FLUID MECHANICS—NON-LINEAR WAVES

STUDIES ON KORTEWEG-DE VRIES EQUATIONS

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STATEMENT

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

N Nirmala Antherjanam

(N.NIRMALA ANTHERJANAM)

CERTIFICATE

This is to certify that this thesis entitled " STUDIES ON KORTEWEG-De VRIES EQUATIONS" that is being submitted by Smt.N.Nirmala Antherjanam, for the award of the Degree of Doctor of Philosophy in Mathematics to the Cochin University of Science and Technology, Cochin 682 022, is a record of bonafide research work carried out by her under my supervision and guidance. The results embodied in this thesis have not been submitted to any other Institute or University for the award of any other degree or diploma.

(Dr. M.JATHAVEDAN)

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Chapter-1

INTRODUCTION

1.1. LINEAR AND NONLINEAR WAVES IN FLUIDS

Wave motion is so wide a subject that any definition will not be sufficient to give a clear picture of the nature of the subject. Yet various definitions are given to the subject. The concept of wave includes the cases of a clearly identifiable disturbance---either localized or non-localized-- that propagatesin space with increasing time, a time-dependent disturbance throughout space that may or may not be repetitive in nature and which frequently has no persistent geometrical feature that can be said to propagate, and even periodic behaviour in space independent of time. The propagation of an acoustic pulse in a solid, the behaviour of a random pattern of waves on the surface of water and the undular pattern of sand bars in an estuary are respectively, physical examples of the above three categories. A basic definition that can be given to a wave is that it is any recognizable signal that is trasferred from one part of the medium to another with a recognizable velocity of propagation, which covers the whole range of wave phenomena. The signal may be any feature of the disturbance, such as a maximum or an abrupt change in some quantity provided that it can be clearly recognised and its location at any time can be determined. The signal may distort, change its magnitude, and change its velocity provided it is still recognizable. Also, different features are important in different types of waves. Though sufficient to our purpose, clearly this definition does not include the third category mentioned above.

Although the basic understanding of waves is provided by the important results from linear theory, most of the theories in physics are nonlinear. Evolution of physical systems are often described by nonlinear ordinary differential equations (ODEs) and partial differential equations (PDEs) depending on whether the system is discrete or continuous. The theory of nonlinear differential equations have gained much from the development in the study of wave propagations.

By nonlinear effects on waves we mean any feature of real wave motions which cannot be reproduced in a linear analysis, ie: in an analysis neglecting the squares of the disturbances. In the case of nonlinear differential equations one cannot often obtain the general solutions because superposition principle is no longer valid in these cases.

Consequently the initial and boundary-value problems associated with nonlinear PDEs are very difficult to handle in a general way. Some specific problems have been tackled from time to time by methods specifically suited to the individual problems.

The role of nonlinearity is to produce progressively more and more deformation in the wave profile as 't' (time) increases. After some time ($t > T$), a physically meaningful solution is the one in which contains a moving jump discontinuity; a socalled weak solution.

Different types of waves can be broadly classified into two main classes, (a) Hyperbolic and (b)Dispersive.

(a) Hyperbolic Waves

Waves formulated mathematically in terms of hyperbolic PDEs are called hyperbolic waves. The most suited governing equation in the case of time dependent wave propagation is a hyperbolic PDE which may be linear or nonlinear.

Nonlinearity in waves manifest itself in a variety of waves, and in the case of waves governed by hyperbolic equations, possibly the most frequently arising is the evolution of discontinuous solutions from arbitrarily well behaved initial data.

(b) Dispersive Waves

The nonhyperbolic wave motions are grouped largely into a main class called dispersive. This class arises from the linear theory. In a medium in which the velocity of progressive waves of small amplitude varies with the wavelength, a disturbance of arbitrary form, which may be regarded as composed of superposed trains of waves of all wavelengths, changes shape as it progresses because the different component wave trains travel with different speeds.

Because of the linearity of the governing equations, the field quantities associated with linear waves may be resolved into Fourier components. Let us consider for simplicity, one-dimensional plane waves. In this case the governing equations have elementary solutions in the form of sinusoidal wave trains $a \exp [i (kx-\omega t)]$, where 'x' denotes a one-dimensional space coordinate, 't' the time, 'a' the amplitude, 'k' the wave number and 'ω' the angular frequency.

The phase velocity V_p defined by $V_p = \omega/k$ represents the speed of propagation of geometrical features of a wave and the group velocity V_g defined by $V_g = \partial\omega/\partial k$ represents the speed of propagation of the energy of the wave or its analogue. In general, both phase velocity and group velocity are functions of 'k' and are not equal, so that the waves of different lengths travel with different group velocities and the disturbance will be spread over a certain length which increases with time. The system is then said to be dispersive. The relation between 'ω' and 'k' is known as the dispersion relation. The role of dispersion is often to take a general disturbance and cause different sinusoidal components of it to be found, at some subsequent instant, at different places. All gravity waves in fluids are dispersive. If 'ω' is a complex function of a real 'k' and $Im(\omega) < 0$, the system is called dissipative.

Unlike the case of linear systems, it is to be noted that a precise definition of dispersion has not yet been established for nonlinear systems. In many cases, however, we can obtain corresponding equations of a linear dispersive system by linearizing the governing equations which are originally nonlinear. In these cases we can consider the dispersion of nonlinear waves. We shall say that the system is dispersive if its linearized form is dispersive in the sense of linear waves. The main nonlinear effect is not the difference in functional form, rather it is the appearance of amplitude dependence in the dispersion relation. Effect of nonlinearity is the steepening of a wave profile and that of the dispersion is the spreading of it by reducing the steepening. A consequence of this is the possible existence of solitary waves, when the two effects are balanced. The nonlinearity of the water wave problem arises from the dynamic and free surface condition.

1.2. KORTEWEG-DE VRIES EQUATION AND WATER WAVES

The simplest model equation describing a nonlinear dispersive non-dissipative phenomenon is the celebrated Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (1.2.1)$$

This equation represents physical phenomena arising out of a balance between a weak nonlinearity and weak dispersion and describes the unidirectional propagation of small but finite amplitude waves in a nonlinear dispersive medium. Korteweg and de Vries (1895) derived the equation from the Euler equation as an approximation to the Navier-Stokes equation assuming that waves being considered have an amplitude which is small and a wavelength which is large when compared to the undisturbed depth. Additionally, they recognized the importance of presuming the Stokes number 'S' ($S = a\lambda^2/h^3$ where 'a' is the typical wave amplitude, 'λ' is the length-scale of the waves and 'h' is the depth) to be neither too large nor too small. The KdV equation is also justifiable as a model for long waves in many physical systems. Because of the range of its potential applications, and because of its very interesting mathematical properties, this equation has been the object of prolific study in the last two decades.

Long waves, which propagate on the free surface of a horizontal layer of fluid with finite depths are known as gravity waves. It is assumed usually that the fluid is inviscid and incompressible. The two-dimensional irrotational motion of such a fluid is described by the harmonic equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad (1.2.2)$$

for the velocity potential $\varphi(x,y,t)$, where 'x' and 'y' are the horizontal and vertical co-ordinates. We shall consider the disturbance of the liquid to be small; ie. we shall assume that all the derivatives $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$ and $\frac{\partial\varphi}{\partial t}$ of the velocity potential and the displacement of the free surface are sufficiently small that we may neglect the squares of these displacements and their products without introducing any significant error into the solution. Under these conditions, we can reduce the problem of the disturbance on the free liquid surface to a boundary-value problem for Laplace's equation (1.2.2). Applying an appropriate perturbation method the original harmonic equation for the velocity potential subject to nonlinear boundary conditions can be reduced to the KdV equation.

