STUDIES ON GAUSSIAN EFFECTIVE POTENTIALS IN SOME MODELS

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Ms. Rose P. Ignatius, under my guidance in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for award of any degree.

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C O N T E N T S

PREFACE

The investigations reported in this thesis have been carried out by the author first as a full time CSIR JRF and later as a part-time student. The thesis comprises six chapters. In chapter 1 a survey of the theory of effective potentials is presented together with concepts and techniques necessary for following the author's work.

In classical field theory, the ordinary potential V is an energy density for that state in which the field assumes the value ϕ. In quantum field theory, the effective potential is the expectation value of the energy density for which the expectation value of the field is ϕ_o. As a result, if V has several local minima, it is only the absolute minimum that corresponds to the true ground state of the theory.

Perturbation theory remains to this day the main analytical tool in the study of Quantum Field Theory. However, since perturbation theory is unable to uncover the whole rich structure of Quantum Field Theory, it is desirable to have some method which, on one hand, must go beyond both perturbation theory and classical approximation in the points where these fail, and at that time, be sufficiently simple that analytical calculations could be performed in its framework.

During the last decade a nonperturbative variational method called Gaussian effective potential, has been discussed widely together with several applications. This concept was described as a means of formalizing our intuitive understanding of zero-point fluctuation effects in quantum mechanics in a way that carries over directly to field theory.

The Gaussian effective potential (GEP) is defined as

$$\overline{V}_G(\phi_o) = \min_{\Omega} V_G(\phi_o,\Omega)$$
$$= \min_{\Omega} (\psi|H|\psi>$$

with $|\psi> = (\frac{\Omega}{\hbar\pi})^{1/4} \exp[-1/2\frac{\Omega}{\hbar}(\phi-\phi_o)^2]$, $\Omega>0$

and Ω a mass parameter.

The width of the Gaussian, governed by the parameter Ω, is left to adjust itself so as to minimize <H> at each ϕ_o. Thus $\overline{V}_G$ can be described as a variational approximation to the conventional effective potential V_{eff} where

$$V_{eff}(\phi_o) = \min_{\{\psi\}} <\psi|H|\psi>$$

wit.. ϕ subject to $<\psi|\phi|\psi> = \phi_o$

In GEP, the global minimum of $\overline{V}_G$ does not give the ground state energy as it was the case with V_{eff}. According to the Rayleigh-Ritz theorem, $\overline{V}_G(\phi_o) \geq V_{eff}(\phi_o)$ at any ϕ_o.

Normally, one can expect a good approximation to E_o for the variational reason that any half-way realistic wave function generally gives a reasonable estimate of the ground state energy. The one loop effective potential is an semiclassical construct, based on adding to the classical potential the order - $\hbar$ quantum corrections, and neglecting terms of order $\hbar^2$ and higher. This method generally breaks down whenever the quantum effects become large.

In chapter II we have computed the Gaussian effective potential for Liouville theory at zero temperature and at finite temperature. The Liouville model field theory is of great current interest. In string dynamics, for example, in order to get a proper quantization for $D<26$, one must examine the quantum Liouville theory. This theory is two dimensional, renormalizable and completely integrable. Polyakov has demonstrated how to express different physical quantities like the spectrum, scattering amplitudes etc. through the correlation functions of quantum Liouville theory. For physical D, one may solve the Liouville theory in order to find the scattering amplitudes. It is shown that even in non-perturbative approach based on GEP, translational invariance remains broken in Liouville theory at zero temperature and is notrestored at a finite temperature, supporting the idea that the breaking of translational symmetry is fundamental to the model both at classical and quantum levels.

We have extended the study to the supersymmetric case. If supersymmetry which predicts boson-fermion multiplets is recognized by nature, the study of finite temperature supersymmetric Grand Unification must provide some insight into the Early Universe scenario. The non-perturbative Gaussian effective potential for the supersymmetric Liouville model both at zero and nonzero temperatures, is obtained in chapter III. It is of some importance to remark that the GEP has not been evaluated for a supersymmetric theory before. It is found that the supersymmetric Liouville theory does not possess a translationally invariant ground state. Here results similar to those obtained in the non-supersymmetric case, have been established indicating that the appearance of the fermionic degrees of freedom has no significant effect on the nature of the core bosonic part.

In chapter IV, following the method of Stancu and Stevenson, we have computed second order corrections to the Gaussian effective potential for the ϕ^6 model in 2+1 dimensions at zero temperature. The ϕ^6 - field theory in 2+1 dimensions is of interest in particle physics as well as solid state physics.

Chapter V introduces a definition as well as evaluation of GEP for coherent states and squeezed states, in analogy with that for excited states. The corresponding effective (renormalized) mass and coupling constant for an anharmonic oscillator are computed. The effective (renormalized) coupling constant exhibits a singularity at $\hbar = o$, which vanishes when the bare mass tends to zero, for both coherent states and squeezed states.

Quantum groups and quantum algebras have been receiving considerable attention in recent years. Some of these investigations focus on quantum group modified quantum mechanics. There is a logical need to apply the nonperturbative approach to such systems that are generically known as quantum oscillators. In chapter VI we have formulated a nonperturbative q-or (q,p)-analogue of GEP with the help of appropriate quantum oscillator commutation relations that depend on a single parameter q or two parameters q,p. When a quantum oscillator algebra is employed, the quantum parameters such as q,p, can serve as additional parameters in the potential, suggesting a more elaborate scheme of minimization. The renormalized mass m_R^2 and coupling constant λ_R are calculated directly from the effective potential. We study three kinds of quantum oscillator systems: quartic coupled quantum oscillators in a single well and in a double well, and sextic coupled quantum oscillators. It is found that for the ground state of a quartic or sextic anharmonic q-oscillator the effective potential is a minimum corresponding to q=1 and a maximum corresponding to $q = -1$. The renormalized mass m_R turns out to be a maximum at $q=1$. Since the m_R has the physical significance of being the first excitation energy, these observations seem to cast ordinary ($q=1$) quantum mechanics in a new perspective. For the X^4-anharmonic (q,p) oscillator, the effective potential yields the minimum only if λ or $\hbar$ vanishes. In the case of quartic q or (q,p) - oscillator in a double well potential, critical values exist for q or q as well as p, for which the double well degenerates into a single well.

Part of the investigations included in this thesis has been included in the following papers:

1. ''Gaussian effective potential for the Liouville model'' Rose P. Ignatius, V.C. Kuriakose and K. Babu Joseph, Phys. Lett. B 220, 181 (1989).
2. ''Non-perturbative calculation of effective potential in supersymmetric Liouville model'', Rose P. Ignatius, K.P. Satheesh, V.C. Kuriakose and K. Babu Joseph. Mod., Phys. Lett. A 5, 2115 (1990).
3. ''Non-perturbative effective potentials of quantum oscillators'', Rose P. Ignatius and K. Babu Joseph, Pramana J. Phys. 42, 285 (1994).

1. INTRODUCTION

1.1 Qualitative concept of effective potential

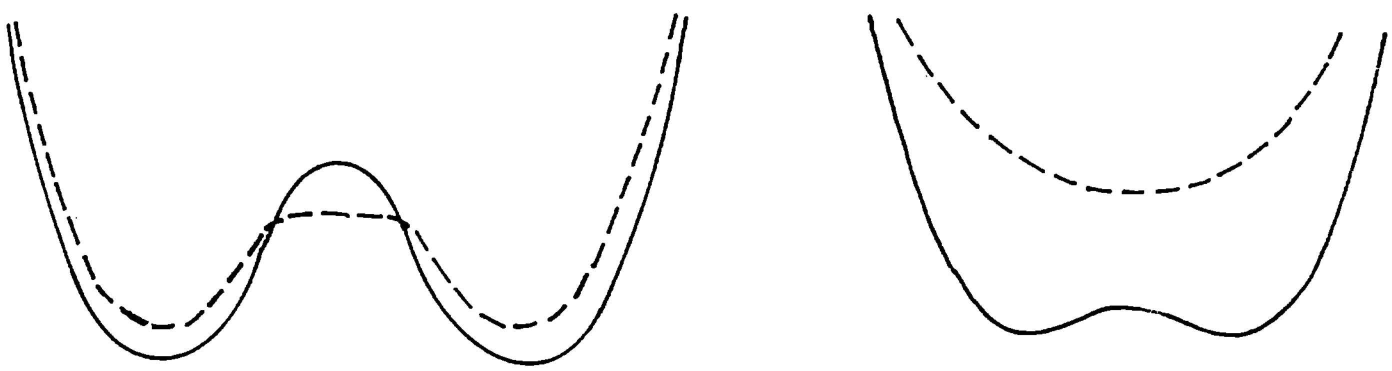
Quantum fluctuations are quantum effects which may modify the classical potential. To quote typical examples of this phenomenon, let us consider the case where the wavefunction is concentrated in a small spatial region ΔX, where the momentum uncertainty is correspondingly large. Here there will be a large contribution to the kinetic energy and to the total energy of the system. This shows that the ground state energy is influenced by the depth as well as the width of the potential well. On account of zero point fluctuations, a quantum mechanical particle behaves as if it does not like to be confined in a narrow potential well or in a small space. The zero point energy $\frac{1}{2}\hbar\omega$ of the harmonic oscillator potential, $V(X) = \frac{1}{2}\omega^2 X^2$, is an important consequence of the uncertainty principle.

The coulomb potential in an atom, $-e^2/r$, is unbounded below, and hence classically, an electron may be expected to fall into the nucleus. But, as in the former case, the electron resists being localised in the small region, and the quantum fluctuations enable it to overcome the attraction of the classical potential. As a result, it occupies a finite energy ground state centred at the origin with a definite spatial extent. A system, which is errant classically, is thus corrected by quantum mechanical fluctuation effects. Such cases can be described in terms of an effective potential which indicates how the quantum fluctuations modify the classical potential.

In the case of a symmetric double well potential, the effective potential is different from the original potential. For small and large quantum effects, the double well potential exhibits the behaviour sketched in Figs. 1.1a and 1.1b respectively.

For small quantum effects the lowest energy state is raised due to the $\frac{1}{2}\hbar\omega$ zero point energy, and the highest energy state is lowered due to the spreading effect. When the quantum effects are large, the particle does not see the two separate wells but is free to move inside the large well with no barrier in between.

For an asymmetric double well potential consisting of a broad well and a slightly deeper but much narrower well, if the quantum effects are small, the effective potential is similar to the classical potential. If the deeper well is made narrower the zero point energy will become very large (according to the



(a) (b)

Fig. 1.1 a. Symmetric double well potential modified by small fluctuations

b. by comparatively large fluctuations

Dotted lines represent the effective potential and solid lines the classical potential uncertainty principle) and hence the particle will prefer to be inside the broader well. The behaviour in this case is as shown in Fig.1.2.

These examples from quantum mechanics illustrate the fact that in order to understand the effect of quantum fluctuations, one has to look for the behaviour of effective potential[1-9]. The effective potential for the ground state of a quantum mechanical system is defined by the relation [10].

$$V_{eff}(X_o) = \min_{[\psi]} \quad \langle\psi|H|\psi\rangle \tag{1.1}$$

where ψ is subject to the conditions

$$\langle\psi|\psi\rangle = 1, \qquad \langle\psi|X|\psi\rangle = X_o \tag{1.2}$$

Here one has to consider the expectation value of the energy obtained with all possible normalized wavefunctions centered at X_o. The effective potential at X_o is then the minimum of the energy expectation value. Its computation involves a functional minimization which is done through the Lagrange multiplier technique of introducing a linear coupling to a local external source [1-3,10]. The global minimum of $V_{eff}(X_o)$ gives the exact ground state energy of the system.

The effective potential $V_{eff}(X_o)$ is convex [11,12,10]:

$$\frac{d^2V_{eff}(X_o)}{dX_o^{\,2}} \geq 0 \tag{1.3}$$

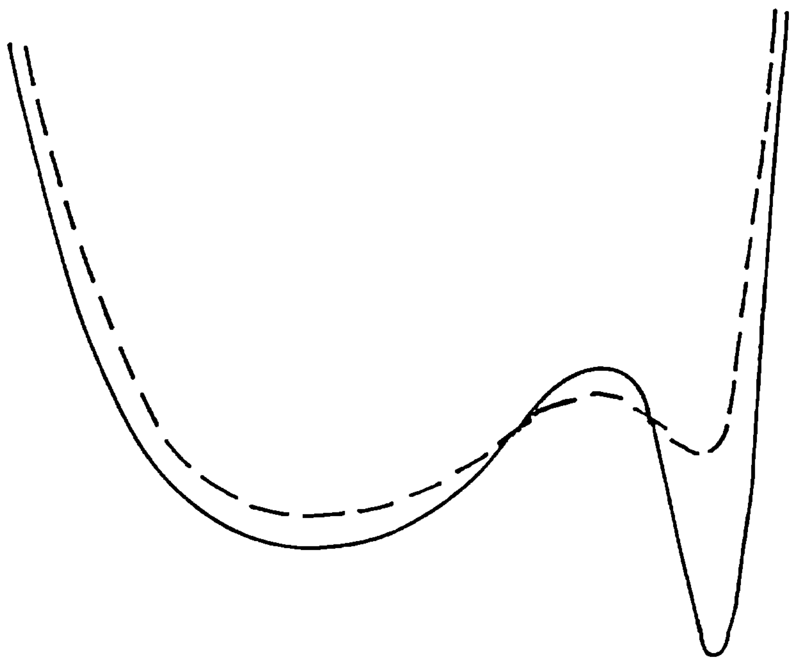


Fig. 1.2 Asymmetric double well potential subjected to large quantum effects.
The dotted curve denotes the effective potential and the solid curve the classical potential.

1.2 Functional method and effective potential

In classical field theory, the ordinary potential $U(\phi)$, is an energy density for that state in which the field assumes the value ϕ. In quantum field theory the effective potential, $V(\phi_c)$, is also an energy density in a certain state for which the expectation value of the field is ϕ_c [11]. If V has several local minima, the effective potential corresponds to the true ground state of the theory.

Consider a single scalar field ϕ whose dynamics is described by a Lagrangian density $\mathcal{L}(\phi,\partial_\mu\phi)$. A linear coupling of ϕ to an external source $j(x)$ which is a c-number function of space and time, is added:

$$\mathcal{L}(\phi,\partial_\mu\phi) \longrightarrow \mathcal{L} + j(x)\,\phi(x) \tag{1.4}$$

The connected generating functional $W(j)$ is defined in terms of the transition amplitude from the vacuum state in the far past to the vacuum state in the far future, in the presence of the source $j(x)$:

$$e^{iW(j)} = \langle 0^+ | 0^- \rangle_j \tag{1.5}$$

W can be expanded in a functional Taylor series:

$$W = \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \; G^{(n)}(x_1 \ldots x_n)\, j(x_1) \ldots j(x_n) \tag{1.6}$$

The successive coefficients in the series are the connected Green's functions; $G^{(n)}$ is the sum of all connected Feynman diagrams with n external lines.

The classical field, ϕ_c in the presence of an external source j(x) is defined by

$$\phi_c(x) = \frac{\delta W}{\delta j(x)} = \left[\frac{\langle 0^+|\phi(x)|0^-\rangle}{\langle 0^+|0^-\rangle}\right]_j \tag{1.7}$$

The effective action, $\Gamma(\phi_c)$, is defined by a functional Legendre transformation

$$\Gamma(\phi_c) = W(j) - \int d^4x \; j(x)\, \phi_c(x) \tag{1.8}$$

From this definition,

$$\frac{\delta \Gamma}{\delta \phi_c(x)} = -j(x), \tag{1.9}$$

the effective action, Γ, can also be expanded in a functional Taylor series:

$$\Gamma = \sum_n \frac{1}{n!}\int d^4x_1 \ldots d^4x_n \, \Gamma^n(x_1 \ldots x_n)\, \phi_c(x_1) \ldots \phi_c(x_n) \tag{1.10}$$

The successive coefficients in this series are one particle irreducible (IPI) Green's functions which are also called the proper vertices. $\Gamma^{(n)}$ is the sum of all IPI Feynman diagrams with n external lines. (By convention a IPI diagram is a connected diagram that cannot be disconnected by cutting a single internal line and it is evaluated with no propagators on the external lines).

Instead of expanding the effective action in powers of ϕ_c, one can also expand it in powers of momentum about the point where all external momenta vanish. In position space such an expansion takes the form

$$\Gamma = \int d^4x[-V(\phi_c) + \tfrac{1}{2}(\partial\mu\phi_c)^2 Z(\phi_c)+\ldots] \tag{1.11}$$

where $V(\phi_c)$ is identified as the effective potential.

To express $V(\phi)$ in terms of IPI Green's functions, we first write $\Gamma^{(n)}$ in momentum space:

$$\Gamma^{(n)}(x_1\ldots x_n) = \int \frac{d^4k_1}{(2\pi)^4} \ldots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4\delta^4(k_1+\ldots k_n) \\ e^{i(k_1x_1+\ldots k_nx_n)} \Gamma^{(n)}(k_1\ldots k_n) \tag{1.12}$$

Putting this into (1.10) and expanding in powers of k_i, we get

$$\Gamma(\phi_c) = \sum_n \frac{1}{n!}\int d^4x_1\ldots d^4x_n \int \frac{d^4k_1}{(2\pi)^4} \ldots \frac{d^4k_n}{(2\pi)^4} \\ \int d^4x\, e^{i(k_1+k_2+\ldots k_n)\cdot x}\, e^{i(k_1x_1+\ldots k_nx_n)} \\ [\Gamma^{(n)}(o, \quad o)\phi_c(x_1)\ldots\phi_c(x_n)+\ldots] \\ = \int d^4x \sum_n \frac{1}{n!}\{\Gamma^{(n)}(o,\ldots o)\,[\phi_c(x)]^n + \quad\} \tag{1.13}$$

Comparing (1.11) and (1.13) we find that the nth derivative of $V(\phi_c)$ is the sum of all IPI graphs with n vanishing external momenta. In the tree approximation V is just the ordinary potential.

To calculate $V(\phi_c)$ we need an approximation scheme which preserves the main advantage of this effective potential formalism, ie, the capability to survey all vacuua at once before deciding which is the tree ground state. Ordinary perturbation theory with its expansion in coupling constants is not appropriate as it is necessary, at each order, to identify the true vacuum state and shift the field. Loop expansion [5,13,14] is an expansion according to the increasing number of independent loops of connected Feynman diagrams. Hence the lowest order graphs will be the Born diagrams or tree graphs. The next order consists of the one loop diagrams which have one integration over the internal momenta, etc. For the effective potential, each loop level still involves an infinite summation corresponding to all possible external lines.

The loop expansion can be identified as an expansion in powers of Planck's constant $\hbar$. This can be seen as follows: Let I be the number of internal lines and V the number of vertices in a given Feynman diagram. The number of independent loops L will be the number of independent internal momenta after the momentum conservation at each vertex is taken into account. Since one combination of these momentum conservations corresponds to the overall conservation of external momenta, the number of independent loops in a given Feynman diagram is given by

$$L = I - (V-1) \tag{1.14}$$

To relate L to the powers of $\hbar$, one has to keep track of the factor $\hbar$ in the standard quantization procedure. First there is one power of $\hbar$ in the canonical commutation relation

$$[\phi(x,t),\ \pi(y,t)] = i\hbar\delta^3(x-y) \tag{1.15}$$

This will give rise to a factor of $\hbar$ in the free propagator in momentum space

$$\langle o|T\phi(x)\phi(o)|o\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx}\, \frac{i\hbar}{k^2-m^2+i\varepsilon} \tag{1.16}$$

The other place where $\hbar$ appears is in the evolution operator exp $[-iHt/\hbar]$ which gives rise to the operator $\exp[\frac{i}{\hbar}\int \mathcal{L}_{int}\,(\phi)\, d^4x]$ in the interaction picture. This means that there will be a factor of $1/\hbar$ for each vertex. Thus for a given Feynman diagram we have P powers of $\hbar$ with

$$P = I - V = L - 1$$

Thus the number of loops and the power of $\hbar$ are directly correlated. The statement that loop expansion corresponds to an expansion in Planck's constant is a statement that it is an expansion in some parameter a that multiplies the total Lagrange density

$$\mathcal{L}(\phi,\partial\mu\phi,a) = a^{-1}\mathcal{L}(\phi,\partial\mu\phi) \tag{1.17}$$

The above counting of the $\hbar$ powers reflects the fact that while every vertex carries a factor a^{-1}, the propagator carries a factor a because it is the inverse of the differential operator occurring in the quadratic terms in $\mathcal{L}$ Because $\hbar$, or a is a parameter that multiplies the total Lagrangian, it is unaffected by shifts of fields and by the redefinition or division of $\mathcal{L}$ into free and interacting parts associated with such shifts [15]. In other words, it allows one to compute $V(\phi_c)$ before the shift.

The loop expansion is certainly not a worse approximation scheme than the ordinary coupling constant expansion perturbation theory, since the loop expansion includes the latter as a subset at a given loop level.

1.3 Nonperturbative approach and Gaussian effective potentials

Actually the one loop effective potential (1LEP) is a semiclassical construct, based on adding to the classical potential the order-$\hbar$ quantum corrections, and neglecting the terms of order $\hbar^2$ or higher. Formally it is $V_{eff}(X_o) \sim V(X_o) + \sum_{n=1}^{\infty} \hbar^n V_n(X_o)$ and $V_{one\ loop} = V(X_o) + \hbar V_1(X_o)$. The one loop approximation generally breaks down whenever the quantum effects become large.

This failure can be seen in the study of one loop effective potential for the potential [16]:

$$V(X) = \sigma + \frac{1}{2} m^2X^2 + \lambda X^4 \tag{1.18}$$

This is the anharmonic oscillator for $m^2 > o$ and the standard double well potential for $m^2 < o$. For the anharmonic oscillator, the one loop approximation for effective potential is accurate for weak coupling but turns out to be unrealistic for $\lambda \geqslant 1$. In the double well case, for small X_o the ILEP contains an imaginary part, and hence, is not defined in that region. These cases illustrate the need for other methods of evaluation of the effective potential.

The effective potential conventionally defined by [10] suffers from several defects. For example it can never have a double well shape. For a double well potential the minimum value of $V_{eff}(X_o)$ lies right in the middle of a real potential barrier (Fig. 1.3a).

The condition $\langle\psi|X|\psi\rangle = X_o$ only requires the wave-function to be centered on X_o in a minimal sense. It could consist of two large peaks on either side of X_o, with $|\psi|^2$ being small in the neighborhood of X_o. The effective potential at a point X_o may not, therefore, reflect the actual conditions there. It may only give an average condition on either side of X_o. Hence the conventional effective potential will behave as if there is no potential barrier at all.

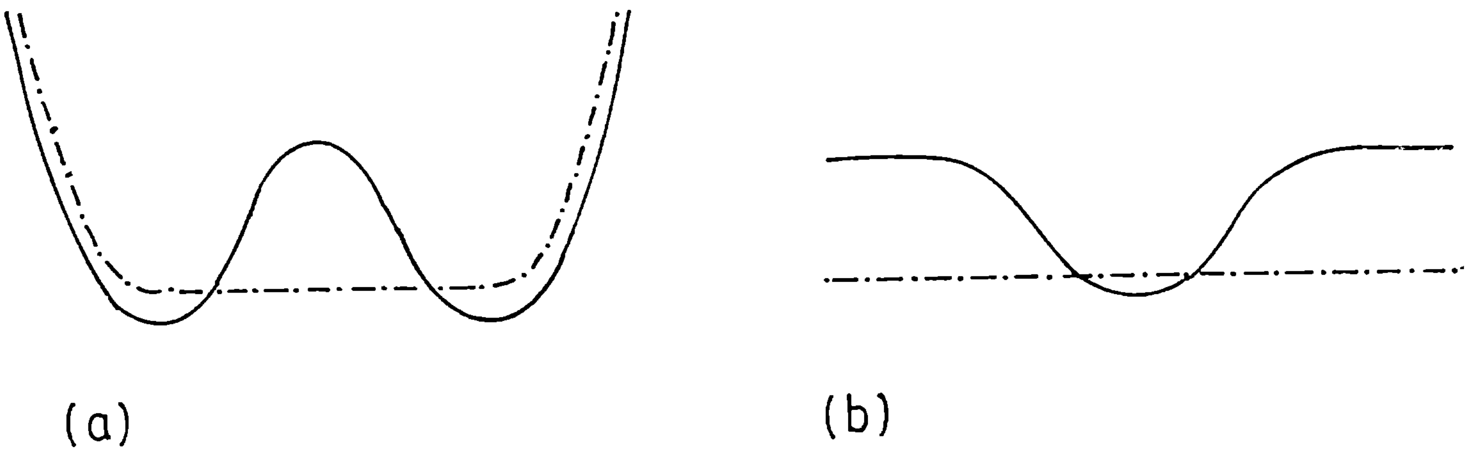


Fig. 1.3 shows the strange behaviour of the conventional effective potential (dotted lines)

a. for the double well potential

b. for a finite depth potential well

Another example is the case of a finite depth potential well which tends to a finite value at infinity. The effective potential in this case equals E_o for all X_o [10] which gives the impression that the particle is free to wander anywhere, as shown in Fig. 1.3b. Actually it will remain localized in the potential well.

The above examples show that the conventional effective potential is unable to give a good picture of the physics. A more realistic effective potential, called the Gaussian effective potential (GEP), has been discussed several times in the literature [16-26]. Here the trial wavefunction is required to be concentrated in the vicinity of X_o. This is done by assuming the admissible wavefunctions to be of Gaussian form centered on X_o. It is, incidentally, the ground state wavefunction of the parabolic potential well. The Gaussian effective potential is then defined as [16]:

$$\bar{V}_G(X_o) \equiv \min_{\Omega} V_G(X_o,\Omega) \equiv \min_{\Omega} \langle\psi|H|\psi\rangle \qquad (1.19)$$

with

$$|\psi\rangle = \left(\frac{\Omega}{\hbar\pi}\right)^{1/4} \exp\left[-\frac{1}{2}\frac{\Omega}{\hbar}(X-X_o)^2\right],\ \Omega>o \qquad (1.20)$$

The width of the Gaussian, governed by the parameter Ω, is left to adjust itself so as to minimize $\langle H\rangle$ at each X_o. Hence the GEP can be considered to be a variational approximation of the ordinary nonperturbative effective potential. The global minimum of the GEP may not give the ground state energy

$V_{eff}(X_o)$. According to Rayleigh-Ritz theorem, $\bar{V}_G(X_o) \geq V_{eff}(X_o)$ ($=E_0$) at any X_o. But in most cases we can expect a good approximation to E_0, due to the fact that any half way realistic wavefunction generally gives a reasonable estimate of the ground state energy.

One can use the Schrödinger representation, $P = -i\hbar \frac{d}{dX}$ and evaluate $\langle\psi|H|\psi\rangle$ as the integral

$$\langle H\rangle = \int_{-\infty}^{+\infty} dX\, \psi^*(X)\left[-\frac{\hbar^2}{2}\frac{d^2}{dX^2} + V(X)\right]\psi(X) \qquad (1.21)$$

where $\psi(X)$ is the Gaussian function. We may also make the substitutions:

$$X = X_o + \hbar\, (2\hbar\Omega)^{-1/2}(a_\Omega + a_\Omega^+) \qquad (1.22)$$

$$P = -\frac{1}{2} i\, (2\hbar\Omega)^{1/2}(a_\Omega - a_\Omega^+) \qquad (1.23)$$

where

$$[a_\Omega, a_\Omega^+] = 1 \qquad (1.24)$$

and

$$a_\Omega |0\rangle_\Omega = 0 \qquad (1.25)$$

a_Ω and a_Ω^+ depend on the frequency of the harmonic oscillator whose ground state $|0\rangle_\Omega$ is the Gaussian trial wavefunction.

GEP in field theory

The field-theoretic generalization of the effective potential is [27]

$$\overline{V}_G(\phi_o) = \min_{\Omega} \; V_G(\phi_o,\Omega) = \min_{\Omega} \; {}_{\phi_o,\Omega}\langle 0|H|0\rangle_{\Omega,\phi_o} \tag{1.26}$$

where $|0\rangle_{\Omega,\phi_o}$ is a renormalized Gaussian wave functional, centered on $\phi = \phi_o$ subject to the conditions:

$$ {}_{\phi_o,\Omega}\langle 0|0\rangle_{\Omega,\phi_o} = 1 \tag{1.27}$$

$$ {}_{\phi_o,\Omega}\langle 0|\phi|0\rangle_{\Omega,\phi_o} = \phi_o \tag{1.28}$$

The calculation can be performed in a Schrödinger wave-functional formalism as indicated in the quantum mechanical examples.

The field ϕ can be written as $\phi_o+\hat{\phi}$ where ϕ_o is a constant classical field and $\hat{\phi}$ is a quantum free field of mass Ω. The state $|0\rangle_{\Omega,\phi_o}$ is the vaccum state of this free field [28]:

$$\phi = \phi_o + \int(dk)_{\Omega} \, [a_{\Omega}(k)e^{-ik.x} + a^{+}_{\Omega}(k)e^{ik.x}] \tag{1.29}$$

Differentiating,

$$\partial_\mu\phi = \int(dk)_{\Omega} \, (-ik_\mu) \, [a_{\Omega}(k)e^{-ik.x} - a^{+}_{\Omega}(k)e^{ik.x}] \tag{1.30}$$

where the energy component of the four vector k_μ is

$$k^o = \omega_{\underline{k}}(\Omega) \equiv (k^2+\Omega^2)^{1/2} \tag{1.31}$$

The integration measure in r spatial dimensions is

$$(dk)_{\Omega} = \frac{d^r k}{(2\pi)^r 2\omega_{\underline{k}}(\Omega)} \tag{1.32}$$

As usual, the creation and annihilation operators obey the commutation relation

$$[a_{\Omega}(k),\; a^{+}_{\Omega}(k')] = \delta_{kk'} = 2\omega_{\underline{k}}(\Omega)(2\pi)^r\delta^r(k - k') \tag{1.33}$$

where $\delta^r(k-k')$ is the r-dimensional Dirac delta function, and $|0\rangle_\Omega$ has the property,

$$a_\Omega(k)|0\rangle_\Omega = 0 \qquad (1.34)$$

The $V_G(\phi_o,\Omega)$ can then be directly evaluated from the Hamiltonian.

For example, in the case of a ϕ^4 model defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2} m_B{}^2\phi^2 - \lambda_B\phi^4 \qquad (1.35)$$

the quantity $V_G(\phi_o,\Omega)$ is obtained [27] as

$$V_G(\phi_o,\Omega) = I_1 + \frac{1}{2}(m_B{}^2 - \Omega^2)I_o + \frac{1}{2}m_B^2\phi_o{}^2 + \lambda_B\phi_o{}^4 + 6\lambda_B I_o\phi_o{}^2 + 3\lambda_B I_o{}^2 \qquad (1.36)$$

where

$$I_N(\Omega) = \int(dk)_\Omega\,[\omega_k{}^2(\Omega)]^N \qquad (1.37)$$

Here N is a positive or negative integer or half-integer. The GEP $V_G(\phi_o)$ is then obtained by minimizing $V_G(\phi_o)$ with respect to the variational parameter Ω, in the range $0<\Omega<\infty$.

