

BIVARIATE BURR DISTRIBUTIONS



THESIS SUBMITTED TO THE
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
UNDER THE FACULTY OF SCIENCE

BY

BISMI.G.NADH

DEPARTMENT OF STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
COCHIN-22

JUNE 2005

CERTIFICATE

Certified that the thesis entitled “Bivariate Burr Distributions” is a bonafide record of work done by Ms. Bismi.G.Nadh under my guidance in the Dept. of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

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June 2005

Dr. V.K. Ramachandran Nair

Professor of Statistics



DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis

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ACKNOWLEDGEMENTS

I wish to express my profound gratitude to Dr. V.K. Ramachandran Nair, Professor, Department of Statistics, CUSAT for his guidance, encouragement and support through out the development of this thesis.

I am deeply indebted to Dr. N. Unnikrishnan Nair, Former Vice Chancellor, CUSAT for the motivation, persistent encouragement and constructive criticism through out the development and writing of this thesis.

The co-operation rendered by all teachers in the Department of Statistics, CUSAT is also thankfully acknowledged.

I also wish to express great happiness at the steadfast encouragement rendered by my family.

Finally I dedicate this thesis to the loving memory of my father Late. Sri. C.K. Gopinath.

BISMI.G.NADH

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CHAPTER I

INTRODUCTION

1.1 Families of Distributions

During the late nineteenth century there had been a tendency to regard all distributions as normal ; the data histograms that displayed multimodality were often fitted with mixtures of normal density curves, and histograms that exhibited skewness were analyzed by transforming the data so that the resulting histograms could be fitted or graduated with normal curve. The incompatibility of normal distribution to explain theoretically and empirically many data situations forced the development of generalized frequency curves. Families of distributions provide functionally simple approximations to observe distributions in situations where it is difficult to derive a model. Since a trial and error approach to find out the appropriate model for the data is clearly undesirable as well as time consuming, flexible systems of distributions must be evolved, which should incorporate, if not all, at least the most common shapes that arise in practice. With this aim in mind many families of distributions have been constructed in literature, such as Pearson system, Burr system, Johnson system etc. Some of them arise as approximation to a wide variety of observed distributions.

Although theoretical arguments which lead to the model is the best way to understand the relevance of a particular system, ultimately their value to be judged primarily on practical requirements. These are

- a) ease of computation.
- b) amenability to algebraic manipulation.
- c) richness in members.

- d) flexibility exhibited through the number of parameters in the system.
- e) easy methods of inferring the parameters.
- f) easy interpretation of the system through a defining equation.

1.2 Burr system

The Burr system of distributions was introduced by Irwing W. Burr (1942), in an attempt to generate frequency functions that could be used in the “traditional attack upon the problem of determining theoretical probabilities and expected frequencies” He developed the system with the aid of a differential equation involving distribution function $F(x)$, as in his opinion distribution functions are theoretically much better than density function (as employed in the Pearson system) as often difficult integration has to be involved in deriving expected frequencies in various class frequencies when the later is employed. The differential equation proposed by him is

$$\frac{dF(x)}{dx} = F(x)[1 - F(x)]g(x, F(x)) \quad (1.2.1)$$

where $g(x, F(x))$ is nonnegative function over $0 \leq F(x) \leq 1$ and x in the range over which the solution is to be used.

If we choose $g(x, F(x)) = \frac{g(x)}{F(x)}$,

$$\frac{dF(x)}{dx} = [1 - F(x)]g(x) \quad (1.2.2)$$

which on integration yields

$$1 - F(x) = e^{-\int_0^x g(t) dt} \quad (1.2.3)$$

It is easy to recognize that $g(x)$ is the failure rate function used in reliability theory. This choice can result in the class of all continuous distributions, which has a failure rate $g(x)$. A restricted class can be obtained if we choose

$$g(x, F(x)) = g(x)$$

so that

$$\frac{dF(x)}{dx} = F(x)[1 - F(x)]g(x) \quad (1.2.4)$$

where $g(x)$ is suitable function nonnegative over the domain of x .

Solution of equation (1.2.4) is

$$F(x) = [1 + e^{-G(x)}]^{-1} \quad (1.2.5)$$

where

$$G(x) = \int_{-\infty}^x g(t) dt \quad (1.2.6)$$

Although Burr (1942) apparently does not provide conditions on $G(x)$ that provide conditions for a distribution function as solution of the differential equation, we note that on $G(x)$ must satisfy the condition (i) $G(-\infty) = -\infty$, (ii) the integral on the right side of equation (1.2.5) diverges to ∞ as $x \rightarrow \infty$ (iii) $G(x)$ is nondecreasing in x .

Burr gave the following twelve solutions in table 1.2.1 for $F(x)$ which is known in literature as the Burr system of distributions.

Table 1.2.1

Type	F(x)	Range
I	x	$0 < x < 1$
II	$(1 + e^{-x})^{-k}$	$-\infty < x < \infty$
III	$(1 + x^{-c})^{-k}$	$x > 0$
IV	$(1 + (\frac{c-x}{x})^{1/c})^{-k}$	$x < c$
V	$(1 + ce^{-\tan x})^{-k}$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
VI	$(1 + e^{-k \sinh x})^{-k}$	$-\infty < x < \infty$
VII	$2^{-k} (1 + \tanh x)^k$	$-\infty < x < \infty$
VIII	$(\frac{2}{\pi} \tan^{-1}(e^x))^k$	$-\infty < x < \infty$
IX	$1 - \frac{2}{2 + c((1 + e^x)^k - 1)}$	$-\infty < x < \infty$
X	$(1 - e^{-x^2})^k$	$0 < x < \infty$
XI	$(x - \frac{1}{2\pi} \sin 2\pi x)^k$	$0 < x < 1$
XII	$1 - (1 + x^c)^{-k}$	$x > 0$

where c and k are positive real numbers.

Burr apparently had in mind, the Pearson family of distributions that was the only popular system in existence at that time, when he proposed the twelve types listed above that are substantially different from the Pearson types. However, we note

that members of the Pearson family can also be embedded in the Burr system through the equation

$$G(x) = -\log\left(\frac{1-F(x)}{F(x)}\right) \quad (1.2.7)$$

Although in most of the basic types of the former, a simple closed to form expression for $F(x)$, other than special functions, is difficult to obtain to make $G(x)$ attractive. The fact that most of the absolutely continuous distributions can be reduced to the form (1.2.5) adds to the importance and relevance of the Burr family in statistical theory.

A close examination of the various types reveals that it is often possible to translate one type to another by means of transformations. For example type II can be converted to type III by the transformation

$$x = -\log y^{-c}$$

As such the transition from one type to another becomes quite handy which is not the case with other families of distributions, where such transformations are quite tedious so that independent inference procedures are required for each. This fact does not take away the richness and flexibility of the Burr system in modeling as several unidentified types belonging to it such as the Weibull distribution with density

$$f(x) = \left(\frac{\lambda}{\sigma}\right) e^{-\left(\frac{x}{\sigma}\right)^\lambda} \left(\frac{x}{\sigma}\right)^{\lambda-1} \quad x > 0, \lambda, \sigma > 0 \quad (1.2.8)$$

where

$$G(x) = \left(\frac{x}{\sigma}\right)^\lambda + \log\left[1 - e^{-\left(\frac{x}{\sigma}\right)^\lambda}\right] \quad (1.2.9)$$

1.2.1 Review of results

Now we present a review of the important results concerning various members of the Burr system, that are useful in this sequel. It is interesting to know that a substantial part of the literature is devoted to the type XII distribution. One reason for this is the useful range of values of the skewness and kurtosis provided by the distribution. Discussion in Hatke (1949), Burr (1968) and Burr and Cislak (1968) considered the range of values of the shape parameters in the (α_3^2, δ) diagram where

$$\delta = \frac{2\alpha_4 - 3\alpha_3^2 - 6}{\alpha_4 + 3} \text{ and } \alpha_3, \alpha_4 \text{ denote the standardized central moments and found that}$$

the type XII covers the curve shape characteristics of Pearson type I, type IV, type VI distributions and their transitional types.

The flexibility of the distribution makes it a useful model in reliability studies. Its role as failure model is discussed in Dubey (1972, 1973) and Evans and Simons (1975). Woo and Ali (1998) calculated the moments and established some simple properties of hazard rate while Gupta et al. (1996) obtained location of critical points for failure rate and mean residual life function.

A study of the type XII distribution based on arguments concerning failure rate and decay rate is provided by Singh and Madalla (1976). The results were extended in Schmittlein (1983), by deriving the large sample properties.

Other than the application to analysis of life time data, the distribution find usefulness in a wide range of areas such as models of income, business failure, duration models, heterogeneity in survival analysis, quality control, life table analysis. For a detailed discussion we refer Burr (1967 a, 1967 b), Zimmer and Burr(1963),

Austin (1973), Tsai (1990), Wingo(1983) Morrison and Schmittlein(1980) and Lancaster(1979 ,1985) , Shankar and Sahani(1994) , Hougaard(1984).

Following the properties of Burr type XII distribution as model for failure time data, several papers have been published on inference procedures relating to reliability, failure time etc using complete and censored sample with the classical, Bayesian and empirical Bayesian approach have been used in the process. This include the work of Papadopoules(1978), Lingappaiah(1979), Evans and Ragab(1983), Nigm(1988), Al-Hussaini and Jaheen(1992), Al-Hussaini and Jaheen and Mousa(1992), Al-Hussaini and Jaheen (1994), Al-Hussaini and Jaheen(1995), Ashour and El-Wakeel (1994), Mousa (1995), Jaheen (1995), Ahmad(1985), Nigm and Abdulwahab (1996), Al-Hussaini and Jaheen (1996), Elshanut M.A.T(1995), Ali Mousa and Jaheen (1997), Woo and Ali (2000), Mohiel El. Din(1991 b), Rasul(1994). Al-Mazoung and Ahmad (1998), Hussain and Nath (1997), Shah and Gokhale (1993), Wingo (1983 , 1993) , Abdelfattah (1996,1997).

An important point to be noted in this connection is that the type XII distribution can be derived independently as a mixture of the Weibull distribution and gamma distributions. If X follows Weibull with density

$$f(x|\theta) = c\alpha\theta x^{c-1} e^{-\alpha\theta x^c} \quad x > 0, \theta, c, \alpha > 0 \quad (1.2.10)$$

where the scale parameter θ follows gamma distribution with density

$$f(\theta) = \frac{\theta^{k-1}}{\Gamma k} e^{-\theta} \quad \theta > 0, k > 0 \quad (1.2.11)$$

the mixing argument gives the unconditional density

$$f(x) = \frac{k\alpha c x^{c-1}}{(1+\alpha x^c)^{k+1}} \quad x > 0, k, c, \alpha > 0 \quad (1.2.12)$$

This approach is useful in extending the distribution to higher dimensions. Another useful approach to obtaining the distribution is by applying the monotonic transformation

$$y = x^{1/c}$$

in the Lomax distribution with density function

$$f(x) = k\alpha^k(x+\alpha)^{-k-1} \quad x > 0, \alpha, k > 0 \quad (1.2.13)$$

It is noted that this distribution is not a member of the Pearson family, exponential family and family of stable distributions.

Few papers have been written about other types of Burr distributions. Surlles and Padjett (1999, 2000) considered the inference on reliability in stress strength model of Burr type X. Jaheen(1996), Sarwati and Abusalih(1991) considered the Bayesian estimation of Burr X model. The use of Burr type II distribution in binary choice model has been cited in Piorer(1980), Fry and orme(1998), Smith(1989). The properties and inference on Burr type III distribution is discussed by AlDayin(1999). Sherrick(1999) used Burr type III distribution in the study of recovering probabilistic information from option market. Shao(2000) investigated the use of Burr type III distribution in estimation of hazardous concentration based on no observed effect concentration toxicity data Lindsay etal.(1996) in their paper, explored the modeling of diameter distribution of forest stands and previous timber volume in a forest using Burr III distribution.

1.2.2 Multivariate Burr system

Compared to the volume of literature in the univariate case, only a few papers have been written about the multivariate version of any Burr distribution. One reason for this is the nonavailability of data relating to real situations when the Burr alternative could thought of and the wide range of applicability of tools based on the multinormal distribution.

The earliest attempt in this direction appears to be a multivariate extensions of Burr type XII distribution demonstrated by Takahasi(1965). He uses the mixing argument similar to be employed in defining the univariate type XII in a multivariate set up. It is assumed that (X_1, X_2, \dots, X_n) is a random vector in R_n having multivariate Weibull distribution with probability density function,

$$f(x_1, x_2, \dots, x_n | \theta) = \prod_{j=1}^n c_j \alpha_j \theta x_j^{c_j-1} e^{-\theta \alpha_j x_j^{c_j}} \quad x_j > 0, c_j, \alpha_j > 0, \theta > 0, j = 1, 2, \dots, n \quad (1.2.14)$$

If the parameter θ with distribution

$$f(\theta) = \frac{\theta^{k-1}}{\Gamma k} e^{-\theta} \quad \theta > 0, k > 0 \quad (1.2.15)$$

then the resultant unconditional density of (X_1, X_2, \dots, X_n) becomes

$$f(x_1, x_2, \dots, x_n) = \frac{\Gamma(k+n)}{\Gamma k} \prod_{j=1}^n \frac{c_j \alpha_j x_j^{c_j-1}}{[1 + \sum_{j=1}^n \alpha_j x_j^{c_j}]^{k+n}} \quad x_j > 0, \alpha_j > 0, c_j > 0, k > 0, j = 1, 2, \dots, n \quad (1.2.16)$$

Takahasi called this distribution “ multivariate Burr distribution” (henceforth, TB_{12}) and derives the following properties possessed by the distribution.

i) Any marginal distribution of TB_{12} is TB_{12} .

ii) Any conditional distribution of TB_{12} is also (multivariate) TB_{12} .

Conditional moments are given in Johnson and Kotz(1972) as

$$E(X_1^r / X_2, \dots, X_n) = \frac{\Gamma(k+n)}{\Gamma(k+n-1)} B(1 + \frac{r_1}{c_1}, k - n - \frac{r_1}{c_1}) [\alpha_1^{-1} [1 + \sum_{j=2}^n \alpha_j x_j]]^{\frac{r_1}{c_1}} \quad (1.2.17)$$

The joint moments may be derived as

$$E(X_1^{r_1}, X_2^{r_2}, \dots, X_n^{r_n}) = \frac{1}{\Gamma k} \prod_{j=1}^n \Gamma(1 + \frac{r_j}{c_j}) \Gamma(k - \sum_{j=1}^n \frac{r_j}{c_j}) \quad (1.2.18)$$

with the following existence condition $1 + \frac{r_j}{c_j} > 0, k > \sum_{j=1}^n \frac{r_j}{c_j} \quad j=1,2,\dots,n$

Since the marginal distributions are univariate Burr type XII, the covariance between any two variables condition X_i and X_j is found to be

$$\begin{aligned} \text{Cov}(X_i, X_j) &= \frac{\alpha_1^{-1/c_1} \alpha_2^{-1/c_2}}{\Gamma k} \Gamma(1 + \frac{1}{c_i}) \Gamma(1 + \frac{1}{c_j}) [\Gamma(k - \frac{1}{c_i} - \frac{1}{c_j}) - \frac{\Gamma(k - \frac{1}{c_i}) \Gamma(k - \frac{1}{c_j})}{\Gamma k}] \quad (1.2.19) \end{aligned}$$

The correlation coefficient between X_i and X_j is a function of c_i, c_j and k given

by

$$\text{Cor}(X_i, X_j)$$

$$\begin{aligned} &= \frac{\frac{\Gamma(1 + \frac{1}{c_i}) \Gamma(1 + \frac{1}{c_j}) \Gamma(k - \frac{1}{c_i} - \frac{1}{c_j})}{\Gamma k} - k^2 B(1 + \frac{1}{c_i}, k - \frac{1}{c_i}) B(1 + \frac{1}{c_j}, k - \frac{1}{c_j})}{\{ [\frac{\Gamma(1 + \frac{2}{c_i}) \Gamma(k - \frac{2}{c_i})}{\Gamma k} - [\frac{\Gamma(1 + \frac{1}{c_i}) \Gamma(k - \frac{1}{c_i})}{\Gamma k}]^2 \} \{ [\frac{\Gamma(1 + \frac{2}{c_j}) \Gamma(k - \frac{2}{c_j})}{\Gamma k} - [\frac{\Gamma(1 + \frac{1}{c_j}) \Gamma(k - \frac{1}{c_j})}{\Gamma k}]^2 \} \}^{\frac{1}{2}}} \quad (1.2.20) \end{aligned}$$

In the bivariate case the joint distribution function of X_1 and X_2 is

$$F(x_1, x_2) = 1 - (1 + \alpha_1 x_1^{c_1})^{-k} - (1 + \alpha_2 x_2^{c_2})^{-k} + (1 + \alpha_1 x_1^{c_1} + \alpha_2 x_2^{c_2})^{-k} \quad x_i > 0, c_i > 0, \alpha_i > 0, k > 0, i = 1, 2 \quad (1.2.21)$$

Normally a bivariate joint distribution must have the property that when the variables are independent the joint distribution must reduce to the product of marginal distribution function. This feature is not satisfied by (1.2.21). However a particular case if this distribution, when $c_1 = c_2 = 1$ with introduction of scale parameter in the form

$$P(X_1, X_2) = \left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2}\right)^{-k} \quad x_i > 0, \sigma_i > 0, k > 0 \quad (1.2.22)$$

is found quite useful in reliability modeling.

Although functionally attractive Takahasi's bivariate Burr type XII form is not generally suitable for fitting bivariate frequency data because the correlation between X_1 and X_2 is completely determined by marginal distribution of X_1 and X_2 .

Durling (1974) ameliorated this deficiency with a slight generalization of Takahasi's bivariate Burr distribution.

$$P(X_1 > x_1, X_2 > x_2) = (1 + \alpha_1 x_1^{c_1} + \alpha_2 x_2^{c_2} + \alpha_1 \alpha_2 \theta x_1^{c_1} x_2^{c_2})^{-k} \quad x_i > 0, c_i > 0, \alpha_i > 0, k > 0, 0 \leq \theta \leq k + 1 \quad i = 1, 2 \quad (1.2.23)$$

The marginal distribution of X_1 and X_2 are Burr type XII. But for fixed c_1, c_2 and k , the conditional parameter θ allows some variations in correlation of X_1 and X_2 . $F(x_1, x_2)$ reduces to Takahasi's bivariate Burr distribution for the limiting

case of $\theta = 0$. For $\theta = 1$, $F(x_1, x_2)$ becomes the product of two independent Burr distribution. Hutchinson (1981) provided a derivation of a version of Durling –Burr distribution which shows that it may derived within a mixing frame work.

Begum and Khan (1998) derived the pdf of r^{th} and s^{th} , $1 \leq r < s \leq n$ concomitants of order statistics for the bivariate Burr type XII distribution. Also their mean and product moments are calculated. Crowder and Kimber (1997) derived statistic to test the independent Weibull model against a p-variate Burr distribution. Null and nonnull properties of the score statistic are investigated with and without nuisance parameters and including the possibility of censoring. Nair(1989) gave a characterization theorem for multivariate Burr type XII distribution using conditional survival function.

The Takahasi's and Durling's distributions have found applications in literature. Johnson and Kotz(1981) constructed a model for time to failure under dependence. They were interested in determining the distribution of time to failure T of a replacement component taken from a stock which has been stored for some time. For example, when the second component is taken from the same product batch as first and there is batch to batch variation such that it is unable to independence between the time to failure of first and second components. When the TB_{12} distribution with $c_1 = c_2 = c$, is used in the model they find that the survival function of the time to failure (T) of the second component is a function which depends upon k, but not upon c. Crowder's (1985) approach to the multivariate Burr distribution was based on standard model for repeated failure time measurements. Suppose that a response time is measured on an individual several occasions; giving a data vector $t = (t_1, t_2, \dots, t_n)$ The joint distribution of t

is defined from the assumption that , conditional upon θ following gamma(1, k) the t_j 's are independent Weibull. The resultant joint distribution is a TB_{12} distribution with scale parameter α_j . Crowder(1985) discussed the properties of TB_{12} distribution in the context of failure time modeling and used the joint and marginal moments of $\ln(t) = (\log(t_1), \log(t_2), \dots, \log(t_n))'$ to suggest a method of moment estimation for n noncentral independent and identically distributed TB_{12} distributed vectors (t_1, t_2, \dots, t_n) . He applied this repeated failure time model to data on the response time of rats and adequate fit to the model. The TB_{12} distribution has also been applied to psychological data by Kimber and Crowder(1989) , and to the breaking strength of fibre by Crowder et al.(1991, P.143-147). When $n= 2$ TB_{12} distribution is a special case of the family of bivariate distributions used by Clayton (1978) to model association in bivariate life table and its application in epidemiological studies of familiar tendency in chronic disease incidence. Durling(1969) suggested the use of TB_{12} and Durling Burr distribution in a model of quantal choice.

A multivariate extension of Burr type II distribution is obtained by extending the extreme value mixture derivation of univariate Burr type II distribution. Assume that (X_1, X_2, \dots, X_n) have conditional upon a common scale parameter θ , independent extreme value distribution with density

$$f(x_1, x_2, \dots, x_n | \theta) = \theta e^{-\theta x_j} e^{-e^{-\theta x_j}} \quad -\infty < x_j < \infty, \theta > 0, j = 1, 2, \dots, n \quad (1.2.24)$$

and θ has gamma distribution with density

$$f(\theta) = \frac{\theta^{k-1}}{\Gamma k} e^{-\theta} \quad 0 < \theta < \infty, k > 0 \quad (1.2.25)$$

Then the usual mixing argument will yield ,

$$f(x_1, x_2, \dots, x_n) = \frac{\Gamma(k+n)}{\Gamma k} \prod_{j=1}^n \frac{e^{-x_j}}{(1 + \sum_{j=1}^n e^{-x_j})^{k+n}} \quad -\infty < x < \infty, k > 0, j = 1, 2, \dots, n \quad (1.2.26)$$

This distribution is known as multivariate Burr type II distribution (henceforth, TB_2) with the following properties

- i) Any marginal distribution of TB_2 is TB_2
- ii) Any conditional distribution of TB_2 is TB_2

Joint moment generating function is

$$M(t_1, t_2, \dots, t_n) = \frac{\Gamma(k + \sum_{j=1}^n t_j)}{\Gamma k} \prod_{j=1}^n \Gamma(1 - t_j) \quad (1.2.27)$$

with existence condition $k + \sum_{j=1}^n t_j > 0, 1 - t_j > 0, j = 1, 2, \dots, n$

The correlation coefficient between X_i and X_j is

$$\rho_{ij} = \frac{\psi'(k)}{\psi'(k) + \psi'(1)} \quad j = 1, 2, \dots, n \quad (1.2.28)$$

where $\psi(\cdot)$ is polygamma function. TB_2 has equicorrelated structure.

Fry(1993) gave a vector notation for the density function

$$f(X) = \frac{\Gamma(k+n)}{\Gamma k} \frac{e^{i'X}}{[1 + iev(-X)]^{k+n}} \quad (1.2.29)$$

where $X = (X_1, X_2, \dots, X_n)'$ $ev(-X)' = [e^{-X_1}, e^{-X_2}, \dots, e^{-X_n}]$ and

i is $n \times 1$ vector of ones. For this distribution

$$E(X) = (\psi(k) - \psi(1))i \quad (1.2.30)$$

$$V(X) = \psi'(1)I_n + \psi(k)ii' \quad (1.2.31)$$

Hutchinson and Satterthwaite(1977) used the Takahasi bivariate Burr type II distribution in fitting of a multifactorial model of disease transmission to data on the clustering of families of hysteria and sociopathy. Fry and Orme(1995) used TB_2 in maximum likelihood estimation in binary data model.