The most important property of the KdV equation is that it admits steady progressive wave solutions called solitary waves. Solitary wave is a long wave of small amplitude travelling without change of form. The first recorded observation of a solitary wave was made in 1834 by the naval architect Sir John Scott Russell (1844, 1845). While riding on horseback along the banks of a canal, he noted the motion of a simple hump of water without change of shape and he followed it for a long distance. He made a careful study of this observed phenomenon and the outcome was reported in 1844, and published in 1845.

Stokes' (1849) investigations in water waves are the starting points for the nonlinear theory of dispersive waves. He discovered the crucial results that periodic wave trains are possible in nonlinear systems and that the dispersion relation involves the amplitude. The dependence on amplitude produces important qualitative changes in the behaviour and introduces new phenomena. He concluded that solitary waves cannot exist. Airy's shallow water theory [Airy (1845)] also does not give the possibility of the existence of solitary waves.

Russell's solitary wave may be regarded as the limiting case of Stokes' oscillatory waves of permanent type, the wavelength being considerably large compared with the depth of the canal, so that the widely separated elevations are independent of one another. But Stokes' theory fails when the wavelength much exceeds the depth and hence it cannot unravel the physical causes leading to the formation of solitary waves.

The speed of propagation of a solitary wave is proportional to its amplitude. Like Burger's shock wave, the KdV solitary wave is also invariant with respect to a Galilean transformation. Relative to their respective values at infinity, a translation and scaling of amplitude of a solitary wave will transform it into another solitary wave. It is remarkable that all solitary waves are similar in this sense.

Boussinesq (1872) and Rayleigh (1876) analyzed mathematically the phenomenon of solitary waves. They could derive approximate results for the shape and velocity of such waves. Rayleigh considered the solitary wave in an Eulerian frame of reference moving at a velocity that brings the wave to rest. Boussinesq's equations include waves moving to both left and right. Going a step beyond Boussinesq's theory, Korteweg and de Vries restricted attention to waves moving to the right only. They modified Rayleigh's theory and derived periodic cnoidal waves. Further they deduced Rayleigh's solitary waves as the limiting case of their cnoidal waves for long wave lengths. Both the solitary waves and the periodic waves described by the KdV equation are found as solutions of constant shape moving with constant velocity.

Solitary waves continued to attract attention in the ensuing decades. References of some works in this direction are given by Weinstein (1926), Lamb (1932) and Stoker (1957). Weinstein developed a systematic theory for determining the velocity of the solitary wave. The works of Keulegan and Patterson (1940) and of Ursell (1953) deserve special mentions. Lavrentieff (1954) and Friedrichs and Hyers (1954) gave rigorous proofs that the Euler equations possess solitary wave solutions of small amplitudes. This result has been refined by Beale (1977). Later Amick and Toland (1979) have shown that the Euler equations have solitary wave solutions of all amplitudes, upto and including a solitary wave of greatest height.

The significance of KdV equation as a basic equation of mathematical physics was brought out in 1960 when Gardner and Morikawa derived the KdV equation as a model for waves in a cold collisionless plasma [Gardner and Morikawa (1960)].

In the famous FPU problem, Fermi, Pasta and Ulam (1955, 1974) studied a one-dimensional lattice of many equal mass particles with weak nonlinear nearest-neighbouring interactions. In computer studies it was observed that for smooth initial conditions, the evolving wave form returned almost to its initial state after many (linear) oscillation periods. This phenomenon is known as FPU recurrence. The current interest in the KdV equation stems from a numerical experiment by Zabusky and Kruskal in which they observed that the solution of KdV equation may exhibit FPU recurrence. It is found that a smooth initial profile evolving under the KdV, developed into a train of solitary waves. The most remarkable thing they noticed was that after interaction two such waves emerged unaffected in shape with amplitude and velocity and suffered only phase shifts. Further the interaction was clean in the sense that no residual disturbance was created. This particle-like behaviour led to these special solitary waves being named 'Solitons' [Zabusky and Kruskal (1965)]. Lax (1968) showed that the soliton is not an approximation but can be derived exactly.

The particle-like behaviour of solitons is of great importance in applications. Let us consider a pulse carrying a bit of information with it. If the pulse suffers heavy dissipation, it may not reach the destination at all. Similarly, if the pulse suffers a significant dispersion, on reaching the destination it may be so spread out and blurred that the information may be totally unintelligible. However, if the pulse travels as a soliton, it can carry the information over long distance without being distorted and without suffering any significant loss in its intensity.

The physical relation between linear theory and soliton theories is the following: If the waves are infinitesimal, the linear theory gives a complete description of their evolution. If the waves have small but finite amplitude, then the linear theory breaks down after a finite time and nonlinear corrections are needed to extend the range of validity of the theory to a long time scale. Typically, soliton theories provide the nonlinear corrections to render the linear theory valid on a longer time scale. There is a short time scale on which the linear theory applies, followed by a longer time scale on which the soliton theory applies, perhaps followed by an even longer time scale on which something else applies.

Many related evolution equations, each of which represents a balance between some form of dispersion (or variation of dispersion in the case of wave-packet evolution) and weak nonlinearity in an appropriate reference frame, have since been found to have properties analogous to those of the KdV equation. Thus there exists an impressive number of rather distinct physical systems for which the KdV equation, or a near relative, has been derived as a model for wave propagation. Some publications regarding this equation are the review articles by Jeffrey and Kakutani (1972), Scott, Chu and Mc Laughlin (1973) and Benjamin (1974).

A major development in the theory of differential equations is the inverse scattering method by Gardner et al. (1967, 1974). An exact solution for KdV equation was obtained by this method and is asymptotically dominated by solitons. In fact the proof of solition property of the KdV equation is one of the triumphs of inverse scattering method for solving PDEs [Miura (1974)].

Apart from the inverse scattering method there are other mathematical methods and results originating from the study of KdV equation. These have led to applications ranging from 'practical' problems of wave propagation to rather 'pure' topics in algebraic geometry. For a discussion of these, we refer to Dubrovin, Matveev and Novikov(1976). The interest in the study of KdV equations and resulting discoveries have its impact in the study of nonlinear wave propagations in water also.

A theory related to the stability of solitary waves has been developed by Benjamin (1972) which is improved by Bona (1975) in order to treat the full non-linear problem without linearization, and a precise formulation and proof of stability for the solutions of the KdV equation is given with more general assumptions concerning the initial data. It is shown by Berrymen and James (1976) that the KdV soliton is stable whereas the Boussinesq solitary wave is unstable to infinitesimal perturbations.

It is to be noted that the KdV equation, inspite of its fame and popularity has not remained unchallenged as a model equation describing the behaviour of (long) water waves in a channel [Peregrine (1966)]. Benjamin, Bona and Mahony (1972) have proposed an alternative model

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad (1.2.3)$$

which they call a regularized KdV equation. An exact relation between equation (1.2.3) and the KdV equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \qquad (1.2.4)$$

exists [Bona and Smith (1975)] in the sense that for the same initial data both equations have unique smooth solutions. But the solitary wave solution of the Benjamin-Bona-Mahony (BBM) equation (1.2.3) may not be a soliton due to the behaviour after interaction with other such waves [Jeffrey (1979)]. The initial boundary-value problem

$$\begin{aligned} u_t + u_x + uu_x - u_{xxt} &= 0, \\ u(x,o) &= g(x), \\ u(o,t) &= h(t) \text{ for } x,t \geqq 0 \end{aligned} \tag{1.2.5}$$

has been analyzed and shown to be well-posed by Bona and Bryant (1973). Specific examples of other sorts of model equation for long waves are given by Bona and Smith(1976) and Bona and Dougalis (1980). Sachs (1984) has investigated the justification of KdV approximation in the case of N-soliton water waves.