<u>Corrections to the GEP</u>

Recently it has been shown that the GEP can be made the starting point for a systematic expansion procedure [16,29,30]. The effective potential has been calculated for $\lambda\phi^4$ theory next to leading order result [31]. The method may be outlined as follows.

The Euclidean action in d dimensions is [1,2,13,32,28].

$$S[\phi] = \int d^dx\, \mathcal{L}(\phi,\partial_\mu\phi) \qquad (1.38)$$

The generating functional for Green's functions is given by the functional integral

$$Z[j] = \int D\phi \exp[-S[\phi]+\int d^d x \; j(x)\phi(x)] \tag{1.39}$$

Let

$$W[j] = \ln Z[j] \tag{1.40}$$

Here W[j] is the generating functional for the connected Green's functions. The effective action $\Gamma[\phi_c]$ is obtained by the Legendre transformation

$$\Gamma[\phi_c] = W[j] - \int d^d x \; j(x)\phi_c(x) \tag{1.41}$$

where

$$\phi_c(x) = \frac{\delta W}{\delta j(x)} = Z^{-1}[j]\int D\phi \; \phi \exp[-S[\phi]+\int d^d x j(x)\phi(x)] \tag{1.42}$$

$\phi_c(X)$ is the vacuum expectation value of the field $\phi(x)$ in the presence of the source $j(x)$. The effective potential $V_{eff}(\phi_c)$ is obtained from $\Gamma[\phi_c]$ by setting $\phi_c(x)$ to a constant ϕ_c [so that j will be x-independent].

$$\Gamma[\phi]|_{\phi_c(x)=\phi_c} = -\mathcal{V} \, V_{eff}(\phi_c) \tag{1.43}$$

where $\mathcal{V} = \int d^d x$, the space-time volume and

$$\hat{\phi}(x) = \phi(x) - \phi_o \tag{1.44}$$

Now to calculate $V_{eff}(\phi_c)$ in the nonstandard kind of perturbation theory, let the Lagrangian be defined as

$$\mathcal{L} = (\mathcal{L}_o+\mathcal{L}_{int})_{\delta = 1} \tag{1.45}$$

where $\mathcal{L}_o$ is the free field Lagrangian with mass Ω for the $\hat{\phi}$ field. An expansion parameter δ is introduced in $\mathcal{L}_{int}$ to keep track of the order of approximation. The approximation consists of a truncated Taylor series in δ about $\delta=0$ and an extrapolation to $\delta=1$.

For calculational convenience, $\emptyset_o$ can be fixed self-consistently to coincide with the classical field $\emptyset_c = \frac{\delta W}{\delta j}$
The result is actually independent of the $\emptyset_o$ used [31].

The mass parameter Ω must be chosen in each order in accordance with the principle of minimal sensitivity [33-35]. The approximation cannot be trusted in a region where it gives a result strongly dependent on Ω. When the approximate result is insensitive to variations in Ω, it is a very good approximation to the exact E_o, which is independent of Ω. Hence the result must be optimized so that it must be as insensitive to Ω as possible. This requires only finding the stationary point. The optimum Ω changes from one order to the next, and this is crucial for the expansion to yield convergent results [36,37,35].

With the usual procedure, the generating functional can be rewritten as

$$Z[j,\emptyset_o]=\exp[\textstyle\int_z j_z \emptyset_o]\exp[-\int_x \hat{\mathcal{L}}_{int}[\frac{\delta}{\delta j_x}]] \cdot \int D\hat{\emptyset} \; \exp[-\int_z \mathcal{L}_{o,z} + \int_z j_z \hat{\emptyset}_z] \tag{1.46}$$

where $\hat{\mathcal{L}}_{int}$ is the functiohal differential operator obtained from $\mathcal{L}_{int}$ by replacing $\hat{\emptyset}$ by $\frac{\delta}{\delta j}$.
The $\hat{\emptyset}$ integration can be done so that

$$Z[j,\emptyset_o] = \exp(j\emptyset_o) \; (\text{Det } G^{-1})^{-1/2} \exp(-\hat{\mathcal{L}}_{int}) \exp(\tfrac{1}{2}jGj) \tag{1.47}$$

Here we havesuppressed the space-time arguments and integrations over them.

The functional determinant is

$$(\text{Det } G^{-1})^{-1/2} = \exp(-\mathcal{V} I_1) \tag{1.48}$$

where

$$I_1(\Omega) = \frac{1}{2}\int_p \ln(p^2+\Omega^2) \tag{1.49}$$

$$\ln Z = W[j,\phi_o] = j\phi_o - \mathcal{V}I_1 + \ln[(1-\hat{\mathcal{L}}_{int}+\frac{1}{2}\hat{\mathcal{L}}^2_{int} + \ldots)\exp(\frac{1}{2}\, jGj)] \tag{1.50}$$

and

$$\hat{\mathcal{L}}^2_{int} = \int_x \hat{\mathcal{L}}_{int,x}\int_y \hat{\mathcal{L}}_{int,y} \tag{1.51}$$

Since $\phi_c(x)$ is a constant, let $\phi_o=\phi_c$; then $\Gamma[\phi_c]$ is given by the above expression for W but without the $j\phi_o$ term. The source j is to be found as a function of ϕ_c by solving (1.42). To zeroth order in δ we have

$$W[j,\phi_o]|_{(o)} = \int_z j_z\phi_o - \mathcal{V}I_1 + \frac{1}{2}\int_z\int_{z'} j_z\, G_{zz'}\, j_{z'} \tag{1.52}$$

so that

$$(\phi_c)_x = \frac{\delta W}{\delta j_x} = \phi_o + (Gj)_x \tag{1.53}$$

where $(Gj)_x \equiv \int_z G_{xz} j_z$. Taking $(\phi_c)_x$ to be x independent, and setting $\phi_o = \phi_c$, j vanishes to this order.

Then

$$\Gamma[\phi_c] = -\mathcal{V}I_1 \tag{1.54}$$

The terms which are first order in δ and second order in δ can then be separately found.

.4 Quantum field theory at finite temperature

It was suggested by Kirzhnits and Linde [38] that the spontaneous symmetry violation in relativistic field theory will disappear above a critical temperature. This motivated other physicists also to study the behaviour of quantum field systems at finite temperature.

The diagramatic functional methods for evaluating effective potentials in field theory can be employed to study finite temperature effects also [39,40].

The finite temperature Green's functions are defined by [41].

$$G_\beta(x_1 \ldots x_j) = \frac{Tr(e^{-\beta H}\ T(\phi(x_1)\ldots\phi(x_j))}{Tr\ e^{-\beta H}} \qquad (1.55)$$

where H is the Hamiltonian governing the dynamics of the field $\phi(x)$, and β^{-1} is proportional to temperature.

The differential equations satisfied by finite-temperature Green's functions are identical with those of the zero temperature theory [39,40]. But when the boundary conditions are imposed, the familiar causal boundary conditions at $t = \pm\infty$ are appropriate at zero temperature and the periodic boundary conditions are considered for imaginary time at finite temperature.

For the finite temperature 2-point functions [42] $D_\beta(x-y)$, we have

$$D_\beta(x-y) = \frac{Tr\ e^{-\beta H} Tr\ \phi(x)\ \phi(y)}{Tr\ e^{-\beta H}} \qquad (1.56)$$

Two diagonal representations for $D_\beta(x-x')$ can be given - one in terms of imaginary time and the other for real time. Here we shall elaborate on the imaginary time technique [39] because this is the approach adopted in the present work.

The operator $e^{-\beta H}$ in the definition of finite-temperature

Green's functions indicates a time translation $t \rightarrow t + i\beta$. This will give rise to periodicity (antiperiodicity) properties for Bose (Fermi) Green's functions in imaginary, ie Euclidean, time.

For a non-interacting field,

$$(\Box_x + m^2)\, D_\beta(x-y) = -\, i\delta^4(x-y) \tag{1.57}$$

To solve this equation we must know the boundary conditions; they are given for imaginary time.

The time argument of D_β can be continued to the Euclidean interval

$0 \leq ix_0, \quad iy_0 \leq \beta$

and the time ordering for imaginary time can be defined as

$$T[\phi(x)\,\phi(y)] = \begin{cases} \phi(x)\,\phi(y) & ix_0 > iy_0 \\ \phi(y)\,\phi(x) & iy_0 > ix_0 \end{cases} \tag{1.58}$$

The two point function $D_\beta(x-x')$ can be transformed using cyclic properties of the trace and transformation properties of the fields under the Poincaré group:

$$(\mathrm{Tr}\, e^{-\beta H})\, D_\beta(x-y)|_{x_0=0} = \mathrm{Tr}[e^{-\beta H}\, T\phi(o,\vec{x})\phi(y^o,\vec{y})] \tag{1.59}$$

$$= \mathrm{Tr}[e^{-\beta H}\, \phi(y^o,\vec{y})\phi(o,\vec{x})]$$

$$= \mathrm{Tr}[e^{-\beta H}\, e^{\beta H}\, \phi(o,\vec{x})\, e^{-\beta H}\phi(y^o,\vec{y})]$$

$$= \mathrm{Tr}[e^{-\beta H}\, \phi(-i\beta,\vec{x})\, \phi(y^o,\vec{y})]$$

$$= \mathrm{Tr}\, e^{-\beta H}\, D_\beta(x-y)|_{x^o=-i\beta}, \tag{1.60}$$

from where we obtain the periodicity condition

$$D_\beta(x-y)|_{x^o=o} = D_\beta(x-y)|_{x^o=-i\beta} \tag{1.61}$$

In the imaginary time domain, D_β may be represented by Fourier series and integrals,

$$D_\beta(x-y) = \frac{1}{-i\beta}\sum_{n=-\infty}^{+\infty} e^{-i\omega_n x^o}\int\frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}.\vec{x}}\,\frac{1}{(-i\beta)} \tag{1.62}$$

$$\sum_{m=-\infty}^{+\infty} e^{i\omega_m y^o}\int\frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}.\vec{y}} D_\beta(\omega_n,\vec{p},\omega_m,\vec{q})$$

where $\omega_n = \frac{2\pi n}{-i\beta}$. The inverse transformation is

$$D_\beta(\omega_n,\vec{p},\omega_m,\vec{q}) = \int_o^{-i\beta}dx^o e^{i\omega_n x^o}\int d^3x\; e^{-i\vec{p}.\vec{x}}$$
$$\int_o^{-i\beta}dy^o\; e^{-i\omega_m y^o}\int d^3y\; e^{i\vec{q}.\vec{y}} D_\beta(x-y) \tag{1.63}$$

and since $D_\beta(x-y)$ depends only on a coordinate difference

$$D_\beta(\omega_n,\vec{p},\omega_m,\vec{q}) = -i\beta\;\delta_{nm}(2\pi)^3\;\delta^3(\vec{p}-\vec{q})\;D_\beta(\omega_n,\vec{p}) \tag{1.64}$$

so that $D_\beta(x-y) = \int_p e^{-ip(x-y)}\, D_\beta(p)$,

$$\int_p \equiv \frac{1}{(-i\beta)}\sum_{m=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3} \tag{1.65}$$

$$D_\beta(p) = \int_x e^{ipx}\, D_\beta(x),\; \int_x \equiv \int_o^{-i\beta}dx^o\int d^3x \tag{1.66}$$

where $p = (\omega_n,\vec{p})$ is never time-like

$$p^2 = \omega_n{}^2-\vec{p}^2 = -\left[\frac{4\pi^2n^2}{\beta^2} + \vec{p}^2\right] \leq 0 \tag{1.67}$$

From (1.57) we have

$$(-p^2+m^2)D_\beta(p) = -i$$

$$D_\beta(p) = \frac{i}{p^2-m^2} \tag{1.68}$$

The finite temperature 2-point function for spin $\frac{1}{2}$ fields is defined by

$$S_\beta(x-y) = \mathrm{Tr}\, e^{-\beta H}\, T(\bar{\emptyset}(x)\, \bar{\emptyset}(y))/\mathrm{Tr}\, e^{-\beta H} \tag{1.69}$$

and for non-inter cting fields

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)S_\beta(x-y) = i\delta^4(x-y) \tag{1.70}$$

The time argument of S_β is continued to the Euclidean region, and we define time ordering by the relation

$$T\,[\emptyset(x)\,\bar{\emptyset}(y)] = \begin{cases} \emptyset(x)\,\bar{\emptyset}(y) & ix^o > iy^o \\ -\bar{\emptyset}(y)\,\emptyset(x) & iy^o > ix^o \end{cases} \tag{1.71}$$

As in the bosonic case, with the similar steps as in (1.59) through (1.60), one obtains the antiperiodic boundary condition:

$$S_\beta(x-y)|_{x^o=o} = -S_\beta(x-y)|_{x^o=-i\beta} \tag{1.72}$$

The imaginary time formalism leads to

$$S_\beta(x-y) = \int_p e^{-ip(x-y)}\, S_\beta(p) \tag{1.73}$$

where

$$S_\beta(p) = \frac{i}{\gamma^\mu p_\mu - m} \tag{1.74}$$

and $p^\mu(\omega_n, \vec{p})$ with

$$\omega_n = \frac{(2n+1)\pi}{-i\beta} \tag{1.75}$$

We can summarise the finite temperature Feynman rules as:

Spin-zero propagator: $\frac{i}{p^2-m^2} = \frac{-i}{\frac{4\pi^2 n^2}{\beta^2} + \vec{p}^{\,2} + m^2}$ (1.76)

Fermion propagator : $\frac{i}{\gamma p - m}$; $p^\mu = [\frac{(2n+1)\pi}{-i\beta}, \vec{p}]$ (1.77)

Loop integral : $\frac{1}{-i\beta} \sum_{n=-\infty}^{+\infty} \frac{\int d^3p}{(2\pi)^3}$ (1.78)

The real time approach [39] is full of ambiguities, because one obtains products of δ functions. In addition, the nice algebraic properties of the covariant I_N integrals occuring in the renormalized field theory are not preserved within the real time formalism.

1.5 Coherent states and squeezed states

The coherent state $|\alpha\rangle$ is defined [43-46] as an eigenfunction of the annihilation operator a with some complex eigenvalue α:

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad (1.79)$$

Like any other state, a coherent state is represented as a linear superposition of number states. We, therefore, write

$$|\alpha\rangle = \sum_{n=o}^{\infty} C_n(t)|n\rangle \quad (1.80)$$

Thus

$$a|\alpha\rangle = \sum_{n=1}^{\infty} C_n\sqrt{n}|n-1\rangle = \alpha \sum_{n=o}^{\infty} C_n|n\rangle \quad (1.81)$$

By matching the coefficients of each number state we have

$$C_1 = \alpha C_o$$
$$C_2 = \alpha C_1|\sqrt{2} \quad (1.82)$$
$$C_n = \alpha C_{n-1}|\sqrt{n}$$

The general coefficient C_n can be expressed as

$$C_n = C_o(\alpha^n|\sqrt{n!}) \quad (1.83)$$

By imposing the normalization condition we have

$$1 = \sum_{n=o}^{\infty} |C_n|^2 = |C_o|^2 \sum_{n=o}^{\infty} (|\alpha|^2)^n/n! = |C_o|^2 e^{|\alpha|^2} \qquad (1.84)$$

which yields $|C_o| = e^{-|\alpha|^2/2}$ In the case of coherent states probability P_n is

$$P_n = |C_n|^2 = e^{-|\alpha|^2}[(|\alpha|^2)^n/n!] = e^{-\langle n\rangle}[\langle n\rangle^n/n!] \qquad (1.85)$$

where we have made the replacement

$$|\alpha|^2 = \langle n\rangle \qquad (1.86)$$

P_n represents a Poisson distribution. In other words, P_n is the probability of detecting n independent events in a fixed time interval, if $\langle n\rangle = |\alpha|^2$ is the average number of events per time interval.

Hence a coherent state $|\alpha\rangle$ is a linear combination of number states whose squared coefficients $|C_n|^2$ represent the probabilities of detecting n quanta in a Poisson distribution with average number of quanta $|\alpha|^2$.

All coherent states are minimum product states with variances equal to those of the vacuum state.

Squeezed states

A state is said to be squeezed if its oscillating variances become smaller than the variances of the vacuum state. The product of the variances attains a minimum value only at the instant that one variance is a minimum and the other is a maximum. If the minimum value of the product is equal to 1/4, then the state is called a 'minimum uncertainty squeezed state'. It is shown that the shape that leads to a minimum uncertainty squeezed state is a Gaussian pulse [47].

Eigenstates of the operator

$$b = \mu a + \nu a^{+} \tag{1.87}$$

defined by the relation

$$b|\beta\rangle = \beta|\beta\rangle \tag{1.88}$$

are called squeezed states. They are also known as photon coherent states [48-50].

If a squeezed state $|\beta\rangle$ is to be an eigenfunction of b, then

$$(\mu a + \nu a^{+}) \sum_{n=o}^{\infty} C_n|n\rangle = \beta \sum_{n=o}^{\infty} C_n|n\rangle \tag{1.89}$$

Here the C_n represent the number state coefficients for the squeezed state at t = o. Operating term by term with $\mu a + \nu a^{+}$ we have

$$\mu \sum_{n=1}^{\infty} \sqrt{n}\, C_n|n-1\rangle + \nu \sum_{n=o}^{\infty} \sqrt{n+1}\, C_n|n+1\rangle = \beta \sum_{n=o}^{\infty} C_n|n\rangle \tag{1.90}$$

Now we have the recursion relations

$$C_1 = \beta C_o/\mu$$

$$C_2 = (\beta C_1 - \nu C_o)/\mu\sqrt{2},$$

and in general

$$C_n = \frac{\beta C_{n-1} - \nu\sqrt{n-1}\, C_{n-2}}{\mu\sqrt{n}} \tag{1.91}$$

For a given set of numerical values of μ, ν and β, we can begin with an arbitrary value of C_o and find the numerical values of the rest of the coefficients recursively. The value of C_o is then adjusted for normalization:

$$\sum_{n=o}^{\infty} |C_n|^2 = 1 \qquad (1.92)$$

From these recursion relations, it is clear that there are only two independent parameters. Therefore, if $|\mu|>|\nu|$ so that the sequence of C_n converges, we can choose

$$|\mu|^2-|\nu|^2=1 \qquad (1.93)$$

This choice of μ and ν results in

$$bb^+ - b^+b = 1 \qquad (1.94)$$

1.6 q-oscillators

The last chapter of this thesis is devoted to a formulation of a non perturbative q-or (q,p)-analogue of GEP for quantum oscillators.

For the last four years much attention has been directed to the study of quantum groups [51-55] and their possible applications [56-78]. Very recently, consequences of introduction of a non commutative algebra due to quantum group in various systems are being subjected to intense studies [59]. Quantum oscillators have already found applications in diverse fields such as molecular spectroscopy [60-62], condensed matter physics [63], quantum optics [64-74] and many body theory [75].

For q oscillators, one can start with an operator a and its adjoint a^+, acting on a Hilbert space with basis $|n\rangle$, n=o,1,2... The ground state $|o\rangle$ is assumed to be annihilated by a:

$$a|o\rangle=o, \qquad |n\rangle = \frac{(a^+)^n}{([n]!)^{1/2}}|O\rangle \tag{1.95}$$

where the q factorial [n]! is

$$[n]! = [n] \ [n-1] \quad [1], \tag{1.96}$$

with $[A] = q^A-q^{-A}/q-q^{-1}$

$$a^+|n\rangle = [n+1]^{1/2}|n+1\rangle \tag{1.97}$$

$$a|n\rangle = [n]^{1/2}\ |n-1\rangle \tag{1.98}$$

$$aa^+|n\rangle = [n+1]|n\rangle \tag{1.99}$$

and the q-commutation relation is

$$aa^+-qa^+a = q^{-N} \tag{1.100}$$

Here N is the number operator which is not assumed to be the same as a^+a. In terms of X and P the q-commutation relation can be read as

$$[X,P] = i\hbar[q^{-N}+(q-1)\ a^+a] \tag{1.101}$$

Although, in principle, q could be real or complex, consistency of the above equation with the assumption that X and P are simultaneously hermitian, constrains q to be a real parameter.

The number operator N is required to satisfy the commutation relations

$$[a,N] = a \qquad [a^+,N] = -a^+ \tag{1.102}$$

$$N|n\rangle = n|n\rangle \tag{1.103}$$

Many other versions of q-oscillator have appeared [76-78]. But we stick to the above formulation in our work.

From the point of view of applicability in concrete physical models, quantum algebras with multiparameter deformations are also of interest [79,80].

II. LIOUVILLE FIELD THEORY

2.1 Introduction

The Liouville field theory is one of the well studied models [81-83]. For particle physicists, the theory has been important in the study of instantons and solitons [84-86]. It finds application in reformulations of the dual string model [87,88] and in the Polyakov [89] approach to string theory. According to him, in order to get a proper quantisation for $D<26$ one must examine the quantum Liouville theory. It also finds application in study of black holes using string theory [90].

The theory describes an exponentially self-interacting scalar field in two dimensions which is renormalizable and completely integrable. In other words, it is exactly solvable just as Sine-Gordon theory is, and hence the explicit evaluation of the partition function of closed surfaces must be possible. Polyakov has demonstrated how to express different physical quantities like the spectrum, scattering amplitude etc. through correlation functions of quantum Liouville theory. For physical D we have to solve the Liouville theory to find the scattering amplitudes.

The Liouville model has been well studied at the classical and quantum levels. Goldstone [91] has computed the exact effective potential of this model using a method that relies on the fact that a shift in the ϕ field is equivalent to a redefinition of the mass parameter m^2 which can in turn be compensated for by normal ordering. Later D' Hoker and Jackiw [92], using loop expansion method, evaluated the effective potential. These calculations have revealed that the translational symmetry broken at the classical level cannot be restored at the quantum level, and the effective potential does not possess a translationally invariant ground state.

In this chapter we calculate the GEP of the Liouville model. It was shown earlier that the GEP formalism works well for ϕ^4, ϕ^6 and Sine-Gordon models [27,93,94]. Interestingly enough, the effective potential obtained for the Liouville model here is exactly identical to the Goldstone form. We also calculate the finite temperature GEP and find that the finite temperature corrections do not restore the translational invariance broken at zero temperature.

2.2 GEP at zero temperature

The Liouville theory is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{\beta^2}\, e^{\beta\phi} \qquad (2.1)$$

where β is a real positive constant.

The Hamiltonian density corresponding to the Lagrangian is

$$\mathcal{H} = \frac{1}{2}\,\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{\beta^2}\, e^{\beta\phi} \qquad (2.2)$$

To calculate the GEP, we use the procedure given in chapter 1.

The ground state expectation value of each term in the Hamiltonian density is obtained as follows:

$$ {}_{\phi_0,\Omega}\langle 0|\tfrac{1}{2}(\dot{\phi}^2+(\nabla\phi)^2)|0\rangle_{\phi_0,\Omega} = \int(dk)_\Omega\,[\omega_{\underline{k}}^{\,2}(\Omega) - \tfrac{1}{2}\Omega^2] \qquad (2.3) $$

The potential energy density term is expanded:

$$ e^{\beta\phi} = e^{\beta(\phi_0+\hat{\phi})} = e^{\beta\phi_0}\,(1+\beta\hat{\phi}+\frac{(\beta\hat{\phi})^2}{2!}+\ldots) \qquad (2.4) $$

where $\hat{\phi} = \int(dk)_\Omega\,[a_\Omega(k)e^{-ik.x} + a^+_\Omega(k)e^{ik.x}]$

Define the integral $I_N(\Omega)$ according to (1.37). Then the ground state expectation value of $\hat{\phi}^{2N}$ (N a positive integer) is given by [93]

$$ {}_{\phi_0,\Omega}\langle 0|\hat{\phi}^{2N}|0\rangle_{\phi_0,\Omega} = \frac{(2N)!}{2^N\,N!}[I_0(\Omega)]^N \qquad (2.5) $$

Applying this result we have computed the following expectation values:

$$ \langle 0|e^{\beta\phi_0}|0\rangle = e^{\beta\phi_0} \qquad (2.6) $$

$$ \langle 0|e^{\beta\phi_0}\frac{\beta^2\hat{\phi}^2}{2!}|0\rangle = e^{\beta\phi_0}\,\frac{I_0(\Omega)}{2}\beta^2 \qquad (2.7) $$

$$ \langle 0|e^{\beta\phi_0}\frac{\beta^4\hat{\phi}^4}{4!}|0\rangle = e^{\beta\phi_0}(\frac{I_0(\Omega)}{2})^2\,\frac{\beta^4}{2!} \qquad (2.8) $$

$$ \langle 0|e^{\beta\phi_0}\frac{\beta^6\hat{\phi}^6|0\rangle}{6!} = e^{\beta\phi_0}\,(\frac{I_0(\Omega)}{2})^3\,\frac{1}{3!}\,\beta^6 \qquad (2.9) $$

Combining relations of the above type,

$$\langle 0|e^{\beta(\phi_o+\hat{\phi})}|0\rangle = e^{\beta\phi_o}\left[1+\frac{I_o(\Omega)\beta^2}{2} + \left(\frac{I_o(\Omega)}{2}\right)^2 \frac{\beta^4}{2!} + \left(\frac{I_o(\Omega)}{2}\right)^3 \frac{\beta^6}{3!}+\ldots\right] = e^{\beta\phi_o}\, e^{\beta^2 I_o(\Omega)/2} \tag{2.10}$$

The odd powers of $\hat{\phi}$ will not contribute to the expectation value [93]. The ground state expectation value of $\mathcal{H}$ is found using the results (2.3) and (2.6) to (2.10).

$$V_G(\phi_o,\Omega) = I_1 - \frac{1}{2}\Omega^2 I_o + \frac{m^2}{\beta^2} e^{\beta\phi_o} e^{\frac{\beta^2 I_o}{2}} \tag{2.11}$$

Using the formal result $\frac{dI_N}{d\Omega} = (2N-1)\Omega\, I_{N-1}$ (2.12)

and minimizing $V_G(\phi_o,\Omega)$ with respect to Ω, we have

$$\frac{dV_G}{d\Omega} = \frac{dI_1}{d\Omega} - \Omega I_o - \frac{1}{2}\Omega^2 \frac{dI_o(\Omega)}{d\Omega} + \frac{m^2}{\beta^2} e^{\beta\phi_o} e^{\beta^2 I_o|2} \frac{\beta^2}{2}\frac{dI_o}{d\Omega}$$

$$= \frac{1}{2}\Omega^2 \Omega I_{-1}(\Omega) + \frac{m^2}{2} e^{\beta\phi_o} e^{\beta^2 I_o/2} (-\Omega I_{-1}(\Omega)) \tag{2.13}$$

The optimal mass parameter $\bar{\Omega}$ is determined in the form;

$$\bar{\Omega}^2 = m^2 e^{\beta\phi_o} e^{\frac{\beta^2 I_o(\bar{\Omega})}{2}} \tag{2.14}$$

The derivative with respect to ϕ_o is

$$\frac{d\bar{V}_G}{d\phi_o} = \frac{\partial V_G}{\partial\phi_o}\Big|_{\Omega=\bar{\Omega}} = \frac{m^2}{\beta} e^{\beta\phi_o} e^{\frac{\beta^2 I_o(\bar{\Omega})}{2}} = \bar{\Omega}^2/\beta \tag{2.15}$$

Let $\bar{\Omega}_o$ be the solution to the $\bar{\Omega}$ equation at ϕ_o=o. Then

$$\bar{\Omega}_o^2 = m^2 e^{\beta^2 I_o(\bar{\Omega}_o)|2} \tag{2.16}$$

$$m^2 = \bar{\Omega}_o^2 e^{\frac{-\beta^2 I_o(\bar{\Omega}_o)}{2}} \tag{2.17}$$

$$\bar{\Omega}^2 = \bar{\Omega}_0^{\,2}\, e^{\beta\phi_o} \exp\left[\tfrac{1}{2}\beta^2\left(I_0(\bar{\Omega}) - I_0(\bar{\Omega}_0)\right)\right] \tag{2.18}$$

In 1+1 dimensions we have by [27]

$$I_0(\Omega) - I_0(\Omega_0) = -\frac{1}{4\pi}\ln\frac{\Omega^2}{\Omega_0^2} \tag{2.19}$$

Thus

$$\bar{\Omega}^2 = \bar{\Omega}_0^{\,2}\exp\left(\frac{\beta}{1+\beta^2/8\pi}\,\phi_o\right) \tag{2.20}$$

$\bar{V}_G$ can be obtained from $\frac{d\bar{V}_G}{d\phi_o}$:

$$\bar{V}_G = \int \frac{\bar{\Omega}_0^{\,2}}{\beta}\exp\left(\frac{\beta}{1+\beta^2/8\pi}\,\phi_o\right)d\phi_o \tag{2.21}$$

$$= \frac{\bar{\Omega}_0^{\,2}}{\beta^2}\left(1+\frac{\beta^2}{8\pi}\right)\exp\left(\frac{\beta}{1+\beta^2/8\pi}\,\phi_o\right) \tag{2.22}$$

The constant of integration is not included here as it is the usual divergent vacuum energy constant which can be subtracted out to obtain a finite result.

The presence of interaction shifts the mass from the bare value m to the renormalized value m_R. Renormalization takes place simply because of the presence of interaction and has nothing to do, a priori, with infinite quantities [95].

Here in the Liouville model infinite quantities do however arise in renormalizing the theory. The renormalized mass m_R is taken to be the true physical mass.

This renormalized mass is defined as

$$m_R^2 = \frac{d^2\bar{V}_G}{d\phi_o^2}\Big|_{\phi_o=0} \qquad (2.23)$$

$$\frac{d^2\bar{V}_G}{d\phi_o^2}\Big|_{\phi_o=o} = \frac{\bar{\Omega}^2_o}{1+\beta^2/8\pi} \qquad (2.24)$$

Now $\bar{V}_G$ can be rewritten in terms of m_R as

$$\bar{V}_G(\phi_o) = \frac{m_R^2}{\beta^2}(1+\beta^2/8\pi)^2 \exp\frac{\beta\phi_o}{1+\beta^2/8\pi} \qquad (2.25)$$

$$\text{Let } \tilde{\beta} = \frac{\beta}{1+\beta^2/8\pi} \qquad (2.26)$$

$$\bar{V}_G(\phi_o) = \left(\frac{m_R^2}{\tilde{\beta}^2}\right) e^{\tilde{\beta}\phi_o} \qquad (2.27)$$

This expression for the GEP of the Liouville model bears close resemblance to one loop effective potential obtained by Goldstone [91].

The quantum equation of motion for the Liouville field is

$$\Box\phi + \frac{m^2}{\beta} e^{\beta\phi} = 0 \qquad (2.28)$$

If the theory possesses a translationally invariant normalizable ground state $|o\rangle$ then

$\langle o|\,\Box\phi|o\rangle = o$ so that

$$\frac{m^2}{\beta}\langle o|e^{\beta\phi}|o\rangle = 0 \qquad (2.29)$$

which violates the formal positivity of the exponential. This suggests that no translationally invariant ground state exists.