By applying mixing argument Rodriguez (1980) derived bivariate Burr type III as a mixture of extreme value type II (θ, c) and gamma(k). The distribution function obtained is

$$F(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2} + \theta x_1^{-c_1} x_2^{-c_2}]^{-k} \quad x_1 > 0, x_2 > 0 \quad (1.2.32)$$

where c_1, c_2 and k are positive marginal shape parameters and $0 \leq \theta \leq k + 1$

The variates X_1 and X_2 are independent if $\theta = 1$. The regression function of X_1 on X_2 is nonlinear.

$$\begin{aligned} E(x_1 | x_2) &= (k+1)B(k+1 + \frac{1}{c_1}, 1 - \frac{1}{c_1}) \left[\frac{(1 + \theta x_2^{-c_2})}{(1 + x_2^{-c_2})} \right]^{1/c_1} \\ &\quad + \theta(k+1)B(k + \frac{1}{c_1}, 2 - \frac{1}{c_1}) \left[\frac{(1 + \theta x_2^{-c_2})}{(1 + x_2^{-c_2})} \right]^{\frac{1-c_1}{c_1}} \\ &\quad - \theta B(k+1 + \frac{1}{c_1}, 1 - \frac{1}{c_1}) \left[\frac{(1 + \theta x_2^{-c_2})}{(1 + x_1^{-c_2})} \right]^{\frac{1-c_1}{c_1}} \quad (1.2.33) \end{aligned}$$

The correlation of X_1 and X_2 exists of $0 \leq \theta \leq k + 1$, $c_1 > 1, c_2 > 1$

For fixed c_1, c_2 and k the correlation is a monotonic decreasing function of θ

$$\text{Cor}(X_1, X_2) = \frac{B(k + \frac{1}{c_1}, 1 - \frac{1}{c_1})B(k + \frac{1}{c_2}, 1 - \frac{1}{c_2})F[-\frac{1}{c_1}, -\frac{1}{c_1}; k, 1 - \theta] - 1}{\left[\frac{B(k + \frac{2}{c_1}, 1 - \frac{2}{c_1})}{k - B^2(k + \frac{1}{c_1}, 1 - \frac{1}{c_1})} \right] \left[\frac{B(k + \frac{2}{c_2}, 1 - \frac{2}{c_2})}{k - B^2(k + \frac{1}{c_2}, 1 - \frac{1}{c_2})} \right]} (1.2.34)$$

In addition to the shape flexibility, a major advantage of fitting Burr III surfaces is the functional simplicity of fitted expression. Both the marginal and joint cumulative distribution function have closed forms. Moreover $X_1^{c_1}$ and X_2 have same marginal Burr type III distributions.

$$P(X_2 \leq x_2 | X_1 = x_1) = (1 + \beta x_2^{-c_2})^{-k-1} (1 + \theta x_2^{-c_2}) \quad (1.2.35)$$

where

$$\beta = \frac{(1 + \theta x_1^{-c_1})}{(1 + x_1^{-c_1})}$$

On the otherhand the practical disadvantage of the Burr type III surfaces are that their moments and correlation are complicated functions of their parameters, and that their marginal distributions must share the same shape parameter k. Rodrigez and Taniguchi(1980) used bivariate Burr type III surfaces to fit bivariate distribution data by method of moments and maximum likelihood. The data consists of gasoline octane requirements for vehicles as determined by customers and expert raters. The fitted surface yield joint distributions of customer and rater requirements. They have shown that bivariate Burr type III model is much more flexible than the bivariate normal model for fitting customer / rater octane requirement data.

1.3 Present Study

The present work is organized into six chapters. Bivariate extension of Burr system is the subject matter of Chapter II. We propose to introduce a general structure for the family in two dimensions and present some properties of such a system. Also in Chapter II we introduce some new distributions, which are bivariate extension of univariate distributions in Burr (1942). In Chapter III, we concentrate on characterization problems of different forms of bivariate Burr system.

A detailed study of the distributional properties of each member of the Burr system has not been undertaken in literature. With this aim in mind in Chapter IV we discussed with two forms of bivariate Burr III distribution. In Chapter V we consider the type XII, type II and type IX distributions.

Present work concludes with Chapter VI by pointing out the multivariate extension for Burr system. Also in this chapter we introduce the concept of multivariate reversed hazard rates as scalar and vector quantity.

CHAPTER II

BIVARIATE BURR SYSTEM OF DISTRIBUTIONS

2.1 Introduction

In continuation of the discussion on the Burr family of distributions in the previous chapter, we propose to introduce a general structure for the family in two dimensions and to present some properties of such a system. The review of work available in literature as described earlier on multivariate Burr distributions reveals that there is much scope for undertaking a study of developing a general bivariate framework. Although multivariate extensions exist for some univariate Burr types, there is no general pattern so far evolved for the development of bivariate family that could be thought of as a natural generalization of univariate set up.

In the present chapter we point out a general method of generating bivariate Burr distributions following the approaches prevalent in literature to extend a univariate distribution to higher dimensions. The common approaches used in this connections are

- i) to generalize the equation defining the univariate family to the multivariate case by involving many variables while keeping the form of equations intact. The extension of Pearson system by Van uven(1947) provides an example of this approach.
- ii) to explicitly specify some relation between the joint distribution and its marginals or conditionals and substitute the desired form of marginals or conditionals to find the required bivariate form. The Morgenstern(1956) , Frechet(1951) and Placket(1965) make use of this approach in defining certain classes of bivariate distributions.

iii) to extend functional form in the univariate set up in some pattern by including more variables and a density can be formed. In such a case it is not necessary that the marginal distributions continue to be one of the form from which generalizations was based. The linear exponential families of distributions by Wani(1961) confirms to this path of generalization.

iv) using the modeling approach to generalize multivariate distributions. In this case the inter relations between the variables are established based on physical properties of the system similar to be one established in the one variable situation and the model is then obtained as the solution of system thus obtained. The derivation of the Freund's (1961) bivariate exponential distribution is based on this methodology.

v) to adopt a characteristic property of univariate distribution to higher dimensions in some meaningful fashion that conveys an equivalent characterization and then derive the distribution possessing such an extended property. The bivariate exponential distributions of Marshall and Olkin(1967) derived from the extended version of famous lack of memory property that characterized the univariate exponential law.

In the present chapter we choose to generate bivariate Burr distributions by extending the defining equation

$$\frac{dF(x)}{dx} = F(x)[1 - F(x)]g(x) \quad (2.1.1)$$

by introducing two random variables (X_1, X_2) into the format and then solve the resulting differential equation to find a general form . However it is to be noted a generalization of this nature to the bivariate case need not be unique can be accomplished in more than one way depending on the interpretation gives to $f(x)$ or $1 - F(x)$. The different cases that will emerge as a result are

$$(a) \quad f(x_1, x_2) = F(x_1, x_2)[1 - F(x_1, x_2)]g(x_1, x_2) \quad (2.1.2)$$

where $f(x_1, x_2)$ is the density function of random vector (X_1, X_2) , replacing the density $f(x)$ in the univariate case.

If we look at $f(x)$ as a derivative of $F(x)$, another alternative to (a) is the system of partial differential equations

$$(b) \quad \frac{\partial F(x_1, x_2)}{\partial x_1} = F(x_1, x_2)[1 - F(x_1, x_2)]g_1(x_1, x_2) \quad (2.1.3)$$

$$\frac{\partial F(x_1, x_2)}{\partial x_2} = F(x_1, x_2)[1 - F(x_1, x_2)]g_2(x_1, x_2) \quad (2.1.4)$$

for some positive functions $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$

Often $1 - F(x)$ in the univariate case is the compliment probability associated with $F(x)$. When this interpretation is attached to $1 - F(x)$ in the defining equation a natural extension the bivariate case calls for the use of $R(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$

This gives the equation

$$(c) \quad f(x_1, x_2) = F(x_1, x_2)R(x_1, x_2)g(x_1, x_2) \quad (2.1.5)$$

Finally we argue as in (b), a fourth possibility is to look at the ststem

$$(d) \quad \frac{\partial F(x_1, x_2)}{\partial x_1} = F(x_1, x_2)R(x_1, x_2)g_1(x_1, x_2) \quad (2.1.6)$$

$$\frac{\partial F(x_1, x_2)}{\partial x_2} = F(x_1, x_2)R(x_1, x_2)g_2(x_1, x_2) \quad (2.1.7)$$

for generating a bivariate Burr family.

Though all the above four definitions could provide bivariate systems in view of the analytical tractability and the nature of solutions, in the present study, we are

concentrate on the set of partial differential equations in (b) in the sequel to generate a bivariate Burr system.

2.2 Bivariate Burr system (Bismi , Nair and Nair , 2005a)

In this section we derive the general forms of the distribution function, density function etc as the solution of the partial differential equations contained in (b) along with boundary conditions on the functions $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$.

Let (X_1, X_2) be a random vector in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i = 1, 2$ admitting absolutely continuous distribution function $F(x_1, x_2)$ and satisfying the differential equations in (2.1.3) and (2.1.4).

To solve the first equation we rewrite it as

$$\frac{1}{F(x_1, x_2)} \frac{\partial F(x_1, x_2)}{\partial x_1} + \frac{1}{1 - F(x_1, x_2)} \frac{\partial F(x_1, x_2)}{\partial x_1} = g_1(x_1, x_2) \quad (2.2.1)$$

$$\frac{\partial}{\partial x_1} \log \frac{F(x_1, x_2)}{1 - F(x_1, x_2)} = g_1(x_1, x_2) \quad (2.2.2)$$

Integrating from a_1 to x_1 ,

$$F(x_1, x_2) = [1 + e^{-G_1(x_1, x_2)}]^{-1} \quad (2.2.3)$$

where

$$G_1(x_1, x_2) = \int_{a_1}^{x_1} g_1(t_1, x_2) dt_1 \quad (2.2.4)$$

Similarly from equation (2.1.4),

$$F(x_1, x_2) = [1 + e^{-G_2(x_1, x_2)}]^{-1} \quad (2.2.5)$$

where

$$G_2(x_1, x_2) = \int_{a_2}^{x_2} g_2(x_1, t_2) dt_2 \quad (2.2.6)$$

Now from the differential equations ,

$$\begin{aligned} \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} &= [1 - F(x_1, x_2)] g_1(x_1, x_2) \frac{\partial F(x_1, x_2)}{\partial x_2} - g_1(x_1, x_2) F(x_1, x_2) \frac{\partial F(x_1, x_2)}{\partial x_2} \\ &\quad + F(x_1, x_2) [1 - F(x_1, x_2)] \frac{\partial g_1(x_1, x_2)}{\partial x_2} \end{aligned} \quad (2.2.7)$$

$$\begin{aligned} \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} &= [1 - F(x_1, x_2)] g_2(x_1, x_2) \frac{\partial F(x_1, x_2)}{\partial x_1} - g_2(x_1, x_2) F(x_1, x_2) \frac{\partial F(x_1, x_2)}{\partial x_1} \\ &\quad + F(x_1, x_2) [1 - F(x_1, x_2)] \frac{\partial g_2(x_1, x_2)}{\partial x_1} \end{aligned} \quad (2.2.8)$$

Substituting for $\frac{\partial F(x_1, x_2)}{\partial x_1}$ and $\frac{\partial F(x_1, x_2)}{\partial x_2}$ we find

$$\frac{\partial g_1(x_1, x_2)}{\partial x_2} = \frac{\partial g_2(x_1, x_2)}{\partial x_1} \quad (2.2.9)$$

Comparing equations (2.2.3) and (2.2.5) ,

$$G_1(x_1, x_2) = G_2(x_1, x_2) = G(x_1, x_2) \quad , \text{ say} \quad (2.2.10)$$

and hence

$$F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1} \quad (2.2.11)$$

The boundary conditions to be satisfied by $G(x_1, x_2)$ are $\lim_{\substack{x_1 \rightarrow b_1 \\ x_2 \rightarrow b_2}} G(x_1, x_2) = \infty$ and

$\lim_{x_i \rightarrow a_i} G(x_1, x_2) = -\infty$ for $i = 1, 2$. Since $F(x_1, x_2)$ has to be monotonic increasing in

(X_1, X_2) we must have $G(x_1, x_2)$ is nondecreasing and have

$$\frac{\partial G(x_1, x_2)}{\partial x_i} \geq 0 \quad \text{for all } x_i, i = 1, 2$$

Thus the functions $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ have to be nonnegative.

2.3 General properties of the bivariate Burr system (Bismi, Nair and Nair, 2005a)

In this section we describe some of the general properties possessed by the bivariate Burr system represented by equation (2.2.11). The marginal distributions are

$$F_1(x_1) = [1 + e^{-G(x_1, b_2)}]^{-1} \quad (2.3.1)$$

and

$$F_2(b_1, x_2) = [1 + e^{-G(b_1, x_2)}]^{-1} \quad (2.3.2)$$

$$\begin{aligned} F_1(x_1, b_2)[1 - F_1(x_1, b_2)] g_1(x_1, b_2) &= [1 + e^{-G(x_1, b_2)}]^{-1} e^{-G(x_1, b_2)} [1 + e^{-G(x_1, b_2)}]^{-1} g_1(x_1, b_2) \\ &= e^{-G(x_1, b_2)} g_1(x_1, b_2) [1 + e^{-G(x_1, b_2)}]^{-2} \\ &= \frac{\partial F_1(x_1, b_2)}{\partial x_1} \end{aligned} \quad (2.3.3)$$

where

$$\begin{aligned} f_1(x_1, b_2) &= \frac{\partial F_1(x_1, b_2)}{\partial x_1} \\ &= [1 + e^{-G(x_1, b_2)}]^{-2} e^{-G(x_1, b_2)} g_1(x_1, b_2) \end{aligned} \quad (2.3.4)$$

$$\begin{aligned} F_2(b_1, x_2)[1 - F_2(b_1, x_2)] g_2(b_1, x_2) &= [1 + e^{-G(b_1, x_2)}]^{-1} e^{-G(b_1, x_2)} [1 + e^{-G(b_1, x_2)}]^{-1} g_2(b_1, x_2) \\ &= e^{-G(b_1, x_2)} g_2(b_1, x_2) [1 + e^{-G(b_1, x_2)}]^{-2} \\ &= \frac{\partial F_2(b_1, x_2)}{\partial x_2} \end{aligned} \quad (2.3.5)$$

where

$$\begin{aligned} f_2(b_1, x_2) &= \frac{\partial F_2(b_1, x_2)}{\partial x_2} \\ &= [1 + e^{-G(b_1, x_2)}]^{-2} e^{-G(b_1, x_2)} g_2(b_1, x_2) \end{aligned} \quad (2.3.6)$$

Equations (2.3.3) and (2.3.5) are univariate Burr type differential equation. Therefore the marginals are of Burr form.

The joint density of (X_1, X_2) takes the form

$$f(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-3} e^{-G(x_1, x_2)} \left[(1 + e^{-G(x_1, x_2)}) \frac{\partial g_1(x_1, x_2)}{\partial x_2} - g_1(x_1, x_2) g_2(x_1, x_2) (1 - e^{-G(x_1, x_2)}) \right] \quad (2.3.7)$$

Conditional densities are

$$f(x_1 | x_2) = \frac{[1 + e^{-G(x_1, x_2)}]^{-3} e^{-G(x_1, x_2)} \left[(1 + e^{-G(x_1, x_2)}) \frac{\partial g_1(x_1, x_2)}{\partial x_2} - g_1(x_1, x_2) g_2(x_1, x_2) (1 - e^{-G(x_1, x_2)}) \right]}{[1 + e^{-G(b_1, x_2)}]^{-2} e^{-G(b_1, x_2)} g_2(b_1, x_2)} \quad (2.3.8)$$

$$f(x_2 | x_1) = \frac{[1 + e^{-G(x_1, x_2)}]^{-3} e^{-G(x_1, x_2)} \left[(1 + e^{-G(x_1, x_2)}) \frac{\partial g_1(x_1, x_2)}{\partial x_2} - g_1(x_1, x_2) g_2(x_1, x_2) (1 - e^{-G(x_1, x_2)}) \right]}{[1 + e^{-G(x_1, b_2)}]^{-2} e^{-G(x_1, b_2)} g_1(x_1, b_2)} \quad (2.3.9)$$

When the variables X_1 and X_2 are independent ,

$$F(x_1, x_2) = F_1(x_1) F_2(x_2)$$

which gives

$$[1 + e^{-G(x_1, x_2)}]^{-1} = [1 + e^{-G(x_1, b_2)}]^{-1} [1 + e^{-G(b_1, x_2)}]^{-1} \quad (2.3.10)$$

which implies

$$e^{-G(x_1, x_2)} = e^{-G(x_1, b_2)} + e^{-G(b_1, x_2)} + e^{-G(x_1, b_2) - G(b_1, x_2)} \quad (2.3.11)$$

Since the univariate Burr distributions have been found to be models of failure times it is of some interest to calculate the concepts useful in failure time analysis in bivariate case also. The survival function corresponding to (2.2.11) is

$$R(x_1, x_2) = 1 - [1 + e^{-G(x_1, b_2)}]^{-1} - [1 + e^{-G(b_1, x_2)}]^{-1} + [1 + e^{-G(x_1, x_2)}]^{-1} \quad (2.3.12)$$

Following the univariate case the vector valued reversed failure rate (Roy (2002)) in the form of (X_1, X_2) as $(\lambda_1(x_1, x_2), \lambda_2(x_1, x_2))$ where

$$\begin{aligned}\lambda_i(x_1, x_2) &= \frac{\partial}{\partial x_i} \log F(x_1, x_2) \quad i = 1, 2 \\ &= \frac{\frac{\partial F(x_1, x_2)}{\partial x_i}}{F(x_1, x_2)} \quad i = 1, 2\end{aligned}\quad (2.3.13)$$

From the defining equation of bivariate Burr system

$$\lambda_1(x_1, x_2) = [1 - F(x_1, x_2)] g_1(x_1, x_2) \quad (2.3.14)$$

and

$$\lambda_2(x_1, x_2) = [1 - F(x_1, x_2)] g_2(x_1, x_2) \quad (2.3.15)$$

providing an interesting relation

$$\frac{\lambda_1(x_1, x_2)}{\lambda_2(x_1, x_2)} = \frac{g_1(x_1, x_2)}{g_2(x_1, x_2)} \quad (2.3.16)$$

Specific expression for $\lambda_1(x_1, x_2)$ and $\lambda_2(x_1, x_2)$ are

$$\lambda_i(x_1, x_2) = \frac{e^{-G(x_1, x_2)} g_i(x_1, x_2)}{[1 + e^{-G(x_1, x_2)}]} \quad i = 1, 2 \quad (2.3.17)$$

Thus each of the ratios in equation (2.3.16) becomes

$$\frac{\lambda_1(x_1, x_2)}{g_1(x_1, x_2)} = \frac{\lambda_2(x_1, x_2)}{g_2(x_1, x_2)} = \frac{e^{-G(x_1, x_2)}}{[1 + e^{-G(x_1, x_2)}]} \quad (2.3.18)$$

We can define the reversed hazard rate as a scalar

$$\lambda(x_1, x_2) = \frac{f(x_1, x_2)}{F(x_1, x_2)} \quad (2.3.19)$$

$$= [1 + e^{-G(x_1, x_2)}]^{-2} e^{-G(x_1, x_2)} [(1 + e^{-G(x_1, x_2)}) \frac{\partial g_1(x_1, x_2)}{\partial x_2} - g_1(x_1, x_2)g_2(x_1, x_2)(1 - e^{-G(x_1, x_2)})] \quad (2.3.20)$$

The marginal reverse hazard rates are

$$\begin{aligned} \lambda_1(x_1) &= \frac{f_1(x_1)}{F_1(x_1)} \\ &= e^{-G(x_1, b_2)} g_1(x_1, b_2) [1 + e^{-G(x_1, b_2)}]^{-1} \end{aligned} \quad (2.3.21)$$

and

$$\begin{aligned} \lambda_2(x_2) &= \frac{f_2(x_2)}{F_2(x_2)} \\ &= e^{-G(b_1, x_2)} g_2(b_1, x_2) [1 + e^{-G(b_1, x_2)}]^{-1} \end{aligned} \quad (2.3.22)$$

The scalar failure rate (Basu(1971)) is

$$h(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)} \quad (2.3.23)$$

$$= \frac{[1 + e^{-G(x_1, x_2)}]^{-3} e^{-G(x_1, x_2)} [(1 + e^{-G(x_1, x_2)}) \frac{\partial g_1(x_1, x_2)}{\partial x_2} - g_1(x_1, x_2)g_2(x_1, x_2)(1 - e^{-G(x_1, x_2)})]}{1 - [1 + e^{-G(x_1, b_2)}]^{-1} - [1 + e^{-G(b_1, x_2)}]^{-1} + [1 + e^{-G(x_1, x_2)}]^{-1}} \quad (2.3.24)$$

The marginal hazard rates are

$$\begin{aligned} h_1(x_1) &= \frac{f_1(x_1)}{R_1(x_1)} \\ &= g_1(x_1, b_2) [1 + e^{-G(x_1, b_2)}]^{-1} \end{aligned} \quad (2.3.25)$$

and

$$\begin{aligned}
 h_2(x_2) &= \frac{f_2(x_2)}{R_2(x_2)} \\
 &= g_2(b_1, x_2)[1 + e^{-G(b_1, x_2)}]^{-1} \quad (2.3.26)
 \end{aligned}$$

The vector failure rate (Johnson and Kotz(1975)) in the form $(h_1(x_1, x_2), h_2(x_1, x_2))$

where

$$\begin{aligned}
 h_i(x_1, x_2) &= -\frac{\partial}{\partial x_i} \log R(x_1, x_2) \quad i = 1, 2 \\
 &= -\frac{\frac{\partial R(x_1, x_2)}{\partial x_i}}{R(x_1, x_2)} \quad i = 1, 2 \quad (2.3.27)
 \end{aligned}$$

$$h_1(x_1, x_2) = \frac{e^{-G(x_1, b_2)} g_1(x_1, b_2)[1 + e^{-G(x_1, b_2)}]^{-2} - e^{-G(x_1, x_2)} g(x_1, x_2)[1 + e^{-G(x_1, x_2)}]^{-2}}{1 - [1 + e^{-G(x_1, b_2)}]^{-1} - [1 + e^{-G(b_1, x_2)}]^{-1} + [1 + e^{-G(x_1, x_2)}]^{-1}} \quad (2.3.28)$$

$$h_2(x_1, x_2) = \frac{e^{-G(b_1, x_2)} g_2(b_1, x_2)[1 + e^{-G(b_1, x_2)}]^{-2} - e^{-G(x_1, x_2)} g(x_1, x_2)[1 + e^{-G(x_1, x_2)}]^{-2}}{1 - [1 + e^{-G(x_1, b_2)}]^{-1} - [1 + e^{-G(b_1, x_2)}]^{-1} + [1 + e^{-G(x_1, x_2)}]^{-1}} \quad (2.3.29)$$

2.4 Members of bivariate Burr system (Bismi , Nair and Nair , 2005a)

In this section we first present the twelve types of bivariate extensions of the univariate types discussed in Burr (1942) in Table 2.4.1 with corresponding $G(x_1, x_2)$ functions in Table 2.4.2.

It is to be noted that most of the absolutely continuous distributions belongs to the present family as it is always possible to identify

$$g_i(x_1, x_2) = \frac{1}{1 - F(x_1, x_2)} \frac{\partial \log F(x_1, x_2)}{\partial x_i} \quad i = 1, 2 \quad (2.4.1)$$

Table 2.4.1

Bivariate Burr distributions

Type	Distribution function	Range	Earlier reference	Marginal Burr
I	$[x_1^{-1/k} + x_2^{-1/k} - 1]^{-k}$	$i=1,2$ $0 < x_i < 1, k > 0$	Cook and Johnson(1986)	I
II	$[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-k}$	$-\infty < x_i < \infty$ $k > 0, 0 \leq \theta \leq k+1$		II
III	$[1 + x_1^{-c_1} + x_2^{-c_2} + \theta x_1^{-c_1} x_2^{-c_2}]^{-k}$	$0 < x_i < \infty, c_i > 0$ $k > 0, 0 \leq \theta \leq k+1$	Rodriguez(1980)	III
IV	$[1 + (\frac{c_1 - x_1}{x_1})^{1/c_1} + (\frac{c_2 - x_2}{x_2})^{1/c_2} + \theta (\frac{c_1 - x_1}{x_1})^{1/c_1} (\frac{c_2 - x_2}{x_2})^{1/c_2}]^{-k}$	$0 < x_i < c_i, c_i > 0$ $k > 0, 0 \leq \theta \leq k+1$		IV
V	$[1 + c_1 e^{-\tan x_1} + c_2 e^{-\tan x_2} + \theta c_1 c_2 e^{-\tan x_1} e^{-\tan x_2}]^{-k}$	$-\frac{\pi}{2} < x_i < \frac{\pi}{2}, c_i > 0$ $k > 0, 0 \leq \theta \leq k+1$		V

Table 2.4.1 continué...

