Models lead time is assumed to be zero. The inventory level at arbitrary time point and its limiting distribution are computed for both Models. Some control problems associated with the Models are investigated.

Some numerical illustrations are provided at the end of chapters 2 and 3.

Chapter 4 considers a bulk demand inventory problem with zero lead time and the server taking vacation each time the inventory becomes dry after the previous replenishment. The system size probabilities and the reliability of the system at arbitrary time epochs are obtained.

Chapter 5 introduces a class of finite capacity single server queueing models in which the server offers a random number of stages of service to each unit depending upon the system size at the onset of its service. A three dimensional Markov chain with the first coordinate representing the system size, the second one representing the number of stages of service given to the unit undergoing service and the third one denoting the number of stages of service completed by the unit underoing service is identified. The system size probabilities and the limiting distributions are computed. Numerical illustration is also provided.

Chapter 6 generalises the $M/G^{a,b}/1$ queueing system with finite capacity. The services are in batches of sizes between a and b and is such that the size of a batch to be served is determined based on the time taken to serve the previous batch. System size probabilities and steady state analysis are carried out. Distribution of the busy period and the busy cycle are studied. Virtual waiting time distribution is also derived. A control problem associated with the model is discussed.

In chapter 7, we consider two cases of single server queueing systems of finite capacity. Model I discusses a $G/E_k/1$ queueing system whereas Model II investigates a queueing system of general bulk service rule with batch size varying from a to b. Expressions for the time dependent system size probabilities at arbitrary time point for Model I and II, Limiting distribution for Model I and virtual waiting time distribution for Model II are obtained.

Chapter 8 discusses a bulk arrival,bulk service queue of finite capacity b. We assume that a service commences only when the system is full and then only a random number of units are taken for service. On completion of the service of a batch if the system is not full, the server goes for vacation of random duration. System size probabilities are computed. In this model the time duration for which the system remains non-empty continuously is defined as the busy period of the system. Expressions for the distribution of the above defined busy period gives an upper bound for the virtual waiting time. By restricting b=2, the virtual waiting time at time t is computed.

Chapter-2

AN INVENTORY MODEL WITH MARKOV DEPENDENT DEMAND QUANTITIES*

2.1 Introduction

In this chapter we deal with a continuous review (s,S) inventory model in which it is assumed that the quantity demanded by each arrival depends on the quantity demanded by the previous arrival and the maximum quantity demanded is $a \leq s$. Specifically, the quantities demanded by the successive arrivals form a Markov chain. Some work have been done earlier in which the assumption of independence on the quantities demanded is relaxed. Karlin and Fabens (1959), Iglehart and Karlin (1962) consider the case of a discrete-time inventory model where the demands are assumed to arise from a Markov process. They assume that at the beginning of each period the system is in one of N states labelled 1,2,...,N which are observed by the inventory manager before he orders. If the demand process is in state j in a period, a demand distribution, φ_j will be operative in the period. The demand process changes state according to known transition probabilities with the transition from period to period governed by a Markov process.

---

* Appeared in Cahiers du C.E.R.O., Vol.33, 1991.

When the demands at each arrival epochs are dependent the structure of the (s,S) optimal policy is not changed; but the main difference is that the choice of the quantity replenished at an order placing epoch will depend upon the demands in the cycle just completed. So the demand process changes the state of the inventory level according to a set of known transition probabilities with the transition at each demand epoch governed by a Markov chain defined over a state space $\{1,2,\ldots,a\}$.

Section 2.2 deals with the description of the model. The various notations used in the sequel are also explained in that section. Section 2.3 discusses the analysis of the model. Limiting distribution of the system is investigated in Section 2.4. The model discussed here can be suitably applied in situations like bonus demands in major companies on recurring basis. The aim of the management is to minimise the total cost by distribution of optimum amount to the satisfaction of both the employees and the employer. An optimisation problem associated with the model is discussed in Section 2.5. A numerical illustration is done in the last section.

2.2. Description of the model

An (s,S) inventory model with the maximum capacity of the warehouse being fixed as S units with zero lead time is is considered. It is assumed that each arrival demands a random number (integer valued) of items; but the maximum quantity that can be demanded is restricted to a with $a \leqslant s$. The basic assumption of our model is that the quantity demanded by each arrival depends on the quantity demanded by the previous arrival so that the quantities demanded by the successive arrivals form a Markov chain defined over the state space $\{1,2,\dots,a\}$. The interarrival times of demands are independent and identically distributed random variables following distribution function $G(.)$ and probability density function $g(.)$ with mean μ (assumed finite). Replenishment is assumed to be instantaneous and such that whenever the inventory drops to s or below for the first time after each replenishment an order is placed to bring the stock level back to S. To avoid perpetual shortage it is assumed that $S > 2s$. The following notations are used in the sequel:

$I(t)$ - On-hand inventory level at time t.

$*$ denotes convolution. For example $(F*G)(t) = \int_{-\infty}^{\infty} F(t)dG(t-u)$

$g^{*k}(.)$ - k-fold convolution of $g(.)$ with itself.

E denotes the set $\{1,2,\dots,a\}$

$E_1 = \{s+1,\ 2+2,\dots,\ S-1,\ S\}$

$N^o = \{0,1,2,\dots\}$

$P_i(n,t)$ = Probability that $I(t)=n$ given that the initial reordering level is i.

$[x]$ denotes the largest integer less than or equal to x.

$$\delta_{[i]} = \begin{cases} 0 \text{ if } i \text{ is a positive integer} \\ 1 \text{ otherwise} \end{cases}$$

$$\sigma_{\{[\frac{S-n}{a}],0\}} = \begin{cases} 1 \text{ if } [\frac{S-n}{a}] = 0 \\ 0 \text{ if } [\frac{S-n}{a}] > 0 \end{cases}$$

2.3 Analysis of the model

Let $0 = T_o < T_1 < T_2 < \ldots$ be the successive demand epochs and $X_o, X_1, X_2, \ldots$ be the quantities demanded by the successive arrivals at these epochs. Then by our assumption $\{X_n, n \in N^o\}$ constitutes a Markov chain defined on the state space E with the initial probability

$$p_i = Pr(X_o=i),\ i \in E.$$

Let us assume without loss of generality that $p_i = 1$ and $p_j = 0$ for $j \neq i$, $j \in E$.

We assume that the Markov chain $\{X_n, n \in N^o\}$ to be irreducible and aperiodic with the one-step transition probability matrix

$$\mathbb{P} = ((\, p_{i,j}\,)),\ i,j \in E \quad \text{where}$$

$$p_{i,j} = \Pr\left\{ X_{n+1}=j \mid X_n=i \right\}$$

Let $Y_o, Y_1, Y_2, \ldots$ be the stock levels just after meeting the demands at $T_o, T_1, T_2, \ldots$. Then

$$Y_n = \begin{cases} Y_{n-1}-X_n & \text{if } Y_{n-1}-X_n > s \\ S & \text{if } Y_n-X_n \leqslant s \end{cases}$$

From the description of X_n and Y_n, $n = 0,1,2,\ldots$ we easily see that the two dimensional stochastic process $\left\{(X_n,Y_n), n \in N^o\right\}$ constitutes a Markov chain defined over the state space $E \times E_1$. The corresponding one-step transition probabilities associated with the Markov chain $\left\{(X_n,Y_n),\ n \in N^o\right\}$ can be generated from the given one-step transition probabilities associated with the demand process.

<u>Theorem 1</u>

The stochastic process $\left\{(X,Y),T\right\} = \left\{(X_n,Y_n),T_n;\ n \in N^o\right\}$ is a Markov renewal process defined over the state space $E \times E_1$ with the corresponding semi-Markov kernel given by

$$\left\{ Q\left\{(i,I),\ (j,J),t\right\},\ i,j \in E;\ I,J \in E_1,\ t \geqslant 0 \right\}$$

where

$$Q\{(i,I),(j,J),t\} = Pr\{(X_{n+1}=j,\ Y_{n+1}=J);\ T_{n+1}-T_n \leqslant t \mid (X_n=i,\ Y_n=I)\}$$

$$= \int_0^t p_{i,j}\, g(u)du$$

$$= p_{i,j}\, G(t)$$

<u>Proof</u>:

The interarrival times of demands are positive, independent and identically distributed random variables. Hence the demand epochs constitute a renewal process. By our basic assumption that the successive quantities demanded forms a Markov chain, the demand magnitude at T_{n+1} depends only on the demand magnitude at T_n and not on T_r, $r = 0,1,2,\ldots,n-1$. Further the demand magnitudes are independent of the stock levels. Hence considering two successive demand epochs say T_n and T_{n+1}

$$Pr\{(X_{n+1}=j,\ Y_{n+1}=J);\ T_{n+1}-T_n \leqslant t \mid (X_o,Y_o),(X_1,Y_1),\ldots,(X_n=i,\ Y_n=I);\ T_o,T_1,\ldots,T_n\}$$

$$= Pr\{(X_{n+1}=j,\ Y_{n+1}=J);\ T_{n+1}-T_n \leqslant t \mid (X_n=i, Y_n= I)\}$$

Since $T_{n+1}-T_n$, $n=0,1,2,\ldots$, are i.i.d random variables with probability density function $g(.)$ and $\{X_n,\ n \in N^o\}$ is a Markov chain which is independent of $\{Y_n,\ n \in N^o\}$,

$$Pr\{(X_{n+1}=j,\ Y_{n+1}=J);\ T_{n+1}-T_n \leqslant t \mid (X_n=i,\ Y_n=I)\}$$

$$= \int_o^t p_{i,j}\ g(u)\ du$$

$$= p_{i,j}\ G(t)$$

$$= Q\{(i,I),\ (j,J),t\}$$ which proves the theorem.

As soon as the stock level falls to an element in $\{s-a+1,\ s-a+2,\ldots,\ s-1,s\}$ for the first time after each replenishment, next order for replenishment is placed so as to bring the inventory level back to S. Initially due to a demand of magnitude $i(i \in E)$ we assume that the inventory level falls to s or below so that $X_o = i$ and $Y_o = S$. Looking at the successive time epochs $0 = T_o^{(1)},\ T_1^{(1)},\ T_2^{(1)},\ \ldots$ at which the inventory level is brought to S, let $F\{(i,S),\ (j,S),t\}$ denote the probability that two consecutive replenishments take place in an amount of time $\leqslant t$ such that the initial demand is for a quantity i and the next demand that leads to replenishment is for a quantity j; $i,j \in E$. Then

$$F\{(i,S),(j,S),t\} = \sum_{n=[\frac{S-s}{a}]+\delta_{[\frac{S-s}{a}]}}^{S-s} Q^{*n}\{(i,S),(j,S),t\}; \quad i,j \in E;\ t \geqslant 0$$

Now define the function $R\{(.),(.),t\}$ by

$$R\{(i,S),(j,S),t\} = \sum_{m=o}^{\infty} F^{*m}\{(i,S),(j,S),t\}$$

with

$$F^{o}\{(i,S),(j,S),t\} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}, \quad i,j \in E,\ t \geqslant 0$$

$F^{*m}\{(i,S),(j,S),t\}$ is obtained from the recursive relation

$$F^{*(m+1)}\{(i,S),(j,S),t\} = \sum_{k\in E} \int_{o}^{t} F\{(i,S),(k,S),du\} F^{*m}\{(k,S),(j,S),t-u\}, \quad i,j \in E;\ t \geqslant 0.$$

Since $I(t)$ denotes the onhand inventory level at time t, $I(t) = Y_n$ for $T_n \leqslant t < T_{n+1}$, so the process $\{I(t), t \geqslant 0\}$ is a semi-Markov process defined on the state space E_1.

Defining $P_i(n,t)$ as $Pr\{I(t)=n|X_o=i\}$ with $n \in E_1$ and $i \in E$ we see that $P_i(n,t)$ satisfies the Markov renewal equations (Cinlar 1975a). Thus

(i) for $n = S$,

$$P_i(S,t) = \Pr\{I(t)=S;\ T_1>t|X_o=i\} + \Pr\{I(t)=S; T_1\leq t|X_o=i\}$$

$$= K_i(S,t) + \sum_{j\in E} \int_o^t F\{(i,S),(j,S),du\}\, P_j(S,t-u)$$

where

$$K_i(S,t) = \Pr\{I(t)=S;\ T_1>t|X_o=i\} = 1-G(t),$$ and

(ii) for $n = s+1,\ s+2, \ldots,\ S-1$

$$P_i(n,t) = K_i^{(1)}(n,t) + \sum_{j\in E} \int_o^t Q\{(i,S),(j,S-j),du\} P_j(n,t-u)$$

where

$$K_i^{(1)}(n,t) = \Pr\{I(t)=n;\ T_1^{(1)} > t|X_o = i\}$$

$$= \int_o^t \sum_{m=[\frac{S-n}{a}]+\sigma\{[\frac{S-n}{a}],0\}}^{S-n} \sum_{j\in E} Q^{*m}\{(i,S),(j,n),du\}\, [1-G(t-u)]$$

The solutions are given by

$$P_i(S,t) = \sum_{j\in E} \int_o^t R\{(i,S),(j,S),du\}\, K_j(S,t-u)$$

and for $n = s+1,\ s+2,\ \ldots,\ S-1$

$$P_i(n,t) = \sum_{j\in E} \int_o^t R\{(i,S),(j,S),du\}\, K_j^{(1)}(n,t-u).$$

2.4 Limiting distribution

To compute the steady state probabilities it is necessary that at each demand epoch, not only the quantities demanded but also the corresponding inventory levels after meeting the demands are to be known. From the given probabilities governing the demand process, the transition probability matrix $((p^{(1)}_{(i,I),(\ell,L)}))$ corresponding to the two dimensional Markov chain $\{(X_n,Y_n),\ n \in N^o\}$ can be derived where

$$p^{(1)}_{(i,I),(\ell,L)} = \Pr\left\{(X_{n+1}=\ell,\ Y_{n+1}=L) \mid (X_n=i, Y_n=I)\right\}$$

$$i,\ell \in E,\ I,L \in E_1$$

The state space of this Markov chain is $\left\{(i_1,I_1) \mid i_1=1,2,\ldots,a;\ I_1 = s+1,s+2,\ldots,S-1\right\} \cup \left\{(1,S),(2,S),\ldots,(a,S)\right\}$ with $i_1+I_1 \leqslant S$. $((p^{(1)}_{(i,I),(\ell,L)}))$ is computed and is given in Table-1.

We have assumed $\{X_n,\ n \in N^o\}$ to be irreducible and aperiodic and hence is the Markov chain $\{(X_n,Y_n), n \in N^o\}$. Let π be the stationary probability vector of the Markov chain $\{(X_n,Y_n), n \in N^o\}$.

That is $\pi = \{\pi(1,s+1),\ \pi(2,s+1),\ldots\pi(a,s+1),\ \pi(1,s+2),\ \pi(2,s+2),\ \ldots,\ \pi(1,S-1),\ \pi(1,S),\ \pi(2,S),\ \ldots,\ \pi(a,S)\}$

$$\begin{array}{c|cccccccccccc|}
 & (1,s+1) & (2,s+1) & \ldots & (a,s+1) & \ldots(a,s+2)\ldots & (a,S-a) & \ldots & (1,S-1) & (1,S) & (2,S) & \ldots & (a,S) \\
\hline
(1,s+1) & 0 & 0 & & \cdot\cdot & \cdot\cdot & \cdot\cdot & & 0 & p_{11} & p_{12} & \cdots & p_{1a} \\
(2,s+1) & 0 & 0 & & \cdot\cdot & \cdot\cdot & \cdot\cdot & & 0 & p_{21} & p_{22} & \cdots & p_{2a} \\
\vdots & & & & & & & & & & & & \\
(a,s+1) & 0 & 0 & & \cdot\cdot & \cdot\cdot & \cdot\cdot & & 0 & p_{a1} & p_{a2} & \cdots & p_{aa} \\
(1,s+2) & p_{11} & 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & & 0 & 0 & p_{12} & \cdots & p_{1a} \\
(2,s+2) & p_{21} & 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & & 0 & 0 & p_{22} & \cdots & p_{2a} \\
\vdots & & & & & & & & & & & & \\
(a,s+2) & p_{a1} & 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & & 0 & 0 & p_{a2} & \cdots & p_{aa} \\
\vdots & & & & & & & & & & & & \\
(a,S-a) & & & & & & & & & & & & \\
\vdots & & & & & & & & & & & & \\
(1,S-1) & & & & & & & & & & & & \\
(1,S) & 0 & 0 & & \cdot\cdot & \cdot\cdot & p_{1a} & \cdots & p_{11} & 0 & \cdot\cdot & \cdots & 0 \\
(2,S) & 0 & 0 & & \cdot\cdot & \cdot\cdot & p_{2a} & \cdots & p_{21} & 0 & \cdot\cdot & \cdots & 0 \\
\vdots & & & & & & & & & & & & \\
(a,S) & 0 & 0 & & \cdot\cdot & \cdot\cdot & p_{aa} & \cdots & p_{a1} & \cdot\cdot & \cdot\cdot & \cdots & 0
\end{array}$$

Table-1: The Transition Probability matrix $((p_{((i,I),(\ell,L))}))$

These satisfy the relation

$$\pi(\ell,L) = \sum_{i=1}^{a} \sum_{I=s+1}^{S} \pi(i,I)\, p^{(1)}_{((i,I),(\ell,L))}$$

with

$$\sum_{i\in E} \sum_{I\in E_1} \pi(i,I) = 1$$

The uniqueness of π follows from Bhat (1984). For, we have assumed $a \leqslant s$ and so the state space E of the Markov chain $\{X_n,\ n \in N^o\}$ is finite. Hence the state space of the Markov renewal process $\{(X,Y),T\}$ has only a finite number of elements in it. Further $\{(X_n,Y_n),\ n \in N^o\}$ is irreducible and aperiodic since $\{X_n, n \in N^o\}$ is irreducible and aperiodic. Hence the invariant measure π is unique.

Since the interarrival times of demands are i.i.d. random variables, the mean sojourn in any state is equal to the mean of the interarrival time distribution of the demands. So the mean sojourn time in state (ℓ,L), $\ell \in E$, $L \in E_1$ is

$$m(\ell,L) = \int_0^\infty (1-G(t))dt = \mu \quad \text{(assumed finite)}$$

Following Cinlar (1975a) the limiting probabilities are obtained as given below.

(i) for $n = S$,

$$\lim_{t \to \infty} P_i(S,t) = \frac{\sum_{j=1}^{a} \pi(j,S)\, m(j,S)}{\sum_{\ell=1}^{s} \sum_{L=s+1}^{S} \pi(\ell,L)\, m(\ell,L)}$$

$$= \sum_{j=1}^{a} \pi(j,S)$$

(ii) for $n = S-a+1, S-a+2, \ldots, S-2, S-1$,

$$\lim_{t \to \infty} P_i(n,t) = \sum_{j=1}^{S-n} \pi(j,n)$$

(iii) for $n = s+1, s+2, \ldots, S-a-1, S-a$

$$\lim_{t \to \infty} P_i(n,t) = \sum_{j=1}^{a} \pi(j,n)$$

We note from the above that the limiting probabilities are independent of the initial state i, as is expected from the theory of Markov chains. Let $\lim_{t \to \infty} P_i(n,t) = \underline{P}(n)$. The following theorem easily follows from the above discussion.

<u>Theorem 2</u>

If the demand quantities are independent and identically distributed random variables on the set E, then the limiting stationary distribution is discrete uniform.

2.5. Optimisation problem

For any inventory model the decision variables are to be so chosen that the objective function associated with that model attains the minimum value at these values of the decision variables. Here the objective function associated with our model is the total expected cost (for any cycle) per unit time under steady state. The decision variables are s and S for a given fixed value of a.

The expected inventory level E(I) at any instant of time is

$$E(I) = \sum_{n=s+1}^{S} n\, \underline{P}(n)$$

$$= \sum_{j=1}^{a} \left\{ \sum_{n=s+1}^{S-a} n\, \pi(j,n) + S\, \pi(j,S) \right\} + \sum_{j=1}^{S-n} \sum_{n=S-a+1}^{S-1} n\, \pi(j,n)$$

We shall call the time elapsed between two successive demands that result in the replenishment of the inventory as the length of a cycle. Suppose in the steady state the quantity replenished at a demand epoch is M and Z denotes the length of the cycle just completed. The joint density function of M and Z be denoted by $f_j(m,z)$. Then

$$f_j(m,z) = P_j\left\{M=m,\ z \leqslant Z < z+dz\right\}$$

where j is the quantity demanded by the last arrival in the previous cycle.

$$f_j(m,z) = \sum_{k=[\frac{S-m}{a}]+\delta_{[\frac{S-m}{a}]}}^{S-s} \Big\{ P_j[(M=m,\ z \leqslant Z < z+dz) \Big| \text{k arrivals demanded totally } m(>(S-s)) \text{ units of which the first } (k-1) \text{ arrivals demanded less than } (S-s) \text{ units }]$$

$$\times \Pr(\text{k arrivals demanded totally } m(>(S-s)) \text{ units of which the first } (k-1) \text{ arrivals demanded less than } (S-s) \text{ units}) \Big\}$$

$$= \sum_{k=[\frac{S-m}{a}]+\delta_{[\frac{S-m}{a}]}}^{S-s} \ \sum_{\substack{i_1, i_2, \ldots, i_k \\ i_1+i_2+\ldots+i_k=m \\ i_1+i_2+\ldots+i_{k-1}<S-s}} p_{j,i_1} p_{i_1,i_2} \cdots p_{i_{k-1},i_k} \ g^{*k}(z)$$

Hence the expected quantity replenished per unit time is

$$E_j\left(\frac{M}{Z}\right) = \int_0^{\infty} \sum_{m=S-s}^{S-s+a-1} \frac{m}{z} f_j(m,z)dz$$

The probability density function of the duration of a cycle is

$$\sum_{m=S-s}^{S-s+a-1} f_j(m,z)$$

Therefore, the expected length of a cycle is

$$E_j(Z) = \int_0^{\infty} z \sum_{m=S-s}^{S-s+a-1} f_j(m,z)dz$$

Let K be the fixed ordering cost, c be the variable procurement cost and h be the holding cost per unit per unit time. So the total expected cost per unit time is $\bar{F}_j(s,S)$ and is given by

$$\bar{F}_j(s,S) = \frac{K}{E_j(Z)} + cE_j\left(\frac{M}{Z}\right) + h\,E(I)$$

where

$E_j(Z)$, $E_j(\frac{M}{Z})$ and $E(I)$ are as already defined. Thus for given K,c,h, one-step transition probability matrix of the demand process and interarrival time distribution, the optimal value of the pair (s,S) can be computed.

2.6. Numerical illustration

Let the one-step transition probability matrix associated with the demand process be

$$\mathbb{P} = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$

and let the interarrival times of demands follow exponential distribution with mean $\lambda = 0.5$.

The stationary probability vector π is computed and $E(I)$ obtained. Then $E_j(Z)$, $E_j(\frac{M}{Z})$ and $\bar{F}_j(s,s)$ for j=1 and K=10, c=1, h=1 are computed and tabulated as follows.

(s,S)	π	E(I)	$E_1(\frac{M}{Z})$	$E_1(Z)$	$\bar{F}_1(s,S)$
(2,5)	{.02, .41, .08, .17, .32}	3.82	1.81	2.06	10.49
(2,6)	{.12, .11, .04, .26, .15, .08, .24}	4.56	2.04	1.84	12.02
(2,7)	{.09, .05, .09, .11, .04, .13, .15, .06, .28}	5.34	1.23	0.95	17.10
(3,7)	{.106, .1, .043, .233, .143, .071, .304}	5.68	2.86	2.97	11.90
(3,8)	{.05, .16, .11, .06, .02, .25, .08, .08, .19}	6.03	1.11	1.21	19.25
(3,9)	{.06, .09, .05, .12, .07, .07, .03, .16, .10, .05, .20}	6.68	0.76	0.49	27.92

From the above table, we see that corresponding to the pairs (2,5), (2,6) and (2,7), the total expected cost per unit time is minimum for the pair (2,5) and corresponding to (3,7),(3,8) and (3,9), the optimal pair is (3,7) and corresponding to (2,7) and (3,7), the optimal pair is (3,7) for the given set of values of K,c,h and λ .

Chapter-3

SOME INVENTORY MODELS WITH MARKOV DEPENDENCE*

3.1 Introduction

In the previous chapter, the assumption of Markov dependence was made on the quantity demanded by successive arrivals. In this chapter the dependence structure is introduced in the (s,S) inventory models in two different ways. In Model I, the successive quantities replenished are dependent- dependence being on the just previous replenished quantity only, whereas in Model II the reorder levels vary according to a Markov chain. Both models deal with zero lead time. Model I considers the case of bulk demands and Model II that of unit demand.

Ever since the book by Arrow, Karlin and Scarf appeared (1958), many researchers have formulated discrete or continuous review inventory problems through (s,S) policy. Sahin (1983) examines an (s,S) inventory model with bulk demands and random lead time. She obtains the binomial moments of the time dependent and limiting distribution of the inventory deficit. This is also analysed by Ramanarayanan and Jacob (1987) where they examine only the time dependent behaviour of the system.

---

* Model II discussed in this chapter appeared in Opsearch, Vol.27, No.1, 1990.

An (s,S) policy with the quantity demanded not exceeding what is available in the stock is examined by Krishnamoorthy and Manoharan (1990). They have derived the system size distribution in the steady state.

In this chapter we consider two Models. In Model I the inventory level is not necessarily brought back to its maximum at a replenishment epoch; instead the successive replenished quantities are assumed to form a Markov chain defined over a state space to be specified. Such a situation arises in the case of financing companies which give loans for building constructions, purchase of vehicles etc. where a fresh loan quantity depends upon the previous loan amount which have been already availed.

Ramanarayanan and Jacob (1986) discuss the case of an (s,S) inventory model with unit demand, random lead time and varying ordering levels. The method suggested by them is not computationally tractable and further, passage to the limit is extremely difficult. Krishnamoorthy and Manoharan (1991) discuss the same model and obtain the correlation between the number of demands during a lead time and the length of the next inventory dry period. Model II is on an inventory policy with Markov dependent reordering levels.

Section 3.2 deals with the description of Model I. System size probability distribution at arbitrary time point and steady state behaviour are obtained. Illustration by a numerical example and cost function over a cycle are examined in the same section.

Section 3.3 is concerned with the description and analysis of Model II. System size probabilities and the limiting distribution are obtained. An optimal decision rule is also discussed. Further a numerical example is also given.

The following notations are used in this chapter:

$I(t)$ - Inventory level at time t ($t \geqslant 0$)

$*$ - Convolution. For example $(F*G)(t) = \int_{-\infty}^{\infty} F(t)\, dG(t-u)$

$f^{*n}(.)$ - n-fold convolution of $f(.)$ with itself.

p_k stands for the probability that k units are demanded by an arrival, $k = a, a+1, \ldots, b-1, b$.

a and b represent the minimum and maximum number of items that will be demanded by an arriving customer. We assume that $0 < a \leqslant b$ and $0 \leqslant s-b+1 \leqslant s$.

$E = \{c, c+1, c+2, \ldots, S-s\}$; $c \geqslant b$.

$A = \{s-b+1, s-b+2, \ldots, s-1, s\}$

$H = \{s+1, s+2, \ldots, S-1, S\}$

$\bar{E} = \{0,1,2,\ldots, s\}$

$\bar{A} = \{1,2,\ldots, S\}$

$N^o = \{0,1,2,\ldots,\}$

$[x]$ - The largest integer less than or equal to x.

$P_{(i,I)}(n,t)$ - Probability that $I(t)=n$ given that the initial quantity replenished is i units and the initial ordering level is I.