This can also be confirmed by using the effective potential herein evaluated using the GEP method. The expression for $\overline{V}_G(\phi_o)$ shows that the effective potential has no minimum except at $\phi = -\infty$. Hence we conclude that the energy spectrum is bounded from below by an unattained vanishing greatest lower bound. Or in other words, the ground state is not attained by the system.

2.3 Liouville theory at finite temperature

To calculate the GEP at finite temperature we follow the imaginary time approach [39]. Here we shall write the $I_N(\Omega)$ integrals in a covariant form and then using the periodic time prescription the required finite temperature integrals are evaluated [96].

These integrals can be reexpressed as covariant integrals over the (r+1) dimensional energy momentum space [27]:

$$I_1(\Omega) = \frac{-i}{2}\,\frac{1}{(2\pi)^{r+1}}\int d^{r+1}k \quad \ln(k^2 - \Omega^2) + \text{constant} \quad (2.30)$$

$$I_o(\Omega) = \frac{i}{(2\pi)^{r+1}}\int d^{r+1}k\,\frac{1}{k^2 - \Omega^2} \quad (2.31)$$

At finite temperature

$$I_1^{FT}(\Omega) = \frac{-i}{2}\int \frac{\ln}{(2\pi)^{r+1}}(k^2 - \Omega^2)d^{r+1}k = \frac{-i}{2(-i\beta)}\sum_{n=-\infty}^{+\infty}\int\frac{d^r k}{(2\pi)^r}\cdot$$

$$\ln\left(\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^{\,2} + \Omega^2\right) \quad (2.32)$$

In order to carry out the summation define [39]

$$v(E) = \sum_{n=-\infty}^{+\infty} \ln \left(\frac{4\pi^2 n^2}{\beta^2} + E^2\right) \tag{2.33}$$

with

$E^2 = \vec{k}^{\,2} + \Omega^2$

Differentiating with respect to E,

$$\frac{\partial v}{\partial E} = \sum_{n=-\infty}^{+\infty} \frac{2E}{\frac{4\pi^2 n^2}{\beta^2} + E^2} \tag{2.34}$$

Using the identity [97]

$$\sum_{n=1}^{\infty} \frac{y}{y^2+n^2} = \frac{-1}{2y} + \frac{1}{2}\pi \coth \pi y \tag{2.35}$$

$$= -\frac{1}{2y} + \frac{\pi}{2} + \frac{\pi\, e^{-2\pi y}}{1-e^{-2\pi y}} \tag{2.36}$$

with $y = \frac{\beta E}{2\pi}$ we obtain

$$\frac{\partial v}{\partial E} = 2\beta\left[\frac{1}{2} + \frac{e^{-\beta E}}{1-e^{-\beta E}}\right] \tag{2.37}$$

Doing the integration, we have

$$v = 2\beta\left[\frac{E}{2} + \frac{1}{\beta}\ln(1-e^{-\beta E})\right] + E \text{ independent terms} \tag{2.38}$$

Now

$$-\frac{i}{2}\int \frac{\ln(k^2-\Omega^2)}{(2\pi)^{r+1}} \qquad d^{r+1}k = \int \frac{d^r k}{(2\pi)^r}\left[\frac{E}{2} + \frac{1}{\beta}\ln(1-e^{-\beta E})\right] \tag{2.39}$$

To evaluate the second integral, with $(\beta\vec{k})^2 = x^2$, we have

$$\int \frac{d^r k}{(2\pi)^r} \frac{1}{\beta}\ln(1-e^{-\beta E})$$

$$= \frac{1}{2\pi^{r-1}\ \beta^{r+1}} \int_0^{\infty} x^{r-1} dx \ln\left(1-e^{-(x^2+\beta^2\Omega^2)^{1/2}}\right) \tag{2.40}$$

$$= I_1^{\,T}(\Omega) \tag{2.41}$$

Now the first integral in (2.39) is just the one loop effective potential at zero temperature [13,98,.5,14]. It can be proved as follows:

Using the identity

$$-\frac{i}{2}\frac{d}{dE}\int_{-\infty}^{+\infty}\frac{dk^o}{2\pi}\ln\left(-k_o^{\,2}+\vec{k}^2+\Omega^2-i\varepsilon\right)=\frac{1}{2}$$

We have

$$\int\frac{d^rk}{(2\pi)^r}\frac{E}{2}=\frac{-i}{2}\int\frac{d^{r+1}k}{(2\pi)^{r+1}}\ln(-k_o^{\,2}+\vec{k}^2+\Omega^2-i\varepsilon\ ) \tag{2.42}$$

$$=\frac{1}{2}\int\frac{d^{r+1}}{(2\pi)^{r+1}}\ln\left(k^2+\Omega^2-i\varepsilon\ \right) \tag{2.43}$$

$$=I_1(\Omega)$$

Hence we get

$$I_1^{FT}=I_1(\Omega)+I_1^{T}(\Omega) \tag{2.44}$$

Similarly, to evaluate $I_o^{FT}(\Omega)$:

$$\frac{i}{k^2-\Omega^2}=\frac{-i}{\frac{4\pi^2n^2}{\beta^2}+\vec{k}^{\,2}+\Omega^2} \tag{2.45}$$

$$I_o^{FT}=\frac{1}{\beta}\sum_n\int\frac{d^rk}{(2\pi)^r}\frac{1}{\frac{4\pi^2n^2}{\beta^2}+\vec{k}^2+\Omega^2} \tag{2.46}$$

Using the identities (2.35) and (2.36) we find

$$I_o^{FT}=\int\frac{d^rk}{(2\pi)^r}\left[\frac{1}{2E}+\frac{1}{E(e^{\beta E}-1)}\right]=I_o+I_o^{T} \tag{2.47}$$

The second term in the integral can be represented in reparameterized form:

$$I_0^T = \int \frac{x^{r-1}dx}{2\pi^{r-1}\beta^{r-1}} \quad \frac{1}{(x^2+\beta^2\Omega^2)^{1/2}\ \exp[(x^2+\beta^2\Omega^2)^{1/2}-1]} \qquad (2.48)$$

The results (2.44) and (2.47) are the same as those obtained from the standard thermodynamics [99].

Now, for the finite temperature case, (2.11) will take the form

$$V_G^T = I_1(\Omega) + I_1^T(\Omega) - \frac{1}{2}\Omega^2(I_0(\Omega) + I_0^T(\Omega)) + \frac{m^2}{\beta^2} e^{\beta\phi_0}\, e^{\frac{\beta^2}{2}(I_0(\Omega)+I_0^T(\Omega))} \qquad (2.49)$$

Minimizing V_G^T with respect to the variational parameter Ω, the finite temperature GEP is evaluated.

Using the relation

$$\frac{dI_N^T}{d\Omega} = (2N-1)\ I_{N-1}^T \qquad (2.50)$$

We have

$$\frac{dV_G}{d\Omega} = \frac{1}{2}\ \Omega^3(I_{-1}(\Omega) + I_{-1}^T(\Omega)) - \frac{m^2}{2}\Omega e^{\beta\phi_0} e^{\frac{\beta^2}{2}(I_0+I_0^T)}(I_{-1}(\Omega) + I_{-1}^T(\Omega)) \qquad (2.51)$$

Hence

$$\bar{\Omega}^2 = m^2\ e^{\beta\phi_0}\ \exp\left[\frac{\beta^2}{2}(I_0+I_0^T)\right] \qquad (2.52)$$

The FTGEP $\bar{V}_G^T(\phi_0)$ is obtained by proceeding as in the zero temperature case.

$$\frac{d\bar{V}_G^T}{d\phi_0} = \frac{\partial V_G^T}{\partial\phi_0} = \frac{m^2}{\beta}\ e^{\beta\phi_0}\ \exp\left\{\frac{1}{2}\ \beta^2(I_0+I_0^T)\right\} = \frac{\bar{\Omega}^2}{\beta} \qquad (2.53)$$

$$\bar{\Omega}_o^2 = \bar{\Omega}^2|_{\phi_o=o} = m^2 \exp\{\tfrac{1}{2}\beta^2(I_o(\bar{\Omega}_o) + I_o^T(\bar{\Omega}_o))\} \tag{2.54}$$

or

$$m^2 = \bar{\Omega}_o^2 \exp - \{\tfrac{1}{2}\beta^2(I_o(\bar{\Omega}_o) + I_o^T(\bar{\Omega}_o))\} \tag{2.55}$$

Furthermore, by (2.52)

$$\begin{aligned}\bar{\Omega}^2 = \bar{\Omega}_o^2 \exp \beta\phi_o \exp\{\tfrac{1}{2}\beta^2[I_o(\bar{\Omega}) - I_o(\bar{\Omega}_o)] \\ + [I_o^T(\bar{\Omega}) - I_o^T(\bar{\Omega}_o)]\}\end{aligned} \tag{2.56}$$

By (2.19) we have

$$\bar{\Omega}^2 = \bar{\Omega}_o^2 \exp\left(\frac{\beta\phi_o}{1+\beta^2/8\pi}\right) \exp\left(\frac{1}{2}\,\frac{\beta^2 \Delta I_o^T}{(1+\beta^2/8\pi)}\right) \tag{2.57}$$

$$\text{where } \Delta I_o^T = I_o^T(\bar{\Omega}) - I_o^T(\bar{\Omega}_o) \tag{2.58}$$

Now the FT GEP works out as

$$\begin{aligned}\bar{V}_G^T(\phi_o) &= \int \frac{d\bar{V}_G^T}{d\phi_o}\, d\phi_o \\ &= \frac{\bar{\Omega}_o^2}{\beta^2}(1+\beta^2/8\pi) \exp\left(\frac{\beta^2}{2(1+\beta^2/8\pi)}\Delta I_o^T\right) \exp\left(\frac{\beta}{1+\beta^2/8\pi}\phi_o\right)\end{aligned} \tag{2.59}$$

The constant of integration is temperature independent and can be subtracted out.

The renormalized mass at finite temperature is defined by the relation

$$\frac{d^2\bar{V}_G^T}{d\phi_o^2}\Big|_{\phi_o=o} = (m_R^T)^2 \tag{2.60}$$

This is obtained as

$$(m_R^T)^2 = \frac{\bar{\Omega}_o^2}{1+\beta^2/8\pi} \exp \frac{\beta^2}{2(1+\beta^2/8\pi)} \quad \Delta I_o^T \qquad (2.61)$$

The FTGEP of the Liouville model is finally expressed in the form

$$\bar{V}_G^T(\emptyset_o) = [\frac{(m_R^T)^2}{\tilde{\beta}^2}] \, e^{\tilde{\beta}\emptyset_o} \qquad (2.62)$$

This effective potential has the minimum at $\emptyset_o=-\infty$; which shows that even in the non-perturbative approach based on GEP, translational invariance remains broken at zero temperature and is not restored at finite temperature. This supports the idea that the breaking of translational symmetry is fundamental to the model both at classical and quantum levels and at all temperatures.

III. SUPERSYMMETRIC LIOUVILLE MODEL

3.1 Introduction

Supersymmetry is a rich theoretical concept which allows one to mix bosons and fermions in the same multiplet which have relevance for particle unification schemes. If supersymmetry is recognized by nature, then the study of finite temperature supersymmetry grand unification theories must provide some insight into the early universe scenario.

For ordinary symmetries at low or zero temperatures, if the symmetry is spontaneously broken, the effective potential has a structure of the kind as shown in Fig.III.1.

In general, an infinite number of degenerate minima occur at $\emptyset \neq o$. As the temperature is raised, the energy increases, the vacuum become symmetric and the Goldstone bosons associated with breaking of continuous symmetries would become massive.

But for the supersymmetry we expect the minimum of the system at $\emptyset = o$. If the supersymmetry is spontaneously broken the expectation value $\langle o|H|o\rangle \neq o$ so that there is a non-zero minimum.

If supersymmetry plays a role in nature it certainly

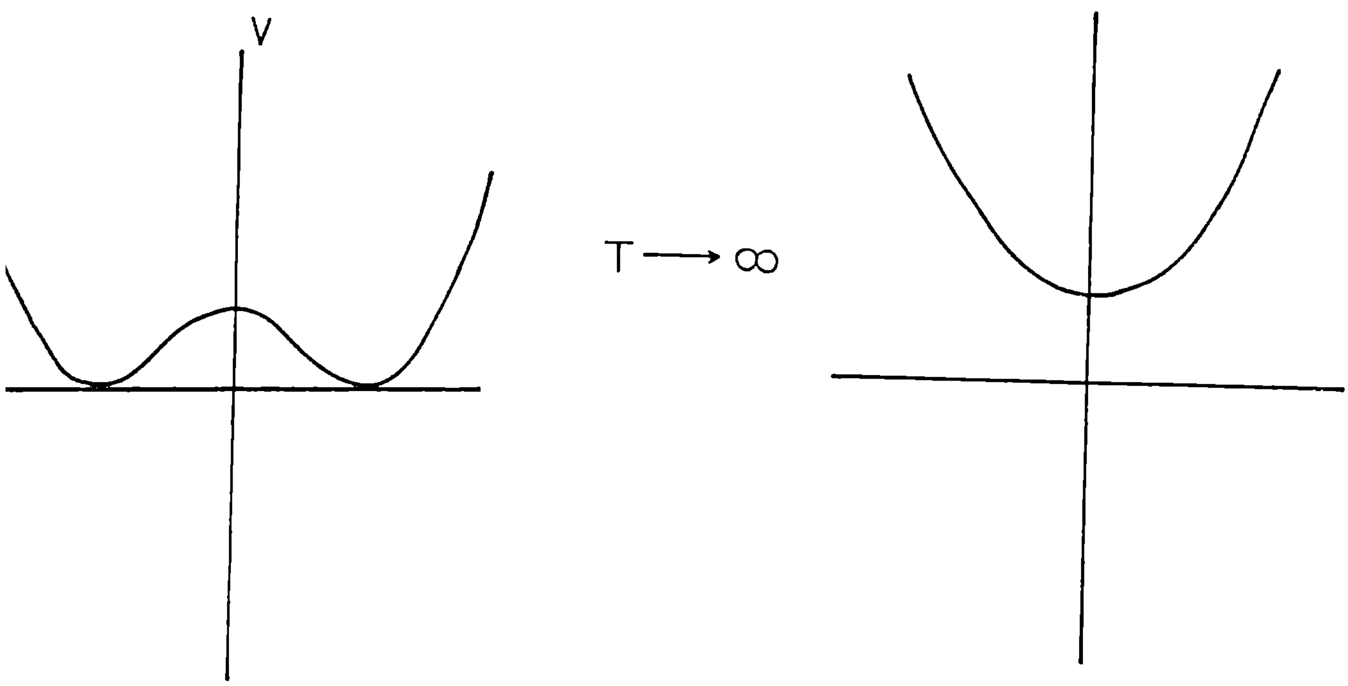


Fig. III.1 (a) shows shape of effective potential at low temperatures

(b) shows shape of effective potential at high temperature is spontaneously broken, because we do not observe degenerate Bose-Fermi multiplets.

A relativistic model of particle physics based on supersymmetry might be a model in which supersymmetry is spontaneously broken at the tree level. The conditions under which supersymmetry is spontaneously broken at the tree level are well understood. On the otherhand a realistic description of particle physics might require a model in which supersymmetry is unbroken at the tree level but broken dynamically by the quantum corrections. Supersymmetry is unbroken if and only if the energy of the vacuum is exactly zero. Even if the vacuum energy appears to be zero in some approximation, tiny corrections that have been neglected may cause the energy to be small but non-zero.

In this chapter we report the computation of the GEP of the supersymmetric Liouville model. This is a theory which sums up fermionic surfaces in string dynamics and is described by the supersymmetric Liouville equation. Polyakov [100] has shown that the proper quantization of the dynamics of the surface spanned by the superstring leads to a supersymmetric Liouville theory for space-time dimension $D<10$.

3.2 GEP at zero temperature

The Lagrangian density describing the supersymmetric Liouville model is

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \phi\, \partial^\mu \phi + \frac{i}{2}\, \bar{\psi} \partial\!\!\!/ \psi - \frac{m^2}{\beta^2}\, e^{\beta\phi} - \frac{m}{2\sqrt{2}}\, e^{\beta\phi/2}\, \bar{\psi}\psi \qquad (3.1)$$

where ϕ and ψ respectively represent a scalar field and a Dirac field in 1+1 dimensions. This model is invariant under supersymmetric transformations [101]. The corresponding Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\, \dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{i}{2}\, \bar{\psi}\, \partial\!\!\!/ \psi + \frac{m^2}{\beta^2}\, e^{\beta\phi} + \frac{m}{2\sqrt{2}}\, e^{\beta\phi/2} \bar{\psi}\psi \qquad (3.2)$$

The expectation value of pure bosonic terms is given by (2.11) of the preceding chapter.

To calculate the GEP for the remaining part of the Lagrangian, we write the fermion field as a free field of variable mass M [102]

$$\psi = \int (dk)_M \sum_\lambda [u_M^\lambda(k)\, b_M(k,\lambda)\, e^{-ik\cdot x} + v_M^\lambda(k)\, d_M^+(k,\lambda) e^{ik\cdot x}] \qquad (3.3)$$

where in r+1 dimensions

$$(dk)_M = \frac{d^r k}{(2\pi)^r 2\omega_k(M)} \qquad (3.4)$$

$$\omega_k(M) = (k^2+M^2)^{1/2} \qquad (3.5)$$

The spinors are normalized to 2M and the b, b^+ and d, d^+ operators obey the usual anticommutation relations; λ is the helicity label; the trial vacuum state $|o\rangle$ is the state annihilated by the b_M and d_M operators as well as by the boson annihilation operator a_Ω. The wave functional $|o\rangle$ is assumed to depend on M, Ω and the boson field shift ϕ_0.

The suggestion to include a shift in the fermion field $\psi = \psi_o + \hat{\psi}$ [19] will leave the spinor ψ with a nonzero expectation and violate Lorentz invariance.

Straightforward calculation of the matrix elements gives

$$\langle o|\tfrac{i}{2}\bar{\psi}\not{\partial}\psi|o\rangle = -2(I_1' - M^2 I_o') \tag{3.6}$$

$$\langle o|\bar{\psi}\psi|o\rangle = -2I_o' M \tag{3.7}$$

$$\langle o|\ \frac{m}{2\sqrt{2}}\, e^{\beta\phi_o/2}\bar{\psi}\psi|o\rangle = \frac{-m}{\sqrt{2}}\, e^{\beta\phi_o/2}\, e^{I_o(\Omega)\beta^2/8}\, I_o' M \tag{3.8}$$

The $I_n(\Omega)$ is given by (1.37) and

$$I_n(M) = I_n' \tag{3.9}$$

The ground state expectation value of the total Hamiltonian is then

$$\begin{aligned} V_G(\phi_o,\Omega,M) = {} & I_1 - \tfrac{1}{2}\Omega^2 I_o + \frac{m^2}{\beta^2}\, e^{\beta\phi_o}\, e^{\beta^2 I_o(\Omega)/2} \\ & - 2(I_1' - M^2 I_o') - \frac{m}{\sqrt{2}}\, e^{\beta\phi_o/2}\, e^{I_o(\Omega)\beta^2/8}\, I_o' M \end{aligned} \tag{3.10}$$

Differentiation of (3.10) yields the optimum values of M and Ω. $\bar{M}$ is obtained from the equation

$$\frac{\partial V_G}{\partial M} = o \tag{3.11}$$

$$\bar{M} = \frac{m}{2\sqrt{2}}\, e^{\beta\phi_o/2}\, e^{I_o(\Omega)\beta^2/8} \tag{3.12}$$

The optimal boson field mass parameter Ω satisfying the relation

$$\left(\frac{\partial V_G}{\partial \Omega}\right)_{\Omega=\bar{\Omega}} = o \tag{3.13}$$

is given by

$$\bar{\Omega}^2 = m^2 e^{\beta\phi_o} e^{\frac{\beta^2 I_o(\bar{\Omega})}{2}} - \frac{m\beta^2}{4\sqrt{2}} e^{\beta\phi_o/2} e^{(I_o(\bar{\Omega})\beta^2/8)} I_o'\bar{M} \quad (3.14)$$

Re-expressing $\bar{\Omega}^2$ in terms of the optimal fermion mass parameter $\bar{M}$,

$$\bar{\Omega}^2 = 8\bar{M}^2 e^{\frac{\beta^2 I_o(\bar{\Omega})}{4}} - \frac{\bar{M}^2\beta^2}{2} I_o'(\bar{M}) \quad (3.15)$$

At $\phi_o = o$, $\bar{\Omega}_o$ is the solution to $\bar{\Omega}$ equation

$$\bar{\Omega}^2_o = m^2 e^{\frac{\beta^2 I_o(\bar{\Omega}_o)}{2}} - \frac{m\beta^2}{4\sqrt{2}} e^{\frac{\beta^2 I_o(\bar{\Omega}_o)}{8}} I_o'(\bar{M}_o)\bar{M}_o \quad (3.16)$$

where $\bar{M}_o = (\bar{M})_{\phi_o=o}$ given by

$$\bar{M}_o = \frac{m}{2\sqrt{2}} e^{\frac{\beta^2}{8} I_o(\bar{\Omega}_o)} \quad (3.17)$$

For I_o' we use cut-off like renormalization [103]

$$I_o' = \frac{1}{4\pi} \ln \frac{4\Lambda^2}{M^2} \quad (3.18)$$

Now

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = \frac{8\bar{M}^2 e^{\frac{\beta^2 I_o(\bar{\Omega})}{4}} - \bar{M}^2 \frac{\beta^2}{2} I_o'(\bar{M})}{8\bar{M}_o^2 e^{\frac{\beta^2 I_o(\bar{\Omega}_o)}{4}} - \bar{M}_o^2 \frac{\beta^2}{2} I_o'(\bar{M}_o)} \quad (3.19)$$

Expressing $\bar{M}^2$ in terms of $\bar{M}_o^2$

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = \frac{8\bar{M}_o^2 e^{\beta\phi_o} e^{\beta^2[\frac{I_o(\bar{\Omega})-I_o(\bar{\Omega}_o)}{4}]} e^{\beta^2 \frac{I_o(\bar{\Omega})}{4}}}{8\bar{M}_o^2 e^{\frac{\beta^2 I_o(\bar{\Omega}_o)}{4}}} \left[\frac{1 - \frac{\beta^2}{16} I_o'(\bar{M}) e^{-\frac{\beta^2}{4} I_o(\bar{\Omega})}}{1 - \frac{\beta^2}{16} I_o'(\bar{M}_o) e^{-\frac{\beta^2}{4} I_o(\bar{\Omega}_o)}} \right] \quad (3.20)$$

or

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\beta\phi_o} e^{\frac{\beta^2}{4}(I_o(\bar{\Omega})-I_o(\bar{\Omega}_o))} \frac{I_o'(\bar{M})}{I_o'(\bar{M}_o)} \tag{3.21}$$

In the preceding step we have assumed that the term $\frac{\beta^2}{16} I_o'(\bar{M}) e^{-\frac{\beta^2}{4} I_o(\bar{\Omega})}$ is sufficiently large compared to unity.

With the help of (3.18) we rewrite the ratio as

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\beta\phi_o} e^{\frac{\beta^2}{4}(I_o(\bar{\Omega})-I_o(\bar{\Omega}_o))} \frac{\ln(4\Lambda^2/\bar{M}^2)}{\ln(4\Lambda^2/\bar{M}_o^2)} \tag{3.22}$$

which is obtained as

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\beta\phi_o} e^{\frac{\beta^2}{4}(I_o(\bar{\Omega})-I_o(\bar{\Omega}_o))} \frac{[\ln \frac{4\Lambda^2}{\bar{M}_o^2} - \beta\phi_o - \frac{\beta^2}{4}(I_o(\bar{\Omega})-I_o(\bar{\Omega}_o))]}{\ln \frac{4\Lambda^2}{\bar{M}_o^2}} \tag{3.23}$$

Using (2.19) we have

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\beta\phi_o} \left(\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2}\right)^{-\frac{\beta^2}{16\pi}} \frac{[\ln \frac{4\Lambda^2}{\bar{M}_o^2} - \beta\phi_o + \frac{\beta^2}{16\pi} \ln \frac{\bar{\Omega}^2}{\bar{\Omega}_o^2}]}{\ln[\frac{4\Lambda^2}{\bar{M}_o^2}]} \tag{3.24}$$

In the weak coupling limit $\beta \ll 1$, and we approximate

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\beta\phi_o} \left(\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2}\right)^{-\frac{\beta^2}{16\pi}} \left[1 - \frac{\beta\phi_o}{\ln \frac{4\Lambda^2}{\bar{M}_o^2}}\right] \tag{3.25}$$

Rearranging, we have

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\frac{\beta\phi_o}{1+\beta^2/16\pi}} \left[1 - \frac{\beta\phi_o}{\ln\frac{4\Lambda^2}{\bar{M}_o^2}}\right]^{\frac{1}{1+\beta^2/16\pi}} \tag{3.26}$$

To find $\bar{V}_G(\phi_o)$

$$\frac{d\bar{V}_G}{d\phi_o} = \frac{\partial V_G}{\partial \phi_o} = \frac{m^2}{\beta} e^{\beta\phi_o} e^{\frac{\beta^2 I_o(\bar{\Omega})}{2}} - \frac{m\beta}{2\sqrt{2}} e^{I_o\beta^2/8} I_o' \bar{M} e^{\beta\phi_o/2} \tag{3.27}$$

This can also be written in terms of $\bar{M}$ and also in terms of $\bar{\Omega}$.

$$\frac{d\bar{V}_G}{d\phi_o} = \frac{8\bar{M}^2 e^{\frac{\beta^2 I_o(\bar{\Omega})}{4}}}{\beta} - \bar{M}^2\beta\, I_o' \tag{3.28}$$

or

$$\frac{d\bar{V}_G}{d\phi_o} = \frac{\bar{\Omega}^2}{\beta} - \frac{\bar{M}^2}{2}\beta\, I_o' \tag{3.29}$$

Expressing $\bar{\Omega}^2$ and $\bar{M}^2$ in terms of $\bar{\Omega}_o^2$ and $\bar{M}_o^2$ and also replacing I_o' by the expression (3.18), we have

$$\frac{d\bar{V}_G}{d\phi_o} = \exp\left(\frac{\beta\phi_o}{1+\beta^2/16\pi}\right)\left(1 - \frac{\beta\phi_o}{\ln\frac{4\Lambda^2}{\bar{M}_o^2}}\right)^{\frac{1}{1+\beta^2/16\pi}} \left[\frac{\bar{\Omega}_o^2}{\beta} - \frac{\bar{M}_o^2\beta}{8\pi}\ln\frac{4\Lambda^2}{\bar{M}_o^2}\right] \tag{3.30}$$

Integrating with respect to ϕ_o, we have the expression for $\bar{V}_G$. For the weak coupling, $\beta<<1$,

$$\bar{V}_G(\phi_o) = \int \exp\frac{\beta\phi_o}{1+\beta^2/16\pi}\left(1 - \frac{\beta\phi_o}{(1+\beta^2/16\pi)\ln\frac{4\Lambda^2}{\bar{M}_o^2}}\right)\left(\frac{\bar{\Omega}_o^2}{\beta} - \frac{\bar{M}_o^2\beta}{8\pi}\ln\frac{4\Lambda^2}{\bar{M}_o^2}\right) d\phi_o \tag{3.31}$$

$$= \left(\frac{\bar{\Omega}_o^2}{\beta^2} - \frac{\bar{M}_o^2}{8\pi} \ln \frac{4\Lambda^2}{\bar{M}_o^2}\right) \beta\bar{\beta}^{-1} \exp \bar{\beta}\, \phi_o$$

$$\left[1 - \frac{\bar{\beta}\phi_o}{\ln \frac{4\Lambda^2}{\bar{M}_o^2}} + \frac{1}{\ln \frac{4\Lambda^2}{\bar{M}_o^2}}\right] \tag{3.32}$$

where

$$\bar{\beta} = \frac{\beta}{1+\beta^2/16\pi} \tag{3.33}$$

The constant of integration is not included here as it is the usual divergent vacuum energy constant which may be subtracted out to obtain a finite result.

In the approximated form $\bar{V}_G$ is

$$\bar{V}_G = \exp \bar{\beta}\phi_o\, \beta\, \bar{\beta}^{-1} \left\{ \left[\frac{\bar{\Omega}_o^2}{\beta^2}\left(1+ \frac{\bar{\beta}\phi_o - 1}{\frac{\bar{M}_o^2}{4\Lambda^2} - 1}\right)\right] + \frac{\bar{M}_o^4}{32\pi\Lambda^2} - \frac{2\bar{M}_o^2}{8\pi} + \frac{\bar{M}_o^2}{8\pi}\bar{\beta}\phi_o \right\} \tag{3.34}$$

The renormalized boson mass is found according to the expression (2.23)

$$m_R^2 = \frac{d^2\bar{V}_G}{d\phi_o^2}\bigg|_{\phi_o=0} = \frac{\bar{\Omega}_o^2}{\beta^2}\bar{\beta}\beta\left[1+ \frac{1}{\left(\frac{\bar{M}_o^2}{4\Lambda^2} - 1\right)}\right] + \frac{\bar{M}_o^4}{32\pi\Lambda^2}\beta\bar{\beta} \tag{3.35}$$

Finally in terms of the renormalized mass, the effective potential reads

$$\bar{V}_G = \frac{m_R^2}{\bar{\beta}^2} \exp \bar{\beta}\phi_o + \left\{ \beta\bar{\beta}^{-1} \exp \bar{\beta}\phi_o(\bar{\beta}\phi_o - 2) \left(\frac{\bar{\Omega}_o^2}{\beta^2\left(\frac{\bar{M}_o^2}{4\Lambda^2} - 1\right)} + \frac{\bar{M}_o^2}{8\pi} \right) \right\} \qquad (3.36)$$

The above expression for effective potential shows that the supersymmetric Liouville model does not possess a translationally invariant ground state, a situation familiar from the ordinary Liouville theory. The one loop calculations [101] give a similar result as

$$V_{eff}(1 \text{ loop}) = \frac{\mu^2}{\beta^2} e^{\beta\phi}[1 + \frac{\hbar\beta^2}{16\pi} (1 - \beta\phi + \ln 2)] \qquad (3.37)$$

3.3 Finite temperature GEP

Here we extend GEP method for supersymmetric Liouville theory to finite temperatures.

