ANALYSIS OF FLUCTUATIONS IN THE EARLY UNIVERSE USING FOKKER-PLANCK FORMALISM

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by
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DECLARATION

I hereby declare that the thesis titled **Analysis of Fluctuations in the Early Universe Using Fokker-Planck Formalism**, submitted for the degree of Doctor of Philosophy is a bonafide record of the research carried out by me under the guidance of Prof. K Babu Joseph, in the Department of Physics, Cochin University of Science and Technology, and that no part of it has been included in any other thesis submitted previously for the award of any degree of any university.

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Date- 5-09-2001

CERTIFICATE

Certified that the thesis titled **Analysis of Fluctuations in the Early Universe Using Fokker-Planck Formalism** is a bonafide record of the research carried out by Mr. Sivakumar C, under my supervision in the Department of Physics, Cochin University of Science and Technology, in partial fulfilment of the requirements for the award of the Degree of Doctor of Philosophy, and no part of it has been included in any other thesis submitted previously for the award of any degree of any university.

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Prof. K Babu Joseph

(Supervising teacher)

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Contents

PREFACE

The work presented in this thesis has been carried out by the author at the Department of Physics, Cochin University of Science and Technology during the period 1996 to 2001.

Einstein's General Theory of Relativity (GTR) laid the foundations of modern theoretical cosmology, which assumes a central role in interpreting astrophysical and cosmological data. It provides a frame-work which is free from the difficulties of Newtonian gravity with respect to special theory of relativity and special theory of relativity with respect to gravity. Unlike astronomy, cosmology is highly speculative in nature and lacked observational evidence in early days. However, during the past decade, the observational support to cosmology has tremendously increased with the advent of the Hubble space telescope and the Cosmic Background Explorer (COBE) satellite. Yet, since we cannot experiment with the cosmos, one can only resort to model making and check how much of the observational data agrees with the predictions of the model. Among the several cosmological models that have been put forward, the standard model or the hot big bang model originated by Friedmann, provides the most successful approximation of the real universe, with maximum consistency with observations. This model is based on following assumptions: (1) the geometry of space is such that at large scales, it is describable by the mathematically simple, spatially symmetric Friedmann-Robertson-Walker metric; (2) the energy-momentum tensor is that which corresponds to a spatially homogeneous and isotropic perfect fluid comprising matter which is either relativistic or non relativistic. This model predicts (1) an early hot phase for the universe, a relic of which is the cosmic microwave background radiation (CMBR), (2) Hubble expansion and (3) the observed abundance of light nuclei in the universe.

However, there are certain problems in the standard model, some of which are directly dependent upon the simplifying assumptions taken. The inflationary model proposed in the early eighties, is a modification of the standard model, which predicts an early exponential expansion for the universe, caused by the potential energy of a scalar or inflation field. This model solves some of the problems in the standard model. The standard model predicts a linear deterministic redshift-apparent magnitude ($z-m$) relation (or a linear Hubble's law). However the Hubble diagram is a scatter diagram with no deterministic Hubble type relation clearly apparent, especially for higher z (in the early universe). The scatter is also found to increase with redshift. Thus the behaviour of the Hubble parameter ($H(t)$) is anomalous in those epochs. Also the uncertainty in the determination of the true value of the Hubble parameter is one of the most intriguing issues in the history of cosmology. In conventional cosmology, the peculiar velocities induced by the observed density fluctuations, are the cause of the randomness in the Hubble diagram. However, peculiar velocities are inadequate for high z, because density fluctuations are evolving phenomena. The observed fractal distribution of galactic clusters over large range of scales is another puzzle within the standard model.

We begin the thesis with a review of basic elements of general theory of relativity (GTR) which forms the basis for the theoretical interpretation of the observations in cosmology. The first chapter also discusses the standard model in cosmology, namely the Friedmann model, its predictions and problems. We have also made a brief discussion on fractals and inflation of the early universe in the first chapter. In the second chapter we discuss the formulation of a new approach to cosmology namely a stochastic approach. In this model, the dynamics of the early universe is described by a set of non-deterministic, Langevin type equations and we derive the solutions using the Fokker-Planck formalism. Here we demonstrate how the problems with the standard model, can be eliminated by introducing the idea of stochastic fluctuations in the early universe. Many recent observations indicate that the present universe may be approximated by a many component fluid and we assume that only the total energy density is conserved. This, in turn, leads to energy transfer between different components of the cosmic fluid and fluctuations in such energy transfer can certainly induce fluctuations in the mean w factor in the equation of state $p = w\rho$, resulting in a fluctuating expansion rate for the universe. We also have made a comparison between theoretical predictions and observations using the Type Ia supernovae data in [25].

The third chapter discusses the stochastic evolution of the cosmological parameters in the early universe, using the new approach. The penultimate chapter is about the refinements to be made in the present model, by means of a new deterministic model [91]. The concluding chapter presents a discussion on other problems with the conventional cosmology, like fractal correlation of galactic distribution. We attempt an explanation for this problem using the stochastic approach.

A part of these investigations has appeared in the form of the following papers published/submitted/presented

1. C Sivakumar, M V John and K Babu Joseph, Pramana-J. Phys. 56, 477 (2001).

2. C Sivakumar, M V john and K Babu Joseph, Int. J. Mod. Phys.D, submitted (2001).

3. M V John, C Sivakumar and K Babu Joseph, Pramana-J. Phys. submitted

(2001).

4. C Sivakumar and K Babu Joseph, to submit (2001).

5. Paper presented at the XIV DAE Sumposium on High energy physics (Hyderabad, 2000) and submitted to the proceedings.

Chapter 1

The Standard Cosmological model

Cosmology is the study of the origin and development of the universe, using the tools of astronomy. Eventhough astronomy started as a branch of science dealing with planets and stars, it now deals with objects which are very far away from us, ie., light from such objects, take billions of years to reach us. On the other hand, cosmology is mainly concerned with the extragalactic world, particularly with the large-scale structure of the universe extending to distances of billions of light years across. Towards the end of the third decade of the twentieth century, Edwin Hubble came up with the discovery that the spectra of galaxies appear to be shifted towards the red end of the spectrum, and that the shift in a given galaxy is proportional to the distance of the galaxy from us (Doppler shift). This striking observation actually laid the foundation for modern observational cosmology. In 1922, Alexander Friedmann had found model solutions of Einstein's equations of General Theory of Relativity(GTR), wherein the property of redshift arose naturally. Friedmann's cosmological models got observational support only in 1929, when Hubble made that remarkable discovery. This model [1, 2, 3] forms the basis of the currently standard picture in cosmology, which is a nearly homogeneous and isotropic expansion of the universe, according to GTR, that traces back to a hot and dense state for the universe. And it was Steven Weinberg who brought the phrase, "the standard model", to cosmology from particle physics. Since the standard model assumes general theory of relativity, let us begin with a brief introduction to it, before going to the details of the Friedmann models.

1.1 Einstein's General Theory of Relativity

General relativity [3-11], the modern theory of gravitation, provides a mathematical model of the physical world and laid the foundations of modern theoretical cosmology. The problem with Newtonian theory of gravity is that it is a theory of instantaneous action at a distance, and hence it fails at very large distances (ie. on cosmological scales). In 1905, Einstein proposed the Special Theory of Relativity(STR), which revolutionised the concepts of space, time and motion on which the Newtonian laws were founded [10]. According to STR there is a limit to the speed beyond which no interaction can propagate. This corresponds to the speed of light which is a constant. Light can never be at rest relative to anything and cannot be acted upon by any force. Then the question arises is, how is light affected by gravitation? The problem with STR is the omnipresence of gravity. No inertial frames and observers exist, which are the basis of STR. Einstein's GTR provides a framework that is free from the difficulties of Newtonian gravity with respect to STR and STR with respect to gravity.

The most distinctive feature of gravity is its permanent character, ie., it is an interaction which cannot be turned on or off at will. Einstein argued that, because of its permanence, gravitation must be related to some intrinsic feature of space and time [3, 11]. He identified this feature as the geometry of space and time(or spacetime) and suggested that the geometry of space (spacetime) is non-Euclidean, ie., spacetime is curved. Non-Euclidean character means that the laws of Euclid are not valid. For example, geometry on the surface of a sphere is non-Euclidean. If we define a straight line on the surface of the sphere as a line of shortest distance between two points, it is easy to see that these lines are great circles, and that any two of these intersect. Thus the laws of Euclid do not hold here.

In GTR the use of scalars, vectors and tensors in non-Euclidean spacetime is important because of the requirement of general covariance of physical laws, independent of coordinate systems. Intrinsic properties of spacetime geometry are described in terms of such geometrical objects, for example, the Riemann-Christoffel curvature tensor.

1.1.1 Contravariant and Covariant tensors

In order to construct physical equations that are invariant under general coordinate transformations, we have to know how the quantities described by the equations behave under these transformations. The simplest transformation rule is that of scalars, which are invariant under general coordinate transformations ($x^i \rightarrow x'^i$). For vectors there are two kinds of transformation rules-contravariant and covariant, and correspondingly we have contravariant and covariant vectors [3, 5]. The transformation law for a contravariant vector (components denoted by A^i) is that, which under a coordinate transformation $x^i \rightarrow x'^i$ transform into

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j, \tag{1.1}$$

where Einstein's summation convention is used. This follows the rules of partial differentiation

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \tag{1.2}$$

The transformation law of a covariant vector (B_i) is

$$B'_i = \frac{\partial x^j}{\partial x'^i} B_j. \tag{1.3}$$

For example, if φ is a scalar field, then $\partial\varphi/\partial x^i$ is a covariant vector,

$$\frac{\partial \varphi}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial \varphi}{\partial x^j}. \tag{1.4}$$

This follows the covariant transformation law. Now we generalize these rules to tensors. In fact a vector is a tensor of rank one and a scalar is a tensor of rank zero. Now we express the transformation rules for second rank contravariant and covariant tensors in the form,

$$T'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l} T^{kl}; \tag{1.5}$$

$$T'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} T_{kl}. \tag{1.6}$$

It is straightforward to write the transformation equation for a mixed tensor having two contravariant and two covariant indices (rank is four):

$$T'^{ij}_{kl} = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l} T^{mn}_{pq} \tag{1.7}$$

The distinction between contravariant and covariant tensors disappears in rectangular Cartesian coordinate systems. In general any vector have distinct sets of contravariant and covariant components in an arbitrary coordinate system.

1.1.2 Metric tensor

In terms of the general coordinates, the line element of a non-Euclidean spacetime is written as the quadratic form [5, 8]

$$ds^2 = \sum_{i,k=0}^{3} g_{ik} dx^i dx^k = g_{ik} dx^i dx^k \qquad (1.8)$$

The space-time coordinates are $x^i = x^0, x^1, x^2$& x^3 ($x^0 = t$, with $c = 1$). g_{ik} is called the metric tensor (from quotient law of tensors, g_{ik} is a second rank covariant tensor so that ds^2 is an invariant). For the flat Minkowski spacetime of STR, g_{ik} has the diagonal form, ie., $g_{ik} = \{1, -1, -1, -1\}$. In general g_{ik} is symmetric in i and k, hence there are at most ten linearly independent components. Eq. (1.8) is said to define the space-time metric and we will assume the signature of eq. (1.8) to be -2. For a curved spacetime, g_{ik} are coordinate dependent. However, sometimes the dependence is a purely coordinate effect and not intrinsic to the curvature of space. For example, the g_{ik} are coordinate-dependent when we use spherical polar coordinates in flat Euclidean space. Thus we clearly have to devise a means of extracting essential geometrical information distinctly from pure coordinate effects.

1.1.3 Contraction and raising and lowering of indices

Contraction means, identifying a covariant index with a contravariant index of a mixed tensor and this will reduce the rank of the resulting tensor by two [5, 6]. For example, A^{ij}_{kl} is a fourth rank tensor, where as A^{ij}_{il} is a second rank tensor, which is evident from its transformation rule. The outer product of two tensors increases the rank, ie., $A^{ij}B^{kl} = C^{ijkl}$ is a tensor of rank four. The inner product is defined as an outer product followed by a contraction. For example, $A^i_j B^j_{lm} = C^i_{lm}$ is an inner product and its rank is three (each contraction reduces the rank by 2).

Now consider associated tensors defined by the following

$$A_i = g_{ik} A^k \qquad A^k = g^{ik} A_i, \qquad (1.9)$$

where g^{ik} is the contravariant form of the metric tensor. These operations define the lowering and raising of indices by the metric tensor, respectively. The significance of such rules is clear from the relation

$$g_{ik}A^iA^k = A_kA^k, \tag{1.10}$$

which is a scalar. Thus from (1.8), it is clear that ds^2 is an invariant interval.

1.1.4 Covariant differentiation and parallel transport

We can readily show that, the ordinary derivative of a vector in an arbitrary space-time does not transform like a vector [5, 8]. Consider the transformation equation for a vector B_i:

$$B_k^{'} = \frac{\partial x^i}{\partial x'^k}B_i \tag{1.11}$$

Here the prime corresponds to a different coordinate system. Differentiating w.r.t. $x^{'m}$ we get

$$\frac{\partial B_k^{'}}{\partial x'^m} = \frac{\partial x^i}{\partial x'^k}\frac{\partial x^n}{\partial x'^m}\frac{\partial B_i}{\partial x^n} + \frac{\partial^2 x^i}{\partial x'^m \partial x'^k}B_i \tag{1.12}$$

Because of the second term on RHS, the ordinary vector derivative does not transform like a tensor. The second derivative $\frac{\partial^2 x^i}{\partial x'^m \partial x'^k}$ is in general non-zero and indicates that the transformation coefficient in eq. (1.11) varies with position in spacetime. This property is not confined to non-Euclidean geometries. It also holds in Euclidean geometry wherein non-Cartesian coordinate systems are used. We have in general

$$\frac{\partial B^i}{\partial x^n} = \lim_{\delta x^n \to 0}\left[\frac{B^i\left(x^k + \delta x^k\right) - B^i\left(x^k\right)}{\delta x^n}\right] \tag{1.13}$$

Here the difference in the numerator is not expected to be a vector, because the two terms in the numerator do not transform like vectors at the same point, due to the variation of the transformation coefficients with position. In order to find the change in the vector between two points, we must somehow measure this difference at the same point. This is achieved by a process known as parallel transport, ie., shifting vectors from an initial to a final point without changing its magnitude and direction. We can express the change in the vector δB^i due to parallel transport as (keeping in mind a simple Euclidean example)

$$\delta B_i = \Gamma^l_{ik} B_l \delta x^k \tag{1.14}$$

where the coefficients Γ^l_{ik} are functions of space-time coordinates x^i The set $\{\Gamma^l_{ik}\}$ constitutes the so-called affine connection on the space-time region. The symbols are sometimes referred to as the Christoffel symbols (II kind). Now the difference between the vector $B_i\left(x^k + \delta x^K\right)$ and the vector obtained by parallel transport, $B_i + \delta B_i$ is a vector at $x^k + \delta x^k$:

$$B_i(x^k + \delta x^k) - [B_i + \delta B_i] = \left[\frac{\partial B_i}{\partial x^k} - \Gamma^l_{ik} B_l\right] \delta x^k \tag{1.15}$$

We may accordingly redefine the derivative of a vector by

$$B_{i;k} = B_{i,k} - \Gamma^l_{ik} B_l \tag{1.16}$$

This derivative must transform as a tensor. It is called a covariant derivative (semicolon represents covariant derivative). Riemannian geometry introduces further simplification[3]:

$$\Gamma^i_{kl} = \Gamma^i_{lk} \; ; g_{ik;l} = 0 \tag{1.17}$$

Γ^i_{kl} and g_{ik} are related by

$$\Gamma^i_{kl} = \frac{g^{im}}{2}\left[\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right] \tag{1.18}$$

The Christoffel symbols of I kind are defined by

$$[ij,k] = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right] \tag{1.19}$$

Combining equations (1.18) and (1.19), we find

$$\Gamma^k_{ij} = g^{mk}\,[ij,m] \tag{1.20}$$

To this end, it may be noted that, by a linear transformation we can arrange to have a coordinate system with

$$g_{ik} = \eta_{ik} = diag\,(1,-1,-1,-1)\,,\ \Gamma^i_{kl} = 0 \tag{1.21}$$

at any chosen point in space-time. Such a coordinate system is called a locally inertial coordinate system (or local inertial frame, LIF). The covariant derivative reduces to ordinary derivative in a LIF. The significance of LIF becomes clear when we discuss the Einstein's equivalence principle [4, 5, 11].

1.1.5 Spacetime curvature and the Riemannian curvature tensor

Consider a triangle (say $\triangle ABC$) on the surface of a sphere (two dimensional). Imagine a vector at the point A being parallel transported to C through two different paths, ie., A to C and A to C through B. It is found that the final directions of the vector are different in the two situations, ie., the outcome depends on the path of transport from A to C [3]. However, if a similar experiment is conducted with a triangle drawn on a flat piece of paper, there will be no change in the directions of final vectors. This is one of the properties that distinguish a curved space from a flat one. The four dimensional curved spacetime may be characterized by means of a fourth rank tensor.

Let us consider the difference between two 2nd order covariant derivatives in the form,

$$B_{p;q;r} - B_{p;r;q} = B_m R^m_{pqr} \tag{1.22}$$

where

$$R^m_{pqr} = \frac{\partial}{\partial x^q}\Gamma^m_{pr} - \frac{\partial}{\partial x^r}\Gamma^m_{pq} + \Gamma^j_{pr}\Gamma^m_{jq} - \Gamma^j_{pq}\Gamma^m_{jr}. \tag{1.23}$$

Also

$$R^m_{pqr} = -R^m_{prq} \tag{1.24}$$

From the quotient law we conclude that R^m_{pqr} are components of a fourth rank tensor. This tensor, known as the Riemann Christoffel curvature tensor, plays an important role in specifying the geometrical properties of the four dimensional spacetime [4]. Spacetime is said to be flat if its Riemann tensor vanishes everywhere. Otherwise it is said to be curved. Thus, in general, a curved spacetime is characterised by (i) coordinate dependent metric tensor components g_{ik}, (ii) non-vanishing Christoffel symbols Γ^i_{jk} and (iii) non-vanishing Riemann curvature tensor.

Properties of Riemann tensor

This tensor can be contracted (equating one covariant index to another contravariant index) in two different ways[4]:

$$R^m_{mqr} = 0 \tag{1.25}$$

and

$$R^m_{pqm} = R_{pq} \tag{1.26}$$

R_{pq} is a second rank tensor called Ricci tensor. From eq. (1.23) the Ricci tensor can be expressed in the following form

$$R_{pq} = \frac{\partial}{\partial x^q}\Gamma^m_{pm} - \frac{\partial}{\partial x^m}\Gamma^m_{pq} + \Gamma^j_{pm}\Gamma^m_{jq} - \Gamma^j_{pq}\Gamma^m_{jm} \tag{1.27}$$

Or equivalently

$$R_{pq} = \frac{\partial}{\partial x^q}\left(\frac{\partial}{\partial x^p}\log\sqrt{g}\right) - \frac{\partial\Gamma^m_{pq}}{\partial x^m} + \Gamma^j_{pm}\Gamma^m_{jq} - \Gamma^j_{pq}\frac{\partial\log\sqrt{g}}{\partial x^j} \tag{1.28}$$

where g is the determinant of the metric tensor. It readily follows from this expression that the Ricci tensor is a symmetric tensor ie., $R_{pq} = R_{qp}$. By further contraction, we get a scalar called Ricci scalar, given by

$$R = g^{pq}R_{pq} \tag{1.29}$$

To see the other symmetry properties of the curvature tensor, we express the Riemann tensor in the covariant form which is more convenient:

$$R_{npqr} = g_{mn}R^m_{pqr} \tag{1.30}$$

The additional symmetries [4] are the following:

$$R_{npqr} = -R_{pnqr} \tag{1.31}$$

$$R_{npqr} = -R_{nprq} \tag{1.32}$$

$$R_{npqr} = R_{qrnp} \tag{1.33}$$

and

$$R_{npqr} + R_{nqrp} + R_{nrpq} = 0 \tag{1.34}$$

In general a tensor of rank r in n-dimensional space has n^r components. However, because of the symmetry relations, the Riemannian tensor has only 20 independent components.