Numerical computations of solitary waves deserve particular mention. Zabusky and Galvin (1971) in their experimental studies have shown that the KdV-like evolution equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0 \tag{1.2.6}$$

of shallow water waves is very accurate even for large nonlinearities and found nontrivial amounts of energy in wave numbers $k > 0.5$. Longuet-Higgins and Fenton (1974) made extensive numerical calculations and found that speed, mass, momentum and potential and kinetic energies for waves of amplitude less than the maximum have the maximum value so that they do not increase monotonically with the wave amplitude. This result has been confirmed by Byatt-Smith and Longuet-Higgins (1976) indicating that the highest and lowest wave profiles intersect at points near the wave crest. Some other numerical results are given by Jeffrey (1979), Knickerbocker and Newell (1980) and Rene (1983).

Bampi and Morro (1979) attempted to shed some light on the physical approximations that are at the basis of the KdV equation. The work of Bona (1983) presented the mathematical details of a rigorous justification of possible experiments to determine the applicability of the KdV equation when nonlinear and dispersive effects are of comparable small order. In such experiments it is assumed that unidirectional waves are generated at one end of the medium in question and then allowed to propagate into an initially undisturbed medium beyond the wave maker.

Miles (1980) has given extensive review of the research works which led to the development of the KdV equation. He has also given detailed analysis of the works in this field [Miles (1981a)]. Another review of the works on the KdV equation is by Cercignani (1977) which gives a historical introduction and a derivation of the KdV equation. The solitary wave solution and conservation laws are also discussed. The existence of infinitely many conservation laws is proved suggesting a Hamiltonian form for the equation. More accounts of the soliton theory and the KdV equation have been given by Faddev and Zakharov (1971), Karpman (1975), Miura (1976a, 1977, 1978), Kakutani and Ono (1978), Johnson (1980), Miles (1981 b), Konno and Jeffrey (1983) and Peregrine (1985).

It is worth mentioning that though the KdV equation was first derived in the context of water waves, the recent revival of the interest in its studies were due to developments in other branches of physics. The knowledge available concerning KdV equation was first employed in the study of water waves by Madsen and Mei (1969). But it was Johnson (1980) who first noted the close relation between KdV equation and water waves. Starting from the basic equations of hydrodynamics he has derived foxr KdV equations: two expressed in cartesian co-ordinates and two in plane polars. Using elementary transformations, he has shown that for a certain class of solutions, only two of them are relevant.

1.3. SCOPE OF THE THESIS

The following chapters contain some results relative to KdV equations. The equations being studies are more general than the equation (1.2.1). As can be noted the equation (1.2.1) has constant coefficients. The equations that are studied in the thesis have coefficients which are not constants. In chapter-II we study the interaction of waves on water of variable depth using derivative-expansion method. Johnson's (1973a) KdV equation is used for this purpose.

The celebrated KdV equation has constant coefficients while there are systems governed by KdV type equations with variable coefficients, Johnson's equation being one of them. Such equations are discussed in chapter-III. A KdV equation with variable coefficients is introduced. The concepts of integrability and Painlevé property (PP) are also discussed.

In the remaining chapters we study the integrability of our model equation.

Using the PP of the PDEs, the auto-Bäcklund transformation (ABT) and Lax pairs (LPs) for this equation are obtained in chapter-IV. LP criterion enables to find some new models of variable coefficients KdV equation that can represent non-soliton dynamical systems also. This can explain the wave breaking phenomenon in variable depth shallow water. In chapter-V similarity transformation for this system is investigated and exact solution in a particular case is obtained. The Ablowitz-Ramani-Segur (ARS) conjecture is used to identify the integrability of the system. In the last chapter we confirm the results already obtained in chapters-IV and V using the concepts of exceptionality and equivalence.

Chapter-II

WAVE INTERACTION ON WATER OF VARIABLE DEPTH

2.1. INTRODUCTION

Wave interaction has been a subject of much interest in continuum mechanics and many of the theoretical predictions on it have been verified by experiments. Interactions between short and long waves have been investigated by means of the coupled equations for a single monochromatic wave and a long wave [Nishikawa et al.(1974), Kawahara, Sugimoto and Kakutani (1975), and Benney (1976, 1977)]. Zakharov (1972) has studied the interaction and the statistics of many localized waves in connection with Langmuir turbulence and Miles (1977a) has discussed the general interaction of two oblique solitary waves using two-dimensional KdV equation. Interactions associated with the parametric end points of the singular regime for two solitary waves have been investigated by Miles (1977b). A survey of results on the problem of soliton interactions in two-dimensions has been given by Freeman (1980).

Numerical studies of solitary wave interactions have been conducted by Sluh (1980) and Bona, Pritchard and Scott (1980). Seabra-Santos et al. (1987) in their numerical and experimental study have described the deformation and fission of a barotropic solitary wave passing over a shelf or an obstacle.

Under certain conditions the interaction between short and long waves are especially important. Kawahara, Sugimoto and Kakutani (1975) have found that a short wave and a long wave can exchange energy in a resonant manner, if the group velocity of the short wave is close to the phase velocity of the long wave. Johnson (1982) has studied the steady oblique interaction of a large and a small solitary wave on the surface of water of constant depth. The phase-shifts of a large and a small solitary wave are obtained assuming that such a two-wave interaction is possible [Johnson (1983)].

Nearly all the investigations of the nonlinear interaction of random dispersive waves are based on asymptotic equations derived by means of suitable approximations. In terms of a systematic perturbation Kawahara and Jeffrey (1979) have derived several asymptotic kinematic equations for a wave system composed of an ensemble of many monochromatic waves having a continuous spectrum together with a long wave. They have applied the method of multiple scales to the Boussinesq equation. As the introduction of multiple scale concept can simplify the order estimation that is necessary in a perturbation analysis, it can systematize the wave packet formalism [Jeffrey and Kawahara (1979)].

In a study by Grimshaw (1979) a slowly varying solitary wave is constructed as an asymptotic solution of a variable coefficients KdV equation. Then the amplitude and phase of the wave to the second order in the perturbation parameter are determined using a multiple scale method. The energy loss of such solitary waves is predicted from a two-time scale expansion [Ko and Kuehl (1982)].

It has been pointed out by Kawahara (1973) that the derivative-expansion method can be applied, in a systematic way, to the analysis of weak nonlinear dispersive waves in uniform media. The weak nonlinear self-interactions of capillary gravity waves [Kawahara (1975a)] and the far-field modulation of stationary water waves [Kakutani and Michihiro (1976)] have been studied using the method. The derivative expansion that avoids the secularity incorporates partial sums, in the sense that the perturbation solution so obtained is not a simple power series solution [Jeffrey and Kawahara (1981)]. This method is also applicable to problems of wave propagation in non homogeneous media [Kawahara (1975b)].

2.2. THREE-WAVE INTERACTION

Let us now consider the resonant interaction among different wave modes. Two primary components of wave numbers k_1 and k_2 and frequencies ω_1 and ω_2 give rise to an interaction term with the magnitudes of the wave numbers lying within the limits $|k_1+k_2|$ and $|k_1-k_2|$. Phillips (1960) has shown that a resonance is possible if the interaction frequencies $\omega_1+\omega_2$ and $\omega_1-\omega_2$ correspond to wave numbers lying within that range. Further it has been pointed out [Phillips(1977)] that exchange of energy among wave modes is analogous to the resonance of a forced linear oscillator. For three-wave interactions the energy exchange is significant only when the conditions

$$k_1 + k_2 \pm k_3 = 0$$

and

$$\omega_1 \pm \omega_2 \pm \omega_3 = 0 \qquad (2.2.1)$$

are satisfied or nearly satisfied simultaneously.