VI	$[1 + c e^{-k \sinh \tau_1} + e^{-k \sinh \tau_2} + \theta e^{-k \sinh \tau_1} e^{-k \sinh \tau_2}]^{-k}$	$-\infty < x_i < \infty, k > 0,$ $0 \leq \theta \leq k + 1$	VI
VII	$[\frac{2}{1 + \tanh x_1} + \frac{2}{1 + \tanh x_2} - 1]^{-k}$	$-\infty < x_i < \infty, k > 0$	VII
VIII	$[\frac{2}{\pi} \tan^{-1}(e^{x_1})]^k + [\frac{2}{\pi} \tan^{-1}(e^{x_2})]^k - [\frac{2}{\pi} \tan^{-1}(e^{x_1} + e^{x_2})]^k$	$-\infty < x_i < \infty, k > 0$	VIII
IX	$[1 - \frac{2}{2 + c_1((1 + e^{x_1})^k - 1)} - \frac{2}{2 + c_2((1 + e^{x_2})^k - 1)} + \frac{2}{2 + c_1((1 + e^{x_1})^k - 1) + c_2((1 + e^{x_2})^k - 1)}]$	$-\infty < x_i < \infty,$ $c_i > 0, k > 0$	IX
Xa	$(1 - e^{-x_1^k})^k + (1 - e^{-x_2^k})^k - (1 - e^{-x_1^k - x_2^k})^k$	$0 < x_i < \infty, k > 0$	X

Table 2.4.1 continue...

X b	$(1 - e^{-x_1^k} - e^{-x_2^k} + e^{-(x_1^k + x_2^k + \theta x_1^k x_2^k)})^k$		$0 < x_i < \infty, k > 0,$ $0 \leq \theta \leq k$		X
XI	$(x_1 x_2 - \frac{1}{2\pi} \sin 2\pi x_1 x_2)^k$		$0 < x_i < 1, k > 0$		XI
XII a	$1 - (1 + a_1 x_1^{\theta_1})^{-k} - (1 + a_2 x_2^{\theta_2})^{-k} + (1 + a_1 x_1^{\theta_1} + a_2 x_2^{\theta_2})^{-k}$		$c_i > 0, k > 0,$ $a_i > 0$ $0 < x_i < \infty$	Takahasi (1965)	XII
XII b	$1 - (1 + a_1 x_1^{\theta_1})^{-k} - (1 + a_2 x_2^{\theta_2})^{-k}$ $+ (1 + a_1 x_1^{\theta_1} + a_2 x_2^{\theta_2} + a_1 a_2 \theta x_1^{\theta_1} x_2^{\theta_2})^{-k}$		$0 < x_i < \infty,$ $c_i > 0, k > 0,$ $0 \leq \theta \leq k + 1,$ $a_i > 0$	Durling (1974)	XII

Table 2.4.2

$G(x_1, x_2)$ functions

Burr Type	$G(x_1, x_2)$
I	$-\log [(x_1^{-1/k} + x_2^{-1/k} - 1)^k - 1]$
II	$-\log [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^k - 1]$
III	$-\log [1 + x_1^{-c_1} + x_2^{-c_2} + \theta x_1^{-c_1} x_2^{-c_2}]^k - 1]$
IV	$-\log [1 + (\frac{c_1 - x_1}{x_1})^{1/c_1} + (\frac{c_2 - x_2}{x_2})^{1/c_2} + \theta (\frac{c_1 - x_1}{x_1})^{1/c_1} (\frac{c_2 - x_2}{x_2})^{1/c_2}]^k - 1]$
V	$-\log [1 + c_1 e^{-\tan^{-1} x_1} + c_2 e^{-\tan^{-1} x_2} + \theta c_1 c_2 e^{-\tan^{-1} x_1} e^{-\tan^{-1} x_2}]^k - 1]$
VI	$-\log [1 + e^{-k \sinh x_1} + e^{-k \sinh x_2} + \theta e^{-k \sinh x_1} e^{-k \sinh x_2}]^k - 1]$
VII	$-\log [\frac{2}{1 + \tanh x_1} + \frac{2}{1 + \tanh x_2} - 1]^k - 1]$
VIII	$-\log [\frac{1 - [\frac{2}{\pi} \tan^{-1}(e^{x_1})]^k - [\frac{2}{\pi} \tan^{-1}(e^{x_2})]^k}{[\frac{2}{\pi} \tan^{-1}(e^{x_1})]^k + [\frac{2}{\pi} \tan^{-1}(e^{x_2})]^k - [\frac{2}{\pi} \tan^{-1}(e^{x_1} + e^{x_2})]^k}]^k$

Table 2.4.2 continue...

IX	$-\log \frac{\frac{2}{2+c_1((1+e^{x_1})^k-1)} + \frac{2}{2+c_2((1+e^{x_2})^k-1)} + \frac{2}{2+c_1((1+e^{x_1})^k-1)+c_2((1+e^{x_2})^k-1)}}{1 - \frac{2}{2+c_1((1+e^{x_1})^k-1)} - \frac{2}{2+c_2((1+e^{x_2})^k-1)} + \frac{2}{2+c_1((1+e^{x_1})^k-1)+c_2((1+e^{x_2})^k-1)}}$
X a	$-\log \left[\frac{1 - (1 - e^{-x_1^2})^k - (1 - e^{-x_2^2})^k + (1 - e^{-x_1^2} e^{-x_2^2})^k}{(1 - e^{-x_1^2})^k + (1 - e^{-x_2^2})^k - (1 - e^{-x_1^2} e^{-x_2^2})^k} \right]$
X b	$-\log[(1 - e^{-x_1^2} - e^{-x_2^2} + e^{-(x_1^2+x_2^2+\theta_1^2 x_1^2)})^{-k} - 1]$
XI	$-\log[(x_1 x_2 - \frac{1}{2\pi} \sin 2\pi x_1 x_2)^{-k} - 1]$
XII a	$\log \left[\frac{1 - (1 + a_1 x_1^{\alpha_1})^{-k} - (1 + a_2 x_2^{\alpha_2})^{-k} + (1 + a_1 x_1^{\alpha_1} + a_2 x_2^{\alpha_2})^{-k}}{(1 + a_1 x_1^{\alpha_1})^{-k} + (1 + a_2 x_2^{\alpha_2})^{-k} - (1 + a_1 x_1^{\alpha_1} + a_2 x_2^{\alpha_2})^{-k}} \right]$
XII b	$\log \left[\frac{1 - (1 + a_1 x_1^{\alpha_1})^{-k} - (1 + a_2 x_2^{\alpha_2})^{-k} + (1 + a_1 x_1^{\alpha_1} + a_2 x_2^{\alpha_2} + a_1 a_2 \theta x_1^{\alpha_1} x_2^{\alpha_2})^{-k}}{(1 + a_1 x_1^{\alpha_1})^{-k} + (1 + a_2 x_2^{\alpha_2})^{-k} - (1 + a_1 x_1^{\alpha_1} + a_2 x_2^{\alpha_2} + a_1 a_2 \theta x_1^{\alpha_1} x_2^{\alpha_2})^{-k}} \right]$

CHAPTER III
CHARACTERIZATIONS OF BIVARIATE BURR SYSTEM

3.1 Introduction

In modeling problems a common approach adopted is that the investigator initially chooses a family of distributions that have a wide variety of members with different shapes and characteristics and then a member of the family that is consistent with the physical properties of the system is chosen as the final model. When using the families of the distributions as the starting point, often the general properties of the family will be of considerable use in identifying the appropriate member. Accordingly there are several investigations concerning the common characteristics pertaining to various systems of distributions and any attempt at unearthing new properties is worthwhile exercise. It also helps to unify the results in the case of individual distributions that are obtained in separate studies. In view of these facts the present chapter contains the characterizations of bivariate Burr system. In terms of versatility and richness in members, Burr system appears to stand out as the best alternative among various systems of distributions.

3.2 Characterization of bivariate Burr system using reliability concepts

(Bismi and Nair 2005b)

In this section we consider characterizations of bivariate Burr system specified by equations (2.1.3) and (2.1.4) as

$$\frac{\partial F(x_1, x_2)}{\partial x_1} = F(x_1, x_2) [1 - F(x_1, x_2)] g_1(x_1, x_2)$$

and

$$\frac{\partial F(x_1, x_2)}{\partial x_2} = F(x_1, x_2) [1 - F(x_1, x_2)] g_2(x_1, x_2)$$

Theorem 3.2.1

Let (X_1, X_2) be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty$, $i = 1, 2$.

Then (X_1, X_2) follows bivariate Burr system specified by equations (2.2.11) if and only if

$$\frac{\lambda_1(x_1, x_2)}{g_1(x_1, x_2)} = \frac{\lambda_2(x_1, x_2)}{g_2(x_1, x_2)} = 1 - F(x_1, x_2) \quad (3.2.1)$$

Proof

If part is clearly proved in equation (2.3.18).

To prove the only if part we note the form (2.2.11).

Then from equations (2.1.3) and (2.1.4) we have

$$\lambda_i(x_1, x_2) = g_i(x_1, x_2) [1 - F(x_1, x_2)] \quad i = 1, 2$$

or

$$\frac{\frac{\partial F(x_1, x_2)}{\partial x_i}}{1 - F(x_1, x_2)} = g_i(x_1, x_2) [1 - F(x_1, x_2)] \quad i = 1, 2$$

whose solution is the Burr distribution given in the theorem.

With the extension of Burr distributions to bivariate case a problem of natural interest is to investigate how far the important properties of univariate Burr distribution can be generalized to appropriate forms in two dimensions. Nair and

Asha (2004) characterize the univariate Burr form using the relationship between hazard rate and reversed hazard rate. Following theorem gives the corresponding results in the bivariate case. In the bivariate case this can lead to four possibilities, the scalar hazard rate $h(x_1, x_2)$, the scalar reversed hazard rate $\lambda(x_1, x_2)$, vector hazard rate $h_i(x_1, x_2)$ and vector reversed hazard rate $\lambda_i(x_1, x_2)$.

Theorem 3.2.2

Let (X_1, X_2) be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1, 2$. Then (X_1, X_2) belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

$$\begin{aligned} \lambda(x_1, x_2) - h(x_1, x_2) & \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \\ & = e^{-G(x_1, x_2)} h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \end{aligned} \quad (3.2.2)$$

Proof

From marginal hazard rate and marginal reversed hazard rate we can write

$$F_i(x_i) = \frac{h_i(x_i)}{h_i(x_i) + \lambda_i(x_i)} \quad i=1, 2 \quad (3.2.3)$$

Solving $F(x_1, x_2)$ from the scalar hazard rate $h(x_1, x_2)$ and scalar reversed hazard rate $\lambda(x_1, x_2)$ and using equation (3.2.3) we have

$$F(x_1, x_2) = \frac{h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{h_1(x_1) + \lambda_1(x_1)} - \frac{h_2(x_2)}{h_2(x_2) + \lambda_2(x_2)} \right]}{\lambda(x_1, x_2) - h(x_1, x_2)} \quad (3.2.4)$$

Let (X_1, X_2) follows bivariate Burr system specified by equation (2.2.11).

Substituting equation (3.2.4) in equation (2.2.11), we get

$$\begin{aligned} \lambda(x_1, x_2) - h(x_1, x_2) & \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \\ & = e^{-G(x_1, x_2)} h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \end{aligned}$$

Conversely suppose that equation (3.2.2) holds.

Substituting $\lambda(x_1, x_2)$ and $h(x_1, x_2)$ from equation (2.3.19) and (2.3.23) and equation (3.2.3) in equation (3.2.2) we get

$$F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}$$

which is the general solution of bivariate Burr system given in the theorem.

Theorem 3.2.3

A continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1, 2$ belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

$$\begin{aligned} & \lambda_i(x_1, x_2) + h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] - \frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} \\ & = e^{-G(x_1, x_2)} \left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2) \left[\frac{h_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \right] \quad i=1, 2 \quad (3.2.5) \end{aligned}$$

Proof

Solving $F(x_1, x_2)$ using equations (2.3.13), (2.3.27) and (3.2.3) we find

$$F(x_1, x_2) = \frac{\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i)+h_i(x_i)} - h_i(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1)+h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2)+h_2(x_2)}\right]}{\lambda_i(x_1, x_2)+h_i(x_1, x_2)} \quad i=1,2 \quad (3.2.6)$$

Assume that (X_1, X_2) follows bivariate Burr system specified by equation (2.2.11) .

Substituting equation (3.2.6) in the general solution of bivariate Burr system specified by equation (2.2.11) we get the equation

$$\begin{aligned} & \lambda_i(x_1, x_2) + h_i(x_1, x_2) \left[\frac{\lambda(x_1)}{\lambda(x_1) + h(x_1)} + \frac{\lambda(x_2)}{\lambda(x_2) + h(x_2)} \right] - \frac{\lambda(x_i)h(x_i)}{\lambda(x_i) + h(x_i)} \\ & = e^{-G(x_1, x_2)} \left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \right] \quad i=1,2 \end{aligned}$$

Conversly starting from equations(3.2.5) and substituting equations (2.3.13), (2.3.27) and (3.2.3) that

$$F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}$$

Theorem 3.2.4

A continuous random vector (X_1, X_2) with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1,2$ belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

$$\begin{aligned} & \lambda_i(x_1, x_2) [\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_2'(x_1, x_2)] \\ & + \lambda(x_1, x_2) [h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] - \frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)}] \\ & = e^{-G(x_1, x_2)} \lambda(x_1, x_2) \left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \right] \quad i=1,2 \quad (3.2.7) \end{aligned}$$

and

$$\begin{aligned}
& \lambda_i(x_1, x_2)[\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_i'(x_1, x_2) \\
& \quad + \lambda(x_1, x_2)[h_i(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right] - \frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)}]] \\
& = e^{-G(x_1, x_2)} \lambda(x_1, x_2)\left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right]\right] i=1,2 \quad (3.2.8)
\end{aligned}$$

where

$$\lambda_i'(x_1, x_2) = \frac{\partial \lambda_i(x_1, x_2)}{\partial x_j} \quad i, j = 1, 2 \quad i \neq j$$

Proof

Solving $F(x_1, x_2)$ from equations (2.3.13), (2.3.19), (2.3.27) and (3.2.3) we find

$F(x_1, x_2)$

$$= \frac{\lambda(x_1, x_2)\left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right]\right]}{\lambda_i(x_1, x_2)[\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_i'(x_1, x_2)] + \lambda(x_1, x_2)h_i(x_1, x_2)} \quad i=1,2 \quad (3.2.9)$$

$$= \frac{\lambda(x_1, x_2)\left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right]\right]}{\lambda_i(x_1, x_2)[\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_i'(x_1, x_2)] + \lambda(x_1, x_2)h_i(x_1, x_2)} \quad i=1,2 \quad (3.2.10)$$

Substituting equation (3.2.9) in equation (2.2.11) we get

$$\begin{aligned}
& \lambda_i(x_1, x_2)[\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_i'(x_1, x_2)] \\
& \quad + \lambda(x_1, x_2)[h_i(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right] - \frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)}] \\
& = e^{-G(x_1, x_2)} \lambda(x_1, x_2)\left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right]\right] \quad i=1,2
\end{aligned}$$

Similarly equation (3.2.10) in equation (2.2.11)

$$\begin{aligned}
& \lambda_1(x_1, x_2)[\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_1'(x_1, x_2)] \\
& + \lambda_1(x_1, x_2)[h_1(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right] - \frac{\lambda_1(x_1)h_1(x_1)}{\lambda_1(x_1) + h_1(x_1)}] \\
= & e^{-G(x_1, x_2)} \lambda_1(x_1, x_2)\left[\frac{\lambda_1(x_1)h_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - h_1(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right]\right] \quad i=1, 2
\end{aligned}$$

Conversely starting from equation (3.2.7) and using equations (2.3.13), (2.3.19), (2.3.27) and (3.2.3) gives

$$F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}$$

Also starting from equation (3.2.8) and proceeding on same way we get

$$F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}$$

Hence the result.

Theorem 3.2.5

A continuous random vector (X_1, X_2) with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty$, $i=1, 2$ belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

$$\begin{aligned}
& \lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_1'(x_1, x_2) - h(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right] \\
= & e^{-G(x_1, x_2)} h(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right] \quad (3.2.11)
\end{aligned}$$

and

$$\begin{aligned}
& \lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_1'(x_1, x_2) - h(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right] \\
= & e^{-G(x_1, x_2)} h(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right] \quad (3.2.12)
\end{aligned}$$

Proof

Solving $F(x_1, x_2)$ using scalar hazard rate in equation (2.3.19) vector valued reversed hazard rate in equation (2.3.23) and equation (3.2.3) we get

$$F(x_1, x_2) = \frac{h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]}{\lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_2'(x_1, x_2) - h(x_1, x_2)} \quad (3.2.13)$$

$$= \frac{h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]}{\lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_1'(x_1, x_2) - h(x_1, x_2)} \quad (3.2.14)$$

Let (X_1, X_2) follows bivariate Burr system specified by equations (2.2.11) .

Substituting equation (3.2.13) in the general solution of bivariate Burr system specified by equation (2.2.11) we get the equation

$$\begin{aligned} & \lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_2'(x_1, x_2) - h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_2(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \\ &= e^{-G(x_1, x_2)} h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \end{aligned}$$

Substituting equation (3.2.14) in equation (2.2.11) we get

$$\begin{aligned} & \lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_1'(x_1, x_2) - h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \\ &= e^{-G(x_1, x_2)} h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \end{aligned}$$

Conversly suppose that equation (3.2.11) and (3.2.12) holds.

Then starting from equation (3.2.11) and using equations(2.3.13) ,(2.3.23) and (3.2.3) gives

$$F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}$$

Starting from equation (3.2.12) and using equations(2.3.13) (2.3.23) and (3.2.3) gives

$$F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}$$

Theorem 3.2.6

Let (X_1, X_2) be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1,2$ Then (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

$$g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda(x_1, x_2) - h(x_1, x_2)]}{\lambda(x_1, x_2) - h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]} \quad i=1,2 \quad (3.2.15)$$

Proof

Let (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4)

Using the identity (3.2.4) and (2.3.13) in equations (2.1.3) and (2.1.4) gives

$$1 - \frac{\lambda_i(x_1, x_2)}{g_i(x_1, x_2)} = \frac{h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]}{\lambda(x_1, x_2) - h(x_1, x_2)} \quad i=1,2 \quad (3.2.16)$$

which on simplification gives

$$g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda(x_1, x_2) - h(x_1, x_2)]}{\lambda(x_1, x_2) - h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]} \quad i=1,2$$

Conversely suppose that equation (3.2.15) holds

Then using equations (2.3.13),(2.3.19) ,(2.3.23) and (3.2.3) we get

$$g_i(x_1, x_2) = \frac{\frac{\partial F(x_1, x_2)}{\partial x_i}}{F(x_1, x_2)[1 - F(x_1, x_2)]} \quad i=1, 2$$

which proves the result.

Theorem 3.2.7

Let (X_1, X_2) be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1, 2$. Then (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

$$g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2) + h_i(x_1, x_2)]}{\lambda_i(x_1, x_2) + h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_2(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] - \frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)}} \quad i=1, 2 \quad (3.2.17)$$

Proof

Suppose that (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4).

Using the identity (3.2.6) and (2.3.15) we find

$$1 - \frac{\lambda_i(x_1, x_2)}{g_i(x_1, x_2)} = \frac{\frac{\lambda(x_i)h(x_i)}{\lambda(x_i) + h(x_i)} - h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]}{[\lambda_i(x_1, x_2) + h_i(x_1, x_2)]} \quad i=1, 2 \quad (3.2.18)$$

which gives

$$g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2) + h_i(x_1, x_2)]}{\lambda_i(x_1, x_2) + h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] - \frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)}} \quad i=1, 2$$

Conversly starting from equation (3.2.17) and using equations (2.3.13), (2.3.27) and (3.2.3) we have

$$g_i(x_1, x_2) = \frac{\frac{\partial F(x_1, x_2)}{\partial x_i}}{F(x_1, x_2)[1 - F(x_1, x_2)]} \quad i=1, 2$$

which proves the result.

Theorem 3.2.8

Let (X_1, X_2) be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1, 2$. Then

(X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

$$g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_2(x_1, x_2)] + \lambda(x_1, x_2)h_i(x_1, x_2)]}{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_2(x_1, x_2)] + \lambda(x_1, x_2)[h_i(x_1, x_2)[\frac{\lambda_1(x_1)}{h_1(x_1) + \lambda_1(x_1)} + \frac{\lambda_2(x_2)}{h_2(x_2) + \lambda_2(x_2)}] - \frac{\lambda_1(x_1)h_1(x_1)}{h_1(x_1) + \lambda_1(x_1)}}} \quad i=1, 2 \quad (3.2.19)$$

and

$$g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_1(x_1, x_2)] + \lambda(x_1, x_2)h_i(x_1, x_2)]}{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_1(x_1, x_2)] + \lambda(x_1, x_2)[h_i(x_1, x_2)[\frac{\lambda_1(x_1)}{h_1(x_1) + \lambda_2(x_1)} + \frac{\lambda_2(x_2)}{h_2(x_2) + \lambda_2(x_2)}] - \frac{\lambda_1(x_1)h_1(x_1)}{h_1(x_1) + \lambda_1(x_1)}}} \quad i=1, 2 \quad (3.2.20)$$

Proof

Let (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and

(2.1.4)

Using equations (3.2.9) and (2.3.13) in equations (2.1.3) and (2.1.4) gives

$$\begin{aligned}
& 1 - \frac{\lambda_i(x_1, x_2)}{g_i(x_1, x_2)} \\
&= \frac{\lambda(x_1, x_2) \left[\frac{\lambda(x_i)h(x_i)}{\lambda(x_i) + h(x_i)} - h_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \right]}{\lambda_i(x_1, x_2) [\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_2^2(x_1, x_2)] + \lambda(x_1, x_2)h_i(x_1, x_2)} \quad i=1,2 \quad (3.2.21)
\end{aligned}$$

which on simplification gives equation (3.2.19).

Similarly using equation (3.2.10) and (2.3.13) in equation (2.1.3) and (2.1.4) gives equation (3.2.20).

Conversely suppose that equations (3.2.19) and (3.2.20) hold.

Then using equations (2.3.13), (2.3.19), (2.3.27) and (3.2.3) we get

$$g_i(x_1, x_2) = \frac{\frac{\partial F(x_1, x_2)}{\partial x_i}}{F(x_1, x_2)[1 - F(x_1, x_2)]} \quad i=1,2$$

which proves the result.

Theorem 3.2.9

Let (X_1, X_2) be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1,2$. Then (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

$$\begin{aligned}
& g_i(x_1, x_2) \\
&= \frac{\lambda_i(x_1, x_2) [\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_2^2(x_1, x_2) - h(x_1, x_2)]}{\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_2^2(x_1, x_2) - h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{h_1(x_1) + \lambda_1(x_1)} + \frac{\lambda_2(x_2)}{h_2(x_2) + \lambda_2(x_2)} \right]} \quad i=1,2 \quad (3.2.22)
\end{aligned}$$

and

$$g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_i'(x_1, x_2) - h(x_1, x_2)]}{\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_i'(x_1, x_2) - h(x_1, x_2)\left[\frac{\lambda_1(x_1)}{h_1(x_1) + \lambda_1(x_1)} + \frac{\lambda_2(x_2)}{h_2(x_2) + \lambda_2(x_2)}\right]} \quad i=1,2 \quad (3.2.23)$$

Proof

Let (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4).