$P_i(n,t)$ - Probability that $I(t)=n$ given that the initial ordering level is i.

3.2. Model I

This model considers an inventory policy where the quantity demanded by an arriving customer lie between a and b with a and b positive integers and $a\leq b$, $s-b+1 \geq 0$. The demand quantities are independent and identically distributed random variables having the discrete distribution p_k, $k=a,a+1,\ldots,b-1,b$. We assume that the time between demands are independent and identically distributed random variables, independent of demand magnitudes, with distribution function $G(.)$ which is absolutely continuous and $g(t)dt = dG(t)$ with first moment μ_1 (assumed finite). Lead time is zero and shortage is not permitted. The maximum capacity of the warehouse is fixed to be S units.

Due to demands that take place the inventory position decreases and as soon as the level falls to A due to a demand for the first time after each replenishment, an order is placed to bring the inventory to its maximum ie. if at the time of ordering the onhand inventory is i, $i \in A$, then the quantity ordered is (S-i) units. Replenishment is instantaneous with the assumption that the successive quantities replenished form a Markov chain defined on the state space E. Let the one-step transition probability matrix associated with this Markov chain be

$$\mathbb{P}_1 = ((\ q_{ij}\ )),\ i,j \in E \tag{1}$$

Analysis

Suppose $0 = T_o < T_1 < T_2 < \ldots < T_n < \ldots$ are the successive time epochs at which the ordering level falls to A for the first time after the previous replenishments. Specifically let $Y_o, Y_1, Y_2, \ldots$ be the ordering levels and $X_o, X_1, X_2, \ldots$ be the quantities replenished at these epochs. Then by our assumption $\{X_n,\ n=0,1,2,\ldots\}$ forms a Markov chain defined over the state space E with the one-step transition probabilities q_{ij} as defined in (1).

Initially at time $T_o = 0$, due to a demand, let $Y_o=s$ so that a replenishment by a quantity say $X_o = S-s$ occurs at the instant of commencement of inventory. (One can as well proceed with the assumption $X_o=i$ with probability q_i, $i \in E$).

Identify $0 = T_o = T_{o,o}, T_{o,1}, \ldots, T_{o,r_o} = T_1 = T_{1,o}, T_{1,1}, T_{1,2}, \ldots, T_{1,r_1} = T_2 = T_{2,o}, \ldots$ as the successive demand epochs. Then $\{T_{n,i} - T_{n,i-1},\ i=1,2,\ldots,r_n\ ;\ n \in N^o\}$ is a sequence of positive, independent and identically distributed random variables and so forms a renewal process. Introduce yet another sequence of random variables $\{Z_{n,i},\ i=1,2,\ldots,r_n;\ n \in N^o\}$ where $Z_{n,i}$ represents the inventory level just after meeting the demand at $T_{n,i}$. The process $\{(X,Y,Z)\} = \{(X_n, Y_n, Z_{n,i}),\ i = 1,2,\ldots, r_n;\ n \in N^o\}$ turns out to be a three dimensional Markov chain. Then we have

<u>Theorem-1</u>

The stochastic process $\{(X,Y,Z),T\} = \{(X_n,Y_n,Z_{n,i}),T_{n,i};\ i = 1,2,\ldots, r_n;\ n \in N^o\}$ is a Markov Renewal Process defined over the state space E x A x H with the semi-Markov kernel defined by $((Q\{(\ell,L,k)(j,J,m),t\}))$ where

(i) between two consecutive demand epochs both of which are not replenishment epochs

$$Q\{(j,J,k),(j,J,m),t\} = \Pr\{X_n=j, Y_n=J, Z_{n,i+1}=m;\ T_{n,i+1}-T_{n,i} \leqslant t \mid X_n=j, Y_n=J, Z_{n,i}=k\}$$

$$= \int_o^t p_{k-m}\, g(u)du,\quad j \in E;\ J \in A;\ m,k \in H;\ t \geqslant 0$$

and

(ii) between two successive demand epochs in which the current demand epoch happens to be a replenishment epoch as well

$$Q\{(\ell,L,k)(j,J,m),t\} = \Pr\{X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=m;\ T_{n+1}-T_{n,r_n-1} \leqslant t \mid X_n=\ell,\ Y_n=L, Z_{n,r_n-1}=k\}$$

$$= \int_0^t p_{k-J}\, q_{\ell j}\, g(u)du,\quad \ell,j \in E;\ L,J \in A;\ k,m \in H;\quad n = 0,1,2,\ldots$$

<u>Proof</u>

The interarrival times of demands $T_{n,i} - T_{n,i-1}$, $i = 1,2,\ldots,r_n$; $n \in N^o$ are assumed to be independent and identically distributed random variables following distribution function G(.) and density function g(.). Further the demand quantities are independent and also does not depend upon the length of the time elapsed between demands. Hence considering time epochs like $T^+_{n,i-1}$ and $T^+_{n,i}$, ($T^+_{n,i}$ represents the time epoch just after meeting a demand) $i = 1,2,\ldots,r_n-1$, $n \in N^o$

$$\Pr\{X_n=j,\ Y_n=J,\ Z_{n,i}=m,\ T_{n,i}-T_{n,i-1} \leqslant t \mid X_o,X_1,\ldots,X_n=j;\ Y_o,Y_1,Y_2,\ldots,Y_n=J;\ Z_{n,1},Z_{n,2},\ldots,Z_{n,i-1}=k;\ T_{n,1},T_{n,2},\ldots,T_{n,i-1}\}$$

$$= \Pr\{(X_n=j,\ Y_n=J,\ Z_{n,i}=m);\ T_{n,i}-T_{n,i-1} \leqslant t \mid (X_n=j, Y_n=J, Z_{n,i-1}=k)\}$$

$$= Q\{(j,J,k),\ (j,J,m),t\},\ j \in E;\ J \in A,\ m,k \in H;\ t \geqslant 0.$$

Here exactly one demand occurs and the demand is for a quantity (k-m) so that

$$Q\{(j,J,k),\ (j,J,m),t\} = \int_0^t p_{k-m}\, g(u)du$$

For case (ii) we consider demand epochs like $T^+_{n,r_{n-1}}$ and T^+_{n+1}. Due to a demand at $T_{n,r_n} = T_{n+1}$, the stock level drops to $J \in A$ so that the demand is for a quantity k-J where k is the inventory level prior to the demand at T_{n+1}. The replenishment at T_{n+1} is by a quantity j and the just previous replenished quantity is ℓ where $j,\ell \in E$. Then by the assumptions of our models,

$$\begin{aligned}
&\Pr\{(X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=m;\ T_{n+1}-T_{n,r_n-1} \leqslant t \mid X_o,X_1,\ldots,X_n=\ell;\\
&\qquad Y_o,Y_1,\ldots,Y_n=L\ ;\ Z_{n,1},\ Z_{n,2},\ldots,Z_{n,r_n-1}=k;T_{n,1},T_{n,2},\ldots T_{n,r_n-1}\\
&= \Pr\{X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=m;\ T_{n+1}-T_{n,r_n-1} \leqslant t \mid X_n= \ ,Y_n=L,Z_{n,r_n-1}=k\}\\
&= Q\{(\ell,L,k),\ (j,J,m),t\}\\
&= \int_o^t p_{k-J}\, q_{\ell j}\, g(u)du
\end{aligned}$$

Hence the theorem.

The next step is to obtain an expression for the Markov renewal function. To this end we proceed as follows.

Consider two successive replenishment epochs T_n and T_{n+1}. Define $F\{(\ell, L, \ell+L), (j, J, j+J), t\}$ as the probability that $T_{n+1}-T_n \leqslant t$ and the replenished quantity and ordering level at T_{n+1} are, respectively, j,J conditional on ℓ and L as the replenished quantity and ordering level respectively at T_n, $\ell, j \in E$ and $L, J \in A$. Thus

$$F\{(\ell,L,\ell+L),\ (j,J,j+J),t\} = \{Pr\ \ X_{n+1}=j,\ Y_{n+1}=J,\ Z_{n+1}=j+J;\ T_{n+1}-T_n \leqslant t \mid X_n=\ell,\ Y_n=L,\ Z_n=\ell+L\}$$

$$\ell, j \in E;\ L, J \in A;\ t \geqslant 0.$$

$$= \sum_{m=[\frac{\ell+L}{b}]}^{\ell+L-(s+1)+1} Q^{*m}\{(\ell,L,\ell+L),\ (j,J,j+J),t\}$$

$$= \int_0^t \sum_{m=[\frac{\ell+L}{b}]}^{\ell+L-(s+1)+1} g^{*m}(u)\ q_{\ell j} \sum_{\substack{i_1+i_2+\ldots+i_m=\ell+L-J \\ i_1+i_2+\ldots+i_{m-1}<\ell+L-s}} p_{i_1} p_{i_2} \ldots p_{i_m}\, du$$

Define

$$R\ \{(\ell,L,\ell+L),(j,J,j+J),t\} = \sum_{n=0}^{\infty} F^{*n}\{(\ell,L,\ell+L),(j,J,j+J),t\}$$

with

$$F^{0}\{(\ell,L,\ell+L),(j,J,j+J),t = \begin{cases} 1 \text{ for } (\ell,L,\ell+L)=(j,J,j+J) \\ 0 \text{ otherwise} \end{cases}$$

and $F^{*n}\{(\ell,L,\ell+L),\ (j,J,j+J),t\}$ is obtained from the recursive relation

$$F^{*(n+1)}\{(\ell,L,\ell+L),(j,J,j+J),t\}=\sum_{i\in E}\sum_{I\in A}\int_0^t F\{(\ell,L,\ell+L),(i,I,i+I),du\}$$

$$F^{*n}\{(i,I,i+I),(j,J,j+J),t-u\},$$

$$\ell,j\in E,\ L,J\in A;\ t\geqslant 0.$$

Since $I(t)$ denotes the inventory level at time t,
$I(t) = Z_{n,i}$ for $T_{n,i}\leqslant t < T_{n,i+1},\ i=1,2,\ldots,r_n;\ n\in N^o$ and so
$\{I(t),t\geqslant 0\}$ is a semi-Markov process defined over H.

Let $P_{(S-s,s)}(n,t) = Pr\{I(t)=n|X_o=S-s,\ Y_o=s\}$ for $n\in H,\ t\geqslant 0$.

Then $P_{(S-s,s)}(n,t)$ satisfies the Markov renewal equations (Cinlar 1975a).

Thus we have

(i) for n=S

$$P_{(S-s,s)}(S,t) = K_{(S-s,s)}(S,t)+\sum_{i\in E}\sum_{I\in A}\int_0^t F\{(S-s,s,S),$$

$$(i,I,i+I),du\}$$

$$P_{(i,I)}(S,t-u)$$

where
$$K_{(S-s,s)}(S,t) = Pr\{I(t)=S,\ T_{o,1}>t|X_o=S-s,\ Y_o=s\}$$

$$= 1-G(t)$$

(ii) for n = S-1, S-2, ..., s+1

$$P_{(S-s,s)}(n,t) = K^{(1)}_{(S-s,s)}(n,t) + \sum_{i\in E}\sum_{I\in A}\int_0^t F\{(S-s,s,S),(i,I,i+I),du\}\, P_{(i,I)}(n,t-u)$$

where

$$K^{(1)}_{(S-s,s)}(n,t) = \int_0^t \sum_{m=[\frac{S-n}{b}]+1}^{S-n} Q^{*m}\{(S-s,s,S),(S-s,s,n),du\}[1-G(t-u)]$$

Hence the solutions are given by

$$P_{(S-s,s)}(S,t) = \int_0^t R\{(S-s,s,S),(S-s,s,S),du\}\, K_{(S-s,s)}(S,t-u)$$

and for n = S-1, S-2,..., s+1

$$P_{(S-s,s)}(n,t) = \sum_{\substack{j\in E\\ j+J \geqslant n}}\sum_{J\in A}\int_0^t R\{(S-s,s,S),(j,J,j+J),du\}\, K^{(1)}_{(j,J)}(n,t-u)$$

where

$$K^{(1)}_{(j,J)}(n,t) = \int_0^t \sum_{m=[\frac{j+J-n}{b}]+1}^{j+J-n} Q^{*m}\{(j,J,j+J),(j,J,n),du\}\,[1-G(t-u)]$$

Steady state analysis

Let $\lim\limits_{t \longrightarrow \infty} P_{(S-s,s)}(n,t) = \underline{P}(n)$ for $n \in H$

To obtain the limiting probabilities of the system size we proceed in the following manner.

We have seen that the three dimensional process $\{(X_n, Y_n, Z_{n,i}),$ $i=1,2,\ldots,r_n;\ n \in N^o\}$ forms a Markov chain with state space

$$\left\{(i,j,k) \,\middle|\, \begin{array}{l} i=c,c+1,\ldots,M;\ j=s-b+1,\ s-b+2,\ldots,s-1,s; \\ \qquad\qquad\qquad\quad k=s+1,\ s+2,\ldots,j+i-1,\ j+i \end{array}\right\}$$

where $k \leqslant S$. From the given one-step transition probabilities associated with the replenished quantities, the one-step transition probability matrix $\mathbb{P}_2 = ((\, p^{(1)}_{(i,j,k),(i',j',k')} \,))$ associated with the three dimensional Markov chain can be obtained. The stationary distributions are then computed as

$$\pi(i',j',k') = \sum_{i\in E} \sum_{j\in A} \sum_{k\in H} \pi(i,j,k)\ p^{(1)}_{(i,j,k),(i',j',k')},$$

$$i' \in E,\ j' \in A;\ \text{and}\ k' \in H.$$

Since the transition to any state takes place at a demand epoch, the mean sojourn time in any state is given by

$$m(i,j,k) = \int_0^{\infty} (1-G(t)dt = \mu_1$$

which is the mean interarrival time between demands. The limiting probabilities are now computed as follows:

$$\underline{P}(S) = \frac{\pi(S-s,s,S)\ m(S-s,s,S)}{\sum_{i\in E}\ \sum_{j\in A}\ \sum_{k\in H} \pi(i,j,k)\ m(i,j,k)}$$

$$= \pi(S-s,\ s,S)$$

Similarly

$$\underline{P}(S-1) = \sum_{j=s-1}^{s} (\pi(S-s,j,S-1)) + \pi(S-s-1,\ s,S-1)$$

$$\underline{P}(S-2) = \sum_{j=s-2}^{s} (\pi(S-s,j,S-2)) + \sum_{j=s-1}^{s} (\pi(S-s-1,j,S-2)) + \pi(S-s-2,s,S-2)$$

$$\vdots$$

$$\underline{P}(s+1) = \sum_{i=c}^{S-s}\ \sum_{j=S-b+1}^{s} \pi(i,j,s+1)$$

Thus

$$\underline{P}(n) = \sum_{i=c}^{S-s}\ \sum_{j=s-b+1}^{s} \pi(i,j,n),\ n=s+1,s+2,\ldots,s-b+1+c$$

and

$$\underline{P}(S-m) = \sum_{j=s-m}^{s} (\pi(S-s,j,S-m)) + \sum_{j=s-(m-1)}^{s} (\pi(S-s-1,j,S-m)) + \ldots$$

$$+ \sum_{j=s-1}^{s} (\pi(S-s-(m-1),j,S-m)) + \pi(S-s-m,s,S-m),$$

$$m=0,1,2,\ldots,S-(s-b+1)-(c+1).$$

<u>Cost function over a cycle</u>

For the inventory model under consideration, a cycle is the length of duration between two successive replenishment epochs. A typical plot of the stock level is shown in Fig.1.

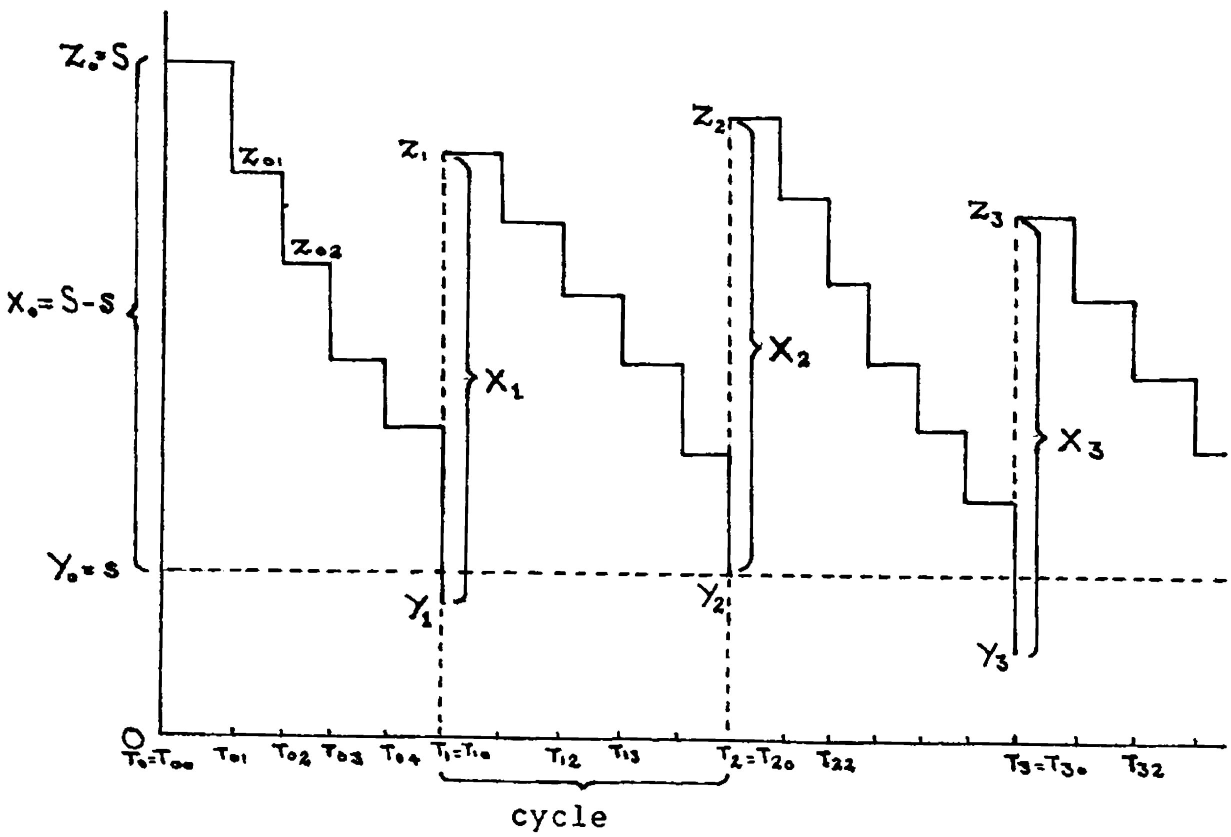


Fig.1.

The total cost over a particular cycle assuming j as the replenishing quantity under steady state is computed as follows: The objective function is the total expected cost per unit time over a cycle under steady state which is so chosen that it attains a minimum value corresponding to the quantity replenished.

Considering two successive replenishment epochs, let Y denotes the ordering level at the current replenishment epoch, Z be the length of the cycle just completed. Then the conditional density function of Y and Z given that X is the reordering level and j is the quantity replenished at the beginning of the cycle is denoted by

$$f_{j,x}(y,z)$$

Hence

$$f_{j,x}(y,z)dz = \sum_{k=[\frac{x+j}{b}]}^{x+j-(s+1)+1} P_{j,x}\left\{Y=y, z\leqslant Z<z+dz \;\middle|\; \begin{array}{l}\text{k demands occurred,}\\ \text{totally consuming}\\ \text{(x+j-y) items and}\\ \text{(k-1) demands}\\ \text{consumed less than}\\ \text{(x+j-s) items.}\end{array}\right\}$$

$$\times \Pr\left\{\begin{array}{l}\text{k demands occurred totally consuming (x+j-y) items}\\ \text{and (k-1) demands consumed less than (x+j-s) items}\end{array}\right\}$$

$$= \sum_{k=[\frac{x+j}{b}]}^{x+j-(s+1)+1} g^{*k}(z) \sum_{\substack{i_1, i_2, \ldots, i_k \\ i_1+\ldots+i_k=(x+j-y) \\ i_1+\ldots+i_{k-1}<(x+j-s)}} p_{i_1} p_{i_2} \ldots p_{i_k}\, dz$$

Hence the conditional expected value of a cycle =

$$E_{j,x}(Z) = \int_0^\infty z\Big(\sum_{y=s-b+1}^{s} f_{j,x}(y,z)\Big)dz$$

The conditional expected inventory level over a cycle =

$$E_{j,x}(I) = \sum_{n=s+1}^{x+j} n \Pr\{I=n \mid j,x\}$$

$$= \sum_{n=s+1}^{x+j} n\, \pi(j,x,n)$$

Hence the conditional expected total cost over a cycle per unit time is

$$\bar{F}_{j,x}(c) = \left\{ \sum_{y=s-b+1}^{s} \frac{K+\gamma j}{E_{j,x}(z)}\, \pi(j,y,y+j) \right\} + h \sum_{n=s+1}^{x+j} n\, \pi(j,x,n)$$

where K is the fixed cost of ordering, γ is the procurement cost per unit and h is the holding cost per unit per unit time.

Clearly the above function is convex.

Numerical example

Let the one-step transition probability matrix associated with the given Markov chain constituted by the successive quantities replenished be given by

$$\mathbb{P}_1 = ((\ q_{ij}\ )) = \begin{bmatrix} 1/2 & 1/2 \\ 1/5 & 4/5 \end{bmatrix}$$

with the maximum capacity of the warehouse being S=4 and let s=1. Let a=1 and b=2 with $p_1 = 1/3$ and $p_2 = 2/3$. Assume that the interarrival times of demands follow exponential distribution with parameter $\lambda = 0.5$.

The set E $= \{2,3\}$ so that $q_{22} = 1/2$; $q_{23} = 1/2$; $q_{32} = 1/5$ and $q_{33} = 4/5$.

From $\mathbb{P}_1$ we compute $\mathbb{P}_2$ defined over the state space $\{(2,0,2), (2,1,3), (2,1,2), (3,0,3), (3,0,2), (3,1,4), (3,1,3), (3,1,2)\}$ as follows:

$$p^{(1)}_{(2,0,2),(2,0,2)} = p_2\, q_{22}$$

$$p^{(1)}_{(2,0,2),(2,1,3)} = p_1\, q_{22}$$

$$\left.\begin{array}{l} p^{(1)}_{(2,0,2),(2,1,2)} \\ p^{(1)}_{(2,0,2),(3,0,2)} \\ p^{(1)}_{(2,0,2),(3,1,3)} \\ p^{(1)}_{(2,0,2),(3,1,2)} \end{array}\right\} = 0$$

$$p^{(1)}_{(2,0,2),(3,0,3)} = p_2\, q_{23}$$

$$p^{(1)}_{(2,0,2),(3,1,4)} = p_1\, q_{23}$$

$$p^{(1)}_{(2,1,3)(2,0,2)}$$

$$\left.\begin{array}{l} p^{(1)}_{(2,1,3),(3,0,3)} \\[2ex] p^{(1)}_{(2,1,3),(3,0,2)} \\[2ex] p^{(1)}_{(2,1,3),(3,1,3)} \\[2ex] p^{(1)}_{(2,1,3),(3,1,2)} \end{array}\right\} = 0$$

$$p^{(1)}_{(2,1,3),(2,1,3)} = p_2\, q_{22}$$

$$p^{(1)}_{(2,1,3),(2,1,2)} = p_1$$

$$p^{(1)}_{(2,1,3),(3,1,4)} = p_2\, q_{23}$$

The other transition probabilities can be obtained in a similar way. Hence

$$\mathbb{P}_2 = \begin{bmatrix} 1/3 & 1/6 & 0 & 1/3 & 0 & 1/6 & 0 & 0 \\ 0 & 1/3 & 1/3 & 0 & 0 & 1/3 & 0 & 0 \\ 1/3 & 1/6 & 0 & 1/3 & 0 & 1/6 & 0 & 0 \\ 0 & 2/15 & 0 & 0 & 1/3 & 8/15 & 0 & 0 \\ 2/15 & 1/15 & 0 & 8/15 & 0 & 4/15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 2/15 & 0 & 0 & 0 & 8/15 & 0 & 1/3 \\ 2/15 & 1/15 & 0 & 8/15 & 0 & 4/15 & 0 & 0 \end{bmatrix}$$

From $\mathbb{P}_2$, the stationary probability vector π and then $\underline{P}(n)$ are computed. Finally $\bar{F}_{j,x}(c)$ for x=1 and j=2,3 are calculated and tabulated as follows.

	π	n	$\underline{P}(n)$	j	$E_{j,1}(z)$	$E_{j,1}(I)$	$\bar{F}_{j,1}(c)$
$K=10,\ \gamma=1,\ h=1,\ S=4$, $s=1,\ a=1,\ b=2,\ p_1=1/3$, $p_2=2/3,\ \lambda=0.5$	{0.11, 0.5, 0.17, 0.12, 0.04, 0.03, 0.01, 0.02}	2	0.34	2	2.67	1.84	4.58
		3	0.63	3	2.93	0.2	0.87
		4	0.03				

For the given inventory problem, the conditional expected total cost per unit time over a cycle is minimum corresponding to the replenishing quantity $j=3$.

3.3. Model-II

Description

Model II deals with a continuous review single commodity inventory problem where we assume that each demand is exactly for one unit. The interarrival times of demands are independent and identically distributed random variables following distribution function $G(.)$ with **density** function $g(.)$ and having finite first moment a . The maximum capacity of the warehouse is fixed as S units. Lead time is zero and no shortage is permitted. Further we assume that the reorder levels vary according to a Markov chain with state space $\{0,1,2,\ldots,s\}$, $s\leqslant S-1$. The quantity replenis ed is always equal to $M=S-s$. In the present analysis we identify a two dimensional Markov chain in the underlying process, thereby gaining more information about the process.

Analysis

The assumption of our model is that the reordering levels are governed by a Markov chain. Denoting $X_o, X_1, X_2, \ldots$ as the initial, first, second, ... reordering levels, $\{X_n, n=0,1,2,\ldots\}$ forms a Markov chain defined over $\{0,1,2,\ldots,s\}$ with initial probability

$$Pr\ (X_o=s) = 1 \quad \text{and}$$

$$Pr\ (X_o=i) = 0, \quad i=0,1,2,\ldots,s-1$$

The one-step transition probability matrix $\mathbb{P}_3$ is given by

$$\mathbb{P}_3 = ((\ p_{ij}\ )) \quad \text{where}$$

$$p_{ij} = \begin{cases} Pr\left\{X_{n+1}=j \mid X_n=i\right\}, & i,j = 0,1,2,\ldots,s \\ 0 \quad \text{for } i,j > s \end{cases}$$

Let $0 = T_o < T_1 < T_2 < \ldots$ be the successive demand epochs and $Y_o, Y_1, Y_2, \ldots$ be the corresponding inventory levels after meeting the demands at $T_o, T_1, T_2, \ldots$. Then $I(T_n+) = Y_n$. The process $\left\{I(t),\ t \geqslant 0\right\}$ is a semi-Markov process defined on $\left\{1,2,\ldots,S\right\}$. The next procedure is to get the embedded Markov Renewal Process. For that we should have the information regarding the most recent reordering level ie. considering the pair (X_n, Y_n), if X_n denotes the last reordering level just prior to T_n, then the process $\left\{(X,Y),T\right\} = \left\{(X_n,Y_n),T_n;\ n \in N^o\right\}$ forms the associated embedded Markov Renewal Process defined over the state space $\bar{E} \times \bar{A}$. The semi-Markov kernel is given by $((\ Q_i(j,k,t)))$ where

$$Q_i(j,k,t) = Pr\left\{Y_{n+1} = k;\ T_{n+1}-T_n \leqslant t \mid Y_n=j,\ X_n=i\right\}$$

$$\begin{aligned} i \quad &= 0,1,2,\ldots,s \\ j,k \quad &= 1,2,\ldots,S,\ t \geqslant 0, \end{aligned}$$

where X_n stands for the reordering level just prior to T_n. The maximum value that j can take is i+M where $i+M \leqslant s+M$ so that $Q_i(j,k,t)$, $i=0,1,2,\ldots,s$; $j,k = 1,2,\ldots,S$ are given by

$$Q_i(j,k,t) = \begin{cases} 1-G(t), & j=i+M;\ k=j \\ G(t), & j=k+1;\ k>s \\ G(t)(1-p_{ik}), & j=s+1,\ k=s \\ G(t)p_{ik}, & j=s+1,\ k=s \\ G(t)\ 1-(p_{ij}+p_{ik}), & j=s,s-1,\ldots,1,\ k=j-1 \\ G(t)\{1-(p_{is}+p_{is-1}+\ldots+p_{ij})\}p_{ij-1}, & \begin{array}{l} j=s,s-1,\ldots,1; \\ k=j-1+M \end{array} \\ G(t)\{1-(p_{is}+p_{is-1}+\ldots+p_{ij+1})\}p_{ij}; & \begin{array}{l} j=s,s-1,\ldots,1; \\ k=j+M-1 \end{array} \end{cases}$$

To obtain the Markov Renewal function, we proceed as follows. Initially at the commencement of inventory the stock level is s and an order is placed; a replenishment occurs instantaneously so that $Y_o=S$ and $X_o=s$ with $p(s) = 1$.