2.3. THE DERIVATIVE-EXPANSION METHOD

We consider the equation

$$L(\partial/\partial x,\ \partial/\partial t) = N(\partial/\partial x, \partial/\partial t)\ [u(x,t)]^{*} \qquad (2.3.1)$$

where $L(\partial/\partial x, \partial/\partial t)$ and $N(\partial/\partial x, \partial/\partial t)$ are differential operators involving spatial and temporal derivatives.

In the derivative-expansion method, the notion of independent variables 'x' and 't' is extended to include a multitude of independent variables like $x_n = \sigma^n x$ and $t_n = \sigma^n t$ ($n = 1,2, \ldots, M$) proportional to the original variables, σ being a suitable small parameter. Accordingly, the dependent variable $u(x,t)$ must be regarded as a function of these extended independent variables.

$$u(x_o,\ x_1,\ \ldots,\ x_M;\ t_o,\ t_1,\ \ldots,\ t_M) \qquad (2.3.2)$$

and is expanded to an asymptotic series in terms of σ by writing

$$u(x_o,\ \ldots,\ x_M,\ t_o,\ \ldots,\ t_M;\ \sigma)$$

$$= \sum_{m=1}^{M} \sigma^m u_m(x_o,\ \ldots,\ x_M,\ t_o,\ \ldots,\ t_M) + 0(\sigma^{M+1}).$$

(2.3.3)

The derivative operators $\partial/\partial x$ and $\partial/\partial t$ are considered to be of the form

$$\frac{\partial}{\partial x} \equiv \sum_{n=0}^{M} \sigma^n \partial/\partial x_n, \tag{2.3.4a}$$

$$\frac{\partial}{\partial t} \equiv \sum_{n=0}^{M} \sigma^n \partial/\partial t_n. \tag{2.3.4b}$$

Hence this name for the method.

Introducing the expansions (2.3.4) into the operators L and N, we obtain

$$L(\partial/\partial x, \partial/\partial t) \equiv \sum_{n=0}^{M} \sigma^n L_n(\partial/\partial x_0, \ldots, \partial/\partial x_M,$$

$$\partial/\partial t_0, \ldots, \partial/\partial t_M) + 0(\sigma^{M+1}), \tag{2.3.5a}$$

$$N(\partial/\partial x, \partial/\partial t) \equiv \sum_{n=0}^{M} \sigma^n N_n(\partial/\partial x_0, \ldots, \partial/\partial x_M,$$

$$\partial/\partial t_0, \ldots, \partial/\partial t_M) + 0(\sigma^{M+1}). \tag{2.3.5b}$$

Substituting (2.3.3) and (2.3.5) into equation (2.3.1), and equating coefficients of like powers of σ, we obtain a set of equations from which it is possible to determine the u_n successively. Each perturbed dependent quantity u_n is to be determined so as to be bounded (non-secular) at each stage of the perturbation.

In this chapter we apply the derivative-expansion method to the study of waves on water of variable depths. The KdV equation derived by Johnson (1973a) is used.

2.4. APPLICATION OF THE DERIVATIVE-EXPANSION METHOD TO WAVE INTERACTIONS.

Here we start with Johnson's (1973a) equation

$$2h^{\frac{1}{2}}u_{,X} + \frac{1}{2}\frac{h_X}{h^{\frac{1}{2}}}u + \frac{3}{h}\,u\,u_{,\xi} + \frac{K}{3}h\,u_{,\xi\xi\xi} = 0 \qquad (2.4.1)$$

where suffix indicates differentiation with respect to the corresponding variable, $h = h(X)$ is the local depth and K is a constant.

The transformation

$$u = h^{-\frac{1}{4}}\eta \qquad (2.4.2)$$

reduces the equation (2.4.1) to a KdV equation with variable coefficients

$$\eta_{,X} + \frac{3}{2} d^7 \eta\eta_{,\xi} + \frac{1}{6} K d^{-2} \eta_{,\xi\xi\xi} = 0 \qquad (2.4.3)$$

where $d = h^{-\frac{1}{4}}$.

Since X is a time-like variable and ξ a space-like variable we write equation (2.4.3) as

$$d^2 \eta_{,t} + \frac{3}{2} d^9 \eta\eta_{,x} + \frac{1}{6} K\eta_{,xxx} = 0 \qquad (2.4.4)$$

We apply the derivative expansion method to the above equation. The variables 'η' and 'd' are regarded as functions of multiple scales and then expanded into asymptotic series as

$$\eta = \sigma\eta_1 + \sigma^2\eta_2 + \sigma^3\eta_3 + \ldots \qquad (2.4.5)$$

and

$$d = d_o + \sigma d_1 + \sigma^2 d_2 + \ldots \qquad (2.4.6)$$

The partial derivatives with respect to 't' and 'x' are also expanded using equations (2.3.4a, b).

Substituting in equation (2.4.4) for η, d, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ using equations (2.4.5), (2.4.6) and (2.3.4a,b) and collecting terms of $O(\sigma)$, $O(\sigma^2)$ etc. we get the following set of equations:

$$O(\sigma) \quad : \quad L_0\eta_1 = 0, \tag{2.4.7a}$$

$$O(\sigma^2) \quad : \quad L_0\eta_2 + L_1\eta_1 = N_0\,[\eta_1^2], \tag{2.4.7b}$$

$$O(\sigma^3) \quad : \quad L_0\eta_3 + L_1\eta_2 + L_2\eta_1 = N_0[2\eta_1\eta_2] + N_1[\eta_1^2], \tag{2.4.7c}$$

and

$$O(\sigma^4) \quad : \quad L_0\eta_4 + L_1\eta_3 + L_2\eta_2 + L_3\eta_1 = N_0\,[\eta_2^2 + 2\eta_1\eta_3] + N_1[2\eta_1\eta_2] + N_2[\eta_1^2], \tag{2.4.7d}$$

where the operators L_j, N_j($j = 0,1,2, \ldots$) are defined by

$$L_0 = d_0^2\frac{\partial}{\partial t_0} + \frac{1}{6}K\frac{\partial^3}{\partial x_0^3}, \tag{2.4.8a}$$

$$L_1 = d_0^2\frac{\partial}{\partial t_1} + 2d_0d_1\frac{\partial}{\partial t_0} + \frac{1}{2}K\frac{\partial^3}{\partial x_0^2\partial x_1}, \tag{2.4.8b}$$

$$L_2 = d_o^2 \frac{\partial}{\partial t_2} + 2d_o d_1 \frac{\partial}{\partial t_1} + (d_1^2 + 2d_o d_2) \frac{\partial}{\partial t_o} + \frac{1}{2} K\left(\frac{\partial^3}{\partial x_o \partial x_1^2} + \frac{3}{\partial x_o^2 \partial x_2} \right), \tag{2.4.8c}$$

$$L_3 = d_o^2 \frac{\partial}{\partial t_3} + 2d_o d_1 \frac{\partial}{\partial t_2} + (d_1^2 + 2d_o d_2) \frac{\partial}{\partial t_1} + (2d_o d_3 + 2d_1 d_2) \frac{\partial}{\partial t_o} + \frac{1}{6} K \left(\frac{\partial^3}{\partial x_1^3} + 3 \frac{\partial^3}{\partial x_o^2 \partial x_3} \right) \tag{2.4.8d}$$

$$N_o = -\frac{3}{4} d_o^9 \frac{\partial}{\partial x_o}, \tag{2.4.9a}$$

$$N_1 = -\frac{3}{4} \left(d_o^9 \frac{\partial}{\partial x_1} + 9d_o^8 d_1 \frac{\partial}{\partial x_o} \right), \text{ and} \tag{2.4.9b}$$

$$N_2 = -\frac{3}{4} \left[d_o^9 \frac{\partial}{\partial x_2} + 9d_o^8 d_1 \frac{\partial}{\partial x_1} + (9d_o^8 d_2 + 36 d_o^7 d_1^2) \frac{\partial}{\partial x_o} \right] \tag{2.4.9c}$$

Though equation (2.4.4) is nonlinear, equation (2.4.7a) is a linear homogeneous equation in η_1. Solving this and substituting in (2.4.7b) we get a linear nonhomogeneous equation in η_2. In the same way equations (2.4.7c) and (2.4.7d) are also linear nonhomogeneous equations.