Using equations (3.2.13) and (2.3.13) in equations (2.1.3) and (2.1.4) gives

$$1 - \frac{\lambda_i(x_1, x_2)}{g_i(x_1, x_2)} = \frac{h(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}\right]}{\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_2'(x_1, x_2) - h(x_1, x_2)} \quad (3.2.24)$$

which on simplification results equation(3.2.22).

Similarly using equation (3.2.14) and (2.3.13) in equation (2.1.3) and (2.1.4) gives the result in equation (3.2.23).

Conversly suppose that equations (3.2.22) and (3.2.23) hold.

Then using equations (2.3.13) , (2.3.19) and (3.2.3) we get

$$g_i(x_1, x_2) = \frac{\frac{\partial F(x_1, x_2)}{\partial x_i}}{F(x_1, x_2)[1 - F(x_1, x_2)]} \quad i=1,2$$

which proves the result.

Next theorem shows that bivariate Burr system satisfies compatibility of conditional densities.

3.3 Characterization using conditional densities

Theorem 3.3.1

Let (X_1, X_2) be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_1, b_1) \times (a_2, b_2)$, $-\infty \leq a_i < b_i \leq \infty, i=1, 2$. Then (X_1, X_2) belongs to the bivariate Burr system specified by equations (2.2.11) if and only if the conditional densities are of the form equation (2.3.8) and (2.3.9)

Proof

Let the random vector (X_1, X_2) belongs to the bivariate Burr family specified by equations (2.2.11). Then conditional densities are of the form (2.3.8) and (2.3.9). Conversely suppose that conditional densities are of the form (2.3.8) and (2.3.9).

Then

$$\begin{aligned} \frac{f(x_1/x_2)}{f(x_2/x_1)} &= \frac{e^{-G(x_1, b_2)} g_1(x_1, b_2) [1 + e^{-G(b_1, x_2)}]^2}{[1 + e^{-G(x_1, b_1)}]^2 e^{-G(b_1, x_2)} g_1(b_1, x_2)} \\ &= \frac{A_1(x_1)}{A_2(x_2)} \end{aligned} \quad (3.3.1)$$

where

$$A_1(x_1) = \frac{e^{-G(x_1, b_2)} g_1(x_1, b_2)}{[1 + e^{-G(x_1, b_1)}]^2} \quad (3.3.2)$$

and

$$A_2(x_2) = \frac{e^{-G(b_1, x_2)} g_2(b_1, x_2)}{[1 + e^{-G(b_1, x_2)}]^2} \quad (3.3.3)$$

$$\int_{a_1}^{b_1} A_1(x_1) dx_1 = \int_{a_2}^{b_2} A_2(x_2) dx_2 \quad (3.3.4)$$

Hence Abraham and Thomas (1984) conditions are satisfied and therefore the bivariate distribution has bivariate Burr form.

CHAPTER IV
BIVARIATE BURR TYPE III DISTRIBUTIONS

4.1 Introduction

In the previous chapter we have considered the bivariate Burr system. A detailed study of the distributional properties of each member of the system has not been undertaken in literature. In model building the first choice is on the family and the second choice specific member there. In order to choose the most appropriate member from the family one should have sufficient understanding of the important characteristic of the members. Present chapter is an attempt in this direction. In the present chapter we have discussed the bivariate Burr III distribution. Rodriguez(1980) derived the bivariate Burr III distribution using mixing argument. The form proposed by him is

$$F(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2} + \theta x_1^{-c_1} x_2^{-c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, 0 \leq \theta \leq k+1, i = 1, 2 \quad (4.1.1)$$

But it can be shown that this distribution can be obtained as solution of set of partial differential equations involving distribution function which we have discussed in second chapter. In that unified approach the bivariate Burr III distribution arises by the choice $G(x_1, x_2)$ as

$$G(x_1, x_2) = -\log[[1 + x_1^{-c_1} + x_2^{-c_2} + \theta x_1^{-c_1} x_2^{-c_2}]^k - 1] \quad (4.1.2)$$

in equation (2.2.11)

In view of analytical tractability in the present study we consider the form

$$F(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (4.1.3)$$

Corresponding density function and survival function are

$$f(x_1, x_2) = \frac{k(k+1)c_1c_2x_1^{-c_1-1}x_2^{-c_2-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]^{k+2}} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (4.1.4)$$

and

$$R(x_1, x_2) = 1 - [1 + x_1^{-c_1}]^{-k} - [1 + x_2^{-c_2}]^{-k} + [1 + x_1^{-c_1} + x_2^{-c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (4.1.5)$$

4.2 General Properties of Type III Model (Bismi and Nair, 2005 d)

In this section we consider some general properties of bivariate Burr III distribution specified in equation (4.1.3)

It is noted that the marginal distributions are

$$F_i(x_i) = [1 + x_i^{-c_i}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (4.2.1)$$

With a choice of

$$g_i(x_i) = \frac{kc_i x_i^{-c_i-1}}{[1 + x_i^{-c_i}]^{-k+1} [[1 + x_i^{-c_i}]^k - 1]} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (4.2.2)$$

equation (4.2.1) satisfies

$$\begin{aligned} F_i(x_i) [1 - F_i(x_i)] g_i(x_i) &= \frac{kc_i x_i^{-c_i-1}}{[1 + x_i^{-c_i}]^{k+1}} \\ &= \frac{dF_i(x_i)}{dx_i} \quad i = 1, 2 \end{aligned} \quad (4.2.3)$$

which is univariate Burr type differential equation .

Thus for the bivariate Burr form (4.1.3) marginals are exactly univariate Burr type III.

With the above marginal distributions, conditional densities of X_i given $X_j = x_j$

arise as

$$f(x_i | X_j = x_j) = \frac{(k+1)c_i x_i^{-c_i-1} (1+x_j^{-c_j})^{k+1}}{[1+x_i^{-c_i} + x_j^{-c_j}]^{k+2}} \quad 0 < x_i < \infty, k, c_i, c_j > 0, i, j = 1, 2 \quad (4.2.4)$$

Using the transformation

$$Y_i = \frac{X_i}{(1+X_j^{-c_j})^{-1/c_j}}, \text{ it can be seen that } Y_i \text{ follows univariate Burr type III with}$$

parameters c_i and $(k+1)$. Hence any property for univariate Burr distribution of X_i can be extended to the conditional distribution of X_i given $X_j = x_j$.

Another type of conditional distribution that of interest especially in reliability modeling is the distribution of X_i given $X_j > x_j$.

Survival function of X_1 given $X_2 > x_2$ is

$$\begin{aligned} R(x_1 | X_2 > x_2) &= P(x_1 | X_2 > x_2) \\ &= \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_2 > x_2)} \\ &= \frac{1 - [1 + x_1^{-c_1}]^{-k} - [1 + x_2^{-c_2}]^{-k} + [1 + x_1^{-c_1} + x_2^{-c_2}]^{-k}}{1 - [1 + x_2^{-c_2}]^{-k}} \quad (4.2.5) \end{aligned}$$

The corresponding density function is calculated as

$$\begin{aligned} f(x_1 | X_2 > x_2) &= \frac{\partial R(x_1 | X_2 > x_2)}{\partial x_1} \\ &= \frac{\frac{kc_1 x_1^{-c_1-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]^{k+1}} - \frac{kc_1 x_1^{-c_1-1}}{[1 + x_1^{-c_1}]^{k+1}}}{1 - [1 + x_2^{-c_2}]^{-k}} \quad (4.2.6) \end{aligned}$$

Similarly

$$R(x_2 | X_1 > x_1) = P(x_2 | X_1 > x_1)$$

$$\begin{aligned}
&= \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_1 > x_1)} \\
&= \frac{1 - [1 + x_1^{-c_1}]^{-k} - [1 + x_2^{-c_2}]^{-k} + [1 + x_1^{-c_1} + x_2^{-c_2}]^{-k}}{1 - [1 + x_1^{-c_1}]^{-k}} \quad (4.2.7)
\end{aligned}$$

and

$$\begin{aligned}
f(x_2 | X_1 > x_1) &= \frac{\partial R(x_2 | X_1 > x_1)}{\partial x_2} \\
&= \frac{\frac{kc_2 x_2^{-c_2-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]^{k+1}} - \frac{kc_2 x_2^{-c_2-1}}{[1 + x_2^{-c_2}]^{k+1}}}{1 - [1 + x_1^{-c_1}]^{-k}} \quad (4.2.8)
\end{aligned}$$

Now we are interested to find the moments and other characteristics.

The $(r_1, r_2)^{th}$ moment of the distribution ,

$$\begin{aligned}
\mu_{r_1, r_2} &= E(X_1^{r_1} X_2^{r_2}) \\
&= \iint x_1^{r_1} x_2^{r_2} f(x_1, x_2) dx_1 dx_2 \\
&= k(k+1)c_1 c_2 \int_0^\infty \int_0^\infty \frac{x_1^{r_1-c_1-1} x_2^{r_2-c_2-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]^{k+2}} dx_1 dx_2 \\
&= \frac{1}{\Gamma k} \Gamma(1-r_1/c_1) \Gamma(1-r_2/c_2) \Gamma(k+r_1/c_1+r_2/c_2) \quad r_i/c_i < 1, i=1,2 \quad (4.2.9)
\end{aligned}$$

In particular the product moment become

$$\begin{aligned}
&E(X_1 X_2) \\
&= \frac{1}{\Gamma k} \Gamma(1-1/c_1) \Gamma(1-1/c_2) \Gamma(k+1/c_1+1/c_2) \quad 1/c_i < 1, k+1/c_1+1/c_2 > 0, i=1,2 \quad (4.2.10)
\end{aligned}$$

There is a recurrence relation connecting the moments of the distribution given by

$\mu_{r_1-c_1, r_2-c_2}$

$$\begin{aligned}
 &= \frac{1}{\Gamma k} \Gamma(1-(r_1-c_1)/c_1) \Gamma(1-(r_2-c_2)/c_2) \Gamma(k+(r_1-c_1)/c_1+(r_2-c_2)/c_2) \\
 &= \frac{1}{\Gamma k} \Gamma(2-r_1/c_1) \Gamma(2-r_2/c_2) \Gamma(k+(r_1/c_1)+(r_2/c_2)-2) \\
 &= \frac{(1-r_1/c_1)(1-r_2/c_2)}{(k+r_1/c_1+r_2/c_2-1)(k+r_1/c_1+r_2/c_2-2)} \mu_{r_1, r_2} \quad (4.3.11)
 \end{aligned}$$

when c_1 and c_2 are positive integers, this relation connects the adjacent moments and is useful to calculate all moments of the distribution devoid of gamma functions.

Covariance becomes

 $\text{Cov}(X_1, X_2)$

$$= \frac{\Gamma(1-1/c_1)\Gamma(1-1/c_2)}{\Gamma k} [\Gamma(k+1/c_1+1/c_2) - \frac{\Gamma(k+1/c_1)\Gamma(k+1/c_2)}{\Gamma k}] \quad 1/c_i < 1, k+1/c_1+1/c_2 > 0, i=1,2 \quad (4.2.12)$$

Then the coefficient of correlation has the expression,

 ρ

$$\begin{aligned}
 &= \frac{\Gamma(1-1/c_1)\Gamma(1-1/c_2) \left[\frac{\Gamma(k+1/c_1+1/c_2)}{\Gamma(k)} - \frac{\Gamma(k+1/c_1)\Gamma(k+1/c_2)}{(\Gamma k)^2} \right]}{\left[\frac{\Gamma(1-2/c_1)\Gamma(k+2/c_1)}{\Gamma k} - \left[\frac{\Gamma(1-1/c_1)\Gamma(k+1/c_1)}{\Gamma k} \right]^2 \right]^{1/2} \left[\frac{\Gamma(1-2/c_2)\Gamma(k+2/c_2)}{\Gamma k} - \left[\frac{\Gamma(1-1/c_2)\Gamma(k+1/c_2)}{\Gamma k} \right]^2 \right]^{1/2}} \quad (4.2.13) \\
 & \quad 1/c_i < 1, k+1/c_1+1/c_2 > 0, i=1,2
 \end{aligned}$$

Regression equations are obtained as

$$\begin{aligned}
 E(x_1 | X_2 = x_2) &= \int_0^{\infty} x_1 f(x_1 | X_2 = x_2) dx_1 \\
 &= (k+1) c_1 \int_0^{\infty} \frac{x_1^{-c_1} (1+x_2^{-c_2})^{k+1}}{[1+x_1^{-c_1} + x_2^{-c_2}]^{k+2}} dx_1 \\
 &= (k+1) (1+x_2^{-c_2})^{-1/c_1} B(1-1/c_1, k+1+1/c_1) \quad (4.2.14)
 \end{aligned}$$

which is decreasing function of x_2 .

Similarly

$$E(x_2 | X_1 = x_1) = (k+1) (1+x_1^{-c_1})^{-1/c_1} B(1-1/c_1, k+1+1/c_1) \quad (4.2.15)$$

which is decreasing function of x_1 .

$$\begin{aligned} \sigma(x_i | X_j = x_j) &= [(k+1) (1+x_j^{-c_j})^{-2/c_j} B(1-2/c_j, k+1+2/c_j) \\ &\quad - [(k+1)(1+x_j^{-c_j})^{-1/c_j} B(1-1/c_j, k+1+1/c_j)]^2]^{1/2} \end{aligned} \quad (4.2.16)$$

The coefficient of variation of X_i given $X_j = x_j$ is

$$cv(x_i | X_j = x_j) = \frac{\sigma(x_i | X_j = x_j)}{E(x_i | X_j = x_j)} \quad (4.2.17)$$

$$= \frac{[(k+1) (1+x_j^{-c_j})^{-2/c_j} B(1-2/c_j, k+1+2/c_j) - [(k+1) (1+x_j^{-c_j})^{-1/c_j} B(1-1/c_j, k+1+1/c_j)]^2]^{1/2}}{(k+1) (1+x_j^{-c_j})^{-1/c_j} B(1-1/c_j, k+1+1/c_j)}$$

$$= \frac{\frac{\Gamma(1+2/c_j)\Gamma(k+1-2/c_j)}{\Gamma(k+1)} - \frac{\Gamma(1+1/c_j)\Gamma(k+1-1/c_j)}{\Gamma(k+1)}}{\frac{\Gamma(1+1/c_j)\Gamma(k+1-1/c_j)}{\Gamma(k+1)}} \quad (4.2.18)$$

This is independent of X_j so is the coefficient of skewness of the conditional distributions.

Now we are interested to find some concepts useful in failure time analysis.

The scalar reversed hazard rate is

$$\begin{aligned} \lambda(x_1, x_2) &= \frac{f(x_1, x_2)}{F(x_1, x_2)} \\ &= \frac{k(k+1)c_1 c_2 x_1^{-c_1-1} x_2^{-c_2-1}}{[1+x_1^{-c_1} + x_2^{-c_2}]^2} \end{aligned} \quad (4.2.19)$$

Vector valued reversed hazard rate (Roy(2002)) is

$$\Delta (\log F(x_1, x_2)) = (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2))$$

where

$$\begin{aligned} \lambda_1(x_1, x_2) &= \frac{\partial}{\partial x_1} \log F(x_1, x_2) \\ &= \frac{k c_1 x_1^{-c_1-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]} \end{aligned} \quad (4.2.20)$$

and

$$\begin{aligned} \lambda_2(x_1, x_2) &= \frac{\partial}{\partial x_2} \log F(x_1, x_2) \\ &= \frac{k c_2 x_2^{-c_2-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]} \end{aligned} \quad (4.2.21)$$

The marginal reverse hazard rate is

$$\begin{aligned} \lambda_i(x_i) &= \frac{f_i(x_i)}{F_i(x_i)} \\ &= \frac{k c_i x_i^{-c_i-1}}{[1 + x_i^{-c_i}]} \quad i = 1, 2 \end{aligned} \quad (4.2.22)$$

Basu's (1971) failure rate is

$$\begin{aligned} h(x_1, x_2) &= \frac{f(x_1, x_2)}{R(x_1, x_2)} \\ &= \frac{k(k+1)c_1 c_2 x_1^{-c_1-1} x_2^{-c_2-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]^{k+2}} \quad (4.2.23) \\ &= \frac{k(k+1)c_1 c_2 x_1^{-c_1-1} x_2^{-c_2-1}}{1 - [1 + x_1^{-c_1}]^{-k} - [1 + x_2^{-c_2}]^{-k} + [1 + x_1^{-c_1} + x_2^{-c_2}]^{-k}} \end{aligned}$$

Gradient hazard rate (Johnson and Kotz(1975)) defined in equation (2.3.27) is given

by

$$\begin{aligned}
 h_i(x_1, x_2) &= -\frac{\partial}{\partial x_i} \log R(x_1, x_2) \quad i = 1, 2 \\
 h_1(x_1, x_2) &= \frac{\frac{k c_1 x_1^{-c_1-1}}{[1+x_1^{-c_1}]^{k+1}} - \frac{k c_1 x_1^{-c_1-1}}{[1+x_1^{-c_1}+x_2^{-c_2}]^{k+1}}}{1 - [1+x_1^{-c_1}]^{-k} - [1+x_2^{-c_2}]^{-k} + [1+x_1^{-c_1}+x_2^{-c_2}]^{-k}} \quad (4.2.24)
 \end{aligned}$$

and

$$h_2(x_1, x_2) = \frac{\frac{k c_2 x_2^{-c_2-1}}{[1+x_1^{-c_1}+x_2^{-c_2}]^{k+1}} - \frac{k c_2 x_2^{-c_2-1}}{[1+x_2^{-c_2}]^{k+1}}}{1 - [1+x_1^{-c_1}]^{-k} - [1+x_2^{-c_2}]^{-k} + [1+x_1^{-c_1}+x_2^{-c_2}]^{-k}} \quad (4.2.25)$$

The marginal failure rate

$$h_i(x_i) = \frac{k c_i x_i^{-c_i-1}}{[1+x_i^{-c_i}]^{k+1} [1 - [1+x_i^{-c_i}]^{-k}]} \quad (4.2.26)$$

4.3 Characterizations of Bivariate Burr Type III Distribution (Bismi and Nair, 2005 d)

In this section we consider some characterization theorems of bivariate Burr type III distribution.

In problem of modeling bivariate data the primary concern is to find an appropriate distribution that explains the data adequately. Partial prior information about the mechanism is some times available in the form of marginal or conditional distributions. The problem is to determine the joint distribution. It is known that the marginal distribution alone is generally insufficient to characterize the joint distribution when the variables are independent. Therefore the specification of the joint distribution through its component densities namely marginals and conditionals have been dealt with many researchers in the past. This include the work of Seshadri

and Patil (1964), Nair and Nair (1990) and Hitha and Nair (1991). Now we consider a characterization theorem using conditional densities and marginals.

Theorem 4.3.1

Let (X_1, X_2) be a random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (4.2.4). Then X_1 is Burr type III if and only if X_2 is Burr type III.

Proof

The conditional density of X_1 given $X_2 = x_2$ is of the form equation (4.2.4).

Assume that X_1 follows univariate Burr type III distribution. Then

$$f_1(x_1) = \frac{kc_1 x_1^{-c_1-1}}{[1+x_1^{-c_1}]^{k+1}} \quad 0 < x_1 < \infty, k, c_1 > 0$$

Also

$$f_1(x_1) = \int f(x_1 | x_2) f_2(x_2) dx_2 \quad (4.3.1)$$

Hence

$$\frac{kc_1 x_1^{-c_1-1}}{[1+x_1^{-c_1}]^{k+1}} = (k+1)c_1 \int_0^\infty \frac{x_1^{-c_1-1} f(x_2) dx_2}{[1+x_1^{-c_1}]^{k+2} [1+\frac{x_2^{-c_2}}{1+x_1^{-c_1}}]^{k+2}} \quad (4.3.2)$$

$$\frac{k}{k+1} [1+x_1^{-c_1}] = \int_0^\infty \frac{[1+x_2^{-c_2}]^{k+1} f(x_2) dx_2}{[1+\frac{x_2^{-c_2}}{1+x_1^{-c_1}}]^{k+2}} \quad (4.3.3)$$

Substituting $u = x_2^{-c_2}$ in equation (4.3.3) gives

$$\begin{aligned}
\frac{k}{k+1} [1 + x_1^{-c_1}] &= \int_0^{\infty} \frac{[1+u]^{k+1} f(u_2^{-1/c_2}) u_2^{-1/c_2-1} du}{\left[1 + \frac{u}{1+x_1^{-c_1}}\right]^{k+2}} \\
&= \int_0^{\infty} H(u) u_2^{1/c_2-1} du \quad (4.3.4)
\end{aligned}$$

Taking inverse Mellin transform (Rhyzik Pa.1194)

$$H(u) = \frac{kc_2 u^{1-1/c_2}}{\left[1 + \frac{u}{1+x_1^{-c_1}}\right]^{k+2}}$$

Hence

$$f_2(x_2) = \frac{kc_2 x_2^{-c_2-1}}{[1+x_2^{-c_2}]^{k+1}} \quad 0 < x_2 < \infty, \quad k, c_2 > 0$$

Thus X_2 is of Burr type III form.

To prove the converse , assume X_2 follows univariate Burr type III.

Then

$$\begin{aligned}
f_1(x_1) &= \int_0^{\infty} f(x_1 | x_2) f_2(x_2) dx_2 \\
&= k(k+1)c_1 c_2 \int_0^{\infty} \frac{x_1^{-c_1-1} x_2^{-c_2-1} dx_2}{[1+x_1^{-c_1} + x_2^{-c_2}]^{k+2}} \\
&= \frac{kc_1 x_1^{-c_1-1}}{[1+x_1^{-c_1}]^{k+1}} \quad 0 < x_1 < \infty, \quad k, c_1 > 0
\end{aligned}$$

Hence Proof.

Apart from the marginal distribution of X_i and the conditional distribution of X_j given $X_i = x_i, i=1,2 \quad i \neq j$ from which the joint distribution can always found , the other quantity that are relevance to the problem is marginal and conditional

distribution of the same component. In the corollary 4.3.1 we consider a characterization on the marginal and conditional distribution of the same component which incidentally also provides a characterization of univariate Burr type III distribution using bivariate Burr type III.

Corollary 4.3.1

Let (X_1, X_2) be a random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (4.2.4). Then (X_1, X_2) is Burr type III if and only if X_2 is Burr type III.

It is well known that a bivariate distribution is not always determined by marginal densities. Many researchers considered the problem of determination of joint density when the conditional distributions are known. Abraham and Thomas (1984), Gouriorex and monfort (1979) have developed the condition under which the densities $f(x_1 | x_2)$ and $f(x_2 | x_1)$ determine the joint density uniquely. According to Abraham and Thomas (1984) if the ratio of the conditional density can be written as

$$\frac{f(x_1 | x_2)}{f(x_2 | x_1)} = \frac{A_1(x_1)}{A_2(x_2)}$$

where

$$\int A_1(x_1) dx_1 = \int A_2(x_2) dx_2$$

then it will uniquely determine the joint density.

Next theorem shows that bivariate Burr III distribution satisfies compatibility of conditional densities.

Theorem 4.3.2

Let (X_1, X_2) be continuous random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) follows bivariate Burr type III distribution if and only if conditional densities are of the form equation (4.2.4).

Proof

Let (X_1, X_2) follows bivariate Burr type III distribution.

Then $f(x_i | x_j)$ $i = 1, 2$ $i \neq j$ is of the form (4.2.4)

Conversly

$$\begin{aligned} \frac{f(x_1 | x_2)}{f(x_2 | x_1)} &= \frac{c_1 x_1^{-c_1 - 1} [1 + x_2^{-c_2}]^{k+1}}{c_2 x_2^{-c_2 - 1} [1 + x_1^{-c_1}]^{k+1}} \\ &= \frac{A_1(x_1)}{A_2(x_2)} \end{aligned} \quad (4.3.5)$$

where

$$A_i(x_i) = \frac{c_i x_i^{-c_i - 1}}{[1 + x_i^{-c_i}]^{k+1}} \quad i = 1, 2 \quad (4.3.6)$$

$$\begin{aligned} \int_0^{\infty} A_1(x_1) dx_1 &= \int_0^{\infty} A_2(x_2) dx_2 \\ &= 1/k \end{aligned} \quad (4.3.7)$$

Hence Abraham and Thomas (1984) condition for unique determination of the joint density using conditional density is satisfied.