The integrals $I_o^{'FT}$ and $I_1^{'FT}$ for the fermionic terms can be obtained using steps similar to those of preceding chapter. The divergent integrals $I_o^{'FT}$ and $I_1^{'FT}$ are expressed in the following forms:

$$I_o^{'FT} = \int \frac{dk}{2\pi} \frac{1}{2(k^2+M^2)^{1/2}} + \frac{1}{2} \int \frac{dk}{2\pi(k^2+M^2)^{1/2}} \frac{1}{\exp \frac{(k^2+M^2)^{1/2}}{T} + 1}$$

$$= I_o'(M) + I_o^{'T}(M) \qquad (3.38)$$

and

$$I_1'^{FT} = \int \frac{dk}{2\pi}\Big[\frac{(k^2+M^2)^{1/2}}{2} + T\ \ln(1+\exp^{-\frac{(k^2+M^2)^{1/2}}{T}})\Big]$$
$$= I_1' + I_1'^T \tag{3.39}$$

where the index T denotes the temperature dependence. For the finite temperature case, the expectation value of the Hamiltonian takes the form;

$$\begin{aligned} V_G^T &= I_1(\Omega) + I_1^T(\Omega) - \frac{1}{2}\Omega^2(I_0(\Omega) + I_0^T(\Omega)) \\ &+ \frac{m^2}{\beta^2} e^{\beta\phi_0} e^{(\beta^2/2)(I_0(\Omega) + I_0^T(\Omega))} \\ &- \frac{m}{\sqrt{2}} e^{\beta\phi_0/2} e^{(\beta^2/8)\ (I_0(\Omega)+I_0^T(\Omega))} M(I_0'+I_0'^T) \\ &- 2\ \{(I_1' + I_1'^T) - M^2(I_0' + I_0'^T)\} \end{aligned} \tag{3.40}$$

Minimizing the above equation with respect to the fermionic mass parameter M we get

$$\bar{M} = \frac{m}{2\sqrt{2}}\ e^{\beta\phi_0/2}\ e^{(\beta^2/8)\ (I_0(\bar{\Omega}) + I_0^T(\bar{\Omega}))} \tag{3.41}$$

or

$$\bar{M} = \bar{M}_0\ \exp(\frac{\beta\phi_0}{2})\ \exp(\frac{\beta^2}{8}\ \Delta I_0^T)\ \exp\ \frac{\beta^2}{8}(I_0(\bar{\Omega})-I_0(\bar{\Omega}\ )) \tag{3.42}$$

ΔI_0^T stands for $I_0^T - I_0^T\ (\bar{\Omega}_0)$

The parameter $\bar{\Omega}$ in this case is

$$\bar{\Omega}^2 = 8\bar{M}^2\ \exp[(I_0(\bar{\Omega})+I_0^T(\bar{\Omega}))\frac{\beta^2}{4}] - \frac{\beta^2}{2}\bar{M}^2(I_0' + I_0'^T) \tag{3.43}$$

which can also be expressed as

$$\bar{\Omega}^2 = 8\bar{M}^2 \, e^{(\beta^2/4)(I_o + I_o^T)} - \frac{\bar{M}^2\beta^2}{2}(I_o'(\bar{M}) + I_o'^T(\bar{M})) \qquad (3.44)$$

Following the same procedure as in the case of zero temperature, we find

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\beta\phi_o} \, e^{(\beta^2/4)(I_o(\bar{\Omega}) - I_o(\bar{\Omega}_o))} \, e^{(\beta^2/4)\Delta I_o^T} \, \frac{I_o' + I_o'^T}{I_o'(\bar{M}_o) + I_o'^T(\bar{M}_o)}$$

$$(3.45)$$

The computation of $I_o'^T$ is given in ref. [39]

Here we quote the result

$$I_o'^T = -\frac{1}{8\pi} \ln \frac{M}{T\pi} - \frac{1}{8\pi}\gamma \qquad (3.46)$$

where γ is the Euler-Mascheroni constant.

Inserting this into (3.45), we have with (2.19)

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = e^{\beta\phi_o} \left(\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2}\right)^{\frac{-\beta^2}{16\pi}} e^{\frac{\beta^2}{4}\Delta I_o^T} \left[1 - \frac{\left(\frac{5\beta\phi_o}{4} + \frac{5\beta^2}{16}\Delta I_o^T - \frac{5\beta^2}{64\pi}\ln\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2}\right)}{\ln\frac{4\Lambda^2}{M_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma}\right]$$

$$(3.47)$$

In the weak coupling limit, $\beta \ll 1$, this is approximated as

$$\frac{\bar{\Omega}^2}{\bar{\Omega}_o^2} = \exp\frac{\beta}{1+\beta^2/16\pi} \, \exp\frac{\beta^2}{4(1+\beta^2/16\pi)}\Delta I_o^T \qquad (3.48)$$

$$\left[1 - \frac{\left(\frac{5\beta\phi_o}{4} + \frac{5}{16}\beta^2\Delta I_o^T\right)}{\ln\frac{4\Lambda^2}{M_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma}\right]^{\frac{1}{1+\beta^2/16\pi}}$$

To find $\bar{V}_G^T(\emptyset_o)$:

$$\frac{d\bar{V}_G^T}{d\emptyset_o} = \frac{\partial V_G^T}{\partial \emptyset_o} = \frac{m^2}{\beta}\, e^{\beta\emptyset_o}\, e^{\frac{\beta^2}{2}(I_o(\bar{\Omega})+I_o^T(\bar{\Omega}))} - \frac{m\beta}{2\sqrt{2}}\, e^{(\beta^2/8)(I_o(\bar{\Omega})+I_o^T(\bar{\Omega}))}(I_o'+I_o'^T)\bar{M}\, e^{\beta\emptyset_o/2} \qquad (3.49)$$

To express the above equation in finite form, we follow the similar technique as in the case of zero temperature((3.28) -(3.30)).

$$\frac{\partial V^T}{\partial \emptyset_o} = e^{\bar{\beta}\emptyset_o}\, e^{\hat{\beta}\,\Delta I_o^T}\left\{1 - \left[\frac{\left(\frac{5\beta\emptyset_o}{4} + \frac{5\beta^2}{16}\Delta I_o^T\right)}{\ln\frac{4\Lambda^2}{\bar{M}_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma}\right]\right\}^{\frac{1}{1+\beta^2/16\pi}} \qquad (3.50)$$

$$\left[\frac{\bar{\Omega}_o^2}{\beta} - \frac{\bar{M}_o^2\beta}{8\pi}\left(\ln\frac{4\Lambda^2}{\bar{M}_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma\right)\right]$$

Integrating this expression with respect to $\emptyset_o$, we get the GEP at finite temperature:

$$\bar{V}_G^T = \left[\frac{\bar{\Omega}_o^2}{\beta} - \frac{\bar{M}_o^2\beta}{8\pi}\left(\ln\frac{4\Lambda^2}{\bar{M}_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma\right)\right]\beta^{-1} \qquad (3.51)$$

$$\exp\bar{\beta}\emptyset_o\,\exp\hat{\beta}\,\Delta I_o^T\left[1 - \frac{\left(\frac{5}{4}\bar{\beta}\emptyset_o + \frac{5}{4}\hat{\beta}\,\Delta I_o^T - \frac{5}{4}\right)}{\ln\frac{4\Lambda^2}{\bar{M}_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma}\right]$$

where

$$\frac{\beta^2}{4\left(1+\frac{\beta^2}{16\pi}\right)} = \hat{\beta} \qquad (3.52)$$

On simplification this becomes

$$\bar{V}_G^T = \frac{\bar{\Omega}_o^2}{\beta^2}\left[1 + \frac{\bar{\beta}\phi_o + \hat{\beta}\,\Delta I_o^T - 1}{\dfrac{\bar{M}_o^2}{(4\Lambda^2)^{4/5}(T\pi)^{2/5}} - 1 + \frac{2}{5}\gamma}\right] + \frac{5\bar{M}_o^4}{32\pi(4\Lambda^2)^{4/5}(\pi T)^{2/5}} + \frac{\bar{M}_o^2\gamma}{16\pi} + \frac{5\bar{M}_o^2}{32\pi}(\hat{\beta}\,\Delta I_o^T + \bar{\beta}\phi_o - 2) \qquad (3.53)$$

The renormalized boson mass at finite temperature is defined by the relation (2.60);

$$m_R^{T^2} = \bar{\beta}\beta\left[\frac{\bar{\Omega}_o^2}{\beta^2} - \frac{\bar{M}_o^2}{8\pi}\left(\ln\frac{4\Lambda^2}{\bar{M}_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma\right)\right] e^{\hat{\beta}\,\Delta I_o^T}\left[1 - \frac{\left(\frac{5}{4}\hat{\beta}\,\Delta I_o^T + 5/4\right)}{\ln\frac{4\Lambda^2}{\bar{M}_o^2} + \frac{1}{2}\ln\frac{T\pi}{\bar{M}_o} - \frac{1}{2}\gamma}\right] \qquad (3.54)$$

With the approximated form as in the case of (3.53),

$$m_R^{T^2} = \beta\bar{\beta}\exp\hat{\beta}\,\Delta I_o^T\left\{\left[\frac{\bar{\Omega}_o^2}{\beta^2}\left(1 + \frac{1 + \hat{\beta}\,\Delta I_o^T}{\left(\dfrac{\bar{M}_o^2}{(4\Lambda^2)^{4/5}(\pi T)^{2/5}} - 1 + \frac{2}{5}\gamma\right)}\right)\right] + \frac{5\bar{M}_o^4}{32\pi(4\Lambda^2)^{4/5}(\pi T)^{2/5}} + \frac{\bar{M}_o^2}{16\pi}\gamma + \frac{5\bar{M}_o^2}{32\pi}\hat{\beta}\,\Delta I_o^T\right\} \qquad (3.55)$$

Re-expressing $\bar{V}_G^{\ T}$ in terms of $m_R^{T^2}$ we have

$$\bar{V}_G^{\ T} = \frac{m_R^{T^2}}{\bar{\beta}^2} \exp \bar{\beta}\emptyset_o + \left\{ (\bar{\beta}\emptyset_o - 2) \left(\frac{\bar{\Lambda}_o^2}{\beta^2 \left(\frac{\bar{M}_o^2}{(4\Lambda^2)^{4/5}(\pi T)^{2/5}} - 1 + \frac{2}{5}\gamma \right)} + \frac{5\bar{M}_o^2}{32\pi} \right) \right\}$$

$$\beta\bar{\beta}^{-1} \exp \bar{\beta}\emptyset_o \qquad (3.56)$$

This is of the form:

$$\bar{V}_G^{\ T} = \frac{(m_R^{\ T})^2}{\bar{\beta}^2} \exp \bar{\beta}\emptyset_o + \exp \bar{\beta}\emptyset_o [k(T)\ (\bar{\beta}\emptyset_o - 2)] \beta\bar{\beta}^{-1} \qquad (3.57)$$

where $k(T)$ stands for $\left(\frac{\bar{\Lambda}_o^2}{\beta^2 \left(\frac{\bar{M}_o^2}{(4\Lambda^2)^{4/5}(\pi T)^{2/5}} - 1 + \frac{2}{5}\gamma \right)} + \frac{5\bar{M}_o^2}{32\pi} \right)$

Here also the minimum of the potential occurs only at $\emptyset_o = -\infty$, and hence it is clear that the translational symmetry which is broken at zero temperature is not restored at high temperature. This behaviour is the same as that of one loop approximation for effective potential [104]. There the effective potential at finite temperature is

$$V_{eff}^{T}(1 \text{ loop}) = \frac{m_R^2}{\beta^2} e^{\tilde{\beta}\emptyset} + \frac{1}{4}MT - \frac{1}{3}MT^2 + \frac{1}{2} \ln \frac{M^2}{4\pi T^2} \qquad (3.58)$$

where m_R^2 is the renormalized mass at zero temperature.

This analysis based on nonperturbative approach leads to conclusions similar to those obtained in the nonsupersymmetric case, indicating that the appearance of the fermionic degrees of freedom have no significant effect on the core bosonic part.

IV. SECOND ORDER CORRECTIONS TO THE GEP FOR ϕ^6 MODEL AT ZERO TEMPERATURE

4.1 Lowest order GEP

The first order GEP for ϕ^6 model has been given earlier [9: Now in this chapter we give the second order GEP for ϕ^6 theory in 2+1 dimensions. The general formalism of higher order correctionsis given in chapter 1 ((1.38) through (1.54)).

Here we define the Lagrangian as

$$\mathcal{L} = (\mathcal{L}_o + \mathcal{L}_{int})_{\delta=1} \tag{4.1}$$

where $\mathcal{L}_o$ is the free field Lagrangian with mass Ω for the $\hat{\phi}$ field,

$$\mathcal{L}_o = \frac{1}{2}\hat{\phi}(x)\ (-\partial^2 + \Omega^2)\hat{\phi}(x) \tag{4.2}$$

where $\hat{\phi}(x)$ is defined by (1.44), and δ is an expansion parameter. The interaction Lagrangian is assumed to be of the form

$$\mathcal{L}_{int}[\hat{\phi}] = \delta(v_o + v_1\hat{\phi} + v_2\hat{\phi}^2 + v_3\hat{\phi}^3 + v_4\hat{\phi}^4 + v_5\hat{\phi}^5 + v_6\hat{\phi}^6) \tag{4.3}$$

where δ is an artificial expansion parameter which is introduced solely for facilitating corrections of various orders.

The coupling constants v_i are defined by the relations

$$\begin{aligned}
v_o &= \frac{1}{2} m_B^2\ \phi_o^2 + \lambda_B \phi_o^4 + \xi_B \phi_o^6 \\
v_1 &= (m_B^2 + 4\lambda_B\phi_o^2 + 6\xi_B\phi_o^4)\phi_o \\
v_2 &= \frac{1}{2}(m_B^2 - \Omega^2) + 6\lambda_B\phi_o^2 + 15\xi_B\phi_o^4 \\
v_3 &= 4\lambda_B\phi_o + 20\xi_B\phi_o^3
\end{aligned} \tag{4.4}$$

$v_4 = \lambda_B + 15\xi_B \phi_o^2$

$v_5 = 6\xi_B \phi_o, \quad v_6 = \xi_B$

It may be noted that these terms arise when $\phi \rightarrow \phi_o + \hat{\phi}$ in the expression $\frac{1}{2} m_B^2 \phi^2 + \lambda_B \phi^4 + \xi_B \phi^6$ which constitutes the ''potential part'' in the standard ϕ^6 - Lagrangian density [31].

The free action can be written as

$$\int_x \mathcal{L}_{o,x} = \int_x \int_y \frac{1}{2} \hat{\phi}(x)\, G^{-1}_{xy}\, \hat{\phi}_y \tag{4.5}$$

in which

$$G^{-1}_{xy} = (-\partial^2 + \Omega^2)\, \delta(x-y) \tag{4.6}$$

The inverse satisfying the condition

$$\int_y G^{-1}_{xy}\, G_{yz} = \delta(x-z) \tag{4.7}$$

is

$$G_{xy} = \int_p \frac{1}{p^2 + \Omega^2}\, e^{-ip(x-y)} \tag{4.8}$$

which is the x space propagator. Now (1.46) - (1.54) follow and we get the zeroth order contribution to W.

The generating functional for connected Green's functions given in (1.50) is reproduced here.

$$W[j,\phi_o] = j\phi_o - \mathcal{V} I_1 + \ln[(1 - \hat{\mathcal{L}}_{int} + \frac{1}{2}\hat{\mathcal{L}}^2_{int} + \ldots) \exp \frac{1}{2}(jGj)] \tag{4.9}$$

where j is the source function, $\mathcal{V}$ the space-time volume and

$I_1 = \frac{1}{2} \int_p \ln(p^2 + \Omega^2) dp$

$$\int_z v_o \exp(\tfrac{1}{2}jGj)_{j=o} = v_o \mathcal{V} \tag{4.10}$$

$$\int_z v_1\hat{\phi} \exp(\tfrac{1}{2}jGj) \equiv v_1 \frac{\delta}{\delta j} \exp \tfrac{1}{2}jGj = v_1\int_z(Gj)_z \tag{4.11}$$

$$\int_z v_2 \left(\frac{\delta}{\delta j}\right)^2 \exp(\tfrac{1}{2}jGj) = v_2 I_o \mathcal{V} \tag{4.12}$$

where

$$I_o = G_{xx} = \int_p \frac{1}{p^2+\Omega^2} \tag{4.13}$$

$$\int_z v_3\left(\frac{\delta}{\delta j}\right)^3 \exp(\tfrac{1}{2}jGj) = 3v_3 I_o \int_z(Gj)_z \tag{4.14}$$

$$\int_z v_4\left(\frac{\delta}{\delta j}\right)^4 \exp \tfrac{1}{2} jGj = 3I_o^2 \mathcal{V} \tag{4.15}$$

$$\int_z v_5\left(\frac{\delta}{\delta j}\right)^5 \exp \tfrac{1}{2} jGj = 15v_5 I_o^2 \int_z(Gj)_z \tag{4.16}$$

$$\int_z v_6\left(\frac{\delta}{\delta j}\right)^6 \exp \tfrac{1}{2} (jGj) = 15 I_o^3 v_6 \mathcal{V} \tag{4.17}$$

where

$$I_n(\Omega) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}2\omega_{\underline{k}}}(\omega_{\underline{k}}^2)^n, \quad \omega_{\underline{k}}^2 = k^2 + \Omega^2 \tag{4.18}$$

Inserting these expressions into (4.9), to first order in δ we have

$$\begin{aligned} W[j,\phi_o]_{(1)} = W|_{(o)} &- \delta[(v_o + v_2 I_o + 3v_4 I_o^2)\mathcal{V} + 15v_6 I_o^3 \\ &+ (v_1 + 3v_3 I_o + 15v_5{}^2 I_o)\int(Gj)_z \end{aligned} \tag{4.19}$$

where $W|_{(o)}$ is the zeroth order contribution.

The effective action is given by

$$\begin{aligned} \Gamma[\phi]_{\phi_c(x)=\phi} &= -\mathcal{V}I_1(\Omega) + \tfrac{1}{2} jGj \\ &- \delta[v_o + v_2 I_o + 3v_4 I_o^2 + 15v_6 I_o^3]\mathcal{V} + o(j) \end{aligned} \tag{4.20}$$

Since j vanishes in zeroth order, and so is $O(\delta)$ bv setting $\phi_c = \phi_o$ (according to (1.53)), it can contribute only to the $o(\delta^2)$ terms. Thus for first order calculations of Γ we can set j = o.

Hence we have

$$\Gamma(\phi_c)_{(1)} = -\nu[I_1(\Omega) + \delta(\tfrac{1}{2} m_B^2 \phi_c^{\ 2} + \lambda_B \phi_c^{\ 4} + (\tfrac{1}{2}(m_B^{\ 2} - \Omega^2) + 6\lambda_B \phi_c^2 + 15\zeta_B \phi_c^4) I_o + 3(\lambda_B + 15\zeta_B \phi_c^{\ 2}) I_o^{\ 2} + 15\zeta_B I_o^{\ 3})] \tag{4.21}$$

For δ=1, the above result is the ordinary GEP for ϕ^6 model [93].

.2 Second order correction

Here we have from (4.19)

$$(\phi_c)_y = (\frac{\delta W}{\delta j})_y = \phi_o + (Gj)_y - \delta(v_1 + 3v_3 I_o + 15 v_5 I_o^{\ 2}) \int_z G_{yz} \tag{4.22}$$

Multiplying throughout by G^{-1}_{xy}, with $\phi_y = \phi_c = \phi_o$, j_x is seen to be independent of x:

$$j = \delta(v_1 + 3v_3 I_o + 15\ I_o^{\ 2}) + O(\delta^2) \tag{4.23}$$

Next we have to compute

$$\frac{1}{2}\hat{\mathcal{L}}^2_{int} \ \exp \frac{1}{2} jGj = \frac{1}{2} \int_x \int_y \hat{\mathcal{L}}_{int,x} \hat{\mathcal{L}}_{int,y} \exp \frac{1}{2} jGj \Big|_{j=o} \tag{4.24}$$

Evaluating term by term we have

$$v_o^{\ 2} \exp \frac{1}{2}(jGj)\big|_{\delta=o} = v_o^{\ 2} \tag{4.25}$$

$$[v_1^{\ 2} (\frac{\delta}{\delta j})_x (\frac{\delta}{\delta j})_y \exp \frac{1}{2} jGj]_{j=o} = v_1^{\ 2}\ G_{xy} \tag{4.26}$$

$$[v_2^2 \left(\frac{\delta}{\delta j}\right)_x^2 \left(\frac{\delta}{\delta j}\right)_y^2 \exp \frac{1}{2} jGj]_{j=o} = v_2^2 G_{xx}G_{yy}+2v_2^2 \ G_{xy}^2$$
$$= v_2^2 I_o^2 +2v_2^2 \ G_{xy}^2 \tag{4.27}$$

$$[v_3^2 \left(\frac{\delta}{\delta j}\right)_x^3 \left(\frac{\delta}{\delta j}\right)_y^3 \exp \frac{1}{2} jGj]_{j=o} = [9I_o^2 G_{xy}+6G_{xy}^3]v_3^2 \tag{4.28}$$

$$[v_4^2 \left(\frac{\delta}{\delta j}\right)_x^4 \left(\frac{\delta}{\delta j}\right)_y^4 \exp \frac{1}{2} jGj]_{j=o} = v_4^2\{9I_o^4+24[3I_o^2(G_{xy})^2+(G_{xy})^4]\} \tag{4.29}$$

$$[v_5^2\left(-\frac{\delta}{\delta j}\right)_x^5 \left(\frac{\delta}{\delta j}\right)_y^5 \exp \frac{1}{2} jGj]_{j=o}$$
$$= 15 \ v_5^2 \ (15 \ I_o^4 G_{xy}+40 \ I_o^2 \ G_{xy}^3 + 8 \ G_{xy}^5) \tag{4.30}$$

$$[v_6^2\left(\frac{\delta}{\delta j}\right)_x^6\left(\frac{\delta}{\delta j}\right)_y^6 \exp \frac{1}{2}jGj]_{j=o} = v_6^2\{225 \ I_o^6 + 90 \ [45I_o^4$$
$$G_{xy}^2 + 60I_o^2 \ G_{xy}^4 + 8 \ G_{xy}^6 \ ]\} \tag{4.31}$$

Now computing the other terms we have

$$[v_o v_2\left(\frac{\delta}{\delta j}\right)_x^2 + v_2 v_o\left(\frac{\delta}{\delta j}\right)_y^2] \exp \frac{1}{2} jGj|_{j=o} = 2v_o v_2 I_o \tag{4.32}$$

$$2v_o v_4[\left(\frac{\delta}{\delta j}\right)_x^4 + \left(\frac{\delta}{\delta j}\right)_y^4] \exp \frac{1}{2} jGj|_{j=o} = 6v_o v_4 I_o^2 \tag{4.33}$$

$$2v_o v_6[\left(\frac{\delta}{\delta j}\right)_x^6 + \left(\frac{\delta}{\delta j}\right)_y^6] \exp \frac{1}{2} jGj|_{j=o} = 30v_o v_6 I_o^3 \tag{4.34}$$

$$v_2 v_4[\left(\frac{\delta}{\delta j}\right)_x^2 \left(\frac{\delta}{\delta j}\right)_y^4 + \left(\frac{\delta}{\delta j}\right)_y^2 \left(\frac{\delta}{\delta j}\right)_x^4]\exp \frac{1}{2}jGj|_{j=o}$$
$$= 6v_2 v_4 I_o^3 + 24v_2 v_4 I_o \ G_{xy}^2 \tag{4.35}$$

$$v_2 v_6[\left(\frac{\delta}{\delta j}\right)_x^2 \left(\frac{\delta}{\delta j}\right)_y^6 + \left(\frac{\delta}{\delta j}\right)_x^6 \left(\frac{\delta}{\delta j}\right)_y^2] \exp \frac{1}{2} jGj \,|_{j=o}$$
$$= 180 \ I_o^2 \ G_{xy}^2 + 30 \ I_o^4 \tag{4.36}$$

$$v_1 v_3 [(\frac{\delta}{\delta j})_x (\frac{\delta}{\delta j})_y^3 + (\frac{\delta}{\delta j})_x^3 (\frac{\delta}{\delta j})_y] \exp \frac{1}{2} jGj \Big|_{j=o}$$

$$= 6\, v_1 v_3 I_o\, G_{xy} \qquad (4.37)$$

$$v_1 v_5 [(\frac{\delta}{\delta j})_x (\frac{\delta}{\delta j})_y^5 + (\frac{\delta}{\delta j})_x^5 (\frac{\delta}{\delta j})_y] \exp \frac{1}{2} jGj \Big|_{j=o}$$

$$= 30\, v_1 v_5 I_o^2\, G_{xy} \qquad (4.38)$$

$$v_3 v_5 [(\frac{\delta}{\delta j})_x^3 (\frac{\delta}{\delta j})_y^5 + (\frac{\delta}{\delta j})_y^5 (\frac{\delta}{\delta j})_x^3] \exp \frac{1}{2} jGj \Big|_{j=o}$$

$$= v_3 v_5 (120\, I_o\, G_{xy}^3 + 90\, I_o^3 G_{xy}) \qquad (4.39)$$

$$v_4 v_6 [(\frac{\delta}{\delta j})_x^4 (\frac{\delta}{\delta j})_y^6 + (\frac{\delta}{\delta j})_x^6 (\frac{\delta}{\delta j})_y^4] \exp \frac{1}{2} jGj \Big|_{j=o}$$

$$= 90\, v_4 v_6 (12\, G_{xy}^2\, I_o^3 + 8\, I_o G_{xy}^4 + I_o^5) \qquad (4.40)$$

On summation we get

$$\frac{1}{2} \int_x \int_y \hat{\mathcal{L}}_{int,x} \hat{\mathcal{L}}_{int,y} \exp (\frac{1}{2} jGj)_{j=o}$$

$$= \frac{1}{2} \delta^2 \int_x \int_y [A^2 + v_1^2\, G_{xy} + 6 v_1 v_3 I_o G_{xy} + 30 v_1 v_5 I_o^2 G_{xy}$$
$$+ 2 v_2^2 G_{xy}^2 + 24 v_2 v_4 I_o G_{xy}^2 + 180\, v_2 v_6 I_o^2 G_{xy}^2$$
$$+ v_3^2 (9\, I_o^2 G_{xy} + 6 G_{xy}^3) + v_3 v_5\, 30 (4\, I_o G_{xy}^3 + 3\, I_o^3 G_{xy})$$
$$+ 24 v_4^2 (3\, I_o^2 G_{xy}^2 + G_{xy}^4) + 90\, v_4 v_6 (12\, G_{xy}^2 I_o^3 + 8\, I_o\, G_{xy}^4)$$
$$+ 15 v_5^2 (15\, I_o^4 G_{xy} + 40\, I_o^2\, G_{xy}^3 + 8\, G_{xy}^5)$$
$$+ 90 v_6^2 (45\, I_o^4\, G_{xy}^2 + 60\, I_o^2\, G_{xy}^4 + 8\, G_{xy}^6)] \qquad (4.41$$

where $A = v_o + v_2 I_o + 3 v_4 I_o^2 + 15 v_6 I_o^3$

Rearranging the terms we obtain the second order terms :

$$\frac{1}{2}\,\delta^2\nu^2A^2 + \frac{1}{2}\,\delta^2\,\nu\{(v_1+3v_3I_o+15v_5I_o^2)^2\ I^{(1)}$$
$$+\ (v_2+6v_4\ I_o\ +\ 45v_6\ I_o^2)^2\ I^{(2)}$$
$$+\ (v_3+10v_5I_o)^2\ I^{(3)}$$
$$+\ (v_4+15v_6I_o)^2\ I^{(4)}$$
$$+\ v_5^2\ I^{(5)}\ +\ v_6^2\ I^{(6)}\} \qquad (4.42)$$

where the $I^{(r)}$ integrals are

$$\frac{1}{1!}\ I^{(1)}(\Omega) = \int_y G_{xy} = \frac{1}{\Omega^2} \qquad (4.43)$$

$$\frac{1}{2!}\ I^{(2)}(\Omega) = \int_y G_{xy}^2 = \int_p \frac{1}{(p^2+\Omega^2)^2} \qquad (4.44)$$

$$\frac{1}{3!}\ I^{(3)}(\Omega) = \int_y G_{xy}^3 = \int_p\int_q \frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)\ [(p+q)^2+\Omega^2]} \qquad (4.45)$$

$$\frac{1}{4!}\ I^{(4)}(\Omega) = \int_p\int_q\int_k \frac{1}{(p^2+\Omega^2)\ (q^2+\Omega^2)(k^2+\Omega^2)[(p+q+k)^2+\Omega^2]} \qquad (4.46)$$

$$\frac{1}{5!}\ I^{(5)}(\Omega) = \int_p\int_q\int_k\int_\ell \frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)(k^2+\Omega^2)(\ell^2+\Omega^2)[(p+q+k+\ell)^2+\Omega^2]}$$

(4.47)

$$\frac{1}{6!}\ I^{(6)}(\Omega) =$$

$$\int_p\int_q\int_k\int_\ell\int_m \frac{1}{(p^2+\Omega^2)(q^2+\Omega^2)(k^2+\Omega^2)(\ell^2+\Omega^2)(m^2+\Omega^2)\ [(p+q+k+\ell+m)^2+\Omega^2]}$$

(4.48)

Combining with the result for the O(δ) case, we have

$$(1\ \hat{\mathcal{L}}_{int} + \frac{1}{2}\hat{\mathcal{L}}^2_{int} + \ldots)\ \exp\frac{1}{2}\ jGj$$
$$= \{1-\delta[\nu A+(v_1+3v_3I_o+15v_5I_o^2)\int_z(Gj)_z]$$
$$+\ \frac{1}{2}\delta^2\nu^2A^2 + \frac{1}{2}\delta^2\nu[(v_1+3v_3I_o+15v_5I_o^2)^2\ I^{(1)}$$

$$+ (v_2+6v_4I_o + 45v_6I_o^2)^2 \; I^{(2)}$$

$$+ (v_3+10v_5I_o)^2 \; I^{(3)} + (v_4+15v_6 \; I_o)^2 \; I^{(4)}$$

$$+ v_5^2 \; I^{(5)} + v_6^2 \; I^{(6)}]\} \quad \exp \tfrac{1}{2} \; jGj \tag{4.49}$$

Taking the logarithm and reexpanding in δ we have the $\delta^2\nu^2A^2$ terms cancel out, which represent the cancellation of disconnected diagrams. Substituting for j from (4.23) we find that the $(v_1+3v_3I_o+15v_5I_o^2)^2 \; I^{(1)}$ terms cancel out. The effective action to second order in δ is

$$\Gamma(\phi_c) = \Gamma_{(1)} + \tfrac{1}{2}\delta^2\nu[(v_2+6v_4I_o+45v_6I_o^2)^2 \; I^{(2)}$$

$$+ (v_3+10 \; v_5I_o)^2 \; I^{(3)} + (v_4+15v_6I_o)^2 \; I^{(4)}$$

$$+ v_5^2 \; I^{(5)} + v_6^2 \; I^{(6)}] \tag{4.50}$$

The effective potential to second order is

$$V^{(2)}(\phi_c,\Omega) = I_1(\Omega) + \delta \; \{\tfrac{1}{2}m_B^2 \; \phi_c^2 + \lambda_B\phi_c^4 + \xi_B\phi_c^6$$

$$+ \tfrac{1}{2} \; I_o(\Omega) \; [m_B^2-\Omega^2 + 12\lambda_B\phi_c^2 + 30\xi_B\phi_c^4]$$

$$+ 3(\lambda_B+15\xi_B\phi_c^2)I_o^2 + 15\xi_BI_o^3 \; \}$$

$$-\delta^2\{\tfrac{I^{(2)}}{8}[m_B^2-\Omega^2 + 12\lambda_B\phi_c^2 + 30\xi_B\phi_c^4$$

$$+ 12(\lambda_B+15\xi_B\phi_c^2) \; I_o + 90\xi_BI_o^2]^2$$

$$+ \tfrac{1}{2}[(4\lambda_B\phi_c + 20\xi_B\phi_c^3) + 60\xi_B\phi_c I_o]^2 \; I^{(3)}$$

$$+ \frac{1}{2}[\lambda_B + 15\xi_B\phi_c^2 + 15\xi_B I_o]^2 I^{(4)}$$

$$+ \frac{1}{2}(6\xi_B\phi_c)^2 I^{(5)} + \frac{1}{2}\xi_B^2 I^{(6)} \Big\} \qquad (4.51)$$

4.3 Renormalization

To renormalize $V^{(2)}(\phi_c, \Omega)$ we express the bare parameters m_B, λ_B and ξ_B as follows:

$$m_B^2 = m_{BG}^2 + \delta \Delta m_B^2 + O(\delta^2) \qquad (4.52)$$

$$\lambda_B = \lambda_{BG} + \delta \Delta\lambda_B + O(\delta^2) \qquad (4.53)$$

$$\xi_B = \xi_{BG} + \delta \Delta\xi_B \qquad (4.54)$$

where m_{BG}^2 and λ_{BG} correspond to the bare parameters obtained in the Gaussian approximation [93].