Following Kawahara and Jeffrey (1979) we consider the nonlinear interaction between a long wave and an ensemble of short waves. For this purpose, to the lowest order of approximation, we consider the linear superposition of wave trains together with a long wave component. Thus, we consider a solution of (2.4.7a) in terms of the Fourier transform

$$\eta_1 = \int_{-\infty}^{\infty} A_1(k; x_1, t_1, \ldots) \exp\left[i(kx_0 - \omega t_0)\right] dk$$
$$+ B_1(x_1, t_1, \ldots), \tag{2.4.10}$$

where $A_1(k)$ represents a slowly varying complex amplitude with the wave number k and B_1 is a slowly varying real function representing the long wave component. The reality of η_1 is assured by the condition

$$A_1^{*}(k) = A_1(-k), \tag{2.4.11}$$

where the asterisk denotes complex conjugation. The dispersion relation satisfied by the linearized equation (2.4.7a) is

$$D(k, \omega) = i\omega d_0^2 + \frac{1}{6} iKk^3 = 0 \tag{2.4.12a}$$

or $$\omega(k) = -\frac{1}{6}\frac{K}{d_o^2}k^3 \tag{2.4.12b}$$

The dispersion relations (2.4.12) admit the three-wave interaction process

$$\omega(k') + \omega(k'') = \omega(k) \text{ for } k' + k'' = k$$

Introducing equation (2.4.10) into equation (2.4.7b) we obtain,

$$L_o\eta_2 = \int_{-\infty}^{\infty}\left\{-d_o^2\left(\frac{\partial}{\partial t_1} + V_g\frac{\partial}{\partial x_1}\right) + 2i\omega d_o d_1 - \frac{3i}{2}d_o^9 kB_1\right\} A_1(k)\exp[i(kx_o-\omega t_o)]dk - \frac{3i}{4}d_o^9\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(k'+k'')A_1(k')A_1(k'')\exp\left\{i[(k'+k'')x_o-(\omega'+\omega'')t_o]\right\}dk'\,dk'' - d_o^2\frac{\partial B_1}{\partial t_1} \tag{2.4.13}$$

where V_g denotes the group velocity and $\omega' = \omega(k')$ and $\omega'' = \omega(k'')$. Because of three-wave interaction, the second term on the right hand side of equation (2.4.13) also contains a secular term. The condition for non secularity is

$$\left\{ \frac{\partial}{\partial t_1} + V_g \frac{\partial}{\partial x_1} - 2i\omega \frac{d_1}{d_o} + \frac{3i}{2} d_o^{\,7} \, kB_1 \right\} A_1(k) +$$

$$\frac{3i}{4} d_o^{\,7} k \int_{-\infty}^{\infty} A_1(k') \; A_1(k-k') \; dk' = 0 \qquad (2.4.14)$$

and

$$\frac{\partial B_1}{\partial t_1} = 0 \qquad (2.4.15)$$

If we assume that $\eta_1 = 0$ and start with the solution

$$\eta_2 = \int_{-\infty}^{\infty} A_2(k) \; \exp\,[i(kx_o - \omega t_o)]dk + B_2, \qquad (2.4.16)$$

where $A_2(k)$ and B_2 are defined similar to A_1 and B_1, we can set, without loss of generality,

$$\eta_3 = \eta_5 = \ldots = 0 \qquad (2.4.17)$$

Then we obtain the non-secularity conditions

$$[-d_o^2 \left(\frac{\partial}{\partial t_2} + V_g \frac{\partial}{\partial x_2} \right) + 2d_o d_1 V_g \frac{\partial}{\partial x_1} +$$

$$i\alpha(2d_o d_2 - 3d_1^2) + \frac{1}{2} i d_o^2 \frac{dV_g}{dk} \frac{\partial^2}{\partial x_1^2} -$$

$$\frac{3i}{2} d_o^9 kB_2] A_2(k) - \frac{3i}{4} kd_o^9$$

$$\int_{-\infty}^{\infty} A_2(k') A_2(k-k')dk' = 0 \qquad (2.4.18)$$

and

$$\frac{\partial B_2}{\partial t_2} + \frac{2d_1}{d_o} \left(\frac{\partial B_2}{\partial t_1} \right) = 0 \qquad (2.4.19)$$

Equation (2.4.14) can be rewritten as

$$\left(\frac{\partial}{\partial t_1} + V_g \frac{\partial}{\partial x_1} \right) |A_1(k)|^2 =$$

$$\frac{3}{2} d_o^7 k \text{ Im} \int_{-\infty}^{\infty} A_1^*(k) A_1(k') A_1(k-k')dk' \qquad (2.4.20)$$

This is an equation for the three-wave interaction. Integrating equation (2.4.20) with respect to k, we get

$$\frac{\partial N}{\partial t_1} + \frac{\partial}{\partial x_1}\left\{NV\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{3}{2}\, d_o^7\, k \,\mathrm{Im}\, A_1^{*}(k)\, A_1(k')$$

$$A_1(k-k')\, dk\, dk', \qquad (2.4.21)$$

where

$$N = \int_{-\infty}^{\infty} \left|A_1(k)\right|^2 dk$$

and

$$V = \frac{1}{N}\int_{-\infty}^{\infty} V_g \left|A_1(k)\right|^2 dk. \qquad (2.4.22)$$

The process can be continued further to the cases in which the non-linearities are of higher order in σ.

2.5. DISCUSSION

The perturbation methods can be applied in the study of a wide range of physical phenomena. The guiding principle for obtaining asymptotic equations is merely the non-secularity of the perturbation.

Johnson's equation is valid for waves propagating over surface of water of variable depth. The depth is varying slowly on the same scale as the initial amplitude of the motion.

The solitary wave solutions of the equation(2.4.1) do not behave as solitons. Johnson has shown that if there is a sudden decrease in the depth, so that a shelf is formed, a solitary wave may break up into a finite number of solitons and if the depth increases, a solitary wave may degenerate into a cnoidal wave.

Ippen and Kulin (1954) has pointed out that a solitary wave cannot maintain the same total energy and volume in water of variable depth. Here, we have seen that the condition for three-wave interaction is satisfied. Since the right hand side of equation (2.4.21) does not vanish, not only there is energy transfer between different wave numbers due to the three-wave interaction, but the total energy of the short waves also is not conserved. This is in agreement with the observations made by Ippen and Kulin.

Chapter-III

A VARIABLE COEFFICIENTS KORTEWEG-DE VRIES EQUATION

3.1. INTRODUCTION

In this chapter we introduce a KdV equation with variable coefficients. First we give an account of different KdV-type equations which arise in the study of water waves.

3.2. KdV TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

Equation (2.4.1) that we have investigated in the previous chapter is an example of a KdV equation with variable coefficients.

It is well-known that the waves reaching a shore can be considered as solitary waves since they are well separated. Thus the development of a solitary wave over a region of varying depth is of great practical importance. Notable contributions in this direction are due to Ippen and Kulin (1970) and the numerical studies of Peregrine (1967) and Madsen and Mei (1969). But it is rather surprising that except in the work of

Madsen and Mei no attempts were made to make use of the knowledge available concerning the KdV equation. Johnson's (1973a) work was perhaps the first serious attempt to fill this gap. It is to be mentioned that Grimshaw (1970) has considered the problem of waves on water of slowly varying depth, investigating the condition for solution to be a solitary wave with slowly varying coefficients.