Define $F(i,j,t)$ as the probability that an order is placed when the level is i and the next order is placed when the level is j, $i,j=0,1,2,\ldots,s$ and the time duration in these is less than or equal to t.

ie. $\quad F(i,j,t) = Q_i^{*i+M-j}(i+M,j+M,t)$, $j \leqslant i+M \leqslant s+M$.

The Markov Renewal function is given by

$$R(s,j,t) = \sum_{m=o}^{\infty} \left\{ \sum_{i=o}^{s} F^{*m}(i,j,t) * F(s,j,t), \right.$$

$$j=0,1,2,\ldots,s, \quad t \geqslant 0.$$

where

$$F^{o}(i,j,t) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$F^{*m}(i,j,t)$ is obtained from the recursive relation

$$F^{*(m+1)}(i,j,t) = \sum_{k=o}^{s} \int_{o}^{t} F(i,k,du)\, F^{*m}(k,j,t-u) \quad \text{(Cinlar 1975a)}$$

Define $P_s(n,t) = \Pr\{I(t)=n \mid X_o=s\}$, $n=S,S-1,\ldots,M+1, M, M-1,\ldots s+1, s, s-1,\ldots,1.$

$P_s(n,t)$ satisfies the Markov renewal equation so that

$$P_s(n,t) = 1-G(t) + \int_{o}^{t} Q_s(S,S-1,du)\, P_s(n,t-u)$$

Hence the solution is given by

$$P_s(S,t) = \int_{o}^{t} R(s,s,du)\,(1-G(t-u))$$

Similarly for n = S-1,

$$P_s(S-1,t) = \int_o^t \sum_{j=s-1}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-(S-1))}(M+j,S-1,v-u) [1-G(t-v)]\, dv$$

$\vdots$

for n = M+1,

$$P_s(M+1,t) = \int_o^t \sum_{j=1}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-(M+1))}(M+j,\ M+1,\ v-u) [1-G(t-v)]\, dv$$

In general

$$P_s(M+i,t) = \int_o^t \sum_{j=1}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-(M+i))}(M+j,M+i,v-u) [1-G(t-v)]dv\ ,\ i=1,2,\ldots,s.$$

and

$$P_s(n,t) = \int_o^t \sum_{j=o}^{s} R(s,j,du) \int_u^t Q_j^{*(M+j-n)}(M+j,n,v-u) [1-G(t-v)]dv\ ,\ n=1,2,\ldots,M$$

<u>Limiting distribution</u>

Let $\mathbb{P}_4$ denotes the one-step transition probability matrix corresponding to the Markov chain $\{(X_n,Y_n),n=0,1,2,\ldots\}$ with

$$\mathbb{P}_4 = ((\ p^{(1)}_{(i,j),(\ell,k)}\ )),\ i,\ell = 0,1,2,\ldots,s;\quad j,k = 1,2,\ldots,S \text{ with}$$

$$p^{(1)}_{(i,j),(\ell,k)} = Pr\{(X_{n+1}=\ell,\ Y_{n+1}=k) \mid (X_n=i,\ Y_n=j)\}$$

$\mathbb{P}_4$ is computed from $\mathbb{P}_3$ in the following way

$$p^{(1)}_{(i,j),(\ell,k)} = \begin{cases} 1 & , i=0,1,2,\ldots,s;\ j=1,;\ \ell=0,\ k=M \\ 1 & , i=0,1,2,\ldots,s;\ j=s+2,\ s+3,\ldots,s+M; \\ & \quad \ell=i,\ k=j-1 \\ \dfrac{p_{i\ell}}{\sum\limits_{r=o}^{s-1} p_{or}} & , i=0,1,2,\ldots,s;\ j=2,3,\ldots,s;\ \ell=j-1; \\ & \quad k=M+j-1 \\ 1-\dfrac{p_{i\ell}}{\sum\limits_{r=o}^{s-1} p_{or}} & , i=0,1,2,\ldots,s;\ j=2,3,\ldots,s;\ \ell=i; \\ & \quad k=j-1 \\ p_{is} & , i=0,1,2,\ldots,s;\ j=s+1;\ell=s;\ k=M+s \\ 1-p_{is} & , i=0,1,2,\ldots,s;\ j=s+1,\ell=i;\ k=s \end{cases}$$

The stationary distributions are given by

$$\pi = \{\pi(0,1),\ \pi(0,2),\ldots,\ \pi(0,M),\ \pi(1,1),\ \ldots,\ \pi(1,M+1),\ldots,$$
$$\pi(s,1)\ \pi(s,2),\ \ldots\ \pi(s,S)\}$$

where

$$\pi(\ell,k) = \sum_{i=o}^{s} \sum_{j=1}^{S} \pi(i,j)\ p^{(1)}_{(i,j),(\ell,k)},\ \ell = 0,1,2,\ldots,s,\ k=1,2,\ldots,S$$

The mean sojourn time in a state (ℓ,k) where ℓ is the last reordering level and k is the stock level after meeting a demand is

$$m(\ell,k) = \int_0^\infty (1-Q_\ell(k,k-1,t))\,dt, \quad \ell = 0,1,2,\ldots,s; \; k= 1,2,\ldots, S.$$

$$= \int_0^\infty (1-G(t))dt$$

$$= a$$

The limiting probabilities are given by

$$\lim_{t\to\infty} \Pr\,(I(t)=S|X_o=s) = \lim_{t\to\infty} P_s(S,t) = \underline{P}(S)\text{(say)}$$

$$= \frac{\pi(s,S)\; m(s,S)}{\sum_{\ell=0}^{s} \sum_{k=1}^{S} \pi(\ell,k)\; m(\ell,k)}$$

$$= \pi(s,S).$$

Similarly $\lim_{t\to\infty} P_s(S-1,t) = \underline{P}(S-1)$

$$= \sum_{j=s-1}^{s} \pi(j,S-1)$$

In general

$$\underline{P}(n) = \sum_{j=o}^{s} \pi(j,n), \; n=1,2,\ldots,M$$

and

$$\underline{P}(M+i) = \sum_{j=i}^{s} \pi(j,M+i), \; i=1,2,\ldots,s.$$

Optimisation

The objective function is the total expected cost per unit time in the steady state. The decision variable, M should be so chosen that the objective function is minimum for that value of M.

The expected time elapsed between two successive demands = $\int_{0}^{\infty} (1-G(t)) = a < \infty$. Under steady state, assuming j, j = 0,1,2,...,s as the reordering level, the expected time elapsed between two successive orders is $(j+M-j)a = Ma$. Therefore the expected number of orders per unit time $= \frac{1}{Ma}$. The expected inventory level at any instant of time is given by

$$E(I) = \sum_{n=1}^{S} n\, \underline{P}(n)$$

Therefore the total expected cost per unit time in the steady state is

$$\bar{F}(M) = \sum_{\ell=o}^{s} \sum_{k=1}^{S} \left\{\frac{K+cM}{Ma}\right\} \pi(\ell,k) + h\, E(I)$$

$$= \left\{\frac{K+cM}{Ma}\right\} \sum_{\ell=o}^{s} \sum_{k=1}^{S} \pi(\ell,k) + h \sum_{n=1}^{S} n\underline{P}(n)$$

where K is the fixed order cost, c is the variable procurement cost per unit, h is the holding cost per unit per unit time.

The optimal value of M is that value of M for which $\bar{F}(M)$ is minimum. It is readily verified that $\bar{F}(M)$ is a convex function in M. The optimal value of M is obtained from the two relations

$$\bar{F}(M) \leqslant \bar{F}(M+1)$$
$$\bar{F}(M) \leqslant \bar{F}(M-1)$$

Illustrations

1) Let $\mathbb{P}_3 = \begin{bmatrix} .7 & .2 & .1 \\ .1 & .7 & .2 \\ .2 & .1 & .7 \end{bmatrix}$

with s=2 and a = 0.5

Keeping s fixed, M is allowed to vary and the conditional probabilities in each case are obtained. For K=50, c=1, h=1, we have the following table.

	Value of M	$\bar{F}(M)$
K=50, c=1, h=1, a=0.5, s=2	2	56.83
	3	38.26
	4	30.78
	5	25.94
	6	23.12
	8	19.59
	10	18.106
	11	17.62
	12	17.99
	13	24.128

The optimal value of M is 11 for the given range of M.

2) For Markov chain $\{X_n, n=0,1,2,...\}$ with state space $\{0,1,2\}$ let

$$\mathbb{P}_3 = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$$

with a = 0.5.

Proceeding on the same line as in Problem 1 with K=10, c=1 and h=1, we obtained the following table.

	Value of M	$\bar{F}(M)$
K=10, c=1, h=1, s=2	3	13.80
	4	10.77
	5	9.34
	6	9.71

The optimal value of M is seen to be 5 for the given range of M.

Chapter-4

BULK DEMAND INVENTORY SYSTEM WITH RANDOM LEAD TIME AND SERVER VACATION

4.1 Introduction

Apart from the previous chapters which deal with zero lead time inventory problems, the present chapter investigates an (s,S) inventory model with random lead time. One more factor that plays a role here is the server vacation which is initiated as soon as the inventory becomes dry. This type of model fits into a number of real situations corresponding to the seller's market.

Random lead time inventory problems as treated as a stochastic process is analogous to a queueing problem (see chapters 15-17 of Studies in Mathematical Theory of Inventory and Production by Arrow, Karlin and Scarf (1958)) with random arrivals (deliveries) and departures (demands). Scarf (1960) treats a dynamic inventory model with random lead times, but under the restriction that a new order may be placed only at a time when there are no outstanding orders. Detailed analysis of continuous review (s,S) inventory systems have been carried out by several other authors and results relating to the probability distribution of the inventory level and the optimal choice of the levels s and S have been discussed. Sahin (1979) treats an inventory model where the demand quantities follow a continuous distribution with lead time remaining a constant.

Daniel and Ramanarayanan (1988) is the first to introduce server vacation to inventory models. They assume that the quantity demanded by each arriving customer is exactly one. Madhusoodanan (1989) considers a model similar to that of our present model in which he extends the technique of Ramanarayanan and Jacob (1987) to the situation where the server goes for vacation. This method has a drawback, namely that it uses the matrix of transition time densities and its convolutions to arrive at the expression for the probability distribution of the inventory level. Here we give an expression for the system size probabilities using a simple technique.

Section 4.2 introduces the model and explains notations and the assumptions of this chapter. Section 4.3 shows how the model is analysed by embedding a Markov renewal process in the random process representing inventory level. The system size probabilities and also the reliability of the system at arbitrary time point are obtained in Sections 4.4 and 4.5 respectively

4.2. Description

We consider a continuous review (s,S) inventory system with quantity demanded by each arriving customer following a discrete distribution on the set $E = \{1,2,\ldots,a\}$ with a as the maximum quantity that can be demanded. The interarrival times of demands are independent and identically distributed random variables having distribution function G(.) which is absolutely continuous with density g(.). The maximum capacity of the store is fixed as S units. Due to demands that take place over time, the level of the inventory falls and when the level reaches s or below for the first time an order is placed for replenishment. If the ordering level is i, then the ordering quantity is S-i. The lead times are assumed to be independent and identically distributed random variables with distribution function F(.) and density function f(.). These are independent of the demand process and the ordering level. If order materialisation does not take place when the inventory level falls to zero, the server goes on vacation for a random duration having distribution function H(.) having density function h(.). On return if the server finds that the order has not materialised he again goes for vacation of random duration which is independent of and having the same distribution as the previous one. This process continues until on return he finds the order having realised. The demands that emanate during a dry period will not be met and therefore will be deemed to be lost. The vacation durationsare also assumed to be independent of the demand process and lead times.

The notations used in this chapter are explained below.

$[x]$ denotes the largest integer less than or equal to x.

$G(.)$ and $g(.)$ respectively, represent the distribution function and density function of interarrival time of demands.

$F(.)$ and $f(.)$ stand, respectively, for the lead time distribution and density.

$H(.)$ and $h(.)$ are the distribution function and density function of vacation times.

p_i = Probability that i units are demanded by an arriving customer $(i=1,2,...,a)$

$*$ denotes convolution

$$\phi(s) = \sum_{i=1}^{a} p_i s^i$$

$p_i^{(b)}$ = Probability of b consecutive demands consuming i units. This is the coefficient of s^i in $[\phi(s)]^{*b}$

$\hat{p}_i$, for $i=1,2,....,a$ denotes the probability of at least (of course not more than a) i units being demanded by a customer.

$\hat{p}_i^{(n)}$ stands for the probability of at least i units being demanded by n customers.

$m(.) = \sum_{n=o}^{\infty} h^{*n}(.)$ ie. the renewal density of vacation.

$A(.)$ = The renewal density of lost demands (during a dry period).

$I(t)$ = Inventory level (onhand inventory) at time t, $t \geq 0$.

B is the set of points $\{s-a+1, s-a+2, \ldots, s-1, s\}$

Assumptions

We assume that the maximum quantity demanded by an arriving customer is a with $1<a<s$. Also it is assumed that $S > 2s$. These assumptions are made to avoid perpetual shortage. Nevertheless, they are not explicitly used. Even when quantity demanded exceeds what is available, the customer goes off with the available number of items.

4.3. Analysis

Suppose $0 = T_o < T_1 < T_2 < \ldots < T_n \ldots$ be the successive time points at which orders are placed for replenishment and $X_o, X_1, X_2, \ldots, X_n, \ldots$ be the corresponding inventory levels (ordering levels), $X(T_n) = X_n$. Assume that $X(o) = i$, for $i = s-a+1, s-a+2, \ldots, s-1, s$ and hence an order is placed at the instant of commencement of inventory. Then we have

Theorem

$(X,T) = \{(X_n,T_n),\ n=0,1,2,\ldots\}$ forms a Markov Renewal process (MRP) with semi Markov kernel

$$Q(i,j,t) = \Pr\{X_{n+1}=j;\ T_{n+1}-T_n \leqslant t \mid X_n=i\},\ i,j \in B \text{ and } t \geqslant 0$$

Proof follows easily from the definition of MRP.

To get $Q(i,j,t)$ we proceed as follows:

The event $\{X_{n+1}=j;\ T_{n+1}-T_n \leqslant t \mid X_n=i\}$ can occur in two mutually exclusive ways (and these are exhaustive):

(i) Before an order materialisation the inventory level drops to zero due to demands and so there is a dry period and hence the server goes on vacation.

(ii) No dry period between order placement and its materialisation.

Hence

$$Q(i,j,t) = Q_1(i,j,t) + Q_2(i,j,t)$$

where,

$Q_1(i,j,t)$ represents the transition probability from i to j in time less than or equal to t with order placed when level is at i, not materialising before the system emptying and $Q_2(i,j,t)$ that without any dry period between transition from i to j, that is, in this case the order which is placed when level reaches $i(i \leq s)$ for the first time after the previous replenishment, materializes before the system becomes empty and then due to a number of demands the next replenishment order is placed when the inventory level reaches $j(j \leq s)$ for the first time after replenishment.

We have,

$$Q_1(i,j,t) = \int_{z=o}^{t} \int_{u=z}^{t} \int_{w=u}^{x} \int_{x=w}^{t} \int_{y=u}^{t} \Big(\sum_{b=1}^{i-1} \sum_{r=\max\{1,[\frac{b}{a}]\}}^{b} g^{*r}(z) p_b^{(r)} g(u-z) \hat{p}_{i-b}$$

$$m(w-u)\ h(x-w)\ \frac{(F(x)-F(u))}{(1-F(w))}\ A(y-u)$$

$$\sum_{n=\max\{1,[\frac{S-i-j}{a}]\}}^{S-i-j} \frac{G^{*n}(t-y)-G^{*(n+1)}(t-y)}{(1-G(x-y))}\ p_{S-i-j}^{(n)}$$

$$dy\ dx\ dw\ du\ dz \tag{1}$$

$$Q_2(i,j,t) = \int_{u=o}^{t} \int_{v=u}^{t} \sum_{b=o}^{i-1} \sum_{r=[\frac{b}{a}]}^{b} g^{*r}(u)$$

$$p_b^{(r)}\ f(v) \sum_{n=\max\{1,[\frac{S-b-j}{a}]\}}^{S-b-j} \frac{G^{*n}(t-u)-G^{*(n+1)}(t-u)}{(1-G(v-u))}$$

$$p_{S-b-j}^{(n)}\ dv\ du \tag{2}$$

where we define $p_o^{(o)}$ as 1 and $p_b^{(o)} = 0$ for $b > 0$.

The right hand side of (1) is arrived at as follows:

The inventory level drops to $i \in B$ (B being visited for the first time after the previous replenishment). We take this as the time origin. Then r demands take place until time z (the r^{th} being at z) which together take away atmost i-1 units and the next demand that take place at time u makes the inventory dry whereupon the server goes on vacation. There are a number of demands lost, the last one taking place at time y (this is represented by $A(y-u)$). The server returns after each vacation to find the inventory dry and hence goes back for a fresh vacation (this is represented by $m(w-u)$). The last vacation is completed at time x since the replenishment takes place in (w,x). Now the inventory level is $(S-i)$. Exactly n demands take place bringing down the inventory to $j \in B$ (this is the first visit to B after the previous replenishment). Hence an order is placed for $(S-j)$ units. A similar argument yields the right hand side of (2) except that in this case there is no dry period.

Now define $R(i,j,t) = \sum_{n=o}^{\infty} Q^{*n}(i,j,t)$ which is the expected number of visits to state j in $(o,t]$ starting initially at i, $i,j \in B$, $t \geq 0$ where

$$Q^{o}(i,j,t) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

4.4 System size probabilities

The stock level $\{I(t), t \geq 0\}$ is a discrete valued stochastic process defined on the state space $\{0,1,2,\ldots,S\}$. Define

$$X(t) = X_n, \quad \text{for} \quad T_n \leq t < T_{n+1}$$

Let $Z(t) = (I(t), X(t))$. Then clearly $\{Z(t), t \geq 0\}$ is a semi-regenerative process with state space $\{0,1,2,\ldots,S\}$ x $\{s-a+1, s-a+2,\ldots,s-1,s\}$ and (X,T) is the Markov renewal process embedded in it. Further assume that (X,T) is irreducible, recurrent and aperiodic.

Let $P((n,j),t) = Pr\{Z(t)=(n,j)\}$, for $n=0,1,2,\ldots,S$ and $j=s-a+1,\ldots,s-1,s$

Then

$$P((n,j),t) = Pr\{Z(t)=(n,j), T_1>t\} + Pr\{Z(t)=(n,j), T_1 \leq t\}$$

$$= K((n,j),t) + \sum_{i \in B} \int_0^t Q(i,j,du)\, P((n,j),t-u) \qquad (3)$$

where for every $(n,j) \in \{0,1,2,\ldots,S\}$ x $\{s-a+1, \ldots, s-1,s\}$ the mapping $t \longrightarrow P((n,j),t)$ is Borel measurable and bounded over finite intervals. Further $K((n,j),t)$ is directly Riemann integrable for every $(n,j) \in \{0,1,2,\ldots,S\}$ x $\{s-a+1,\ldots,s-1,s\}$. Therefore the Markov renewal equation (3) has one and only one solution given by

$$P((n,j),t) = \sum_{i \in B} \int_{o}^{t} R(i,j,du)\ K((n,j),t-u), \quad n=0,1,2,\ldots,S; \quad j=s-a+1,\ldots,s \text{ and } t \geqslant 0.$$

From this we can compute $P((n,j),t)$ for different values of n and j.

$K((n,j),t)$ for different values of n and j are computed as follows:

<u>Case (i):</u> When $n=S$, $K((S,j),t) = (1-G(t))\ F(t),\ j \in B$

<u>Case (ii):</u> For $S-j+1 \leqslant n < S$

$$K((n,j),t) = \int_{v=o}^{t} \int_{z=o}^{v} \sum_{c=o}^{j-1} \sum_{r=\max\{0,[\frac{c}{a}]\}}^{b} (g^{*r}(z)p_c^{(r)})f(v)$$

$$\sum_{b=\max\{1,[\frac{S-c-n}{a}]\}}^{S-c-n} \frac{G^{*b}(t-z)-G^{*(b+1)}(t-z)}{1-G(v-z)}\ p_{S-c-n}^{(b)}\ dv\ dz$$

It is to be noted that in this case, there can be no dry period and this is taken care of by the probability of atmost $c(\leqslant j-1)$ units being sold off until replenishment takes place.

<u>Case (iii):</u> For $s+1 \leqslant n \leqslant S-j$

$$K((n,j),t) = \int_{z=o}^{y} \int_{v=z}^{y} \int_{x=v}^{t} \int_{w=v}^{x} \int_{y=v}^{x} \sum_{c=1}^{j-1} \sum_{r=\max\{1,[\frac{c}{a}]\}}^{c} g^{*r}(z)$$

$$p_c^{(r)} g(v-z)\ \hat{p}_{j-c}\ m(w-v)\ h(x-w)\ \frac{F(x)}{1-F(w)}$$

$$A(y) \sum_{b=\max\{1,[\frac{S-j-n}{a}]\}}^{S-j-n} \frac{G^{*b}(t-y)-G^{*(b+1)}(t-y)}{1-G(x-y)}\ p_{S-j-n}^{(b)}$$

$$dy\ dw\ dx\ dv\ dz$$

$$+ \int_{z=o}^{t} \int_{v=o}^{t} \sum_{c=o}^{j-1} \sum_{r=\max\{o,[\frac{c}{a}]\}}^{c} g^{*r}(z)\ p_c^{(r)} f(v)$$

$$\frac{(G^{*b}(t-z)-G^{*(b+1)}(t-z))}{1-G(v-z)}\ p_{S-c-n}^{(b)}\ dv\ dz$$

In the above the first term on the right hand side represents the situation where there is a dry period and the second term is for the case with no dry period. The way in which they are arrived at is on the same lines as that used for arriving at (1) and (2) of Section 4.3.

<u>Case (iv)</u>: For n satisfying the condition $n=j=s-a+1, s-a+2, \ldots, s-1, s$.

$$K((j,j),t) = (1-G(t))(1-F(t))$$

<u>Case (v)</u>: For $0<n<j$

$$K((n,j),t) = (1-F(t)) \sum_{b=\max\{1,[\frac{j-n}{a}]\}}^{j-n} (G^{*b}(t)-G^{*(b+1)}(t))\, p_{j-n}^{(b)}$$

and finally,

<u>Case (vi)</u>: For $n=0$, we get

$$K((o,j),t) = (1-F(t)) \sum_{b=\max\{1,[\frac{j}{a}]\}}^{\infty} (G^{*b}(t)-G^{*(b+1)}(t))\, p_j^{(b)}$$

In the expression for $P((o,j),t)$ we allow b to take arbitrary large values. This only means that demands take place even when inventory is dry and hence they are not met.

4.5. <u>System Reliability</u>

System reliability at time t denoted by $R_1(t)$, is defined as the probability that the system is working at time t. In this case $R_1(t)$ is the probability that the server is available and hence inventory level is larger than zero.

$$R_1(t) = \Pr\{Z(t) \neq (o,j)\}$$

The event $\{Z(t) \neq (o,j)\}$ can happen in three mutually exclusive ways as follows:

(i) Last order is placed in (u,u+du]; but no replenishment until t and no dry period until time t.

(ii) Last order is placed in (u,u+du]; replenishment takes place before inventory level becomes zero

(iii) Last order is placed in (u,u+du]; replenishment takes place during a dry period; the server returns after vacation before time t.

Therefore

$$R_1(t) = \sum_{i \in B} \int_0^t R(i,j,du) \Big\{ (1-F(t-u)) \int_{v=u}^{t} \sum_{b=o}^{j-1} \sum_{r=[\frac{b}{a}]}^{b} g^{*r}(v-u) p_b^{(r)} (1-G(t-v))\, dv \Big]$$

$$+\Big\{ \int_{v=u}^{t} \sum_{b=o}^{j-1} \sum_{r=[\frac{b}{a}]}^{b} g^{*r}(v-u)\ p_b^{(r)}\ f(v-u)dv \Big\}$$

$$+\Big\{ \int_{v=u}^{t} \int_{z=v}^{t} \int_{w=z}^{x} \int_{x=w}^{t} \int_{y=z}^{x} \sum_{b=1}^{j-1} \sum_{r=\max\{1,[\frac{b}{a}]\}}^{b} g^{*}(v-u) p_b^{(r)} g(z-v)$$

$$\hat{p}_{j-b}\ m(w-z)\ h(x-w)\ A(y-z)\ \left(\frac{F(x)-F(u)}{1-F(w)} \right) dx\ dy\ dw\ dz\ dv \Big\} \Big\}$$

Chapter-5

FINITE CAPACITY QUEUEING SYSTEMS WITH STATE DEPENDENT NUMBER OF STAGES OF SERVICE

5.1. Introduction

This chapter introduces a class of finite capacity single server queueing models in which the server offers a random number of stages of service say k, k=1,2,...,m to each unit depending upon the system size at the onset of its service. First we examine Model I in which the arrival pattern follows Poisson process of rate λ. The distribution of service times in all stages are independent and identically distributed (i.i.d) random variables following distribution function G(.) and probability density function g(.) with mean μ_1(assumed to be finite). Thus the system under consideration generalises the truncated (truncated at m) bilateral Phase-type distribution (see Shanthikumar 1985) in which arrival takes place according a Poisson process.

In Model II, the interarrival times follow a general distribution F(.) with probability density function f(.) and mean a (finite). Service times in each stage are i.i.d. random variables following exponential distribution with parameter μ(fini

Consider a service facility manned by a single server and providing m different types of services. The services are given one by one by the same server with each unit demanding m stages of service. The system is of finite capacity b so that customers arriving when the system is full are lost to the system. However, each unit at the commencement of its service is offered a random number of stages of service (at least equal to one and atmost m) depending upon the number of units in the system at that epoch. Specifically, we assume that the number k of stages of service offered when the number of units in the system at the commencement of service is j has probability q_{jk}, $j=1,2,...,b$, $k=1,2,...,m$. We encounter such a situation as modelled in this chapter in workshops where machines of identical nature are brought for overhauling.