Bianchi identity

$$R_{mpqr;n} + R_{mprn;q} + R_{mpnq;r} = 0 \tag{1.35}$$

The significance of the Bianchi identity is that it leads to a zero divergence tensor called the Einstein tensor defined as

$$G^{mp} = R^{mp} - \frac{1}{2}g^{mp}R. \tag{1.36}$$

1.1.6 The Principle of Equivalence(PE)

The principle of equivalence [4, 5, 11] played a key role in GTR. The principle of equivalence states that gravitational effects are identical in nature to those arising through acceleration. The seed for this idea goes back to the observation by Galileo that bodies fall at a rate independent of mass. In Newtonian terms, the acceleration of a body in a gravitational field g is

$$m_I a = m_G g \tag{1.37}$$

and no experiment has ever been able to detect a difference between the inertial and gravitational masses m_I and m_G (Inertial mass m_I occurs in Newton's second law and gravitational mass m_G occurs in Newton's universal law of gravitation). These considerations led Einstein to suggest that inertial and gravitational forces were indeed one and the same. This leads to equivalence principle which is usually stated in two different forms.

The weak principle of equivalence states that the effects of gravitation can be transformed away locally and over small intervals of time by using suitably accelerated reference frames. The strong principle of equivalence states that, any physical interaction (other than gravitation) behaves in a locally inertial frame (for example, freely falling lift) as if gravitation were absent. In other words, in a small laboratory falling freely in a gravitational field, mechanical phenomena are the same as those observed in a Newtonian inertial frame in the absence of a gravitational field. In 1907, Einstein replaced the phrase 'mechanical phenomena' by the phrase 'laws of physics' and the resulting statement is the principle of equivalence. These freely falling frames covering the neighbourhood of an event are very important in relativity, they are called local inertial frames (LIFs). These local inertial frames are characterized by

$$g_{ik} = \eta_{ik} = diag\,(1, -1, -1, -1)\,, \ \Gamma^i_{kl} = 0$$

Thus in the LIF, gravitation has been transformed away momentarily and in a small neighbourhood of any point in a curved spacetime. However, gravitation cannot be removed globally and a curved spacetime is characterised by a non-vanishing Riemann tensor.

1.1.7 Einstein's field equations

Energy momentum tensor and the action principle

The energy momentum tensor plays a vital role in general relativity. In electrodynamics the electromagnetic energy momentum tensor T^{ik} describes essentially the conservation of energy through $T^{ik}_{;k} = 0$. In general relativity T^{ik} acts as a source of Einstein's field equations [1].

The famous action principle was introduced in 1834 by Hamilton to obtain the generalized laws of dynamics. The action [12] is defined through an integral

$$A = \int_{t_1}^{t_2} L(q_r, \dot{q}_r, t)dt \qquad (1.38)$$

The scalar function L is called Lagrangian, which is a function of generalized coordinates q_r, their time derivatives and time coordinate t. When the system makes a transition from an initial state to a final state, the actual path is that particular path for which A is stationary for small displacements of the path (ie., $\delta A = 0$)

A more general form of action principle

Let a system be described by means of a set of functions $\phi^A (A = 1, 2, \cdot)$ of spacetime coordinates x^i [8]. From ϕ^A and its derivatives $\phi^A_{,i}$, construct a Lagrangian density (scalar function)

$$L' = L'\left(\phi^A, \phi^A_{,i}, x^i\right) \qquad (1.39)$$

The action integral is defined as

$$A = \int_V L' \sqrt{-g} d^4x, \tag{1.40}$$

where V is the volume of the specified space-time manifold with the boundary surface $\sum$. The equations satisfied by ϕ^A are such as to make $\delta A = 0$, for small variations $\delta\phi^A$ ($\phi^A \to \phi^A + \delta\phi^A$) which vanish on $\sum$. The spacetime geometry is specified by the metric tensor g_{ik}. If we demand that the g_{ik} are also dynamical variables and that the action A remains stationary for small variations of the type

$$g_{ik} \to g_{ik} + \delta g_{ik}. \tag{1.41}$$

Then the variation in action is expressed as

$$\delta A = -\frac{1}{2} \int T^{ik} \, \delta g_{ik} \sqrt{-g} \, d^4x \tag{1.42}$$

and this yields a definition of the energy-momentum tensor for the entire physical system described by the action principle [4, 8]:

$$T^{ik} = \frac{-2}{\sqrt{-g}} \left[\frac{\partial}{\partial g_{ik}} L' \sqrt{-g} - \left(\frac{\partial L' \sqrt{-g}}{\partial g_{ik,l}} \right)_{,l} \right] \tag{1.43}$$

The variations of A w.r.t. g_{ik} leads to energy-momentum tensor of various interactions. The energy-momentum tensor of a fluid with density ρ and pressure p in the generally covariant form is

$$T^{ik} = (\rho + p) \, u^i u^k - p \, g^{ik} \tag{1.44}$$

where u^i is the four-velocity of the fluid. It can be readily shown that

$$T^{ik}_{;k} = 0 \tag{1.45}$$

This represents the conservation law for the energy momentum tensor T^{ik}. Hilbert derived the field equations of relativity from an action principle. The action is given by

$$A = \frac{1}{2\kappa}\int_V R\sqrt{-g}\, d^4x + \int_V L'\sqrt{-g}\, d^4x \tag{1.46}$$

The variation of A with respect to g_{ik} leads to the following equation called the field equation of GTR [3, 4, 8].

$$R_{ik} - \frac{1}{2}g_{ik}R = -\kappa T_{ik} \tag{1.47}$$

where the coupling constant κ is given by (from Newtonian approximation) $\kappa = \frac{8\pi G}{c^4} = 8\pi G$ (with $c = 1$). Before this derivation, Einstein, however formed his equations of general relativity from some general considerations. According to him, the energy tensor acts as the source of gravity. However, in order to get a stationary universe, Einstein modified the first term of the action in (1.46) by adding a constant term (cosmological constant). ie.,

$$A = \frac{1}{2\kappa}\int_V (R + 2\lambda)\sqrt{-g}\, d^4x + \int_V L'\sqrt{-g}\, d^4x \tag{1.48}$$

Then the modified field equation is

$$R_{ik} - \frac{1}{2}g_{ik}R + \lambda g_{ik} = -\kappa T_{ik} \tag{1.49}$$

When Einstein came to know about Hubble's discovery of the expansion of the universe, he abandoned the cosmological constant term in his field equation. However, this $\lambda-$term is one of the most intriguing factors in current theoretical physics. In view of its application to cosmology, the $\lambda-$term is usually taken to the RHS of this equation, after making the substitution

$$\rho_\lambda = \frac{\lambda}{8\pi G} \tag{1.50}$$

so that

$$R_{ik} - \frac{1}{2} g_{ik} R = -\kappa \left[T_{ik} + \rho_\lambda \, g_{ik} \right] \tag{1.51}$$

In the contravariant form we have

$$R^{ik} - \frac{1}{2} g^{ik} R = \kappa \left[T^{ik} + \rho_\lambda \, g^{ik} \right] \tag{1.52}$$

From (1.44), we can see that the term $\rho_\lambda g^{ik}$ is identical to the energy-momentum tensor for a perfect fluid having density ρ_λ and pressure $p_\lambda = -\rho_\lambda$.

1.1.8 Earlier solutions

The Schwarzschild solution

Karl Schwarzschild in 1916, solved the Einstein's field equation to describe the geometry of spacetime in the empty space outside a spherically symmetric distribution of matter. The Schwarzschild metric is [1, 3]

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 \left[d\theta^2 + \sin^2\theta \, d\varphi^2 \right] \tag{1.53}$$

Most of the traditional tests of GTR [8] are based on the Schwarzschild solution, and they seek to measure the fine differences between the predictions of Newtonian gravitation and those of general relativity. The gravitational redshift, precession of the perihelion of mercury, the bending of light, existence of black holes etc. are some of the predictions of GTR. Among these, the precession of the perihelion of mercury is perhaps the most impressive test in favour of GTR. The theory predicts a precession of 43 seconds of arc per century, and there is a good agreement with observations. Eventhough the Schwarzschild solution represents the first physically significant solution of the field equations of relativity, this is a local solution in the sense that the distortions of spacetime geometry from Minkowski geometry gradually diminish to zero as we move away from the gravitating mass. ie., spacetime is asymptotically flat and also static.

The Einstein solution

Einstein realized that the Schwarzschild solution cannot provide the correct spacetime geometry of the universe, since the universe is filled with a continuous distribution of matter.

In order to solve field equations of general relativity (which are an interlinked set of nonlinear partial differential equations), it is essential to introduce certain simplifying assumptions, just like the assumption of spherical symmetry in the Schwarzschild solution. Einstein assumed homogeneity and isotropy in his cosmological problem. He further assumed that spacetime is static. Under these assumptions, the line element of the spacetime could be described by [3]

$$ds^2 = dt^2 - a^2\{\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\}. \tag{1.54}$$

The constant a is called radius of the universe. However Einstein failed to obtain static, homogeneous, isotropic dense model of the universe from the field equations, using the above line element. Hence he introduced a cosmological constant term in his field equation, which introduces a force of repulsion between objects, to obtain a closed model.

Einstein believed that GTR can yield only matter filled spacetimes as solutions of the field equations. However, it was proved wrong shortly after the publication of his paper in 1917. W. de Sitter, a Dutch astronomer, published another solution of the field equation which predicts an empty spacetime, but expanding (ie., expansion without matter). It had the remarkable property of predicting a redshift proportional to the distance. However, the de Sitter model fails to meet Mach's criterion that there should be a background of distant matter against which local motion can be measured. But the observations of Hubble and Humason (1929) indicated that the universe is not static but expanding. Einstein, then abandoned the cosmological constant term in his field equations, remarking that it was the biggest blunder of his life.

The combined effect of deSitter's notion of expansion and Einstein's notion of non-emptiness is obtained in the Friedmann model (1922). In 1922, the general homogeneous and isotropic solution of the original Einstein equations was found by the Russian mathematician Alexandre Friedmann. It is these Friedmann models based on the original Einstein field equations, and not the Einstein or de Sitter models, that provide the mathematical background for most modern cosmological theories. In 1929, Hubble's observations regarding the redshifts of galaxies established a linear relation between velocities and distances, indicating that the universe is expanding, ie., galaxies are receding away from each other (Hubble's law)

1.2 The Standard Cosmological model

This model forms the most successful approximation of the real universe, with maximum consistency with observations. The Friedmann models are the simplest ones and are based on the following simplifying assumptions [1-4,13,14]:

1. The geometry of space is such that at large scales, it is describable by the mathematically simple, spatially symmetric Friedmann-Robertson-Walker (FRW) metric, which is based on the cosmological principle, ie., matter distribution in the universe is homogeneous (independent of location) and isotropic (same in all directions) on very large scales.

2. The energy-momentum tensor is that which corresponds to a spatially homogeneous and isotropic perfect fluid comprising matter which is either relativistic or non-relativistic.

3. The world lines of matter form a highly ordered non-intersecting bundle of geodesics, which can be parameterized by three space like coordinates x^μ, $\mu = 1, 2, 3$. Thus x^μ =constant, along a world line. Also, there exists a set of space-like hypersurfaces given by t =constant, orthogonal to this set of world lines. The time t may called the cosmic time. The observers whose world lines follow the above equation are called fundamental observers (Weyl postulate).

1.2.1 Robertson-Walker line element

The rigorous derivation of FRW metric, which is used by Friedmann and others, had to await the work of H P Robertson in 1935 and A G Walker in 1936. Independently, these authors showed that there are only three kinds of such spacetime, denoted below by the parameter values $k = 0, +1, \& -1$. The line element, whose derivation is given in many standard text books [4, 13], is given by $(c = 1)$

$$ds^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 \left[d\theta^2 + \sin^2\theta \, d\varphi^2 \right] \right\} \tag{1.55}$$

Here r, θ, φ are the three coordinates x^μ. The function $a(t)$ sets the scale of the spacelike sections spanned by r, θ, φ. $a(t)$ is called the scale factor or expansion factor for the universe. For $k = 0$, the expression in the bracket is simply the line element for three dimensional Euclidean geometry, ie., for flat space. For this reason this case is often referred to as the flat Robertson-Walker model. For $k = 1$, the space is finite but unbounded with $0 \leq r \leq 1$ (space curvature is +ve). This is the closed model, while for $k = -1$, the model is said to be open (spatial curvature is $-$ve). Thus for $k = 1$, 0 or -1, the three dimensional spatial part of the metric is hyperspherical, hyperplanar or pseudo-hyperspherical, respectively. For both $k = 0$ and $k = -1$, the coordinate r goes over the range $0 \leq r \leq \infty$. The θ, φ coordinates range over the intervals $-\pi \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$ in all three cases.

1.2.2 The Redshift

Consider a galaxy G_1 at $r = r_1$ emiting light and is received by the observer on the galaxy G_0 at $r = 0$. According to GTR, light travels along a null geodesic $ds = 0$, which in this situation turns out to be a path along which θ and φ are constants. From the FRW metric [3],

$$\frac{dr}{\sqrt{1-kr^2}} = \frac{-dt}{a(t)} \tag{1.56}$$

where the minus sign on the RHS indicates that along the path of the light ray, r decreases as t increases. Let G_1 emit monochromatic light of wavelength λ_1. Consider two epochs at t_1 and $t_1 + \delta t_1$ when two successive wave crests leave G_1, reaching G_0 at t_0 and $t_0 + \delta t_0$ respectively. Then,

$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} \tag{1.57}$$

Assuming that the function $a(t)$ varies only slowly so that it does not change significantly over the small intervals δt_0 and δt_1, we get

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)} \tag{1.58}$$

Let the wavelength perceived by G_0 be λ_0. Then $\lambda_1 = c\delta t_1$ and $\lambda_0 = c\delta t_0$. Hence

$$\frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)} = 1 + z \tag{1.59}$$

The parameter z is called redshift and a redshift of wavelengths indicates $z > 0$ and hence $a(t_0) > a(t_1)$ (provided $t_0 > t_1$). So, to be able to explain the observed galactic redshift, it is necessary to have $a(t)$ increasing with t.

1.2.3 Hubble's law

The linear relationship between the distance to a galaxy and its observed redshift, may be deduced from the FRW metric, without specific knowledge of the dynamics of the expansion [1]. The expansion of the universe means that the proper physical distance between a pair of well separated galaxies is increasing with time, ie., the galaxies are receding from each other. A gravitationally bound system such as the local group is not expanding, and this gravitational instability tends to collect galaxies into increasingly more massive systems that break away from the general expansion to form a hierarchy of clusters.

Consider a galaxy having an absolute luminosity L and let the measured flux be F It emits light at $r = r_1$, $t = t_1$ and we receive it at $r = 0$, $t = t_0$. Due to the expansion of the universe,

$$F = \frac{L}{4\pi a^2(t_0)\, r_1^2 (1+z)^2} = \frac{L}{4\pi D_L^2} \tag{1.60}$$

where

$$D_L = r_1 a(t_0)(1+z) \tag{1.61}$$

D_L is called the luminosity distance of the galaxy and $a(t_0)$ is the scale factor for the present universe. If $r << 1$, we can approximate the integral in eq. (1.57) to write

$$r_1 \simeq \frac{(t_0 - t_1)}{a\,(t_0)} \tag{1.62}$$

By Taylor expansion near t_0 we get

$$a\,(t_1) \simeq a\,(t_0) + (t_1 - t_0)\;\dot{a}\,(t_0) \tag{1.63}$$

and hence

$$\frac{a\,(t_1)}{a\,(t_0)} \simeq 1 - r_1 a\,(t_0)\,\frac{\dot{a}\,(t_0)}{a\,(t_0)} \tag{1.64}$$

Using eq. (1.59) we have

$$(1+z)^{-1} = \frac{a\,(t_1)}{a\,(t_0)} \simeq 1 - r_1 a\,(t_0)\,\frac{\dot{a}\,(t_0)}{a\,(t_0)} \tag{1.65}$$

For $z << 1$ and for small r_1,

$$1 - z = 1 - D_L \left(\frac{\dot{a}}{a}\right)_{t_0} \tag{1.66}$$

and so

$$z = H_0 D_L \tag{1.67}$$

ie., with Doppler velocity $v = cz = z$ $(c = 1)$

$$v = H_0 D_L \tag{1.68}$$

where $H_o = \frac{\dot{a}(t_0)}{a(t_0)}$ is the present value of Hubble constant. This is Hubble's velocity-distance relation. The Hubble parameter H in general is a function of time.

1.2.4 The Friedmann models

In the preceding sections, the dynamics of the expanding universe only appeared implicitly in the time dependence of the scale factor $a(t)$. To make this time dependence explicit, one must solve for the evolution of the scale factor [14] using the Einstein's equations.

$$G^i_k = R^i_k - \frac{1}{2}\delta^i_k R = 8\pi G T^i_k, \tag{1.69}$$

where T^i_k is the energy-momentum tensor (or stress-energy tensor) for the source including matter, radiation, vacuum energy etc. The assumption of homogeneity and isotropy of the cosmic fluid, implies that T^μ_o s ($\mu = 1, 2, 3$) must be zero, and the spatial components T^α_β must have a diagonal form with $T^1_1 = T^2_2 = T^3_3$. It is convenient to write [14]

$$T^i_k = diag\left[\rho(t), -p(t), -p(t), -p(t)\right], \tag{1.70}$$

where $\rho(t)$ and p(t) are the energy density and pressure of the cosmic fluid (if we treat it as an ideal fluid). From the Einstein's equations, using the FRW metric, we get the following equations which are known as the Friedmann equations of cosmology:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho, \tag{1.71}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G p \tag{1.72}$$

These two equations together with the equation of state

$$p = p(\rho) = w\rho \tag{1.73}$$

completely determine the three functions $a(t), \rho(t)$ and $p(t)$. In the dust approximation (as done by Friedmann), we take $p = 0$ (because the dust matter is assumed to be collissionless) in the above equations.