Let us discuss Johnson's problem of a solitary wave moving onto a shelf. We consider a small amplitude motion defined by the amplitude parameter ε. The depth is allowed to vary slowly on the same scale ε. The far-field (distance $O(\varepsilon^{-1})$) approximation then incorporates the effects of changing depth and the near-field first approximation is unaltered since the depth approaches a constant as $\varepsilon \longrightarrow 0$. If we assume that the nonlinear and dispersive effects are of the same order, the resulting equation has terms depending on the depth, nonlinearity and dispersion, all being of order unity and can be written in terms of the far-field distance co-ordinate

$$X = \varepsilon x \qquad (2.2.1a)$$

and the appropriate characteristic co-ordinate

$$\xi = \int_0^x d^{-\frac{1}{2}} (\epsilon x) dx - t = O(1), \qquad (3.2.1b)$$

where 'x' and 't' are the original (non-dimensional) space and time variables respectively. When the attenuation factor '$d^{-1/4}$' is removed the final equation takes the form

$$u_X + d^{-\frac{7}{4}} u u_\xi + d^{\frac{1}{2}} u_{\xi\xi\xi} = 0,$$

$$d = d(X) \qquad (3.2.2)$$

where $u(\xi, X)$ is proportional to the elevation of the wave. It is to be noted that in (3.2.2) the region of changing depth ($d=O(1)$, $X = O(1)$) and the 'period' ($\xi = O(1)$) of the wave are of the same order of magnitude. However, in the original non-dimensional variables the region of changing depth is extended (having length of $O(\epsilon^{-1})$, $\epsilon \longrightarrow 0$) and the 'wave length' is still $O(1)$. The change in depth need not be sudden even as a function of the far-field co-ordinate X. In fact, it may occur asymptotically rapidly or slowly. Also it was proved that if a solitary wave moves over the uniform depth ($d=1$) without changing shape before reaching the shelf, it breaks up into a finite number of solitons (n) on the shelf provided

$$d_o = [\ \tfrac{1}{2}\ n(n+1)\ ]^{-\frac{4}{9}}, \qquad (3.2.3)$$

where 'd_o' is the depth of the shelf and 'n' is an integer ($n \geq 1$). In a subsequent paper [Johnson(1972)] this result was confirmed and some numerical solutions of (3.2.2) for various shelf depths were presented. But the problem of ultra-slowly varying depth ($\varepsilon \longrightarrow 0$) was not examined. An approach suitable for dealing with such problems was developed by Johnson (1973b) and an asymptotic solution to (3.2.2) as $\varepsilon \longrightarrow 0$, with a solitary-wave initial condition was constructed.

Evolution of a wave should be determined according to the relative importance of non-linearity, dispersion and nonhomogeneity. Problems involving small and slowly varying nonhomogeneities, in a sense that a perturbation method in terms of a small parameter is applicable, lead to equations with slowly varying coefficients [Kakutani (1971) and Jeffrey and Kawahara (1982)]. Kakutani has shown that a modification of the KdV equation can describe shallow-water wave propagation over gently slopping bottoms. The generalized KdV equation derived by Jeffrey and Kawahara

$$\frac{\partial B}{\partial t_3} + c\frac{\partial B}{\partial x_3} + \mu \frac{\partial^3 B}{\partial \otimes^3} + \gamma B\frac{\partial B}{\partial \otimes} + \delta B = 0, \qquad (3.2.4)$$

where $\otimes = k(x_2, t_2, \ldots)\, x_1 - \Omega\, (x_2, t_2, \ldots)\, t_1$ is a phase variable, μ, γ and δ are functions of slow variables x_3 and t_3, B is a real function covers the result obtained by Kakutani. Another modification of the KdV equation was given by Grimshaw (1978).

$$\eta_X + \delta\, \gamma^{-1}\, \eta\, \eta_\xi - c_1 c_0^{-4}\, \eta_{\xi\xi\xi} = 0, \qquad (3.2.5)$$

where $X = \varepsilon^3 x$ (ε is a measure of weak dispersion),

$\xi = \varepsilon^{-2} \int_0^X c_0(X')^{-1}\, dX' - \varepsilon t$, $\eta = \gamma\eta^{(o)}$ ($\eta^{(o)}$ is the height of the interface), γ is an appropriate 'Green's law' factor, and δ, c_0, c_1 are functions of X.

The amplitude of a solitary wave in a channel of gradually varying depth would vary inversely as depth [Miles (1980)]. A balance between geometry of depth and geometry of waves can be thought to exist

[Brugarino and Pantano (1981)]. Approximate solutions of variable coefficients KdV equation for one-dimensional waves over a bottom of variable depth show how the wave shape changes as it moves into shallower water [Cramer et al. (1985)]. Some other works related to this are due to Peregrine (1968), Clements and Rogers (1975), Kawahara (1976), Miles (1979) and Watanabe and Yajima (1984).

3.3. INTEGRABLE AND NON-INTEGRABLE SYSTEMS

One of the important developments in mathematical physics was the discovery of inverse scattering transform (IST) method whereby the initial-value problem for a nonlinear wave system can be solved exactly through a succession of linear calculations. This method can be viewed as a generalization of Fourier analysis in the sense that it provides the exact solution to certain nonlinear evolution equations, just as the Fourier transform does for certain linear evolution equations. For any dynamical system, there exist true connections between solvability and integrability conditions. Nonlinear evolution equations which are exactly solvable by IST are said to satisfy the integrability condition. The term "integrable" is more commonly referred to as " completely integrable " but this latter term has very different connotations in the study of Hamiltonian systems, which motivates our choice of the former. The existence theorem on the solution of ODEs indicate that the integrability of the dynamical system cannot influence the local character of the solution, as long as the analytic region is concerned. Thus the integrability is usually discussed in connection with the global or long time behaviour of the solution.

The existence of infinitely many conservation laws, the existence of multi-soliton solutions and solvability by inverse scattering are closely related. Miura, Gardner and Kruskal (1968) have discovered the existence of infinite sequence of explicit conservation laws for the KdV equation. The existence of infinite number of conserved quantities clearly added confidence that explicit solutions would be found.

Let us first consider the integrability of a system of ODEs. The solutions of a system of ODEs are regarded as (analytic) functions of a complex (time) variable 't'. The " movable" singularities [Ince(1956), Hille (1976)] of the solution are the singularities of the solution (as a function of complex t) whose location depends on the initial conditions, and are hence, movable (fixed singularities occur at points where the coefficients of the equation are singular). The system is said to possess the PP when all the movable singularities are single-valued (simple poles). When the system possesses PP it is integrable [Tabor and Weiss (1981)].

It was Kowalevskaya (1889) who first used PP to completely integrate a dynamical system of physical significance. It was shown that when the system is integrable there exists a converging power series expansion of solution in the neighbourhood of the singularities. With respect to the value of the Kowalevskaya exponents one can prove the existence of a number of first integrals that make the system integrable or non integrable. With a widely growing interest in dynamical systems and non-linear evolution equations in the 1970's these classical results were revived in a somewhat unexpected way.