In the queueing literature one can find that the service characteristics change dynamically to accommodate variations in the queue size. Hillier et. al. (1964), Gupta (1967) and Rosenshine (1967) have examined queueing systems in which the service rates are an instantaneous functions of the system state Shantikumar (1979) discusses a class of queueing models in which the service time of a customer at a single server facility is dependent on the queue size at the onset of its service. Bertsimas (1990) analyses the $C_k/C_m/s$ system where C_k is the class of Coxian probability density functions of order k.

Whereas he does not assume the state-dependent number of stages of service we do make this assumption and further assume that the service time in each stage has a general distribution in Model I. Further service times in all stages are i.i.d random variables independent of the system size.

The following are the notations that are used in the sequel.

Let $\Lambda_j(x)$ denote the probability that j arrivals take place in an interval of duration x.

When interarrival times follow exponential distribution with parameter λ, we have

$$\Lambda_j(x) = e^{-\lambda x} (\lambda x)^j/j!, \quad j = 0,1,2,\ldots,b-1$$

and let $\bar{\Lambda}_b(x)$ be defined as

$$\bar{\Lambda}_b(x) = \sum_{j \geq b} e^{-\lambda x}(\lambda x)^j/j!$$

E denotes $\{(o,o,o)\} \cup \{(i,j,k): i \in (1,2,\ldots,b);\ j \in (1,2,\ldots,m);\ k \in (0,1,2,\ldots,j-1)\}$

$X(t)$ - the system size at time t

$Y(t)$ - the number of stages of service offered at the onset of service to the unit undergoing service at time t.

$Z(t)$ - the number of stages of service completed by the unit undergoing service at time t.

The system is in (i,j,k) means $X(t)=i$, $Y(t)=j$, $Z(t)=k$, $(i,j,k) \in E$ when $X(t)=0$, then $Y(t)=0$ and $Z(t)=0$.

$*$ - convolution

N - the set of natural numbers

N^o - $\{0\} \cup N$

$\gamma_{\mu,r}(u)$ - is the gamma density with parameters μ and r, $\mu > 0$ and $r = 1,2,\ldots$

Analysis, system size probabilities and the limiting distribution of Model I are provided in Section 5.2. Section 5.3 deals with the analysis, system size probabilities and the limiting distribution of Model II. The last section illustrates computational problem associated with the models.

5.2. <u>Model I</u>

<u>Analysis</u>

Let $0 = T_o < T_1 < T_2 < \ldots < T_n \ldots$ be the successive stage service completion epochs, $X_o, X_1, X_2, \ldots, X_n, \ldots$ be the number of units in the system just after $T_o, T_1, T_2, \ldots, T_n, \ldots$. Let $Y_o, Y_1, Y_2, \ldots$ and $Z_o, Z_1, Z_2, \ldots$ respectively denote the number of stages of service offered and the number of stages of service completed by the unit undergoing service at time T_n, $n=0,1,2,\ldots$ Then the process $\{(X,Y,Z),T\} = \{(X_n, Y_n, Z_n), T_n; n \in N^o\}$ is easily seen to constitute a Markov renewal process defined over the state space E.

Consider the epochs of successive stage completion (not service completion) of a unit. Define

$$Q\,\{(i_1,j,k_1),(i_2,j,k_2=k_1+1),t\}$$

$$= \Pr\{(X_{n+1}=i_2,\ Y_{n+1}=j,\ Z_{n+1}=k_2),\ T_{n+1}-T_n \leq t \mid (X_n=i_1,\ Y_n=j,\ Z_n=k_1)\}$$

$$= \begin{cases} \int_0^t \wedge_{i_2-i_1}(u)\, g(u)du, & \text{for } i_1=1,2,\ldots,b-1,\ i_2=i_1+r_1 \\ & \text{where } r_1=0,1,2,\ldots,b-i_1;\ j=1,2,\ldots,m \text{ and } k_1=0,1,2,\ldots,j-2 \\ \int_0^t \bar{\wedge}_{b-i_1}(u)\, g(u)du, & \text{for } i_1=1,2,\ldots,b;\ i_2=b \\ & j=1,2,\ldots,m,\ k_1=0,1,2,\ldots,j-2 \end{cases}$$

In the case when the service commencement epoch of a stage is such that it is the last stage for the unit undergoing service, then

$$Q\{(i_1,j_1,j_1-1),(i_2,j_2,0),t\}= Pr\{(X_{n+1}=i_2,\ Y_{n+1}=j_2,\ Z_{n+1}=0 \\ T_{n+1}-T_n \leqslant t \mid (X_n=i_1, Y_n=j_1, Z_n=j_1-1)\}$$

$$= \begin{cases} \int_0^t \Lambda_{i_2-i_1+1}(u)\ q_{i_2 j_2}\ du, & \text{for } i_1=1,2,\ldots,b-1, \\ & i_2=i_1+r_1-1 \\ \quad \text{where } r_1=0,1,2,\ldots,b-i_1 & \\ & \\ \int_0^t \overline{\Lambda}_{b-i_1+1}(u)\ g(u)\ q_{i_2 j_2} du, & \text{for } i_1=1,2,\ldots,b; \\ & i_2=b,\ j_1,j_2=1,2,\ldots,m \end{cases}$$

with the provision that if $i_1=1$ and $i_2=0$, then

$$Q\{(1,j_1,j_1-1),(0,0,0),t\} = \int_0^t e^{-\lambda u}\ g(u)du$$

For all $n \in N$ define

$$Q^n\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}= Pr[(X_n,Y_n,Z_n)=(i_2,j_2,k_2);T_n \leqslant t \\ (X_o,Y_o,Z_o)=(i_1,j_1,k_1)\}$$

Then we have the recursive relation

$$Q^{n+1}\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \sum_{i=i_1-1}^{b} \sum_{j=o}^{m} \sum_{k=o,k_1+1}$$

$$\int_o^t Q\{(i_1,j_1,k_1),(i,j,k),du\}\ Q^n\{(i,j,k),(i_2,j_2,k_2),t-u\}$$

Finally define the Markov renewal function

$$R\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \sum_{n=o}^{\infty} Q^{*n}\{(i_1,j_1,k_1)(i_2,j_2,k_2),t\},\ t\geq 0$$

$$\text{and } (i_1,j_1,k_1),(i_2,j_2,k_2) \in E$$

$$\text{with } Q^o\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \begin{cases} 1 \text{ for } (i_1,j_1,k_1)=(i_2,j_2,k_2) \\ 0 \text{ for } (i_1,j_1,k_1)\neq(i_2,j_2,k_2) \end{cases}$$

System size probabilities

Without loss of generality we may assume that at time $T_o=0$ a stage service completion has just taken place so that the state of the system is $(X_o,Y_o,Z_o) = (i_o,j_o,k_o)$ (assumed fixed). Consider the three dimensional process $L(t)=\{X(t),Y(t),Z(t)\}$. Then the process $\{L(t),\ t\geq 0\}$ is the associated semi-regenerative process with the Markov renewal process $\{(X,Y,Z),T\}$ embedded in it.

Define

$$P_{(i_o,j_o,k_o)}((i,j,k),t) = \Pr\left\{L(t)=(i,j,k)\,|\,L(o)=(i_o,j_o,k_o)\right\}$$

Then $P_{(i_o,j_o,k_o)}((i,j,k),t) = \Pr\{L(t)=(i,j,k);T_1>t\,|\,L(o)=(i_o,j_o,k_o)\}$

$$\Pr\{L(t)=(i,j,k);T_1\leqslant t\,|\,L(o)=(i_o,j_o,k_o)\}$$

Let $\Pr\{L(t) = (i,j,k);\ T_1 > t\,|\ L(o) = (i_o,j_o,k_o)\}$ represents $K_{(i_o,j_o,k_o)}((i,j,k),t)$. Then

$$K_{(i_o,j_o,k_o)}((i,j,k),t) = K_{(i_o,j_o,k_o)}((i,j_o,k_o),t)$$

$$= \begin{cases} \Lambda_{i-i_o}(t)\ [1-G(t)], & i \neq b \\ \bar{\Lambda}_{b-i_o}(t)\ [1-G(t)], & i=b \end{cases}$$

Now $K_{(.)}(.,t)$ is bounded over finite intervals and directly Riemann integrable. So $P_{(i_o,j_o,k_o)}((i,j,k),t)$ satisfies Markov renewal equation (Cinlar 1975 a).

Hence the solutions are given by

(i) for $(i,j,k) \neq (o,o,o)$,

$$P_{(i_o,j_o,k_o)}((i,j,k),t) = \int_o^t \sum_{i_1 \leq i} R\{(i_o,j_o,k_o),(i_1,j,k),du\}$$

$$K_{(i_1,j,k)}((i,j,k),t-u)$$

(ii) Probability that the system size is zero at time t is

$$P_{(i_o,j_o,k_o)}((o,o,o),t) = \int_o^t R\{(i_o,j_o,k_o),(o,o,o),du\} \wedge_o(t-u)$$

Limiting Behaviour

We start with a given set of q_{ij}'s, $i=1,2,\ldots,b-1$; $j=1,2,\ldots,m$. From the given q_{ij}'s, the one-step transition probabilities corresponding to the Markov chain $\{(X,Y,Z)\} = \{(X_n,Y_n,Z_n),\ n \in N^o\}$ is evaluated as follows.

Define

$$p_{((i_1,j_1,k_1),(i_2,j_2,k_2))} = Pr\{(X_{n+1}, Y_{n+1}, Z_{n+1})=(i_2,j_2,k_2) \mid (X_n,Y_n,Z_n) = (i_1,j_1,k_1)\}$$

For $i_1=0$ and $i_2=0$

$$p_{((i_1,j_1,k_1),(i_2,j_2,k_2))} = \int_o^\infty \lambda e^{-\lambda u} \int_u^\infty q_{11}\ g(v-u) \wedge_o(v-u)dv\ du$$

For $i_1=o$, $i_2=1$, $j_2=1,2,\ldots,m$ and $k_2=0$

- G4979 -

$$p_{((i_1,j_1,k_1),(i_2,j_2,k_2))} = \int_o^\infty \lambda e^{-\lambda u} \int_u^\infty q_{11}\; g(v-u)\Lambda_1(v-u) q_{1j_2} dv\; du$$

Likewise the various transition probabilities can be computed for different values of $(i_1,j_1,k_1),(i_2,j_2,k_2) \in E$. From these the stationary probabilities $\pi(i,j,k),(i,j,k) \in E$ associated with the Markov chain $\{(X,Y,Z)\}$ are computed from the solution of the equations

$$\pi(i,j,k) = \sum_{i_1=o}^{b} \sum_{j_1=o}^{m} \sum_{k_1=o}^{j_1-1} \pi(i_1,j_1,k_1)\; p_{((i_1,j_1,k_1),(i,j,k))}$$

$$\text{with} \quad \sum_{i=o}^{b} \sum_{j=o}^{m} \sum_{k=o}^{j-1} \pi(i,j,k)=1$$

The mean sojourn time in any state (i,j,k) is $m(i,j,k)$ and

$$\begin{aligned} m(i,j,k) &= E[T_{n+1}-T_n \leqslant t \mid (X_n=i,Y_n=j,Z_n=k)] \\ &= \mu_1 \text{ (assumed finite)} \end{aligned}$$

Define

$$\int_o^\infty K_{(i_1,j,k)}((i,j,k),t)dt \text{ as } n((i_1,j,k),(i,j,k))$$

Let $\lim\limits_{t \to \infty} P_{(i_o,j_o,k_o)}((i,j,k),t) = \underline{P}(i,j,k)$

Then $\underline{P}(i,j,k)$ is obtained as

$$\underline{P}(i,j,k) = \frac{\sum\limits_{i_1 \leq i} \pi(i_1,j,k)\ n((i_1,j,k),(i,j,k)}{\sum\limits_{i_1=o}^{b} \sum\limits_{j_1=o}^{m} \sum\limits_{k_1=o}^{j_1-1} \pi(i_1,j_1,k_1)\ m(i_1,j_1,k_1)} \qquad \text{(Cinlar 1975b)}$$

Hence for $i \neq 0$,

$$\underline{P}(i,j,k) = (\mu_1)^{-1} \sum_{i_1 \leq i} \pi(i_1,j,k)\ n((i_1,j,k)(i,j,k))$$

and for $i=0$

$$\underline{P}(o,o,o) = (\mu_1)^{-1}\ \pi(o,o,o) \int_o^\infty \wedge_o(t)dt$$

5.3. <u>Model-II</u>

<u>Analysis</u>

Let $0 = T_o, T_1, T_2, \ldots$ be the successive arrival instants; $X_o, X_1, X_2, \ldots$ be the system size just prior to $T_o, T_1, T_2, \ldots$ and Y_i, Z_i, $i=0,1,2,\ldots$ are as defined in Model I. Then the process $\{(X,Y,Z),T\} = \{(X_n,Y_n,Z_n);T_n\ n \in N^o\}$ constitute a Markov renewal process defined over the state space E with

$$Q_1\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \Pr\{(X_{n+1}=i_2,Y_{n+1}=j_2,Z_{n+1}=k_2);$$
$$T_{n+1}-T_n\leqslant t\,|\,(X_n=i_1,Y_n=j_1,Z_n=k_1)\}$$

$$= \int_o^t \sum_{\ell_2+\ell_3+\ldots+k_2=i_1-i_2}^{(i_1-i_2)m} \gamma_{\mu,j_1-k_1+\ell_2+\ell_3+\ldots+k_2}(u)$$

$$q_{i_1\ell_2}\, q_{i_1-1\ell_3}\cdots q_{i_2j_2}\, e^{-\mu(t-u)}\, dF(u)$$

for $t\geqslant 0$ and $(i_1,j_1,k_1),(i_2,j_2,k_2) \neq (o,o,o)$

and

$$Q_1\{(o,o,o),(o,o,o),t\} = \sum_{j_1=1}^{m} \int_o^t dF(u)\, q_{1j_1}\, \gamma_{\mu,j_1}(t-u)$$

The Markov renewal function is given by

$$R_1\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \sum_{n=o}^{\infty} Q^{*n}\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\},\ t\geqslant$$

and $(i_1,j_1,k_1),(i_2,j_2,k_2) \in E.$

System Size Probabilities

Initially at time $T_o=0$, we assume that an arrival is just taking place so that the state of the system is (i_o,j_o,k_o) (fixed) As before defining $L(t) = \{(X(t),Y(t),Z(t))\}$, the stochastic process $\{L(t),t\geqslant 0\}$ is the semi-regenerative process defined over the state space E. In this case also it is readily seen that

$$P^{(1)}_{(i_o,j_o,k_o)}((i,j,k),t) = Pr\{L(t)=(i,j,k)|L(o)=(i_o,j_o,k_o)\}$$

satisfies the Markov renewal equation so that the solution is given by

(i) for $(i,j,k) \neq (o,o,o)$

$$P^{(1)}_{(i_o,j_o,k_o)}((i,j,k),t) = \int_o^t \sum_{i_1 \geq i} \sum_{j_1=1}^{m} \sum_{k_1=o}^{j_1-1} R_1\{(i_o,j_o,k_o),$$

$$(i_1,j_1,k_1),du\} \, K^{(1)}_{(i_1,j_1,k_1)}((i,j,k),t-u$$

where

$$K^{(1)}_{(i_1,j_1,k_1)}((i,j,k),t) = Pr\{L(t)=(i,j,k),T_1>t|L(o)=(i_1,j_1,k_1)\}$$

$$= \begin{cases} [1-F(t)] \displaystyle\int_o^t \sum_{\ell_2+\ell_3+\ldots+k=i-i_1}^{(i-i_1)m} \gamma_{\mu,j_1-k_1+\ell_2+\ell_3+\ldots+k}(u) \\ \qquad e^{-\mu(t-u)} \, du, \text{ for } (i_1,j_1,k_1) \neq (i,j,k) \\ [1-F(t)]e^{-\mu t} \qquad \text{for } (i_1,j_1,k_1)=(i,j,k) \end{cases}$$

(ii) The probability that the system size is zero at time t is given by

$$P^{(1)}_{(i_o,j_o,k_o)}((o,o,o),t) = \int_o^t \sum_{i_1=o}^{b} \sum_{j_1=1}^{m} \sum_{k_1=o}^{j_1-1} R_1\{(i_o,j_o,k_o),(i_1,j_1,k_1),du\} \, K^{(1)}_{(i_1,j_1,k_1)}((o,o,o),$$

where

$$K^{(1)}_{(i_1,j_1,k_1)}((o,o,o),t) = [1-F(t)] \int_o^t \sum_{\ell_2+\ell_3+\ldots+\ell_{i_1}=i_1}^{i_1 m} \gamma_{\mu,j_1-k_1+\ell_2+\ldots+\ell_{i_1}}(u)du$$

Limiting Behaviour

For this model also, given q_{ij}'s, $i=1,2,\ldots,b$ and $j=1,2,\ldots,m$, the one-step transition probabilities corresponding to the Markov chain $\{(X,Y,Z)\}$ are computed as in the case of Model-I. Also the stationary distributions $\pi^{(1)}(i,j,k)$, $(i,j,k) \in E$ are also evaluated as before. The mean sojourn time in any state is

$$m^{(1)}(i,j,k) = \int_o^\infty [1-F(t)]e^{-\mu t}\, dt, \quad i>0 \quad \text{and}$$

$$m^{(1)}(o,o,o) = \int_o^\infty [1-F(t)]dt = a$$

Let $\lim_{t \to \infty} P^{(1)}_{(i_o,j_o,k_o)}((i,j,k),t) = \underline{P}^{(1)}(i,j,k)$.

Then the limiting probabilities are now computed as, for all $(i,j,k) \in E$,

$$\underline{P}^{(1)}(i,j,k) = \frac{1}{m^{(1)}(i,j,k)} \Big\{ \pi^{(1)}(i,j,k)\, n^{(1)}((i,j,k),(i,j,k)) + \sum_{i_1=i+1}^{b} \sum_{j_1=1}^{m} \sum_{k_1=o}^{j_1-1} \pi^{(1)}(i_1,j_1,k_1) n^{(1)}((i_1,j_1,k_1),(i,j,k))$$

where $n^{(1)}(i_1,j_1,k_1),(i_2,j_2,k_2) = \int_o^\infty K^{(1)}_{(i_1,j_1,k_1)}((i_2,j_2,k_2),t)dt$,

for $(i_1,j_1,k_1),(i_2,j_2,k_2) \in E$

5.4. Numerical Illustrations

1. Consider the case when b=2 and m=2. For Model I, let the parameter λ of the arrival process be equal to 1. Also assume that the service time in each stage follows exponential distribution with parameter μ=1. Let the probabilities determining the number of stages be q_{11} = .3 and q_{12} = .7.

From the given q_{ij}'s, the transition probability matrix $\mathbb{P}$ associated with the Markov chain $\{(X,Y,Z)\}$ defined on the state space $\{(0,0,0), (1,1,0), (1,2,0), (1,2,1), (2,2,1)\}$ is computed as

$$\mathbb{P} = \begin{bmatrix} 0.15 & 0.045 & 0.105 & 0.35 & 0.35 \\ 0.5 & 0.15 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.15 & 0.35 & 0 & 0 \\ 0 & 0.3 & 0.7 & 0 & 0 \end{bmatrix}$$

The stationary probabilities are obtained as

$$\pi = \{\pi(0,0,0) = 0.109,\ \pi(1,1,0) = 0.129,\ \pi(1,2,0) = .232,$$
$$\pi(1,2,1) = 0.265,\ \pi(2,2,1) = 0.265\}$$

The limiting probabilities are given by

$$\underline{P}(0,0,0) = 0.109$$
$$\underline{P}(1,1,0) = 0.1707$$
$$\underline{P}(1,2,0) = 0.1923$$
$$\underline{P}(1,2,1) = 0.1325$$
$$\underline{P}(2,2,1) = 0.3975$$

2. Considering Model II let the given set of probabilities q_{ij}, $i=1,2,3$ be $q_{11}=0.1$, $q_{12}=0.2$; $q_{21}=0.2$; $q_{22}=0.1$; $q_{31}=0.2$; $q_{32}=0.2$ for $b=3$ and the maximum number of stages offered $m=2$.

For simplicity assume that the interarrival times follow exponential distribution with parameter $\lambda=1$ and the service rate μ in each stage be also equal to 1. The transition probability matrix associated with the Markov chain $\{(X,Y,Z)\}$ defined over the state space $\{(0,0,0),\ (1,1,0),\ (1,2,0),\ (1,2,1),\ (2,1,0),\ (2,2,0),\ (2,2,1),\ (3,1,0),\ (3,2,0),\ (3,2,1)\}$ computed from the given q_{ij}'s is as follows.

$$P^{(1)} = \begin{bmatrix}
.5 & .1 & .2 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\
.25 & .05 & .1 & .1 & .5 & 0 & 0 & 0 & 0 & 0 \\
.05 & .025 & .05 & .05 & 0 & .5 & .325 & 0 & 0 & 0 \\
.25 & .05 & .1 & .1 & 0 & 0 & .5 & 0 & 0 & 0 \\
.2375 & .0125 & .025 & .025 & .1 & .05 & .05 & .5 & 0 & 0 \\
.0125 & .0125 & .025 & .025 & .05 & .025 & .025 & 0 & .5 & .325 \\
.2375 & .0125 & .025 & .025 & .1 & .05 & .05 & 0 & 0 & .5 \\
.475 & .025 & .05 & .05 & .2 & .1 & .1 & 0 & 0 & 0 \\
.025 & .0125 & .025 & .025 & .1 & .05 & .05 & 0 & 0 & .7125 \\
.475 & .025 & .05 & .05 & .2 & .1 & .1 & 0 & 0 & 0
\end{bmatrix}$$

The stationary distributions are given by

$\pi^{(1)}=\{\pi^{(1)}(0,0,0)=0.32,\ \pi^{(1)}(1,1,0)=.056,\ \pi^{(1)}(1,2,0)=0.11,$
$\pi^{(1)}(1,2,1)=0.13,\ \pi^{(1)}(2,1,0) = .009,\ \pi^{(1)}(2,2,0)=0.08,$
$\pi^{(1)}(2,2,1)=0.13,\ \pi^{(1)}(3,1,0)=0.004,\ \pi^{(1)}(3,2,0)=0.04,\ \pi^{(1)}(3,2,1)=$

The mean sojourn time in any state

$$m^{(1)}(i,j,k) = \frac{1}{\lambda+\mu}\ , \quad i > 0 \quad \text{and}$$

$$m^{(1)}(0,0,0) = \frac{1}{\lambda}\ , \quad i = 0$$

The limiting probabilities are computed and tabulated as follows:

$n^{(1)}((i_1,j_1,k_1)(i,j,k))$			$\underline{P}^{(1)}(i,j,k)$
$n^{(1)}((0,0,0),\ (0,0,0))$	=	1	$\underline{P}(0,0,0) = 0.32$
$n^{(1)}((1,1,0),\ (1,1,0))$	=	0.5	
$n^{(1)}((2,1,0),\ (1,1,0))$	=	0.009	
$n^{(1)}((2,2,0),\ (1,1,0))$	=	0.0125	
$n^{(1)}((2,2,1),\ (1,1,0))$	=	0.05	$\underline{P}^{(1)}(1,1,0) = 0.06$
$n^{(1)}((3,1,0),\ (1,1,0))$	=	0.003	
$n^{(1)}((3,2,0),\ (1,1,0))$	=	0.002	
$n^{(1)}((3,2,1),\ (1,1,0))$	=	0.003	
$n^{(1)}((1,2,0),\ (1,2,0))$	=	0.5	
$n^{(1)}((2,1,0),\ (1,2,0))$	=	0.1	
$n^{(1)}((2,2,0),\ (1,2,0))$	=	0.025	
$n^{(1)}((2,2,1),\ (1,2,0))$	=	0.1	$\underline{P}^{(1)}(1,2,0) = 0.14$
$n^{(1)}((3,1,0),\ (1,2,0))$	=	0.006	
$n^{(1)}((3,2,0),\ (1,2,0))$	=	0.003	
$n^{(1)}((3,2,1),\ (1,2,0))$	=	0.006	

$n^{(1)}((1,2,1)(1,2,1))$	$= 0.5$	
$n^{(1)}((2,1,0)(1,2,1))$	$= 0.0125$	
$n^{(1)}((2,2,0)(1,2,1))$	$= 0.0125$	
$n^{(1)}((2,2,1)(1,2,1))$	$= 0.025$	$\underline{P}(1,2,1) = 0.13$
$n^{(1)}((3,1,0)(1,2,1))$	$= 0.003$	
$n^{(1)}((3,2,0)(1,2,1))$	$= 0.001$	
$n^{(1)}((3,2,1)(1,2,1))$	$= 0.003$	
$n^{(1)}((2,1,0)(2,1,0))$	$= 0.5$	
$n^{(1)}((3,1,0)(2,1,0))$	$= 0.1$	$\underline{P}(2,1,0) = .02$
$n^{(1)}((3,2,0)(2,1,0))$	$= 0.025$	
$n^{(1)}((3,2,1)(2,1,0))$	$= 0.1$	
$n^{(1)}((2,2,0)(2,2,0))$	$= 0.5$	
$n^{(1)}((3,1,0)(2,2,0))$	$= 0.05$	$\underline{P}(2,2,0) = 0.08$
$n^{(1)}((3,2,0)(2,2,0))$	$= 0.0125$	
$n^{(1)}((3,2,1)(2,2,0))$	$= 0.05$	
$n^{(1)}((2,2,1)(2,2,1))$	$= 0.5$	
$n^{(1)}((3,1,0)(2,2,1))$	$= 0.0125$	$\underline{P}(2,2,1) = 0.12$
$n^{(1)}((3,2,0)(2,2,1))$	$= 0.0125$	
$n^{(1)}((3,2,1)(2,2,1))$	$= 0.006$	
$n^{(1)}((3,1,0)\ (3,1,0))$	$= 0.5$	$\underline{P}(3,1,0) = 0.004$
$n^{(1)}((3,2,0)(3,2,0))$	$= 0.5$	$\underline{P}(3,2,0) = 0.03$
$n^{(1)}((3,2,1)(3,2,1))$	$= 0.5$	$\underline{P}(3,2,1) = 0.11$

Chapter-6

A BULK SERVICE QUEUE WITH SERVICE TIME DEPENDENT BATCH SIZE

6.1. Introduction

The present chapter generalises the $M/G^{a,b}/1$ queueing system with finite capacity. It is natural to expect a dependence structure between the batch size to be served and the time spent on serving the previous batch that has just departed in various situations of business transactions.

Bulk service queues with infinite waiting room (W.R) capacity is investigated by Bhat (1967) and he obtains the busy period and the busy cycle distributions of the queue length process in a GI/M/1 queue with service in groups of random size. Easton and Chaudhary (1982) analyse bulk service queueing systems with Erlang input. Manoharan (1990) discusses a bulk service queueing system with Erlang input and server vacation and obtains the time dependent behaviour of the system. However, these works do not introduce any dependence structure in the basic process. Shanthikumar (1979) deals an M/G/1 queue with two types of service time distributions depending on the system size and obtains the Laplace transform of the waiting time distribution. Krishnamoorthy and Rajan Varughese (1990 a,b) introduce Markov dependence on the sizes of batches to be served in succession in

$M/G^{a,b}/1$ and $G/M^{a,b}/1$ queues with finite capacity. They obtain the limiting distributions of the system size. Certain measures of effectiveness are also discussed by them.