From eq. (1.71), it follows that

$$\frac{k}{a^2} = \frac{8\pi G}{3}\rho - \frac{\dot{a}^2}{a^2}, \tag{1.74}$$

and

$$\frac{k}{a^2} = \frac{\dot{a}^2}{a^2}\left[\frac{\rho}{(3H^2/8\pi G)} - 1\right] \tag{1.75}$$

where we used the result that $\dot{a}/a = H(t)$, called the Hubble parameter, which measures the expansion rate of the universe.

$$\frac{k}{a^2} = \frac{\dot{a}^2}{a^2}\left[\frac{\rho}{\rho_c} - 1\right] \tag{1.76}$$

where $\rho_c = 3H^2/8\pi G$, called the critical density of the universe (critical because it corresponds to the flat case). If we use $\rho/\rho_c = \Omega$, then

$$\frac{k}{a^2} = \frac{\dot{a}^2}{a^2}[\Omega - 1] \tag{1.77}$$

For the present universe

$$k = H_0^2\, a_0^2\, (\Omega_0 - 1) \tag{1.78}$$

Since $H_0^2 a_0^2 \geq 0$, there is a correspondence between the sign of k and the sign of $\Omega - 1$. ie., $k = +1$ corresponds to $\Omega > 1$ $(\rho > \rho_c)$, $k = 0$ corresponds to $\Omega = 1$

($\rho = \rho_c$) and $k = -1$ corresponds to $\Omega < 1$ ($\rho < \rho_c$). Correspondingly, we have closed, flat, and open models. Equations (1.71) and (1.72) can be combined into a single equation for $\ddot{a}$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3\, p) \tag{1.79}$$

and in terms of the Hubble parameter $H(t)$, it leads to the Raychaudhuri equation [13, 17]

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = -H^2 - \frac{4\pi G}{3}(\rho + 3p) \tag{1.80}$$

For matter (non-relativistic), $\rho + 3p > 0$, implying that $\ddot{a} < 0$. The $a(t)$ curve (which has positive $\dot{a}$ at the present epoch t_0) must be convex. ie., a will have been smaller in the past and becomes zero at some time in the past (say $t = 0$). The time span t_0 must be less than $1/H_o$. As $a \to 0$, $\rho \to \infty$ and the components of the curvature tensor diverge. This is the singularity problem of the standard model, which is an artefact of the theory. Also, when the radius of curvature of spacetime becomes comparable to the Planck length ($l_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-33} cm$), quantum effects of gravity [see 90] become significant, and classical Einstein's GTR becomes invalid. For a resolution of this problem, a quantum theory of gravity is needed. Equations (1.71) and(1.72) can be combined to yield another equation

$$\frac{d}{dt}\left[a\,\left(\dot{a}^2 + k\right)\right] = \dot{a}\,\left[2a\ddot{a} + \dot{a}^2 + k\right] \tag{1.81}$$

or

$$\frac{d}{da}\left(\rho a^3\right) + 3\, p\, a^2 = 0 \tag{1.82}$$

This result is a consequence of the conservation law implicit in Einstein equations

$$T^{i}_{k;i} = 0 \tag{1.83}$$

Eq. (1.82) is called the conservation law, because it corresponds to the conservation of the energy-momentum tensor of the universe. For an equation of state of the form by eq. (1.73),

$$\rho \propto a^{-3(1+w)} \tag{1.84}$$

and from conservation law,

$$\dot{\rho} = -3H\,(\rho + p) \tag{1.85}$$

For non-relativistic matter ($w = 0$) $\Longrightarrow \rho \propto a^{-3}$, for radiation ($w = 1/3$) $\Longrightarrow \rho \propto a^{-4}$ and if $p = -\rho$ (for example, vacuum) $w = -1$ and $\rho =$ constant. If $p = -\rho$, pressure is negative (since $\rho > 0$, to maintain $\frac{\dot{a}^2}{a^2} > 0$) and the negative pressure allows for the energy inside the volume to increase even when the volume expands (the case of inflation). From Friedmann equations

$$\frac{\dot{a}^2}{a^2} \propto a^{-3(1+w)}, \tag{1.86}$$

$$\dot{a} \propto a^{-\frac{1}{2}(1+3w)} \tag{1.87}$$

Integrating

$$a(t) \propto t^{2/(3(1+w))} \quad w \neq -1, \tag{1.88}$$

$$a\,(t) \propto e^{\lambda\, t},\; w = -1 \tag{1.89}$$

where λ is some constant. For $w = 0$, $a \propto t^{2/3}$, for $w = 1/3$, $a \propto t^{1/2}$ and for $w = 1$, $a \propto t^{1/3}$ If we assume matter and radiation are the main components of ρ, and each is conserved separately (the assumption in the standard model), then the present universe is matter dominated. There is an epoch (recombination or decoupling epoch, t_d) at which $\rho_r = \rho_m$, and before that the universe is radiation dominated. At t_d, matter gets decoupled from the radiation background, and the universe becomes transparent. There is also another parameter used in FRW models, called the deceleration parameter $q(t)$, given by

$$\frac{\ddot{a}}{a} = -qH^2 \tag{1.90}$$

A +ve value for q indicates decelerating expansion of the universe. If the present value of the deceleration parameter is $q_0 = 1/2$, it corresponds to a flat universe ($k = 0$). $q_0 > 1/2$ leads to closed universe and $q_0 < 1/2$ corresponds to open model.

1.3 Structure formation

One of the outstanding problems in cosmology today, is undoubtedly the origin and evolution of large scale structures [13, 15, 16, 17]. The basic framework for structure formation requires that small density perturbations, formed in an otherwise uniform distribution of matter and radiation in the very early universe, grow under the influence of gravity until gravitational instabilities develop and the structure collapses and galaxies, clusters of galaxies etc. that we see today are formed. The formation of structure (or galaxy formation) began when the universe became matter dominated, or we can say that the time of matter-radiation equality is the initial epoch for structure formation. Thus the structures we see today, are formed by a process known as gravitational instability, from primordial fluctuations in the cosmic fluid (which is evident from the small anisotropy in CMBR). But, because the strength of clustering is expected to increase with time (ie., the evolution of the density contrast is proportional to some power of scale factor in the linear approximation of the standard model), the galaxies must deviate from the smooth Hubble expansion. These deviations away from uniform Hubble flow are known as peculiar velocities. According to standard model, $\delta v \propto \Omega_0^{0.6}\delta\rho$, where $\delta\rho$ is the density perturbation, and Ω_0 is the present value of the ratio between critical density and density of the universe.

1.3.1 Linear perturbation theory

We assume that at some time in the past, there were small deviations from homogeneity in the universe. As long as these inhomogeneities are small, their growth can be studied by the linear perturbation theory. On each hypersurface (x^μ = constant, $\mu = 1, 2, 3$), one can define an average plus a perturbation [17]

$$\rho\left(x,t\right) = \rho_b\left(t\right) + \delta\rho\left(x,t\right), \tag{1.91}$$

$$p\left(x,t\right) = p_b\left(t\right) + \delta p\left(x,t\right), \tag{1.92}$$

$$H\left(x,t\right) = H_b\left(t\right) + \delta H\left(x,t\right) \tag{1.93}$$

Here t is the time coordinate labelling the hypersurfaces, and $x = (x^1, x^2, x^3)$ are space coordinates. Density contrast is defined by

$$\delta\left(x,t\right) = \frac{\delta\rho\left(x,t\right)}{\rho_b} \tag{1.94}$$

In the first order linear perturbation approximation, the conservation equation (1.85) remains the same, but the Raychaudhuri equation becomes, to first order

$$\dot{H} = -H^2 - \frac{4\pi G}{3}(\rho + 3p) - \frac{1}{3}\frac{\nabla^2 \delta p}{(p+\rho)} \tag{1.95}$$

∇^2 is the Laplacian on a comoving hypersurface, given in terms of comoving coordinates by

$$\nabla^2 = a^{-2}\delta^{ij}\partial_i\partial_j \tag{1.96}$$

The energy conservation equation (1.85) and Raychaudhuri equation (1.95) determine the evolution of the energy density and the Hubble parameter along each world line, including first order perturbations away from homogeneity and isotropy. The perturbation equations [15, 16] obtained are the following

$$(\delta\dot{\rho}) = -3(\rho_b + 3p_b)\,\delta H - 3H_b\delta\rho, \tag{1.97}$$

$$(\delta\dot{H}) = -2H_b\delta H - \frac{4\pi G}{3}\delta\rho - \frac{\nabla^2 \delta p}{(p_b + \rho_b)}, \tag{1.98}$$

where an overdot denotes differentiation with respect to time. In the linear approximation (ie., $\delta < 1$), when perturbation from the background density is small, the evolution of density contrast is given by

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} = \frac{\nabla^2 p}{\rho_b a^2} + 4\pi G\rho_b\delta \tag{1.99}$$

For dust models, one can solve the above equation to yield δ proportional to some power of the scale factor a, ie., $\delta \propto a$ in the matter dominated case and $\delta \propto a^2$ for radiation dominated epoch. When we take two or more component cosmic fluid, instead of treating them separately, one can still find solutions for δ, but the process gets highly complicated.

The linear perturbation theory fails when the density contrast becomes nearly unity. Since most of the observed structures in the universe, for example, galaxies, clusters etc. have density contrasts far in excess of unity, their structure can be understood only by a fully nonlinear theory [15-19]. Although local extensions of linear theory do provide a qualitative first step in comparing theory with observations, a deeper insight into gravitational clustering is provided by the dynamical approximations, like the Zeldovich approximation, adhesion approximation etc. Several approximations have been suggested to model gravitational instability for the strongly non-linear regime ($\delta \geq 1$).

1.4 Predictions and problems of the standard model

The Friedmann models are the most successful approximation of the real universe, with maximum consistency with observations. Now we consider the validity and the problems of this theory through cosmological observations.

1.4.1 Hubble's expansion and the redshift-apparent magnitude $(z - m)$ relation

Modern observational cosmology began with Hubble's observations. He obtained a value for the Hubble constant, $H_0 = 500 km\ s^{-1} Mpc^{-1}$, from his observations. However, present day observations suggest that H_0 lies in the range $50 \leq H_0 \leq 100\ km\ s^{-1} Mpc^{-1}$[21, 22]. It is usually expressed in the following form, $H_0 = 100 h_0\ km\ s^{-1} Mpc^{-1}$, with $0.5 \leq h_0 \leq 1$ being the uncertainty in the measured value of H_0. The Hubble constant relates the redshift z of a nearby galaxy to its distance D by $v = cz = H_0 D$. Thus if we measure z and D for a number of galaxies, we should be able to estimate H_0. However there is difficulty in estimating D. How is it possible to measure the distance D? All the distance measurements are based on the assumption that recognizable types of distant objects are similar to nearby objects of the same type. Let L and l be the absolute and apparent luminosity of an object, respectively, then [3, 7]

$$l = \frac{L}{4\pi D_L^2} \tag{1.100}$$

where D_L is the luminosity distance of the galaxy given by $D_L = r_j a(t_0)(1+z)$, with r_j the radial coordinate of the galaxy emitting light at some time t_j in the past and z its redshift. $a(t_0)$ is the present value of the scale factor (Our galaxy is at $r = 0$ and receiving light at t_0). In flat FRW models

$$r_j = \int_{t_j}^{t_0} \frac{dt}{a(t)} \tag{1.101}$$

From (1.88) it can be shown that

$$r_j = \frac{t_0^{2/3}}{a(t_0)} \int_{t_j}^{t_0} \frac{dt}{t^{2/3}} = \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] \tag{1.102}$$

so that

$$D_L = \frac{2}{H_0}\left[(1+z) - \sqrt{1+z}\right] \tag{1.103}$$

A more general form of luminosity distance [3] is

$$D_L = \frac{1}{H_0 q_0^2}\left\{q_0 z + (q_0 - 1)\left[(1 + 2zq_0)^{1/2} - 1\right]\right\}, \tag{1.104}$$

where q_0 is the present deceleration parameter. $\left|q_0 - \frac{1}{2}\right| > 0$ for $k = \pm 1$ models and $q_0 \to 1/2$ leads to the flat case (1.103). Astronomers, instead of using the power l as a measure of apparent brightness, use a logarithmic measure, the apparent magnitude m. This is greater, the fainter the object, and is defined so that two objects whose luminosities (l_1 and l_2) differ by a factor of 100 differ in apparent magnitude by 5, that is

$$\frac{l_1}{l_2} = 100^{(m_2-m_1)/5} \tag{1.105}$$

Hence

$$m_2 - m_1 = 2.5 \log_{10}(l_1/l_2) \tag{1.106}$$

The absolute magnitude M of an object is the apparent magnitude that the object would have at a distance $10pc$ [7]. Thus measuring D in pc (pc or parsec means parallax second. One pc is the distance to an object whose parallax is one second of arc with respect to a baseline which is usually the diameter of the earth's orbit around the sun for astronomical purposes. $1pc$ = 3.26 light years = $3.08 \times 10^{18} cm$)

$$100^{\frac{M-m}{5}} = \frac{10^2}{D^2} \tag{1.107}$$

and

$$m - M = 5 \log D_{pc} - 5 \tag{1.108}$$

However cosmologists measure distance in megaparsecs ($1Mpc = 10^6 pc$)

$$m - M = 5 \log D_{Mpc} + 25 = \mu \tag{1.109}$$

μ is called the distance modulus of the object. Substituting for D_L, we get the redshift-apparent magnitude relation. Using $H_0 = 100h_0\ kms^{-1}Mpc^{-1}$ one can arrive at a useful formula [3]

$$\mu_0 = 5\log\left\{\frac{1}{h_0}\left(\frac{z}{q_0} + \frac{(q_0-1)}{q_0^2}\left[(1+2zq_0)^{\frac{1}{2}} - 1\right]\right)\right\} - 2.5\log(1+z) + K(z) + 42.39 \tag{1.110}$$

where $K(z)$ is called the $K-$correction term, which allows us to obtain the relevant absolute magnitude corresponding to zero redshift [20]. Observationally H_0 is measured from the $z - m$ diagram. The FRW models predict a linear redshift-magnitude relation. However observations of distant extragalactic objects like supernovae, quasars etc. indicate that the diagram is a scatter diagram, which increases with z, with no Hubble type relation clearly apparent [20, 23-28]. The Hubble's law is the foundation on which the expanding universe models rest. If the law is known to be valid for all extragalactic objects, then only can we use it to claim that an object at high z is farther away from us and is being viewed at an earlier epoch than an object of low redshift. This problem and a new model, which provide an alternative explanation for the puzzle are discussed in detail in Chapter 2 of this thesis.

1.4.2 Fractal distribution of galaxies

The basic assumption of standard cosmological model is the Einstein's cosmological principle which, in fact, is the hypothesis that the universe is spatially homogeneous and isotropic on large scales. The earlier large-scale surveys of galaxy distribution are based on visual counts of galaxy angular positions. Such surveys extend to depths nearly 10% of the present Hubble radius. Redshifts are much more powerful tracers of structures than angular positions alone, because the redshifts reduce the ambiguity in distance. These surveys show clumpy small-scale distribution. One measure of this is the two-point correlation function $\xi(r)$ [15], defined by the joint probability of finding a galaxy in each of the volume elements dV_1 and dV_2 at a separation r,

$$dP = n^2 dV_1 dV_2[1 + \xi(r)] \quad (1.111)$$

where n is the mean number density of galaxies. It is found that [1, 15, 16], for $r < 10\ h^{-1}Mpc$, $\xi(r) \propto r^{-1.8}$. Such correlation functions indicate that the small-scale galaxy distribution approximates a nested clustering hierarchy, or fractal with dimension $D_1 = 1.2$. In the following subsections, we will discuss briefly the topics, fractals and fractal dimension.

Fractals

Fractals are geometrical objects [29-32] that are self-similar (or scale-invariant) under a change of scale, for example, magnification. This means that, if we cut out a portion and then we blow this piece up, the resulting object will look the same as the original one. It was B B Mandelbrot, who coined the term, fractals, for those complex structures to express that they can be characterized by a non-integer (fractal) dimensionality. Although Euclidean geometry and the theory of smooth functions can describe regular shapes and forms (e.g., lines, planes and differentiable functions), the concept of fractal geometry is needed for describing irregular shapes and forms as well as the behaviour of extremely irregular mathematical functions. The branching of trees and their roots, blood vessels, nerves in the human body etc. have fractal properties. Other examples include a landscape with peaks and valleys of all sizes, a coastline with its multitude of inlets and peninsulas, the mass distribution within a galaxy, the distribution of galaxies and clusters in the universe and so forth.

Fractals are either mathematical or natural ones. The Koch curve [32] is a good example of a mathematical fractal, which can be used to mimic a coastline.

The shapes and patterns found in nature are usually random fractals (for e.g. the galactic distribution). The reason is that, they consist of random shapes or patterns that are formed stochastically at any length scale. Because of the randomness, the self-similarity of natural fractals is only statistical, ie., given a sufficiently large number of samples, a suitable magnification of a part of one sample can be matched closely with some members of the ensemble of samples.

Fractal dimension

A correct definition of a fractal set is a 'mathematical object' whose fractal or Hausdorff dimension (D_1) is strictly larger than its topological dimension and less than the dimension of the embedding Euclidean space [29, 31]. For example, in the case of a straight line, a magnification by a factor 3, increases its length by $3 = 3^1$. For a square, when the side is magnified by 3, its area is magnified by $9 = 3^2$. For a cube the volume increases by 3^3 In general the magnification is 3^{D_1}, where D_1 is the dimension which is integer for all these three cases. However, for a Koch curve, it is found that the magnification is by $3^{D_1} = 4$, which gives a fractal dimension $D_1 = \ln 4/\ln 3 = 1.26$. The fractal dimension of a typical coastline is 1.2, which is different from the dimension of the embedding space, but is close to that of a Koch curve.

Measuring the volume of a fractal embedded in a $d-$dimensional Euclidean space, leads to the conclusion that they are objects having no integer dimension. To determine the volume V of a fractal structure of linear size L, the structure is covered by $N(L)$ number of boxes of unit volume, hence $V(L) = N(L)$. For a fractal, $N(L)$ diverges as $L \to \infty$, according to a non-integer exponent,

$$N(L) \propto L^{D_1} \tag{1.112}$$

hence the dimension is

$$D_1 = \lim_{L\to\infty} \frac{\ln N(L)}{\ln L}. \tag{1.113}$$

Now consider a fractal of finite size. Let $N(l)$ be the number of $d-$dimensional boxes of side l needed to cover the structure. Then $N(l)$ diverges as $l \to 0$ according to

$$N(l) \propto l^{-D_1} \tag{1.114}$$

Therefore

$$D_1 = \lim_{l\to 0} \frac{\ln N(l)}{\ln(1/l)}. \tag{1.115}$$

For non-fractals $D_1 = d$.

Since fluctuations are always present in physical processes, they never lead to structures with perfect symmetry. For instance, the random walk of particles, diffusion limited aggregation of particles etc. lead to fractal geometry. In the case of natural fractals it is more effective to calculate the so called density-density correlation function [31] (since natural fractals are scale-invariant only in a statistical sense),

$$\xi(r) = \frac{1}{V} \sum_{r'} \rho\left(r + r'\right) \rho\left(r'\right), \tag{1.116}$$

which gives the probability of finding a particle at $r + r'$, provided there is one at r' An object is non-trivially scale invariant, if its correlation function is unchanged upto a constant under rescaling of lengths by an arbitrary factor q:

$$\xi(qr) \propto q^{-\alpha} \xi(r), \tag{1.117}$$

where α is some non-integer number greater than zero and less than d. It can be shown that the only function which satisfies this equation is the power law dependence of $\xi(r)$ on r ie.,

$$\xi(r) \propto r^{-\alpha} \tag{1.118}$$

Hence

$$N(L) \propto \int_0^L \xi(r)\, d^d r \propto L^{d-\alpha} \tag{1.119}$$

Using (1.112), we have $D_1 = d - \alpha$ which is the fractal dimension. The two point correlation function of galactic distribution leads to a fractal dimension of ~ 1.8 [15, 16] for $r \leq 10\ h^{-1}Mpc$. However, many recent redshift surveys have revealed that, the three dimensional distribution of galaxies and clusters are characterised by fractal behaviour [33-42]. This has confirmed the de-Vaucouleurs power-law density-distance relation, $\rho(r) \propto (r/r_0)^{D_1-3}$, with a fractal dimension $D_1 \approx 2$ at least in the range of scales 1 to 200 $h^{-1}Mpc$. This fractal behaviour of galaxy distribution within a scale of $\sim 200h^{-1}Mpc$ (this scale may even deeper and the switch over scale to homogeneity is not yet identified) is a challenge to standard cosmology. Also, for a homogeneous distribution of galaxies, Hubble's count law is [1]

$$N(<m) \propto 10^{0.2D_1 m} \tag{1.120}$$

N is the number of galaxies brighter than the magnitude m. $D_1 = 3$ corresponds to the standard case. Observations show relative, persistent fluctuations in the number count versus m relation, which cannot be accounted for by the homogeneous distribution of the standard model. This problem is discussed in the final chapter of the thesis.