Now substituting in (4.51) we have

$$V^{(2)}(m_B^2, \lambda_B) = V^{(2)}[m_{BG}^2, \lambda_{BG}, \xi_{BG}]$$
$$+ \delta^2 \Big\{ \frac{1}{2} \Delta m_B^2[\phi^2 + I_o(\Omega)]$$
$$+ \Delta\lambda_B[\phi_c^4 + 6\phi_c^2 I_o(\Omega) + 3I_o^2(\Omega)]$$
$$+ \Delta\xi_B[\phi_c^6 + 15\phi_c^4 I_o + 45\phi_c^2 I_o^2 + 15 I_o^3] \Big\}$$
$$+ O(\delta^3) \qquad (4.55)$$

We impose the optimization condition for $\bar{\Omega}$ as

$$\frac{\partial V^{(2)}}{\partial \Omega^2} = -\frac{1}{8}F^2(\bar{\Omega}, \phi_c)\frac{dI_{-1}}{d\Omega^2} + \frac{3}{2}(\lambda_{BG} + 15\xi_{BG}\phi_c^2 + 15\xi_{BG} I_o)$$
$$- \frac{1}{2}G^2 \cdot \frac{dI^{(3)}}{d\Omega^2} + 30G\xi_{BG}\phi_c\, I_{-1}\, I^{(3)}$$
$$- \frac{1}{2}H^2 \frac{dI^{(4)}}{d\Omega^2} + \frac{15}{2} H \xi_{BG} I_{-1}\, I^{(4)}$$

$$- \frac{1}{2}(6\xi_{BG}\phi_c)^2 \frac{dI^{(5)}}{d\Omega^2} - \frac{1}{2}\xi_{BG}^2 \frac{dI^{(6)}}{d\Omega^2}$$

$$- \frac{1}{4} I_{-1}(\bar{\Omega}) \Delta m_B^2 - 3\Delta\lambda_B I_{-1}(\phi_c^2 + I_o)$$

$$- 15 I_{-1} \Delta\xi_B[\phi_c^4 + 6\phi_c^2 I_o + 3I_o^2] = o \qquad (4.56)$$

where

$$F = m_{BG}^2 - \Omega^2 + 12\lambda_{BG}(\phi_c^2 + I_o(\Omega)) + 30\xi_{BG}(\phi_c^4 + 6\phi_c^2 I_o + 3I_o^2) \qquad (4.57)$$

$$G = 4\lambda_{BG}\phi_c + 20\xi_{BG}\phi_c^3 + 60\xi_{BG}\phi_c I_o \qquad (4.58)$$

$$H = \lambda_{BG} + 15\xi_{BG}\phi_c^2 + 15\xi_{BG}I_o \qquad (4.59)$$

Here we have made use of the result [31], $I^{(2)} = I_{-1}$,

and

$$\frac{dI_n}{d\Omega^2} = (n - \frac{1}{2}) I_{n-1} \qquad (4.60)$$

The first order expressions for the bare parameters [93] are

$$m_{BG}^2 = m_R^2 - 12\lambda_{BG}I_o(m_R) - 90\xi_{BG}[I_o(m_R)]^2 \qquad (4.61)$$

$$\lambda_R = \lambda_r \frac{[1 - 12\lambda_r I_{-1}(m_R)]}{1 + 6\lambda_r I_{-1}(m_R)} \qquad (4.62)$$

where

$$\lambda_r = \lambda_{BG} + 15\xi_{BG}I_o(m_R) \qquad (4.63)$$

Since the 6th derivative of GEP at the origin is found to be finitely related to ξ_{BG}, λ_r and m_R, the parameter ξ_{BG} needs no renormalisation [93].

Introducing cutoff like renormalization [103,105] in 2+1 dimensions we find from (4.13)

$$I_o(m_R) = \frac{1}{2\pi}[\frac{\Lambda}{\pi} - \frac{m_R}{2}] \tag{4.64}$$

Now

$$m_R^2 = m_{BG}^2 + 12\lambda_{BG}[\frac{\Lambda}{2\pi^2} - \frac{m_R}{4\pi}] + 90\xi_{BG}\{(\frac{\Lambda}{2\pi^2})^2 + (\frac{m_R}{4\pi})^2 - \frac{\Lambda}{4\pi^3}m_R\} \tag{4.65}$$

For 2+1 dimensions I_{-1} is convergent:

$$I_{-1}(m_R) = \frac{1}{4\pi\ m_R} \tag{4.66}$$

Now from (4.62) we have

$$\lambda_R = \frac{\lambda_r[1 - \frac{12\lambda_r}{4\pi\ m_R}]}{1 + \frac{6\lambda_r}{4\pi\ m_R}} \tag{4.67}$$

where

$$\lambda_r = \lambda_{BG} + 15\xi_{BG}\ (\frac{\Lambda}{2\pi^2} - \frac{m_R}{4\pi}) \tag{4.68}$$

Solving for λ_{BG} we have

$$\lambda_{BG} = - 15\xi_{BG}(\frac{\Lambda}{2\pi^2} - \frac{m_R}{4\pi}) - (\frac{\lambda_R}{4} - \frac{\pi m_R}{6}) \pm [\frac{\lambda_R^2}{16} + (\frac{\pi m_R}{6})^2 - \frac{5\pi m_R\lambda_R}{12}]^{1/2} \tag{4.69}$$

λ_{BG} is obtained in a finite form and hence we assume that the corrections to the bare parameters are also finite in 2+1 dimensions [31].

For the $I^{(n)}$ integrals in 2+1 dimensions, we have a small x-cut off [31] like renormalization. In general, in coordinate space

$$\frac{1}{n!}\ I^{(n)}(\Omega) = \int_x G^n(x) = \frac{(2\pi)^{d/2}}{\Gamma(d/2)} \int_0^\infty dx\ x^{d-1}\ G^{(n)}(x) \tag{4.70}$$

In 2+1 dimensions,

$$G(x) = \frac{e^{-\Omega x}}{4\pi x} \tag{4.71}$$

Now
$$\frac{1}{3!}I^{(3)}(\Omega) = \frac{1}{16\pi^2}\int_a^\infty dx\,\frac{e^{-3\Omega x}}{x}$$
$$= \frac{1}{16\pi^2}\,E_1(3\Omega a)$$
$$= \frac{-1}{16\pi^2}[\ln(\Omega a) + \ln 3 + \gamma + O(a)] \tag{4.72}$$

where γ is Euler's constant, and a denotes the x-cutoff.

Similarly

$$\frac{1}{4!}\,I^{(4)}(\Omega) = \frac{\Omega}{16\pi^3}[\frac{e^{-4\Omega a}}{4\Omega a} - E_1(4\Omega a)]$$
$$= \frac{\Omega}{16\pi^3}[\frac{1}{4\Omega a} + \ln(\Omega a) + \ln 4 - 1 + \gamma] \tag{4.73}$$

$$\frac{1}{5!}\,I^{(5)}(\Omega) = \frac{1}{(4\pi)^4}[\frac{-(5\Omega)^2}{2!}(\gamma + \ln 5 - 3/2)$$
$$- \frac{(5\Omega)^2}{2!}\ln\Omega a + \frac{1}{2a^2} - \frac{5\Omega}{a}] \tag{4.74}$$

$$\frac{1}{6!}\,I^{(6)}(\Omega) = \frac{1}{(4\pi)^5}[36\Omega^3(\gamma + \ln 6 - 11/6)$$
$$+ 36\Omega^3\ln\Omega a + \frac{1}{3a^3} - \frac{3\Omega}{a^2} + \frac{18\Omega^2}{a}] \tag{4.75}$$

Rewriting the quantity F defined by (4.57) in terms of renormalized parameters

$$F = m_R^2 - 12\lambda_{BG}\,\Delta I_o - 180\xi_{BG}\,\Delta I_o \cdot I_o(m_R)$$
$$+ 90\xi_{BG}(\Delta I_o)^2 - \Omega^2 - 180\phi_c^2\xi_{BG}\,\Delta I_o$$
$$- 12\phi_c^2\,\lambda_{BG} + 30\xi_{BG}\phi_c^4 \tag{4.76}$$

where ΔI_o in 2+1 dimensions [27] is given by

$$\Delta I_o = \frac{1}{4\pi}(\Omega - m_R) = I_o(m_R) - I_o(\Omega) \tag{4.77}$$

and λ_{BG} and $I_o(m_R)$ are given by (4.69) and (4.64)

Now

$$\frac{d\bar{V}^{(2)}}{d\phi_c^2} = \left.\frac{\partial V^{(2)}}{\partial \phi_c^2}\right|_{\Omega=\bar{\Omega}}$$

$$= \frac{1}{2}\,[m_{BG}^2 + 12\lambda_{BG} I_o + 90\xi_{BG} I_o{}^2 + 4\lambda_{BG}\phi_c^2$$

$$+ 60\xi_{BG}\phi_c^2 I_o + 6\xi_{BG}\phi_c{}^4]$$

$$- I_{-1}F[3\lambda_{BG} + 15\xi_{BG}\phi_c^2 + 45\xi_{BG}I_o]$$

$$- 8[\lambda_{BG}^2 + 75\xi_{BG}^2\phi_c^4 + 225\xi_{BG}^2 I_o^2$$

$$+ 20\lambda_{BG}\xi_{BG}\phi_c^2 + 300\xi_{BG}^2 I_o\phi_c^2$$

$$+ 30\xi_{BG}\lambda_{BG}I_o]\ I^{(3)}$$

$$- 15H\xi_{BG}I^{(4)} - 18\xi_{BG}^2\ I^{(5)}$$

$$+ \frac{1}{2}[\Delta m_B^2 + 12\Delta\lambda_B I_o + 90I_o^2\Delta\xi_B]$$

$$+ 2(\Delta\lambda_B\phi_c^2 + 15\Delta\xi_B I_o\phi_c^2) + 3\Delta\xi_B\phi_c^4 \tag{4.78}$$

The renormalized mass parameter in the second order is obtained as

$$m^2 = 2\left.\frac{d\bar{V}}{d\phi_c^2}\right|_{\phi_c=0}$$

$$= m_R^2 - 12\lambda_{BG}(\Delta I_o)_o$$

$$- 180\xi_{BG} I_o(m_R)\Delta I_o + 90\xi_{BG}(\Delta I_o)_o^2$$

$$- 2I_{-1}(3\lambda_{BG} + 45\xi_{BG} I_o(\bar{\Omega}_o))\ \ (m_R{}^2 - 12\ \lambda_{BG}(\Delta I_o)_o$$

$$- 180\xi_{BG}(\Delta I_o)_o\ I_o(m_R) + 90\xi_{BG}(\Delta I_o)_o^2 - \bar{\Omega}_o^2)$$

$$-16[\lambda_{BG}^2 + 225\xi_{BG}^2\, I_o^2(\bar{\Omega}_o) + 30\xi_{BG}\,\lambda_{BG} I_o(\bar{\Omega}_o)]$$
$$(I^{(3)})_o - 30\xi_{BG}(I^{(4)})_o(\lambda_{BG}+15\xi_{BG} I_o(\bar{\Omega}_o))$$
$$- 36\xi_{BG}^2(I^{(5)})_o + [\Delta m_B^2 + 12\Delta\lambda_B I_o(\bar{\Omega}_o)$$
$$- 90\, I_o^2(\bar{\Omega}_o)\,\Delta\xi_B] \qquad (4.79)$$

Here $(\Delta I_o)_o = \frac{1}{4\pi}(\bar{\Omega}_o - m_R)$

and $(I^{(3)})_o = I^{(3)}_{\bar{\Omega}=\bar{\Omega}_o}$

$(I^{(4)})_o = I^{(4)}_{\bar{\Omega}=\bar{\Omega}_o}$

$(I^{(5)})_o = I^{(5)}_{\bar{\Omega}=\bar{\Omega}_o}$

and $I_o(\bar{\Omega}_o) = \frac{1}{2\pi}[\frac{\Lambda}{\pi} - \frac{\bar{\Omega}_o}{2}]$

Here $\bar{\Omega}_o$ is the solution of (4.56) at $\phi_c = o$.

The renormalized coupling constant λ_s is given by

$$\lambda_s = \frac{1}{2}\,\frac{d^2\bar{V}}{d\phi_c^2}\Big|_{\phi_c=o}$$

$$=\lambda_{BG} + 15\xi_{BG}\, I_o(\bar{\Omega}_o) + \frac{1}{16\pi\,\bar{\Omega}_o^3}\,(m_R^2 - 12\lambda_{BG}\,(\Delta I_o)_o$$
$$- 90\xi_{BG}(\Delta I_o)_o^2\, I_o(m_R)$$
$$+ 90\xi_{BG}(\Delta I_o)_o^2 - \bar{\Omega}_o^2)\,(3\lambda_{BG}+45\xi_{BG}\, I_o(\bar{\Omega}_o))\,\frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}$$
$$- \frac{1}{8\pi\bar{\Omega}_o}(m_R^2 - 12\lambda_{BG}(\Delta I_o)_o - 90\xi_{BG}(\Delta I_o)_o\, 2\, I_o(m_R)$$
$$+ 90\xi_{BG}(\Delta I_o)_o^2 - \bar{\Omega}_o^2)\,(15\xi_{BG} - 45\xi_{BG}\,\frac{1}{8\pi}\,\bar{\Omega}_o\,\frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})$$
$$- \frac{1}{8\pi\bar{\Omega}_o}(12\lambda_{BG} - 12\lambda_{BG}\,\frac{1}{8\pi\,\bar{\Omega}_o}\,\frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}$$
$$+ 30\xi_{BG}\, 6I_o(\bar{\Omega}_o) - 90\xi_{BG}\, I_o\,\frac{1}{4\pi\bar{\Omega}_o}\,\frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})$$
$$(3\lambda_{BG}+45\xi_{BG}\, I_o(\bar{\Omega}_o))$$

$$+ 4(225\zeta_{BG}^2 \, I_o(\bar{\Omega}_o) \frac{1}{4\pi\bar{\Omega}_o} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}$$

$$+ 20\lambda_{BG} \zeta_{BG} + 300\zeta_{BG}^2 \, I_o(\bar{\Omega}_o)$$

$$- \frac{15}{4\pi\bar{\Omega}_o} \zeta_{BG} \lambda_{BG} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}) \, I^{(3)} \, (\bar{\Omega}_o)$$

$$- 4[\lambda_{BG}^2 + 225\zeta_{BG}^2 \, I_o^2(\bar{\Omega}_o) + 30\zeta_{BG} \lambda_{BG} \, I_o(\bar{\Omega}_o)] \frac{dI^{(3)}}{d\Omega^2}(\frac{d^2\bar{\Omega}}{d\phi_c^2}) \, \phi_{c=o}$$

$$- \frac{15}{2} \zeta_{BG} \frac{dI^{(4)}}{d\Omega^2}\Big|_{\Omega=\bar{\Omega}_o} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o} (\lambda_{BG} + 15\zeta_{BG} I_o(\bar{\Omega}_o))$$

$$- \frac{15}{2} \zeta_{BG} \, I^{(4)} (15\zeta_{BG} - 15\zeta_{BG} \frac{1}{4\pi\bar{\Omega}_o} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})$$

$$- 9\zeta_{BG}^2 \frac{dI^{(5)}}{d\Omega^2} \frac{d\Omega^2}{d\phi_c^2}\Big|_{\phi_c=o}$$

$$- \frac{1}{4}[6\Delta\lambda_B \frac{1}{4\pi\bar{\Omega}_o} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o} + 90 I_o \frac{1}{4\pi\bar{\Omega}_o} \frac{d\Omega^2}{d\phi_c^2}\Big|_{\phi_c=o} \Delta\zeta_B]$$

$$+ \Delta\lambda_B + 15 \Delta\zeta_B I_o(\bar{\Omega}_o) \qquad (4.80)$$

where $\frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}$ satisfies the equation

$$\frac{1}{32\pi\bar{\Omega}_o^3}(m_R^2 - 12\lambda_{BG}(\Delta I_o)_o - 180\zeta_{BG}(\Delta I_o)_o \, I_o(m_R) + 90\zeta_{BG}(\Delta I_o^2)_o$$

$$- \bar{\Omega}_o^2) (\frac{-d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o} + 12\lambda_{BG}(1 - \frac{1}{8\pi\bar{\Omega}_o} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}) + 30\zeta_{BG}(6 I_o$$

$$- \frac{3 I_o}{4\pi\bar{\Omega}_o} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})) - \frac{3}{128\pi\bar{\Omega}_o^5} (m_R^2 - 12\lambda_{BG} \Delta I_o$$

$$- 90\zeta_{BG} \Delta I_o \cdot 2 I_o(m_R) + 90\zeta_{BG}(\Delta I_o)^2$$

$$- \bar{\Omega}_o^2) \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o} + \frac{3}{2} (15\zeta_{BG} - \frac{15\zeta_{BG}}{8\pi\bar{\Omega}_o} \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})$$

$$- (\lambda_B + 15\xi_{BG} I_o(\bar{\Omega}_0))\ (15\xi_{BG} - \frac{15\xi_{BG}}{8\pi\bar{\Omega}_0}\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})$$

$$\frac{3}{4\pi^3\bar{\Omega}_0}(\ln\bar{\Omega}_0 a + \ln 4 + \gamma)$$

$$+ \frac{3}{16\pi^3\bar{\Omega}_0^3}(\lambda_{BG} + 15\xi_{BG} I_o(\bar{\Omega}_0))^2\ (\ln\bar{\Omega}_0 a + \ln 4 + \gamma - 1)\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}$$

$$+ (15\xi_{BG} - \frac{15\xi_{BG}}{8\pi\bar{\Omega}_0}\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})\ \frac{15\xi_{BG}}{8\pi\bar{\Omega}_0} I^{(4)}(\bar{\Omega}_0)$$

$$- (\lambda_{BG} + 15\xi_{BG} I_o(\bar{\Omega}_0))\ \frac{15\xi_{BG}}{2}\ \frac{1}{8\pi\bar{\Omega}_0^3}\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}\ I^{(4)}(\bar{\Omega}_0)$$

$$+ (\lambda_{BG} + 15\xi_{BG} I_o)\ \frac{15\xi_{BG}}{8\pi\bar{\Omega}_0}\frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}\ (\frac{3}{4\pi^3\bar{\Omega}_0}(\ln\bar{\Omega}_0 a + \ln 4 + \gamma))$$

$$- \frac{1}{2}(6\xi_{BG})^2\ \frac{dI^{(5)}}{d\bar{\Omega}^2} - \frac{1}{2}\xi_{BG}^2\frac{d\bar{\Omega}_0^2}{d\phi_c^2}\ \frac{45}{4^3\pi^5}[\frac{27}{\bar{\Omega}_0}(\gamma + \ln 6$$

$$- \frac{1}{2} + \ln\bar{\Omega}_0 a + \frac{1}{36\bar{\Omega}_0^2\ a^2})] + \frac{1}{32\pi\bar{\Omega}_0^3}\ \Delta m_B^2\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}$$

$$+ \frac{3\Delta\lambda_B}{8\pi\bar{\Omega}_0^3}\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}\ I_o(\bar{\Omega}_0) - \frac{3\Delta\lambda_B}{4\pi\bar{\Omega}_0}(1 - \frac{1}{8\pi\bar{\Omega}_0}\frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o})$$

$$+ \frac{15}{8\pi\bar{\Omega}_0^3}\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}\ \Delta\xi_B(3I_o(\bar{\Omega}_0))^2 - \frac{15}{4\pi\bar{\Omega}_0}\ (\Delta\xi_B)$$

$$(6I_o - \frac{3I_o}{4\pi\bar{\Omega}_0}\ \frac{d\bar{\Omega}^2}{d\phi_c^2}\Big|_{\phi_c=o}) = 0 \qquad (4.81)$$

Integrating the equation (4.78) we have the GEP [31]:

$$\bar{V}^2(\phi_c) = \frac{1}{2}\ m_R^2\phi_c^2 + \lambda_{BG}\phi_c^4 + \xi_{BG}\phi_c^6$$

$$- 6\lambda_{BG}\ \Delta I_o\phi_c^2 + 45\xi_{BG}(\Delta I_o)^2\phi_c^2$$

$$- 90 I_o(m_R)\ \xi_{BG}\ \Delta I_o\phi_c^2 + 15\xi_{BG}\phi_c^4\ I_o$$

$$- \frac{1}{4\pi\bar{\Omega}_0}(3\lambda_{BG} + 45\xi_{BG}I_0)\,(m_R^2\phi_c^2 - 12\lambda_{BG}\,\Delta I_0\phi_c^2$$

$$- 180\xi\Delta I_0 \cdot I_0\,(m_R)\phi_c^2 + 90\xi_{BG}(\Delta I_0)^2\phi_c^2 - \bar{\Omega}^2\phi_c^2$$

$$- 90\xi_{BG}\,\Delta I_0\phi_c^4 - 6\phi_c^4\,\lambda_{BG} + 10\xi_{BG}\phi_c^6)$$

$$- \frac{1}{4\pi\bar{\Omega}_0}15\xi_{BG}\,(\frac{m_R^2\,\phi_c^4}{2} - 6\,\lambda_{BG}\Delta I_0\phi_c^4 - 90\xi_{BG}\,\Delta I_0\;I_0(m_R)\phi_c^4$$

$$+ 45\xi_{BG}(\,\Delta I_0)^2\phi_c^4 - \frac{\Omega^2\phi_c^4}{2} - 60\xi_{BG}\,\Delta I_0\phi_c^6 - 4\lambda_{BG}\phi_c^6$$

$$+ \frac{15\xi_{BG}\phi_c^8}{2}) - 8[\lambda_{BG}^2\phi_c^2 + 75\xi_{BG}^2\,\frac{\phi_c^6}{3} + 225\xi_{BG}^2\,I_0^2\,\phi_c^2$$

$$+ 10\lambda_{BG}\,\xi_{BG}\,\phi_c^4 + 150\xi_{BG}^2\,I_0\phi_c^4 + 30\xi_{BG}\,\lambda_{BG}\,I_0\,\phi_c^2]\,I^{(3)}$$

$$- 15(\lambda_{BG}\phi_c^2 + \frac{15\xi_{BG}\phi_c^4}{2} + 15\xi_{BG}I_0\phi_c^2)\xi_{BG}\,I^{(4)}$$

$$- 18\xi_{BG}^2\,\phi_c^2 I^{(5)} + \frac{1}{2}[\Delta m_B^2 + 12\Delta\lambda_B I_0 + 90 I_0^2\,\Delta\xi_B\,]\phi_c^2 \qquad (4.82)$$

$$+ \Delta\lambda_B\,\phi_c^4 + 15\,\Delta\xi_B\;\;I_0\phi_c^4 + \Delta\xi_{B}\,\phi_c^6$$

We have shown that (2+1) dimensional ϕ^6 theory is a renormalizable theory and its second order GEP terms are also finite. Unlike in the case of the ϕ^4 theory [31], ϕ^8 - term occur in $V^2(\phi_c,\Omega)$. This is an artifact of the method. 2+1 dimensional ϕ^6 scalar field theory finds applications in the study of vortex solutions of the abelian Chern-Simons theory [106] and in the study of soliton black holes at finite temperature [107]. Besides its importance in particle physics as a model scalar field theory, the ϕ^6 self interacting model finds applications in solid state physics also, where it has been used to explain the first order phase transition from the ferroelectric to paraelectric state and the structural phase transitions observed in crystals [108-110].

V. GEP FOR COHERENT STATES AND SQUEEZED STATES OF ANHARMONIC OSCILLATOR

5.1 Introduction

For the past three decades, developments in the field of coherent states and their applications have been widely discussed. The history of study of coherent states goes back to the early days of quantum mechanics, when around 1926 Schrödinger [43] reported the existence of a certain class of states that display the classical behaviour of the oscillator. Later Glauber [44] called these states coherent states and applied them to the radiation field. In the literature the coherent states have been called the minimum uncertainty product states in the sense that the relation var(x) var(p) $= \frac{1}{4}$ holds for the $|o\rangle$ or vacuum state.

The minimum uncertainty method has been applied to general Hamiltonian potential systems, to obtain both generalized coherent states and generalized squeezed states [111-114]. Recently, to produce the wave function of a quantum anharmonic oscillator, a variational procedure has been proposed [115] and found that the wave function looks like a coherent state constructed from the solution of the classical equations of motion with a semiclassical construction. In this chapter we calculate the GEP of an anharmonic oscillator in coherent states and squeezed states.

5.2 Coherent states

We choose the Hamiltonian:

$$H = \frac{1}{2}P^2 + \frac{1}{2}m^2X^2 + \lambda X^4 \tag{5.1}$$

where

$$X = X_\alpha + \left(\frac{\hbar}{2\Omega}\right)^{1/2} (a_\Omega + a_\Omega^+) \tag{5.2}$$

$$P = -\frac{1}{2}i(2\hbar\Omega)^{1/2} (a_\Omega - a_\Omega^+), \quad X_\alpha = \langle\alpha|X|\alpha\rangle \tag{5.3}$$

and Ω is the mass parameter. The subscript Ω is a reminder that a_Ω and a_Ω^+ depend on the frequency of the harmonic oscillator.

The state $|\alpha\rangle_\Omega$ is the coherent state which can be taken as a Gaussian trial wave function which depends on Ω. To evaluate GEP, we determine the minimum of the expectation value of the Hamiltonian in the coherent state:

$$\bar{V}_G(X_\alpha) = \min_\Omega V_G(X_\alpha,\Omega) = \min_\Omega \langle\alpha|H|\alpha\rangle \tag{5.4}$$

Term by term, we have

$$\langle\alpha|\frac{1}{2}P^2|\alpha\rangle = \frac{-\hbar\Omega}{4}(\alpha^2 - 1 - 2|\alpha|^2 + \alpha^{*2}) \tag{5.5}$$

$$\langle\alpha|\frac{1}{2}m^2X^2|\alpha\rangle = \frac{1}{2}m^2\Big(X_\alpha^2 + \left(\frac{\hbar}{2\Omega}\right)^{1/2} 2X_\alpha(\alpha+\alpha^*) + \frac{\hbar}{2\Omega}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)\Big) \tag{5.6}$$

$$\langle\alpha|\lambda X^4|\alpha\rangle = \lambda X_\alpha^4 + 4X_\alpha^3\lambda\left(\frac{\hbar}{2\Omega}\right)^{1/2}(\alpha+\alpha^*) + 6X_\alpha^2\lambda\left(\frac{\hbar}{2\Omega}\right)(\alpha^2 + \alpha^{*2} + 1 + 2|\alpha|^2)$$

$$+ 4\lambda X_\alpha (\frac{\hbar}{2\Omega})^{3/2} (\alpha^3+\alpha^{*3}+3\alpha+3\alpha|\alpha|^2$$

$$+ 3\alpha^* + 3\alpha^*|\alpha|^2) + \lambda(\frac{\hbar}{2\Omega})^2(\alpha^4+\alpha^{*4} + 6(\alpha^2+\alpha^{*2})$$

$$+ |\alpha|^2\, 4(\alpha^2+\alpha^{*2}) + 6|\alpha|^4+12|\alpha|^2+3) \qquad (5.7)$$

On summation we get the expectation value of the Hamiltonian as

$$_\Omega\langle\alpha|H|\alpha\rangle_\Omega = \sum_{l=0}^{4} C_l\, X_\alpha^l = V_G(X_\alpha,\Omega) \qquad (5.8)$$

where

$$C_0 = \frac{-\hbar\Omega}{4}(\alpha^2-1-2|\alpha|^2+\alpha^{*2}) + \frac{\hbar m^2}{4\Omega}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)$$

$$+ \lambda(\frac{\hbar}{2\Omega})^2(\alpha^4+\alpha^{*4}+6(\alpha^2+\alpha^{*2})$$

$$+ 4|\alpha|^2(\alpha^2+\alpha^{*2})+6|\alpha|^4+12|\alpha|^2+3)$$

$$C_1 = m^2(\frac{\hbar}{2\Omega})^{1/2}(\alpha+\alpha^*) + 4\lambda(\frac{\hbar}{2\Omega})^{3/2}(\alpha^3+\alpha^{*3}+3\alpha+3\alpha^*+3|\alpha|^2(\alpha+\alpha^*))$$

$$C_2 = \frac{1}{2}m^2 + \frac{3\lambda\hbar}{\Omega}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)$$

$$C_3 = 4\lambda(\frac{\hbar}{2\Omega})^{1/2}(\alpha+\alpha^*)$$

$$C_4 = \lambda$$

Minimizing the quantity $_\Omega\langle\alpha|H|\alpha\rangle_\Omega$, we get the optimum condition for Ω as

$$\bar{\Omega}^3(\alpha^2-1-2|\alpha|^2+\alpha^{*2}) + (\frac{\hbar}{2})^{-1/2}\, \bar{\Omega}^{3/2}(\alpha+\alpha^*)$$

$$(m^2X_\alpha+4\lambda X_\alpha^3) + 2\bar{\Omega}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)\,(\frac{1}{2}m^2+6\lambda X_\alpha^2)$$

$$+ (\frac{\hbar}{2})^{1/2}\, \bar{\Omega}^{1/2}\, 12\lambda X_\alpha(\alpha^3+\alpha^{*3}+3(\alpha+\alpha^*) + 3|\alpha|^2(\alpha+\alpha^*)) \qquad (5.9)$$

$$+ 2\hbar\lambda(\alpha^4+\alpha^{*4}+6(\alpha^2+\alpha^{*2}) + 4|\alpha|^2(\alpha^2+\alpha^{*2}) + 6|\alpha|^4+12|\alpha|^2+3) = 0$$

This equation can have six roots and the largest positive root, designated as $\bar{\Omega}$, is to be employed by convention [16], when the effective potential is calculated.