Section 6.2 sketches a complete description of the two models under consideration. Notations used in the chapter are explained at the end of the same section. Analysis of the Models are carried out in Section 6.3. Section 6.4 deals with the system size probabilities and Section 6.5 the steady state analysis of the Models. Distribution of the busy period and the busy cycle are studied in Section 6.6. Virtual waiting time distribution is derived in Section 6.7. A control problem associated with Model II is discussed in the last section.

6.2. Description of the models

An $M/G^{a,b}/1$ queueing system with a waiting room (W.R) of finite capacity c is discussed. The arrival process follows a Poisson process of rate λ. There is a service station (S.S) whose capacity is b. We assume that $c \geq b$. Services are in batches whose sizes vary between a and b ($a<b$). The size of the next batch to be served is determined on the basis of the time spent on serving the batch that has just departed. Let $b_k(t)$ be the probability that the size of the next batch to be served is k given that the time taken to serve the present batch is t, $t>0$ and $k=a,a+1,\ldots,b$.

Two models are discussed here.

In Model I if at a departure epoch the number waiting for service is at least a and at most equal to the number say $k(a \leqslant k \leqslant b)$ determined by the specified rule, then all of them are transferred to the S.S. If the number waiting is larger than k, then the first k among the waiting units are taken for service. Finally, if the number waiting is less than a the server remains idle until there are a in the W.R.

In Model II, if at a departure epoch only less than the batch size k determined by the rule is available in the W.R., then the server remains idle until there are k in the W.R. If the number waiting is larger than k, the first k alone are admitted to the S.S. In both the models service times of batches are independent and identically distributed random variables, independent of the batch size (this is assumed for simplicity) following distribution function G(.) which is assumed to be absolutely continuous and let $g(t)dt = dG(t)$.

Let $\mu = \int_0^\infty t\, g(t)dt$ (assumed finite)

The models described above find application in production process, storage theory and so on.

Notations

* denotes convolution. For eg. $(F*G)(t) = \int_{-\infty}^{\infty} F(t)\, dG(t-u)$

$\Lambda_i(x)$ denotes the probability that i arrivals take place in an interval of duration x.

Thus $\Lambda_i(x) = [e^{-\lambda x}(\lambda x)^i]/i!, \; i = 0,1,2,\ldots,c$

and $\bar{\Lambda}_c(x) = \sum_{i \geqslant c}^{\infty} [e^{-\lambda x}(\lambda x)^i]/ i!$

γ denotes the Gamma density where

$$\gamma_{\lambda,k}(x) = [e^{-\lambda x}\lambda^k x^{k-1}]/(k-1)!, \; k=1,2,\ldots; \lambda>0$$

$$\Gamma_{\lambda,k}(x) = \int_0^x \gamma_{\lambda,k}(u)du, \; k=1,2,\ldots; \lambda>0$$

N^0 is the set $\{0,1,2,3,\ldots\}$

$$x^+ = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

$b_k(t)=$ The probability that the size of the batch to be served next equal to k given that the batch that has just been served out was served an amount of time t; t>0; k=a, a+1,...,b.

Obviously $\sum_{k=a}^{b} b_k(t) = 1$

$X(t)$ - Number of units in the W.R at time t.

$$Y(t) = \begin{cases} 0 \text{ if the server is idle at time } t \\ 1 \text{ if the server is busy at time } t \end{cases}$$

$$Z(t) = \begin{cases} \text{Number of units in the S.S at time } t \text{ if } Y(t)=1 \\ \text{Number to be served as determined by the rule} \\ \text{if } Y(t)=0 \end{cases}$$

By saying that the system is in state (i,j,k) at time t, we mean $X(t)=i$, $Y(t)=j$, $Z(t)=k$.

6.3. Analysis

Model-1

Let $0 = T_o < T_1 < T_2 < \ldots T_n < \ldots$ be the successive departure epochs of the initial $1,2,\ldots,n,\ldots$ batches. The number of units in the W.R at $0+$, T_1+, $T_2+ \ldots$, $T_n+,\ldots$ be specified by $X_o, X_1, X_2, \ldots, X_n, \ldots$. Suppose $Z_o, Z_1, Z_2, \ldots, Z_n$, ... are the number of units in the S.S. | the number to be served next as determined by the specified rule at these epochs if service cannot start at this point of time. We introduce another sequence $Y_o, Y_1, \ldots, Y_n, \ldots$, where $Y_n=1$ if a service can start immediately after the departure of the previous batch and $Y_n=0$ otherwise. Thus in Model I, for example, if at the n^{th} departure epoch at least a are in the W.R. then Z_n denotes the number transferred to the S.S. and so $Y_n=1$. If the number waiting at the n^{th} departure epoch is less than a then Z_n denotes the number to be served in the next batch as determined by the specified rule and then $Y_n=0$. More precisely if Y_n takes the value 1 at time T_n+, then X_n and Z_n respectively represent the number of units in the W.R. and that in the S.S. at T_n+. Then we have

Theorem-1

The process $\{(X,Y,Z),T\} = \{(X_n,Y_n,Z_n),\ T_n;\ n \in N^o\}$ forms a Markov renewal process defined over the state space

$$E = \{(i,j,k)\,|\,i=0,1,2,\ldots,a-1;\ j=0;\ k=a,a+1,\ldots,b\} \cup \{(i,j,k)\,|\ i=0,1,2,\ldots,c;\ j=1;\ k=a,a+1,\ldots,b\}$$

with semi-Markov kernel as described below.

Define,

$$Q\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = Pr\{(X_{n+1}=i_2,Y_{n+1}=j_2,Z_{n+1}=k_2);\ T_{n+1}-T_n \leqslant t\,|\,(X_n=i_1,Y_n=j_1,Z_n=k_1)\}$$

Then (i) for $j_1=0$; $i_1=0,1,2,\ldots,a-1$; $j_2=1$, $i_2=0,1,2,\ldots,c-k_2$; $a \leqslant k_1,\ k_2 \leqslant b$,

$$Q\{(i_1,o,k_1),(i_2,1,k_2),t\} = \int_o^t \int_u^t \gamma_{\lambda,a-i_1}(u) \Lambda_{i_2+k_2}(v-u)\ g(v-u)\ b_{k_2}(v-u)dv\ du$$

(ii) For $0 \leq i_1 < a$, $j_1=0$, $i_2=0$ and $j_2=1$, $a \leqslant k_1, k_2 \leqslant b$

$$Q\{(i_1,o,k_1),(0,1,k_2),t\} = \int_o^t \int_u^t \sum_{\ell=a}^{k_2} \gamma_{\lambda,a-i_1}(u) \wedge_\ell(v-u)\, g(v-u) \sum_{k=k_2}^{b} b_k(v-u)\,dv\,du$$

(iii) For $i_1=i_2=0$, $j_1=j_2=0$, $a \leqslant k_1$, $k_2 \leqslant b$

$$Q\{(o,o,k_1),(o,o,k_2),t\} = \int_o^t \int_u^t \gamma_{\lambda,a}(u)\, e^{-\lambda(v-u)}\, g(v-u)\, b_{k_2}(v-u)\,dv\,du$$

(iv) For $i_1=i_2=0$; $j_1=1$, $j_2=0$, $a \leqslant k_1, k_2 \leqslant b$

$$Q\{(o,1,k_1),(o,o,k_2),t\} = \int_o^t g(u)\, e^{-\lambda u}\, b_{k_2}(u)\,du$$

(v) For $i_1=0,1,2,\ldots,c-k_1$; $j_1=j_2=1$; $a \leqslant k_1, k_2 \leqslant b$

$$Q\{(i_1,1,k_1),(i_2,1,k_2),t\} = \begin{cases} \int_o^t g(u) \wedge_{i_2+k_2-i_1}(u)\, b_{k_2}(u)\,du & \text{when } i_2+k_2 < c-k_2 \\ \int_o^t g(u)\, \bar{\wedge}_{i_2+k_2-i_1}(u)\, b_{k_2}(u)\,du, & \text{when } i_2+k_2=c-k_2 \end{cases}$$

(vi) For $i_1=i_2=0$, $j_1=j_2=1$, $a\leqslant k_1,k_2\leqslant b$

$$Q\{(o,1,k_1),(o,1,k_2),t\} = \int_o^t g(u) \sum_{\ell=a}^{k_2} \Lambda_\ell(u) \sum_{k=k_2}^{b} b_k(u)du$$

(vii) For $i_1=1,2,\ldots,c-k_1$; $i_2=0$, $j_1=j_2=1$; $a\leqslant k_1,k_2\leqslant b$

$$Q\{(i_1,1,k_1),(o,1,k_2),t\} = \int_o^t g(u)\Lambda_{k_2-i_1}(u) \sum_{k=k_2}^{b} b_k(u)du$$

The $Q(.,.,t)$ defined above gives the semi-Markov kernel of the Markov renewal process.

Further define

$$R\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} = \sum_{n=o}^{\infty} Q^{*n}\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}$$

This gives the expected number of visits to state (i_2,j_2,k_2) up to time t starting from (i_1,j_1,k_1) initially.

<u>Model-II</u>

For Model II we continue to use the same notations (X_n,Y_n,Z_n) to represent the state of the system at time T_n, $n=0,1,2,\ldots$ However there is a difference in the meaning now to be given to Z_n if $Y_n=1$, namely that now Z_n stands for the number in the S.S as determined by the specified rule. The state space for this model is given by

$E_1=\{(i,o,k) \mid i=0,1,2,\ldots,k-1;\ k=a,a+1,\ldots,b\} \cup \{(i,1,k) \mid i=0,1,2,\ldots,c;\ k=a,a+1,\ldots,b\}$. Here again the $\{(X_n,Y_n,Z_n),\ T_n;\ n \in N^o\}$ constitutes a Markov renewal process on E_1. But there is a slight difference in the semi-Markov kernel of the process. The semi-Markov kernel in this case is given as follows:

(i) For $i_1=0,1,2,\ldots,k_1-1;\ i_2=0,1,2,\ldots,k_2-1,\ a\leqslant k_1,k_2\leqslant b, j_1=j_2$

$$Q_1\{(i_1,o,k_1),(i_2,o,k_2),t\} = \int_o^t \gamma_{\lambda,k_1-i_1}(u) \int_u^t g(v-u) \Lambda_{i_2}(v-u)\ b_{k_2}(v-u)dv\ du$$

(ii) For $i_1=0,1,2,\ldots,c-k_1;\ i_2=i_1,i_1+1,\ldots,k_2-1;\ j_1=1,\ j_2=0,\ a \leqslant k_1,\ k_2 \leqslant b$.

$$Q_1\{(i_1,1,k_1),\ (i_2,o,k_2),t\} = \int_o^t \Lambda_{i_2-i_1}(u)\ b_{k_2}(u)\ g(u)du$$

(iii) For $i_1=0,1,2,\ldots,c-k_1;\ j_1=j_2=1,\quad a \leqslant k_1,\ k_2 \leqslant b$

$$Q_1\{(i_1,1,k_1),(i_2,1,k_2),t\}=\begin{cases} \int_o^t \Lambda_{i_2+k_2-i_1}(u)b_{k_2}(u)g(u)du, & \text{when } i_2+k_2< c-k_2 \\ \int_o^t \bar{\Lambda}_{i_2+k_2-i_1}(u)\ b_{k_2}(u)g(u)du & \text{when } i_2+k_2=c-k_2 \end{cases}$$

(iv) Finally for $i_1=0,1,2,\ldots,k_1-1$; $i_2=0,1,2,\ldots,c-k_2$;

$j_1=0$, $j_2=1$, $a\leqslant k_1$, $k_2\leqslant b$

$$Q_1\{(i_1,o,k_1),(i_2,1,k_2),t\}=\int_o^t \gamma_{\lambda,k_1-i_1}(u)\int_u^t \Lambda_{i_2+k_2}(v-u)\, b_{k_2}(v-u)\quad g(v-u)dv\, du$$

Define $R_1\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}=$

$$\sum_{n=o}^{\infty} Q_1^{*n}\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\} \text{ with}$$

$$Q_1^{o}\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}=\begin{cases}0 \text{ if } i_1\neq i_2,\ j_1\neq j_2,\ k_1\neq k_2\\ 1 \text{ if } i_1=i_2,\ j_1=j_2,\ k_1=k_2\end{cases}$$

$R_1\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}$ gives the expected number of visits in time t to the state (i_2,j_2,k_2) starting from the state (i_1,j_1,k_1).

The following theorem can be easily proved.

<u>Theorem-2</u>

A necessary and sufficient condition for the Markov chain $\{(X_n,Y_n,Z_n),\ n \in N^o\}$ with state space E_1 to be recurrent is that $1 > b_k(t) > 0$ for all $t > 0$ and $k = a,a+1,\ldots,b$.

We assume that $1>b_k(t) > 0$ for all $t > 0$ and $k = a,a+1,\ldots,b$.

6.4. System size probabilities

In both Models we assume that initially ie. at time $T_o=0$, $X_o=0$, $Y_o=1$ and $Z_o=a$. Consider the three dimensional process $L(t) = \{X(t), Y(t), Z(t)\}$. Then the process $\{L(t), t\geq 0\}$ is a semi-regenerative process with the Markov renewal process $\{(X_n, Y_n, Z_n), T_n, n \in N^o\}$ embedded in it.

Model-1

Let $P_{(o,1,a)}((i,j,k),t) = Pr\{L(t)=(i,j,k) | L(o)=(o,1,a)\}$,
for $i=0,1,2,\ldots,c$;
$j=0,1$; $k=a,a+1,\ldots,b$.

The function $P_{(o,1,a)}((i,j,k),t)$ satisfies the Markov renewal equation (Cinlar 1975a)

For $i=0,1,2,\ldots,c$; $j=1$ and $k=a$, write

$Pr\{L(t)=(i,1,a);\ T_1>t | L(o)=(0,1,a)\}$ as $K_{(o,1,a)}((i,1,a),t)$

The solution can be written as

(i) For $i=0,1,2,\ldots,c$; $j=1$ and $k=a+1, a+2,\ldots,b$

$$P_{(o,1,a)}((i,1,k),t) = \int_o^t \sum_{i_1=o}^{\min\{c-k,i\}} R\{(o,1,a),(i_1,1,k),du\}$$

$$K_{(i_1,1,k)}((i,1,k),t-u)$$

(ii) For $i=0,1,2,\dots,a-1$; $j=0$, $k=a,a+1,\dots,b$

$$P_{(o,1,a)}((i,o,k),t) = \int_o^t \sum_{i_1=o}^{(a-1)-i} R\{(o,1,a),(i_1,o,k),du\} \, K_{(i_1,o,k)}((i,o,k),t-u)$$

(iii) For $i=0,1,2,\dots,c$; $j=1$ and $k=a$, we get

$$P_{(o,1,a)}((i,1,a),t) = \int_o^t \Bigg\{ \sum_{i_1=o}^{\min\{c-a,i\}} [R\{(o,1,a),(i_1,1,a),du\} \, K_{(i_1,1,a)}((i,1,a),t-u)]$$

$$+ \sum_{i_1=o}^{a-1} [R\{(o,1,a),(i_1,o,a),du\} \, K^{(1)}_{(i_1,o,a)}((i,1,a),t-u)] \Bigg\}$$

where

$$K_{(i_1,1,k)}((i,1,k),t) = Pr\{L(t)=(i,1,k),T_1>t \mid L(o)=(i_1,1,k)\}$$

$$= \Lambda_{i-i_1}(t) \; [1-G(t)]$$

Similarly

$$K_{(i_1,o,k)}((i,o,k),t) = \Lambda_{i-i_1}(t)$$

and

$$K^{(1)}_{(i_1,o,a)}((i,1,a),t) = \int_o^t \gamma_{\lambda,a-i_1}(u) \Lambda_{i-i_1}(t-u)[1-G(t-u)]du$$

Model-II

Here also the function $P_{(o,1,a)}((i,j,k),t)$ satisfies the Markov renewal equation . Thus the solution can be written as

(i) $$P_{(o,1,a)}((o,o,k),t) = \int_o^t R\{(o,1,a),(o,o,k),du\} \; K_{(o,o,k)}((o,o,k),t-u)$$

(ii) For $i=0,1,2,\ldots,k-1$; $j=0$, $k=a,a+1,\ldots,b$

$$P_{(o,1,a)}((i,o,k),t) = \int_o^t \sum_{i_1=o}^{i} R\{(o,1,a),(i_1,o,k),du\} \; K_{(i_1,o,k)}((i,o,k),t-u)$$

(iii) For $i=0,1,2,\ldots,c$; $j=1$, $k=a,a+1,\ldots,b$

$$P_{(o,1,a)}((i,1,k),t) = \int_o^t \Bigg\{ \sum_{i_1=o}^{\min\{c-k,i\}} R\{(o,1,a),(i_1,1,k),du\} \; K_{(i_1,1,k)}((i,1,k),t-u) + \sum_{i_1=o}^{k-1} [R\{(o,1,a),(i_1,o,k)du\} \; K^{(1)}_{(i_1,o,k)}((i,1,k),t-u)] \Bigg\}$$

where $K(.,.,t)$ are as defined earlier and are as follows:

(i) $K_{(o,o,k)}((o,o,k),t) \quad = \quad e^{-\lambda t}$

(ii) $K_{(i_1,o,k)}((i,o,k),t) = \Lambda_{i-i_1}(t)$

(iii) $K_{(i_1,1,k)}((i,1,k),t) \quad = \Lambda_{i-i_1}(t)\ [1-G(t)]$

(iv) $K^{(1)}_{(i_1,o,k)}((i,1,k),t) \quad = \quad \int_o^t \gamma_{\lambda,k-i_1}(u)\,\Lambda_i(t-u)$

$[1-G(t-u)]du$

6.5. Steady state analysis

To obtain the limiting distribution of the system size, first we compute the one-step transition probability matrix $\mathbb{P}$ associated with the Markov chain $\{(X_n,Y_n,Z_n),\ n \in N^o\}$. The one-step transition probabilities are obtained as

$$p^{(1)}_{((i_1,j_1,k_1),(i_2,j_2,k_2))} = Pr\{(X_{n+1}=i_2,\ Y_{n+1}=j_2,\ Z_{n+1}=k_2)\,|\,(X_n=i_1,\ Y_n=j_1,\ Z_n=k_1)\}$$

$$= \int_o^\infty Q\{(i_1,j_1,k_1),(i_2,j_2,k_2),t\}\,dt,$$

for (i_1,j_1,k_1) and $(i_2,j_2,k_2) \in E\ (E_1)$ for Model I(Model II).

Using this we can easily compute the stationary distributions

$\pi(i_1,j_1,k_1)$ for $i_1=0,1,2,\ldots,c$; $j_1=0,1$; $a \leqslant k_1 \leqslant b$ where

$$\pi(i,j,k) = \lim_{n \to \infty} \Pr\left\{(X_n=i,\ Y_n=j,\ Z_n=k) \mid (X_o=o, Y_o=1, Z_o=a)\right)$$

Denote $\int_o^\infty K_{(i_1,j_1,k_1)}((i_2,j_1,k_1),t)\,dt$ as $\eta((i_1,j_1,k_1),(i_2,j_1,k_1))$

and

$$\int_o^\infty K^{(1)}_{(i_1,j_1,k_1)}((i_2,j_2,k_2),t)\,dt \text{ as } \eta^{(1)}((i_1,j_1,k_1),(i_2,j_2,k_2))$$

Write

$$\lim_{t \to \infty} P_{(o,1,a)}((i,j,k),t) = q((i,j,k)) \text{ for } (i,j,k) \in E(E_1)$$

<u>Model I</u>

The mean sojourn time in state (i,o,k) is

$$m((i,o,k)) = \int_o^\infty e^{-\lambda t}\,dt = \frac{1}{\lambda}\ ;\ i=0,1,2,\ldots,a-1;\ a \leqslant k \leqslant b$$

and that in state $(i,1,k)$ is

$$m((i,1,k)) = \int_o^\infty [1-G(t)]\,e^{-\lambda t}\,dt,\ i=0,1,2,\ldots,c-k;\ a \leqslant k \leqslant b$$

Hence

(i) for $i=0,1,2,\ldots,a-1$, $k=a,a+1,\ldots,b$

$$q((i,o,k)) = \frac{\sum_{i_1=o}^{a-1} \pi(i_1,o,k)\ \eta((i_1,o,k),(i,o,k))}{[\sum_{k_1=a}^{b} \{ \sum_{i_1=o}^{a-1} \pi(i_1,o,k_1) m((i_1,o,k_1)) + \sum_{i_1=o}^{c-k_1} \pi(i_1,1,k_1) m((i_1,1,k_1))\}}$$

(ii) for $i=0,1,2,\ldots,c$; $k=a,a+1,\ldots,b$

$$q((i,1,k)) = \frac{\sum_{i_1=o}^{a-1} \pi(i_1,o,k)\ \eta^{(1)}((i_1,o,k),(i,1,k)) + \sum_{i_1=o}^{c-k} \pi(i_1,1,k)\eta((i_1,1,k)(i,1}{[\sum_{k_1=a}^{b} \{ \sum_{i_1=o}^{a-1} \pi(i_1,o,k_1)\ m((i_1,o,k_1)) + \sum_{i_1=o}^{c-k_1} \pi(i_1,1,k_1) m((i_1,1,k_1)}$$

Model-II

The mean sojourn time in state (i,o,k) is

$$m((i,o,k)) = \int_0^\infty e^{-\lambda t}\, dt = \frac{1}{\lambda}, \quad i=0,1,2,\ldots,k-1;\ a\leqslant k\leqslant b$$

and that in state $(i,1,k)$ is

$$m((i,1,k)) = \int_0^\infty (1-G(t))e^{-\lambda t}\, dt, \quad i=0,1,2,\ldots,c-k;\ a\leqslant k\leqslant b$$

Hence

(i) for $i=0,1,2,\ldots,k-1$; $k=a,a+1,\ldots,b$

$$q((i,o,k)) = \frac{\sum_{i_1=o}^{k-1} \pi(i_1,o,k)\ \eta((i_1,o,k),(i,o,k))}{[\sum_{k_1=a}^{b} \{ \sum_{i_1=o}^{k-1} \pi(i_1,o,k_1) m(i_1,o,k_1) + \sum_{i_1=o}^{c-k_1} \pi(i_1,1,k_1) m(i_1,1,k_1)\}]}$$

and (ii) for $i=0,1,2,\ldots,c$; $k=a,a+1,\ldots,b$

$$q((i,1,k)) = \frac{\sum_{i_1=1}^{k-1} \pi(i_1,o,k)\eta^{(1)}((i_1,o,k),(i,1,k)) + \sum_{i_1=o}^{c-k} \pi(i_1,1,k)\eta((i_1,1,k),(i,1,k))}{\left[\sum_{k_1=a}^{b} \left\{\sum_{i_1=o}^{k-1} \pi(i_1,o,k_1)m((i_1,o,k_1)) + \sum_{i_1=o}^{c-k_1} \pi(i_1,1,k_1)m((i_1,1,k_1))\right\}\right]}$$

6.6 Distribution of the busy cycle and busy period

A busy period for Model I is the time elapsed since the transfer of a batch of size a to the S.S, thus terminating an idle period, until at a departure epoch the system reaches a state (i,o,j). For Model II, a busy period starts with the number of waiting customers reaching the prescribed level until at a departure epoch the specified number of customers are not found in the W.R.

Let $0 = T_o^{(1)}, T_1^{(1)}, T_2^{(1)}, \ldots$ be the successive busy period termination epochs for Model I/Model II. Correspondingly $X_n^{(1)}$ and $Z_n^{(1)}$, $n=0,1,2,\ldots$ respectively denote the number available in the W.R and the number specified by the rule with $Y_n^{(1)}$, $n=0,1,2,\ldots$ taking the value 0. Again let T,B and I, respectively represent the length of the busy cycle, busy period and idle period. The corresponding distributions can be easily computed by refering Table I for Model and Table II for Model II. Assume that n departures occur in a busy cycle.

Table-I

Batch No.	Batch size of the present one	Batch size determined by the rule for the next batch	No.of arrivals during service	No. in the system at a service completion epoch
1	a	j_2	ℓ_1	$\underset{\sim}{\ell_1} \geq a$; $\underset{\sim}{\ell_1} = \min(c, \ell_1)$
2	$\underset{\sim}{\ell_1}$ when $j_2 > \underset{\sim}{\ell_1}$ j_2 when $j_2 \leq \underset{\sim}{\ell_1}$	j_3	ℓ_2	$\underset{\sim}{\ell_2} \geq a$; $\underset{\sim}{\ell_2} = \min(c, \underset{\sim}{\ell_1} + \ell_2) - \underset{\sim}{\ell_1}$ $\underset{\sim}{\ell_2} \geq a$; $\underset{\sim}{\ell_2} = \min(c, \underset{\sim}{\ell_1} + \ell_2) - j_2$
3	$\underset{\sim}{\ell_2}$ for $j_3 > \underset{\sim}{\ell_2}$ j_3 when $j_3 \leq \underset{\sim}{\ell_2}$	j_4	ℓ_3	$\underset{\sim}{\ell_3} \geq a$; $\underset{\sim}{\ell_3} = \min(c, \underset{\sim}{\ell_2} + \ell_3) - \underset{\sim}{\ell_2}$ $\underset{\sim}{\ell_3} \geq a$; $\ell_3 = \min(c, \underset{\sim}{\ell_2} + \ell_3) - j_3$
⋮	⋮	⋮	⋮	
n	$\underset{\sim}{\ell_{n-1}}$ when $j_n > \underset{\sim}{\ell_{n-1}}$ j_n when $j_n \leq \underset{\sim}{\ell_{n-1}}$	j_{n+1}	ℓ_n	$\underset{\sim}{\ell_n} = \min(c, \underset{\sim}{\ell_{n-1}} + \ell_n) - \underset{\sim}{\ell_{n-1}} < a$ $\underset{\sim}{\ell_n} = \min(c, \underset{\sim}{\ell_{n-1}} + \ell_n) - j_n < a$

The distribution of the busy cycle is given by (refer Table I)

$$F(x) = \Pr(T \leqslant x) = \sum_{j_1=a}^{b} \sum_{i=o}^{a-1} \Pr(T \leqslant x \mid \text{the state of the system is } (i,o,j_1)) \times \Pr(\text{the state of the system is } (i,o,j_1))$$

$$= \sum_{i=o}^{a-1} \sum_{j_1=a}^{b} \pi(i,o,j_1) \sum_{n=1}^{\infty} \sum_{j_2 \ldots j_{n+1}=a}^{b} \sum_{\ell_1,\ell_2,\ldots\ell_n} \int_{u_1} \int_{u_2} \cdots \int_{u_{n+1}} \gamma_{\lambda,a-i}(u_1)$$

$$g(u_2-u_1) \Lambda_{\ell_1}(u_2-u_1)\; b_{j_2}(u_2-u_1) \ldots g(u_{n+1}-u_n) \Lambda_{\ell_n}(u_{n+1}-u_n)$$

$$b_{j_{n+1}}(u_{n+1}-u_n)\; du_1\, du_2 \ldots du_{n+1}$$

where the ranges of $\ell_1, \ell_2, \ldots, \ell_n$ can be obtained from Table I.