1.4.3 The abundance of light nuclei

One of the fundamental problems of cosmology is to explain the primary creation of matter in the universe. It is generally understood that nuclei with atomic number $A \geq 12$ are synthesized in stars through nuclear reactions. The nuclei Li^6, Be^9, B^{10} and B^{11} could be produced in galactic cosmic rays by the breakup of heavy nuclei as they travel through the interstellar medium. It is believed that the observed abundances of deuterium(D) and helium(He) is through the process of nucleosynthesis [3, 13, 43, 44] of elementary particles, beginning with that of the neutron and proton. The pioneering work in this field was done by George Gamow in the mid 1940s. Gamow was concerned with the problem of the origin of elements. He described the formation of nuclei of He and D starting from protons and neutrons by nuleosynthesis in the early universe, when temperature of the universe was of the order of 10^9K. However, by that time, Burbidge et al. [45] demonstrated that such nucleosynthesis can take place in stars also. Instead, if stars are able to achieve the objective of explaining the abundances of all elements observed, we need not consider this a cosmological problem. However, there is some doubt whether stars can do this entirely on their own, and that is why cosmology becomes important. The doubt centers around the relative abundance of He, D and H. The ratios are approximately of the order He/H $\simeq$ 0.3 and D/H $\simeq$ few times 10^{-6} [8]. The ratios are by mass densities; the stellar nucleosynthesis is unable to explain this. It is found that for our galaxy, unless the stars were much brighter in the past, the process can account for only at most 30 percent of the above value. In standard cosmology, Gamow successfully explained the nucleosynthesis of He and D (not heavier elements), in the first second or so after the big bang. The high temperature of the radiation dominated early phase is just right for this process.

1.4.4 The microwave background

The cosmic microwave background radiation (CMBR) provides the most fundamental evidence that the universe began from a hot early phase [1, 3, 13] (big bang cosmology). Gamow and his collegues Alpher and Herman predicted that the photons of the early hot era would have cooled down to provide a thermal radiation background in the microwave region of the spectrum at present. In 1965 Arno A Penzias and Robert W Wilson of Bell Telephone Laboratory at Holmdel, New Jersey, detected the CMBR, with a black body spectrum.

In the subsequent phases after nucleosynthesis, the universe cooled as it expanded, and the temperature falls as [3]

$$T \propto \frac{1}{a}. \tag{1.121}$$

However in those phases, electrons act as scattering centres for radiation and the universe was quite opaque. As temperature lowered, H atoms are formed and electrons are slowly removed from the cosmological brew, and as a result, the main agent responsible for the scattering of radiation disappears from the scene. The universe becomes transparent and this epoch is called the recombination epoch. This also corresponds to the decoupling epoch, since matter gets decoupled from radiation (this corresponds to $z \sim 10^3$). The radiation temperature falls from this as the universe expanded, and at present its value is $T_\gamma = 2.736 \pm 0.017$. This CMBR across the sky is highly isotropic and uniform. The temperature of CMBR across the sky, is reasonably uniform with $\frac{\Delta T}{T} \simeq 10^{-5}$ on angular scales ranging from 10 arc seconds to 180° [13]. The primeval density inhomogeneities necessary to initiate structure formation result in predictable temperature fluctuations in the CMBR, and so the anisotropies of CMBR provide a powerful test of theories of structure formation. The discovery of CMBR 35 years ago, had a profound effect on the direction and pace of research in physical cosmology.

1.4.5 Other problems with the standard model and the inflationary model

Apart from the problems we have already pointed out, there are other serious problems with the standard picture. Eventhough, the discovery of CMBR made a widespread acceptance of the standard model, there are several puzzles to be solved, with the standard picture. The most outstanding among them are the following.

Singularity problem

From equations (1.84) and (1.88), it is evident that, in the standard model, the scale factor vanishes at some time $t = 0$, and the matter density at that time becomes infinite (also the temperature). It can be shown that at that time, the curvature tensor R_{ijkl} goes to infinity, ie., geometry itself breaks down at that instant. This is unavoidable in the theory. This point $t = 0$ is known as the cosmological singularity or big bang. ie., spacetime was singular at that epoch. One of the most puzzling questions facing cosmologists is whether anything existed before $t = 0$. The universe came into existence at this instant, violating the law of conservation of energy. This is the singularity problem [46, 47].

Flatness problem

From the Friedmann model we have (see eq. (1.77),

$$|\Omega - 1| = \frac{|\rho(t) - \rho_c|}{\rho_c} = [\dot{a}(t)]^{-2} \quad (1.122)$$

ρ_c is the critical density for flat universe. The present value of Ω is not known exactly ($0.1 \leq \Omega_0 \leq 2$). But $\dot{a}^2 \propto 1/t$ in the early evolution of the universe, $|\Omega - 1| = \left|\frac{\rho}{\rho_c} - 1\right|$ was extremely small. For the present Ω_0 to be in the given range, Ω at early times was equal to 1, to a very high precision (extremely high fine tuning). It can be shown that

$$\Omega(10^{-43}s) = 1 \pm O(10^{-57}). \quad (1.123)$$

$$\Omega(1s) = 1 \pm O(10^{-16}). \quad (1.124)$$

This means that if Ω at Planck time ($t_p = 5.4 \times 10^{-44}s$) was slightly greater than 1, say $\Omega(10^{-43}s) = 1 + 10^{-55}$, the universe would be closed, and would have collapsed millions of years ago. If Ω is slightly less than 1, ie., $\Omega(10^{-43}s) = 1 - 10^{-55}$, the present energy density in the universe would be negligibly small and the life would not exist. In the standard model, it is not clear why the universe was created almost flat, with such accuracy (fine tuning). This is the flatness problem [3, 46].

Horizon problem

An observer at $r = 0$ at a given time t can communicate to the maximum distance $2ct$ in a time interval t (c is the speed of light, which is the limiting speed). This represents the radius of the observer's particle horizon. For the GUT era (where unification of three basic forces, namely strong, electromagnetic and weak, take place), the temperature is $T \sim 10^{15} Gev$ and $t = 10^{-36}s$ after the big bang, the particle horizon is of the order of $6 \times 10^{-26}cm$. Suppose the universe expanded as in Friedmann models till the temperature has dropped from $10^{15} Gev$ to $3K \sim 3 \times 10^{-4} ev$ (present epoch). From thermodynamical considerations we can show that the temperature varies inversely as the scale factor for the universe($a(t)$). Therefore the scale factor increased by a factor 3×10^{27}, and the particle horizon becomes $180cm$ only. Since no physical interaction travels faster than light, the particle horizon sets limits on the range of causal influences. Therefore one can not expect homogeneity to be established beyond this range. Then how come the universe is homogeneous on large scales observed today. This is called the horizon problem [3, 46].

Problem of small scale inhomogeneity

The universe is not exactly homogeneous (because hierarchical structures are present). However, on very large scales$\sim$ 4000 $h^{-1}Mpc$, it is believed to be homogeneous and isotropic. In the early epochs, before recombination (matter-radiation equality) the universe is assumed to be homogeneous and isotropic, which is quite reasonable, due to the remarkable discovery that CMBR has uniform temperature on all angular scales. However, recent measurements show anisotropies in CMBR ($\frac{\Delta T}{T} \sim 10^{-5}$) and this indicates inhomogeneities in the matter distribution of the universe at the time of decoupling (recombination). After decoupling, these irregularities grow under gravitational instability. The density contrast is usually expressed [13] in a Fourier expansion

$$\frac{\delta\rho(\vec{x})}{\rho_b} = \frac{1}{(2\pi)^3} \int \delta_k \exp(-i\vec{k}.\vec{x}) d^3k. \tag{1.125}$$

ρ_b is the background density. k is the comoving wavenumber associated with a given mode, and δ_k is its amplitude. So long as $\frac{\delta\rho}{\rho_b} << 1$ (linear regime), its physical wavenumber and wavelength scale simply with $a(t)$: $k_{phys} = k/a(t)$,

$\lambda_{phys} = a(t)\frac{2\pi}{k}$. Once $\frac{\delta\rho}{\rho_b}$ becomes ≥ 1 (nonlinear), it separates from the general expansion and maintains an approximately constant physical size (For example, on the scale of galaxy $\delta\rho/\rho_b \sim 10^5$). And also from the fact that in the linear regime, $\delta\rho/\rho_b$ grows as $a(t)$ during matter dominated epoch, we can infer that perturbations of amplitude 10^{-5} or so must have existed on these scales at the epoch of decoupling. It should be possible to account for the anisotropies in the CMBR (observed by COBE satellite) on this basis. However, the problem in the standard model, is that at the time of decoupling ($z \sim 10^3$), the Hubble radius (c/H) subtends an angle of only 0.8° on the sky today, while CMBR shows anisotropies on all angular scales. The difficulty is that, if one imagines causal, microphysical processes acting during the earliest moments of the universe and giving rise to primordial density fluctuations, the existence of particle horizons in the standard model precludes production of inhomogeneities on the scales of interest [13].

Entropy problem

The total entropy (S) of the observable part of the universe is of the order $(a_0 T_\gamma)^3 \sim 10^{87}$, where a_0 is the present scale factor and $T_\gamma = 2.7K$ is the temperature of CMBR. The entropy at Planck epoch will also be of the same order in the standard picture. In a different way, we can say that the ratio of the number of photons to baryons is 10^8 to 10^{10} in the universe. Why is this number so large? This is referred to as the entropy problem [46, 47].

The monopole problem

This is related to the horizon problem. The grand unified theories predict that as the universe cools down and the temperature reaches $\sim 10^{28}K$, a spontaneous symmetry breaking occurs and as a result, magnetic monopoles are produced. However no such monopoles have yet been detected. This is the monopole problem [3, 46]. Maxwell's electrodynamics does not permit the existence of monopoles, but they do arise in the above scenario. The mass of the monopoles so produced is $\sim \frac{10^{16}Gev}{c^2} \sim 10^{-8}g$. If one monopole is created in the horizon size region, ie., in a spherical volume of radius $6 \times 10^{-26}cm$, its density is

$$\rho_{monopole} = \frac{10^{-8}g}{(\frac{4\pi}{3})(6 \times 10^{-26})^3 cm^3}. \tag{1.126}$$

The present density in Friedmann model is

$$\rho_{monopole} = \frac{10^{-8}g}{(\frac{4\pi}{3})(6 \times 10^{-26})^3(3 \times 10^{27})^3 cm^3} \cong 3 \times 10^{-15} g\ cm^{-3} \tag{1.127}$$

This value of the monopole density is much larger than the cosmological density ($10^{-29}g\ cm^{-3}$) of the present universe, and if monopoles existed at this rate, the universe would have collapsed much earlier.

Cosmological constant problem

The cosmological constant, first introduced by Einstein as an arbitrary parameter, is seen in contemporary physics to have a quantum origin. It is related to the vacuum energy of the universe, with an equation of state $p_v = -\rho_v$. From the cosmological observations, it follows that the vacuum energy density in the present universe can not be much greater than the critical density of the universe. ie., $\rho_v \leq 10^{-29}g\ cm^{-3}$ If ρ_v is viewed as arising from the potential energy of a scalar field $(V(\phi))$, then quantum field theory can not account for this vanishingly small value for ρ_v compared with Planck density $\rho_{pl} = 0.5 \times 10^{94} g\ cm^{-3}$ This is one of the most difficult problems of unified theories with spontaneous symmetry breaking. This is called cosmological constant problem [46, 48].

Inflation

Many new models [49-56] were proposed during 1980s to overcome some of these problems with the standard picture. In 1981, Alan Guth proposed [57, 58] and later improved [59-63] by Linde and Albrecht and Steinhardt, the so called inflationary model as the solution to these problems. Inflation means rapid or exponential expansion ie., $a\,(t) \propto e^{\lambda\, t}$ where λ is a constant. According to the inflationary universe [46, 57-65] scenario, the universe in the very early stages of its evolution was exponentially expanding in the unstable vacuum-like state (of quantum mechanical nature). At the end of the exponential expansion (inflation) the energy of the unstable vacuum (of a classical scalar field or inflation field) transforms into the energy of hot dense matter and the subsequent evolution of the universe is described by the usual hot big bang theory (or standard model). The inflationary universe scenario makes it possible to obtain a simple solution to many long-standing cosmological problems and leads to a crucial modification of the standard picture of the large-scale structure of the universe.

The basic idea is that, there was an epoch when vacuum energy ($p_v = -\rho_v$) was the dominant component of the energy density of the universe, so that scale factor grew exponentially (see eq. (1.89)), ie.,

$$a(t) \propto \exp\left(\sqrt{\frac{8\pi G}{3} V(\phi)}\; t\right) \tag{1.128}$$

where $V\,(\phi)$ is the potential energy of the scalar field or inflation field. During such an epoch, a small, smooth and causally coherent patch can grow to such a size that it easily encompasses the entire observable universe today. The size, during a small interval of time rises by a factor of $\sim 10^{50}$ This inflationary scenario solves flatness, horizon, entropy and monopole problems of standard model

However, there are still problems with this inflationary models. One of them is the age problem. According to inflationary models, the present value of the density parameter Ω_0 lies close to unity. This in turn leads to $H_0 t_0 \sim 2/3$. However, present estimates [21, 25, 70, 74] put this value in the range $0.85 < H_0 t_0 < 1.91$ contrary to the above prediction. This is called the age problem in the standard and inflationary models.

If we postulate a non-zero relic cosmological constant in the present universe with density (ρ_λ) comparable to matter density (ρ_m) to overcome the age crisis, then the problem is that this cosmological constant is indistinguishable from the vacuum energy which produces inflation, and the model is bound to explain how this vacuum energy does manage to change from large initial magnitude in the early universe to a very small value at present. This will require extreme fine tuning, which is not different from the fine tuning problems in the standard model [3, 46].

Inflation does not solve the small-scale inhomogeneity problem. It also shed no light on the singularity problem of the big bang cosmology, because the inflationary stage is expected to occur at a time many orders of magnitude greater than the Planck time. In order to overcome these problems several new cosmological models were proposed [49-56] during the eighties and nineties.

Chapter 2

The New approach

2.1 Introduction

The majority of cosmologists have by now taken for granted that the standard hot big bang model of the universe is the correct starting point for the study of cosmology. The hot big bang model, which is the most successful approximation of the real universe, is based on the assumption that the matter distribution in the universe is homogeneous and isotropic on the large scale average. This interpretation leads to the prediction of Hubble's law that the apparent recession velocity of a galaxy is proportional to its distance, the constant of proportionality being the Hubble parameter $H(t)$. Observationally, the Hubble parameter is found from the redshift-apparent magnitude $(z - m)$ diagram or Hubble diagram (Fig. 1).

The uncertainty in the determination of the Hubble parameter [24, 25, 26, 27, 66], which is a measure of the expansion rate of the universe, is one of the most intriguing issues in the history of cosmology. The origin of the uncertainty is obvious from the Hubble diagram; despite rigorous attempts to control random errors and systematic effects (for example, the effect of dust grain in the region between stars and galaxies, effect of metallicity or chemical composition etc.

which are common to all types of measurements) in measurements [66], there is a clear scatter in it, which is evident in the latest Type Ia supernovae data (from supernovae cosmology project, in papers by Perlmutter et al. [25] and Riess et al. [26]) with no deterministic Hubble type relation clearly apparent. The same is true for a collection of high redshift quasars also [20, 23, 24], with $1 < z < 5$ (see Fig. 1), where the points are widely scattered[1]. However, the distance measurements are extremely difficult for quasars, due to the difficulty in identifying the standard candles, the scatter may be caused by the variation of intrinsic luminosities of quasars of the same z. Once this is admitted, the cosmologists can no longer claim that a faint object is necessarily far away. Some authors have proposed non-cosmological contributions to the redshift as a possible explanation for the scatter [20, 24, 28]. The vertical scatter in the Hubble diagram could be due to objects of varying redshifts with the same luminosity. This possibility admits an intrinsic non-cosmological component in the redshift z, like $(1 + z) = (1 + z_i)(1 + z_c)$ with cosmological component z_c obeying Hubble's law. The intrinsic component z_i may have origin in Doppler and gravitational effects or due to some difference in the age of the objects. The data is more accurate for supernova (SNa), and the scatter in its diagram needs explanation. It is also found that the scatter increases with redshift. The aim of this work is to offer an alternative explanation for the above puzzle.

Variations in the expansion rate due to peculiar velocities are a cause of error in measuring the true value of H_0 (present Hubble parameter) [66]. These are supposed to be induced by observed density fluctuations. Since the density fluctuations are evolving phenomena, peculiar velocities induced by them cannot lead to the large randomness observed at early epochs, though it is a feasible

[1]The figure is taken from Ref [20], page 407

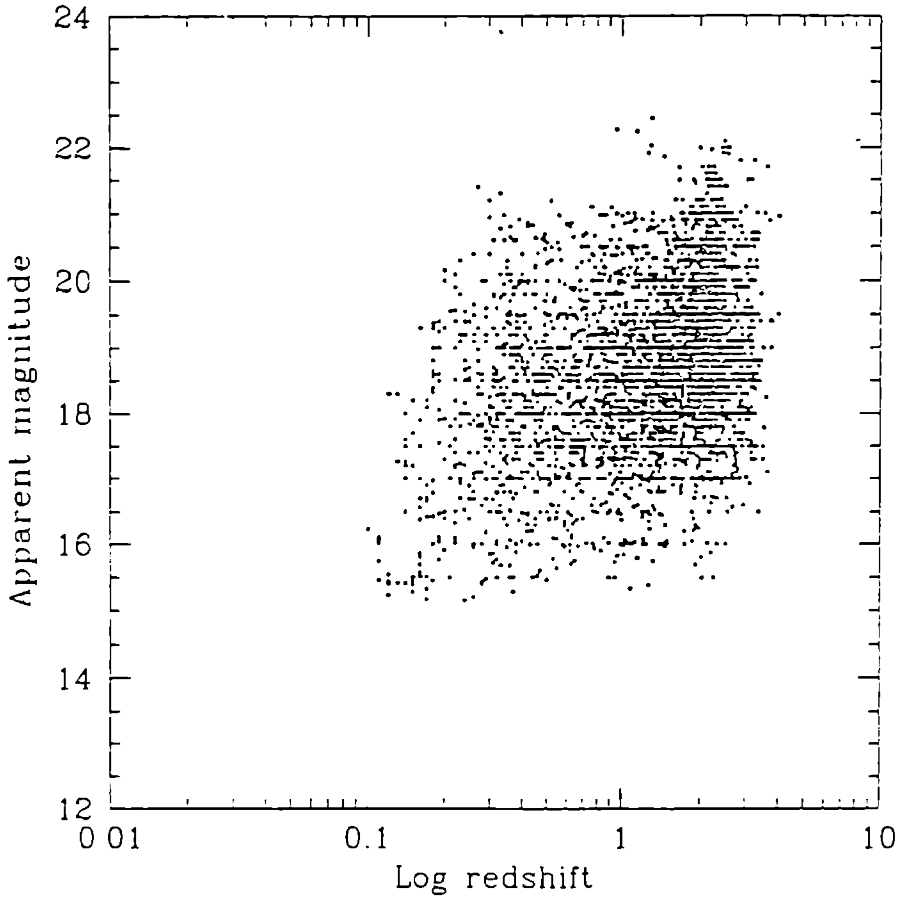

Fig. 1 The redshift–magnitude plot for quasars from the Hewitt and Burbidge catalogue.

physical process in the late universe.