Hence proof.

Next we consider some characterization theorems using the relationship between scalar hazard rate, scalar reversed hazard rate, gradient hazard rate and gradient reversed hazard rate.

Theorem 4.3.3

Let (X_1, X_2) be continuous random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) belongs to the bivariate Burr type III distribution if and only

$$\lambda(x_1, x_2) - h(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^k \lambda(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \quad (4.3.8)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type III distribution .

Then using equation (3.2.4) in equation (4.1.3) we have equation (4.3.8).

Conversely starting from (4.3.8) and using (2.3.19) , (2.3.23) and (3.2.3) we get

$$F(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^k \quad 0 < x_i < \infty, k, c_i > 0 \quad i = 1, 2$$

Theorem 4.3.4

Let (X_1, X_2) be continuous random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) belongs to the bivariate Burr type III distribution if and only

$$\lambda_i(x_1, x_2) + h_i(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^k \left[\frac{\lambda_i(x_i) h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_1, x_2) \right] \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \quad i = 1, 2 \quad (4.3.9)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type III distribution .

Then using equation (3.2.6) in equation (4.1.3) we have equation (4.3.9).

Conversely starting from (4.3.9) and using (2.3.13) , (2.3.27) and (3.2.3) we get

$$F(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0 \quad i = 1, 2$$

Theorem 4.3.5

A continuous random vector (X_1, X_2) in the support of R_2^+ with distribution function $F(x_1, x_2)$ belongs to the bivariate Burr type III distribution if

$$\lambda(x_1, x_2) = \frac{k+1}{k} \lambda_1(x_1, x_2) \lambda_2(x_1, x_2) \quad (4.3.10)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type III distribution .

Then by equation (4.2.19), (4.2.20) and (4.2.21) we have equation (4.3.10).

4.4 Relation between Burr Type III and Other Distributions

Let (X_1, X_2) follows to the bivariate Burr type XII distribution .Table 4.5.1 gives relation between this distribution and other distributions.

Table 4.4.1

Transformation	Distribution function
$Y_i = [1 + X_i^{c_i}]^{-k}$	$[y_1^{-1/k} + y_2^{-1/k} - 1]^{-k} \quad 0 < y_i < 1, k > 0 \quad i = 1, 2$ (Cook and Johnson (1986))
$U_i = -\log X_i^{c_i}$	$[1 + e^{-u_1} + e^{-u_2}]^{-k} \quad -\infty < u_i < \infty, k > 0$ (Burr II)
$V_i = \frac{1}{X_i}$	$[1 + v_1^{-c_1} + v_2^{-c_2}]^{-k} \quad 0 < v_i < \infty, k, c_i > 0 \quad i = 1, 2$ (Burr XII)
$W_i = X_i^{-c_i}$	$[1 + w_1 + w_2]^{-k} \quad 0 < w_i < \infty, k > 0, i = 1, 2$ (Mardia(1960))

4.4 Bivariate Burr Type III Distribution form II (Bismi and Nair, 2005 e)

We can develop another bivariate form for the type III distribution using mixing argument.

Suppose the variables X_i , 's $i = 1, 2$ have conditional upon a common scale parameter θ , independent transformed gamma distribution and θ follows Weibull distribution. Then

$$f(x_i | \theta) = \frac{c\theta^{ck_i} x_i^{ck_i-1} e^{-\theta^c x_i^c}}{\Gamma k_i} \quad 0 < x_i < \infty, \theta, c, k_i > 0 \quad i = 1, 2 \quad (4.4.1)$$

and

$$f(\theta) = c\theta^{c-1} e^{-\theta^c} \quad \theta, c > 0 \quad (4.4.2)$$

Then the unconditional density is of the form

$$\begin{aligned} f(x_1, x_2) &= \int_0^\infty f(x_1 | x_2) f(\theta) d\theta \\ &= \frac{c^3}{\Gamma k_1 \Gamma k_2} \int_0^\infty \theta^{c(k_1+k_2+1-1/c)} e^{-\theta^c [1+x_1^c+x_2^c]} x_1^{ck_1-1} x_2^{ck_2-1} d\theta \\ &= \frac{c^2 \Gamma(k_1 + k_2 + 1)}{\Gamma k_1 \Gamma k_2} \frac{x_1^{ck_1-1} x_2^{ck_2-1}}{[1 + x_1^c + x_2^c]^{k_1+k_2+1}} \quad 0 < x_i < \infty, c, k_i > 0 \quad i = 1, 2 \quad (4.4.3) \end{aligned}$$

We define the distribution as bivariate Burr type III distribution.

Corresponding distribution function is

$F(x_1, x_2)$

$$= \frac{c^2 \Gamma(k_1 + k_2 + 1)}{\Gamma k_1 \Gamma k_2} \int_0^{x_1} \int_0^{x_2} \frac{t_1^{ck_1-1} t_2^{ck_2-1}}{[1 + t_1^c + t_2^c]^{k_1+k_2+1}} dt_1 dt_2 \quad 0 < x_i < \infty, c, k_i > 0 \quad i = 1, 2 \quad (4.4.4)$$

Also this distribution can be derived under the unified approach which we have considered in chapter II by choosing

$$G(x_1, x_2) = -\log \left[\left[\frac{c^2 \Gamma(k_1 + k_2 + 1)}{\Gamma k_1 \Gamma k_2} \int_0^{x_1} \int_0^{x_2} \frac{t_1^{ck_1-1} t_2^{ck_2-1}}{[1+t_1^c + t_2^c]^{k_1+k_2+1}} dt_1 dt_2 \right]^{-1} - 1 \right] \quad (4.4.5)$$

Marginal densities are

$$f(x_i) = ck_i \frac{x_i^{ck_i-1}}{[1+x_i^c]^{k_i+1}} \quad 0 < x_i < \infty, c, k_i > 0 \quad i = 1, 2 \quad (4.4.6)$$

Hence X_i , $i = 1, 2$ follows univariate Burr type III with parameters c and k_i

Conditional density of X_i given $X_j = x_j$ $i, j = 1, 2$ is

$$f(x_i | X_j = x_j) = \frac{c \Gamma(k_i + k_j + 1)}{\Gamma k_i \Gamma(k_j + 1)} \frac{\left[\frac{x_i^c}{1+x_j^c} \right]^{k_i} x_i^{-1}}{\left[1 + \frac{x_i^c}{1+x_j^c} \right]^{k_i+k_j+1}} \quad 0 < x_i < \infty, c, k_i > 0 \quad i, j = 1, 2 \quad (4.4.7)$$

Conditional moments are given by

$$\begin{aligned} E(x_i^r | X_j = x_j) &= \int_0^\infty x_i^r f(x_i | X_j = x_j) dx_i \\ &= \frac{c \Gamma(k_i + k_j + 1)}{\Gamma k_i \Gamma(k_j + 1)} \int_0^\infty x_i^r \frac{\left[\frac{x_i^c}{1+x_j^c} \right]^{k_i} x_i^{-1}}{\left[1 + \frac{x_i^c}{1+x_j^c} \right]^{k_i+k_j+1}} dx_i \\ &= [1+x_j^c]^{r/c} \frac{\Gamma(k_i + r/c) \Gamma(k_j + 1 - r/c)}{\Gamma k_i \Gamma(k_j + 1)} \quad i, j = 1, 2 \quad (4.4.8) \end{aligned}$$

Regression function of X_i given $X_j = x_j$ $i, j = 1, 2$ is

$$E(x_i | X_j = x_j) = [1+x_j^c]^{1/c} \frac{\Gamma(k_i + 1/c) \Gamma(k_j + 1 - 1/c)}{\Gamma k_i \Gamma(k_j + 1)} \quad i, j = 1, 2 \quad (4.4.9)$$

which is increasing in x_j

Point of intersection of two regression lines is $(c_1(x_1, x_2) \quad c_2(x_1, x_2))$ where

$$c_1(x_1, x_2) = \frac{\frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1 - 1/c)}{\Gamma k_1 \Gamma(k_2 + 1)} [1 + \frac{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma k_2 \Gamma(k_1 + 1)}]^c]}{[1 - \frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1 - 1/c)}{\Gamma k_1 \Gamma(k_2 + 1)} \frac{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma k_2 \Gamma(k_1 + 1)}]^c]} \quad (4.4.10)$$

$$c_2(x_1, x_2) = \frac{\frac{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma k_2 \Gamma(k_1 + 1)} [1 + \frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1 - 1/c)}{\Gamma k_1 \Gamma(k_2 + 1)}]^c]}{[1 - \frac{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma k_2 \Gamma(k_1 + 1)} \frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1 - 1/c)}{\Gamma k_1 \Gamma(k_2 + 1)}]^c]} \quad (4.4.11)$$

and the product moment

$$E(X_1 X_2) = \frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1/c)\Gamma(1 - 2/c)}{\Gamma k_1 \Gamma k_2} \quad (4.4.12)$$

correlation coefficient

$$\rho = \frac{\frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1/c)}{\Gamma k_1 \Gamma k_2} [\Gamma(1 - 2/c) - [\Gamma(1 - 1/c)]^2]}{[\frac{\Gamma(k_1 + 2/c)\Gamma(1 - 2/c)}{\Gamma k_1} - [\frac{\Gamma(k_1 + 1/c)\Gamma(1 - 1/c)}{\Gamma k_1}]^2]^{1/2} [\frac{\Gamma(k_2 + 2/c)\Gamma(1 - 2/c)}{\Gamma k_2} - [\frac{\Gamma(k_2 + 1/c)\Gamma(1 - 1/c)}{\Gamma k_2}]^2]^{1/2}} \quad (4.4.13)$$

Correlation tends to zero as c tends to ∞ .

Let (X_1, X_2) has the form (4.4.3). Then table 4.4.1 gives relation between this distribution and other distribution.

Table 4.4.1

Transformation	Corresponding density
$c = 1$	$\frac{\Gamma(k_1 + k_2 + 1)}{\Gamma k_1 \Gamma k_2} \frac{x_1^{k_1-1} x_2^{k_2-1}}{[1 + x_1 + x_2]^{k_1+k_2+1}} \quad 0 < x_i < \infty, k_i > 0 \quad i = 1, 2$ (Inverted Dirichlet(Tio and Guttman(1965)))
$f_i = \frac{X_i^c}{k_i}$	$\frac{\Gamma(k_1 + k_2 + 1)}{\Gamma k_1 \Gamma k_2} \frac{2(2k_1)^{k_1} (2k_2)^{k_2} f_1^{k_1-1} f_2^{k_2-1}}{[2 + 2k_1 f_1 + 2k_2 f_2]^{k_1+k_2+1}} \quad 0 < f_i < \infty, k_i > 0 \quad i = 1, 2$ (Bivariate F)
$k_1 = 1, k_2 = 1, Y_i = -\log X_i^c$	$\frac{2e^{-y_1} e^{-y_2}}{[1 + e^{-y_1} + e^{-y_2}]^3} \quad 0 < y_i < \infty, k_i > 0 \quad i = 1, 2$ (Bivariate logistic (Gumbel(1961)))

Next we consider some characterizations using conditional densities.

Theorem 4.4.1

Let (X_1, X_2) be a random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (4.4.7). Then X_1 is Burr type III if and only if X_2 is Burr type III.

Proof

The conditional density of X_1 given $X_2 = x_2$ is of the form equation(4.4.7).

Assume that X_1 follows univariate Burr type III distribution. Then

$$f_1(x_1) = \frac{k_1 c x_1^{c k_1 - 1}}{[1 + x_1^{c k_1}]^{k_1 + 1}} \quad 0 < x_1 < \infty \quad k_1, c > 0$$

Also

$$f_1(x_1) = \int f(x_1 | x_2) f_2(x_2) dx_2$$

Hence

$$\frac{k_1 c x_1^{c k_1 - 1}}{[1 + x_1^{c k_1}]^{k_1 + 1}} = \frac{\Gamma(k_1 + k_2 + 1) c^2}{\Gamma k_1 \Gamma(k_2 + 1)} \int_0^\infty \frac{[1 + x_2^{-c}]^{k_2 + 1} x_1^{-1} f(x_2) dx_2}{[1 + x_1^{-c} + x_2^{-c}]^{k_1 + k_2 + 1}} \quad (4.4.14)$$

Substituting $u = x_2^c$ in equation (4.4.14) gives

$$\begin{aligned} \frac{\Gamma(k_1 + 1) \Gamma k_2}{\Gamma(k_1 + k_2 + 1)} [1 + x_1^{-c}]^{k_2} &= \frac{1}{k_2 c} \int_0^\infty \frac{[1 + u]^{k_2 + 1} f(u^{1/c}) u^{1/c - 1} du}{\left[1 + \frac{u}{1 + x_1^{-c}}\right]^{k_1 + k_2 + 1}} \\ &= \frac{1}{k_2 c} \int_0^\infty H(u) u^{1/c - 1} du \end{aligned} \quad (4.4.15)$$

Taking inverse Mellin transform (Rhyzik Pa. 1194)

$$H(u) = \frac{k_2 c u^{k_2-1/c}}{\left[1 + \frac{u}{1+x_1^c}\right]^{k_1+k_2+1}}$$

Hence

$$f_2(x_2) = \frac{k_2 c x_2^{k_2-1}}{[1+x_2^c]^{k_2+1}} \quad 0 < x_2 < \infty \quad k_2, c > 0$$

Thus X_2 is of Burr type III form.

To prove the converse , assume X_2 follows univariate Burr type III.

Then

$$\begin{aligned} f_1(x_1) &= \int_0^{\infty} f(x_1 | x_2) f_2(x_2) dx_2 \\ &= \frac{\Gamma(k_1 + k_2 + 1) c^2}{\Gamma k_1 \Gamma(k_2 + 1)} \int_0^{\infty} \frac{x_1^{k_1-1} x_2^{k_2-1} dx_2}{[1 + x_1^{-c} + x_2^{-c}]^{k_1+k_2+1}} \\ &= \frac{k c x_1^{k_1-1}}{[1 + x_1^c]^{k_1+1}} \quad 0 < x_1 < \infty \quad , c, k_1 > 0 \end{aligned}$$

Corollary 4.4.1

Let (X_1, X_2) be a random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (4.4.7). Then (X_1, X_2) is Burr type III if and only if X_1 is Burr type III .

Next we show that the bivariate Burr III with form (4.4.1) satisfies compatibility of conditional densities.

Theorem 4.4.2

Let (X_1, X_2) be continuous random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) follows bivariate Burr type III specified by equation (4.4.1) if and only if the conditional densities are of the form in equation (4.4.7)

Proof

Let (X_1, X_2) follows bivariate Burr type III distribution.

Then $f(x_i | x_j)$ $i = 1, 2$ $i \neq j$ is of the form (4.4.7)

Conversly

$$\begin{aligned} \frac{f(x_1 / x_2)}{f(x_2 / x_1)} &= \frac{k_1 x_1^{ck_1-1} [1 + x_2^c]^{k_2+1}}{k_2 x_2^{ck_2-1} [1 + x_1^c]^{k_1+1}} \\ &= \frac{A_1(x_1)}{A_2(x_2)} \end{aligned} \quad (4.4.16)$$

where

$$A_i(x_i) = \frac{k_i x_i^{ck_i-1}}{[1 + x_i^{-c}]^{k_i+1}} \quad i = 1, 2 \quad (4.4.17)$$

$$\begin{aligned} \int_0^{\infty} A_1(x_1) dx_1 &= \int_0^{\infty} A_2(x_2) dx_2 \\ &= 1/c \end{aligned} \quad (4.4.18)$$

Hence Abraham and Thomas (1984) condition for unique determination of the joint density using conditional density is satisfied.

Hence proof.

CHAPTER V

BIVARIATE BURR TYPE XII , IX AND II DISTRIBUTIONS

5.1 Introduction

The work on the univariate and bivariate Burr distributions were mainly centered on the type XII distribution. In two dimensional case, the type XII introduced by Takahasi (1965) was later studied by Durling(1969, 1974) , Johnson and Kotz (1981), Crowder (1985), Crowder and Kimber (1997) and Begum and Khan (1998). In view of the importance of this distribution has enjoyed and the volume of work it has produced in literature we take up a detailed study of the type XII distribution under the new frame work, introduced in chapter II. The logic used in the derivation of bivariate Burr type XII distribution was the mixing argument. But it was pointed out in the second chapter that under a unified frame work the entire Burr system of distributions can be conceived as the solution of a set of partial differential equations , involving distribution function , so that the system contains , besides the generalization of the twelve types in the univariate case , many more absolutely continuous distributions.

It may be noted that apart from specifying the distribution functions much work has not been under taken on the type IX distribution in the univariate set up. As a new probability model with potential for application we discuss this model and bring about some of its salient characteristics. Also in this chapter we have discussed the type II distribution which we have introduced in chapter II.

5.2 General Properties of Type XII Model

In the unified approach the bivariate Burr XII distribution arises by the choice of $G(x_1, x_2)$ as

$$G(x_1, x_2) = \log \frac{1 - [1 + x_1^{c_1}]^{-k} - [1 + x_2^{c_2}]^{-k} + [1 + x_1^{c_1} + x_2^{c_2}]^{-k}}{[1 + x_1^{c_1}]^{-k} + [1 + x_2^{c_2}]^{-k} - [1 + x_1^{c_1} + x_2^{c_2}]^{-k}} \quad (5.2.1)$$

in equation (2.2.11). We can write the distribution function of bivariate Burr XII distribution in the form

$$F(x_1, x_2) = 1 - [1 + x_1^{c_1}]^{-k} - [1 + x_2^{c_2}]^{-k} + [1 + x_1^{c_1} + x_2^{c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (5.2.2)$$

Corresponding density function and survival function are

$$f(x_1, x_2) = \frac{k(k+1)c_1c_2x_1^{c_1-1}x_2^{c_2-1}}{[1 + x_1^{c_1} + x_2^{c_2}]^{k+2}} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (5.2.3)$$

and

$$R(x_1, x_2) = [1 + x_1^{c_1} + x_2^{c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (5.2.4)$$

A closely related form of the distribution is discussed in Johnson and Kotz (1972) which is obtained by replacing $x_i^{c_i}$ in equation (5.2.3) by $\alpha_i x_i^{c_i}$ for $i = 1, 2$. Our derivation of the distribution based on the choice of the functional form $G(x_1, x_2)$ ensures that the marginals are exactly Burr type XII.

Next we consider some general properties of bivariate Burr XII distribution specified in equation (5.2.3)

It is noted that the marginal distributions are

$$F_i(x_i) = 1 - [1 + x_i^{c_i}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \quad (5.2.5)$$

with a choice of

$$g_i(x_i) = \frac{kc_i x_i^{c_i-1}}{[1+x_i^{c_i}][1-[1+x_i^{c_i}]^{-k}]} \quad (5.2.6)$$

Equation (5.2.5) satisfies

$$\begin{aligned} F_i(x_i) [1 - F_i(x_i)] &= [1 - [1 + x_i^{c_i}]^{-k}] [1 + x_i^{c_i}]^{-k} \\ &= \frac{dF_i(x_i)}{dx_i} \quad i = 1, 2 \\ &= \frac{dF_i(x_i)}{g_i(x_i)} \end{aligned} \quad (5.2.7)$$

which is univariate Burr type differential equation in which marginal densities can be written as

$$\begin{aligned} \frac{dF_i(x_i)}{dx_i} &= f_i(x_i) \\ &= \frac{kc_i x_i^{c_i-1}}{[1+x_i^{c_i}]^{k+1}} \quad 0 < x_i < \infty, k, c > 0, i = 1, 2 \end{aligned} \quad (5.2.8)$$

Thus for the bivariate Burr form (5.2.3) marginals are exactly univariate Burr type XII

With the above marginal distributions, conditional densities of X_i given $X_j = x_j$ arise as

$$f(x_i | X_j = x_j) = \frac{(k+1)c_i x_i^{c_i-1} (1+x_j^{c_j})^{k+1}}{[1+x_i^{c_i} + x_j^{c_j}]^{k+2}} \quad 0 < x_i < \infty, k, c_i, c_j > 0, i, j = 1, 2 \quad (5.2.9)$$

Using the transformation

$$Y_i = \frac{X_i}{(1+X_j^{c_j})^{1/c_i}}$$

it can be seen that Y_i follows univariate Burr type XII with parameters c and $(k+1)$.

Hence any property for univariate Burr distribution of X_i can be extended to the conditional distribution of X_i given $X_j = x_j$.

Another type of conditional distribution that of interest especially in reliability modeling is the distribution of X_1 , given $X_2 > x_2$.

Survival function of X_1 given $X_2 > x_2$ is

$$\begin{aligned}
 R(x_1 | X_2 > x_2) &= P(x_1 | X_2 > x_2) \\
 &= \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_2 > x_2)} \\
 &= \frac{[1 + x_1^{c_1} + x_2^{c_2}]^{-k}}{[1 + x_2^{c_2}]^{-k}} \\
 &= \left[1 + \frac{x_1^{c_1}}{1 + x_2^{c_2}}\right]^{-k} \tag{5.2.10}
 \end{aligned}$$

$$= \left[1 + \left[\frac{x_1}{a_1(x_2)}\right]^{c_1}\right]^{-k} \tag{5.2.11}$$

where

$$a_1(x_2) = [1 + x_2^{c_2}]^{1/c_1} \tag{5.2.12}$$

The corresponding density function is calculated as

$$\begin{aligned}
 f(x_1 | X_2 > x_2) &= \frac{\partial R(x_1 | X_2 > x_2)}{\partial x_1} \\
 &= \frac{kc_1}{a_1(x_2)} \frac{\left[\frac{x_1}{a_1(x_2)}\right]^{c_1-1}}{\left[1 + \left[\frac{x_1}{a_1(x_2)}\right]^{c_1}\right]^{k+1}} \tag{5.2.13}
 \end{aligned}$$

Similarly

$$\begin{aligned}
 R(x_2 | X_1 > x_1) &= P(x_2 | X_1 > x_1) \\
 &= \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_1 > x_1)}
 \end{aligned}$$

$$= \left[1 + \left[\frac{x_2}{a_2(x_1)}\right]^{c_2}\right]^{-k} \quad (5.2.14)$$

where

$$a_2(x_1) = [1 + x_1^{c_1}]^{1/c_2} \quad (5.2.15)$$

and

$$\begin{aligned} f(x_2 | X_1 > x_1) &= \frac{\partial R(x_2 | X_1 > x_1)}{\partial x_2} \\ &= \frac{kc_2}{a_2(x_1)} \frac{\left[\frac{x_2}{a_2(x_1)}\right]^{c_2-1}}{\left[1 + \left[\frac{x_2}{a_2(x_1)}\right]^{c_2}\right]^{k+1}} \end{aligned} \quad (5.2.16)$$

An interesting point to be noted is the relationship between the conditional distribution of X_i given $X_j = x_j$ and that of X_i given $X_j > x_j$. The former has Burr form with parameters $(c, k+1)$ while the latter is Burr with parameters (c, k) respectively. This enables us to write variety properties involving X_i given $X_j = x_j$ and X_i given $X_j > x_j$, like the mean, variance, coefficient of variation, skewness etc, some which is independent of the condition involved.