The distribution of the busy period is $B(x)=\Pr(B \leqslant x)$

$$B(x)= \sum_{i=o}^{c-j_1} \sum_{j_1=a}^{b} \sum_{n=1}^{\infty} \sum_{j_2,j_3,\ldots j_{n+1}} \sum_{\ell_1,\ell_2,\ldots,\ell_n} \int_{u_1} \int_{u_2} \cdots \int_{u_n}$$

$$g(u_1)\Lambda_{\ell_1}(u_1)\; b_{j_2}(u_1)\; g(u_2-u_1)\Lambda_{\ell_2}(u_2-u_1) b_{j_3}(u_2-u_1) \ldots$$

$$g(u_n-u_{n-1}) \Lambda_{\ell_n}(u_n-u_{n-1})\; b_{j_{n+1}}(u_n-u_{n-1})\; du_1\, du_2 \ldots du_n$$

Table-II

Batch No.	Size of the batch undergoing service	Size of the next batch determined by the specified rule	Number of arrivals during the service of the present batch	No. in the system at a service completion epoch
1	j_1	j_2	ℓ_1	$\underset{\sim}{\ell}_1 \geqslant j_2$; $\underset{\sim}{\ell}_1=\min(c,\ell_1)$
2	j_2	j_3	ℓ_2	$\underset{\sim}{\ell}_2 \geqslant j_3$; $\underset{\sim}{\ell}_2=\min(c,\underset{\sim}{\ell}_1+\ell_2)-j_2$
3	j_3	j_4	ℓ_3	$\underset{\sim}{\ell}_3 \geqslant j_4$; $\underset{\sim}{\ell}_3=\min(c,\underset{\sim}{\ell}_2+\ell_3)-j_3$
⋮				
(n-1)	j_{n-1}	j_n	ℓ_{n-1}	$\underset{\sim}{\ell}_{n-1} \geqslant j_n$; $\underset{\sim}{\ell}_{n-1}=\min(c,\underset{\sim}{\ell}_{n-2}+\ell_{n-1})-$
n	j_n	j_{n+1}	ℓ_n	$\underset{\sim}{\ell}_n=\min(c,\underset{\sim}{\ell}_{n-1}+\ell_n)-j_n<j_{n+1}$

$$F(x) = Pr(T \leqslant x) = \sum_{i=o}^{j_1-1} \sum_{j_1=a}^{b} \pi(i,o,j_1) \sum_{n=1}^{\infty} \sum_{j_2,j_3,\ldots j_{n+1}=a}^{b} \sum_{\ell_1,\ell_2,\ldots\ell_n}$$

$$\int_{u_1} \int_{u_2} \cdots \int_{u_{n+1}} \gamma_{\lambda,(j_1-i)}(u_1) g(u_2-u_1) \Lambda_{\ell_1}(u_2-u_1)$$

$$b_{j_2}(u_2-u_1)\ g(u_3-u_2)\Lambda_{\ell_2}(u_3-u_2)\ b_{j_3}(u_3-u_2)\ldots g(u_{n+1}-u_n)$$

$$\Lambda_{\ell_n}(u_{n+1}-u_n)\ b_{j_{n+1}}(u_{n+1}-u_n)du_1\ du_2\ \ldots\ du_{n+1}$$

Distribution of the busy period B is $B(x) = Pr(B \leqslant x)$

$$B(x) = \sum_{i=o}^{c-j_1} \sum_{j_1=a}^{b} \sum_{n=1}^{\infty} \sum_{j_2,j_3,\ldots j_{n+1}=a}^{b} \sum_{\ell_1,\ell_2,\ldots\ell_n} \int_{u_1} \int_{u_2} \cdots$$

$$\int_{u_n} g(u_1)\Lambda_{\ell_1}(u_1)\ b_{j_2}(u_1)\ g(u_2-u_1)\Lambda_{\ell_2}(u_2-u_1)\ b_{j_3}(u_2-u_1)\ldots$$

$$g(u_n-u_{n-1})\Lambda_{\ell_n}(u_n-u_{n-1})\ b_{j_{n+1}}(u_n-u_{n-1})du_1\ du_2\ \ldots\ du_n$$

where the ranges of values of $\ell_1, \ell_2, \ldots \ell_n$ are given in Table II.

From the two distributions, a measure of effectiveness can be obtained as follows:

Define the server utilisation factor U as the ratio of the busy period to the busy cycle

That is $U = B/T$

Hence a measure of effectiveness is

$E(B)/E(T)$ which is always < 1

since $E(T) = E(B) + E(I)$.

From the distributions of B and T, E(B) and E(T) can be computed and hence E(B)/E(T).

6.7. Virtual waiting time distribution

Let W be the virtual waiting time of a customer in the queue in the steady state, for Model II. Then

$$Pr(W \leq w) = Pr(W=0) + Pr(0<W \leq w)$$

$$Pr(W=0) = \sum_{j=a}^{b} \pi(j-1,o,j) \qquad (1)$$

$$Pr(0<W\leq w) = \sum_{i=o}^{j-2} \sum_{j=a}^{b} \{Pr(0<W\leq w) \mid \text{the state of the system is } (i,o,j))$$

$$\times\ Pr\ (\text{the state of the system is } (i,o,j))\}$$

$$+ \sum_{i=o}^{c-1} \sum_{j=a}^{b} \{Pr(0<W\leq w \mid \text{the state of the system is } (i,1,j$$

$$Pr(\text{the state of the system is } (i,1,j))\}$$

The first term on the R.H.S is $\sum_{i=o}^{j-2} \sum_{j=a}^{b} \pi(i,o,j) \Gamma_{\lambda, j-(i+1)}(w)$

To find $P(0<W\leqslant w|$ the state of the system is $(i,1,j))$ we proceed as follows.

Assume that at the 'time of arrival' of the virtual customer state of the system is $(i,1,j)$ and that the present batch has already been served for an amount of time u. Further assume that n departures occur before the particular unit is taken for service. During the service of the n batches all together ℓ arrivals take place in such a way that at each departure epoch, the W.R contains $\geqslant$ the number specified and just after the departure of the n^{th} batch the state of the system is $(i_1,1,k_1)$ in which the particular unit arrived is being taken for service. If just after the departure of the n^{th} batch, the state of the system is (i_2,o,k_2) the server remains idle till (k_2-i_2) customers arrive and the particular unit is taken for service.

Now $Pr(0<W\leqslant w|$system in $(i,1,j))$

$$= \sum_{n=1}^{\infty} \underset{\substack{u_1\ u_2 \ \cdots\ u_n \\ u_n-u\leqslant w}}{\int \int \cdots \int} g(u_1-u)\ b_{j_2}(u_1)\ g(u_2-u_1)$$

$$b_{j_3}(u_2-u_1) \cdots g(u_n-u_{n-1})\ b_{j_{n+1}}(u_n-u_{n-1})$$

$$\sum_{\ell} \Lambda_{\ell}(u_n-u) \left[\sum_{i_1=o}^{c-k_1} \sum_{k_1=a}^{b} \pi(i_1,1,k_1) + \right.$$

$$\left. \sum_{i_2=o}^{k_2-1} \sum_{k_2=a}^{b} \pi(i_2,o,k_2)\ \gamma_{\lambda,k_2-i_2}(w-u_n) \right] du_1\ du_2 \cdots du_n$$

Hence

$$\Pr(0<W\leqslant w) = \sum_{i=o}^{j-2}\sum_{j=a}^{b} \pi(i,o,j)\Gamma_{\lambda,j-(i+1)}(w) + \sum_{i=o}^{c-j}\sum_{j=a}^{b} \pi(i,1,j)$$

$$\sum_{n=1}^{\infty} \underset{u_n-u\leqslant w}{\int_{u_1}\int_{u_2}\cdots\int_{u_n}} g(u_1-u)\, b_{j_2}(u_1) g(u_2-u_1) b_{j_3}(u_2-u_1)\cdots$$

$$g(u_n-u_{n-1})\, b_{j_{n+1}}(u_n-u_{n-1}) \sum_{\ell} \Lambda_{\ell}(u_n-u)$$

$$\Big[\sum_{i_1=o}^{c-k_1}\sum_{k=a}^{b} \pi(i_1,1,k_1) + \sum_{i_2=o}^{k_2-1}\sum_{k_2=a}^{b} \pi(i_2,o,k_2)$$

$$\gamma_{\lambda,(k_2-i_2)}(w-u_n)\Big]du_1\, du_2 \cdots du_n \qquad (2)$$

Hence $\Pr(W\leqslant w)$ is the sum of the right hand sides of (1) and (2). A similar expression can be written for the virtual waiting time for Model I with the difference that here the service is initiated even when the W.R contains less than the number determined by the rule provided that at least a units are available at that epoch.

6.8. Control problem

In this section we indicate how to obtain the optimal value of c for given values of a and b. We shall take the objective function as the total expected cost per unit time in the steady state. The decision variable is c which should be so chosen that the objective function value is minimum for that value of c.

The mean of the service time distribution is $\int_0^\infty [1-G(t)]dt = \mu$. Suppose that at the departure epoch of a batch, the system is in state (i,j,k). Conditioned on this, we proceed to find the optimal value of c between this and the next departure epochs.

Let h be the cost of waiting per unit per unit time when service is going on; h_1 be the cost per unit loss due to finite W.R capacity c; h_2 be the cost of waiting per unit per unit time due to no service; h_3 be the cost associated with the server's idleness.

Assume that the system is in state $(i,1,k)$, $i=0,\ldots,c-k$ immediately after the departure of a batch. This means $(c-i)$ more units can be admitted to the W.R during a service. The first among these $(c-i)$ units arrive on the average after $\frac{1}{\lambda}$ units of time since the commencement of service. The next after $\frac{2}{\lambda}$ units of time, ... Thus these units are to wait for atleast $(\mu - \frac{1}{\lambda})$, $(\mu - \frac{2}{\lambda})$,...., $(\mu - \frac{(c-i)}{\lambda})$ units of time provided at least $(c-i)$ arrivals take place. Initially i units are waiting. These are to wait on the average for μ units of time to get a chance to enter the S.S after the completion of the present service. Hence cost towards waiting over a service time provided the state of the system is $(i,1,k)$ just after a departure is

$$h\left\{i\mu + [\mu - \tfrac{1}{\lambda}]^+ + [\mu - \tfrac{2}{\lambda}]^+ + \ldots + [\mu - \tfrac{(c-i)}{\lambda}]^+\right\}$$

The expected number of units lost to the system during the service is $[\mu - \frac{(c-i)}{\lambda}]^+\lambda$. Then the expected loss to the system during a service due to finite waiting room is $[\mu - \frac{(c-i)}{\lambda}]^+\lambda . h_1$.

Assume that the system is in state (i,o,k) just after the service of a batch, i=0,1,2,...,a-1 for Model I or i=0,1,2,...,k-1 for Model II. In particular consider Model I. If the system is in (i,o,k) then (a-i) more arrivals should take place in order that a service commences. Hence expected cost due to servers idlensss is $h_3(\frac{a-i}{\lambda})$.

Since the system is in (i,o,k), (a-i) more units have to arrive to start service. These units arrive on the average after $\frac{1}{\lambda}$, $\frac{2}{\lambda}$, ... $\frac{a-i}{\lambda}$ units of time after last departure epoch at the end of which the system is in state (i,o,k). Initially i units are waiting for service. Hence cost towards waiting due to no service is

$$\left\{ i(\tfrac{a-i}{\lambda}) + (\tfrac{a-i-1}{\lambda}) + \ldots + \tfrac{2}{\lambda} + \tfrac{1}{\lambda} \right\} h_2$$

As soon as the $(a-i)^{th}$ arrival takes place, these a units are transferred to the S.S and service commences. The expected loss to the system during the service of the batch (of size a units), due to finite W.R capacity is $[\mu - \frac{c}{\lambda}]^{+}\lambda h_1$.

Thus the expected total cost to the system per unit time, evaluated between two successive departure epochs, under Model I, is

$$\bar{F}(c) = \sum_{i=o}^{c-k} \sum_{k=a}^{b} q((i,1,k)) \frac{1}{\mu}\{ h(i\mu + [\mu-\frac{1}{\lambda}]^{+} + [\mu-\frac{2}{\lambda}]^{+} + \ldots + [\mu-\frac{(c-i)}{\lambda}]^{+})$$

$$+ ([\mu - \frac{(c-i)}{\lambda}]^{+}\lambda . h_1)\} + \sum_{i=o}^{a-1} \sum_{k=a}^{b} q((i,o,k))\{h_3(\frac{a-i}{\lambda}) +$$

$$h_2[i(\frac{a-i}{\lambda}) + (\frac{a-i-1}{\lambda}) + \ldots + \frac{2}{\lambda} + \frac{1}{\lambda}] + \frac{1}{\mu}[\mu - \frac{c}{\lambda}]^{+}\lambda h_1\}$$

An expression for the cost function can be obtained for Model-II in a similar fashion.

Now the optimal value c^* of c can be calculated from the relations

$$\bar{F}(c^*) \leqslant \bar{F}(c^*+1)$$

and

$$\bar{F}(c^*) \leqslant \bar{F}(c^*-1)$$

Chapter-7

FINITE CAPACITY $G/E_k/1$ AND $M/G^{a,b}/1$ QUEUEING SYSTEMS

7.1 Introduction

This chapter deals with two models of finite capacity, single server queueing systems. Model-I is about $G/E_k/1$ queueing system in which the interarrival times of customers are assumed to follow a general distribution. Each arrival to the system induces an additional k phases and the service times in each phase follow exponential distribution. Model-II investigates a queueing system with general bulk service rule with batch sizes varying from a to b. The system capacity is assumed to be finite in both the Models.

There are many situations in real-life where we encounter such types of queueing models as described above. The transportation process involving buses, trains, aero-planes etc. all deal with such type of queueing systems.

Jacob, Krishnamoorthy and Madhusoodanan (1988) obtain the time dependent solution to $M/G^{a,b}/1$ model whereas the same model with vacation is studied by Jacob and Madhusoodanan (1987). Their approach is based on matrix convolution method. Here we make use of a much more simplified tool- Markov Renewal theoretic approach.

Section 7.2 describes the models and notations used in the chapter. Analysis, time-dependent system size probabilities and the steady state distributions of the $G/E_k/1$ queueing system are carried out in Section 7.3. Analysis of $M/G^{a,b}/1$ queueing model, its system size probabilities, steady state behaviour and virtual waiting time distribution are derived in Section 7.4.

7.2. Description of the Models

Model I- The $G/E_k/1$ system

The interarrival times of customers to the single server queueing system are assumed to be independent and identically distributed random variables following distribution function G(.) with density g(.). The system is of finite capacity b. The service pattern follows Erlang distribution of order $k(k \geqslant 2)$. The queue discipline is FCFS. Here the arrival of each customer induces additional k phases into the system provided there are atmost (b-1) units in the system. Service in each phase is exponentially distributed with same rate μ. Arrivals taking place when the system is full are lost. The system size at any time point is the number of phases in the system at that instant. The service mechanism is such that the server becomes idle and stops serving only when there is no unit in the system.

Model II- The $M/G^{a,b}/1$ Queue

We assume that the model under consideration has a waiting room (W.R) and a service station (S.R) each of capacity b. Customers arrive one by one according to a homogeneous Poisson process of rate λ and join the W.R subject to the capacity restriction. All arrivals taking place when the W.R. is full are lost to the system. Service times are independent and identically distributed random variables following distribution function H(.) and density function h(.). Services are in batches with at least a customers and a maximum of b in each batch. When the service of a batch is completed the server scans the W.R. and transfers all those in the W.R. provided there are at least a customers, to the S.S. On the other hand if the server finds less than a customers waiting for service, he goes for vacation for a random length of time following distribution function F(.) and density function f(.). On return, if the system size is still less than a, he takes another vacation immediately which is independent of and identically distributed as the previous one. This process continues until on return he finds at least a units waiting for service.

Notations

* denotes convolution. $g^{*n}(x)$ denotes the n-fold convolution of g(x) with itself.

N denotes $\{1,2,3,4,\ldots\}$

N^o denotes the set $\{0,1,2,\ldots\}$

E is the set $\{0,1,2,\ldots, k,k+1,\ldots,bk\}$

$(Q(i,j,t))$ for $i,j \in E$ is a $(bk+1)x(bk+1)$ matrix whose $(i,j)^{th}$ entry is $Q(i,j,t)$

$P_i(j,t)$ is the probability that at time t there are j phases in the system given that the system has started with i phases initially.

Let $\Lambda_n(x) = \dfrac{e^{-\lambda x}(\lambda x)^n}{n!}$, $n=0,1,2,\ldots,b-1$ and

$$\overline{\Lambda}_b(x) = \sum_{n \geqslant b} \frac{e^{-\lambda x}(\lambda x)^n}{n!}$$

γ denotes the Gamma density where

$$\gamma_{\lambda,k}(x) = (e^{-\lambda x}\lambda^k x^{k-1})/(k-1)!, \quad k=1,2,3,\ldots$$

E_1 is the set $\{0,1,2,\ldots,a-1\}$

E_2 is the set $\{0,1,2,\ldots,b\}$

$\hat{g}(\alpha)$ - the Laplace transform of g(t).

That is $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha t} g(t)dt$

$(Q_1(i,j,t))$ is a square matrix of order a whose $(i,j)^{th}$ entry is $Q_1(i,j,t)$.

$P_{(o,\ell)}((i,j),t)$ is the probability that i units are undergoing service and j units are in the W.R at time t under the condition that at time zero the server went on vacation as there were only ℓ ($\leqslant a-1$) units in the system, with either $i=0$ or $a\leqslant i\leqslant b$; $j=0,1,2,\ldots,b$.

7.3. Model I: $G/E_k/1$ Queue

Let Y_t denotes the number of phases in the system at time t. Then the process $\{Y_t, t\geqslant 0\}$ is a semi-regenerative process with state space E. Identify $0=T_o, T_1, T_2, \ldots$ as the initial, first, second, ... arrival instants. Let X_n denote the number of phases present in the system just prior to the n^{th} arrival. Then the process $\{(X,T)\}=\{(X_n,T_n), n\in N^o\}$ is the embedded Markov renewal process.

Define

$$Q(i,j,t) = Pr\{X_{n+1}=j; T_{n+1}-T_n\leqslant t | X_n=i\},\ i,j\in E, t\geqslant 0$$

Then (i) for $0\leqslant i\leqslant (b-1)k$; $j=0,1,2,\ldots,i,\ldots bk$

$$Q(i,j,t) = \int_0^t \{(e^{-\mu u}(\mu u)^{(i+k-j)}) | (i+k-j)!\}\, g(u)du$$

(ii) for $(b-1)k < i \leqslant bk$, $j \leqslant i$

$$Q(i,j,t) = \int_0^t \{(e^{-\mu u}(\mu u)^{i-j}/(\ i-j\ )!\}\, g(u)du$$

The semi-Markov kernel over E is

$$\mathbb{Q} = \{Q(i,j,t),\ i,j \in E,\ t \geqslant 0\}$$

Let $Q(t) = (Q(i,j,t))$, $i,j \in E$

For all $n \in N$ define

$$Q^n(i,j,t) = \Pr\{X_n=j;\ T_n \leqslant t \mid X_o=i\}, \quad i,j \in E,\ t \geqslant 0$$

$$\text{with} \quad Q^o(i,j,t) = I(i,j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

For all $t \geqslant 0$, $n \in N$ we have the recursive relation,

$$Q^{n+1}(i,m,t) = \sum_{j \in E} \int_o^t Q(i,j,du)\, Q^n(j,m,t-u)$$

The Markov renewal function is given by

$$R(i,j,t) = \sum_{n=o}^{\infty} Q^{*n}(i,j,t), \quad i,j \in E,\ t \geqslant 0$$

with the Markov renewal kernel $\mathbb{R}$.

Let $\mathbb{R}(t) = (R(i,j,t))$ for $i,j \in E$, $t \geqslant 0$

Since E is finite, the Markov renewal kernel can be computed by taking Laplace transforms.

System size probabilities

Assume that initially there are a units (ak phases) in the system. Then $P_{ak}(i,t)$ satisfies the Markov renewal equations so that

$$P_{ak}(i,t) = \sum_{m \in E} \int_0^t R(ak,m,du)\, K_m(i,t-u)$$

where $K_m(i,t) = \Pr\{Y_t=i,\ T_1>t \mid X_o=m\}$ and is obtained as follows:

(i) for $0 \leqslant m \leqslant bk$, $i=0$

$$K_m(i,t) = \{(e^{-\mu t}(\mu t)^m)/m!\}\,[1-G(t)]$$

(ii) for $0 \leqslant m \leqslant (b-1)k$, $1 \leqslant i \leqslant m+k$

$$K_m(i,t) = \frac{e^{-\mu t}(\mu t)^{m+k-i}}{(m+k-i)!}\,[1-G(t)]$$

(iii) finally for $(b-1)k < m \leqslant bk$, $1 \leqslant i \leqslant m$

$$K_m(i,t) = \frac{e^{-\mu t}(\mu t)^{m-i}}{(m-i)!}\,[1-G(t)]$$

Steady state distribution

To obtain the limiting distribution of the system size, first we verify whether (X,T) is irreducible,recurrent and aperiodic. For this we assume that the expected number of phases L completed during an interarrival time is greater than k. With this assumption (X,T) becomes irreducible and recurrent.

That is $E(L) = \int_0^\infty \sum_{\ell \in E} \ell \frac{e^{-\mu u}(\mu u)^\ell}{\ell!} g(u)du > k$

Also the sojourn time in state 0 has an exponentially distributed component and so Q(o,j,t) is not a step function. Hence all states are aperiodic in (X,T). So (X,T) is an irreducible aperiodic and recurrent process. Further the state space E is finite.

Let π be an invariant measure for X which gives the stationary distribution of the process (X,T). This π is obtained from the solution of the set of linear equations $\pi\mathbb{P}=\pi$ subject to the condition $\pi e=1$ where π is a (bk+1) component row vector and e = (1,1,...,1) which is a (bk+1)-component column vector.

Further $\mathbb{P} = ((p_{i,j}))$, $i,j \in E$ is a matrix of order (bk+1) x (bk+1) where

$$p_{i,j} = \Pr\{X_{n+1}=j \mid X_n=i\} = Q(i,j,\infty)$$

$$\pi \mathbb{P} = \pi \text{ implies that}$$

$$\sum_{i\in E} \pi(i)\, p_{i,j} = \pi(j),\ j \in E$$

The expected sojourn time in state j is

$$\tau(j) = \int_0^\infty [1-\sum_{i\in E} Q(i,j,t)]dt$$

Let $\tau = \{\tau(0), \tau(1), \ldots, \tau(bk)\}$

We compute the limiting probabilities as

$$\lim_{t\to\infty} P_{ak}(i,t) = \underline{P}(i) = \frac{\sum_{j=0}^{i} \pi(j)\, n(j,i)}{\sum_{j\in E} \pi(j)\, \tau(j)}$$

where $n(j,i) = \int_0^\infty K_j(i,t)dt$

The limiting distribution of thesystem size probabilities are also obtained by the method of Laplace transformations as follows:

We have already obtained

$$P_{ak}(i,t) = \sum_{m\in E} \int_0^t R(ak,m,du)\, K_m(i,t-u) \qquad \text{(A)}$$

Let

$$\hat{P}_\alpha(ak,i) = \int_0^\infty e^{-\alpha t} P_{ak}(i,t)dt$$

$$\hat{R}_\alpha(ak,m) = \int_0^\infty e^{-\alpha t} R(ak,m,t)dt$$

and

$$\hat{K}_\alpha(m,i) = \int_0^\infty e^{-\alpha t} K_m(i,t)dt$$

Taking Laplace transform of both sides of (A) and applying Tauberian theorem (Widder (1948)) we have

$$\lim_{t \to \infty} P_{ak}(i,t) = \lim_{\alpha \to 0} \hat{P}_\alpha(ak,i)$$

$$= \lim_{\alpha \to 0} \left(\sum_{m \in E} \hat{R}_\alpha(ak,m) \hat{K}_\alpha(m,i) \right)$$

7.4. <u>Model-II</u>- <u>$M/G^{a,b}/1$ Queue</u>

Let $g(u)du$ be the probability that the service of a batch which has started at time zero is completed in the interval $(u,u+du]$ and during this service time at least a arrivals have taken place. Thus

$$g(u)du = \sum_{j=a}^{\infty} \wedge_j(u) \; h(u)du \tag{1}$$

where $h(.)$ is the service time density function. Let $X(t)$ be the number of units underoing service and $Y(t)$ be the number of units waiting for service at time t.

Define $Z(t) = \{X(t), Y(t)\}$. Then the stochastic process $\{Z(t), t \geq 0\}$ is a semi-regenerative process defined over the state space $E_1 \times E_2$.

The time epochs at which the server goes for vacation after a service with less than a units waiting for service are the busy period termination epochs. Let $0=T_o, T_1, T_2, \ldots$ be the time epochs of successive busy period terminations and Y_n, $n=0,1,2,\ldots$ be the number of units in the system (ie. in the W.R.) at time T_n+. Then the process $\{(Y,T)\} = \{(Y_n, T_n), n \in N^o\}$ is a time homogeneous Markov renewal process defined over the set E_1. The kernal of the semi-Markov process is

$$\{Q_1(i,j,t),\ i,j \in E_1,\ t \geq 0\}$$

where the $Q_1(i,j,t)$ are given by

$$Q_1(i,j,t) = \Pr\{Y_{n+1}=j; T_{n+1}-T_n \leq t \mid Y_n=i\},\ i,j \in E_1,\ t \geq 0$$

$$= \int_o^t \int_u^t \int_v^t \sum_{n=o}^{\infty} f^{*n}(u)\ f(v-u) \sum_{r=o}^{a-1-i} \wedge_r(u)\, \overline{\wedge}_{a-(i+r)}^{(v-u)}$$

$$\sum_{m=1}^{\infty} g^{*(m-1)}(w-v)\ h(t-w)\wedge_j(t-w)dw\ dv\ du \quad (2)$$

The Markov Renewal function is given by

$$R_1(i,j,t) = \sum_{n=o}^{\infty} Q_1^{*n}(i,j,t),\ i,j \in E_1,\ t \geq 0.$$

System size probabilities

We assume that initially, ie. at time $T_o=0$, the server just enters a vacation period after completing a busy period so that the initial state of the process is

$$Z(o) = \{X(o), Y(o)\} = (o,\ell) \text{ for some } \ell \in E_1$$

Let
$$P_{(o,\ell)}((i,j),t) = Pr\{Z(t)=(i,j)|Z(o)=(o,\ell)\} , \ell \in E_1.$$

Then
$$P_{(o,\ell)}((i,j),t) = Pr\{Z(t)=(i,j),T_1>t|Z(o)=(o,\ell)\}$$
$$+ Pr\{Z(t)=(i,j),T_1\leqslant t|Z(o)=(o,\ell)\}$$

Now
$$Pr\{Z(t)=(i,j), T_1\leqslant t|Z(o)=(o,\ell)\} = \sum_{k\in E_1}\int_o^t Q_1(\ell,k,du)$$
$$P_{(o,k)}((i,j),t-u)$$

and

$$Pr\{Z(t)=(i,j);T_1>t|Z(o)=(o,\ell)\} = K_{(o,\ell)}((i,j),t)$$

Then $P_{(o,\ell)}((i,j),t)$ satisfies the Markov renewal equation. Hence the solutions are given by

$$P_{(o,\ell)}((i,j),t) = \sum_{k\in E_1}\int_o^t R_1(\ell,k,du)\ K_{(o,k)}((i,j),t-u)$$

where

(i) for $a<i\leqslant b$; $j=0,1,2,\dots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \int_u^t \sum_{n=o}^{\infty} f^{*n}(u)f(v-u) \sum_{r=o}^{a-1-k} \wedge_r(u)\bar{\wedge}_{a-(k+r)}(v-u)$$

$$\int_v^t \sum_{m=1}^{\infty} g^{*m-1}(w-v) \int_w^t h(x-w)\wedge_i(x-w)\ [1-H(t-x)]$$

$$\wedge_j(t-x)dx\ dw\ dv\ du$$

(ii) for $i=a$; $j=0,1,2,\dots,b$

$$K_{(o,k)}((i,j),t)) = \int_o^t \int_u^t \sum_{n=o}^{\infty} f^{*n}(u)f(v-u) \sum_{r=o}^{a-1-k} \wedge_r(u)$$

$$\{\wedge_{a-(k+r)}(v-u)[1-H(t-v)]\wedge_j(t-v)$$

$$+ \bar{\wedge}_{a-(k+r)}(v-u) \int_v^t \int_w^t \sum_{m=1}^{\infty} g^{*(m-1)}(w-v)\ h(y-w)$$

$$\wedge_a(y-w)\ [1-H(t-y)]\wedge_j(t-y)dy\ dw\}\ dv\ du$$

(iii) for $i=0$, $j=k,k+1,\dots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \sum_{n=o}^{\infty} f^{*n}(u) \sum_{m=o}^{a-1-k} \wedge_m(u)[1-F(t-u)]\bar{\wedge}_{j-(k+m)}(t-u)du$$

and $\qquad$ 0 otherwise

Virtual waiting time distribution

Let W(t) be the virtual waiting time of a customer in the queue.