In the stochastic model herein reported, we have attempted to explain this scatter as arising from an inherent stochastic or non-deterministic nature of the Hubble parameter, which provides a possible potential source of uncertainty in the measured value of H_0. In the following section we present the assumptions that lead to a stochastic nature for the Hubble parameter and derive the probability distribution function (PDF) for H using the Fokker-Planck formalism.

2.2 Stochastic equation of state

Since high values of z means we are probing into early epochs, the scatter diagram mentioned above indicates that the behaviour of the Hubble parameter is anomalous in the early universe. The scatter indicates the uncertainty in the measured value of H_0. Here we formulate an alternative scenario in which the Hubble parameter or the expansion rate of the universe is a stochastic variable, where the dynamics of the universe is described by non-deterministic Langevin type equations.

It may be noted that in the standard cosmology, the cosmological fluid is not uni-component. In fact, it is known that matter and radiation (with equations of state $p_m = 0$ and $p_r = \frac{1}{3}\rho_r$, respectively) in disequilibrium coexist in many elementary subvolumes of the universe [67]. Some recent measurements [25, 26, 48, 68-80] on the age of the universe, Hubble parameter, deceleration parameter, gravitational lensing etc. point to the need of extending the standard model by including more components other than matter and radiation to the energy momentum tensor of the present universe. Recent observations suggest that a large fraction of the energy density (ρ) of the universe has negative pressure. One explanation is in terms of vacuum energy or cosmological constant (with equation of state $p_v = -\rho_v$) while another is in terms of quintessence in the form of a scalar field slowly evolving down a potential (with equation of state $p_q = w_q \rho_q$, $-1 < w_q < 0$). The gravitational effects of such components are opposite to that of non-relativistic matter, that is, they push things apart and may even lead to an accelerating expansion of the universe. Thus more components are present in the cosmic fluid and thus our universe may be approximated by a perfect fluid having many components, each with an equation of state of the form

$$p_i = w_i \rho_i \tag{2.1}$$

with $-1 \leq w_i \leq +1$, $i = 1, 2, \cdots$ If we denote the total energy density due to all such components as ρ, then

$$\rho = \rho_m + \rho_r + \rho_v + \cdot \tag{2.2}$$

where ρ_m, ρ_r, ρ_v etc. are the average densities of matter, radiation, vacuum energy etc. In general,

$$\rho = \sum_i \rho_i, \tag{2.3}$$

where ρ_is represent energy densities of various components. From the energy-momentum conservation law (here it is assumed that only the total energy density is conserved), we have [14]

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) = -3\frac{\dot{a}}{a}\rho(1 + w) \tag{2.4}$$

where $p = \sum_i p_i$ is the total pressure and the ratio $w = p/\rho$ should lie between -1 and $+1$. Splitting ρ and p into individual components, the above equation becomes

$$\dot{\rho}_1 + \dot{\rho}_2 + \dot{\rho}_3 + \quad = -3\frac{\dot{a}}{a}\left[\rho_1(1+w_1) + \rho_2(1+w_2) + \cdots\right] \tag{2.5}$$

Equating the RHS of equations (2.4) and (2.5), we will get

$$w(t) = \frac{-\left[\dot{\rho}_1 + \dot{\rho}_2 + \cdot\cdot\right]}{3(\dot{a}/a)\rho} - 1 = \frac{\sum_i \rho_i(1+w_i)}{\rho} - 1 \tag{2.6}$$

The conservation of individual components, which may be expressed as $\dot{\rho}_i = -3\frac{\dot{a}}{a}\rho_i(1+w_i)$, is only an extra assumption, since it does not follow from the Einstein field equation. Equivalently, it can be stated that in a many component fluid as in the above case, the Einstein equations, along with the equations of state of individual components, are insufficient to determine the creation of individual components, ie., individual $\dot{\rho}_i$'s. Thus it is more general to assume that the total energy density is conserved, and this will lead to creation of one component at the expense of other components (like particle creation from vacuum energy) [55]. Since they are not uniquely determined by the field equations, such creations can be considered sporadic events, like those occurring in active galactic nuclei, which can result in fluctuations in the ratio ρ_i/ρ. In [48], Weinberg discusses some phenomenological proposals made by some authors, of the exchange of energy between vacuum and matter, or vacuum and radiation, in such a way that either ρ_v/ρ_m or ρ_v/ρ_r remains constant. He also considers the possibility of creation of radiation from vacuum energy, keeping a fixed ratio ρ_v/ρ_m. Here, as in the case of other stochastic processes like Brownian motion, a complete solution of the macroscopic system (the universe) would consist in solving all the microscopic equations, describing the individual creation processes, but such a rigorous derivation will be very complicated or even impossible. In this context, a stochastic approach is more reasonable, in which we consider the creation rates to be fluctuating, leading to fluctuations in the ratios ρ_i/ρ. This, in turn, will lead to a stochastic equation of state (fluctuating w), as can be seen from eq. (2.6) above. Consequently, the expansion rate also will be fluctuating, and the equation for the Hubble parameter will appear as a Langevin type equation (or stochastic differential equation, SDE). Physically motivated interaction models, which lead to energy transfer between various components, are proposed in the literature [77, 81], but we propose this new phenomenological model to explain the scatter in the Hubble diagram as arising from the possible fluctuations in such energy transfer.

2.3 Stochastic approach to the standard model

2.3.1 Stochastic Hubble parameter

Let us now formulate the stochastic theory, starting from the basic equation in FRW model for the Hubble parameter: [13]

$$\dot{H} = -H^2 - \frac{4\pi G}{3}\rho(1+3w) \tag{2.7}$$

For the sake of simplicity, we assume that the effect of the curvature factor appearing in the field equation is negligible, as in inflationary models, so that the background is approximately flat. Hence above equation becomes

$$\dot{H} = -\frac{3}{2}H^2(1+w(t)) \tag{2.8}$$

If w is a constant, we are back to the deterministic equation with solution $H \propto 1/t$. For the Friedmann dust approximation of the present universe, with $w = 0$, we have $H = 2/3t$. With a fluctuating w, eq. (2.8) is a Langevin type equation, describing the evolution of the stochastic variable H; ie., the fluctuating character of w leads to a random behaviour (or evolution) for the Hubble parameter. We use stochastic methods [82, 83] for the analysis of the above problem, and the probability distribution function (PDF) of the stochastic variable is calculated by solving the corresponding Fokker-Planck Equation (FPE).

To simplify the problem further, we make use of the transformation,

$$x \equiv \frac{2}{3H} \tag{2.9}$$

so that eq. (2.8) becomes

$$\dot{x} = 1 + w \tag{2.10}$$

Here x is a measure of Hubble radius in the flat FRW model. This equation is a non-deterministic, stochastic, first order differential equation. When $w = 0$, this equation is analogous to that of a particle moving in a medium with constant velocity. With a fluctuating w, the analogous particle is subjected to random forces as it moves. Now we make certain simplifying assumptions, which may be stated explicitly as follows. We consider eq. (2.10) as a stochastic Langevin equation with w as a Gaussian $\delta-$correlated Langevin force term, whose mean is zero. Though the assumption of $\delta-$correlation and that of zero mean value for w are taken for the sake of simplicity, we expect that they are reasonable, considering the time-scales involved. The corresponding FPE, describing the evolution of the probability distribution function $W(x, t)$ is obtained from eq. (2.10) by finding the drift and diffusion coefficients (from the Kramers-Moyal expansion coefficients) [82].

2.3.2 The Fokker-Planck equation

If we have a general Langevin type equation [82] of the form

$$\dot{y} = h(y,t) + g(y,t)\,\Gamma(t) \tag{2.11}$$

where $\Gamma(t)$ is a fluctuating quantity, with zero mean and is Gaussian $\delta-$correlated, then the corresponding Fokker-Planck equation (FPE) describing the time-evolution of the PDF, $W(y,t)$ can be written as

$$\frac{\partial W(y,t)}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial y}\right)^n D^{(n)}(y)\, W(y,t) \tag{2.12}$$

where the $D^{(n)}(y)$ are the Kramers-Moyal expansion coefficients, which are generally defined by

$$D^{(n)}(y,t) = \frac{1}{n!}\left[\lim_{\tau\to 0}\frac{1}{\tau}\left\langle [y(t+\tau) - x]^n\right\rangle\right]_{y(t)=x} \tag{2.13}$$

Here $y(t+\tau)$ $(\tau > 0)$ is a solution of eq. (2.11) which at time t has the sharp value $y(t) = x$. Under the assumption of $\delta-$correlation and zero mean of $\Gamma(t)$, all coefficients vanish for $n \geq 3$ and we retain only the coefficients $D^{(1)}$ and $D^{(2)}$, called drift and diffusion coefficients, respectively. In the one variable case, the drift and diffusion coefficients are

$$D^{(1)}(y,t) = h(y,t) + \frac{\partial g(y,t)}{\partial y} g(y,t) \tag{2.14}$$

and

$$D^{(2)}(y,t) = g^2(y,t) \tag{2.15}$$

In the present case these coefficients are found to be constants. The one variable FPE [82] is

$$\frac{\partial W(y,t)}{\partial t} = \left[-\frac{\partial}{\partial y}D^{(1)}(y) + \frac{\partial^2}{\partial y^2}D^{(2)}(y)\right] W(y,t) \tag{2.16}$$

If we extend the procedure to the $N-$variable ($\{y\} = y_1, y_2 \cdots y_N$) case, then the Langevin equation is

$$\dot{y}_i = h_i(\{y\}, t) + g_{ij}(\{y\}, t)\Gamma_j(t) \tag{2.17}$$

with $\langle \Gamma_j(t) \rangle = 0$ and $\left\langle \Gamma_i(t)\Gamma_j(t') \right\rangle = \delta_{ij}\delta(t - t')$. The drift and diffusion terms are

$$D_i(\{y\}, t) = h_i(\{y\}, t) + g_{kj}(\{y\}, t)\frac{\partial}{\partial y_k} g_{ij}(\{y\}, t) \tag{2.18}$$

$$D_{ij}(\{y\}, t) = g_{ik}(\{y\}, t) g_{jk}(\{y\}, t) \tag{2.19}$$

and the FPE becomes

$$\frac{\partial W(\{y\}, t)}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial y_i} D_i^{(1)}(\{y\}) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial y_i \partial y_j} D_{ij}^{(2)}(\{y\}) \right] W(\{y\}, t). \tag{2.20}$$

The FPE is just an equation of motion for the distribution function of fluctuating macroscopic variables. For a deterministic treatment, we neglect the fluctuations of the macroscopic variables. For the FPE (2.20) this would mean that we neglect the diffusion term. We come across such equations when dealing with the Brownian motion of a particle. By solving the FPE one will get the distribution function, from which averages of various macroscopic variables are obtained by integration. Now let as return to the SDE (2.10). The corresponding FPE can be written as

$$\frac{\partial W(x, t)}{\partial t} = \left[-D^{(1)} \frac{\partial}{\partial x} + D^{(2)} \frac{\partial^2}{\partial x^2} \right] W(x, t). \tag{2.21}$$

We find the drift coefficient $D^{(1)} = 1$, and the diffusion coefficient $D^{(2)} = D$, is assumed to be a constant. These coefficients follow from eq. (2.10). It will be noted that this diffusion term arises from fluctuations in w alone. We can solve the FPE by first assuming an ansatz

$$W(x, .t) = \phi_n(x)\, e^{-\lambda_n t}, \tag{2.22}$$

where we treat $\phi_n(x)$ and λ_n as the eigenfunctions and eigenvalues of the Fokker-Planck operator

$$L_{FP} = \left[-\frac{\partial}{\partial x} D^{(1)}(x) + \frac{\partial^2}{\partial x^2} D^{(2)}(x) \right], \tag{2.23}$$

with appropriate boundary conditions. Now we define two more functions in order to get a solution for the FPE:

$$\Phi(x) = -\int \frac{D^{(1)}}{D} dx' \;=\; -\frac{x}{D}, \tag{2.24}$$

and

$$\psi_n(x) = \exp\left[\frac{\Phi}{2}\right] \phi_n(x) = e^{-x/2D} \phi_n(x), \tag{2.25}$$

where $\Phi(x)$ is treated as a stochastic potential, and $\psi_n(x)$ is an eigenfunction of the Hermitian operator L_H

$$L_H = \exp(\frac{\Phi}{2}) L_{FP} \exp(-\frac{\Phi}{2}). \tag{2.26}$$

Making use of (2.22) and (2.25), the time independent part of FPE becomes

$$\frac{\partial^2 \psi_n(x)}{\partial x^2} = \left[\frac{1}{4D^2} - \frac{\lambda_n}{D}\right] \psi_n(x) = -k^2 \psi_n(x). \tag{2.27}$$

Here

$$k = \pm \left[\frac{\lambda_n}{D} - \frac{1}{4D^2}\right]^{1/2} \tag{2.28}$$

The most general solution to eq. (2.21) is

$$W(x,t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \phi_n(x), \tag{2.29}$$

where c_n can be real or complex, but $W(x,t)$ is always real. When a stationary solution exists, $\lambda_0 = 0$. In the above situation, we see that for $\lambda_n < 1/4D$, k^2 is negative and the solution $\psi_n(x)$ is exponentially diverging, which is not an admissible solution. Thus we conclude that $\lambda_n \geq \frac{1}{4D}$ so that k is real, though there is no stationary solution existing in this case. This is a constraint equation for the eigenvalue parameter λ_n, which has the dimensions of frequency. Thus the only physically reasonable solution existing, is with

$$\psi_n(x) = A \exp(ikx), \tag{2.30}$$

$-\infty < k < +\infty$, which gives

$$\phi_k(x) = A \exp\left(\frac{x}{2D} + ikx\right) \tag{2.31}$$

A is a normalisation constant. This situation is justifiable since it precisely corresponds to the deterministic solution $x = t$ of eq. (2.10) when we calculate the PDF One point we have to note here is that this eigenfunction formulation of finding solution to FPE is applicable only if the drift and diffusion coefficients are independent of time, as in our case. Following the standard procedure, we make use of the completeness relation for $\psi(x)$ to specify the initial condition

$$\delta(x - x') = \int_{-\infty}^{+\infty} \psi_k^*(x) \; \psi_k(x') \; dk \tag{2.32}$$

We evaluate the transition probability for the stochastic variable to change from the state x' at time t' to x at time t as

$$\begin{aligned} P\left(x,t \mid x',t'\right) &= \exp\left[L_{FP}(t-t')\delta(x-x')\right] \\ &= \exp\left[\frac{\Phi(x')}{2} - \frac{\Phi(x)}{2}\right] \int_{-\infty}^{+\infty} \psi_k^*(x)\, \psi_k(x')\, dk\, e^{-\lambda(t-t')} \\ &= \frac{1}{2\sqrt{\pi D(t-t')}} \exp\left[-\frac{[(x-x')-(t-t')]^2}{4D(t-t')}\right] \end{aligned} \tag{2.33}$$

The probability distribution functions at two times are related by

$$W(x',t') = \int P(x',t' \mid x,t)\, W(x,t)\, dx \tag{2.34}$$

Since here the transition probability has the initial value

$$P(x,t \mid x',t) = \delta(x-x'), \tag{2.35}$$

the PDF $W(x,t)$ is the same as the transition probability. Thus we get the distribution function as

$$W(x,t) = \frac{1}{2\sqrt{\pi D t}} \exp\left[-\frac{(x-t)^2}{4Dt}\right], \tag{2.36}$$

which is Gaussian in form and is real. Note that we have chosen $A = 1/\sqrt{2\pi}$ in eq. (2.30) for normalisation purpose. We can immediately replace this distribution function in terms of the stochastic Hubble parameter H as $W'(H,t)$. Dropping the prime, we can write this PDF as

$$W(H,t) = \frac{1}{3H^2}\frac{1}{\sqrt{\pi D t}} \exp\left[-\frac{(2-3Ht)^2}{36H^2Dt}\right] \tag{2.37}$$

The Gaussian in (2.36) has its peak moving along in such a way that the expectation value of the variable is $\langle x \rangle = t$ and this corresponds to the deterministic solution of (2.10). The width of the Gaussian is found from the variance $\sigma^2 =$

$\langle (x - \langle x \rangle)^2 \rangle = 2\ D\ t$ and $\sigma \geq \langle x \rangle$ till $t = 2D$. With $H = 100\ h\ kms^{-1}Mpc^{-1}$, $t = t_{17} \times 10^{17}s$ and $D = D_{17} \times 10^{17}s$, the PDF can be written as

$$W(h,t) = \frac{3.0856}{3h^2} \frac{1}{\sqrt{\pi D_{17} t_{17}}} \exp\left[- \frac{(6.1712 - 3ht_{17})^2}{36h^2 D_{17} t_{17}} \right] \tag{2.38}$$

For the range of values of interest, $1 < t_{17} < 5$ and $D_{17} \sim 10^{-3}$, $W(h,t)$ is approximately a Gaussian.

2.4 Comparison with data and conclusions

In section 2.2, we have shown that fluctuations in the creation rates are physical processes which can lead to a stochastic equation of state. A fluctuating w factor, in turn, will lead to fluctuations in the time evolution of the Hubble parameter; ie., the expansion rate of the universe becomes a stochastic quantity, instead of remaining a deterministic variable. We argue that such a fluctuating expansion rate might have led to randomness in the recession velocities of objects (galaxies and other extra galactic objects) in addition to peculiar velocities. The effect of this randomness of recession velocities is similar to that of peculiar velocities, and will produce scatter in the Hubble diagram.

From the Hubble diagram, one can find the PDF for the present Hubble parameter H_0, which arises from the point-to-point variance of the measured Hubble flow, in the following way. We use the data of 42 Type Ia supernovae (the supernova cosmology project) given in a paper by Perlmutter et al [25]. The traditional measure of distance to a supernova (SN) is its observed distance modulus $\mu_o = m_{bol} - M_{bol}$, the difference between its bolometric apparent and absolute magnitude. In FRW cosmology, the distance modulus is predicted from the source's redshift z, according to $\mu_p = 5 \log \left(\frac{D_L}{1Mpc} \right) + 25$, where $D_L = r_j a(t_0)(1+z)$ is given by equations (1.103) and (1.104) for the present model. Thus one can obtain the predicted distance modulus of an object with redshift z. Conventionally, assuming that the observed and predicted distance moduli coincide, one can find a value of H_0. For a collection of objects, one can find the likelihood for H_0, from a χ^2 statistic:

$$\chi^2 = \sum_i \frac{(\mu_{p,i} - \mu_{o,i})^2}{\sigma_i^2} \tag{2.39}$$

where σ_i is the total uncertainty in the corrected peak magnitude of SN Ia, which includes dispersion in galactic redshift due to peculiar velocities, uncertainty in galaxy redshift and other effects. For the special model we are considering, h is the only parameter and the normalised PDF can now be obtained as (we use the methods given in the paper by Riess et al. [26])

$$p(h \mid \mu_o) = \frac{\exp(-\chi^2/2)}{\int_{-\infty}^{+\infty} dh\ \exp(-\chi^2/2)} \tag{2.40}$$

The Hubble constant, as derived by these authors [26] from the MLCS and TF approaches, are $65.2 \pm 1.3\ km\ s^{-1} Mpc^{-1}$ and $63.8 \pm 1.3\ km\ s^{-1} Mpc^{-1}$ respectively. This, they claim, are extremely robust and do not include any systematic errors like the uncertainty in the absolute magnitude of SN Ia. In the present case, we have computed $p(h \mid \mu_o)$ (where $H_0 = 100h\ km\ s^{-1} Mpc^{-1}$) using the supernova data in [25], which corresponds to their Fit C and attempt to compare $p(h \mid \mu_o)$ with the theoretical PDF $W(h, t_0)$ of the present Hubble parameter, to evaluate the diffusion constant D appearing in the expression. It is found that the two curves, shown in Fig. 2, coincide for a value of $D = 3.77 \times 10^{13} s$ (This corresponds to an age $3.029 \times 10^{17} s$). The best fit value is $h = 0.679$ and a 68.3% credible region has a half width $\sigma_h = 0.011$. Since our primary objective is to make an order of magnitude evaluation of D, we choose a fiducial absolute magnitude for

SN Ia in computing μ_o, equal to -19.3 mag. Slight variation in this quantity will not significantly affect D, though the best fit value for h may change.