A second useful feature of the above conditional distributions is that they satisfy the differential equations

$$\frac{\partial R(x_1 | X_2 > x_2)}{\partial x_1} = R(x_1 | X_2 > x_2)[1 - R(x_1 | X_2 > x_2)]g_1(x_1, x_2) \quad (5.2.17)$$

$$\frac{\partial R(x_2 | X_1 > x_1)}{\partial x_2} = R(x_2 | X_1 > x_1)[1 - R(x_2 | X_1 > x_1)]g_2(x_1, x_2) \quad (5.2.18)$$

with a choice of

$$g_i(x_i, x_j) = \frac{kc_i \left[\frac{x_i}{a_i(x_j)} \right]^{c_i-1} \left[1 + \left[\frac{x_i}{a_i(x_j)} \right]^{c_i} \right]^{-1}}{a_i(x_j) \left[1 - \left[1 + \left[\frac{x_i}{a_i(x_j)} \right]^{c_i} \right]^{-k} \right]} \quad i, j = 1, 2 \quad (5.2.19)$$

Now we are interested to find the moments and other characteristics.

5.3 Moments and Other Characteristics of Burr Type XII Distribution

Because of the transformation pointed out in the previous section, that induces a relation between the marginal and conditional distributions, many properties of the bivariate distribution can be established with out appealing to the bivariate density function. Apart from the mathematical convenience the approach also brings about some results that are useful in reliability context. For example when (X_1, X_2) represents the random life times of a two component system

$$m_i(x_j) = E(X_i | X_j > x_j) \quad i, j = 1, 2 \quad i \neq j \quad (5.3.1)$$

represents the mean true to failure (MTTF) of the i^{th} component when the j^{th} component has survivor time x_j

The expression for $m_1(x_2)$ is

$$\begin{aligned} m_1(x_2) &= \int_0^{\infty} f(x_1 | X_2 > x_2) dx_1 \\ &= \int_0^{\infty} R(x_1 | X_2 > x_2) dx_1 \\ &= \int_0^{\infty} \left[1 + \frac{x_1^{c_1}}{1 + x_2^{c_2}} \right]^{-k} dx_1 \\ &= \int_0^{\infty} \left[1 + \left[\frac{x_1}{[1 + x_2^{c_2}]^{1/c_1}} \right]^{c_1} \right]^{-k} dx_1 \end{aligned}$$

$$= \frac{(1+x_2^{c_2})^{1/c_1}}{c_1} B(1/c_1, k-1/c_1) \quad (5.3.2)$$

which is increasing function of x_2 .

Means can be directly calculated from $m_i(x_j)$ as

$$\begin{aligned} E(x_i) &= m_i(0^+) \\ &= \frac{\Gamma(k-1/c_i)\Gamma(1+1/c_i)}{\Gamma k} \quad c_i > 1/k \quad i=1,2 \end{aligned} \quad (5.3.3)$$

Further we have following expression for the variance

$$V(x_i) = \frac{\Gamma(k-2/c_i)\Gamma(1+2/c_i)}{\Gamma k} - \left[\frac{\Gamma(k-1/c_i)\Gamma(1+1/c_i)}{\Gamma k} \right]^2 \quad i=1,2 \quad (5.3.4)$$

Similarly

$$\begin{aligned} m_2(x_1) &= \int_0^{\infty} f(x_2 | X_1 > x_1) dx_2 \\ &= \int_0^{\infty} R(x_2 | X_1 > x_1) dx_2 \\ &= \frac{(1+x_1^{c_1})^{1/c_2}}{c_2} B(1/c_2, k-1/c_2) \end{aligned} \quad (5.3.5)$$

which is increasing function of x_1 .

This means that the mean life time of component X_i can be increased by increasing the value of component j.

The $(r_1, r_2)^{th}$ moment of he distribution ,

$$\begin{aligned} \mu_{r_1, r_2} &= E(X_1^{r_1} X_2^{r_2}) \\ &= \iint x_1^{r_1} x_2^{r_2} f(x_1, x_2) dx_1 dx_2 \end{aligned}$$

$$\begin{aligned}
&= k(k+1)c_1c_2 \int_0^\infty \int_0^\infty \frac{x_1^{r_1+c_1-1} x_2^{r_2+c_2-1}}{[1+x_1^{c_1}+x_2^{c_2}]^{k+2}} dx_1 dx_2 \\
&= \frac{1}{\Gamma k} \Gamma(1+r_1/c_1) \Gamma(1+r_2/c_2) \Gamma(k-r_1/c_1-r_2/c_2) \\
&= \frac{r_1 r_2}{c_1 c_2} \frac{1}{\Gamma k} \Gamma(r_1/c_1) \Gamma(r_2/c_2) \Gamma(k-r_1/c_1-r_2/c_2) \quad r_i/c_i > 0, k > r_1/c_1+r_2/c_2, i=1,2 \quad (5.3.6)
\end{aligned}$$

In particular the product moment become

$$\begin{aligned}
&E(X_1 X_2) \\
&= \frac{1}{c_1 c_2} \frac{1}{\Gamma k} \Gamma(1/c_1) \Gamma(1/c_2) \Gamma(k-1/c_1-1/c_2) \quad 1/c_i > 0, k > 1/c_1+1/c_2, i=1,2 \quad (5.3.7)
\end{aligned}$$

There is a recurrence relation connecting the moments of the distribution given by

$$\begin{aligned}
\mu_{r_1+c_1, r_2+c_2} &= \frac{1}{\Gamma k} \Gamma(1+(r_1+c_1)/c_1) \Gamma(1+(r_2+c_2)/c_2) \Gamma(k-(r_1+c_1)/c_1-(r_2+c_2)/c_2) \\
&= \frac{1}{\Gamma k} \Gamma(2+r_1/c_1) \Gamma(2+r_2/c_2) \Gamma(k-r_1/c_1-r_2/c_2-2) \\
&= \frac{(1+r_1/c_1)(1+r_2/c_2)}{(k-r_1/c_1-r_2/c_2-1)(k-r_1/c_1-r_2/c_2-2)} \frac{1}{\Gamma k} \Gamma(1+r_1/c_1) \Gamma(1+r_2/c_2) \Gamma(k-r_1/c_1-r_2/c_2) \\
&= \frac{(1+r_1/c_1)(1+r_2/c_2)}{(k-r_1/c_1-r_2/c_2-1)(k-r_1/c_1-r_2/c_2-2)} \mu_{r_1, r_2} \quad (5.3.8)
\end{aligned}$$

When c_1 and c_2 are positive integers, this relation connects the adjacent moments and is useful to calculate all moments of the distribution devoid of gamma functions.

Covariance becomes

$$\begin{aligned} & \text{Cov}(X_1, X_2) \\ &= \frac{\Gamma(1/c_1)\Gamma(1/c_2)}{c_1 c_2} \frac{1}{\Gamma k} [\Gamma(k-1/c_1-1/c_2) - \frac{\Gamma(k-1/c_1)\Gamma(k-1/c_2)}{\Gamma k}] \quad 1/c_i > 0, k > 1/c_1 + 1/c_2, i=1,2 \quad (5.3.9) \end{aligned}$$

Then the coefficient of correlation has the expression ,

ρ

$$\begin{aligned} &= \frac{\Gamma(1+1/c_1)\Gamma(1+1/c_2) \left[\frac{\Gamma(k-1/c_1-1/c_2)}{\Gamma(k)} - \frac{\Gamma(k-1/c_1)\Gamma(k-1/c_2)}{(\Gamma k)^2} \right]}{\left[\frac{\Gamma(1+2/c_1)\Gamma(k-2/c_1)}{\Gamma k} - \left[\frac{\Gamma(1+1/c_1)\Gamma(k-1/c_1)}{\Gamma k} \right]^2 \right]^{1/2} \left[\frac{\Gamma(1+2/c_2)\Gamma(k-2/c_2)}{\Gamma k} - \left[\frac{\Gamma(1+1/c_2)\Gamma(k-1/c_2)}{\Gamma k} \right]^2 \right]^{1/2}} \quad (5.3.10) \\ & \quad 1/c_i > 0, k > 1/c_1 + 1/c_2, i=1,2 \end{aligned}$$

Regression equations are obtained by using the transformation

$$Y_i = \frac{X_i}{(1 + X_j^{c_j})^{1/c_j}} \quad \text{discussed earlier.}$$

$$\begin{aligned} E(x_1 | X_2 = x_2) &= \int_0^{\infty} x_1 f(x_1 | X_2 = x_2) dx_1 \\ &= \int_0^{\infty} y_1 f(y_1 | X_2 = x_2) dy_1 \\ &= \int_0^{\infty} \frac{y_1^{c_1} (1 + x_2^{c_2})^{1/c_1}}{[1 + y_1^{c_1}]^{k+2}} dy_1 \\ &= (k+1) (1 + x_2^{c_2})^{1/c_1} B(1+1/c_1, k+1-1/c_1) \quad (5.3.11) \end{aligned}$$

which is increasing function of x_2 .

Similarly

$$E(x_2 | X_1 = x_1) = (k+1) (1 + x_1^{c_1})^{1/c_2} B(1+1/c_2, k+1-1/c_2) \quad (5.3.12)$$

which is increasing function of x_1 .

Further we note that

$$E(x_1 | X_2 > x_2) = \frac{k}{(k-1/c_1)} E(x_1 | X_2 = x_2) \quad (5.3.13)$$

and similarly

$$E(x_2 | X_1 > x_1) = \frac{k}{(k-1/c_2)} E(x_2 | X_1 = x_1) \quad (5.3.14)$$

The coefficient of variation of X_i given $X_j = x_j$ is

$$\begin{aligned} \text{cv}(x_i | X_j = x_j) &= \frac{\sigma(x_i | X_j = x_j)}{E(x_i | X_j = x_j)} \\ &= \frac{\sigma(y_i | X_j = x_j)}{E(y_i | X_j = x_j)} \\ &= \frac{\left[\frac{\Gamma(1+2/c_i)\Gamma(k+1-2/c_i)}{\Gamma(k+1)} - \left[\frac{\Gamma(1+1/c_i)\Gamma(k+1-1/c_i)}{\Gamma(k+1)} \right]^2 \right]^{1/2}}{\frac{\Gamma(1+1/c_i)\Gamma(k+1-1/c_i)}{\Gamma(k+1)}} \quad (5.3.15) \end{aligned}$$

This is independent of X_j , so is the coefficient of skewness of the conditional distributions.

It has been pointed out earlier that most of the applications of the Burr type XII law is in reliability analysis. Hence we consider the role of bivariate model in explaining the reliability aspect of a two component system.

The scalar reversed hazard rate is

$$\begin{aligned} \lambda(x_1, x_2) &= \frac{f(x_1, x_2)}{F(x_1, x_2)} \\ &= \frac{k(k+1)c_1 c_2 x_1^{c_1-1} x_2^{c_2-1}}{1 - [1 + x_1^{c_1}]^{-k} - [1 + x_2^{c_2}]^{-k} + [1 + x_1^{c_1} + x_2^{c_2}]^{-k}} \quad (5.3.16) \end{aligned}$$

Vector valued reversed hazard rate (Roy(2002)) defined in equation (2.3.13) is given by

$$\begin{aligned}
\lambda_1(x_1, x_2) &= \frac{\partial}{\partial x_1} \log F(x_1, x_2) \\
&= \frac{\frac{k c_1 x_1^{c_1-1}}{[1+x_1^{c_1}]^{k+1}} - \frac{k c_1 x_1^{c_1-1}}{[1+x_1^{c_1}+x_2^{c_2}]^{k+1}}}{1-[1+x_1^{c_1}]^{-k} - [1+x_2^{c_2}]^{-k} + [1+x_1^{c_1}+x_2^{c_2}]^{-k}} \quad (5.3.17)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_2(x_1, x_2) &= \frac{\partial}{\partial x_2} \log F(x_1, x_2) \\
&= \frac{\frac{k c_2 x_2^{c_2-1}}{[1+x_2^{c_2}]^{k+1}} - \frac{k c_2 x_2^{c_2-1}}{[1+x_1^{c_1}+x_2^{c_2}]^{k+1}}}{1-[1+x_1^{c_1}]^{-k} - [1+x_2^{c_2}]^{-k} + [1+x_1^{c_1}+x_2^{c_2}]^{-k}} \quad (5.3.18)
\end{aligned}$$

The marginal reverse hazard rate is

$$\begin{aligned}
\lambda_i(x_i) &= \frac{f_i(x_i)}{F_i(x_i)} \\
&= \frac{k c_i x_i^{c_i-1}}{[1+x_i^{c_i}]^{k+1} [1-[1+x_i^{c_i}]^{-k}]} \quad i=1,2 \quad (5.3.19)
\end{aligned}$$

Basu's (1971) failure rate is

$$\begin{aligned}
h(x_1, x_2) &= \frac{f(x_1, x_2)}{R(x_1, x_2)} \\
&= \frac{k(k+1)c_1 c_2 x_1^{c_1-1} x_2^{c_2-1}}{[1+x_1^{c_1}+x_2^{c_2}]^2} \quad (5.3.20)
\end{aligned}$$

$$\begin{aligned}
&h(x_1+t_1, x_2+t_2) - h(x_1, x_2) \\
&= \frac{k(k+1)c_1 c_2 (x_1+t_1)^{c_1-1} (x_2+t_2)^{c_2-1}}{[1+(x_1+t_1)^{c_1}+(x_2+t_2)^{c_2}]^2} - \frac{k(k+1)c_1 c_2 x_1^{c_1-1} x_2^{c_2-1}}{[1+x_1^{c_1}+x_2^{c_2}]^2} \quad (5.3.21)
\end{aligned}$$

This expression is negative when $c_i \leq 1$. Hence bivariate Burr XII has decreasing failure rate when $c_i \leq 1$.

When $c_i > 1$, small values of x_i the above expression is positive and large values of x_i it tends to zero.

Gradient hazard rate (Johnson and Kotz(1975)) is

$$h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))$$

where

$$h_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \log R(x_1, x_2) \quad i = 1, 2$$

$$h_1(x_1, x_2) = \frac{k c_1 x_1^{c_1-1}}{[1 + x_1^{c_1} + x_2^{c_2}]} \quad (5.3.22)$$

and

$$h_2(x_1, x_2) = \frac{k c_2 x_2^{c_2-1}}{[1 + x_1^{c_1} + x_2^{c_2}]} \quad (5.3.23)$$

When $c_i \leq 1$,

$$h_1(x_1 + t_1, x_2 + t_2) = \frac{k c_1 (x_1 + t_1)^{c_1-1}}{[1 + (x_1 + t_1)^{c_1} + (x_2 + t_2)^{c_2}]} \leq h_1(x_1, x_2) \quad (5.3.24)$$

$$h_2(x_1 + t_1, x_2 + t_2) = \frac{k c_2 (x_2 + t_2)^{c_2-1}}{[1 + (x_1 + t_1)^{c_1} + (x_2 + t_2)^{c_2}]} \leq h_2(x_1, x_2) \quad (5.3.25)$$

$$h_1(x_1 + t, x_2 + t) = \frac{k c_1 (x_1 + t)^{c_1-1}}{[1 + (x_1 + t)^{c_1} + (x_2 + t)^{c_2}]} \leq h_1(x_1, x_2) \quad (5.3.26)$$

$$h_2(x_1 + t, x_2 + t) = \frac{k c_2 (x_2 + t)^{c_2-1}}{[1 + (x_1 + t)^{c_1} + (x_2 + t)^{c_2}]} \leq h_2(x_1, x_2) \quad (5.3.27)$$

$$h_1(x_1 + t, x_2) = \frac{k c_1 (x_1 + t)^{c_1-1}}{[1 + (x_1 + t)^{c_1} + (x_2)^{c_2}]} \leq h_1(x_1, x_2) \quad (5.3.28)$$

$$h_2(x_1, x_2 + t) = \frac{k c_2 (x_2 + t)^{c_2-1}}{[1 + (x_1)^{c_1} + (x_2 + t)^{c_2}]} \leq h_2(x_1, x_2) \quad (5.3.29)$$

Hence bivariate Burr XII has decreasing failure rate when $c_i \leq 1$.

When $c_1 > 1$, small values of x_1 , the above expressions is positive and large values of x_1 , it tends to zero.

5.4 Characterizations of Bivariate Burr Type XII Distribution

In this section we consider some characterization theorems of bivariate Burr type XII distribution.

Theorem 5.4.1

Let (X_1, X_2) be a random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (5.2.9). Then X_1 is Burr type XII if and only if X_2 is Burr type XII.

Proof

The conditional density of X_1 given $X_2 = x_2$ is of the form equation (5.2.9).

Assume that X_1 follows univariate Burr type XII distribution. Then

$$f_1(x_1) = \frac{kc_1 x_1^{c_1-1}}{[1+x_1^{c_1}]^{k+1}} \quad 0 < x_1 < \infty \quad k, c_1 > 0$$

Also

$$f_1(x_1) = \int f(x_1 | x_2) f_2(x_2) dx_2 \quad (5.4.1)$$

Hence

$$\frac{kc_1 x_1^{c_1-1}}{[1+x_1^{c_1}]^{k+1}} = (k+1)c_1 \int_0^\infty \frac{x_1^{c_1-1} [1+x_2^{c_2}]^{k+1} f(x_2) dx_2}{[1+x_1^{c_1}]^{k+2} [1+\frac{x_2^{c_2}}{1+x_1^{c_1}}]^{k+2}} \quad (5.4.2)$$

$$\frac{k}{k+1} [1 + x_1^{c_1}] = \int_0^{\infty} \frac{[1 + x_2^{c_2}]^{k+2} f(x_2) dx_2}{[1 + \frac{x_2^{c_2}}{1 + x_1^{c_1}}]^{k+2}} \quad (5.4.3)$$

Substituting $u = x_2^{c_2}$ in equation (5.4.3) gives

$$\begin{aligned} \frac{k}{k+1} [1 + x_1^{c_1}] &= \int_0^{\infty} \frac{[1 + u]^{k+1} f(u^{1/c_2}) u^{1/c_2-1} du}{[1 + \frac{u}{1 + x_1^{c_1}}]^{k+2}} \\ &= \int_0^{\infty} H(u) u^{1/c_2-1} du \end{aligned} \quad (5.4.4)$$

Taking inverse Mellin transform (Rhyzik P.1194)

$$H(u) = \frac{kc_2 u^{1-1/c_2}}{[1 + \frac{u}{1 + x_1^{c_1}}]^{k+2}}$$

Hence

$$f_2(x_2) = \frac{kc_2 x_2^{c_2-1}}{[1 + x_2^{c_2}]^{k+1}} \quad 0 < x_2 < \infty \quad k, c_2 > 0$$

Thus X_2 is of Burr type XII form.

To prove the converse, assume X_2 follows univariate Burr type XII.

Then

$$\begin{aligned} f_1(x_1) &= \int_0^{\infty} f(x_1 | x_2) f_2(x_2) dx_2 \\ &= k(k+1) c_1 c_2 \int_0^{\infty} \frac{x_1^{c_1-1} x_2^{c_2-1} dx_2}{[1 + x_1^{c_1} + x_2^{c_2}]^{k+2}} \\ &= \frac{kc_1 x_1^{c_1-1}}{[1 + x_1^{c_1}]^{k+1}} \quad 0 < x_1 < \infty \quad k, c_1 > 0 \end{aligned}$$

Apart from the marginal distribution of X_i and the conditional distribution of X_j given $X_i = x_i, i = 1, 2, i \neq j$ from which the joint distribution can always be found, the other quantity that is relevant to the problem is marginal and conditional distribution of the same component. In the corollary 5.4.1 we consider a characterization on the marginal and conditional distribution of the same component which incidentally also provides a characterization of univariate Burr type XII distribution using bivariate Burr type XII.

Corollary 5.4.1

Let (X_1, X_2) be a random vector in the support of R_2^+ having absolutely continuous distribution function with respect to Lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (5.2.9). Then (X_1, X_2) is Burr type XII if and only if X_2 is Burr type XII.

Theorem 5.4.2

Let (X_1, X_2) be continuous random vector in the support of R_2^+ having absolutely continuous distribution function with respect to Lebesgue measure. Then (X_1, X_2) follows bivariate Burr type XII distribution if and only if conditional densities are of the form equation (5.2.9).

Proof

Let (X_1, X_2) follow bivariate Burr type XII distribution.

Then $f(x_i / x_j), i = 1, 2, i \neq j$ is of the form (5.2.9)

Conversely

$$\begin{aligned} \frac{f(x_1 | x_2)}{f(x_2 | x_1)} &= \frac{c_1 x_1^{c_1-1} [1 + x_2^{c_2}]^{k+1}}{c_2 x_2^{c_2-1} [1 + x_1^{c_1}]^{k+1}} \\ &= \frac{A_1(x_1)}{A_2(x_2)} \end{aligned}$$

where

$$A_i(x_i) = \frac{c_i x_i^{c_i-1}}{[1 + x_i^{c_i}]^{k+1}} \quad i = 1, 2 \quad (5.4.6)$$

$$\begin{aligned} \int_0^\infty A_1(x_1) dx_1 &= \int_0^\infty A_2(x_2) dx_2 \\ &= 1/k \end{aligned} \quad (5.4.7)$$

Abraham and Thomas condition for unique determination of the joint density using conditional density is satisfied. Hence proof.

Next we consider some characterization theorems using the relationship between scalar hazard rate , scalar reversed hazard rate , gradient hazard rate and gradient reversed hazard rate

Theorem 5.4.3

Let (X_1, X_2) be continuous random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) belongs to the bivariate Burr type XII distribution if and only

$$\lambda(x_1, x_2) - h(x_1, x_2) = [1 + x_1^{c_1} + x_2^{c_2}]^k \lambda(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \quad (5.4.8)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type XII distribution.

Solving $R(x_1, x_2)$ from equations (2.3.19) , (2.3.23) and (3.2.3) we have

$$R(x_1, x_2) = \frac{\lambda(x_1, x_2) \left[\frac{\lambda_1(x_1)}{h_1(x_1) + \lambda_1(x_1)} - \frac{h_2(x_2)}{h_2(x_2) + \lambda_2(x_2)} \right]}{\lambda(x_1, x_2) - h(x_1, x_2)} \quad (5.4.9)$$

Then using equation (5.4.9) in equation (5.2.4) we have equation (5.4.8).

Conversely starting from (5.4.8) and using (2.3.19) , (2.3.23) and (3.2.3) we get

$$R(x_1, x_2) = [1 + x_1^{c_1} + x_2^{c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2$$

Theorem 5.4.4

Let (X_1, X_2) be continuous random vector in the support of R_2^+

having absolutely continuous distribution function with respect to lebesgue measure.

Then (X_1, X_2) belongs to the bivariate Burr type XII distribution if and only

$$\begin{aligned} & \lambda_i(x_1, x_2) + h_i(x_1, x_2) \\ & = [1 + x_1^{c_1} + x_2^{c_2}]^k \left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} + \lambda_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \right] i = 1, 2 \end{aligned} \quad (5.4.10)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type XII distribution .

Solving $R(x_1, x_2)$ from equations (2.3.13) , (2.3.27) and (3.2.3) we find

$$R(x_1, x_2) = \frac{\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - \lambda_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]}{\lambda_i(x_1, x_2) + h_i(x_1, x_2)} \quad i = 1, 2 \quad (5.4.11)$$

Then using equation (5.4.11) in equation (5.2.4) we have equation (5.4.10).

Conversely starting from (5.4.10) and using (2.3.13) , (2.3.29) and (3.2.3) we get

$$R(x_1, x_2) = [1 + x_1^{c_1} + x_2^{c_2}]^{-k} \quad 0 < x_i < \infty, k, c_i > 0 \quad i = 1, 2$$

Theorem 5.4.5

Let (X_1, X_2) be continuous random vector in the support of R_2^+ having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) belongs to the bivariate Burr type XII distribution if

$$h(x_1, x_2) = \frac{k+1}{k} h_1(x_1, x_2) h_2(x_1, x_2) \quad (5.4.12)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type XII distribution .

Then by equation (5.3.20) ,(5.3.22) and (5.3.23)we have equation (5.4.12).

5.5 Relation Between Burr Type XII and Other Distributions

Let (X_1, X_2) follows to the bivariate Burr type XII distribution .Table 5.5.1 gives relation between this distribution and other distributions.