If α is real, we have the $\bar{\Omega}$ equation as

$$\bar{\Omega}^3 - 2(\frac{\hbar}{2})^{-1/2}\ \alpha\ \bar{\Omega}^{3/2}(m^2X_\alpha+4\lambda X_\alpha^3) - 2\bar{\Omega}(4\alpha^2+1)\ (\frac{1}{2}m^2+6\lambda X_\alpha^2)$$
$$- 12(\frac{\hbar}{2})^{1/2}\ \bar{\Omega}^{1/2}\ \lambda X_\alpha(8\alpha^3+6\alpha)$$
$$- 2\hbar\lambda(16\alpha^4+24\alpha^2+3) = 0 \qquad (5.10)$$

$$\frac{\partial V_G(X_\alpha,\Omega)}{dX_\alpha}\Big|_{\Omega=\bar{\Omega}} = \frac{d\bar{V}}{dX_\alpha} \qquad (5.11)$$

Since $\frac{\partial V_G}{\partial\Omega}$ vanishes at $\Omega=\bar{\Omega}$,

$$\frac{d\bar{V}}{dX_\alpha} = m^2(\frac{\hbar}{2\bar{\Omega}})^{1/2}(\alpha+\alpha^*) + 4\lambda(\frac{\hbar}{2\bar{\Omega}})^{3/2}(\alpha^3+\alpha^{*3}+3\alpha+3\alpha^*+3|\alpha|^2(\alpha+\alpha^*))$$
$$+ 2X_\alpha(\frac{1}{2}m^2 + \frac{3\lambda\hbar}{\bar{\Omega}}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2))$$
$$+ 3X_\alpha^2\ (4\lambda(\frac{\hbar}{2\bar{\Omega}})^{1/2}(\alpha+\alpha^*)) + 4\lambda X^3 \qquad (5.12)$$

We define the effective mass $m_c^2(\alpha)$ of the anharmonic oscillator by the relation

$$m_c^2\ (\alpha,\alpha^*) = \frac{d^2\bar{V}}{dX_\alpha^2}\Big|_{X_\alpha=0} = -\frac{m^2}{2}(\frac{\hbar}{2})^{1/2}\bar{\Omega}_0^{-3/2}(\alpha+\alpha^*)\ \frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=0}$$
$$- 6\lambda(\frac{\hbar}{2})^{3/2}\bar{\Omega}_0^{-5/2}(\alpha^3+\alpha^{*3}+3(\alpha+\alpha^*)$$
$$+ 3|\alpha|^2(\alpha+\alpha^*))\ \frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=0}$$
$$+ m^2 + \frac{6\lambda\hbar}{\bar{\Omega}_0}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2) \qquad (5.13)$$

where $\frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=0}$ can be obtained from (5.9) as

$$\frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=0} = -[\ \frac{1}{2}(\frac{\hbar}{2})^{-1/2}\bar{\Omega}_0^{3/2}m^2(\alpha+\alpha^*) + 12\lambda(\frac{\hbar}{2})^{1/2}\bar{\Omega}_0^{1/2}(\alpha^3+\alpha^{*3}+3\alpha$$

$$+ 3|\alpha|^2(\alpha+\alpha^*) + 3\alpha^*)]/(3\bar{\Omega}_0^2(\alpha^2-1-2|\alpha|^2+\alpha^{*2})+m^2(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)) \quad (5.14)$$

Here $\bar{\Omega}_0 = \bar{\Omega}$ at $X_\alpha = o$

For the ground state expectation value of the Hamiltonian [16], $\frac{d\bar{\Omega}}{dX_0}\Big|_{X_0=0} = 0$, and hence (5.14) gives marked difference in the behaviour of $\bar{\Omega}$ from the ordinary ground state and excited state results.

Substituting (5.14) into (5.13), we have

$$m_c^2(\alpha) = m^2 + \frac{6\lambda\hbar}{\bar{\Omega}_0}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2) + [\frac{m^4}{4}(\alpha+\alpha^*)^2$$
$$+ 9m^2\lambda(\frac{\hbar}{2})\bar{\Omega}_0^{-1}(\alpha+\alpha^*) + 18\lambda^2\hbar^2\bar{\Omega}_0^{-2}$$
$$(\alpha^3+\alpha^{*3}+3(\alpha+\alpha^*)+3|\alpha|^2(\alpha+\alpha^*))^2]/[3\bar{\Omega}_0^2(\alpha^2-1-2|\alpha|^2+\alpha^{*2})+$$
$$m^2(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)] \quad (5.15)$$

We may also introduce an effective coupling constant $\lambda_c(\alpha,\alpha^*)$ for the coherent state $|\alpha\rangle$ through the relation

$$\lambda_c(\alpha,\alpha^*) = \frac{1}{4!}\left(\frac{d^4\bar{V}_G}{dX_\alpha}\right)^4_{X_\alpha=o} \quad (5.16)$$

Then on calculation, we find

$$\frac{1}{4!}\frac{d^4\bar{V}_G}{dX_\alpha^4}\Big|_{X_\alpha=o} = \frac{1}{4!}[-[\frac{m^2}{2}(\frac{\hbar}{2})^{1/2}\bar{\Omega}_0^{-3/2}(\alpha+\alpha^*)$$
$$+ 6\lambda(\frac{\hbar}{2})^{3/2}\bar{\Omega}_0^{-5/2}(\alpha^3+\alpha^{*3}+3\alpha+3\alpha^*$$
$$+ 3|\alpha|^2(\alpha+\alpha^*))]\frac{d^3\bar{\Omega}}{dX_\alpha^3}\Big|_{X_\alpha=o}$$

$$- \frac{18\lambda\hbar}{\bar{\Omega}_0^{\,2}}(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)\ \frac{d^2\bar{\Omega}}{dX_\alpha^2}\Big|_{X_\alpha=0}$$

$$+ 3[\tfrac{3}{4}m^2(\tfrac{\hbar}{2})^{1/2}\ \bar{\Omega}_0^{-5/2}(\alpha+\alpha^*)$$

$$+ 15\lambda(\tfrac{\hbar}{2})^{3/2}\bar{\Omega}_0^{-7/2}\ (\alpha^3+\alpha^{*3}+3\alpha+3\alpha^*$$

$$+ 3|\alpha|^2(\alpha+\alpha^*))]\ \frac{d^2\bar{\Omega}}{dX_\alpha{}^2}\Big|_{X_\alpha=0}\ \ \frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=0}$$

$$+ \frac{36\lambda\hbar}{\bar{\Omega}_0^3}(\alpha^2+\alpha^{*2}\ \ \ +2|\alpha|^2)\ (\frac{d\bar{\Omega}}{dX_\alpha})^2\Big|_{X_\alpha=0}$$

$$- 36\lambda(\tfrac{\hbar}{2})^{1/2}\ \bar{\Omega}_0^{-3/2}(\alpha+\alpha^*)\ \frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=0}$$

$$- [m^2(\tfrac{\hbar}{2})^{1/2}\ \frac{15}{8}\ \bar{\Omega}_0^{-7/2}(\alpha+\alpha^*)$$

$$+ \frac{105}{2}\ \lambda(\frac{\hbar}{2})^{3/2}\ \bar{\Omega}_0^{-9/2}(\alpha^3+\alpha^{*3}+3\alpha+3\alpha^*$$

$$+ 3|\alpha|^2(\alpha+\alpha^*)]\ (\frac{d\bar{\Omega}}{dX_\alpha})^3\Big|_{X_\alpha=0} + 24\lambda] \qquad (5.17)$$

where

$$\frac{d^2\bar{\Omega}}{dX_\alpha^2}\Big|_{X_\alpha=0} = 0 = \{-\ [3m^2(\tfrac{\hbar}{2})^{-1/2}\ \bar{\Omega}_0^{1/2}(\alpha+\alpha^*)$$

$$+ (\tfrac{\hbar}{2})^{1/2}\bar{\Omega}_0^{-1/2}\ 12\lambda(\alpha^3+\alpha^{*3}+3\alpha+3\alpha^*+3|\alpha|^2(\alpha+\alpha^*)]\frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=0}$$

$$- [6\bar{\Omega}_0(\alpha^2-1-2|\alpha|^2+\alpha^{*2})\ (\frac{d\bar{\Omega}}{dX_\alpha})^2\Big|_{X_\alpha=0}$$

$$+ 24\lambda\bar{\Omega}_0(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)]\}\ /$$

$$[3\bar{\Omega}_0{}^2(\alpha^2-1-2|\alpha|^2+\alpha^{*2})+m^2(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)] \qquad (5.18)$$

and

$$\frac{d^3\bar{\Omega}}{dX_\alpha^3}\Big|_{X_\alpha=0} = \{-\ [\tfrac{9}{2}\ (\tfrac{\hbar}{2})^{-1/2} m^2\ \bar{\Omega}_0^{1/2}\ \ (\alpha+\alpha^*)$$

$$+\ 18(\tfrac{\hbar}{2})^{1/2}\bar{\Omega}_o^{-1/2}\ \lambda(\alpha^3+\alpha^{*3}+3(\alpha+\alpha^*)$$

$$+\ 3|\alpha|^2(\alpha+\alpha^*))]\ \frac{d^2\bar{\Omega}}{dX_\alpha^2}\Big|_{X_\alpha=o}$$

$$-\ 18\bar{\Omega}_o(\alpha^2-1-2|\alpha|^2+\alpha^{*2})\ \frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=o}\ \frac{d^2\bar{\Omega}}{dX_\alpha^2}\Big|_{X_\alpha=o}$$

$$-\ [\tfrac{9}{4}m^2\ (\tfrac{\hbar}{2})^{-1/2}\bar{\Omega}_o^{1/2}\quad(\alpha+\alpha^*)$$

$$-\ (\tfrac{\hbar}{2})^{1/2}9\,\bar{\Omega}_o^{-3/2}\lambda(\alpha^3+\alpha^{*3}+3(\alpha+\alpha^*)$$

$$+\ 3|\alpha|^2(\alpha+\alpha^*))]\ (\frac{d\bar{\Omega}}{dX_\alpha})^2\Big|_{X_\alpha=o}$$

$$-\ 72(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)\lambda\,\frac{d\bar{\Omega}}{dX_\alpha}\Big|_{X_\alpha=o}$$

$$-\ 6(\alpha^2-1-2|\alpha|^2+\alpha^{*2})\ (\frac{d\bar{\Omega}}{dX_\alpha})^3\Big|_{X_\alpha=o}$$

$$-\ (\tfrac{\hbar}{2})^{-1/2}\bar{\Omega}_o^{3/2}(\alpha+\alpha^*)24\,\lambda\}/$$

$$(3\bar{\Omega}_o^2(\alpha^2-1-2|\alpha|^2+\alpha^{*2})\ +\ m^2(\alpha^2+\alpha^{*2}+1+2|\alpha|^2)) \qquad (5.19)$$

To obtain the effective potential, we integrate the equation (5.12) with respect to X_α and get

$$\bar{V}_G = \tfrac{1}{2}m^2X_\alpha^2\ +\ m^2X_\alpha(\tfrac{\hbar}{2\bar{\Omega}})^{1/2}(\alpha+\alpha^*)$$

$$+\ \lambda X_\alpha^4\ +\ 4\lambda X_\alpha^3(\tfrac{\hbar}{2\bar{\Omega}})^{1/2}(\alpha+\alpha^*)$$

$$+\ \lambda 3X_\alpha^2\ (\tfrac{\hbar}{\bar{\Omega}})\ (\alpha^2+\alpha^{*2}+1+2|\alpha|^2)\ +\ 4\lambda X_\alpha(\tfrac{\hbar}{2\bar{\Omega}})^{3/2}(\alpha^3+\alpha^{*3}+3(\alpha+\alpha^*)$$

$$+\ 3|\alpha|^2(\alpha+\alpha^*)) \qquad (5.20)$$

Here the effective potential $\bar{V}_G$ is real even though the coherent state parameter α may be complex. On inspection of the expression for the effective coupling constant $\lambda_c(\alpha,\alpha^*)$ it is seen that it contains some non-analytic terms in $\hbar$, namely those in negative powers of $\hbar$. As a result λ_c diverges in the limit $\hbar \to o$. However λ_c remains finite in this limit if the 'bare mass parameter' m is taken to zero. It is also noted that there is no similar problem in the ground state $|o\rangle$.

5.3 Squeezed states

The generalizations of coherent states namely, squeezed states, have become of more and more interest in recent times [48,116-122]. A wide range of applications have been suggested, ranging from gravitational wave detection and polariton theory [123] to low-noise optical communications, to the inhibition of atomic phase decay [124]. Recently, there have been several attempts at generalizing the notion of squeezing. Very recently, Nieto [125] generalized the notion of squeezed states to arbitrary symmetry systems and discussed its relationship to squeezed states obtained for general potentials. It is shown that the coherent light interacting with a nonlinear nonabsorbing medium modelled as anharmonic oscillator can also give rise to the amplitude-squared squeezing effect [126]. This model system has previously been shown to give rise to usual second order squeezing in terms of the field amplitude [127]. In this section we extend the GEP method to define the effective potential of anharmonic oscillator for squeezed states $|\beta\rangle$ defined by (1.87) and (1.88).

Expressing X and P in the form

$$X = X_\beta + \left(\frac{\hbar}{2\Omega}\right)^{1/2}(a_\Omega + a_\Omega^+)$$

$$P = -\frac{i}{2}(2\hbar\Omega)^{1/2}(a_\Omega - a_\Omega^+),$$

where $X_\beta = \langle\beta|X|\beta\rangle$,

we calculate the expectation value for each term of the Hamiltonian given by (5.1) for a squeezed state $|\beta\rangle$ with the operator

$b = \mu a + \nu a^+$;

$$\langle\beta|\tfrac{1}{2}P^2|\beta\rangle = -\tfrac{1}{4}\hbar\Omega[\mu^{*2}\beta^2 + \nu^2\beta^{*2} - \mu^*\nu(2|\beta|^2+1) + \mu^2\beta^{*2}$$
$$- \mu\nu^*(2|\beta|^2+1) + \nu^{*2}\beta^2 - 2(|\mu|^2+|\nu|^2)|\beta|^2 - |\nu|^2$$
$$- 2\nu^*\mu^*\beta^2 + 2\mu\nu\beta^{*2} - |\mu|^2] \qquad (5.21)$$

$$\langle\beta|\tfrac{1}{2}m^2X_\beta^2|\beta\rangle = \tfrac{1}{2}m^2X_\beta^2 + m^2X_\beta + \left(\frac{\hbar}{2\Omega}\right)^{1/2}(\mu^*\beta - \nu\beta^* + \mu\beta^* - \nu^*\beta)$$
$$+ \frac{\hbar}{2\Omega}(\mu^{*2}\beta^2 + \nu^2\beta^{*2} - \mu^*\nu(2|\beta|^2+1)$$
$$+ \mu^2\beta^{*2} + \nu^{*2}\beta^2 - \mu\nu^*(2|\beta|^2+1)$$
$$+ 2(|\mu|^2+|\nu|^2)|\beta|^2 + |\mu|^2 + |\nu|^2$$
$$- 2\mu^*\nu^*\beta^2 - 2\mu\nu\beta^{*2}) \qquad (5.22)$$

$$\langle\beta|\lambda X_\beta^4|\beta\rangle = \lambda X_\beta^4 + 4\lambda X_\beta^3\left(\frac{\hbar}{2\Omega}\right)^{1/2}A + 6\lambda X_\beta^2\left(\frac{\hbar}{2\Omega}\right)B$$
$$+ 4\lambda X_\beta\left(\frac{\hbar}{2\Omega}\right)^{3/2}C + \left(\frac{\hbar}{2\Omega}\right)^2 D \qquad (5.23)$$

where
$$A = \mu^*\beta - \nu\beta^* + \mu\beta^* - \nu^*\beta$$
$$B = \mu^{*2}\beta^2 + \nu^2\beta^{*2} - \mu^*\nu(2|\beta|^2+1)$$
$$+ \mu^2\beta^{*2} + \nu^{*2}\beta^2 - \mu\nu^*(2|\beta|^2+1)$$
$$+ 2(|\mu|^2+|\nu|^2)|\beta|^2 + |\mu|^2 + |\nu|^2$$
$$- 2\mu^*\nu^*\beta^2 - 2\mu\nu\beta^{*2}$$

$$C = \beta^3(\mu^{*3} - \nu^{*3} - 3\mu^{*2}\nu^* + 3\nu^{*2}\mu^*)$$
$$+ \beta^{*3}(\mu^3 - \nu^3 - 3\mu^2\nu + 3\nu^2\mu)$$
$$+ (|\beta|^2 3\beta + 3\beta)\ (\nu^{*2}\mu - \mu^{*2}\nu$$
$$+ \mu^{*2}\mu - \nu^{*2}\nu + 2\mu^*\nu^*\nu - 2\nu^*\mu^*\mu)$$
$$+ [|\beta|^2[3\beta^*] + 3\beta^*]\ [\nu^2\mu^* - \mu^2\nu^* + \mu^2\mu^* - \nu^2\nu^* + 2\mu\nu\nu^*$$
$$- 2\nu\mu\mu^*]$$

and $$D = \beta^4(\mu^{*4} + \nu^{*4} - 4\mu^{*3}\nu^* - 4\nu^*\mu^* + 6\mu^{*2}\nu^{*2})$$
$$+ \beta^{*4}(\mu^4 + \nu^4 - 4\nu^3\mu - 4\mu^3\nu + 6\mu^2\nu^2)$$
$$+ 6\beta^2 + 4\beta^2|\beta|^2[-\mu^{*3}\nu - \nu^{*3}\mu$$
$$+ \mu\mu^{*3} + \nu\nu^{*3} + 3\mu^{*2}\nu^*\nu + 3\nu^{*2}\mu^*\mu - 3\mu^{*2}\nu^*\mu - 3\nu^{*2}\mu^*\nu]$$
$$+ 6\beta^{*2} + 4\beta^{*2}|\beta|^2\ [-\nu^3\mu^* - \mu^3\nu^* + \nu^*\nu^3 + \mu^*\mu^3 + 3\nu^2\mu\mu^*$$
$$+ 3\mu^2\nu^*\nu - 3\nu^2\mu\nu^* - 3\mu^2\nu\mu^*] + 12|\beta|^2 + 6|\beta|^4 + 3[\mu^{*2}\nu^2$$
$$+ \mu^2\nu^{*2} + (\nu^*\nu)^2 + (\mu^*\mu)^2 - 2\mu^{*2}\mu\nu - 2\nu^2\mu^*\nu^* - 2\mu^2\nu^*\mu^*$$
$$- 2\nu^{*2}\mu\nu]$$

Putting together the expressions given by (5.21),(5.22) and (5.23) the expectation value of the Hamiltonian in a squeezed state $|\beta\rangle$ is

$$\langle\beta|H|\beta\rangle = \frac{-\hbar\Omega}{4}B' + \frac{1}{2}m^2X_\beta^2 + m^2X_\beta\left(\frac{\hbar}{2\Omega}\right)^{1/2} A + \frac{1}{2}m^2\left(\frac{\hbar}{2\Omega}\right)B$$
$$+ \lambda X_\beta^4 + 4\lambda X_\beta^3\left(\frac{\hbar}{2\Omega}\right)^{1/2}A + 6X_\beta^2\lambda\left(\frac{\hbar}{2\Omega}\right)B + 4\lambda X_\beta\left(\frac{\hbar}{2\Omega}\right)^{3/2}C$$
$$+ \lambda\left(\frac{\hbar}{2\Omega}\right)^2 D \qquad (5.24)$$

where $$B' = \mu^*\beta^2 + \nu^2\beta^{*2} - \mu^*\nu(2|\beta|^2+1)$$
$$- \mu\nu^*(2|\beta|^2+1) + \mu^2\beta^{*2} + \nu^{*2}\beta^2 - 2(|\mu|^2+|\nu|^2)|\beta|^2$$
$$- |\mu|^2 - |\nu|^2 + 2\beta^{*2}\mu\nu + 2\beta^2\mu^*\nu^*$$

Minimizing the expectation value of H with respect to Ω, we have

$$\frac{d\langle H\rangle}{d\Omega} = \frac{\hbar}{4} B' + m^2 X_\beta (\frac{\hbar}{2})^{1/2} A \frac{1}{2} \bar{\Omega}^{-3/2}$$
$$+ \frac{1}{2} m^2 \frac{\hbar}{2\bar{\Omega}^2} B + 2\lambda X_\beta^3 (\frac{\hbar}{2})^{1/2} \bar{\Omega}^{-3/2} A$$
$$+ 6X_\beta^2 \lambda \frac{\hbar}{2\bar{\Omega}^2} B + 6\lambda X_\beta (\frac{\hbar}{2})^{3/2} C \bar{\Omega}^{-5/2}$$
$$+ \lambda (\frac{\hbar}{2})^2 \frac{2}{\bar{\Omega}^3} D = 0 \qquad (5.25)$$

The effective mass is calculated for the squeezed state $|\beta\rangle$ as in the case of coherent states:

$$m_s^2 = \frac{d^2\bar{V}}{dX_\beta^2}\Big|_{X_\beta=0} = m^2 + \frac{6\hbar\lambda B}{\bar{\Omega}_0} + [\frac{m^4}{4} A^2 + 9m^2 (\frac{\hbar}{2}) \bar{\Omega}_0^{-1} A C \lambda$$
$$+ 72\lambda^2 C^2 (\frac{\hbar}{2})^2 \bar{\Omega}_0^{-2}]/[3\bar{\Omega}_0^2 B' + m^2 B]$$

To find the effective coupling constant we have to evaluate the fourth derivative of the GEP:

$$\lambda_s = \frac{1}{4!} \frac{d^4 V}{dX_\beta^4}\Big|_{X_\beta=0} = \frac{1}{4!} \Big\{- [\frac{m^2}{2} (\frac{\hbar}{2})^{1/2} \bar{\Omega}_0^{-3/2} A$$
$$+ 6\lambda (\frac{\hbar}{2})^{3/2} \bar{\Omega}_0^{-5/2} C] \frac{d^3\bar{\Omega}}{dX_\beta^3}\Big|_{X_\beta=0}$$

$$+ [\frac{9}{4} m^2 (\frac{\hbar}{2})^{1/2} \bar{\Omega}_0^{-5/2} A + 45\lambda (\frac{\hbar}{2})^{3/2} \bar{\Omega}_0^{-7/2}]$$

$$\frac{d\bar{\Omega}}{dX_0}\Big|_{X_\beta=0} \qquad \frac{d^2\bar{\Omega}}{dX_\beta^2}\Big|_{X_\beta=0}$$

$$- \frac{18\lambda\hbar}{\bar{\Omega}^2} B \frac{d^2\bar{\Omega}}{dX_\beta^2}\Big|_{X_\beta=o}$$

$$- 36\lambda\left(\frac{\hbar}{2}\right)^{1/2} \bar{\Omega}_o^{-3/2} A \frac{d\bar{\Omega}}{dX_\beta}\Big|_{X_\beta=o}$$

$$+ 36\lambda \frac{\hbar}{\bar{\Omega}_o^3} B \left(\frac{d\bar{\Omega}}{dX_\beta}\right)^2\Big|_{X_\beta=o}$$

$$- [m^2\left(\frac{\hbar}{2}\right)^{1/2} \frac{15}{8} \bar{\Omega}_o^{-7/2} A$$

$$+ \frac{105}{2}\lambda\left(\frac{\hbar}{2}\right)^{3/2} \bar{\Omega}_o^{-9/2} \quad] \left(\frac{d\bar{\Omega}}{dX_\beta}\right)^3\Big|_{X_\beta=o}$$

$$+ 24\lambda\} \tag{5.26}$$

where

$$\frac{d\bar{\Omega}}{dX_\beta}\Big|_{X_\beta=o} = - [\left(\frac{\hbar}{2}\right)^{-1/2} \bar{\Omega}_o^{3/2} A \frac{1}{2} m^2$$

$$+ \left(\frac{\hbar}{2}\right)^{1/2} \bar{\Omega}_o^{1/2} 12\lambda C]/[3\bar{\Omega}_o^2 B' + m^2 B] \tag{5.27}$$

$$\frac{d^2\bar{\Omega}}{dX_\beta^2}\Big|_{X_\beta=o} = \{-[3m^2\left(\frac{\hbar}{2}\right)^{-1/2} \bar{\Omega}_o^{1/2} A$$

$$+ \left(\frac{\hbar}{2}\right)^{1/2} \bar{\Omega}_o^{-1/2} 12\lambda C] \frac{d\bar{\Omega}}{dX_\beta}\Big|_{X_\beta=o}$$

$$- [6\bar{\Omega}_o B' \left(\frac{d\bar{\Omega}}{dX_\beta}\right)^2\Big|_{X_\beta=o}]$$

$$- 24\lambda\bar{\Omega}_o B\}/[3\bar{\Omega}_o^2 B' + m^2 B] \tag{5.28}$$

and

$$\frac{d^3\bar{\Omega}}{dX_\beta^3} = \{-[\frac{9}{2}m^2\left(\frac{\hbar}{2}\right)^{-1/2} \bar{\Omega}_o^{1/2} A$$

$$+ 18\left(\frac{\hbar}{2}\right)^{1/2} \bar{\Omega}_o^{-1/2} \lambda C] \frac{d^2\bar{\Omega}}{dX_\beta^2}\Big|_{X_\beta=o}$$

$$- 18\bar{\Omega}_o B' \frac{d\bar{\Omega}}{dX_\beta}\Big|_{X_\beta=o} \frac{d^2\bar{\Omega}}{dX_\beta^2}\Big|_{X_\beta=o}$$

$$-[\frac{9}{4}m^2(\frac{\hbar}{2})^{-1/2}\bar{\Omega}_o^{-1/2}\ A -(\frac{\hbar}{2})^{1/2}\ 9\ \bar{\Omega}_o^{-3/2}\ \lambda C]$$

$$(\frac{d\bar{\Omega}}{dX_\beta})^2\Big|_{X_\beta=o}$$

$$- 72\lambda B\ \frac{d\bar{\Omega}}{dX_\beta}\Big|_{X_\beta=o} \qquad (5.29)$$

$$- 6B'(\frac{d\bar{\Omega}}{dX_\beta})^3\Big|_{X_\beta=o} - (\frac{\hbar}{2})^{-1/2}\bar{\Omega}_o^{3/2}A\ 24\lambda\}/(3\bar{\Omega}_o^{2}B'+m^2B)$$

The effective potential in the squeezed state is obtained by integrating the $\frac{\partial\bar{V}_G}{\partial X_\beta}$ with respect to X_β.

$$\bar{V}_G = \frac{1}{2}m^2X_\beta^2 + m^2X_\beta(\frac{\hbar}{2\bar{\Omega}})^{1/2}A$$
$$+ \lambda X_\beta^4 + 4\lambda X_\beta^3\ (\frac{\hbar}{2\bar{\Omega}})^{1/2}A$$
$$+ 6X_\alpha^2\lambda(\frac{\hbar}{2\bar{\Omega}})B$$
$$+ 4\lambda X_\alpha(\frac{\hbar}{2\bar{\Omega}})^{3/2}C \qquad (5.30)$$

where $\bar{\Omega}$ is given by (5.25).

The effective coupling constant λ_s for a squeezed state has a singularity at $\hbar$ = o, which disappears for zero bare mass m, an effect already encountered with coherent states. Since the bare mass m can be arbitrary, it might as well be set equal to zero. The singular behaviour in λ_c and λ_s for $m \neq o$ is perhaps an artifact of the GEP method.

VI. QUANTUM OSCILLATORS

6.1 Introduction

Some of the investigations on quantum groups and quantum algebras focus on quantum group modified quantum mechanics. In [58], for example, the spectrum of a q-anharmonic oscillator with quartic interaction has been studied using first order perturbation theory. There is a logical need to apply the nonperturbative approach to such systems that are generically known as quantum oscillators.

In this chapter we formulate a nonperturbative q - or (q,p) - analogue of GEP with the help of appropriate quantum oscillator commutation relations that depend on a single parameter q, or two parameter q,p.

For q-deformed quantum mechanics, we seek generalizations of position and momentum operators, X and P, in the form

$$X = X_o + \left(\frac{\hbar}{2\Omega}\right)^{1/2}(a_\zeta + a_\zeta^+) \qquad (6.1)$$

$$P = \frac{-i}{2}(2\hbar\Omega)^{1/2}(a_\zeta - a_\zeta^+) \qquad (6.2)$$

where X_0 is a classical c-number and a_ζ and a_ζ^+ are the annihilation and creation operators respectively with the set $\zeta = [\Omega,q]$. Here Ω denotes a variational parameter having the dimension of mass or frequency that appears in the original q=1 formulation of the GEP [16], and q is the quantum deformation parameter. For q=1 the standard definitions of X and P are recovered.

We impose the q-commutation relation analogous to (1.100):

$$a_\xi a_\xi^+ - q\, a_\xi^+ a_\xi = q^{-N_\xi} \qquad (6.3)$$

where N_ξ is the number operator which is not assumed to be the same as $a_\xi^+ a_\xi$

Now, GEP is customarily defined as in (1.19) and (1.20). When q is real and q>1, a q-analogue of the Gaussian function has been defined [128]. But its explicit form is not required here. We merely assume that the lowest variational trial state $|\psi\rangle_\xi$ depends on Ω as well as q, and the excited states can be generated therefrom by applying a_ξ^+ as many times as necessary. The nonperturbative q analogue of GEP is called nonperturbative q effective potential (NP_qEP) and is defined with respect to any state $|\psi\rangle_\xi$ as follows:

$$\begin{aligned} V_q(X_o) &= \min_{\Omega} V_q(X_o, \Omega) \\ &\equiv \min_{\Omega} {}_\xi\langle\psi|H|\psi\rangle_\xi \qquad (6.5) \end{aligned}$$

Studies in quantum group phenomenology [59] indicate the possibility of a given system being associated with a particular q value. This motivates one to define the system-specific q-effective potential (SS_qEP). Here, the quantum parameter q can serve as additional parameter in the potential, suggesting a more elaborate scheme of minimization. Formally we define SS_qEP as follows:

$$V(X_o) = \min_{\xi} V(X_o,\xi)$$
$$\equiv \min_{\Omega,q} {}_{\xi}\langle\psi|H|\psi\rangle_{\xi} \qquad (6.6)$$

If the two parameter quantum algebra characterising a (q,p)-oscillator is used [80], then the relevant commutation relations are

$$a_\eta a_\eta^+ - q\ a_\eta^+\ a_\eta = p^{-N_\eta} \qquad (6.7)$$

$$a_\eta a_\eta^+ - p^{-1}\ a_\eta^+\ a_\eta = q^{N_\eta} \qquad (6.8)$$

where the set $\eta=[\Omega,q,p]$. A (q,p)-deformed number $[A]_{q,p}$ is defined by the relation

$$[A]_{q,p} = \frac{q^A - p^{-A}}{q - p^{-1}} \qquad (6.9)$$

It is clear that the q-oscillator corresponds to the particular case where q = p. By analogy with the NP_qEP, one defines the $NP_{qp}EP$ by the relation,

$$V_{q,p}(X_o) = \min_{\Omega} V_{q,p}(X_o,\Omega)$$
$$\equiv \min_{\Omega} {}_{\eta}\langle\psi|H|\psi\rangle_{\eta} \qquad (6.10)$$

The system-specific $V(X_o)$, denoted $SS_{qp}EP$, is defined as

$$V(X_o) = \min_{\eta} V(X_o,\eta)$$
$$= \min_{\Omega,q,p} {}_{\eta}\langle\psi|H|\psi\rangle_{\eta} \qquad (6.11)$$

The renormalized mass m_R and renormalized coupling constant λ_R (in the quartic case) can be obtained as explained in the preceding chapters. It is clear that both m_R^2 and λ_R depend on the quantum parameter(s). m_R can be interpreted as the difference between the first excited state and ground state energies while λ_R denotes the amplitude for a transition from the state $|1\rangle$ to $|3\rangle$ under the action of coupling λX^4 [19].