The stochastic nature of the expansion rate provides a cause of systematic error in measuring the true value of the Hubble constant H_0, apart from the peculiar velocities induced by the observed density fluctuations. The conventional explanation may be adequate to account for the observed peculiar velocities of objects, in the range of 100 $km\ s^{-1}Mpc^{-1}$ – 400 $km\ s^{-1}Mpc^{-1}$. The scatter in the Hubble diagram at low redshifts may be explained on this basis. But, since the amplitude of density fluctuations in the early universe was very low, the scatter at very high redshifts remains unexplained, and it is desirable to look for some alternative mechanisms which can induce random motions, like the one presented here.

Moreover, since a fluctuating w factor (in the equation of state) can lead to fluctuations in ρ, our stochastic approach can be extended to include the density fluctuations also. We will discuss in detail the stochastic approach to the evolution of cosmological parameters including ρ in the next chapter. Once it is possible to identify some standard candles for quasars, we can apply this formalism to estimate the value of the diffusion constant D. A stochastic theory of density fluctuations also helps us to find this value from observations (if provided with the data), and to check whether these different estimations give identical results. If there are some explicit examples of models where a stochastic w emerges, the predicted value of D may be compared with our estimation, but here, we have not made any attempts in this regard. Thus the scatter in the Hubble diagram is an indication of non-deterministic behaviour of H, whose randomness is explained on the basis of a stochastic theory.

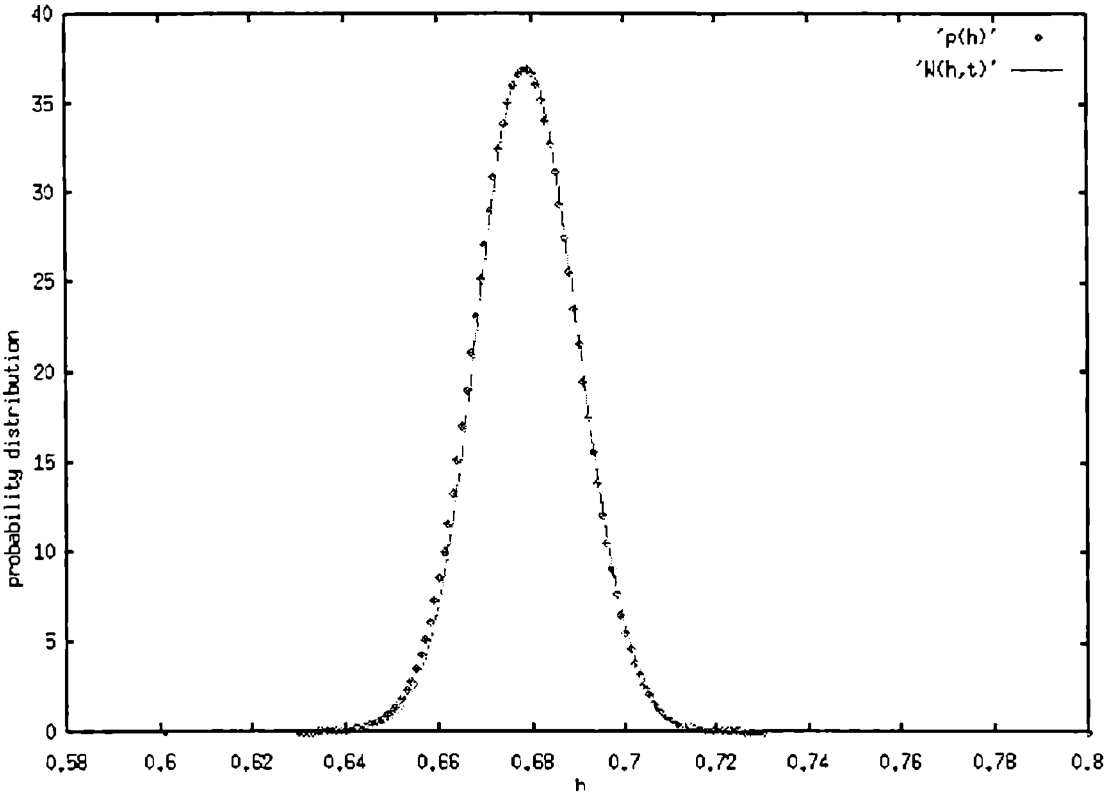

Fig. 2 The PDF vs h. The dotted line is the PDF calculated using observational data and the solid curve is the theoretical one, obtained by plotting W(h,t)

Chapter 3

Evolution of cosmological parameters in the new model

3.1 Introduction

In this chapter we develop a stochastic formulation of cosmology in the early universe, after considering the scatter in the redshift-apparent magnitude $(z - m)$ diagram in the early epochs as a piece of observational evidence for a non-deterministic evolution of the early universe. The standard model is based on the assumption of homogeneity and isotropy of matter distribution on very large scales (cosmological principle), which leads to the interpretation of deterministic and linear Hubble's law, and one expects a scatter-free $z - m$ diagram for galaxies and other extra galactic objects. For nearer galaxies (low z), the scatter is small, and this can be accounted by the conventional peculiar velocities. However, the scatter increases with z and the large randomness observed for high redshift objects (early epochs) is not due to peculiar velocities alone, but due to some other mechanisms. In the preceding chapter we proposed that a stochastic equation of state or a fluctuating mean w factor in the equation of state led to a non-deterministic or stochastic Hubble parameter and argued that such a fluctuating expansion rate in the early universe might have led to randomness in the recession velocities of objects, in addition to peculiar velocities, and will produce the scatter in the redshift-apparent magnitude diagram in those epochs. The other consequences of such a fluctuating expansion rate will be discussed in the concluding chapter. Here we formulate a more general description of the stochastic dynamics of the early universe in the Fokker-Planck formalism and discuss the non-deterministic evolution of the total density of the universe. Since the evolution of the scale factor for the universe depends on the energy density, from the coupled Friedmann equations (modified) we calculate the two-variable probability distribution function (PDF) using the Fokker-Planck equation (FPE).

Such a stochastic approach is necessary when the mean w factor in the equation of state of the cosmic fluid is a fluctuating quantity. We can say that the evolution of the universe in those epochs is non-deterministic (becomes nearly deterministic for the present epoch) and the corresponding dynamical equations are of Langevin type where one can evaluate the PDFs of the stochastic variables. We also make clear that the fluctuations in the ratio ρ_i/ρ (which leads to a stochastic equation of state) that we are taking into account here (see section 2.2), are classical; ie., our stochastic model is a modification to the classical Friedmann model of the early universe ($z \sim 1$ to 10 and may be even higher), when fluctuations are significant. In [84, 85] Fang et al. discuss a stochastic approach to early universe (before the recombination epoch), where the cosmic fluid consisting of primeval plasma and radiation, is not perfect, but has dissipations due to differences in the adiabatic cooling rates of the components of the cosmic fluid and the possible energy transfer between them. However, once we probe into still earlier epochs (stages of inflation etc.), quantum fluctuations become very important. Many authors [86-89] discuss the need for a stochastic approach to inflation, when the quantum fluctuations of the scalar field are significant, and try to get a PDF for the scalar field after solving the quantum Langevin equation (or FPE) describing the evolution of the scalar field. However we adopt the stochastic approach in the classical regime, where fluctuations in the creation rates and also in the possible energy transfer between different components of the cosmic fluid, lead to a stochastic equation of state. This causes a non-deterministic (stochastic) expansion rate for the universe, and the dynamics of the early universe is described by a set of stochastic differential equations instead of the deterministic Friedmann equations of standard model.

3.2 Stochastic evolution of the cosmological parameters

3.2.1 Density parameter

Suppose the universe is approximated by a many component fluid in the early epochs, with a fluctuating w term in the equation of state. Now we write the evolution equation for the total density in those epochs (assuming that total energy density is conserved), immediately after inflation, when the curvature factor appearing in the field equation is negligible. Therefore,

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left[1 + w(t)\right]\rho. \tag{3.1}$$

Using Friedmann equations [14] we have

$$\dot{\rho} = -24\pi G\left[1 + w(t)\right]\rho^{3/2} \tag{3.2}$$

This equation is a SDE of the Langevin type. Since w is a fluctuating 'force' term, ρ is a stochastic variable, ie., its evolution is non-deterministic. The random behaviour of ρ in the early universe is due to fluctuations in the factor w alone. If fluctuations are zero, we are back to the deterministic standard model. We apply the standard stochastic methods to this equation and the PDF is calculated using the FPE.

By making use of the transformation,

$$\sigma = \frac{1}{\sqrt{6\pi G\rho}}, \tag{3.3}$$

eq. (3.2) becomes

$$\dot{\sigma} = 1 + w(t), \tag{3.4}$$

which is again a non-deterministic, Langevin type equation, we come across such equations in Brownian motion etc. (here $\sigma \propto t$ for a pure deterministic situation as in the standard model). To solve eq. (3.4) we use certain simplifying assumptions we already made in section 2.3.1. The FPE formed from eq. (3.4) is

$$\frac{\partial W(\sigma,t)}{\partial t} = \left[-\frac{\partial}{\partial\sigma} + D\frac{\partial^2}{\partial\sigma^2}\right] W(\sigma,t). \tag{3.5}$$

Here again diffusion coefficient $D^{(2)} = D$ (D is a constant with dimensions of time, which is introduced for the purpose of generality). In order to obtain non-stationary solutions of eq. (3.5), we use the following separation ansatz for $W(\sigma,t)$,

$$W(\sigma,t) = \phi(\sigma)\exp(-\lambda t). \tag{3.6}$$

Substituting this into (3.5) and solving for $\phi(\sigma)$ we get

$$\phi(\sigma) = A\exp\left[\frac{\sigma}{2D} + ik\sigma\right], \tag{3.7}$$

where

$$k^2 = \frac{\lambda}{D} - \frac{1}{4D^2}. \tag{3.8}$$

Thus we see that for $\lambda < 1/4D$, k^2 is negative and the solution is exponentially diverging, which is not physically admissible. Hence we conclude that $\lambda \geq 1/4D$, so that k is real. We write the most general solution as

$$W(\sigma, t) = \sum_n c_n \phi_n(\sigma) \exp(-\lambda_n t), \tag{3.9}$$

where c_n can be real or complex. For a continuous parameter k, from equations (3.6) and (3.7) the general solution or the distribution function is given by

$$W(\sigma, t) = A \int_{-\infty}^{+\infty} \exp\left[\frac{\sigma}{2D} + ikD - k^2 Dt - \frac{t}{4D}\right] dk. \tag{3.10}$$

We choose $A = 1/2\pi$ as a normalisation constant. On evaluating the above integral, we get the PDF as

$$W(\sigma, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left[\frac{-(\sigma - t)^2}{4Dt}\right], \tag{3.11}$$

which is Gaussian. The average value of the stochastic variable σ is $\langle\sigma\rangle = t$ (corresponds to the deterministic situation). The variance $v = \langle(\sigma - \langle\sigma\rangle)^2\rangle = 2Dt$. Once $W(\sigma, t)$ is known, it is straight forward to write the distribution function $W(\rho, t)$ as

$$W(\rho, t) = \frac{1}{\sqrt{96\pi^2 DG\rho^3 t}} \exp\left[\frac{-(1 - t\sqrt{6\pi G\rho})^2}{24\pi GD\rho t}\right] \tag{3.12}$$

We can also find the transition probability using an equation similar to (2.33) for the stochastic variable to change from an initial state (σ', t') to a final state (σ, t) as

$$P(\sigma,t \mid \sigma^{'},t^{'}) \;=\; \frac{1}{\sqrt{4\pi D(t-t^{'})}}\exp\left[\frac{-[(\sigma-\sigma^{'})-(t-t^{'})]^2}{4D(t-t^{'})}\right], \tag{3.13}$$

with the initial value

$$P(\sigma,t \mid \sigma^{'},t) = \delta\left(\sigma-\sigma^{'}\right), \tag{3.14}$$

indicating a Markovian nature [82] for the stochastic variable σ and the PDF $W(x,t)$ is same as the transition probability $P(x,t \mid 0,0)$. In terms of ρ eq. (3.13) becomes

$$P(\rho,t \mid \rho^{'},t^{'}) \;=\; \frac{1}{\sqrt{4\pi D(t-t^{'})}}\exp\left[\frac{-[(\sqrt{\rho^{'}}-\sqrt{\rho})-\sqrt{6\pi G\rho\rho^{'}}(t-t^{'})]^2}{24\pi G\rho\rho^{'}D(t-t^{'})}\right] \tag{3.15}$$

This represents the probability for the energy density to change from an initial value ρ to a final value $\rho^{'}$ during a time interval $(t-t^{'})$ in the early epochs. This characterizes the stochastic behaviour of density evolution in the early universe.

3.2.2 Scale factor and density parameter together as a two variable Fokker-Planck problem

Under the assumption that the factor w is fluctuating during the early epochs, the evolution of the scale factor also becomes non-deterministic, since the time evolution of $a(t)$ is determined by the total density of the universe. So we have a system of coupled SDEs derived from Friedmann equations [14]. We have

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\rho}{3}}, \tag{3.16}$$

and

$$\dot{\rho} = -\sqrt{24\pi G}[1 + w(t)]\rho^{3/2} \tag{3.17}$$

Here we are considering the dynamics of the universe after the inflationary stage. With the transformation (3.3), the above system of equations reduces to

$$\dot{a} = \frac{2a}{3\sigma}, \tag{3.18}$$

and

$$\dot{\sigma} = 1 + w(t). \tag{3.19}$$

Following the standard procedure [82] (consider the equations (2.17) to (2.20)) we have the drift coefficients $D_a^{(1)} = 2a/3\sigma$, $D_\sigma^{(1)} = 1$ and the diffusion coefficient $D_{\sigma\sigma}^{(2)} = D$ is assumed to be a constant. It may be noted that the diffusion term arises due to fluctuations in w alone. The two variable FPE for the distribution function $W(a, \sigma, t)$ can be written as

$$\frac{\partial W}{\partial t} = -\frac{2}{3\sigma}\left[W + a\frac{\partial W}{\partial a}\right] - \frac{\partial W}{\partial \sigma} + D\frac{\partial^2 W}{\partial \sigma^2} \tag{3.20}$$

To solve this equation, we assume an ansatz

$$W(a, \sigma, t) = U(a)\ V(\sigma)\ \exp(-\lambda t) \tag{3.21}$$

and substituting into eq. (3.20), we obtain

$$\frac{\sigma}{V}\,D\,\frac{d^2V}{d\sigma^2} - \frac{\sigma}{V}\frac{dV}{d\sigma} + \lambda\sigma \;=\; \frac{2}{3}\left[\frac{a}{U}\frac{dU}{da} \;+\; 1\right] \tag{3.22}$$

Each side of this equation can be equated to a constant m, since the LHS depends only on σ and the RHS only on a. When $m = 0$, $\frac{dU}{U} \propto -\frac{da}{a}$, which on integration gives

$$U(a) \propto \frac{1}{a}, \tag{3.23}$$

and

$$V(\sigma) \propto \exp\left[\frac{\sigma}{2D} + ik\sigma\right], \tag{3.24}$$

with k given by eq. (3.8). A physically reasonable solution exists for $\lambda \geq 1/4D$, which is

$$W(a,\sigma,t) = \frac{B}{a}\exp\left[\frac{\sigma}{2D} + ik\sigma - \lambda t\right] \tag{3.25}$$

Here B is a normalisation constant, chosen to be $1/2\pi$. One point to be noted is that, the most general solution to eq. (3.22), when $m \neq 0$, is a series solution owing to the essential singularity at $\sigma = 0$. One can find a limiting solution as $\sigma \to 0$, in the following form

$$W(a,\sigma) \to a^{(\frac{2}{3}m-1)}\frac{\exp(\frac{\sigma}{2D})}{(m/D)}\sum_{n=1}^{\infty}\frac{(m\sigma/D)^n}{n!(n-1)!}. \tag{3.26}$$

However, we will get a real general solution in a compact form after integrating eq. (3.25) in the range $-\infty < k < +\infty$,

$$W(a,\sigma,t) = \frac{1}{\sqrt{4\pi Dt}}(a)^{-1}\exp\left[-\frac{(\sigma-t)^2}{4Dt}\right] \tag{3.27}$$

In terms of ρ, it becomes

$$W(a,\rho,t) = \frac{1}{\sqrt{96\pi^2 GD\rho^3 a^2 t}}\exp\left[-\frac{\left(1 - t\sqrt{6\pi G\rho}\right)^2}{24\pi GD\rho t}\right] \tag{3.28}$$

The two variable PDF is Gaussian in σ, and diverges as $a \to 0$, where classical approach fails and quantum theory [90] takes over. Now we write the expression for the transition probability (see equations (2.33) and (2.34)):

$$P(a,\sigma,t \mid a',\sigma',t') = \frac{(aa')^{-1}}{\sqrt{4\pi D(t-t')}} \exp\left[\frac{-[(\sigma-\sigma')-(t-t')]^2}{4D(t-t')}\right] \quad (3.29)$$

In terms of ρ and a it becomes

$$P(a,\rho,t \mid a',\rho',t') = \frac{(aa')^{-1}}{\sqrt{4\pi D(t-t')}} \exp\left[\frac{-[(\sqrt{\rho'}-\sqrt{\rho})-\sqrt{6\pi G\rho\rho'}(t-t')]^2}{24\pi G\rho\rho' D(t-t')}\right], \quad (3.30)$$

which represents the transition probability for the variables to change from the state (a',ρ') to (a,ρ). Thus the scale factor a together with the density ρ evolves in a non-deterministic way, which in turn, strongly influences the formation of large scale structures in the universe, since the evolution of the density perturbations also depends on w. This we consider in detail in the final chapter. In all these cases we get Gaussian distributions, which are sharply peaked initially, but spread out with time.

3.3 Conclusion

To conclude, we note that the stochastic approach presented in the last two chapters is a modification to the standard model, when fluctuations are present. In the standard model where the cosmological principle is strictly valid, we have a deterministic evolution of the universe. However, when we probe into the early epochs (possibly after inflation), the dynamics of the universe can not be described by the purely deterministic equations of Friedmann, since fluctuations are present. The observations show a scatter diagram for redshift-magnitude relation for supernovae and high redshift objects. This scatter diagram is an indication of the non-deterministic behaviour of the Hubble parameter, whose randomness is explained on the basis of a stochastic theory, after introducing

w as a fluctuating quantity in the equation of state (we pointed out in section 2.2 that fluctuations in the creation rates are physical processes which can lead to a stochastic equation of state). This further leads to stochastic fluctuations in the time evolution of the total energy density of the universe, as well as in the expansion factor for the universe. Thus both parameters become stochastic quantities, instead of remaining deterministic variables. We have shown in Chapter 2 that the time-evolution of the Hubble parameter is fluctuating, ie., the expansion rate of the universe also becomes a stochastic quantity, instead of being a deterministic variable.