The connection between ODEs of the Painlevé type and the integrable PDEs has been pointed out by Ablowitz et al.(1977). Ablowitz, Ramani and Segur (1978, 1980 a,b) have conjectured that PDEs solvable by IST are closely connected with the six types of Painlevé equations (PI-P VI) [Ince (1956)]. Also, if the similarity reduced equation is any one of the six Painlevé equations then the given PDE is integrable and the respective dynamical system is fully deterministic, otherwise chaotic [Bountis (1985)]. The works of Jimbo, Kruskal and Miwa (1982), Weiss, Tabor and Carnevale (1983), Weiss (1983, 1984), Chudnovsky and Chudnovsky, and Tabor (1983); Ramani, Dorizzi and Grammaticos (1983); and Steeb et al. (1983) led to a conjecture that the integrability is related to the PP for PDEs also. A given PDE is said to be integrable if it possesses the PP or can be transformed to a PDE of Painlevé type. We shall note here that the last statement assumes a definition of PP in the case of PDEs. This will be discussed in the following chapters. Steeb and Grauel (1984) in their " Singular Point Analysis" for PDEs demonstrated that the Kadomstev-Petviashvili (K-P) equation has the PP.

3.4. A KdV EQUATION WITH VARIABLE COEFFICIENTS

We introduce a variable coefficients KdV equation

$$u_{,t} + \alpha t^n \, u \, u_{,x} + \beta t^m \, u_{,xxx} = 0, \qquad (3.4.1)$$

where α and β are constant parameters and n and m are real numbers. The celebrated KdV equation is obtained when $n = m = o$. For $\alpha = 3/2$, $\beta = 1/6$ and $m = o$, $n = 1/2$, we can transform (3.4.1) to the well-known purely concentric KdV equation.

$$2 \, v_{,t} + v/t + 3 \, vv_{,x} + \tfrac{1}{3} v_{,xxx} = 0 \qquad (3.4.2)$$

through a nonlinear transformation

$$u = v \sqrt{t} \, , \qquad (3.4.3)$$

Equation (3.4.2) is studied by several authors [Calogero and Degasperis (1978 a,b), Nakumara (1980), Steeb et al. (1983), and Knickerbocker and Newell (1985)]. Some soliton like solutions of (3.4.2) in terms of Airy functions have also been developed.

Such equations like (3.4.1) is particularly significant in the study of the development of a steady solitary wave as it enters a region where the bottom is no longer level [Maxon and Viecelli (1974), Miles(1978) and Johnson and Thompson (1978)].

In terms of the transformation

$$u = \frac{12\beta}{\alpha} t^{m-n} (\log F)_{xx} , \qquad (3.4.4)$$

equation (3.4.1) can be rewritten into the bilinear form

$$\frac{m-n}{t} FF_{,x} + F(F_{,t} + \beta t^m F_{,xxx})_x -$$

$$F_{,x}(F_{,t} + \beta t^m F_{,xxx}) + 3\beta t^m (F^2_{,xx} - F_{,x} F_{,xxx}) = 0$$

$$(3.4.5)$$

We shall note that for $m = n$, equation (3.4.5) reduces to a bilinear equation which can be exactly solved by using a kind of perturbation method [Whitham (1974)] as in the case of KdV equation with constant coefficients. But the method fails when $m \neq n$.

The following chapters are devoted to study the integrability of equation (3.4.1).

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Chapter-IV

AUTO-BÄCKLUND TRANSFORMATIONS, LAX PAIRS AND PAINLEVÉ PROPERTY OF A VARIABLE COEFFICIENTS KORTEWEG-De VRIES EQUATION

4.1. INTRODUCTION

In this chapter we discuss the Painlevé analysis of KdV equation with variable coefficients. The PP is used to identify the values of 'm' and 'n' in equation (3.4.1) for which the system is integrable. We have found these parameter values using a property of LPs obtained from the Painlevé analysis. The possible ABT is also developed, when the system is integrable.

4.2. LAX PAIRS, AUTO-BÄCKLUND TRANSFORMATIONS AND PAINLEVÉ PROPERTY OF PARTIAL DIFFERENTIAL EQUATIONS.

Lax (1968) has obtained the following criterion

Some results of this chapter find place in a paper published in the Journal of Mathematical Physics 27(11) November 1986, pp. 2640-2643.

for the integrability of an equation of the form

$$u_{,t} = K(u) \cdot \tag{4.2.1}$$

Suppose B is some space of functions such that to each 't', u(t) belongs to B. Again let to each $u \in B$, we can associate a self adjoint operator L over some Hilbert space with the property that as 'u' changes according to the equation (4.2.1), L(t) remain unitarily equivalent. Then eigen values of L constitute a set of integrals of equation (4.2.1).

Since L is unitarily equivalent there exists a one parameter family of unitary operators U(t) such that $U(t)^{-1} L(t) U(t)$ is independent of 't' and U(t) satisfies an equation of the form

$$U_{,t} = B U, \tag{4.2.2}$$

where B(t) is antisymmetric. This leads to an equation of the form

$$L_{,t} = [B, L]. \tag{4.2.3}$$

Thus the problem of integrability reduces ultimately to the existence of an antisymmetric solution B(t) to equation (4.2.3). Equation (4.2.3) is called a LP representation and L and B are called LPs.

The existence of Bäcklund transformations (BTs) is another important characteristic feature of solvable nonlinear equations [Miura (1976b)]. These transformations were introduced originally as generalizations of contact transformations and in particular in studies of the geometry of surfaces. BT is a transformation between solutions of solvable differential equations. The basic idea can be stated for second order PDEs for which the BTs were originally derived. Given such a second order equation the BT consists of a pair of first order PDEs relating a solution of the given equation to another solution of the same equation or to a solution of another second order equation. Transformations which relate solutions of the same equation are called ABTs. Using this transformation new solution of an equation can be derived from a given solution. For a third order PDE the BT consists of an equation of first order of Riccati form plus an equation of second order.

BTs are closely associated with the existence of IST solutions. The pair of linear equations that are introduced in the course of effecting the solution by the inverse method are transformable to the BTs that are now known to be associated with certain of the evolution equations. Conversely, the BTs for the above mentioned evolution equations each contains an equation with Riccati-type nonlinearity. If these Riccati-type equations are replaced by a pair of first order equations one finds that the resulting equations are of the type first introduced by Zakharov and Shabat (1972) in their application of the inverse method to the nonlinear Schrödinger equation. There is no general procedure for finding the BTs [Forsyth (1959), Wahlquist and Estabrook (1973), Lamb (1974), Chen (1974), Dodd et al. (1982) and Hlavaty (1983)].

Ward (1984) has extended the study of PP, well known in the context of ODEs, to PDEs: A system of PDEs in n independent variables are considered in the complex domain, the coefficients being analytic on C^n. If S is an analytic noncharacteristic complex hypersurface in C^n, then the PDE which is analytic on S is mereomorphic on C^n.

A weaker form of the PP was suggested by Weiss, Tabor and Carnevale (1983) while studying the Lorentz Series expansion of single valued solutions of a PDE in the neighbourhood of a movable singularity: Let a solution $u(x_1, \ldots, x_n)$ of a PDE be represented in some domain of C^n as

$$u = \sum_{j=0}^{\infty} u_j \varphi^{j-\alpha}, \qquad (4.2.4)$$

where α is a positive integer, φ is a function determining an analytic manifold

$$\varphi(x_1, \ldots, x_n) = 0 \qquad (4.2.5)$$

in C^n, along which the poles of 'u' occur, and 'φ' and $u_j(x_1, \ldots, x_n)$ are analytic functions in a neighbourhood of the manifold $\varphi = 0$. If the expansion (4.2.4) satisfies the given PDE and contains as many arbitrary functions as it should (in a general solution of the PDE) due to the Cauchy-Kovalesvskaya theorem, then this PDE is considered to have the PP.