We study three kinds of quantum oscillator systems: quartic coupled quantum oscillators in a single well and in a double well, and sextic coupled quantum oscillators.

6.2 Quartic quantum oscillators - Single well

The Hamiltonian representing a quartic quantum single well oscillator is

$$H = \frac{1}{2}P^2 + \frac{1}{2}m^2X^2 + \lambda X^4 \tag{6.12}$$

If the system is a q-oscillator, then its NP_qEP can be evaluated. The expectation value of H for the nth eigenstate is

$$\langle n|H|n\rangle = \langle n|\frac{1}{2}P^2 + \frac{1}{2}m^2X^2 + \lambda X^4|o\rangle \tag{6.13}$$

Using (6.1), (6.2) and (1.95)-(1.100) we can evaluate each term:

$$\langle n|\tfrac{1}{2}P^2|n\rangle = \langle n|\tfrac{1}{2}[-\tfrac{i}{2}(2\hbar\Omega)^{\frac{1}{2}}\ (a_{\mathfrak{z}}-a_{\mathfrak{z}}^{+})]^2|n\rangle$$

$$= \tfrac{1}{4}\hbar\Omega\langle n|[a_{\mathfrak{z}}\,a_{\mathfrak{z}}^{+} + a_{\mathfrak{z}}^{+}\ a_{\mathfrak{z}}]|n\rangle$$

$$= \frac{\hbar\Omega}{4}\{[n+1] + [n]\} \tag{6.14}$$

$$\langle n|\tfrac{1}{2}\ m^2X^2|n\rangle = \frac{\hbar}{4}\ \frac{m^2}{\Omega}([n] + [n+1]) + \frac{1}{2}\ m^2X_o^2 \tag{6.15}$$

$$\langle n|\lambda X^4|n\rangle = (\frac{\hbar}{2\Omega})^2\lambda\{\langle n|aaa^+a^+ + aa^+aa^+ + a^+aaa^+ \tag{6.16}$$

$$+ aa^+a^+a + a^+aa^+a + a^+a^+aa|n\rangle\}$$

$$+ 6\lambda X_o^2\ \frac{\hbar}{2\Omega}\ \{\langle n|aa^+ + a^+a|n\rangle\} + \lambda X_o^4$$

$$= \lambda(\frac{\hbar}{2\Omega})^2\{[n+1]\ [n+2] + [n+1]^2 + 2[n]\ [n+1]$$

$$+ [n]^2 + [n]\ [n-1]\} + 3\lambda X_o{}^2\ \frac{\hbar}{\Omega}\ ([n]+[n+1]) + \lambda X_o^4$$

Hence

$$\langle n|H|n\rangle = \sum_{\ell=o,2,4} k_\ell X_o^\ell \tag{6.17}$$

where $k_o = \frac{\hbar\Omega}{4}([n] + [n+1]) + \frac{\hbar m^2}{4\Omega}([n] + [n+1])$

$$+\lambda(\frac{\hbar}{2\Omega})^2\ ([n+1]\ [n+2] + [n+1]^2 + 2[n]\ [n+1]$$

$$+ [n]^2 + [n]\ [n-1])$$

$$k_2 = \tfrac{1}{2}m^2 + \frac{3\lambda\hbar}{\Omega}([n] + [n+1])$$

$$k_4 = \lambda$$

The condition for the potential to be a minimum with respect to Ω, is the cubic equation

$$A\,\bar{\Omega}^3 + B\bar{\Omega} + C = o \tag{6.18}$$

where $A = [n] + [n+1]$

$B = -(m^2 + 12\lambda X_o{}^2)$

$$C = -2\hbar\lambda([n+1]\,[n+2] + [n+1]^2 + 2[n+1]\,[n] + [n]^2 + [n]\,[n-1])$$

of the three roots of (6.18), the largest positive one, designated as $\bar{\Omega}$, is to be employed for setting up the effective potential. This procedure is an extrapolation from the usual q=1 bosonic theory [16].

The NP_qEP for the ground state is obtained as

$$\begin{aligned} V_q(X_o)_g &= \langle o|H|o\rangle \\ &= \tfrac{1}{4}\hbar\Omega + \tfrac{1}{2}m^2\left(X_o^2 + \frac{\hbar}{2\Omega}\right) \\ &\quad + \lambda\left\{X_o^4 + \frac{6X_o^2\hbar}{2\Omega} + \frac{\hbar^2}{(2\Omega)^2}([1]+[2])\right\} \end{aligned} \tag{6.19}$$

The optimum $\bar{\Omega}$ is the largest root of the equation

$$\bar{\Omega}^3 - (m^2+12\lambda X_o^2)\bar{\Omega} - 2\hbar\lambda([1] + [2]) = o \tag{6.20}$$

Assuming $\bar{\Omega} \neq o$, (6.19) may be rewritten in the form

$$V_q(X_o)_g = \frac{\hbar}{2}\bar{\Omega} + \frac{1}{2}m^2X_o^2 + \lambda X_o^4 - \frac{\lambda\hbar^2}{(2\bar{\Omega})^2}([1]+[2]) \tag{6.21}$$

The renormalized mass m_R, which is equal to the first excitation energy E_1-E_o, is obtained from $V_q(X_o)_g$ in the following manner.

We have

$$\frac{d^2V_q}{dX_o^2} = m^2 + 12\lambda X_o^2 + \frac{6\hbar}{\bar{\Omega}}\lambda - \frac{6\hbar\lambda}{\bar{\Omega}^2}X_o \;\frac{d\bar{\Omega}}{d\psi_o} \tag{6.22}$$

$$\left(\frac{d^2V_q}{dX_o^2}\right)_{X_o=0} = m_R^2 = m^2 + \frac{6\hbar}{\bar{\Omega}_o}\lambda \tag{6.23}$$

where $\bar{\Omega}_o = \bar{\Omega}|_{X_o=o}$

We have

$$\frac{d\bar{\Omega}}{dX_o} = \frac{24\lambda X_o\bar{\Omega}}{3\bar{\Omega}^2 - m^2 + 12\lambda X_o{}^2} \tag{6.24}$$

Hence

$$\frac{d\bar{\Omega}}{dX_o}\Big|_{X_o=0} = 0 \tag{6.25}$$

But

$$\frac{d^2\bar{\Omega}}{dX_o^2}\Big|_{X_o=0} = \frac{24\,\lambda\,\bar{\Omega}}{3\bar{\Omega}_o^2 - m^2} \tag{6.26}$$

and

$$\frac{d^4V}{dX_o^4}\Big|_{X_o=0} = 24\lambda - \frac{18\hbar\lambda}{\bar{\Omega}_o{}^2}\left(\frac{d^2\bar{\Omega}}{dX_o^2}\right)_{X_o=0} \tag{6.27}$$

The renormalized coupling con[illegible]t λ_R is having the form:

$$\frac{1}{4!}\,\frac{d^4V}{dX_o^4} = \lambda_R = \left(1 - \frac{12\lambda\hbar}{3\bar{\Omega}_o^3 - m_R^2\bar{\Omega}_o}\right) / \left(1 + \frac{6\hbar\lambda}{3\bar{\Omega}_o^3 - m_R^2\bar{\Omega}_o}\right) \tag{6.28}$$

We have studied the variation of m_R^2 given by (6.23) as a function of q in Fig. (VI.1) and found that for $q>o$, $m_R^2>o$ and that m_R^2 has a maximum at q=1, the ordinary bosonic limit. If m^2 is not very much larger than λ, for most of the negative q values examined, $\bar{\Omega}_o$ becomes negative. Recalling that in the ordinary bosonic theory the mass parameter $\bar{\Omega}$ for the ground state GEP, is kept positive for convergence reasons [16], we are prompted to retain this proviso in the q-boson|(q,p)-boson theory, as well. [illegible] and is the positivity of m_R^2. However, [illegible] small [illegible]ive λ (compared to m^2), we obtain

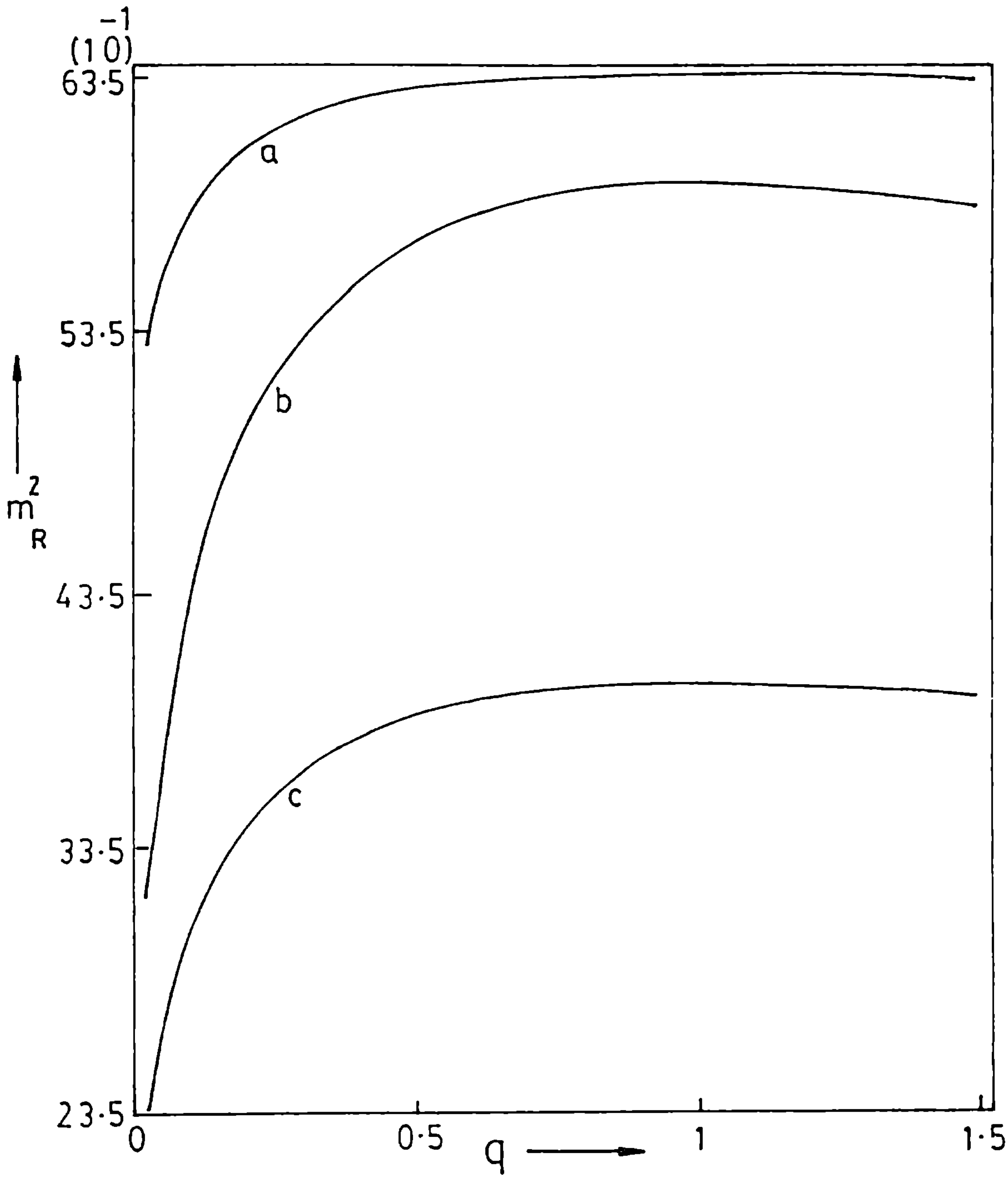


Fig. VI.1 Variation of the renormalized mass m_R^2 with the q parameter for a q-oscillator moving in a quartic potential well

a) m=2 λ=1; b) m=1 λ=2; c) m=1 λ=1.

positive $\bar{\Omega}_o$ and positive m_R^2.

The values of $\bar{\Omega}_o$, m_R^2 and λ_R for various q values for a single well quartic well are presented in Table VI.1.

In order to evaluate the ground state SS_qEP, we extremise the ground state potential (6.19) with respect to q:

$$\frac{\hbar^2\lambda}{(2\bar{\Omega})^2}(1-q^{-2}) = 0$$

Since $\lambda \neq o, \bar{\Omega} \neq \infty$, it follows that the extrema correspond to $q=\pm 1$. One readily checks that for positive λ, the ground state potential is a minimum for q=1 (giving SS_qEP=GEP) and a maximum for q=-1.

Since we have

$$\frac{d^2V_q(X_o)_g}{dq^2} = \frac{\hbar^2\lambda}{(2\bar{\Omega})^2}\, 2q^{-3}, \tag{6.29}$$

this expression is >0 for q=1, and <0 for q= -1. However, the extremal condition for the potential for the nth excited state, when written out in full, is a complicated algebraic equation.

For the nth excited state, from (6.17)

$$\frac{dV_q}{dq} = A_1\left\{\frac{\hbar\bar{\Omega}}{4} + \frac{m^2\hbar}{4\bar{\Omega}} + \frac{3\lambda\hbar\phi_o^2}{\bar{\Omega}}\right\} + B_1\lambda\left(\frac{\hbar}{2\bar{\Omega}}\right)^2 = 0 \tag{6.30}$$

where

$$A_1 = \frac{d}{dq}[n] + \frac{d}{dq}[n+1]$$

Table VI.1

Values of optimum mass parameter $\bar{\Omega}_0$, renormalized mass m_R^2 and renormalized coupling constant λ_R for various bare-values of m and λ

Bare values of parameters	q	$\bar{\Omega}_0$	m_R^2	λ_R
m=2	0.2	2.881538	6.082222	1.624577
λ=1	0.4	2.637647	6.27455	1.420569
	0.6	2.559752	6.343977	1.385415
	0.8	2.531691	6.369958	1.374637
	1.0	2.525102	6.376142	1.37223
	1.2	2.529499	6.372011	1.373831
	1.4	2.540098	6.362114	1.377775
	-0.2	-2.672633	1.755023	1.41822
	-0.4	-2.367502	1.465684	1.626652
	-0.6	1.53168	7.917268	1.227186
	-0.8	1.651885	7.632214	1.234678
	-1.0	1.675131	7.58181	1.236247
	-1.2	1.659775	7.614949	1.235206
	-1.4	1.619974	7.703764	1.232594
	-1.6	1.55805	7.850967	1.228752
	-1.8	1.466909	8.090235	1.223485

Bare values of parameters	q	$\bar{\Omega}_o$	m_R^2	λ_R
m = 2	-2.0	1.302779	8.605541	1.214684
λ = 1	-2.2	-2.328351	1.423069	1.662928
	-2.4	-2.354402	1.451582	1.638482
	-2.6	-2.380612	1.47964	1.615109

For all the lower negative values $\bar{\Omega}_o$ is negative

m=1	0.2	3.030445	4.959814	-0.7189545
λ=2	0.4	2.631953	5.559353	3.457643
	0.6	2.496712	5.806321	2 [illegible]12747
	0.8	2.446694	5.904578	2.226091
	1.0	2.434839	5.928458	2.189219
	1.2	2.442756	5.912484	2.213577
	1.4	2.461755	5.874573	2.276614

For all negative values of q, $\bar{\Omega}_o$ is negative.

m=1	0.2	2.458426	3.440586	0.1617446
λ=1	0.4	2.150903	3.789527	-2.625644
	0.6	2.047257	3.93075	-162.5988
	0.8	2.009047	3.986491	12.24415
	1.0	1.999999	4.000002	9.999777
	1.2	2.006041	3.990966	11.38229
	1.4	2.020547	3.969493	17.51249

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λ = 1	-2.2	-2.328351	1.423069	1.662928
	-2.4	-2.354402	1.451582	1.638482
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	1.0	1.999999	4.000002	9.999777
	1.2	2.006041	3.990966	11.38229
	1.4	2.020547	3.969493	17.51249

For all negative values of q, $\bar{\Omega}_o$ is negative.

Bare values of parameters	q	$\bar{\eta}_0$	m_R^2	λ_R
m=4	-0.01	-6.7373211	15.10944	0.9712756
λ =1	-0.04	-5.051526	14.81224	0.9752612
	-0.07	-4.661561	14.71288	0.9746904
	-0.08	2.756399	18.17675	1.021381
	- 0.1	3.215603	17.8659	1.019552
	- 0.4	3.875497	17.54819	1.017773
	- 0.7	3.927506	17.52769	1.017667
	- 0.9	3.935242	17.52469	1.017652
	- 1.0	3.935374	17.52463	1.01765
	- 1.2	3.933782	17.52525	1.017654
	- 1.4	3.928449	17.52732	1.017665
	- 5	3.705849	17.61906	1.01815
	-10	3.215603	17.8659	1.019552
	-13	2.567778	18.33665	1.022316
	-14	-4.649964	14.70967	0.974666
	-15	-4.690233	14.72075	0.974749

For lower negative q values, $\bar{\eta}_0$ is negative. For positive q values corresponding to m=4, λ=1, the behaviour is similar to that of the values of m and λ considered above.

$$= \sum_{k=1}^{n} (-n+1)q^{-n}q^{2k-2} + \sum_{k=1}^{n} q^{-n+1}(2k-2)q^{2k-3}$$

$$+ \sum_{k=1}^{n+1} (-n)\, q^{-n-1}\, q^{2k-2} + \sum_{k=1}^{n+1} q^{-n}(2k-2)\, q^{2k-3} \qquad (6.31)$$

$$B_1 = \frac{d}{dq}([n+1][n+2] + [n+1]^2 + 2[n][n+1] + [n]^2 + [n][n-1])$$

$$= [\sum_{k=1}^{n+2} (-n-1)q^{-n-2}\, q^{2k-2} + \sum_{k=1}^{n+2} q^{-n-1}(2k-2)q^{2k-3}]$$

$$[\sum_{r=1}^{n+1} q^{-n} q^{2r-2}]$$

$$+[\sum_{k=1}^{n+2} q^{-n-1} q^{2k-2}]\ [\sum_{r=1}^{n+1} (-n)q^{-n-1} q^{2r-2}$$

$$+ \sum_{r=1}^{n+1} q^{-n}(2r-2)q^{2r-3}] + 2[\sum_{k=1}^{n+1} q^{-n}\, q^{2k-2}]$$

$$[\sum_{r=1}^{n+1} (-n)\, q^{-n-1} q^{2r-2} + \sum_{r=1}^{n+1} q^{-n}(2r-2)\, q^{2r-3}]$$

$$+ 2 \{ [\sum_{k=1}^{n+1} q^{-n}\, q^{2k-2}]\ [\sum_{r=1}^{n} (-n+1)\, q^{-n} q^{2r-2}$$

$$+ \sum_{r=1}^{n} q^{-n+1}(2r-2)\, q^{2r-3}] + [\sum_{k=1}^{n+1} (-n)$$

$$q^{-n-1} q^{2k-2} + \sum_{k=1}^{n+1} q^{-n}(2k-2)\, q^{2k-3}]$$

$$[\sum_{r=1}^{n} q^{-n+1}\, q^{2r-2}]\} + 2[\sum_{k=1}^{n} q^{-n+1} q^{2k-2}]$$

$$[\sum_{r=1}^{n} (-n+1)q^{-n}\, q^{2r-2} + \sum_{r=1}^{n} q^{-n+1}(2r-2)q^{2r-3}]$$

$$+ \sum_{k=1}^{n} q^{-n+1}\, q^{2k-2} [\sum_{r=1}^{n-1} (-n+2)\, q^{-n+1}$$

$$q^{2r-2} + \sum_{r=1}^{n} q^{-n+2}(2r-2)\, q^{2r-3}]$$

$$+ \left[\sum_{k=1}^{n} (-n+1)\, q^{-n}\, q^{2k-2} + \sum_{k=1}^{n} q^{-n+1} (2k-2)\, q^{2k-3}\right]$$

$$\left[\sum_{r=1}^{n-1} q^{-n+2}\, q^{2r-2}\right] \qquad (6.32)$$

The above equation is of 2(2n+1) degree in q. Some of its roots may represent minima while others, maxima, corresponding to $q \neq \pm 1$.

By invoking the (q,p) commutation relations (6.7) and (6.8), one obtains the $NP_{qp}EP$ for any state $|n\rangle_{q,p}$. The potential is formally the same as NP_qEP with all the q-deformed [ ] replaced by corresponding (q,p) - deformed ones $[\ \ ]_{q,p}$. The $\bar{\Omega}$ equations are same as (6.18) and (6.20) with the above mentioned replacements of brackets.

For a(q,p) oscillator the ground state effective potential is a minimum only if

$$\frac{\partial}{\partial q} V_{q,p} = 0$$

$$= \frac{\hbar^2 \lambda}{(2\bar{\Omega})^2} \frac{\partial}{\partial q}\left[[1]_{q,p} + [2]_{q,p}\right] \qquad (6.33)$$

Also

$$\frac{\partial V_{qp}}{\partial p} = 0 = \frac{\hbar^2 \lambda}{(2\bar{\Omega})^2} \frac{\partial}{\partial p}\left[[1]_{q,p} + [2]_{q,p}\right] \qquad (6.34)$$

We have

$$\frac{\lambda \hbar^2}{(2\bar{\Omega})^2} = 0 \text{ and } \frac{\lambda \hbar^2}{(2\bar{\Omega}_p)^2} = 0 \qquad (6.35)$$

These relations imply that either $\lambda = 0$ or $\hbar = 0$. Since the latter condition can be easily ruled out, one is confronted with the possibility of a trivial non-interacting (q,p) oscillator theory. The message is clear: The ground state $SS_{q\bar{p}}EP$ for a quartic (q,p) oscillator cannot be found by the variational method herein presented. These remarks however, need not apply to the excited states.

6.3 Double well potentials

The Hamiltonian for a double well quartic potential is chosen in the form

$$H = \frac{1}{2}P^2 + \frac{1}{2}m^2X^2 + \lambda X^4 + \frac{m^4}{16\lambda} \qquad (6.36)$$

where $m^2 < o$, $\lambda > o$. As this differs from the single well Hamiltonian only by the presence of the constant term, the evaluation of the nonperturbative effective potential is not a novel exercise.

The equations (6.17), (6.19) and (6.21) will be repeated herewith an additional term $\frac{m^4}{16\lambda}$. The equations (6.18) and (6.20) will be the same in this case also. The renormalized parameters will be represented by the equations (6.23) and (6.28).

The only interesting point that crops up is the existence of critical parameter values that separate the double well region from the single well region. This becomes evident from a numerical study.

Fixing $\lambda = 1$ and $\frac{m^2}{\lambda} = -2$ it is seen that for a q oscillator, there is a critical value for q, denoted by $q_{crit\ 1}$, above which the double well shape degenerates into a single well shape and another critical value $q_{crit\ 2}$ at which the double well shape is regained. In the present case, $q_{crit\ 1} \approx 0.16$ and $q_{crit\ 2} \approx 6.27$. Again with $\lambda = 2$, $q_{crit\ 1} \approx 0.26$ and the second critical value $q_{crit\ 2} \approx 3.66$ at which the double well shape is recovered. The variation of V_q vs X_0 for various values of the q parameter is represented in Table VI.2 and Fig.VI.2.

Critical behaviour is exhibited also by (q,p) oscillators moving in a double well potential. In this case one varies both q and p. For instance, taking $\lambda = 2$, $\frac{m^2}{\lambda} = -2$, $q = 0.2$, we have obtained double well behaviour in the domain $-0.83 \leqslant p \leqslant 0.26$. However, for $-0.83 \leqslant p \leqslant 0$, $\bar{\Omega}_0$ is negative and for $0 < p \leqslant 0.25$ $\bar{\Omega}_0$ is positive. In the single well region corresponding to $p \leqslant -0.83$, $\bar{\Omega}_0$ is positive. Similar critical behaviour may be monitored, alternatively, by keeping p fixed and tuning q.

6.4 Sextic quantum oscillators

A general sextic anharmonic oscillator is modelled by the Hamiltonian

$$H = \frac{1}{2}P^2 + \sum_{j=1}^{6} C_j X^j \qquad (6.37)$$

Table VI.2

Variation of $\bar{V}_q$ a double well potential vs X_o for different q values

Bare values of parameters	q=0.2		q=4	
	X_o	V_q	X_o	V_q
$\frac{m^2}{\lambda} = -2$	0	0.6049316	0	0.566014
$\lambda = 2$	0.05	0.6043606	0.05	0.5658453
	0.1	0.6026885	0.1	0.565358
	0.15	0.6000521	0.15	0.564612
	0.2	0.5966906	0.2	0.5637358
	0.25	0.5929851	0.25	0.5629642
	0.3	0.5894944	0.3	0.5626902
	0.35	0.5869968	0.35	0.5635233
	0.4	0.586243	0.4	0.5661694
	0.45	0.5894805	0.45	0.5724065
	0.5	0.5975545	0.5	0.583334
	0.55	0.6128652	0.55	0.6011693

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	0.5	0.5975545	0.5	0.583334
	0.55	0.6128652	0.55	0.6011693

Bare values of parameters	q=0.26		q=3.7		q=0.3		q=3	
	X_o	V_q	X_o	V_q	X_o	V_q	X_o	V_q
$\frac{m^2}{\lambda}=-2$	0	0.5594239	0	0.5529971	0	0.5361255	0	0.5198197
$\lambda=2$	0.05	0.5593281	0.05	0.5529738	0.05	0.5362983	0.05	0.520189
	0.1	0.5590536	0.1	0.5529106	0.1	0.5368088	0.1	0.5212796
	0.15	0.5586484	0.15	0.5528418	0.15	0.537653	0.15	0.5230435
	0.2	0.5582175	0.2	0.5528523	0.2	0.5388541	0.2	0.5254427
	0.25	0.5579678	0.25	0.55312	0.25	0.5405144	0.25	0.5284971
	0.3	0.5582635	0.3	0.5539759	0.3	0.5428785	0.3	0.5323617
	0.35	0.5596815	0.35	0.5559711	0.35	0.5464026	0.35	0.5374151
	0.4	0.56299	0.4	0.5599312	0.4	0.5516995	0.4	0.544399
	0.45	0.5696836	0.45	0.5670588	[illegible]	0.5603685	0.45	0.5541792
	0.5	0.581081	0.50	0.578929	0.5	0.5734778	0.5	0.5684806
	0.55	0.5993396	0.55	0.5975971	0.55	0.5932103	0.55	0.5892264

Bare values of parameters	q=0.1		q=0.2		q=7	
	X_o	V_q	X_o	V_q	X_o	V_q
$\frac{m^2}{\lambda}=-2$	0	0.4746122	0	0.3802683	0	0.424101
$\lambda=1$	0.05	0.4741456	0.05	0.3804073	0.05	0.4239276
	0.1	0.4728031	0.1	0.3808506	0.1	0.4234543
	0.15	0.4707307	0.15	0.3816738	0.15	0.422797
	0.2	0.4681809	0.2	0.3830187	0.2	0.4221562
	0.25	0.46552	0.25	0.3850886	0.25	0.4218242
	0.3	0.463213	0.3	0.3881817	0.3	0.4221882
	0.35	0.4618702	0.35	0.3927014	0.35	0.4237655
	0.4	0.4619786	0.4	0.3992691	0.4	0.4275082
	0.45	0.4649722	0.45	0.4081855	0.45	0.4331793
	0.5	0.4713113	0.5	0.4206476	0.5	0.4426523
	0.55	0.4822873	0.55	0.437527	0.55	0.4567147

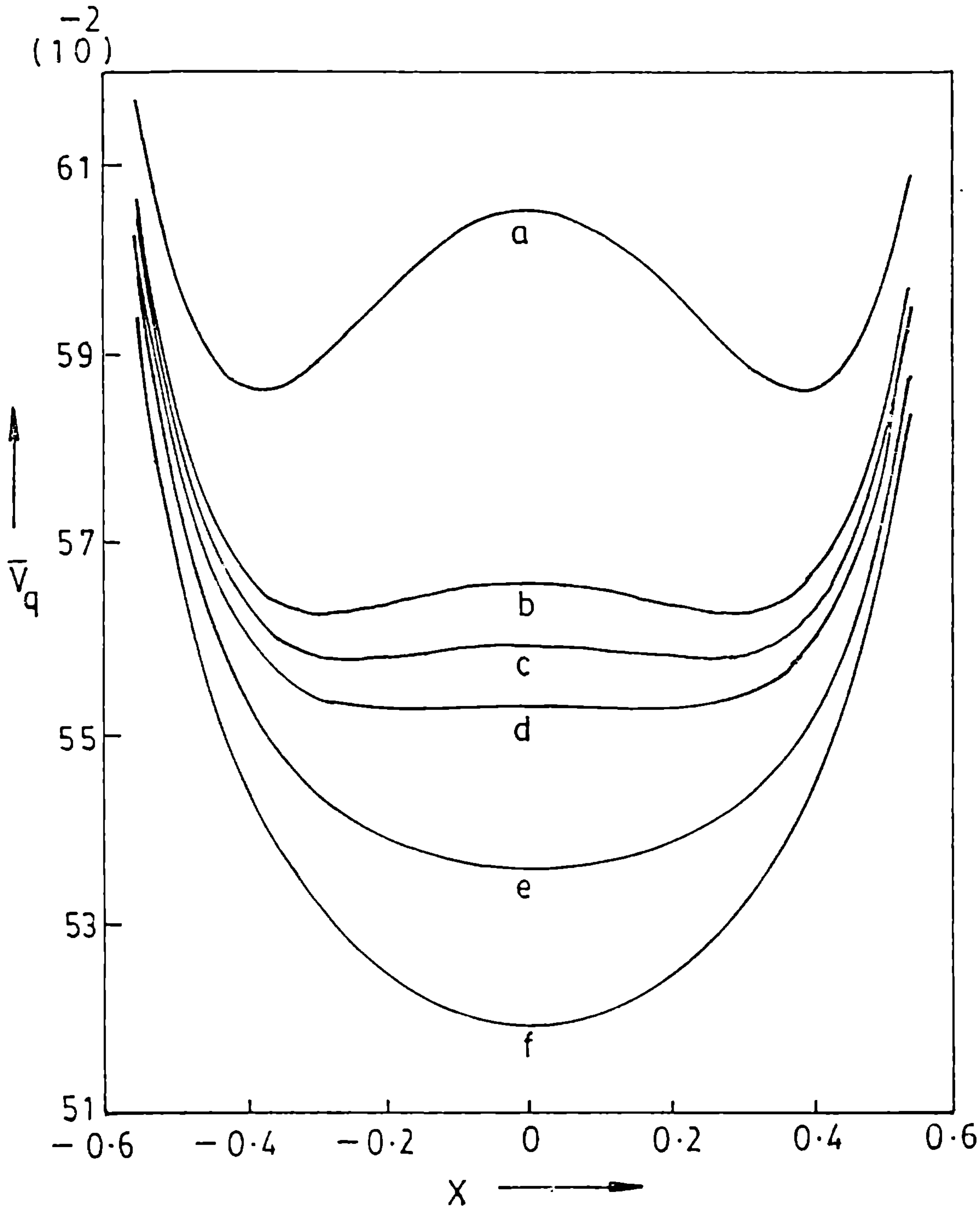

Fig. VI.2 Variation of V_q of a q-oscillator moving in a double well potential, with position X.
a) q=0.2; b) q=4; c) q=0.26; d) q=3.7;
e) q=0.3; f) q=3