Let $\Pr\{W(t) \leqslant x \mid Z(t) = (i,j),\ Z(o) = (o,\ell)\} = B\{W(t) \leqslant x\}$

Then

(i) for $a \leqslant i \leqslant b$; $a-1 \leqslant j \leqslant b-1$

$$B\{W(t)\leqslant x\} = \int_o^t \int_u^t \sum_{k\in E_1} R_1(\ell,k,du) \sum_{n=o}^{\infty} f^{*n}(v-u) \int_v^t f(w-v) \sum_{r=o}^{a-1-k} \Lambda_r($$

$$\Lambda_{i-(r+k)}(w-v)\,\Lambda_j(t-w)\,[1-H(t-w)] \int_t^{t+x} h(\tau)\,d\tau\ dw\ dv$$

(ii) for $a \leqslant i \leqslant b$; $o \leqslant j < a-1$

$$B\{W(t)\leqslant x\} = \int_o^t \int_u^t \sum_{k\in E_1} R_1(\ell,k,du) \sum_{n=o}^{\infty} f^{*n}(v-u)$$

$$\int_v^t f(w-v) \sum_{r=o}^{a-1-k} \Lambda_r(v-u)\Lambda_{i-(r+k)}(w-v)$$

$$\Lambda_j(t-w)\ \ [1-H(t-w)] \int_t^{t+x} h(\tau)\bar{\Lambda}_{a-(j+1)}(\tau-t)\,d\tau\ dw\ dv$$

(iii) for $i=0$, $0 \leqslant j < b-1$

$$B\{W(t)\leqslant x\} = \int_o^t \sum_{k\in E_1} R_1(\ell,k,du) \int_t^{t+x} f(v-u)\Lambda_{j-k}(t-u)\bar{\Lambda}_{a-(j+1)}(v-t$$

(iv) for $a \leq i \leq b$; $j=b$

$$B\{W(t) \leq x\} = \int_{o}^{t} \sum_{k \in E_1} R_1(\ell,k,du) \int_{u}^{t} \sum_{n=o}^{\infty} f^{*n}(v-u)$$

$$\int_{v}^{t} f(w-v) \sum_{r=o}^{a-1-k} \wedge_r(v-u) \wedge_{i-(r+k)}(w-v)$$

$$\bar{\wedge}_b(t-w) \int_{t}^{t+x} h(\tau-w) \int_{\tau}^{t+x} h(z-\tau) \bar{\wedge}_{a-1}(z-\tau) dz\, d\tau\, dw\, dv$$

finally for $i=0$, $j=b$

$$B\{W(t) \leq x\} = \int_{o}^{t} \sum_{k \in E_1} R_1(\ell,k,du) \int_{t}^{t+x} f(v-u) \bar{\wedge}_{b-k}(t-u)$$

$$\int_{v}^{t+x} h(w-v) \bar{\wedge}_{a-1}(w-v) dw\, dv$$

Next consider the case in which the server does not go for vacation.

As in the previous case, the semi-regenerative process is $\{Z(t), t \geq 0\}$ defined over the state space $E_1 \times E_2$. Let $0 = T_o, T_1, T_2, \ldots$ be the successive time points at which the server becomes idle with less than a customers waiting for service and Y_n be the number of customers waiting for service at time T_n so that $Y_n = Y(T_n+)$. Here also

$$\{(Y,T)\} = \{(Y_n, T_n),\ n \in N^o\}$$

is the time homogeneous Markov renewal process defined over E_1 with

$$Q_2(i,j,t) = \int_o^t \gamma_{\lambda,a-i}(u) \int_u^t \sum_{m=1}^{\infty} g^{*(m-1)}(v-u)\; h(t-v)\wedge_j(t-v)dv\; du$$

and

$$R_2(i,j,t) = \sum_{n=o}^{\infty} Q_2^{*n}(i,j,t) \tag{3}$$

The system size probabilities in this case are given by

$$P_{(o,\ell)}((i,j),t) = \sum_{k\in E_1} \int_o^t R_2(\ell,k,du)\; K_{(o,k)}(i,j),t-u)$$

where $K_{(o,k)}((i,j),t)$ is as defined earlier and the expressions are given by

(i) for $a<i\leqslant b$, $j=0,1,2,\ldots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \int_u^t [\gamma_{\lambda,a-k}(u) \sum_{m=1}^{\infty} g^{*m-1}(v-u)$$

$$\int_v^t h(w-v)\wedge_i(w-v)[1-H(t-w)]\wedge_j(t-w)]dw\; dv\; du$$

(ii) for $i=a$, $j=0,1,2,\ldots,b$

$$K_{(o,k)}((i,j),t) = \int_o^t \gamma_{\lambda,a-k}(u)\{[1-H(t-u)]\wedge_j(t-u) +$$

$$\int_u^t \int_v^t \sum_{m=1}^{\infty} g^{*(m-1)}g(v-u)h(w-v)$$

$$\wedge_a(w-v)[1-H(t-w)]\wedge_j(t-w)dw\; dv\,\}\, du$$

finally (iii) for $i=0$, $j\leq a-1$

$$K_{(o,k)}((i,j),t) = \wedge_{j-k}(t)$$

The virtual waiting time distribution in this case conditioned on the system size are given below:

(i) for $a\leq i\leq b$; $a-1\leq j\leq b-1$

$$B\{W(t)\leq x\} = \int_o^t \int_u^t \int_v^t \sum_{k=o}^{a-1} R_2(\ell,k,du)\ \gamma_{\lambda,a-k}(v-u) \sum_{m=1}^{\infty} g^{*m-1}(w-v)$$

$$\int_w^t h(y-w)\wedge_i(y-w)[1-H(t-w)] \int_t^{t+x} h(\tau)\wedge_j(t-y)\ d\tau\ dy\ dw\ dv$$

(ii) for $a\leq i\leq b$; $0\leq j<a-1$

$$B\{W(t)\leq x\} = \int_o^t \int_u^t \sum_{k=o}^{a-1} R_2(\ell,k,du)\ \gamma_{\lambda,a-k}(v-u) \int_v^t \sum_{m=1}^{\infty} g^{*(m-1)}(w-v)$$

$$\int_w^t h(y-w)\wedge_i(y-w)[1-H(t-w)]$$

$$\int_t^{t+x} h(\tau)\wedge_j(t-y)\ \gamma_{\lambda,a-1-j}(\tau-t)d\tau\ dy\ dw\ dv$$

finally for $i=0$; $o\leq j<a-1$

$$B\ \{W(t)\leq x\} = \int_o^t \int_u^t \sum_{k\in E_1} R_2(\ell,k,du)\ \gamma_{\lambda,a-k}(v-u) \int_v^t \sum_{m=1}^{\infty} g^{*(m-1)}(w-v)$$

$$\int_w^t h(y-w) \sum_{r=o}^{j} \wedge_r(y-w)\wedge_{j-r}(t-y) \int_t^{t+x} \gamma_{\lambda,a-(j+1)}(z)dz\ dy\ dw\ dv$$

Chapter-8

TRANSIENT SOLUTION TO $M^X/G^Y/1/b$ QUEUE WITH VACATION*

8.1. Introduction

This chapter discusses a vacation queueing model in which the system can be operated only when it is full, but only a random number of units are taken in a batch for service. Unlike the previous chapters, matrix convolution technique is adopted here to arrive at the time-dependent system size probabilities.

Chaudhary (1979) obtains the limiting probabilities of queue lengths at random and departure epochs in the case of an $M^X/G/1$ queueing system. The transient and stationary behaviour of the M/G/1/k queue, with a fixed maximum number of customers, k, in the system at any time is studied by Cohen (1969). Bagchi and Templeton (1973) make use of Cohen's method to generalise his results to $M^X/G^Y/1/k$ queue.

Section 8.2 deals with the description of the model together with the notations and preliminaries used in this chapter. Transition time densities and expressions for renewal density are given in Section 8.3. Time dependent system size probabilities and distribution of the busy period

*Appeared in Calcutta Statistical Association Bulletin, Vol.39, March and June 1990, Nos. 153-154, pp.21-29.

are derived in Sections 8.4 and 8.5 respectively. Virtual waiting time distribution is derived in the last section.

8.2 Description of the model

A single server queueing system with the arrival pattern following a compound Poisson process is considered. The random variable X represents the number of customers arriving in a batch for service with the distribution of X defined as

$$\Pr\{X=i\} = p_i, \quad i=1,2,\ldots,b.$$

Let $$\phi_1(s) = \sum_{i=1}^{b} p_i s^i$$

Again customers in batch arrive at Poisson rate λ and joins the queue in the waiting room (W.R). The system is of finite capacity b and arrivals occuring when the W.R. is full are lost to the system. The service commences only when the waiting room (W.R) is full (b) and then a random number of units Y are taken in a batch to the service station (S.S) for service. The service pattern follows a general distribution $G_Y(.)$ with density $g_Y(.)$ where Y is the batch size taken for service with the distribution of Y given by $\Pr\{Y=r\}= q_r$, $r=1,2,\ldots,b$ with $\phi_2(s) = \sum_{k=1}^{b} q_k s^k$. On completion of the service of a batch, if the waiting room (W.R) contains less than b customers, the server immediately takes a vacation for a random duration having general distribution H(.) with density function h(.). The vacation policy is of the exhaustive type- every time the server returns after vacation, if the system size is less than b the server again goes on vacation whose duration has the same distribution H(.). The random variables X and Y are assumed to be independent. Further the sizes of the arriving batches are independent, so are the sizes of the batches taken for service.

<u>Notations and preliminaries</u>

p_j^{*i} - the coefficient of s^j in $[\phi_1(s)]^i$

N(t) represents the number of arrival instants upto time t so that

$$\Pr\{X_1+X_2+\ldots+X_{N(t)}=j\} = \sum_{i=1}^{j} p_j^{*i}(e^{-\lambda t}\,(\lambda t)^i/i!)$$

Denote $\Pr\{X_1+X_2+\ldots+X_{N(t)}=j\}$ as $\Lambda_j(t)$

Let $\bar{\Lambda}_k(t)$ represents the probability that at least k arrivals occur upto time t.

Let $f_{ij}(x)$ be the probability that a batch of size i taken for service at time zero completes the service in $(x,x+dx]$ and j units arrive in $(o,x]$ such that $j+(b-i)\geq b$. Hence

$$f_{ij}(x) = \sum_{k=1}^{j} p_j^{*k} \frac{e^{-\lambda x}(\lambda x)^k}{k!} g_i(x)\; q_i$$

$$= \Lambda_j(x)\; g_i(x)\; q_i, \quad j=i,i+1,\ldots,b;\; i=1,2,\ldots,b$$

$P_{ji}(t)$ represents the probability that at time t there are i units in the W.R and j units in the S.S.

$$i = b-j,\; b-j+1,\ldots,b;\; j=1,2,\ldots,b$$

8.3 Transition time densities

For $j<i$ and $i=1,2,\ldots,b$, define $f_{ij}(x)=0$

Write

$$\mathbb{F}(x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) & .. & .. & f_{1b}(x) \\ 0 & f_{22}(x) & .. & .. & f_{2b}(x) \\ 0 & 0 & f_{33}(x) & .. & f_{3b}(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & f_{bb}(x) \end{bmatrix}$$

Introduce

$$\mathbb{H}(x) = \begin{bmatrix} f_{10}(x) & 0 & 0 & 0 & \cdots & 0 \\ f_{20}(x) & f_{21}(x) & 0 & 0 & \cdots & 0 \\ f_{30}(x) & f_{31}(x) & f_{32}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ f_{bo}(x) & f_{b1}(x) & f_{b2}(x) & . & \cdots & f_{bb-1}(x) \end{bmatrix}$$

where $f_{ij}(x) = 0$ for $j \geqslant i$; $i=1,2,\ldots,b$.

Let $\underline{f}_i(x) = (0,0,0,\ldots\ldots,0,f_{ii}(x),\ f_{ii+1}(x),\ldots\ldots,f_{ib}(x))$, for $i=1,2,\ldots\ldots,b$. Then $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ is a b-component column vector. Taking the ℓ^{th} co-ordinate of the above vector and naming it as $K_i^{\ell}(x)$ we see that the probability that the system starting with the service of a batch of size $i(1 \leqslant i \leqslant b)$ units initially, continues to work uninterruptedly and finally the service of a batch of size ℓ has been completed in $(x, x+dx]$ and ℓ arrivals have occured during this (last) service time to make the system size full again.

Now $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{H})(x)$ is a b-component row vector.

Let $F_i(x) = (\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{H})(x)$. This stands for the probability that a busy period that has started with the service of a batch of size i units initially and after serving n more batches, has ended in (x,x+dx] with atmost (b-1) units waiting for service. Let the ℓ^{th} coordinate of the vector $F_i(x)$ be denoted as $F_i^{\ell}(x)$. Hence $F_i^{\ell}(x)$ represents the probability that the system starting with the service of a batch of size $i(1\leqslant i\leqslant b)$ units, the busy period ends in (x,x+dx] with ℓ units waiting for service, $\ell=0,1,2,\ldots,b-1$.

Renewal density

The time points at which a busy period is initiated after each vacation are regeneration points. Let $T_i, i=1,2,\ldots$ represent the successive vacation completion points.

Let a busy period be initiated with the service of a batch of size i $(1\leqslant i\leqslant b)$ units and z be the time epoch at which the busy period has ended with atmost b-1 units in the W.R. Assume that m vacations complete in (z,v), the last being completed in (v,v+dv) at which the system size is again less than b and so the server goes on vacation which is completed in (u,u+du). The service now starts as there are b units waiting. Hence

$$A(u) = Pr\{u < T_n \leqslant u+du\}$$

$$= \int_o^u \sum_{i=1}^{b} \sum_{\ell=o}^{b-1} F_i^{\ell}(z) \int_z^u \Big(\sum_{m=o}^{\infty} h^{*m}(v-z)\Big) \sum_{j=o}^{b-1-\ell} \wedge_j(v-z)h(u-v)$$

$$\bar{\wedge}_{b-(\ell+j)}(u-v)dv\ dz$$

Therefore the renewal density is given by

$$M(u) = Pr\{u < T_1+T_2+ \ldots T_n \leqslant u+du\}$$

$$= \sum_{n=1}^{\infty} A^{*n}(u)$$

8.4 <u>System size probabilities</u>

$P_{oi}(t)$ is the probability that there is no unit in the S.S and i units are there in the W.R, i=0,1,2,...,b-1,b.

(i) For i=1,2,...,b-1

$$P_{oi}(t) = \int_o^t \Big[\sum_{a=1}^{b} \sum_{\ell=o}^{i} F_a^{\ell}(u) \sum_{m=o}^{\infty} h^{*m}(t-u)\wedge_{i-\ell}(t-u) +$$

$$M(u) \int_u^t \sum_{a=1}^{b} \sum_{\ell=o}^{i} F_a^{\ell}(z-u) \sum_{m=o}^{\infty} h^{*m}(t-z)\wedge_{i-\ell}(t-z)dz\Big]du \quad (1)$$

(ii) For i=b.

$$P_{ob}(t) = \int_o^t \int_u^t \Big[\sum_{a=1}^{b} \sum_{\ell=o}^{b-1} F_a^{\ell}(u) \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{j=o}^{b-1-\ell} \wedge_j(v-u)$$

$$\bar{\wedge}_{b-(\ell+j)}(t-v)[1-H(t-v)]$$

$$+ M(u)\Big(\sum_{a=1}^{b} \sum_{\ell=o}^{b-1} F_a^{\ell}(v-u) \int_v^t \sum_{m=o}^{\infty} h^{*m}(z-v) \sum_{j=o}^{b-1-\ell} \wedge_j(z-v)\bar{\wedge}_{b-(\ell+j)}(t-z)$$

$$(1-H(t-z))dz\Big]dv\ du \qquad (2)$$

(iii) For j=1,2,...,b; i=0,1,2,...,b

$$P_{ji}(t) = (1-G_j(t))q_j \wedge_{i-(b-j)}(t) + \int_o^t (M(u) + \sum_{a=1}^{b} \sum_{\ell=1}^{b} K_a^{\ell}(u))$$

$$(1-G_j(t-u))q_j \wedge_{i-(b-j)}(t-u)\ du \qquad (3)$$

and (iv) for j=i=o

$$P_{oo}(t) = G_b(t)\ q_b \wedge_o(t) + \int_o^t \int_u^t (M(u) + \sum_{a=1}^{b} \sum_{\ell=1}^{b} K_a^{\ell}(u))$$

$$\{g_b(v-u)\ q_b \wedge_o(t-u) \sum_{m=o}^{\infty} h^{*m}(t-v)\}\ dv\ du \qquad (4)$$

8.5. <u>Busy Period Distribution</u>

For the present model, define the busy period of the system as the time duration for which the system continuously remains non-empty. To obtain the so-defined busy period distribution we proceed as follows:

Delete the first column of $\mathbb{H}(.)$ and represent the corresponding $((\, b \times (b-1)\,))$ matrix by $\mathbb{H}_1(.)$.

$$\mathbb{H}_1(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ f_{21}(x) & 0 & 0 & \cdots & 0 \\ f_{31}(x) & f_{32}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \cdot & \cdot & \cdot & & \cdot \\ f_{b1}(x) & f_{b2}(x) & \cdot & \cdots & f_{bb-1}(x) \end{bmatrix}$$

Hence $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{H}_1)(x)$ is a (b-1) component row vector. Denote this by $F_i^{(1)}(x)$ whose ℓ^{th} coordinate we write as $F_i^{(1)\ell}(x)$. Then $F_i^{(1)\ell}(x)$ represents the probability that the system starting with the service of a batch of size i units $(1 \leqslant i \leqslant b)$ initially, the busy period ends in $(x, x+dx]$ with ℓ units waiting for service, $\ell = 1,2,\ldots,b-1$.

Define $\quad {}_oA(u) = \Pr\{u < T_n \leqslant u+du\}$

Then $\quad {}_oA(u) = \int_o^u \int_z^u \sum_{i=1}^{b} \sum_{\ell=1}^{b-1} F_i^{(1)\ell}(z) \sum_{m=o}^{\infty} h^{*m}(v-z)h(u-v)$

$$\sum_{j=o}^{b-1-\ell} \wedge_j(v-z)\, \overline{\wedge}_{b-(\ell+j)}(u-v)\; dv\, dz$$

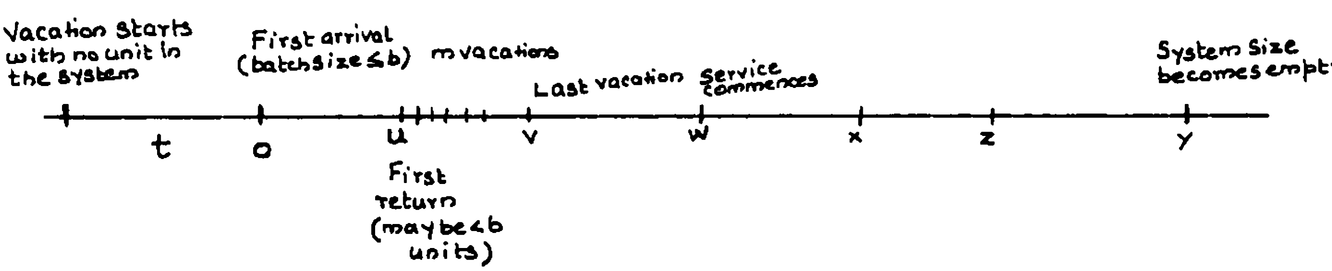
Further let

$$ {}_oM(u) = \Pr\{u < T_1+T_2+\ldots+T_n \leqslant u+du\} $$

Then

$$ {}_oM(u) = \sum_{n=1}^{\infty} {}_oA^{*u}(u) $$

Now we derive the distribution of the busy period



Suppose at time O the first arrival has occured (with a batch size $\leqslant$ b units) during the vacation where we assume that the vacation has started with no unit in the system. The distribution of the above defined busy period is given by B(y) where

$$ B(y) = \Pr\{y < Y \leqslant y+dy\} $$

$$ = \int_{(t)} \int_o^y \int_u^y \int_v^y \sum_{j+\ell_1+\ell_2 \leqslant b-1} \Big\{(p_j \lambda e^{-\lambda t}(h(t+u) * \sum_{m=o}^{\infty} h^{*m}(v-u)h(w-v)) $$

$$ \Big(\sum_{k=1}^{\ell_1} p_{\ell_1}^{*k} \frac{e^{-\lambda u}(\lambda u)^k}{k!} \Big(\sum_{r=1}^{\ell_2} e^{-\lambda(v-u)} \frac{(\lambda(v-u))^r}{r!} p_{\ell_2}^{*r} \Big) $$

$$\Big(\sum_{\ell_3=b-(j+\ell_1+\ell_2)}^{b}\ \sum_{n=1}^{\ell_3} \Big(e^{-\lambda(w-v)}\ \frac{(\lambda(w-v))^n}{n!}\ p_{\ell_3}^{*n}\ \Big)$$

$$(g_b(y-w)q_b \wedge_o(y-w)+ \int_w^y \int_x^y [\,{}_oM(x-w)\Big(\sum_{i=1}^{b}\sum_{\ell=1}^{b} K_i^{\ell}(z-x)\Big)f_{bo}(y-z)$$

$$dz\ dx])\Big\}dw\ dv\ du\ dt \qquad (5)$$

8.6 <u>Virtual waiting timedistribution</u>

By virtual waiting time at time t in the queue, we mean the amount of time an arrival has to wait in the queue before it being taken for service if it were to arrive at time t (Takac's (1962)).

Let the virtual waiting time at time t be W_t. Expression (5) is an upper bound for W_t. We get sharper bounds in (o,b-1). In this case,

$P\{W_t \leqslant x\}$ is the probability that vacation is completed at or prior to t+x and a batch of size b is taken for service + probability that a batch of size (b-1) is taken for service after completing the vacation and during its service time which ends at or prior to t+x at least (b-1) units have arrived + Probability that a batch of size (b-2) is taken for service and .... + ... We illustrate this by restricting b=2. The different possibilities for the state of the system at time t in this case are $\{(0,0),\ (0,1),\ (0,2),\ (1,1),\ (1,2),\ (2,0),\ (2,1),\ (2,2)\}$

(i) For (0,0):

$$P\{W_t \leqslant x\} = \int_0^t g_2(u)\ q_2\Lambda_o(t) \int_t^{t+x} h(v-u) \quad \bar{\Lambda}_1(v-t)dv\ du +$$

$$\int_0^t \sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u) \int_u^t g_2(w-u)q_2\Lambda_o(t-u)$$

$$\int_t^{t+x} h(v-w) \quad \bar{\Lambda}_1(v-t)dv\ dw\ du + \int_0^t \int_0^t M(u)g_2(w-u)q_2\Lambda_o(t-u)$$

$$\int_t^{t+x} h(v-w) \quad \bar{\Lambda}_1(v-t)dv\ dw\ du \tag{1}$$

(ii) For (0,1):

$$P\{W_t \leqslant x\} = \int_0^t (g_2(u)q_2\Lambda_1(t)+g_1(u)q_1\Lambda_o(t)) \int_t^{t+x} h(v-u)\sum_{i=o}^{\infty}\Lambda_i(v-t)$$

$$\{q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) + \int_s^{t+x} f_{10}(s-v)h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)\}$$

$$dz\ ds\ dv\ du + \int_0^t \left(\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)\right)$$

$$\int_u^t (g_2(w-u)q_2\Lambda_1(t-u)+g_1(w-u)q_1\Lambda_o(t-u)) +$$

$$\int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \Lambda_i(v-t) \{q_2 + \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v) h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)\} \, dz \, ds \, dv \, dw \, du +$$

$$\int_o^t M(u) \int_u^t (g_2(w-u) q_2 \Lambda_1(t-u) + g_1(w-u) q_1 \Lambda_o(t-u))$$

$$\int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \Lambda_i(v-t) \{q_2 + \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v) \, h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)\} \, dz \, ds \, dv \, dw \, du$$

(iii) <u>For (O,2)</u>

$$P\{W_t \leqslant x\} = \int_o^t [g_2(u) q_2 \Lambda_2(t) + g_1(u) q_1 \Lambda_1(t)] \int_t^{t+x} h(v-u) \sum_{i=o}^{\infty} \Lambda_i(v-t)$$

$$\int_v^{t+x} [\sum_{j=1}^{\infty} f_{2j}(s-v) + \sum_{j=o}^{\infty} f_{1j}(s-v) [\int_s^{t+x} q_2 + \sum_{j=o}^{\infty} f_{1j}(z-s)] +$$

$$\int_s^{t+x} f_{20}(s-v) \, h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s) dz \, ds \, dv \, du +$$

$$\int_o^t [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)] \int_u^t [g_2(w-u) q_2 \Lambda_2(t-u) +$$

$$g_1(w-u) \, q_1 \Lambda_1(t-u)] \int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \Lambda_i(v-t)$$

$$\{\int_v^{t+x} [\sum_{j=1}^{\infty} f_{2j}(s-v) + \sum_{j=o}^{\infty} f_{1j}(s-v) [\int_s^{t+x} q_2 + \int_s^{t+x} \sum_{j=o}^{\infty} f_{ij}(z-s)]$$

$$\int_s^{t+x} f_{20}(s-v) \, h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)]\} \, dz \, ds \, dv \, dw \, du$$

$$+ \int_o^t M(u) \int_u^t [g_2(w-u)q_2 \wedge_2(t-u)+g_1(w-u)q_1 \wedge_1(t-u)]$$

$$\int_t^{t+x} h(v-w) \sum_{i=o}^{\infty} \wedge_i(v-t)\{ \int_v^{t+x} [\sum_{j=1}^{\infty} f_{2j}(s-v)+ \sum_{j=o}^{\infty} f_{1j}(s-v)$$

$$[\int_s^{t+x} (q_2+ \sum_{j=o}^{\infty} f_{1j}(z-s))] + \int_s^{t+x} f_{20}(s-v)\ h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)]\}$$

$$dz\ ds\ dv\ dw\ du$$

<u>For (1,1)</u>

$$P\ \{W_t \leqslant x\} = \wedge_o(t) \int_t^{t+x} g_1(u)\ q_1[q_2+ \int_u^{t+x} \sum_{j=1}^{\infty} f_{1j}(v-u)$$

$$+ \int_v^{t+x} f_{10}(v-u)\ h(z-v) \sum_{i=1}^{\infty} \wedge_i(z-v)dz\ dv\ du$$

$$+ \int_o^t [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell} (u)] \wedge_o(t-u) \int_t^{t+x} g_1(v-u)$$

$$q_1 \left\{ q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) + \int_t^{t+x} f_{10}(s-v)h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)\right\}$$

$$dz\ ds\ dv\ du$$

$$+ \int_o^t M(u) \wedge_o(t-u) \int_t^{t+x} g_1(v-u)\ q_1 \left\{ q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v)\right.$$

$$\left. + \int_s^{t+x} f_{10}(s-v)\ h(z-s) \sum_{i=1}^{\infty} \wedge_i(z-s)\right\} dz\ ds\ dv\ du$$