An estimate of D is also possible with the distribution function of the density parameter, provided we have data for the density parameter. Thus, if w is a fluctuating quantity, the evolution of the early universe becomes stochastic or non-deterministic, and the dynamical equations of those epochs are Langevin type equations, where one can evaluate the PDFs of the variables. In [85], Berera and Fang describe dynamically how the stochastic fluctuations arising from various dissipations generate seeds of density perturbations in the early universe, apart from the quantum fluctuations of the standard inflationary model. Here we attempted to adopt a stochastic approach to the early universe due to fluctuations in the mean equation of state. Since the stochastic equation of state leads to fluctuations in the time evolution of the total density of the universe, the density contrast will also become a stochastic quantity. We will discuss this aspect in Chapter 5.

Thus we have formulated a stochastic model and developed a set of Langevin equations for the cosmological parameters ($H(t)$, $\rho(t)$, $a(t)$ etc.), in early epochs, when the universe is approximated by a many component fluid. It is expected that the diffusion coefficient is a crucial factor in the evolution of the universe, especially in the early phase, where it influences the time evolution of the cosmological parameters. As fluctuations die out with time ($D \to 0$), the evolution becomes deterministic and we recover the standard FRW models.

Chapter 4

Application of the stochastic approach to the generalized Chen-Wu type cosmological model

4.1 Introduction

In this chapter we reconsider the classical stochastic model of cosmology developed in the last two chapters. The uncertainty in the determination of the Hubble parameter, which is a measure of the expansion rate of the universe, is one of the most intriguing issues in the history of cosmology. The origin of uncertainty is obvious from the redshift-apparent magnitude diagram; despite rigorous attempts to control random errors in measurement, there is a clear scatter in it, though it is now possible to narrow down this to a great extent. But, now we will show that by using the $z - m$ data for Type Ia supernovae [25], the scatter increases as we go to higher redshifts. In Chapter 2, we have attempted to explain this scatter as arising from an inherent stochastic or non-deterministic nature of the Hubble parameter. It was shown that a fluctuating $w-$factor in the equation of state $p = w\rho$ will lead to this kind of behaviour for H, and the equation for the Hubble parameter will appear as a Langevin type equation. There we assumed for the sake of simplicity that space sections are flat and w is a Gaussian $\delta-$correlated stochastic force with zero mean. With these assumptions, we have written the FPE, whose solution gives the theoretical PDF for H_0 at time t_0, denoted as $W(h, t_0)$ (where $H_0 = 100h\ km\ s^{-1}Mpc^{-1}$; The subscript 0 denotes the present epoch). Using the $z - m$ data μ_o (o corresponds to observational distance modulus) for SN Ia used in [25, 26], we computed the observational PDF $p(h \mid \mu_o)$ for h in the present universe, again assuming its space sections to be flat. This PDF arises from the point to point variance of the Hubble flow. We compared the two plots for the present universe (see Fig. 2) and found them to agree well, for a value of the diffusion constant, appearing in the FPE for the stochastic Hubble parameter, equal to $3.77 \times 10^{13}s$.

This result is a first step towards an understanding of the anomalous scatter in the Hubble diagram at high redshifts. However, there are certain refinements to be made in our analysis. One drawback of the above scheme of comparing these two PDFs is that when we derived $W(h, t)$, the assumption was made that w has mean value zero, whereas the observational PDF $p(h \mid \mu_o)$ was evaluated for a model which contains matter and vacuum energy, which has mean total pressure negative. Instead, if we had used in this evaluation the expression for the distance modulus for a flat universe, which is matter dominated (ie., with $w = 0$), an observational PDF would have been obtained, but the best fit value for h would be ridiculously low. But most of the present observations are incompatible with an $\Omega_\Lambda = 0$ flat model.

Another shortcoming is that though in both cases we take the PDF for h, it remains to be explained how legitimate is the comparison of $W(h, t_0)$ for the present universe with a PDF $p(h \mid \mu_o)$ evaluated using the data that include high z objects, which belong to the distant past.

Now we try to rectify these two defects and to make a more rigorous test of the stochastic assumptions using observational data by (1) comparing both the theoretical and observational PDFs evaluated for the same model, which is an alternative flat model [91], and (2) evaluating the observational PDF $p(h_j \mid \mu_{oj})$ for the Hubble parameter at the same epoch t_j as that in the theoretical PDF $W(h_j, t_j)$. This procedure helps us to compare the theoretical and observational PDFs for the Hubble parameter for the same model and at the same epoch. The value of the diffusion constant evaluated at any time is obtained as nearly a constant, in agreement with our assumptions. A novel feature in our new approach is that we evaluate the observational PDF for the Hubble parameter at various instants in the past, also with an objective of justifying our assertion that the scatter increases as we go into the past.

4.2 Stochastic approach to the new model

In all FRW models, the Einstein equations, when combined with the conservation of total energy density, can be written in terms of the Hubble parameter as

$$\dot{H} = -H^2 - \frac{4\pi G}{3}(\rho + 3p). \tag{4.1}$$

If we restrict ourselves to flat models, then (with $p = w\rho$),

$$\dot{H} = -\frac{3}{2}H^2(1 + w). \tag{4.2}$$

In Chapter 2, we considered this flat case and assumed that w is a Gaussian $\delta-$correlated Langevin 'force' term with mean value zero. This means that the mean total pressure of the universe is zero, the same as that for dust. But many recent observations [68-80] are incompatible with this model and hence, as mentioned in the introduction, we look for a more observationally correct, but simple model to apply our stochastic approach.

4.2.1 Generalized Chen-Wu type cosmological model

The deterministic model [91] we propose to use is the one in which the total energy density obeys the condition $\rho + 3p = 0$, and hence having a coasting evolution ($a \propto t$). On the basis of some dimensional considerations in line with quantum cosmology, Chen and Wu [50] have argued that an additional component which corresponds to an effective cosmological constant Λ, must vary as $1/a^2$ in the classical era. Their decaying-Λ model assumes inflation and yields a value for q_0 (deceleration parameter). Their model alleviates some of the problems of the standard model, but their results were found to be incompatible with observations. In [91], the authors generalize this model by arguing that the Chen -Wu ansatz is applicable to the total energy density of the universe and not to Λ alone. If we assume that the energy components in this model are ordinary matter and vacuum, then the condition $\rho + 3p = 0$, gives $\rho_m/\rho_v = 2$ and if it is only radiation and vacuum, then $\rho_r/\rho_v = 1$. In [91], it was shown that in this model, most outstanding cosmological problems such as flatness, horizon, monopole, entropy, size and age of the universe, cosmological constant etc. are absent. It was also shown that this model can solve the problem of generation of density perturbations at scales well above the present Hubble radius and that it can generate such density perturbations even after the era of nucleosynthesis.

4.2.2 Evolution of the Hubble parameter in the new model

The new deterministic model has $w = -1/3$ in the deterministic case, we rewrite eq. (4.2) with $w' = w + \frac{1}{3}$, as

$$\dot{H} = -\frac{3}{2}H^2(\frac{2}{3} + w^{'}). \tag{4.3}$$

Now we assume that $w^{'}$ fluctuates about its zero mean value and is $\delta-$correlated. Making the substitution

$$x = \frac{1}{H}, \tag{4.4}$$

the above equation becomes

$$\dot{x} = 1 + \frac{3}{2}w^{'} \tag{4.5}$$

When $w^{'} = 0$, this is a deterministic equation, and the solution is straight forward. However, with a fluctuating $w^{'}$, the variable x becomes stochastic and one can find only the PDF of such a variable. This can be done through the Fokker-Planck formalism as in the previous cases. With the drift coefficient $D^{(1)}$ set equal to unity (follows from the above equation), and the diffusion coefficient $D^{(2)} = D$ being assumed to have some constant value, to be determined from observations, we can write the FPE as

$$\frac{\partial W^{'}(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} + D\frac{\partial^2}{\partial x^2}\right] W^{'}(x,t). \tag{4.6}$$

To solve the FPE, we assume the ansatz:

$$W^{'}(x,t) = \phi_n(x)\exp(-\lambda_n t). \tag{4.7}$$

The remaining procedure to find the PDF is the same as that described in section 2.3.2. The transition probability for the variable to make a transition from $(x^{'}, t^{'})$ to (x,t) is

$$P(x,t \mid x',t') = \frac{1}{2\sqrt{\pi D(t-t')}} \exp\left[-\frac{[(x-x')-(t-t')]^2}{4D(t-t')}\right] \tag{4.8}$$

For the special initial value

$$P(x,t \mid x',t) = \delta(x-x'), \tag{4.9}$$

the transition probability $P(x,t \mid x',t')$ is the distribution function $W'(x,t)$. In our case, we have the initial condition $x = x' = 0$, at $t = t' = 0$, so that

$$W'(x,t) = P(x,t \mid 0,0) = \frac{1}{2\sqrt{\pi D t}} \exp\left[-\frac{(x-t)^2}{4Dt}\right] \tag{4.10}$$

This also a Gaussian distribution function with $\langle x \rangle = t$. In terms of stochastic Hubble parameter (in the new model), the distribution function becomes

$$W''(H,t) = \frac{1}{2H^2}\frac{1}{\sqrt{\pi D t}} \exp\left[-\frac{(1-Ht)^2}{4\,H^2 D\,t}\right] \tag{4.11}$$

With $H = 100h\ km\ s^{-1} Mpc^{-1}$, $t = t_{17} \times 10^{17} s$ and $D = D_{17} \times 10^{17} s$ the PDF $W(h,t)$ can be written as

$$W(h,t) = \frac{3.0856}{2\,h^2}\frac{1}{\sqrt{\pi D_{17} t_{17}}} \exp\left[-\frac{(3.0856 - h\,t_{17})^2}{4\,h^2 D_{17}\,t_{17}}\right] \tag{4.12}$$

For the range of values of interest, $1 < t_{17} < 5$ and $D_{17} \sim 10^{-3}$, $W(h,t)$ is approximately Gaussian. For fixed D, the half width of the Gaussian is found to increase as we go to lower values of t.

4.2.3 PDF for H from observational data

Conventionally, assuming that the observed and predicted value of distance moduli coincide, one can make an estimate of the present Hubble constant, where

$$\mu_o = m_{bol} - M_{bol},$$

and

$$\mu_p = 5\log\left[\frac{D_L}{1Mpc}\right] + 25.$$

The luminosity distance is $D_L = r_j\, a(t_0)\, (1+z)$ ($a(t_0)$ is the present scale factor and r_j is the radial coordinate of the SN Ia which emitted the light at some time t_j in the past). Here

$$r_j = \int_{t_j}^{t_0} \frac{dt}{a(t)}.$$

For the coasting model ($a \propto t$) discussed in the previous sections, for curvature $k = 0$, r_j can be evaluated as

$$r_j = \frac{t_0}{a(t_0)} \int_{t_j}^{t_0} \frac{dt}{t} = \frac{t_0}{a(t_0)} \ln(1+z), \tag{4.13}$$

so that

$$D_L = \frac{(1+z)}{H_0} \ln(1+z). \tag{4.14}$$

One can substitute this into the expression for μ_p to obtain the predicted distance modulus of an object with redshift z. For a collection of objects, the likelihood for H_0 can also be found from a χ^2 statistic (see eq. (2.39)). The normalised PDF for h can be found from [26]

$$p(h \mid \mu_o) = \frac{\exp\left(-\chi^2/2\right)}{\int_{-\infty}^{+\infty} dh\ \exp\left(-\chi^2/2\right)}. \tag{4.15}$$

As in Chapter 2, we compute $p(h \mid \mu_o)$ for the new model, using the SN Ia data in [25], which corresponds to their Fit C and attempt to compare $p(h \mid \mu_o)$

with the PDF $W(h, t_0)$, to evaluate the diffusion constant D appearing in this expression. It is found that the two curves, shown in Fig. 3, coincide for a value of $D \approx 2.36 \times 10^{13} s$ (This corresponds to an age $4.8583 \times 10^{17} s$). Here also we choose the absolute magnitude appearing in the expression for observed distance modulus to be -19.3, a slight variation in this quantity does not significantly affect D.

In the above, we compared the theoretical and observational PDFs for the same alternative model and thus it does not have the first shortcoming mentioned in the introduction. The other incompatibility which still exists can be explicitly stated as follows: $W(h, t_0)$ is the PDF for the Hubble parameter of the present universe, and it contains the diffusion constant D. But $p(h \mid \mu_o)$, which we try to identify with $W(h, t_0)$, depends on the scatter in the Hubble diagram for all ranges of z. For instance, if we include more high redshift objects in our sample, the scatter would be larger and hence the half-width of the distribution $p(h \mid \mu_o)$ will be larger. This, in turn, will affect the computed value of D, which is quite unreasonable.

This problem can, however, be overcome if we agree to compute $p(h_j \mid \mu_{oj})$ for each value of redshift z (or for small enough redshift intervals centred about such values), and compare these with $W(h_j, t_j)$ that corresponds to the same epoch t_j. To do this, we modify eq. (4.14) by re-evaluating r_j in (4.13) in a different way. One can also write, for the new deterministic model

$$r_j = \frac{t_j}{a(t_j)} \int_{t_j}^{t_0} \frac{dt}{t} = \frac{1}{H_j\, a(t_j)} \ln(1+z), \tag{4.16}$$

so that

$$D_L = \frac{(1+z)^2}{H_j}\ln(1+z) \tag{4.17}$$

Evaluating μ_p using this expression, we can evaluate χ^2 and hence also $p(h_j \mid \mu_{oj})$, which is the PDF for the Hubble parameter at some particular value of z. We divide the data in [25] for various redshift intervals around $z = 0.05,\ 0.15,\ 0.35,\ 0.45,$ 0.55 and 0.65, each with $\triangle z = 0.05$. The PDF for the average Hubble parameter for such intervals is calculated with an expression identical to (4.15). The results are plotted in Fig. 4 along with the corresponding theoretical PDF $W(h_j, t_j)$ which overlaps with them. The relevant parameters are given in Table1.

4.3 Conclusions

It is noted from Fig. 4 and Table1 that, for the intervals with larger values of z, the 68.3% credible region of $p(h \mid \mu_{0j})$ has a halfwidth σ_h, which also increases. This behaviour is the one expected from theory, as noted while plotting the theoretical PDF (4.12). Physically, this means that the scatter increases as we go to higher redshifts. The intervals with centre at $z = 0.15$ and 0.35 are exceptions to this, but this may be due to the fact that these intervals contain only very few objects. As more SN Ia are observed in these redshift intervals, an accurate picture will emerge.

The value of the diffusion constant D evaluated for various intervals, however, does not show any dependence on z. This justifies our assumption that D is some constant.

A pitfall, even in the present analysis, is that the intervals we consider are with $\triangle z = 0.05$ and this value may not be small enough to give correct answers. This, again, can be overcome only in the future, when the number of observed SN Ia becomes large.

This chapter is a modification to the theoretical investigation on the origin of the random motions that cause large scatter in the Hubble diagram at high redshifts. Conventionally, the random motions are viewed as peculiar velocities induced by the observed density fluctuations. Given the fact that density fluctuations are evolving phenomena, peculiar velocities induced by them can not lead to the large randomness observed at early epochs, though it is a feasible physical process in the late universe. A fluctuating expansion rate arising from the stochastic nature of the equation of state, on the other hand, provides a natural explanation for the large scatter in the Hubble diagram at high redshifts.

Table 1. Diffusion constant for various epochs

Redshift z	No. of SNe in the interval $z \pm 0.05$	Best fit value of h	Standard deviation σ_h	Age in units of 10^{17} s t_{17}	Diffusion constant D s
0.05	15	0.693	0.011	4.4502	0.5775×10^{14}
0.15	3	0.772	0.025	3.9987	2.147×10^{14}
0.35	5	0.830	0.024	3.7204	1.5×10^{14}
0.45	15	0.875	0.017	3.528	1.655×10^{14}
0.55	7	0.985	0.026	3.1345	1.12×10^{14}
0.65	6	1.043	0.030	2.959	1.226×10^{14}

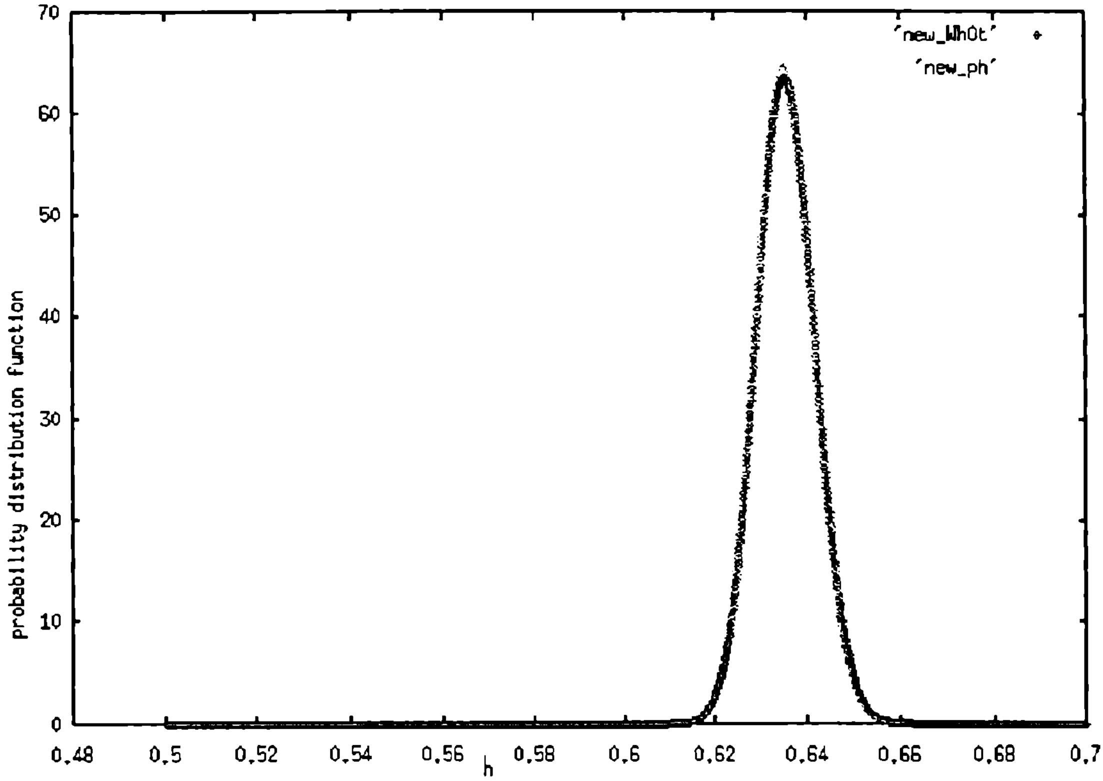

Fig. 3 Observational and theoretical PDFs vs h, using the redshift-apparent magnitude (z-m) data for Type Ia Supernovae as given in [25], which corresponds to their Fit C. The continuous line is for the theoretical PDF, whereas the dotted line gives the observational PDF.