Assuming q-commutation relations, the expectation value of the Hamiltonian for the nth state is obtained:

$$\langle n|H|n\rangle = \sum_{\ell=o}^{6} C_\ell g_\ell \qquad (6.38)$$

where C_o is a constant that depends only on Ω and q parameters:

$$C_o = \frac{\hbar\Omega}{4}([n] + [n+1])$$

The remaining coefficients C_ℓ ($\ell \neq o$) have the same significance as in (6.37). The functions g_ℓ are given by the following set of relations:

$$g_o = 1$$

$$g_1 = X_o$$

$$g_2 = X_o^2 + \frac{\hbar}{2\Omega}([n] + [n+1])$$

$$g_3 = X_o^3 + \frac{3\hbar}{2\Omega}X_o([n] + [n+1])$$

$$g_4 = X_o^4 + \frac{3\hbar}{\Omega} X_o^2([n]+[n+1]) + (\frac{\hbar}{2\Omega})^2$$
$$([n+2]\ [n+1] + [n+1]^2 + 2[n+1]\ [n] + [n]^2 + [n][n-1])$$

$$g_5 = X_o^5 + \frac{10X_o^3\hbar}{2\Omega}([n]+[n+1]) + \frac{5X_o\hbar^2}{(2\Omega)^2}([n+2]\ [n+1]$$
$$+ [n+1]^2 + 2[n+1]\ [n] + [n]^2 + [n]\ [n-1])$$

$$g_6 = X_o^6 + \frac{15X_o^4\hbar}{2\Omega}([n] + [n+1]) + 15X_o^2(\frac{\hbar}{2\Omega})^2 ([n+2]\ [n+1] +$$
$$+ [n+1]^2 + 2[n+1]\ [n] + [n]^2$$
$$+ [n]\ [n-1]) + (\frac{\hbar}{2\Omega})^3([n+3]\ [n+2]$$
$$[n+1] + [n+2]^2\ [n+1] + 2[n+2]\ [n+1]^2$$

$$+ 2[n+2]\,[n+1]\,[n] + [n+1]^3 + 2[n+1]\,[n-1]\,[n]$$
$$+ 3[n+1]^2\,[n] + 3[n+1]\,[n]^2 + [n]^3$$
$$+ 2[n]^2[n-1] + [n]\,[n-1]^2 + [n]\,[n-1]\,[n-2])$$

The condition for the optimum mass parameter $\bar{\Omega}$ is expressed as

$$\bar{\Omega}^4 + D\,\bar{\Omega}^2 + \hbar E\,\bar{\Omega} + \hbar^2 F = 0 \qquad (6.39)$$

where the coefficients are given by

$$D = -(2C_2+6C_3X_o + 12C_4X_o^2 + 20C_5X_o^3 + 30C_6X_o^4)$$

$$E = -(C_4+5C_5X_o+15C_6X_o^2)\,([n+2][n+1] + [n+1]^2+2[n+1][n]$$
$$+ [n]^2 + [n]\,[n-1])/([n] + [n+1])$$

$$F = -\frac{3}{2}\,C_6([n+3][n+2][n+1] + [n+2]^2[n+1] + 2[n+2][n+1]^2$$
$$+ 2[n+2][n+1][n] + [n+1]^3 + 2[n+1][n-1][n]$$
$$+ 3[n+1]^2[n] + 3[n+1][n]^2 + [n]^3 + 2[n]^2[n-1]$$
$$+ [n][n-1]^2 + [n]\,[n-1]\,[n-2])$$

The ground state expectation value of the Hamiltonian is

$$\langle o|H|o\rangle = f_o + \sum_{k=1}^{6} C_k\, f_k \qquad (6.40)$$

where $f_o = \frac{\hbar\Omega}{4} + \left(\frac{\hbar}{2\Omega}\right)^3 C_6([1] + 2[2] + [2]^2 + [2]\,[3])$

$$f_1 = X_o$$
$$f_2 = X_o^2 + \frac{\hbar}{2\Omega}$$
$$f_3 = X_o^3 + \frac{3\hbar}{2\Omega}X_o$$
$$f_4 = X_o^4 + \frac{6\hbar}{2\Omega}X_o^2 + \left(\frac{\hbar}{2\Omega}\right)^2([1] + [2])$$

$$f_5 = X_o^5 + \frac{10\hbar}{2\Omega} X_o^3 + 5X_o(\frac{\hbar}{2\Omega})^2([2] + [1])$$

$$f_6 = X_o^6 + \frac{15\hbar}{2\Omega} X_o^4 + 15(\frac{\hbar}{2\Omega})^2([1] + [2])X_o^2$$

The optimization condition reads

$$\bar{\Omega}^4 + G\bar{\Omega}^2 + H\bar{\Omega} + I = 0 \qquad (6.41)$$

with the symbols standing for the following:

$$G = -(2C_2 + 6C_3X_o + 12C_4X_o^2 + 20C_5X_o^3 + 30C_6X_o^4)$$

$$H = -\hbar(2C_4+10C_5X_o+30C_6X_o^2)([1]+[2])$$

$$I = \frac{-3}{2} C_6\hbar^2([1] + 2[2] + [2]^2 + [2]\,[3])$$

Denoting the largest positive root of (6.41) at $X_o=0$ by $\bar{\Omega}_o$, one writes an expression for the renormalized mass:

$$m_R^2 = 2C_2+12C_4(\frac{\hbar}{2\bar{\Omega}_o}) + 30C_6([1] + [2])\,(\frac{\hbar}{2\bar{\Omega}_o})^2$$
$$-\{3C_3 \frac{\hbar}{2\bar{\Omega}_o^2} + 5C_5 \frac{\hbar^2}{2\bar{\Omega}_o^3}([2] + [1])\} \frac{d\bar{\Omega}}{dX_o}\Big|_{X_o=o} \qquad (6.42)$$

where

$$\frac{d\bar{\Omega}}{dX_o}\Big|_{X_o=o} = \frac{6C_3\bar{\Omega}_o^2 + 5C_5\hbar\bar{\Omega}_o([2] + [1])}{4\bar{\Omega}_o^3 - 4C_2\bar{\Omega}_o - C_4\hbar([2]+[1])} \qquad (6.43)$$

Numerical computations show (Table VI.3 and Figs.VI.3) that when q is negative, for positive coefficients $C_1 \ldots C_6$ and C_6 not very much smaller than C_2, all the roots of the

Table VI.3

m_R^2 versus q for 3 different sets of coefficients

Bare values of parameters	q	$\bar{\Omega}_0$	m_R^2
$C_1 = 6$	0.2	4.2323	14.2740
$C_2 = 5$	0.4	2.9695	5.5108
$C_3 = 4$	0.6	2.5682	− 48.0659
$C_4 = 3$	0.8	2.4211	583.5098
$C_5 = 2$	1.0	2.3863	210.988
$C_6 = 1$	1.2	2.4095	359.639
	1.4	2.4654	−327.4481
	1.6	2.5404	− 67.9981
	1.8	2.6269	− 25.4698
$C_1 = 1$	0.2	6.1826	14.3384
$C_2 = 2$	0.4	4.1309	17.1876
$C_3 = 3$	0.6	3.5175	17.7071
$C_4 = 4$	0.8	3.3021	17.6001
$C_5 = 5$	1.0	3.2520	17.5327
$C_6 = 6$	1.2	3.2854	17.5798
	1.4	3.3663	17.6599
	1.6	3.4763	17.7054
	1.8	3.6051	17.6881

Bare values of parameters	q	$\bar{\Omega}_0$	m_R^2
C_1=C_2=	0.2	3.9903	6.1044
C_3=C_4=	0.4	2.6982	7.1824
C_5=C_6=1	0.6	2.3156	7.3933
	0.8	2.1818	7.3757
	1.0	2.1508	7.3586
	1.2	2.1715	7.3707
	1.4	2.2217	7.3895
	1.6	2.2900	7.3957
	1.8	2.3700	7.381

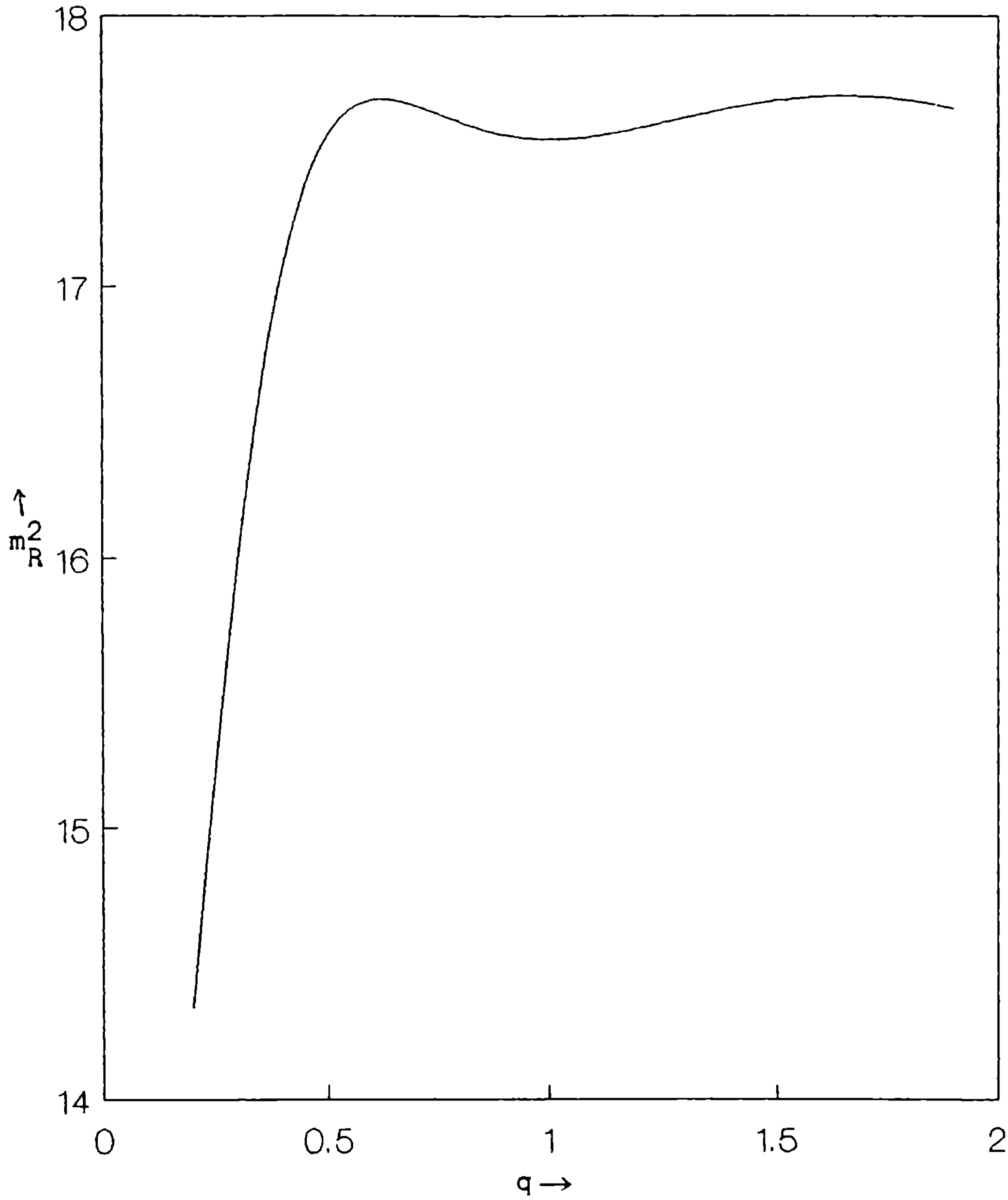

Fig. VI.3 a) Variation of the renormalized mass m_R^2 with the q parameter for a q-oscillator moving in a sextic potential well with $C_1=1$, $C_2=2$, $C_3=3$, $C_4=4$, $C_5=5$, $C_6=6$.

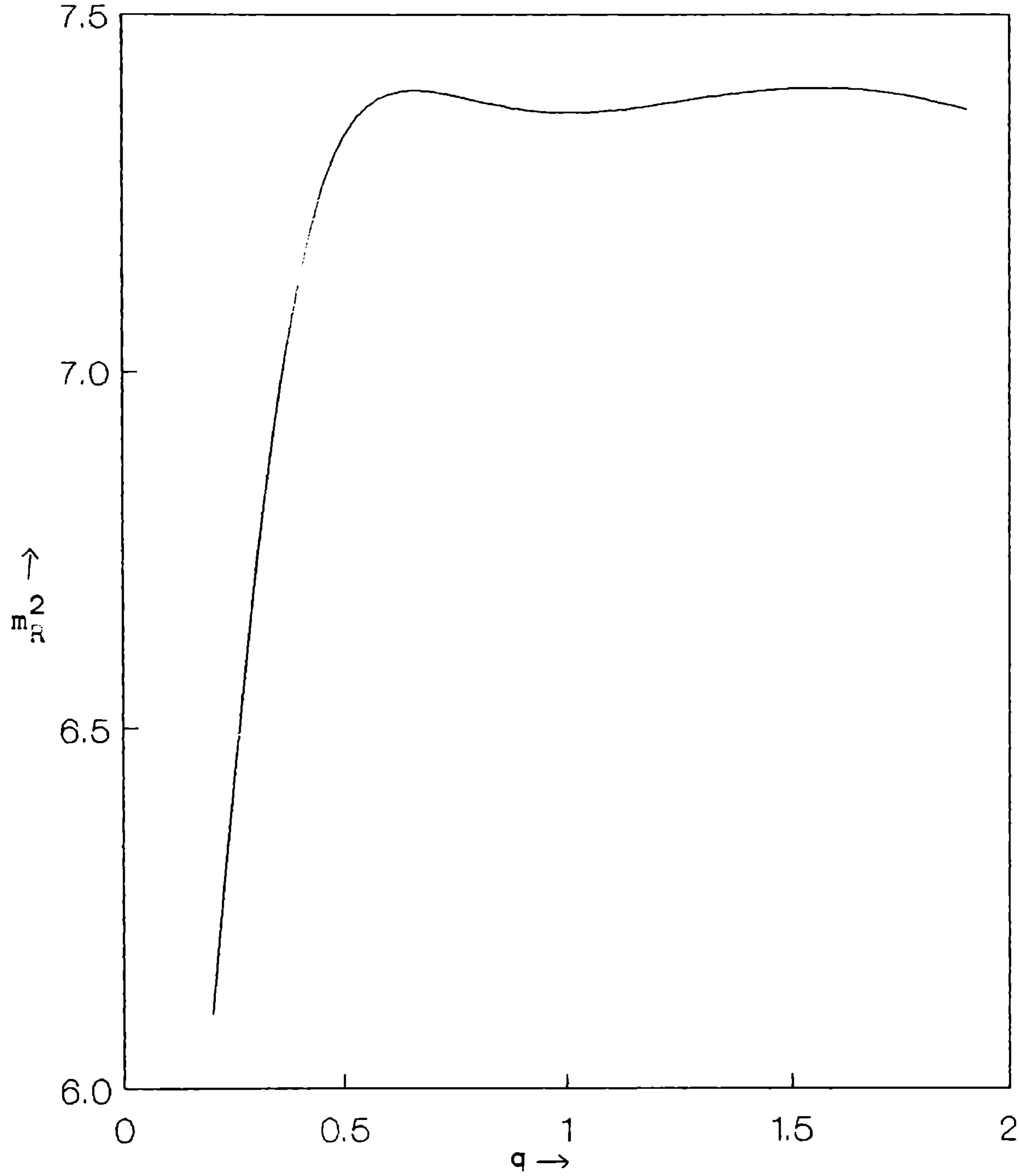

Fig. VI.3.b) Variation of renormalized mass m_R^2 for a q-oscillator moving in a sextic potential well with $C_1=C_2=C_3=C_4=C_5=C_6=1$.

quartic equation (6.41) are imaginary. Under the same conditions, and for positive q values, two real roots exist, of which one is positive and the other negative.

Setting the odd order coefficients equal to zero, we obtain a sextic oscillator with only even powers of X in the potential. In this case the renormalised mass is

$$m_R^2 = 2C_2 + \frac{6\hbar C_4}{\bar{\Omega}_0} + 30C_6 \left(\frac{\hbar}{2\bar{\Omega}_0}\right)^2 ([1] + [2]) \tag{6.44}$$

A plot of m_R^2 vs positive q values, is quite similar to that for the X^4 theory (Table VI.4 and Fig.VI.4). Taking only even order coefficients as positive, and C_6 not very much smaller than C_2, for negative q values the theory is not defined, because all the roots of the $\bar{\Omega}$ equation (6.41) become imaginary. In a general setting with all the coefficients $C_1 \ldots C_6$ not equal to zero, the behaviour is different from that for the X^4- model, at least in the typical cases herein studied. This shows that the quantum sextic model possesses physical significance only in selected parameter domains.

One can define renormalized quartic (C_{4R}) and renormalized sextic (C_{6R}) coupling constants in respect of the even power sextic q-oscillator model. Thus

$$C_{4R} = \frac{1}{4!} \frac{d^4 V_q}{dX_o^4}\bigg|_{X_o = 0} \tag{6.45}$$

Table VI.4

Values of optimum mass parameter $\bar{\Omega}_0$, renormalized mass m_R^2 and renormalized coupling constants C_{4R} and C_{6R} for various values of bare parameters

Bare values of parameters	q	$\bar{\Omega}_0$	m_R^2	C_{4R}	C_{6R}
$C_1=C_3=$	0.2	4.1233	11.6452	1.7608	48.3782
C_5 =0	0.4	2.8728	13.7212	- 3.4733	129.5493
C_2 =3	0.6	2.4938	14.7511	-12.9765	337.6716
C_4 =2	0.8	2.3588	15.1984	-22.8249	649.5359
C_6 =1	1.0	2.3272	15.3107	-26.6451	797.1662
	1.2	2.3483	15.2354	-23.9995	693.3396
	1.4	2.3992	15.0593	-19.0162	517.201
	1.6	2.4681	14.8325	-14.3267	374.586
	1.8	2.5484	14.5839	-10.6122	277.479

Bare values of parameters	q	$\bar{\Omega}_o$	m_R^2	C_{4R}	C_{6R}
$C_1=C_3$	0.2	3.9903	6.4240	1.3020	31.0291
$=C_5=0$	0.4	2.6982	8.2411	-1.4532	57.5225
$C_2=C_4$	0.6	2.3150	9.1601	-4.6462	88.3192
$=C_6=1$	0.8	2.1818	9.5550	-6.6484	109.8417
	1.0	2.1508	9.6533	-7.2249	116.392
	1.2	2.1715	9.5847	-6.8350	111.9436
	1.4	2.2217	9.4326	-5.9772	102.4172
	1.6	2.2900	9.2323	-4.9781	91.7590
	1.8	2.3700	9.0116	-4.0061	81.8259
$C_1=C_3$	0.2	5.1895	9[illegible]21	2.7650	115.1282
$=C_5=0$	0.4	3.4572	12.8123	- 3.6784	205.8184
$C_2=1$	0.6	2.9388	14.5929	-11.3716	306.1851
$C_4=2$	0.8	2.7569	15.3817	-16.2420	374.702
$C_6=3$	1.0	2.7145	15.5805	-17.6473	395.3591
	1.2	2.7428	15.4472	-16.6966	381.339
	1.4	2.8111	15.1357	-14.6072	351.1839
	1.6	2.9040	14.7360	-12.1776	317.2088
	1.8	3.0128	14.3001	- 9.8201	285.2893

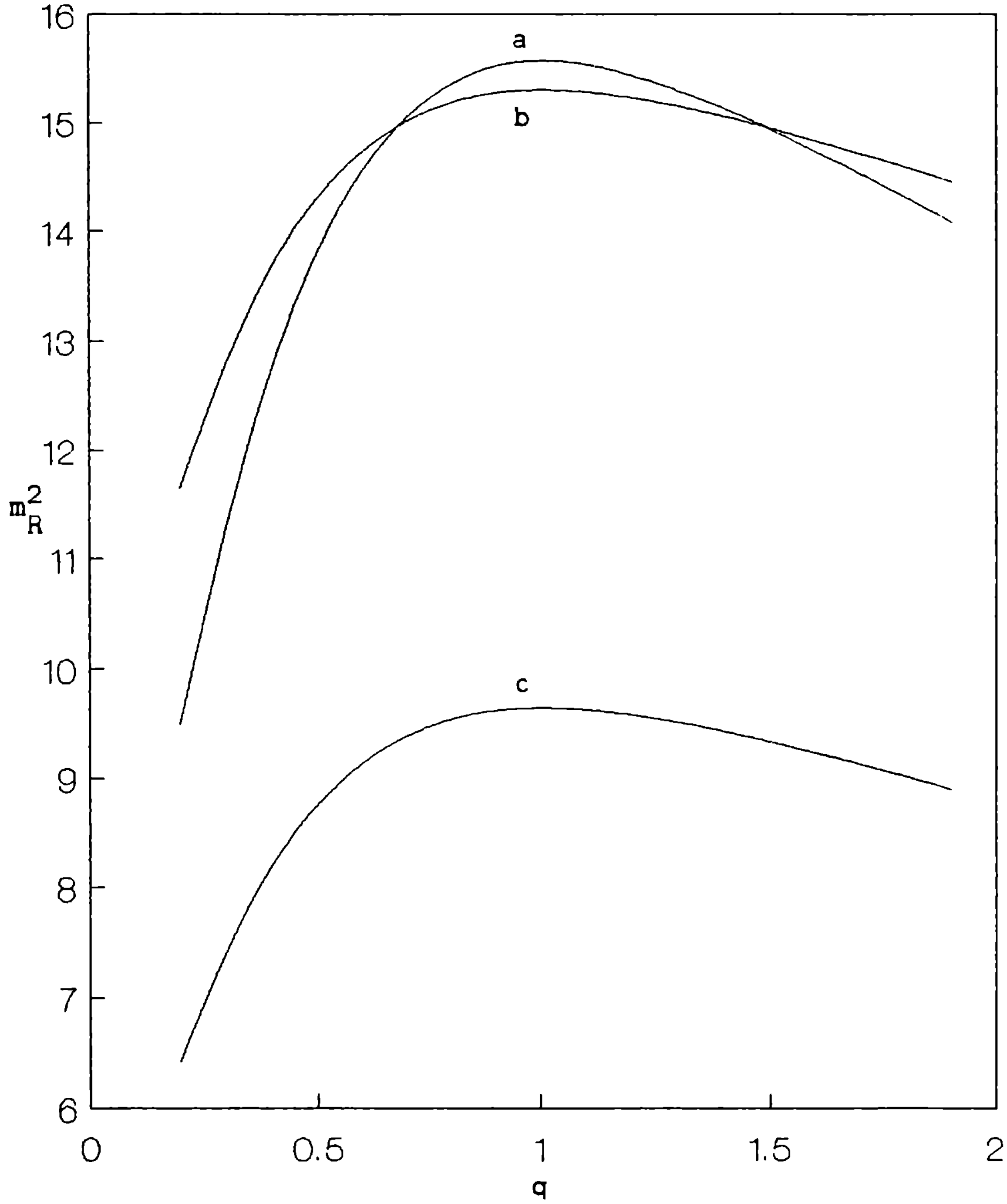

Fig. VI.4 Variation of the renormalized mass m_R^2 with the q-parameter for a q-oscillator moving in a sextic potential well

a) $C_1=C_3=C_5=o$, $C_2=1$, $C_4=2$, $C_6=3$

b) $C_1=C_3=C_5=o$, $C_2=3$, $C_4=2$, $C_6=1$

c) $C_1=C_3=C_5=o$, $C_2=C_4=C_6=1$

$$= C_4 - \frac{3C_4}{24}\frac{\hbar}{\bar{\Omega}_o^2}\frac{d^2\bar{\Omega}}{dX_o^2}\Big|_{X_o=o} + 15C_6\frac{\hbar}{2\bar{\Omega}_o}$$

$$- \frac{15}{8}C_6\frac{\hbar^2}{\bar{\Omega}_o^3}([1] + [2])\frac{d^2\bar{\Omega}}{dX_o^2}\Big|_{X_o=o} \qquad (6.46)$$

and

$$\frac{d^2\bar{\Omega}}{dX_o^2}\Big|_{X_o=o} = \frac{24\bar{\Omega}_o^2C_4 + 60C_6\hbar\bar{\Omega}_o([1] + [2])}{4\bar{\Omega}_o^3 - 4C_2\bar{\Omega}_o - 2C_4\hbar([1]+[2])}$$

Expressing the fourth derivative of $\bar{\Omega}$ at $X_o=o$, in the form

$$\frac{d^4\bar{\Omega}}{dX_o^4}\Big|_{X_o=o} = \Big\{-720C_6\bar{\Omega}_o^2 - 2[144C_4\bar{\Omega}_o + 180C_6([1]+[2])]\frac{d^2\bar{\Omega}}{dX_o^2}\Big|_{X_o=o} \qquad (6.47)$$

$$+ (36\bar{\Omega}_o^2 - 12C_2)\left(\frac{d^2\bar{\Omega}}{dX_o^2}\right)_{X_o=o}\Big\}/(4\bar{\Omega}_o^3 - 4\bar{\Omega}_oC_2$$

$$- 2C_4\hbar([1] + [2]))$$

$$C_{6R} = \frac{1}{6!}\frac{d^6V_g}{dX_o^6}\Big|_{X_o=o}$$

$$= C_6 - \frac{5C_6\hbar}{2\bar{\Omega}_o^2} - \frac{\hbar}{2\bar{\Omega}_o^2}\Big[\frac{1}{12}C_4 + \frac{\hbar}{2\bar{\Omega}_o}([1]+[2])\frac{10C_6}{24}\Big]\left(\frac{d^4\bar{\Omega}}{dX_o^4}\right)_{X_o=o}$$

$$+ \frac{\hbar}{2\bar{\Omega}_o^3}\Big[\frac{C_4}{2} + \frac{\hbar}{2\bar{\Omega}_o}\frac{15C_6}{4}([1]+[2])\Big]\left(\frac{d^2\bar{\Omega}}{dX_o^2}\right)_{X_o=o} \qquad (6.48)$$

To get the system specific q effective potential (SS_qEP), one has to determine the optimization condition for q also.

Differentiating <o|H|o> given by (6.40) with respect to q, we have

$$(1-\bar{q}^2)\ (\frac{\hbar}{2\Omega})^2\{C_4 + 5C_5X_o + C_6(15X_o^2 + \frac{\hbar}{2\Omega}(q^{-2} - q^{-1}+1)$$
$$(1+q^2+q^{-2}+2)\} = o \qquad (6.49)$$

This equation has the roots $q = \pm 1$

But

$$\frac{d^2V_q}{dq^2} = (1+2q^{-3})\ (\frac{\hbar}{2\Omega})^2\ \{C_4 + 5C_5\ X_o$$
$$+ C_6(15X_o^2 + \frac{\hbar}{2\Omega}\ (q^{-4} - q^{-3} + 2q^{-2} - q^{-1}$$
$$+ q^2-q+2))\} \qquad (6.50)$$

For positive coefficients, this equation becomes positive at q=1, showing that it is a minimum of the potential. If q=-1 it becomes negative and hence corresponds to the maximum of the effective potential. q can have another set of six values corresponding to the roots of the factored out expression, which depend on the coefficients C_k. The possibility of some of them representing true minima cannot be ruled out.

As for the (q,p) analogue of the sextic oscillator, we have equations (6.38)-(6.48) with [A] replaced by $[A]_{q,p}$. In order to get the $SS_{qp}EP$ for the groundstate, the conditions are

$$\frac{\partial V_{q,p}}{\partial p} = (\frac{\hbar}{2\Omega})^2\ (-p^{-2})\ [C_4 + 5C_5X_o + 15C_6X_o^2$$

$$+ \frac{C_6\hbar}{2\Omega}(3p^{-2} + 2q^2 + 4qp^{-1}$$

$$+ 2q + 2p^{-1} + 2)] = o \qquad \ldots\ldots$$

$$\frac{\partial V_{q,p}}{\partial q} = \left(\frac{\hbar}{2\Omega}\right)^2 [C_4 + 5C_5X_o + 15X_o^2C_6$$

$$+ C_6 \frac{\hbar}{2\Omega}(3q^2 + 2p^{-2} + 4qp^{-1} + 2q + 2p^{-1} + 2)]$$

$$= o \qquad (6.52)$$

besides equation (6.41).

As $p \to \infty$ the conditions (6.51) and (6.52) imply $q = o$. Nontrivial solutions of (6.51) and (6.52) correspond to $p^{-2} \neq o$. The $SS_{qp}EP$ corresponds to $q = p^{-1}$ subject to the condition that the second deriva[illegible] of $V_{q,p}$ are greater than or equal to zero.

6.5 Concluding remarks

In this chapter we have addressed the question of formulating q and (q,p) analogues of the GEP. A direct generalization of GEP gives the non-perturbative effective potentials, namely NP_qEP and $NP_{qp}EP$, applicable to q-oscillators and (q,p)-oscillators, respectively. The SS_qEP is seen to correspond to q=1, at least in the ground state of q oscillators, showing that the ordinary bosonic theory appears to have a natural significance in the variational approach. Such uniqueness is not necessarily shared by the excited states of the system.

The potential shape transitions exhibited by double well oscillators at critical values of the parameter(s), is a novel phenomenon which may have implications in the study of spontaneous symmetry breaking in q and (q,p)-quantum field theories.

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