For (1,2)

$$P\{W_t \leqslant x\} = \Lambda_1(t) \int_t^{t+x} g_1(u)\ q_1 \int_u^{t+x} \sum_{j=1}^{\infty} f_{2j}(v-u) +$$

$$[\sum_{j=o}^{\infty} f_{1j}(v-u)\ [q_2 + \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v)\ h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)]] +$$

$$\int_v^{t+x} f_{20}(v-u)\ h(z-v) \sum_{i=1}^{\infty} \Lambda_i(z-v)dz\ ds\ dv\ du +$$

$$\int_o^t [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)]\ \Lambda_1(t-u) \int_t^{t+x} g_1(v-u)\ q_1 \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(w-v) +$$

$$[\sum_{j=o}^{\infty} f_{1j}(w-v)(q_2 + \int_w^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-w) +$$

$$\int_s^{t+x} f_{10}(s-w)\ h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)] +$$

$$\int_w^{t+x} f_{20}(w-v)\ h(z-w) \sum_{i=1}^{\infty} \Lambda_i(z-w)dz\ ds\ dw\ dv\ du +$$

$$\int_o^t M(u)\ \Lambda_1(t-u) \int_t^{t+x} g_1(v-w)\ q_1 \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(w-v)$$

$$\int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(w-v) + [\sum_{j=o}^{\infty} f_{ij}(w-v)(q_2 + \int_w^{t+x} \sum_{j=1}^{\infty} f_{ij}(s-w) +$$

$$\int_s^{t+x} f_{10}(s-w)\ h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)] +$$

$$\int_{w}^{t+x} f_{20}(w-v)\ h(z-w) \sum_{i=1}^{\infty} \Lambda_i(z-w)dz\ ds\ dw\ dv\ du$$

For (2,0)

$$P\{W_t \leqslant x\} = \Lambda_o(t) \int_{t}^{t+x} g_2(u)\ q_2\ [\Lambda_o(u-t) \int_{u}^{t+x} h(v-u) \sum_{i=1}^{\infty} \Lambda_i(v-u)]$$

$$+ \sum_{i=1}^{\infty} \Lambda_i(u-t)\ dv\ du +$$

$$\int_{o}^{t} [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)]\Lambda_o(t-u) \int_{t}^{t+x} g_2(v-u)q_2\{[\Lambda_o(v-t)$$

$$\int_{v}^{t+x} h(z-v) \sum_{i=1}^{\infty} \Lambda_i(z-v)] + \sum_{i=1}^{\infty} \Lambda_i(v-t)\}\ dz\ dv\ du +$$

$$\int_{o}^{t} M(u)\ \Lambda_o(t-u) \int_{t}^{t+x} g_2(v-u)\ q_2\{[\Lambda_o(v-t) \int_{v}^{t+x} h(z-v)$$

$$\sum_{i=1}^{\infty} \Lambda_i(z-v)] + \sum_{i=1}^{\infty} \Lambda_i(v-t)\}\ dz\ dv\ du$$

For (2,1)

$$P\{W_t \leqslant x\} = \Lambda_1(t) \int_{t}^{t+x} g_2(u)\ q_2[q_2 + \int_{u}^{t+x} \sum_{j=1}^{\infty} f_{1j}(v-u)\ +$$

$$\int_{v}^{t+x} f_{10}(v-u)\ h(z-v) \sum_{i=1}^{\infty} \Lambda_i(z-v)]\ dz\ dv\ du$$

$$\int_{o}^{t} [\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u)]\ \Lambda_1(t-u) \int_{t}^{t+x} g_2(v-u)q_2[q_2 + \int_{v}^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_{s}^{t+x} f_{10}(s-v)h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)]dz\ ds\ dv\ du +$$

$$\int_o^t M(u)\ \Lambda_1(t-u) \int_t^{t+x} g_2(v-u)\ q_2[q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{ij}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v)\ h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)]dz\ ds\ dv\ du$$

For (2,2):

$$P\{W_t \leqslant x\} = \Lambda_2(t) \int_t^{t+x} g_2(u)\ g_2\Big\{ \int_u^{t+x} \sum_{j=1}^{\infty} f_{2j}(v-u) +$$

$$\sum_{j=o}^{\infty} f_{1j}(v-u)\ [q_2+ \int_v^{t+x} \sum_{j=1}^{\infty} f_{1j}(s-v) +$$

$$\int_s^{t+x} f_{10}(s-v)\ h(z-s) \sum_{i=1}^{\infty} \Lambda_i(z-s)]\Big\}\ dz\ ds\ dv\ du +$$

$$\int_o^t (\sum_{a=1}^{2} \sum_{\ell=1}^{2} K_a^{\ell}(u))\Lambda_2(t-u) \int_t^{t+x} g_2(v-u)\ q_2\Big\{ \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(s-v)+$$

$$\sum_{j=o}^{\infty} f_{1j}(s-v)[q_2+ \int_s^{t+x} \sum_{j=1}^{\infty} f_{1j}(w-s)+ \int_w^{t+x} f_{10}(w-s)h(z-w)$$

$$\sum_{i=1}^{\infty} \Lambda_i(z-w)]\Big\}dz\ dw\ ds\ dv\ du +$$

$$\int_o^t M(u)\Lambda_2(t-u) \int_t^{t+x} g_2(v-u)\ q_2+\Big\{ \int_v^{t+x} \sum_{j=1}^{\infty} f_{2j}(s-v) +$$

$$\sum_{j=o}^{\infty} f_{1j}(s-v)\ [q_2+ \int_s^{t+x} \sum_{j=1}^{\infty} f_{1j}(w-s)+ \int_w^{t+x} f_{10}(w-s)\ h(z-w)$$

$$\sum_{i=1}^{\infty} \Lambda_i(z-w)]\Big\}\ dz\ dw\ ds\ dv\ du.$$

REFERENCES

1. Ali, O.M.E. and M.F. Neuts (1984), A service system with two stages of waiting and feedback of customers, J.Appl.Prob., 21, 404-413.

2. Arrow, K.J., T. Harris and J. Marschak (1951), Optimal inventory policy, Econometrica, 19 No.3, 250-272.

3. Arrow, K.J., S. Karlin and H. Scarf (1958), Studies in the Mathematical theory of Inventory and Production, Standford University Press, Standford.

4. Asmussen, S (1987), Applied Probability and Queues, Wiley, New York.

5. Avi-Itzhak, B., W.L. Maxwell and L.W.Miller (1965), Queueing with alternating priorities, Oper.Res., 13, 306-318.

6. Bagchi, T.P. and J.G.C.Templeton (1972), Numerical Methods in Markov chains and Bulk Queues, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, New York.

7. Bagchi, T.P. and J.G.C.Templeton (1973), A note on the $M^X/G^Y/1,k$ bulk queueing system, J.Appl.Prob., 10,901-906.

8. Bailey, N.T.J. (1954a), On the queueing process with bulk service, J.Roy.Statist. Soc. B, 16, 80-87.

9. Bailey, N.T.J. (1954b), A continuous time treatment of a simple queue using generating function, J.Roy.Statist. Soc. B, 16, 288-291.

10. Beckmann, M.J. and S.K.Srinivasan (1987), An (s,S) inventory system with Poisson demands and exponential lead time, O.R. Spectrum, 9, No.4, 213-217.

11. Bellmann, R., I. Glicksberg and O. Gross (1955), On the optimal inventory equation, Mgmt. Sci., 2, 83-104.

12. Bertsimas Dimitris (1990), An analytic approach to a general class of G/G/s queueing systems, Oper. Res. 38, No.2, 139-155.

13. Bhat, U.N. (1964), Imbedded Markov chain analysis of bulk queues, J. Austral.Math.Soc. 4, 244-263.

14. Bhat, U.N. (1967), Some explicit results for the queue GI/M/1 with group service, Sankhya, Series A, 29, Part 2, 199-206.

15. Bhat, U.N. (1968), A study of the queueing systems, M/G/1 and GI/M/1, Lecture Notes in O.R. and Math. Eco. 2, Springer-Verlag, Berlin.

16. Bhat, U.N. (1972), Elements of Applied Stochastic Processes, Wiley, New York.

17. Bhat, U.N. (1984), Elements of Applied Stochastic Processes, 2nd Edition, John Wiley and Sons, New York.

18. Borthakur, A. (1971a), A Poisson queue with a general bulk service rule, J.Assam.Se.Soc. XIV, 162-167.

19. Borthakur, A. (1971b), An additional special channel Poisson queueing system with a general rule for bulk service, Inst. of Math. Statist. Bulletin, 72 t-8.

20. Champernowne, D.G. (1956), An elementary method of solution of the queueing problem with a single server and continuous parameter, J.Roy.Statist.Soc.,B, 18, 125-128.

21. Chaudhary, M.L. (1979), The queueing system $M^X/G/1$ and its ramification, Naval Res. Logist. Quarterly, 26, No.4.

22. Chaudhary, M.L. and J.G.C. Templeton (1981), The queueing system $M/G^B/1$ and its ramifications, Euro.J.Opnl.Res. 6, 56-60.

23. Chaudhary, M.L. and J.G.C. Templeton (1984), A first course in bulk queues, John Wiley and Sons, New York.

24. Cinlar, E. (1969), Markov renewal theory, Adv.Appl.Prob.1, 123-187.

25. Cinlar, E. (1975a), Markov renewal theory: A Survey, Mgmt. Sci., 21, 727-752.

26. Cinlar, E. (1975b), Introduction to Stochastic Processes, Prentice-Hall, Inc. Englewood Cliffs, New Jersey.

27. Cohen, J.W. (1969), The single server queue, North Holland Publishing Co., Amsterdam.

28. Cohen, J.W. (1982), The single server queue, 2nd revised ed. New York, Elsevier, North Holland.

29. Conolly, B.W. (1958), A difference equation technique applied to a simple queue with arbitrary arrival interval distribution, J.Roy.Statist.Soc.B, 21, 168-175.

30. Cooper, R.B. (1970), Queues served in cyclic order: waiting times, Bell System Tech.J. 49, 399-413.

31. Cooper, R.B. (1972), Introduction to Queueing Theory, Macmillan, New York.

32. Cooper, R.B. (1981), Introduction to Queueing Theory, North Holland (Elsevier), New York.

33. Cox, D.R. (1955), The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables, Proc. Cambridge Phil. Soc. 51, 433-441.

34. Cox, D.R. (1962), Renewal theory, Methuen, London.

35. Cox, D.R. and W.L. Smith (1961), Queues, Methuen, London.

36. Daniel, J.K. (1985), Queueing and Inventory models with rest periods, Ph.D. dissertation, University of Cochin.

37. Daniel, J.K. and R. Ramanarayanan (1987), An (s,S) inventory system with two servers and rest periods, Cahiers du C.E.R.O., 29.

38. Daniel, J.K. and R. Ramanarayanan (1988), An (s,S) inventory system with rest to the server, Naval Res. Logist., **35**, 119-123.

39. De Kok, A.G., H.C. Tijms and F.A. Vander Duyn Schouten (1984), Approximations for the single product production-inventory problem with compound Poisson demand and service level constraints, Adv. Appl. Prob. 16, 378-401.

40. Doshi, B.T. (1985), A note on stochastic decomposition in a GI/G/1 queue with vacation or set up times, J.Appl. Prob., 22, 419-428.

41. Doshi, B.T. (1986), Queueing systems with vacations- A survey, Queueing Systems, 1, 29-66.

42. Dvoretzky, A., J. Keifer and J. Wolfowitz (1952), The inventory problem I, II, Econometrika, 20, 187-222, 450-460.

43. Easton, G.D. and M.L. Chaudhary (1982), The queueing system $E_k/M^{a,b}/1$ and its numerical analysis, Comp.Oper.Res., 9, 197-205.

44. Erlang, A.K. (1909), Probability and Telephone calls, Nyt.Tidsskr, Math. Ser. B, 20, 33-39.

53. Gross, D. and C.M. Harris (1974), Fundamentals of Queueing Theory, John Wiley and Sons, New York.

54. Gross, D. and C.M. Harris (1984), Fundamentals of Queueing Theory, (revised edition), John Wiley and Sons, New York.

55. Gupta, S.R. (1967), Queues with Hyper-Poisson input and exponential service time distribution with state dependent arrival and service rates, Oper. Res. 15, 847-856.

56. Hadley, G. and T.M. Whitin (1963), Analysis of Inventory Systems, Prentice-Hall Inc. Englewood Cliffs, New Jersey.

57. Harris, F. (1915), Operations and Cost, Factory Management Series, A.W. Shaw Co., Chicago.

58. Harris, C.M. (1967), Queues with state-dependent stochastic service rates, Oper. Res. 15, 117-130.

59. Harris, C.M. (1970), Some results of bulk-arrival queues with state dependent service times, Mgmt. Sci., 16, 313-326.

60. Heyman, D.P. (1977), The T-policy for the M/G/1 queue, Mgmt. Sci. 23, 775-778.

61. Hillier, F.S., R.W. Conway and W.L.Maxwell (1964), A multiple server queueing model with state dependent service rates, J.Ind.Eng. 15, 153-157.

62. Holman, D.P., M.L. Chaudhary and A. Ghosal (1981), Some results for the general bulk service queueing system, Bull. Austral. Math. Soc. 25, 161-179.

63. Hurter, A.P. and F.C. Kaminsky (1967), An application of stochastic processes to a problem in inventory control, Oper. Res. 15, 467-472.

64. Iglehart, D.L. and S. Karlin (1962), Optimal policy to dynamic inventory process with non-stationary stochastic demands, Arrow, K.J., S. Karlin and H. Scarf (eds.), Studies in Applied Probability and Management Science, Standford Univ. Press, Standford.

65. Jacob, M.J. (1988), Probabilistic analysis of some queueing and inventory systems, Ph.D. thesis, Cochin University of Science and Technology.

66. Jacob, M.J., A. Krishnamoorthy and T.P.Madhusoodanan (1988), Transient solution of finite capacity $M/G^{a,b}/1$ queueing system, Naval Res. Logist. Quart. 35, 437-441.

67. Jacob, M.J. and T.P.Madhusoodanan (1987), Transient solution for a finite capacity $M/G^{a,b}/1$ queueing system with vacations to the server, Queueing Systems, 2, 381-386.

68. Jaiswal, N.K. (1960), Time-dependent solution of the bulk service queueing problem, Oper.Res.8, No.6, 773-781.

69. Jaiswal, N.K. (1962), Distribution of busy periods for the bulk service queueing problems, Defence Science Journal, October.

70. Kalpakam, S and G. Arivarignan (1985a), Analysis of an exhibiting inventory system, Stoch. Analy. Appl., 3(4), 447-465.

71. Kalpakam, S. and G. Arivarignan (1985 b), A continuous review inventory system with arbitrary interarrival times between failure, Opsearch, 22, No.3, 153-168.

72. Kambo, N.S. and M.L. Chaudhary (1982), Distribution of the busy period for the bulk service queueing system $E_k/M^{a,b}/1$, Comp. Opns. Res. 9, 86-1 to 86-7.

73. Kaplan, R.S. (1970), A dynamic inventory model with stochastic lead times, Mgmt. Sci., 16, 491-507.

74. Karlin, S and A. Fabens (1959), A stationary inventory model with Markovian demand, Chap. 11 in Arrow K.J., S. Karlin and P. Suppes (eds), Mathematical Methods in the Social Sciences, Standford Univ. Press, 1960, Standford.

75. Kendall, D.G. (1951), Some problems in the theory of queues, J. Roy. Statist. Soc.B, 13, 151-185.

76. Kendall, D.G. (1953), Stochastic Processes occurring in the theory of queues and their analysis by the method of imbedded Markov chains, Ann.Math.Statist., 24, 338-354.

77. Krishnamoorthy, A. and M. Manoharan (1990), An (s,S) inventory system with state dependent demands, Opsearch, 27, No.2, 93-101.

78. Krishnamoorthy, A. and M. Manoharan (1991), An inventory system with unit demand and varying ordering levels, Optimization, No.1, 7-15.

79. Krishnamoorthy, A. and Rajan Varughese (1990 a), $M/G^{a,b}/1$ queue with Markov dependent batch size,(Communicated to Queueing Systems).

80. Krishnamoorthy, A. and Rajan Varughese (1990 b), $G/M^{a,b}/1$ queue with Markov dependent batch size (Communicated to Stochastic Analysis and Applications).

81. Lavenberg, S.S. (1975), The steady state queueing time distribution for the M/G/1 finite capacity queue, Mgmt.Sci. 21, No.5, 501-506.

82. Ledermann, W. and G.E.H. Reuter (1956), Spectral theory for the differential equation of simple birth and death processes, Phil. Trans. Roy. Soc. London, A, 246, 321-369.

83. Lee, T.T. (1984), M/G/1/N queue with vacation time and exhaustive service discipline, Oper. Res. 32, 774-784.

84. Levy, P. (1954), Processus Semi-Markoviens, Proc. Int.Congr. Math. (Amsterdam), 3, 416-426.

85. Levy, H. and U. Yechiali (1975), Utilization of idle time in an M/G/1 queueing system, Mgmt. Sci. 22, 202-211.

86. Lindley, D.V. (1952), The theory of queues with a single server, Proc. Cambr. Phil. Soc. 48, 277-289.

87. Liu, L., B.R.K. Kashyap and J.G.C. Templeton (1987), The service system $M/M^R/\infty$ with impatient customers, Queueing systems, 2, 363-372.

88. Madhusoodanan, T.P. (1989), Time-dependent solutions for some queueing and inventory models, Ph.D. dissertation, Cochin University of Science and Technology.

89. Manoharan, M. (1990), Analysis of some stochastic models in Inventories and Queues, Ph.D. dissertation, Cochin University of Science and Technology.

90. Manoharan, M. and A. Krishnamoorthy (1989a), A finite capacity M/G/1 queue with vacation, Tamkang Journal of Mgmt.Sci., 10, No.2, 37-45.

91. Manoharan, M. and A.Krishnamoorthy (1989b), Markov renewal theoretic analysis of a perishable inventory system, Tamkang Journal of Mgmt. Sci., 10, No.2, 47-57.

92. Manoharan, M, A.Krishnamoorthy and T.P.Madhusoodanan (1987), On the (s,S) inventory policy with independent non-identically distributed interarrival demand times and random lead time, Cahiers du C.E.R.O., 29, 239-248.

93. Masse, P. (1946), Les Reserver et. la Regulation de i
Avenier dans la vie Economique, 2 Vols, Herman and Cie, Paris.

94. Medhi, J. (1975), Waiting time distribution in a Poisson queue with general bulk service rule, Mgmt. Sci., 21, 777-782.

95. Medhi, J. (1984), Recent developments in bulk queueing models, Wiley Eastern Ltd., New Delhi.

96. Miller, R.G. (1959), A contribution to the theory of bulk queues, J. Roy. Statist. Soc. Ser.B, 21, 320-337.

97. Miller, L.W. (1975), A note on the busy period of an M/G/1 finite queue, Oper. Res. 23, 1179-1182.

98. Morse, P.M. (1958), Queues, Inventories and Maintenance, Wiley, New York.

99. Murari, K. (1969), A queueing problem with correlated arrivals and general service time distribution, Z. Angew Math. Mech. 49, 151-159.

100. Naddor, E. (1966), Inventory Systems, John Wiley and Sons, Inc. New York.

101. Nahmias, S. (1982), Perishable inventory theory: a review, Oper. Res. 30, 680-708.

102. Neuts, M.F. (1966 a), Semi-Markov analysis of a bulk queue, Bull. Soc. Math. Belg. 18, 28-42.

103. Neuts, M.F. (1986 b), The single server queue with Poisson input and semi-Markov service times, J.Appl. Prob. 3, 722-734.

104. Neuts, M.F. (1967), A general class of bulk queues with Poisson input, Ann. Math. Statist., 38, 759-770.

105. Neuts, M.F. (1977), Some explicit formulas for the steady state behaviour of the queue with semi-Markovian service times, Adv.Appl.Prob.9, 141-157.

106. Neuts, M.F. (1979), Queues Solvable without Rouches theorem, Oper.Res. 27, 767-781.

107. Parthasarathy, P.R. (1987), A transient solution to an M/M/1 queue: A simple approach, Adv.Appl.Prob. 19, 997-998.

108. Parthasarathy, P.R. and M. Sharafali (1989), Transient solution to the many-server Poisson queue: A simple approach, Adv.Appl.Prob., 21, 584-594.

109. Pegden, C.D. and M. Rosenshine (1982), Some new results for the M/M/1 queue, Mgmt.Sci., 28, 821-828.

110. Pollaczck, F (1964), Concerning an analytic method for the treatment of queueing problems, Proc.Symp. on Congestion theory, Univ. of North Carolina, 1-42.

111. Prabhu, N.U. (1965), <u>Queues and Inventories</u>, John Wiley and Sons, Inc. New York.

112. Prabhu, N.U. (1980), <u>Stochastic Storage Processes</u>, Springer-Verlag, New York.

113. Ponser, M. (1973), Single-Server queues with service time dependent on waiting time, Oper.Res. 21, 610-616.

114. Pyke, R. (1961a), Markov renewal processes: definitions and preliminary properties, Ann.Math.Statist. 32, 1231-1242.

115. Pyke, R. (1961b), Markov renewal processes with finitely many states, Ann.Math.Statist. 32, 1243-1259.

116. Ramachandran Nair, V.K. (1988), Queueing models with rest to the server after serving a random number of units, Ph.D. thesis, Cochin University of Science and Technology.

117. Ramanarayanan, R. and M.J. Jacob (1986), An inventory system with random lead times and varying ordering levels, (unpublished).

118. Ramanarayanan, R. and M.J. Jacob (1987), General analysis of (s,S) inventory systems with random lead times and bulk demand, Cahiers du C.E.R.O.,29, Nos. 3-4, 249-254.

119. Ramaswami, V. (1981), Algorithms for a continuous review (s,S) inventory system, J.Appl.Prob. 18, 461-472.

120. Rosenshine, M. (1967), Queues with state-dependent service times, Trans. Res. 1, 97-104.

121. Richards, F.R. (1975), Comments on the distribution of inventory position in a continuous review (s,S) policy inventory system, Oper.Res. 23, 366-371.

122. Ross, S.M. (1970), Applied Probability Models with Optimization Applications, Holden-Day, San Francisco.

123. Ryshikov, J.I. (1973), Lagerhattung, Akademic Verlag, Berlin.

124. Saaty, T.L. (1961), Elements of Queueing Theory with Applications, McGraw Hill, New York.

125. Sahin, I. (1979), On the stationary analysis of continuous review (s,S) inventory systems with constant lead times, Oper.Res. 27, 717-729.

126. Sahin, I. (1983), On the continuous review (s,S) inventory model under compound renewal demand and random lead times, J.Appl.Prob. 20, 213-219.

127. Scarf, H. (1960), The optimality of (s,S) policies in the dynamic inventory proble, Chapter 13 in Mathematical Methods in the Social Sciences, Arrow, K.J., S.Karlin and P. Suppes (eds.), Standford University Press, Standford.

128. Schal, M. (1971), The analysis of queues with state-dependent parameters by Markov renewal processes, Adv.Appl.Prob., 3, 155-175.

129. Scholl, M. and Kleinrock (1983), On the M/G/1 queue with rest periods and certain service-independent queue disciplines, Oper.Res. 31, 705-719.

130. Shantikumar, J.G. (1979), On a single server queue with state dependent service, Naval.Res.Logist.Quart. 26, No.2, 305-309.

131. Shantikumar, J.G. (1980), Some analysis of the control of queues using level crossing of regeneration processes, J.Appl.Prob. 17, 814-821.

132. Shantikumar, J.G. (1982), Analysis of a single server queue with time and operation dependent server failures, Adv.in Mgmt.Sci.1, 339-359.

133. Shantikumar, J.G. (1985), Bilateral Phase-type distribution, Naval Res.Logist.Quart. 32, 119-136.

134. Sivazlian, B.D. (1974), A continuous review (s,S) inventory system with arbitrary interarrival distribution between unit demands, Oper.Res. 22, 65-71.

135. Smith, W.L. (1955), Regenerative Stochastic Processes, Proc. Roy.Soc.Ser.A., 232, 6-31.

136. Smith, W.L. (1958), Renewal theory and its ramifications, J.Roy.Statist.Soc. Ser.B. 20(27), 243-302.

137. Srinivasan, S.K. (1979), General analysis of S-s inventory systems with random lead times and unit demands, J.Math.Phy.Sci. 13, 107-129.

138. Syski, R. (1988), Further comments on the solution of the M/M/1 queue, Adv.Appl.Prob. 20, 693.

139. Takacs, L. (1961), Transient behaviour of single server queueing processes with Erlang input, Trans. Amer. Math.Soc., 100, 1-28.

140. Takacs, L. (1962), Introduction to the theory of Queues, Oxford Univ. Press, New York.

141. Teghum, J., J.Loris-Teghum, J.P. Lambotte (1969), Modèles d' Attente M/G/1 et GI/M/1 à Arrivées et services en Groupes, Springer-Verlag Berlin, Heidelberg, New York.

142. Thangaraj, V. and R. Ramanarayanan (1983), An operating policy in inventory systems with random lead times and unit demands, Math.Operations forsch.U. Statist.Ser.Optim. 14, 111-124.

143. Tijms, H. (1972), Analysis of (s,S) inventory models, Mathematical Centre Tracts 10, Amsterdam.

144. Tijms, H. (1986), Stochastic Modelling and Analysis, John Wiley and Sons, Inc. New York.

145. Veinott, A.F. Jr. and H.M. Wagner (1965), Computing optimal (s,S) inventory policies, Mgmt. Sci. 11, No.5, 525-552.

146. Veinott, A.F. Jr. (1966), The status of mathematical inventory theory, Mgmt. Sci. 12, 745-777.

147. Welch, P.D. (1964), On a generalised M/G/1 queueing process in which the first customer of each busy period receives exponential service, Oper.Res., 12, 736-752.

148. Whitin, T.M. (1953), The Theory of Inventory Management, Princeton University Press, Princeton.

149. Widder, D.V. (1946), The Laplace Transform, Princeton University Press, Princeton.

45. Fabens, A.J. (1961), The solution of queueing and inventory models by semi-Markov processes, J.Roy. Statist. Soc. B, 23, 113-117.

46. Fabens, A.J. and A.G.A.D. Perera (1963), A correction to the above, J. Roy. Statist. Soc. B., 25, 455.

47. Feldmann, R.M. (1975), A continuous review (s,S) inventory system in a random environment, J.Appl.Prob. 15, 654-659.

48. Feller, W. (1965), An Introduction to Probability Theory and its Applications, Vol.1 and II, John Wiley and Sons, Inc. New York.

49. Gani, J. (1957), Problems in the probability theory of storage systems, J. Royal Statist.Soc.B, 19, 181-200.

50. Gaver, D.P. Jr. (1959), Imbedded Markov chain analysis of a waiting line process in continuous time, Ann. Math. Statist. 30, 698-720.

51. Gnedenko, B.V. and I.V. Kovalenko (1968), Introduction to Queueing Theory, Israel Program for Scientific Translations, Jerusalem.

52. Gross, D. and C.M. Harris (1971), On one-for-one ordering inventory policies with state dependent lead times, Oper. Res. 19, 735-760.