Table 5.5.1

Transformation	Distribution function
$Y_i = [1 + X_i^{c_i}]^{-k}$	$[y_1^{-1/k} + y_2^{-1/k} - 1]^{-k} \quad 0 < y_i < 1, k > 0 \quad i = 1, 2$ (Cook and Johnson (1986))
$U_i = -\log X_i^{c_i}$	$[1 + e^{-u_1} + e^{-u_2}]^{-k} \quad -\infty < u_i < \infty, k > 0$ (Burr II)
$V_i = \frac{1}{X_i}$	$[1 + v_1^{-c_1} + v_2^{-c_2}]^{-k} \quad 0 < v_i < \infty, k, c_i > 0 \quad i = 1, 2$ (Burr III)
$W = \frac{X_i^{c_i}}{a_i}$	$[1 + a_1 w_1 + a_2 w_2]^{-k}$ (Nayak (1987))
$c_i = 1$	$[1 + x_1 + x_2]^{-k} \quad 0 < x_i < \infty, c_i > 0, k > 0, i = 1, 2$ (Mardia(1960))

5.6 Bivariate Burr IX Distribution (Bismi and Nair, 2005 f)

In this section we consider general properties, characterizations of bivariate Burr IX distribution. The distribution arises by the choice of $G(x_1, x_2)$ as

$$G(x_1, x_2) = \log \frac{1 - \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1]} - \frac{2}{2 + c_2[(1 + e^{x_2})^k - 1]} + \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]}}{\frac{2}{2 + c_1[(1 + e^{x_1})^k - 1]} + \frac{2}{2 + c_2[(1 + e^{x_2})^k - 1]} - \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]}} \quad (5.6.1)$$

in equation (2.2.11)

The distribution function is

$$F(x_1, x_2) = 1 - \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1]} - \frac{2}{2 + c_2[(1 + e^{x_2})^k - 1]} + \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]} \quad (5.6.2)$$

$-\infty < x_i < \infty, k > 0, c_i > 0, i = 1, 2$

$$f(x_1, x_2) = \frac{4k^2 c_1 c_2 e^{x_1} e^{x_2} (1 + e^{x_1})^{k-1} (1 + e^{x_2})^{k-1}}{[2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]]^3} \quad -\infty < x_i < \infty, k > 0, c_i > 0, i = 1, 2 \quad (5.6.3)$$

and

$$R(x_1, x_2) = \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]} \quad -\infty < x_i < \infty, k > 0, c_i > 0, i = 1, 2 \quad (5.6.4)$$

The marginal distributions are specified by

$$F_i(x_i) = 1 - \frac{2}{2 + c_i[(1 + e^{x_i})^k - 1]} \quad -\infty < x_i < \infty, k > 0, c_i > 0, i = 1, 2 \quad (5.6.5)$$

$$F_i(x_i) [1 - F_i(x_i)] g_i(x_i)$$

$$= \frac{2c_i k e^{x_i} (1 + e^{x_i})^{k-1}}{[2 + c_i[(1 + e^{x_i})^k - 1]]^2} \quad -\infty < x_i < \infty, k > 0, c_i > 0, i = 1, 2 \quad (5.6.6)$$

$$= \frac{dF_i(x_i)}{dx_i} \quad i = 1, 2 \quad (5.6.7)$$

which is univariate Burr type differential equation where

$$g_i(x_i) = \frac{ke^{x_i}(1+e^{x_i})^{k-1}}{[(1+e^{x_i})^k - 1]} \quad i = 1, 2 \quad (5.6.8)$$

Thus the marginals are exactly univariate Burr type IX distribution.

Conditional density of X_i given $X_j = x_j$ is

$$f(X_i | X_j = x_j) = \frac{2kc_i e^{x_i} (1+e^{x_i})^{k-1} [2+c_j[(1+e^{x_j})^k - 1]]^2}{[2+c_i[(1+e^{x_i})^k - 1] + c_j[(1+e^{x_j})^k - 1]]^3} \quad (5.6.9)$$

$$i, j = 1, 2, i \neq j$$

In view of the closed form expression for the survival function of the distribution, it is handy to compute the reliability characteristics such as failure rate, reversed failure rate etc.

The Basu's (1971) failure rate is given by

$$\begin{aligned} h(x_1, x_2) &= \frac{f(x_1, x_2)}{R(x_1, x_2)} \\ &= \frac{2k^2 c_1 c_2 e^{x_1} e^{x_2} (1+e^{x_1})^{k-1} (1+e^{x_2})^{k-1}}{[2+c_1[(1+e^{x_1})^k - 1] + c_2[(1+e^{x_2})^k - 1]]^2} \quad (5.6.10) \end{aligned}$$

The vector valued failure rate (Johnson and Kotz (1975)) is given by

$$\begin{aligned} h(x_1, x_2) &= (h_1(x_1, x_2), h_2(x_1, x_2)) \\ h_i(x_i, x_j) &= \frac{-\partial \log R(x_i, x_j)}{\partial x_i} \\ &= \frac{kc_i e^{x_i} (1+e^{x_i})^{k-1}}{2+c_i[(1+e^{x_i})^k - 1] + c_j[(1+e^{x_j})^k - 1]} \quad i, j = 1, 2, i \neq j \quad (5.6.11) \end{aligned}$$

The scalar reversed hazard rate is given by

$$\lambda(x_1, x_2) = \frac{f(x_1, x_2)}{F(x_1, x_2)}$$

$$= \frac{4k^2 c_1 c_2 e^{x_1} e^{x_2} (1+e^{x_1})^{k-1} (1+e^{x_2})^{k-1}}{[2+c_1[(1+e^{x_1})^k-1]+c_2[(1+e^{x_2})^k-1]]^3} \quad (5.6.11)$$

$$1 - \frac{2}{2+c_1[(1+e^{x_1})^k-1]} - \frac{2}{2+c_2[(1+e^{x_2})^k-1]} + \frac{2}{2+c_1[(1+e^{x_1})^k-1]+c_2[(1+e^{x_2})^k-1]}$$

The vector valued reversed hazardrate (Johnson and Kotz (1975)) is given by

$$h(x_1, x_2) = (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2))$$

$$\lambda_i(x_i, x_j) = \frac{\partial \log F(x_i, x_j)}{\partial x_i}$$

$$= \frac{\frac{2kc_i e^{x_i} (1+e^{x_i})^{k-1}}{[2+c_i[(1+e^{x_i})^k-1]]^2} - \frac{2kc_i e^{x_i} (1+e^{x_i})^{k-1}}{[2+c_i[(1+e^{x_i})^k-1]+c_j[(1+e^{x_j})^k-1]]^2}}{1 - \frac{2}{2+c_i[(1+e^{x_i})^k-1]} - \frac{2}{2+c_j[(1+e^{x_j})^k-1]} + \frac{2}{2+c_i[(1+e^{x_i})^k-1]+c_j[(1+e^{x_j})^k-1]}} \quad (5.6.12)$$

$i, j = 1, 2 \quad i \neq j$

5.7 Characterizations of Burr Type IX Distribution (Bismi and Nair, 2005 f)

In this section we consider some characterizations of Burr type IX distribution.

The following theorem characterizes the Burr type IX distribution using conditional densities.

Theorem 5.7.1

Let (X_1, X_2) be a random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (5.6.9). Then X_1 is Burr type IX if and only if X_2 is Burr type IX.

Proof

The conditional density of X_1 given $X_2 = x_2$ is of the form equation (5.6.9).

Assume that X_1 follows univariate Burr type IX distribution. Then

$$f_1(x_1) = \frac{2c_1 e^{x_1} (1 + e^{x_1})^{k-1}}{[2 + c_1 [(1 + e^{x_1})^k - 1]]^2} \quad -\infty < x_1 < \infty, k > 0, c_1 > 0$$

Then using the relation

$$f_1(x_1) = \int f(x_1 | x_2) f_2(x_2) dx_2$$

we have

$$\frac{2kc_1 e^{x_1} (1 + e^{x_1})^{k-1}}{[2 + c_1 [(1 + e^{x_1})^k - 1]]^2} = 2k c_1 \int_{-\infty}^{\infty} \frac{2c_1 e^{x_1} (1 + e^{x_1})^{k-1} [2 + c_2 [(1 + e^{x_2})^k - 1]]^2 f_2(x_2) dx_2}{[2 + c_1 [(1 + e^{x_1})^k - 1] + c_2 [(1 + e^{x_2})^k - 1]]^2} \quad (5.7.1)$$

Substituting $u = c_2 [(1 + e^{x_2})^k - 1]$ in equation (5.7.1) gives

$$2 + c_1 [(1 + e^{x_1})^k - 1] = \int_0^{\infty} \frac{[2 + u]^2 f_2(\log(\frac{u}{c_2} + 1)^{1/c} - 1) du}{\left[1 + \frac{u}{2 + c_1 [(1 + e^{x_1})^k - 1]}\right]^3 \left(\frac{u}{c_2} + 1\right)^{1/c-1}} \quad (5.7.2)$$

$$= \int_0^{\infty} H(u) u^{c-1} du \quad (5.7.3)$$

Taking inverse Mellin transform (Rhyzik P.1194)

$$H(u) = \frac{u^{c-1}}{\left[1 + \frac{u}{2 + c_1 [(1 + e^{x_1})^k - 1]}\right]^3}$$

Hence

$$f_2(x_2) = \frac{2c_2 e^{x_2} (1 + e^{x_2})^{k-1}}{[2 + c_2 [(1 + e^{x_2})^k - 1]]^2} \quad -\infty < x_2 < \infty, k > 0, c_2 > 0$$

Thus X_2 is of Burr type IX form.

To prove the converse , assume X_2 follows univariate Burr type IX.

Then

$$\begin{aligned}
 f_1(x_1) &= \int_{-\infty}^{\infty} f(x_1 | x_2) f_2(x_2) dx_2 \\
 &= 2k c_1 \int_{-\infty}^{\infty} \frac{2c_1 e^{x_1} (1 + e^{x_1})^{k-1} [2 + c_2 [(1 + e^{x_2})^k - 1]]^2 dx_2}{[2 + c_1 [(1 + e^{x_1})^k - 1] + c_2 [(1 + e^{x_2})^k - 1]]^3} \\
 &= \frac{2c_1 e^{x_1} (1 + e^{x_1})^{k-1}}{[2 + c_1 [(1 + e^{x_1})^k - 1]]^2} \quad -\infty < x_1 < \infty, k > 0, c_1 > 0
 \end{aligned}$$

Corollary 5.7.1

Let (X_1, X_2) be a random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of X_1 given $X_2 = x_2$ is of the form equation (5.6.9). Then (X_1, X_2) is Burr type IX if and only if X_2 is Burr type IX.

Theorem 5.7.2

Let (X_1, X_2) be random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) follows bivariate Burr type IX distribution if and only if conditional densities are of the form equation (5.6.9).

Proof

Let (X_1, X_2) follows bivariate Burr type IX distribution.

Then $f(x_i | x_j)$ $i=1, 2$ $i \neq j$ is of the form (5.6.9)

Conversly

$$\begin{aligned} \frac{f(x_1 | x_2)}{f(x_2 | x_1)} &= \frac{c_1 e^{x_1} (1 + e^{x_1})^{k-1} [2 + c_2 [1 + e^{x_2}]^k - 1]}{c_2 e^{x_2} (1 + e^{x_2})^{k-1} [2 + c_1 [1 + e^{x_1}]^k - 1]} \\ &= \frac{A_1(x_1)}{A_2(x_2)} \end{aligned}$$

where

$$A_i(x_i) = \frac{c_i e^{x_i} (1 + e^{x_i})^{k-1}}{[2 + c_i [1 + e^{x_i}]^k - 1]} \quad i = 1, 2 \quad (5.7.4)$$

$$\begin{aligned} \int_{-\infty}^{\infty} A_1(x_1) dx_1 &= \int_{-\infty}^{\infty} A_2(x_2) dx_2 \\ &= 2k \end{aligned} \quad (5.7.5)$$

Abraham and Thomas condition for unique determination of the joint density using conditional density is satisfied. Hence proof.

Next we consider some characterization theorems using the relationship between scalar hazard rate , scalar reversed hazard rate , gradient hazard rate and gradient reversed hazard rate

Theorem 5.7.3

Let (X_1, X_2) be continuous random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) belongs to the bivariate Burr type IX distribution if and only

$$\begin{aligned} &2[\lambda(x_1, x_2) - h(x_1, x_2)] \\ &= [2 + c_1 [(1 + e^{x_1})^k - 1] + c_2 [(1 + e^{x_2})^k - 1]] \lambda(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \end{aligned} \quad (5.7.6)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type IX distribution.

Then using equation (5.4.9) in equation (5.6.4) we have equation (5.7.6).

Conversely starting from (5.7.6) and using (2.3.19) , (2.3.23) and (3.2.3) we get

$$R(x_1, x_2) = \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]} \quad -\infty < x_i < \infty, k, c_i > 0 \quad i = 1, 2$$

Theorem 5.7.4

Let (X_1, X_2) be continuous random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2)

belongs to the bivariate Burr type XII distribution if and only

$$2[\lambda_i(x_1, x_2) + h_i(x_1, x_2)] \\ = [2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]] \left[\frac{\lambda_i(x_i)h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} + \lambda_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \right] \quad i = 1, 2 \quad (5.7.7)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type IX distribution.

Then using equation (5.4.11) in equation (5.6.4) we have equation (5.7.7).

Conversely starting from (5.7.7) and using (2.3.13), (2.3.27) and (3.2.3) we get

$$R(x_1, x_2) = \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]} \quad -\infty < x_i < \infty, k, c_i > 0 \quad i = 1, 2$$

Theorem 5.7.5

Let (X_1, X_2) be continuous random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2)

belongs to the bivariate Burr type IX distribution if

$$h(x_1, x_2) = 2h_1(x_1, x_2)h_2(x_1, x_2) \quad (5.7.8)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type IX distribution .

Then by equation (5.6.10)and (5.6.11)we have equation (5.7.8).

5.8 Bivariate Burr II Distribution (Bismi and Nair, 2005 f)

In this section we consider general properties, characterizations of bivariate Burr II distribution. The distribution arises by the choice of $G(x_1, x_2)$ as

$$G(x_1, x_2) = -\log[[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^k - 1] \quad (5.8.1)$$

in equation (2.2.11)

The distribution function of bivariate Burr type II distribution is

$$F(x_1, x_2) = [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-k} \quad -\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k+1 \quad i = 1, 2$$

Corresponding density function and survival function is

$$f(x_1, x_2) = \frac{k(k+1)e^{-x_1}e^{-x_2}(1+\theta e^{-x_1})(1+\theta e^{-x_2})}{[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{k+2}} - \frac{k\theta e^{-x_1} e^{-x_2}}{[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{k+1}} \quad (5.8.3)$$

$$-\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k+1 \quad i = 1, 2$$

and

$$R(x_1, x_2) = 1 - [1 + e^{-x_1}]^{-k} - [1 + e^{-x_2}]^{-k} + [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-k} \quad (5.8.4)$$

$$-\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k+1 \quad i = 1, 2$$

The marginal distributions are specified by

$$F_i(x_i) = [1 + e^{-x_i}]^{-k} \quad -\infty < x_i < \infty, k > 0, i = 1, 2 \quad (5.8.5)$$

$$F_i(x_i)[1-F_i(x_i)] g_i(x_i) = \frac{ke^{-x_i}}{[1+e^{-x_i}]^{k+1}} \quad -\infty < x_i < \infty, k > 0, i=1,2 \quad (5.8.6)$$

$$= \frac{dF(x_i)}{dx_i} \quad i=1,2 \quad (5.8.7)$$

which is univariate Burr type differential equation where

$$g_i(x_i) = \frac{ke^{-x_i}[1+e^{-x_i}]^{k-1}}{[[1+e^{-x_i}]^k - 1]} \quad i=1,2 \quad (5.8.8)$$

Hence marginals are exactly univariate Burr type II distribution.

Conditional density of X_i given $X_j = x_j$ is

$$f(X_i | X_j = x_j) = \frac{k(k+1)e^{-x_i}(1+\theta e^{-x_i})(1+\theta e^{-x_j})}{\left[1 + \frac{e^{-x_i}(1+\theta e^{-x_j})}{(1+e^{-x_j})}\right]^{k+2}} - \frac{k\theta e^{-x_i}}{\left[1 + \frac{e^{-x_i}(1+\theta e^{-x_j})}{(1+e^{-x_j})}\right]^{k+1}} \quad (5.8.9)$$

$$-\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k+1 \quad i, j = 1, 2, i \neq j$$

Now we are interested to find concepts useful in failure time analysis.

The Basu's(1971) failure rate is given by

$$h(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)}$$

$$= \frac{\frac{k(k+1)e^{-x_1}e^{-x_2}(1+\theta e^{-x_1})(1+\theta e^{-x_2})}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]^{k+2}} - \frac{k\theta e^{-x_1}e^{-x_2}}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]^{k+1}}}{1 - [1+e^{-x_1}]^{-k} - [1+e^{-x_2}]^{-k} + [1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]^{-k}} \quad (5.8.10)$$

The vector valued failure rate (Johnson and Kots (1975)) is given by

$$h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))$$

$$h_i(x_1, x_j) = \frac{-\partial \log R(x_i, x_j)}{\partial x_i}$$

$$= \frac{\frac{ke^{-x_i}}{[1+e^{-x_i}]^{k+1}} - \frac{ke^{-x_i}e^{-x_j}(1+\theta e^{-x_j})}{[1+e^{-x_i}+e^{-x_j}+\theta e^{-x_i}e^{-x_j}]^{k+2}}}{1-[1+e^{-x_i}]^{-k}-[1+e^{-x_j}]^{-k}+[1+e^{-x_i}+e^{-x_j}+\theta e^{-x_i}e^{-x_j}]^{-k}} \quad i, j = 1, 2 \quad (5.8.11)$$

The scalar reversed hazard rate is given by

$$\begin{aligned} \lambda(x_1, x_2) &= \frac{f(x_1, x_2)}{F(x_1, x_2)} \\ &= \frac{2k(k+1)e^{-x_1}e^{-x_2}(1+\theta e^{-x_1})(1+\theta e^{-x_2})}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]^2} - \frac{2k\theta e^{-x_1}e^{-x_2}}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]} \quad (5.8.12) \end{aligned}$$

The vector valued reverse hazardrate (Johnson and Kotz (1975))is given by

$$\begin{aligned} \Delta \log F(x_1, x_2) &= (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2)) \\ \lambda_i(x_i, x_j) &= \frac{\partial \log F(x_i, x_j)}{\partial x_i} \\ &= \frac{ke^{-x_i}(1+\theta e^{-x_j})}{[1+e^{-x_i}+e^{-x_j}+\theta e^{-x_i}e^{-x_j}]} \quad i, j = 1, 2 \quad i \neq j \quad (5.8.13) \end{aligned}$$

5.9 Characterizations of Burr Type II Distribution (Bismi and Nair,2005 f)

Theorem 5.9.1

Let (X_1, X_2) be continuous random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure. Then

(X_1, X_2) belongs to the bivariate Burr type II distribution if and only if

$$\begin{aligned} &\lambda(x_1, x_2) - h(x_1, x_2) \\ &= [1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]^k h(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1)+h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2)+h_2(x_2)} \right] \quad (5.9.1) \end{aligned}$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type II distribution.

Then using equation (3.2.4) in equation (5.8.2) we have equation (5.9.1).

Conversely starting from (5.9.1) and using (2.3.19) , (2.3.23), and (3.2.3) we get

$$F(x_1, x_2) = [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-k} \quad -\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k+1 \quad i = 1, 2$$

Theorem 5.9.2

Let (X_1, X_2) be continuous random vector in the support of R_2 having absolutely continuous distribution function with respect to lebesgue measure. Then (X_1, X_2) belongs to the bivariate Burr type II distribution if and only

$$\lambda_i(x_1, x_2) + h_i(x_1, x_2)]$$

$$= [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^k \left[\frac{\lambda_i(x_i) h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} + \lambda_i(x_1, x_2) \left[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] \right] \quad i = 1, 2 \quad (5.9.2)$$

Proof

Let (X_1, X_2) follows to the bivariate Burr type II distribution.

Then using equation (3.2.6) in equation (5.8.2) we have equation (5.9.2).

Conversely starting from (5.9.2) and using (2.3.13) , (2.3.27) and (3.2.3) we get

$$F(x_1, x_2) = [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-k} \quad -\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k+1 \quad i = 1, 2$$

CHAPTER VI
SOME MULTIVARIATE EXTENSIONS

6.1 Introduction

In connection with study of Burr systems in two dimensions , the main bivariate forms encountered were ,

$$(a) \quad f(x_1, x_2) \quad = \quad F(x_1, x_2)[1 - F(x_1, x_2)]g(x_1, x_2)$$

$$(b) \quad \frac{\partial F(x_1, x_2)}{\partial x_1} \quad = \quad F(x_1, x_2)[1 - F(x_1, x_2)]g_1(x_1, x_2)$$

$$\frac{\partial F(x_1, x_2)}{\partial x_2} \quad = \quad F(x_1, x_2)[1 - F(x_1, x_2)]g_2(x_1, x_2)$$

$$(c) \quad f(x_1, x_2) \quad = \quad F(x_1, x_2)R(x_1, x_2)g(x_1, x_2)$$

$$(d) \quad \frac{\partial F(x_1, x_2)}{\partial x_1} \quad = \quad F(x_1, x_2)R(x_1, x_2)g_1(x_1, x_2)$$

$$\frac{\partial F(x_1, x_2)}{\partial x_2} \quad = \quad F(x_1, x_2)R(x_1, x_2)g_2(x_1, x_2)$$

It is of natural interest to explore the extensions of the bivariate concepts so far discussed and of the corresponding distributions in the general multivariate cases. Though most of these multivariate generalizations can be obtained as straight forward extensions , in some cases the conditions attached to them become more restrictive. In this chapter we briefly sketch the multivariate forms of the defining equations and the definitions , since the ideas were already conveyed in the bivariate case The explanations in the more general cases are only touched up in the following discussions.

6.2 Multivariate Burr System (Bismi and Nair, 2005 c)

Let $\underline{X} = (X_1, X_2, \dots, X_n)$ be continuous random vector in the support of R_n admitting absolutely continuous distribution function $F(x_1, x_2, \dots, x_n)$, probability density function $f(x_1, x_2, \dots, x_n)$, and survival function $R(x_1, x_2, \dots, x_n)$.

The different forms for the Multivariate Burr System are ,

$$(a) \quad f(x_1, x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n)[1 - F(x_1, x_2, \dots, x_n)]g(x_1, x_2, \dots, x_n) \quad (6.2.1)$$

$$(b) \quad \frac{\partial F(x_1, x_2, \dots, x_n)}{\partial x_1} = F(x_1, x_2, \dots, x_n)[1 - F(x_1, x_2, \dots, x_n)]g_1(x_1, x_2, \dots, x_n) \quad (6.2.2)$$

$$\frac{\partial F(x_1, x_2, \dots, x_n)}{\partial x_2} = F(x_1, x_2, \dots, x_n)[1 - F(x_1, x_2, \dots, x_n)]g_2(x_1, x_2, \dots, x_n) \quad (6.2.3)$$

$$\frac{\partial F(x_1, x_2, \dots, x_n)}{\partial x_n} = F(x_1, x_2, \dots, x_n)[1 - F(x_1, x_2, \dots, x_n)]g_n(x_1, x_2, \dots, x_n) \quad (6.2.4)$$

for some functions $g_1(x_1, x_2, \dots, x_n)$, $g_2(x_1, x_2, \dots, x_n)$ $g_n(x_1, x_2, \dots, x_n)$. In this

case we are considering a set of n partial differential equations

$$(c) \quad f(x_1, x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n)R(x_1, x_2, \dots, x_n)g(x_1, x_2, \dots, x_n) \quad (6.2.5)$$

$$(d) \quad \frac{\partial F(x_1, x_2, \dots, x_n)}{\partial x_1} = F(x_1, x_2, \dots, x_n)R(x_1, x_2, \dots, x_n)g_1(x_1, x_2, \dots, x_n) \quad (6.2.6)$$

$$\frac{\partial F(x_1, x_2, \dots, x_n)}{\partial x_2} = F(x_1, x_2, \dots, x_n)R(x_1, x_2, \dots, x_n)g_2(x_1, x_2, \dots, x_n) \quad (6.2.7)$$

$$\frac{\partial F(x_1, x_2, \dots, x_n)}{\partial x_n} = F(x_1, x_2, \dots, x_n)R(x_1, x_2, \dots, x_n)g_n(x_1, x_2, \dots, x_n) \quad (6.2.8)$$

In view of the analytical tractability we are concentrate on the set of n partial differential equations in (b) to generate multivariate Burr system .