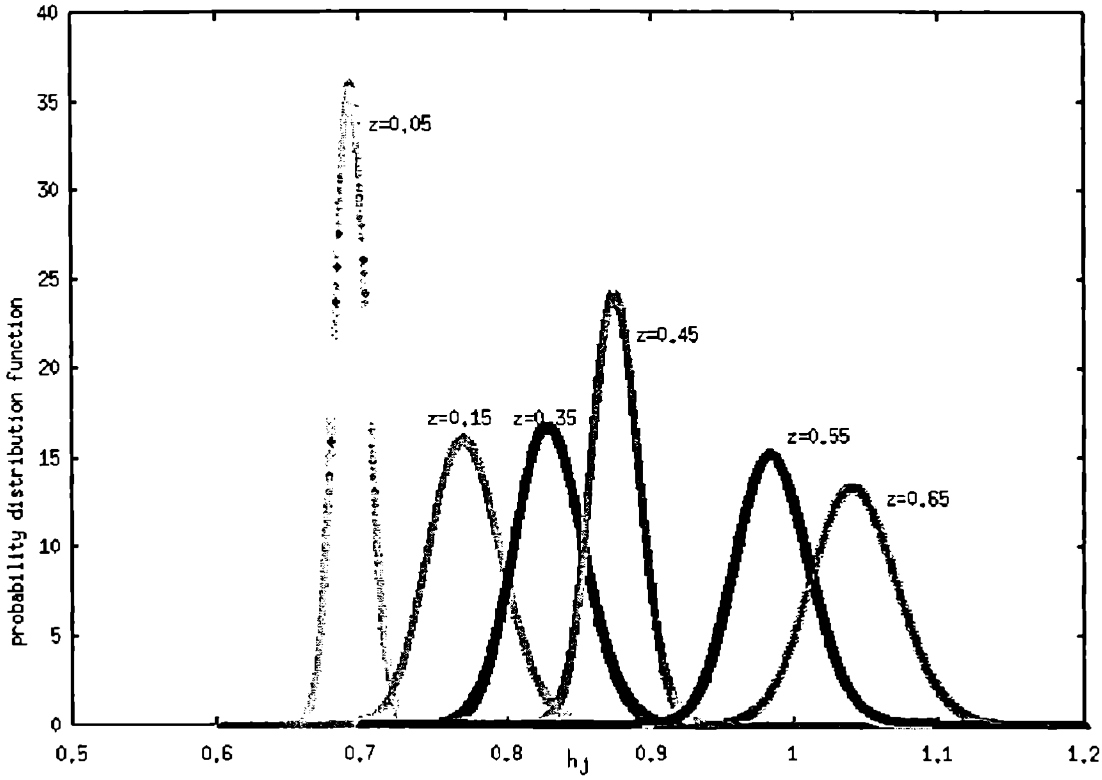

Fig. 4 Observational and theoretical PDFs vs h for various epochs centred about z = 0.05, 0.15, 0.35, 0.45, 0.55, 0.65, using the z-m data for Type Ia SNe as given in [25], which corresponds to their Fit C and which lies in the interval $\Delta z = z \pm 0.05$

Chapter 5

Other problems with the standard model and discussions

5.1 Introduction

The basic assumption of the standard cosmological model is Einstein's cosmological principle which, in fact, is the hypothesis that the universe is spatially homogeneous and isotropic on large scales. It is thus assumed that the large scale $3D^l$ geometry of the universe is isotropic and homogeneous. Many of the problems of the standard model are a direct consequence of such simplifying assumptions. The cosmological principle implies linear deterministic Hubble's law, $v = Hr$ (or a linear, scatter-free redshift-apparent magnitude relation), which is valid at scales where matter distribution can be considered on an average uniform, and is well established within local scales (Since the early 1980s, multi-object spectrographs, CCD detectors etc. have allowed the mass production of galactic redshifts). However, many recent analyses (redshift surveys such as CfA, SSRS, LEDA, IRAS, Perseus-Pisces and ESP for galaxies and Abell and Aco for galactic clusters) have revealed that the three dimensional distribution of galaxies and clusters of galaxies are characterized by large-scale structures (hierarchical) and huge voids [33-42]. Such a distribution shows fractal correlation upto to the limit of available samples. This has confirmed the de-Vaucouleurs power-law density-distance relation [33],

$$\rho(r) \propto (\frac{r}{r_0})^{D_1-3} \tag{5.1}$$

with fractal dimension $D_1 \approx 2$ at least in the range of scales 1 to 200 $h^{-1}Mpc$, ie., a sheet-like distribution of galaxies. In the above expression $D_1 = 3$ corresponds to perfect homogeneous distribution of galaxies and a perfect linear Hubble expansion.

A fractal [29] is a geometric shape that is not homogeneous, yet preserves the property that each part is a reduced scale-version of the whole. That is, fractals are self-similar structures or possess scale-invariant properties. If the matter in the universe were actually distributed like a pure fractal on all scales, then the cosmological principle would be invalid, and the standard model in trouble.

Thus the universe is inhomogeneous and essentially fractal-on the scales of galaxies and clusters of galaxies, but most cosmologists believe that on much larger scales it becomes isotropic and homogeneous [37, 38], eventhough the cross-over scale to homogeneity is not yet identified [33]. According to the standard model of cosmological structure formation, such a transition should occur on scales of a few hundred Mpc. The main source of controversy is that the most available three-dimensional maps of galaxy positions are not large enough to encompass the expected transition to homogeneity. Distances must be inferred from redshifts, and it is difficult to construct these maps from redshift surveys, which require spectroscopic studies of large numbers of galaxies. Sylos Labini etal. [42] have analysed a number of redshift surveys and find $D_1 = 2$ for all the data they look at, and argue that there is no transition to homogeneity for scales upto 4000 Mpc, way beyond the expected turn over. A controversy exists among cosmologists regarding this switch over scale to homogeneity. The fractal behaviour of galaxy distribution within a scale of $\sim 200\ h^{-1}Mpc$ (this scale may be even deeper), is a challenge for standard cosmology, where a linear Hubble's law is a strict consequence of the homogeneity of the expanding universe. The presence of dark matter (distributed homogeneously) may save the cosmological principle even at small scales [39]. In this way, one may save the usual FRW metric (which needs a homogeneous density), while a substantial revision to the models of galaxy formation is required. On the contrary, if the dark matter is found to have the same distribution of luminous one, then a basic revision of the theory must be considered. In fact, from a theoretical point of view, one would like to identify the dynamical processes which can lead to such a fractal distribution

5.2 A stochastic evolution of density perturbations

The structures we see today are formed by a process known as gravitational instability, from primordial fluctuations in the cosmic fluid [15, 16]. But, because the strength of clustering is expected to increase with time (evolution of density contrast being proportional to some power of scale factor in the linear approximation according to the standard model), the galaxies must deviate from the smooth Hubble expansion. These deviations away from uniform Hubble's flow are known as peculiar velocities. According to the standard Friedmann model, $\delta v \propto \Omega_0^{0.6} \delta\rho$, where $\delta\rho$ is the density perturbation and Ω_0 is the present value of the ratio between critical density and density of the universe. This peculiar velocity is one of the independent probes of inhomogeneities in the gravitational field, induced by the density fluctuations. Another probe is the fluctuations in the background radiation. Observations show a very nearly linear Hubble expansion for local scales, and deviations from this deterministic Hubble's flow increase with redshift, as is obvious from the Hubble diagram, which is a scatter diagram and the scatter increases as we go to early epochs [20, 24, 28]. The conventional explanation for the scatter is in terms of the peculiar velocities alone, induced by observed density fluctuations. But density fluctuations are evolving phenomena, they cannot induce the large randomness observed at high redshifts. The scatter in the Hubble diagram or deviations from the linear Hubble expansion may also arise from an inherent stochastic nature of the Hubble parameter, apart from the peculiar velocities which are significant only in the late universe. We have explained the anomalous scatter in the Hubble diagram at high redshifts on the basis of a fluctuating or random expansion rate of the universe, thanks to a stochastic equation of state. Under a stochastic equation of state, dynamical equations describing the evolution of density and its perturbations must be stochastic and only a PDF can be found for these quantities. The stochastic evolution of density and other cosmological parameters are described in the previous chapters. Here we will show that under the above circumstances, the time-evolution of density perturbations is described by Langevin type equations

In the early universe, when cosmic fluid is not uni-component, a stochastic equation of state emerges. ie., the w factor becomes a fluctuating quantity. Hence the evolution of density perturbations is a stochastic process. In the following we will show that the PDF of the stochastic density contrast is approximately Gaussian. In the early universe, the energy density in any region can be written as a perturbation equation [16]:

$$\rho(x,t) = \rho_b(t) + \delta\rho(x,t), \tag{5.2}$$

where ρ_b, is the background density, which at any time t is independent of location. However, ρ at different regions is slightly different in the early universe, and hence $\delta\rho$ also. The evolution of density contrast ($\delta = \delta\rho/\rho_b$) according to FRW model, is a deterministic one, proportional to some power of the scale factor in the linear approximation. In the stochastic approach, due to a fluctuating equation of state, its evolution is a stochastic process. We assume that the total energy density is conserved, so

$$\dot{\rho} = -3H(\rho + p). \tag{5.3}$$

After some simplifications, we get the evolution equation of $\delta\rho$ [16, 17] in the form

$$\delta\dot{\rho} \;=\; -3(\rho_b + p_b)\delta H \;-\; 3H_b\delta\rho. \tag{5.4}$$

Here the suffix b stands for the background. Using $\delta\rho = \rho_b\delta$, the above equation becomes

$$\dot{\delta} \;=\; 3H_b w(t)\delta \;-\; 3[1 + w]\delta H. \tag{5.5}$$

Now using the relation

$$H_b^2 = \frac{8\pi G}{3}\rho_b \tag{5.6}$$

we have

$$(H - \delta H)^2 = \frac{8\pi G}{3}(\rho - \delta\rho). \tag{5.7}$$

Equating both sides, we have

$$H^2 = \frac{8\pi G}{3}\rho \tag{5.8}$$

and

$$2\, H\, \delta H \;=\; \frac{8\pi G}{3}\delta\rho. \tag{5.9}$$

Using equations (5.8), (5.9) and (5.2), we get

$$\delta H = \frac{1}{2}\sqrt{\frac{8\pi G}{3}}\delta\rho\;(\rho_b)^{-1/2}\left[1 + \frac{\delta\rho}{\rho_b}\right]^{-1/2} \tag{5.10}$$

Retaining only first order terms, this becomes

$$\delta H \;=\; \frac{1}{2}\sqrt{\frac{8\pi G}{3\rho_b}}\delta\rho. \tag{5.11}$$

Substituting for H_b and δH in eq. (5.5), it becomes

$$\dot{\delta} = \sqrt{6\pi G\rho_b}[w - 1]\delta. \tag{5.12}$$

Using the transformation

$$y = \ln\delta, \tag{5.13}$$

eq. (5.12) becomes

$$\dot{y} \;=\; \sqrt{6\pi G\rho_b}[w - 1]. \tag{5.14}$$

This is a Langevin equation. Since it has another stochastic variable (ρ_b), we have to write the stochastic equation for ρ_b also, and the resulting system of equations has to be treated together, which leads to a two variable Fokker-Planck problem and a two variable PDF. For this we define a transformation

$$x = \frac{1}{\sqrt{6\pi G\rho_b}}. \tag{5.15}$$

Therefore from eq. (5.14) and the conservation equation for density, we have

$$\dot{y} = -\frac{1}{x} + \frac{w(t)}{x}, \tag{5.16}$$

and

$$\dot{x} = 1 + w(t). \tag{5.17}$$

To write the FPE, we need the drift and diffusion coefficients. From the general equation [82]

$$\dot{x}_i = h_i + g_{ij}\Gamma_j, \tag{5.18}$$

the drift coefficients are

$$D_i^{(1)} = h_i + g_{kj}\frac{\partial}{\partial x_k}g_{ij}, \tag{5.19}$$

and the diffusion terms are

$$D_{ij}^{(2)} = g_{ik}\, g_{jk}. \tag{5.20}$$

From eqs. (5.16) and (5.17) the drift terms are obtained:

$$D_y^{(1)} = -\frac{1}{x} - \frac{1}{x^2} \approx -\frac{1}{x^2};$$

$$D_x^{(1)} = 1 \tag{5.21}$$

where we have used the fact that in the early universe ρ_b is very high. The diffusion terms are

$$
\begin{aligned}
D^{(2)}_{yy} &= \frac{1}{x^2} \\
D^{(2)}_{yx} = D_{xy} &= \frac{1}{x} \\
D^{(2)}_{xx} &= D.
\end{aligned} \tag{5.22}
$$

Here the diffusion terms arise due to fluctuations in the mean w factor alone, and D is a constant. Now we write the FPE for the distribution function $W(y, x, t)$:

$$
\frac{\partial W}{\partial t} = \left[-\frac{\partial D^{(1)}_y}{\partial y} - \frac{\partial D^{(1)}_x}{\partial x} + \frac{\partial^2 D^{(2)}_{yy}}{\partial y^2} + \frac{\partial^2 D^{(2)}_{xx}}{\partial x^2} + 2\frac{\partial}{\partial y}\frac{\partial}{\partial x} D_{yx} \right] W(y, x, t). \tag{5.23}
$$

Substituting for the drift and diffusion coefficients

$$
\frac{\partial W}{\partial t} = \left[-\frac{1}{x^2}\frac{\partial}{\partial y} - \frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial y^2} + D\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial^2}{\partial y \partial x} \right] W(y, x, t). \tag{5.24}
$$

Neglecting the crossed term (to obtain an approximate solution) and applying a separation ansatz,

$$
W(y, x, t) = u(x) v(y) \exp(-\lambda t) \tag{5.25}
$$

we get by variable separation

$$
\frac{1}{v}\frac{d^2 v}{dy^2} - \frac{1}{v}\frac{dv}{dy} = x^2 \left[\frac{1}{u}\frac{du}{dx} - \frac{D}{u}\frac{d^2 u}{dx^2} - \lambda \right] \tag{5.26}
$$

Here both sides can be equated to a constant (say c). We have a set of equations for $u(x)$ and $v(y)$. They are

$$
\frac{d^2 v}{dy^2} - \frac{dv}{dy} - cv = 0, \tag{5.27}
$$

and

$$\frac{d^2u}{dx^2} - \frac{1}{D}\frac{du}{dx} + \frac{1}{D}\left[\lambda + \frac{c}{x^2}\right] u = 0. \tag{5.28}$$

Due to the essential singularity of the eq. (5.28) at $x = 0$, a series solution is possible. However, if we take $c = 0$, then we will get a compact solution (in order to understand the general behaviour). After solving equations (5.27) and (5.28), we can write the complete solution as

$$W(y, x, t) = A \exp\left[y + \frac{x}{2D} + ikx - \lambda t\right], \tag{5.29}$$

where

$$k^2 = \frac{\lambda}{D} - \frac{1}{4D^2}. \tag{5.30}$$

Physically reasonable solution exists for $\lambda \geq \frac{1}{4D}$. A is the normalisation constant, chosen as $1/2\pi$. The general solution is obtained by integrating eq. (5.29) in the range $-\infty \leq k \leq \infty$, and we get

$$W(y, x, t) = \frac{1}{\sqrt{4\pi Dt}} e^y \exp\left[-\frac{(x-t)^2}{4Dt}\right] \tag{5.31}$$

This is Gaussian in x. We can write the distribution function in terms of the original variables δ and ρ_b ($y = \ln\delta$ and $x = 1/\sqrt{6\pi G\rho_b}$):

$$W(\delta, \rho_b, t) = \frac{1}{\sqrt{96\pi^2 GD\rho_b^3 t}} \exp\left[-\frac{(1 - t\sqrt{6\pi G\rho_b})^2}{24\pi G\rho_b Dt}\right] \tag{5.32}$$

This characterizes the non-deterministic (stochastic) evolution of density perturbations in the linear approximation and such an analysis becomes important when we consider the expansion rate to be fluctuating.

5.3 Discussions and conclusions

When the universe is approximated by a many component fluid, the fluctuations in creation rates are certainly physical processes which can lead to stochastic fluctuations in the mean w factor of the equation of state (ie., a stochastic equation of state). The evolution of the universe becomes stochastic (or the expansion rate of the universe fluctuates), where the time-evolution of the cosmological parameters is described by the Langevin equations or SDEs. We argue that such dynamical processes may lead to a fractal distribution (or a scale invariant inhomogeneous distribution) of galaxies, since the fluctuations in the evolution process never lead to structures with perfect symmetry and most natural fractals were formed through stochastic processes.

For a homogeneous distribution of galaxies, Hubble's count law is [1]

$$N(< m) \propto 10^{0.2D_1 m}, \tag{5.33}$$

where m is the apparent magnitude of the object and N represents number of galaxies brighter than the magnitude m. $D_1 = 3$ corresponds to standard model. Equivalently, one can express (5.33) in terms of redshift (z) also, ie., $N(< z)$. The apparent magnitude is related to z in the following way

$$m - M = 5\log\left(\frac{D_L}{1Mpc}\right) + 25, \tag{5.34}$$

where M is the absolute magnitude of the galaxy. The luminosity distance D_L is $r_j a(t_0)[1+z]$. In flat FRW models, $r_j$is calculated from

$$r_j = \int_{t_j}^{t_0} \frac{dt}{a(t)} = \frac{t_j^{2/3}}{a(t_j)} \int_{t_j}^{t_0} \frac{dt}{a(t)}. \tag{5.35}$$

Integrating

$$r_j = \frac{2}{a(t_j)H_j}\left[\sqrt{1+z}-1\right], \tag{5.36}$$

and

$$D_L = \frac{2}{H_j}(1+z)^2\left[\sqrt{1+z}-1\right], \tag{5.37}$$

where $H(t)$ corresponds to the epoch t_j. Thus the luminosity distance is related to redshift, which depends on $H(t)$. Fluctuations in the number counts around the average behaviour as a function of m or z can discriminate between a genuine fractal distribution and a homogeneous one [40]. Number counts versus apparent magnitude can be used to test whether the large scale distribution of galaxies (or clusters) can be compatible with a fractal or with a homogeneous behaviour [41, 42]. In a fractal distribution, one expects to find persistent scale invariant fluctuations around the average behaviour, which do not decay with m or z. On the other hand, in a homogeneous distribution, on large enough scales, the relative variance of the counts should decrease exponentially with m [1, 40]. Labini et al., [40, 41] claim that, the relative fluctuations in the counts as a function of m has a constant magnitude (for $z \geq 0.1$), which can not be due to any smooth correction to the data as evolution effects, but they can be the outcome of an inhomogeneous distribution of galaxies.

In Chapter 2, we have shown that a fluctuating w factor, in turn, will lead to fluctuations in the time-evolution of the Hubble parameter; ie., the expansion rate of the universe becomes a stochastic quantity. We argue that such a fluctuating expansion rate might have led to a randomness in the recession velocities of objects, in addition to peculiar velocities ($\delta v \propto \Omega_0^{0.6}\delta\rho$) induced by density inhomogeneities. We also argue that, this randomness in the recession velocities of galaxies led to an inhomogeneous (fractal) distribution of galaxies and clusters of galaxies, since the dynamical equations containing stochastic quantities describe fractal growth [85, 92]. From (5.33) and (5.37), a fluctuating or stochastic expansion rate ($H(t)$) may also provide the constant fluctuations observed in the number count versus m (or z) relation.

Thus a stochastic evolution of the universe may provide the dynamical process leading to the self-similar structures observed in the universe and also produce the scatter observed in the Hubble diagram at high redshifts, where peculiar velocities of standard model are inadequate. The present fluctuations found in the number count versus apparent magnitude relation, which is a characteristic of fractal distribution of galaxies, may also be due to the stochastic nature of the Hubble parameter (See eqs. (5.34) and (5.37)). Since both density and Hubble parameter are stochastic in the early epochs, the time-evolution of density perturbation also must be non-deterministic, and hence described by SDEs, as we have done in section 5.2. In this chapter we have not made any attempt to characterize the statistically scale invariant structures observed in the range of scales 1 to $200h^{-1}Mpc^{-1}$ by measuring either the correlation function or power spectrum, we have attempted only to provide a possible stochastic process which can produce the observed inhomogeneous distribution of galaxies. The correlation function for galactic distribution $\xi(r) \propto r^{-1.8}$, for $r < 10\ h^{-1}Mpc$ is well established [15, 16]. However such statistical methods are based on the assumption of homogeneity and hence are not appropriate to test the scale invariance of structures for a large range of scales [36, 42]. Here we have only argued that a stochastic evolution of the universe may lead to a scale invariant large scale structure.

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