Let (X_1, X_2) be a random vector in the support of $(a_1, b_1) \times (a_2, b_2), \dots, \times (a_n, b_n)$ $-\infty \leq a_i < b_i \leq \infty, i = 1, 2, \dots, n$ admitting absolutely continuous distribution function $F(x_1, x_2, \dots, x_n)$ and satisfying the n partial differential equations in (b)

To solve the first equation in (b) rewrite it as

$$\frac{\partial}{\partial x_1} \left[\log \frac{F(x_1, x_2, \dots, x_n)}{1 - F(x_1, x_2, \dots, x_n)} \right] = g_1(x_1, x_2, \dots, x_n) \quad (6.2.9)$$

Integrating from a_1 to x_1 ,

$$F(x_1, x_2, \dots, x_n) = [1 + e^{-G_1(x_1, x_2, \dots, x_n)}]^{-1} \quad (6.2.10)$$

where

$$G_1(x_1, x_2, \dots, x_n) = \int_{a_1}^{x_1} g_1(t_1, x_2, \dots, x_n) dt_1 \quad (6.2.11)$$

To solve the i^{th} equation in (b) rewrite it as

$$\frac{\partial}{\partial x_i} \left[\log \frac{F(x_1, x_2, \dots, x_n)}{1 - F(x_1, x_2, \dots, x_n)} \right] = g_i(x_1, x_2, \dots, x_n) \quad (6.2.12)$$

Integrating from a_i to x_i ,

$$F(x_1, x_2, \dots, x_n) = [1 + e^{-G_i(x_1, x_2, \dots, x_n)}]^{-1} \quad (6.2.13)$$

where

$$G_i(x_1, x_2, \dots, x_n) = \int_{a_i}^{x_i} g_i(x_1, x_2, \dots, t_i, \dots, x_n) dt_i \quad (6.2.14)$$

Proceed in a similar way for all equations in (b) and comparing expressions for

$F(x_1, x_2, \dots, x_n)$ we have,

$$\begin{aligned}
 G_1(x_1, x_2, \dots, x_n) &= G_2(x_1, x_2, \dots, x_n) = \dots = G_i(x_1, x_2, \dots, x_n) \\
 &= \dots = G_n(x_1, x_2, \dots, x_n) = G(x_1, x_2, \dots, x_n) \quad , \text{ say}
 \end{aligned}$$

and hence

$$F(x_1, x_2, \dots, x_n) = [1 + e^{-G(x_1, x_2, \dots, x_n)}]^{-1} \quad (6.2.15)$$

Also

$$\begin{aligned}
 \frac{\partial g_1(x_1, x_2, \dots, x_n)}{\partial x_2 \partial x_3 \dots \partial x_n} &= \frac{\partial g_2(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_3 \dots \partial x_n} = \dots = \frac{\partial g_i(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_2 \dots \partial x_{i-1} \partial x_{i+1} \dots \partial x_n} \\
 &= \dots = \frac{\partial g_n(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_2 \partial x_3 \dots \partial x_{n-1}} \quad (6.2.16)
 \end{aligned}$$

The boundary conditions to be satisfied by $G(x_1, x_2, \dots, x_n)$ are

$$\lim_{\substack{x_1 \rightarrow b_1 \\ x_2 \rightarrow b_2 \\ \vdots \\ x_n \rightarrow b_n}} G(x_1, x_2, \dots, x_n) = \infty \quad \text{and} \quad \lim_{x_i \rightarrow a_i} G(x_1, x_2, \dots, x_n) = -\infty \quad \text{for} \quad i = 1, 2, \dots, n.$$

Since $F(x_1, x_2, \dots, x_n)$ has to be monotonic increasing in (X_1, X_2, \dots, X_n) we must have

$G(x_1, x_2, \dots, x_n)$ is nondecreasing and have

$$\frac{\partial G(x_1, x_2, \dots, x_n)}{\partial x_i} \geq 0 \quad \text{for all } x_i, i = 1, 2, \dots, n$$

Thus the functions $g_i(x_1, x_2, \dots, x_n)$, $i = 1, 2, \dots, n$ have to be nonnegative.

By this approach every absolutely continuous multivariate distributions belongs to this family as it is possible to choose

$$g_i(x_1, x_2, \dots, x_n) = \frac{1}{1 - F(x_1, x_2, \dots, x_n)} \frac{\partial}{\partial x_i} [\log F(x_1, x_2, \dots, x_n)] \quad (6.2.17)$$

Table 6.2.1 gives multivariate extension of the univariate types discussed in Burr (1942).

Table 6.2.1
Multivariate Burr Distributions

Type	$F(x_1, x_2, \dots, x_n)$	Range $i=1,2,\dots,n$
I	$[\sum_{i=1}^n x_i^{-1/k} - (n-1)]^{-k}$	$0 < x_i < 1$
II	$(1 + \sum_{i=1}^n e^{-x_i})^{-k}$	$-\infty < x_i < \infty$
III	$(1 + \sum_{i=1}^n x_i^{-c_i})^{-k}$	$x_i > 0$
IV	$(1 + \sum_{i=1}^n (\frac{c_i - x_i}{x_i})^{1/c_i})^{-k}$	$x_i < c_i$
V	$(1 + \sum_{i=1}^n c_i e^{-\tan x_i})^{-k}$	$-\frac{\pi}{2} < x_i < \frac{\pi}{2}$
VI	$(1 + \sum_{i=1}^n e^{-k \sinh x_i})^{-k}$	$-\infty < x_i < \infty$
VII	$[\sum_{i=1}^n \frac{2}{1 + \tanh x_i} - (n-1)]^{-k}$	$-\infty < x_i < \infty$
VIII	$[\sum_{i=1}^n [\frac{\pi}{2 \tan^{-1}(e^{x_i})}] - (n-1)]^{-k}$	$-\infty < x_i < \infty$
IX	$[\sum_{i=1}^n [1 - \frac{2}{2 + c_i [(1 + e^{x_i})^k - 1]]} - (n-1)]^{-1}$	$-\infty < x_i < \infty$
X	$[\sum_{i=1}^n \frac{1}{[1 - e^{-x_i^2}]} - (n-1)]^{-k}$	$0 < x_i < \infty$
XI	$(\prod_{i=1}^n x_i - \frac{1}{2\pi} \sin 2\pi \prod_{i=1}^n x_i)^k$	$0 < x_i < 1$
XII	$1 - \sum_{i=1}^n [1 + x_i^{c_i}]^{-k} + [1 + \sum_{i=1}^n x_i^{c_i}]^{-k}$	$x_i > 0$

where all c_i and k are positive real numbers.

A direct extension of the scalar reversed hazard rate and vector reversed hazard rate in the multivariate case has the following definition.

Definition 6.2.1

Vector valued reverse hazard rate

$$\Delta \log F(x_1, x_2, \dots, x_n) = (\lambda_1(x_1, x_2, \dots, x_n), \lambda_2(x_1, x_2, \dots, x_n), \dots, \lambda_n(x_1, x_2, \dots, x_n)) \quad (6.2.18)$$

where

$$\lambda_i(x_1, x_2, \dots, x_n) = \frac{\partial}{\partial x_i} \log F(x_1, x_2, \dots, x_n) \quad i=1, 2, \dots, n \quad (6.2.19)$$

Definition 6.2.2

The scalar reversed hazard rate can be defined as

$$\lambda(x_1, x_2, \dots, x_n) = \frac{f(x_1, x_2, \dots, x_n)}{F(x_1, x_2, \dots, x_n)} \quad (6.2.20)$$

Using the above definitions all characterizations in section 3.2 can be extended directly to the multivariate case.

REFERENCES

1. Abdelfattah, A-A.M. (1996) A correlation type goodness of fit for the Burr distribution, *Egypt. Statist. J.*, 40, 182-190.
2. Abdelfattah, A-A.M. (1997) On the goodness of fit for the Burr failure model, *Egypt. Statist. J.*, 41, 10-27.
3. Abraham, J. and Thomas, J.B. (1984) A note on the characterization of bivariate densities by conditional densities *Commun. Statist. Theor. Meth.*, A, 1B (3), 395-400.
4. Ahmad, M. (1985) Estimation of parameters of Burr distribution based on certain order statistics, *Stud. Scient. Math. Hungaria*, 20, 130-137.
5. Al-Dayin, G.R. (1999) Burr type III distribution Properties and estimation, *Egypt. Statist. J.* 43, 107-116.
6. Al-Hussaini, E.K. and Jaheen, Z.F. (1992) Bayesian estimation of the parameters, reliability and failure rate functions of Burr type XII failure model, *J. Statist. Comput. Simul.*, 41, 31-40.
7. Al-Hussaini, E.K., Jaheen, Z.F. and Mousa, M.A.M.H. (1992) Estimation under the Burr type XII failure model based on censored data A comparative study, *Test*, 1, 47-60.
8. Al-Hussaini, E.K. and Jaheen, Z.F. (1994) Approximate Bayes estimates applied to the Burr model, *Commun. Statist. Theor. Meth.*, 23, 99-121.
9. Al-Hussaini, E.K. and Jaheen, Z.F. (1995) Bayesian prediction bounds for the Burr XII failure model, *Commun. Statist. Theor. Meth.*, 24, 1829-1842.

10. Al-Hussaini, E.K. and Jaheen, Z.F. (1996) Bayesian prediction bounds for the Burr type XII in the presence of outliers, *J. Statist. Planning Infer.*, 55, 23-27.
11. Alimousa, M.A.M, Jaheen, Z.F. (1997) Bayesian prediction for the Burr type XII model based on doubly censored data, *Statistics*, 29, 285-294.
12. Al-Marzoung, A.M. and Ahmad, M. (1995) Estimation of parameters of Burr probability model using fractional moments, *Pakistan J. Statist.*,1, 67-77.
13. Ashour, S.K and El. Wakeel, M.A.M.H. (1994) Bayesian prediction of the median of the Burr distribution with fixed and random sample sizes, *Statistics*, 25, 113-122.
14. Austin, M.(1973) Control chart constants for largest and smallest in sampling from a normal distribution using the generalized Burr distribution, *Technometrics*, 15, No.4, 931-933.
15. Basu, A.P.(1971): Bivariate failure rate, *J. Amer. Statist. Ass.*, 66, 103-104
16. Begum, A.A and Khan, A.H.(1998) Concomitants of order statistics from bivariate Burr distribution, *J. Appl. Statist. Sci.*, 7, 255-265.
17. Bismi, G., Nair, N.U. and Nair, V.K.R. (2005 a) Bivariate Burr system of distributions (Communicated)
18. Bismi, G.and Nair, V.K.R. (2005 b) Characterization of bivariate Burr system of distributions using reliability concepts. (Communicated)
19. Bismi, G. and Nair, V.K.R. (2005 c) Multivariate Burr system of distributions. (Communicated)

20. Bismi, G. and Nair, V.K.R. (2005 d) Bivariate Burr III distribution. (Communicated)
21. Bismi, G. and Nair, V.K.R. (2005 e) An alternative derivation of Bivariate Burr III distribution. (Communicated)
22. Bismi, G. and Nair, V.K.R. (2005 f) Bivariate Burr II and IX distributions. (Communicated)
23. Burr, I.W. (1942) Cumulative frequency function, *Ann. Math. Statist.*, 13, 215-232.
24. Burr, I.W. (1967 a) A useful approximation to the normal distribution function with application for simulation, *Technometrics*, 9, 647-651.
25. Burr, I.W. (1967 b) The effects of nonnormality on constants for X and R charts, *Indus. Quality. Control*, 24, 563-569.
26. Burr, I.W. (1968) On a general system of distributions: III. characteristics The sampling range, *J. Amer. Statist. Ass.*, 63, 636-643.
27. Burr, I.W. and Cislak, P.J. (1968) On a general system of distributions: I. Its curve shape characteristics. II The sample median, *J. Amer. Statist. Ass.*, 63, 627-635.
28. Clayton, D. (1978) A model for association in bivariate life tables and its application in epidemiological studies of familiar tendency in chronic disease incidence, *Biometrika*, 65, 141-151.
29. Cook, J and Johnson, N.L. (1986) Generalized Burr Pareto logistic distributions with application to uranium exploration data set, *Technometrics*, 28, 123-131.

30. Crowder, M.J. (1985) A distributional model for repeated failure time measurements, *J. Roy. Statist. Soc. ser. B*, 47, 447-452.
31. Crowder, M.J. (1990) A multivariate distribution with Weibull connections, *J. Roy. Statist. Soc. ser. B*, 51, 93- 107.
32. Crowder, M.J., Kimber, A.C., Smith, R.L. and Sweeting, T.J. (1991) The statistical analysis of reliability data, Chapman & Hall, London.
33. Crowder, M.J. and Kimber, A.C. (1997) : A score test for multivariate Burr and other Weibull mixture distributions, *Scand. J. Statist.*, 24, 419-432.
34. Dubey, S.D. (1972) Statistical contribution to reliability engineering, ARL TR 72-120, AD 751261.
35. Dubey, S.D. (1973) Statistical treatment of certain life testing and reliability problems, ARL TR 73-105, AD 774837.
36. Durling, F.C. (1969) Bivariate probit, logit and burrit analysis, Technical report 41, Southern Methodist university, Dullas, Texas.
37. Durling, F.C. (1974) The bivariate Burr distribution in G.P.Patil etal(eds.), *Statistical distributions in scientific work*, vol 1, Dordrecht, Reidel., P. 329-335.
38. Elshanut, M.A.T. (1995) Bayesian estimation of the parameters of the Burr distribution with progressively censored data, *Egypt. Statist. J.*, 39, 190-206.
39. Evans, I.G. and Ragab, A.S. (1973) Bayesian inference given a type -2 censored samples from Burr distribution, *Commu. Statist. Theor. Meth.*, 12, 1569-1580.

40. Evans, R.A and Simons, G.(1975) Research on statistical procedures in reliability engineering, ARL TR, 75-154, ADAO 13687.
41. Frecht, M. (1951) Sureles tableaux de correlation dont les maryes sont donnees, Annales de l'university de lyon, section A, series 3, 14, 53-77.
42. Freund, J. (1961) A bivariate extension of the exponential distribution, J. Amer. Statist. Ass., 56, 971-977.
43. Fry, T.R.L. (1993) Univariate and multivariate Burr distributions A survey, Pakistan j. Statist., vol 9(1) A, 1-24.
44. Fry, T.R.L. and Orme, C.D. (1995) Maximum likelihood estimation in binary data models using panel data under alternative distributional assumptions, Economic letters, 49, 359-366.
45. Fry, T.R.L. and Orme,C.D. (1998) A generalized logistic tobit model, J. Quantitative Economics,14,23-32.
46. Gupta, R.L, Gupta, R.C. and Lvin, S.J (1996) Analysis of failure time data by Burr distribution, Commun. Statist. Theor. Meth., 25, 2013-2024.
47. Gumbel, E.J.(1961) Bivariate logistic distribution, J. Amer. Statist. Ass., 56, 335-349.
48. Gourieroux, C. and Monfort, M. (1979) On the characterization of joint distribution by conditional distributions, J. Econometrics,10, 115-118.
49. Hatke, M.A. (1949) A certain cumulative probability function, Ann. Math. Statist., 28, 263-271.
50. Hitha, N and Nair, N.U. (1989) A characterization of bivariate lomax and finite range distributions, Culcutta Statist. Ass. Bull.,163-167.

51. Houggard,P. (1984) Life table methods for heterogeneous population distributions describing heterogeneity, *Biometrika*, 71, 75-83.
52. Hussain, A.M and Nath, S.K. (1997) Estimation of parameters in the presence of outliers of Burr type XII distribution, *Commun. Statist. Theor. Meth.*, 26, 637-652.
53. Hutchinson, T.P. and Satterthwaite, S.P. (1977) Mathematical models for describing the clustering of sociopathy and hysteria in families, *British J. Psychiatry*, 130, 294-297.
54. Hutchinson, T.P. (1981) Compound gamma bivariate distributions, *Metrika*, 28, 263-271.
55. Jaheen, Z.F. (1996) Empirical Bayes estimation of the reliability and failure rate functions of Burr X models, *J. appl. Statist. Sci.*,3, 281-288.
56. Johnson, N.L and Kotz, S (1972) *Distributions in statistics:Continuous multivariate distributions*, New York, John wiley & sons.
57. Johnson ,N.L and Kotz, S. (1975) A vector valued multivariate hazard rate, *J. Multivariate Anal.*,5, 53-66
58. Johnson, N.L and Kotz, S. (1981) Dependent revolutions time to failure under dependence, *Amer. J. Math. And Manage. Sci.*,1, 155-165.
59. Lancaster, T. (1979) Econometric methods for duration of unemployment, *Econometrica*, 47, 939-956.
60. Lancaster, T. (1985) Generalized residuals and heterogeneous duration models with application to weibull model, *J. Econometrics*, 28, 155-169.

61. Lindsay, S.R, Wood, J. and Woolens, R.C. (1996) Modelling diameter distribution of forest stands using Burr distribution, *J. Appl. Stat.*, 23, 609-616.
62. Lingappaiah, G.S. (1979) Bayesian approach to the estimation parameters in the Burr's distribution with outliers, *J. Orissa Math. Soc.*, 1, 55-59.
63. Mardia, K.V. (1960) Multivariate pareto distribution, *Annals. Math. Statist.* 33, 1008-1015.
64. Marshal, A.W. and Olkin, I. (1967 a) A multivariate exponential distribution, *J. Amer. Statist. Ass.*, 62, 30-44.
65. Mohie, El-Dil (1991b),: Maximum likelihood weights of the location of Burr distribution, *Computers. Math. Appl.*, 21, 33-38.
66. Morgestern, D. (1956) Einfache Beispiele zweidimensionaler verteilungen, *Mitteilungsblati fur Mathematische Statistik*, 8, 234-235.
67. Morrison and Schemittler (1980) Jobs, strikes and wars, *Probability models for duration, organization, behaviours and human performance*, 25, 224-251.
68. Mousa, M.A.M.A. (1995) Empirical Bayes estimators for the Burr type XII accelerated life testing model based on type-2 censored data, *J. Statist. Comp. Simul.*, 52, 95-103.
69. Nair, N.U. (1989) Characterizations of multivariate distributions, *J. Korean Statist. Soc.*, 18, 72-79.
70. Nair, N.U and Nair, K.R.M (1990) Characterization of bivariate Geometric distribution, *J. Indian Statist. Ass.*, 26, 45-49.

71. Nair, N.U and Asha, G. (2004) Characterization using failure rate and reversedb failure rate, J. Indian Society of Probability and Statistics (Preprint).
72. Nayak, T.K. (1987) Multivariate Lomax distribution properties and usefulness in reliability theory, J. Appl. Prob, 24, 170-177.
73. Nigm, A.M. (1988) Prediction bounds for the Burr model, Commun. Statist. Theor. Meth., 287-297.
74. Nigm, A.M and Al-Wabab, N.Y.N. (1996) Bayesian prediction with a random sample size for the Burr life time distribution, Commun. Statist. Theor. Meth., 25, 1289-1303.
75. Piorer, D.J. (1980) A Lagrange multiplier test for the skewness in binary logit models, Economic Letters, 5, 141-143.
76. Pappadopoules, A.S. (1978), The Burr distribution as a failure model from Bayesian approach, IEEE. Trans. Reli., R-27, 369-371.
77. Plackett, R.L. (1965) A class of bivariate distributions, J Amer. Statist. Ass., 60, 516-522.
78. Rasul, M. (1994) Efficiency of quaslikelihood estimation for the Burr log linear model. Pak. J. Statist, 10, 633-640.
79. Rhyzik, I.M. and Gradshteyn, I.S. (1994) Tables of integrals, series and products, Accademic press inc., Harcourt Bruce & company publishers, NewYork, P.1194.
80. Rodriguez, R.N. (1980) Multivariate Burr III distribution, Part I, Theoretical properties, Research publications, GMR-3232, Warren.

81. Rodriguez, R.N. and Taniguchi, B.Y. (1980) A new statistical model for predicting customer octane satisfaction using trained rator observations, Transactions of society of automotive engineers, INC-Warren.
82. Roy, D. (2002) A characterization model approach for generating bivariate life time distributions using revered hazard rates, J. Japan. Statist. Soc., vol 32, No 7, 239-245.
83. Sarwati, A. and Abusalih, M.S. (1991) Bayesian prediction bounds for the Burr X model, Commun. Statist. Theor. Meth., 20, 2307-2330.
84. Sankar, G. and Sahani, V. (1994) The study of maintenance model with Burr failure distribution, Microelectronics Reli., 34, 1513-1517.
85. Schmittlein, D.C. (1983) Some sampling properties of a model for income distribution, J. Business and Economics., Statist., 1, 147-153.
86. Seshadri, V. and Patil, G.P. (1964) A characterization of bivariate distribution by marginal and conditional distribution of the same component, Ann. Instist. Statist. Math., 15, 215-221.
87. Shah, A. and Gokhale, A.V. (1993) On mps estimation for the Burr type XII model, Commun. Statist. Simul. Comp., 22, 615-664.
88. Shao, Q. (2000) Estimation for hazardous concentration based on NOEC toxicity data, An alternative approach, Enviornmetrics, 11, 583-585.
89. Sherrick, S.J., Castagliola, P. and Viswanath Thirupattur (1996) Recovering probabilistic information from option markets, The J. Future markets, 16, No.5, 545-560.
90. Singh, S.K and Madalla, G.S. (1976) A function of size distribution of incomes, Econometrika, 44, 5, 963-970.

91. Smith, R.J. (1989) On the use of distributional misspecification checks in limited dependent variable models, *Econometric J.*, 99, 178-192.
92. Surles, J.G. and Padgett, W.J (1998) Inference for $P[Y < X]$ in the Burr type X model, *J. Appl. Statist. Sci.* 7, 225-238.
93. Surles, J.G. and Padgett, W.J (2001) Inference for reliability and stress strength for a scaled Burr X model, *Life time Data Analysis*, 7, 187-200.
94. Takahasi, K. (1965) Note on multivariate Burr distribution, *Annals. Inst. Statist. Math.*, 17, 257-260.
95. Tio, G.C. and Guttman, I.J. (1965) The inverted Dirichlet distribution with application, *J. Amer. Statist. Ass.*, 60, 793-805, Correction, 60, 1251-1252.
96. Tsai, H. (1990) Probabilistic tolerance design for a subsystem under Burr distribution using Taguchi's loss function., *Commun. Statist. Theor. Meth.*, 19, 4679-4696.
97. Van uven, M.J. (1947-1948) Extension of Pearson probability distribution of two variables, I-IV, *Proceedings of the Royal academy of sciences, Amsterdam*, 50, 1063-1070, 1252-1264, 51, 41-52, 191-196.
98. Wani, J.K. (1968) On multivariate linear exponential family, *Proceedings of the Cambridge philosophical soc.*, 64, 481-485.
99. Wingo, D.R. (1983) Maximum likelihood methods for fitting Burr type XII distribution to life test data, *Biom J.*, 25, 77-84.
100. Wingo, D.R. (1993) Maximum likelihood methods for fitting Burr type XII distribution to multiply (progressive) censored life test data, *Metrika*, 40, 203-210.

101. Woo J. and Ali, M.M. (1998) Right tail probability estimation in a Burr distribution, *J. Statist. Research*, 33, 51-55.
102. Woo J. and Ali, M.M. (2000) : M.S.E comparison of Bayes reliability estimates in a Burr distribution, *J. Infer. Opt. Sci.*, 161-177.
- 103 Zimmer, W. and Burr, I.W. (1963) Variable sampling plan based on Non- Normal population, *Indus. Quality Control.*, 20, 18-26.