4.3. PAINLEVÉ PROPERTY OF VARIABLE COEFFICIENTS KdV EQUATION

The equation (3.4.1) has the PP when its solutions

$u(x,t)$ are "single valued" about the movable singularity manifolds, determined from the singularity analysis of the Lorentz series expansion,

$$u(x,t) = \varphi^{\eta}(x,t) \sum_{j=0}^{\infty} u_j(x,t)\, \varphi^j(x,t), \qquad (4.3.1)$$

where $\varphi(x,t)$ and $u_j(x,t)$ are analytic functions in a neighbourhood of the manifold

$$\varphi\,(x,t) = 0 \qquad (4.3.2)$$

and η is an integer to be determined. Substituting (4.3.1) in equation (3.4.1), a leading-order terms analysis uniquely determines the possible values of η. The resulting series expansion of (3.4.1) gives the required ABT and LP for the IST.

The leading-order terms analysis gives the value $\eta = -2$. The recursion relations for $u_j(x,t)$ are found to be

$$u_{j-3,t} + (j-4)\, u_{j-2}\, \varphi_{,t}$$

$$+ \alpha t^n \sum_{k=0}^{j} u_{j-k}\, (u_{k-1,x} + (k-2) u_k\, \varphi_{,x})$$

$$+ \beta t^m \Big\{ u_{j-3,xxx} + 3(j-4)\, u_{j-2,xx}\, \varphi_{,x}$$

$$+ 3(j-3)\,(j-4)\, u_{j-1,x}\, \varphi_{,x}^{\,2}$$

$$+ 3(j-4)\, u_{j-2,x}\, \varphi_{,xx} + (j-2)(j-3)(j-4) u_j\, \varphi_{,x}^{\,3}$$

$$+ 3(j-3)(j-4)\, u_{j-1}\, \varphi_{,x}\, \varphi_{,xx} + (j-4) u_{j-2}\, \varphi_{,xxx} \Big\}$$

$$= 0, \qquad (4.3.3)$$

where

$$\varphi_{,x} = \frac{\partial \varphi}{\partial x}, \; u_{j,x} = \frac{\partial u_j(x,t)}{\partial x}, \text{ etc.} \qquad (4.3.4)$$

Collecting terms involving u_j, it is readily found that

$$\beta t^m\, \varphi_{,x}^{\,3} (j-6)(j-4)(j+1) u_j$$

$$= F\,(u_{j-1}, \ldots, u_o, \varphi_{,t}, \varphi_{,x}, \ldots), \qquad (4.3.5)$$

for $j = 0,1,2, \ldots$.

We note that the recursion relations (4.3.5) are not defined when $j = -1$, 4 and 6. These values of 'j' are called the " resonances" of the recursion relation and, corresponding to these values of 'j', we can insert arbitrary functions of x and t instead of $u_j(x,t)$ into the series expansion (4.3.1). But for $j = -1$, the series expansion (4.3.1) is not defined and so the admissible values of resonances are $j=4$ and $j=6$ only.

Putting $j = 0,1,2, \ldots$ in (4.3.3), we get

$$j = 0, \quad u_o = -(12\beta/\alpha)t^{m-n} \varphi_{,x}^2 , \tag{4.3.6}$$

$$j = 1, \quad u_1 = (12\beta/\alpha)t^{m-n} \varphi_{,xx} , \tag{4.3.7}$$

$$j = 2, \quad (t^{-n}/\alpha) \varphi_{,x} \varphi_{,t} + u_2 \varphi_{,x}^2 -(3\beta/\alpha)t^{m-n} \varphi_{,xx}^2 +(4\beta/\alpha)t^{m-n} \varphi_{,x} \varphi_{,xxx} = 0, \tag{4.3.8}$$

$$j = 3, \quad (t^{-n}/\alpha)\varphi_{,xt} + (m-n)(t^{-n-1}/\alpha)\varphi_{,x} +u_2 \varphi_{,xx} - u_3 \varphi_{,x}^2 + (\beta t^{m-n}/\alpha) \varphi_{,xxxx} = 0, \tag{4.3.9}$$

and for $j=4$ we get

$$\frac{\partial}{\partial x}\left\{\frac{t^{-n}}{\alpha}\varphi_{,xt} + (m-n)\frac{t^{-n-1}}{\alpha}\varphi_{,x}\right.$$

$$\left. + u_2\varphi_{,xx} - u_3\varphi_{,x}^2 + (\beta t^{m-n}/\alpha)\varphi_{,xxxx}\right\} = 0, \quad (4.3.10)$$

which is a compatibility condition. The compatibility condition at $j=6$ involves extensive calculations.

When we assign $u_4 = u_6 = 0$ and $u_3 = 0$, we can find that

$$u_j = 0, \text{ for all } j \geq 3, \quad (4.3.11)$$

provided u_2 is a solution of (3.4.1), which implies that

$$u_{2,t} + \alpha t^n u_2\, u_{2,x} + \beta t^m\, u_{2,xxx} = 0. \quad (4.3.12)$$

From equation (4.3.1) and equations (4.3.6)-(4.3.12), we get

$$u_0 = -(12\beta/\alpha)t^{m-n}\,\varphi_{,x}^2, \quad (4.3.13)$$

$$u_1 = (12\beta/\alpha)t^{m-n}\,\varphi_{,xx}, \qquad (4.3.14)$$

$$(t^{-n}/\alpha)\varphi_{,x}\,\varphi_{,t} + u_2\,\varphi_{,x}^2 - (3\beta/\alpha)t^{m-n}\,\varphi_{,xx}^2$$

$$+ (4\beta/\alpha)t^{m-n}\,\varphi_{,x}\,\varphi_{,xxx} = 0, \qquad (4.3.15)$$

$$(t^{-n}/\alpha)\varphi_{,xt} + [(m-n)/\alpha]t^{-n-1}\varphi_{,x} + u_2\,\varphi_{,xx}$$

$$+ (\beta/\alpha)t^{m-n}\,\varphi_{,xxxx} = 0, \qquad (4.3.16)$$

$$u_{2,t} + \alpha t^n u_2\,u_{2,x} + \beta t^m\,u_{2,xxx} = 0 \qquad (4.3.17)$$

and

$$u_j = 0, \quad \text{for } j \geq 3 \qquad (4.3.18)$$

Substituting from equations (4.3.13) to (4.3.18) in equation (4.3.1) we get

$$u(x,t) = \frac{-12\beta}{\alpha}\,t^{m-n}\,\frac{\varphi_{,x}^2}{\varphi^2} + \frac{12\beta}{\alpha}\,t^{m-n}\,\frac{\varphi_{,xx}}{\varphi} + u_2, \qquad (4.3.19)$$

or

$$u(x,t) = \frac{12\beta}{\alpha}\,t^{m-n}\,\frac{\partial^2}{\partial x^2}(\log\varphi) + u_2, \qquad (4.3.20)$$

where $u(x,t)$ and u_2 exact solutions of (3.4.1) and (4.3.12) respectively.

Equations (4.3.13)-(4.3.20) define the ABT for the variable coefficients KdV equation (3.4.1) provided (4.3.15) and (4.3.16) are consistent. If any one of the solutions $u_2(x,t)$ is known then another solution $u(x,t)$ of equation (3.4.1) can be determined using the ABT. The consistency of equations (4.3.15) and (4.3.16) can be verified by using a property of the LPs.

The LPs are obtained from the equations (4.3.15) and (4.3.16) by using a transformation

$$\varphi_{,x} = V^2 . \qquad (4.3.21)$$

Substituting (4.3.21) in (4.3.16) yields

$$\frac{t^{-n}}{\alpha} V_{,t} + \frac{(m-n)}{2\alpha} t^{-n-1} V + u_2 V_{,x} +$$

$$\frac{\beta}{\alpha} t^{m-n} V_{,xxx} + \frac{3\beta}{\alpha} t^{m-n} V_{,x} \frac{V_{,xx}}{V} = 0. \qquad (4.3.22)$$

Equation (4.3.15) is also transformed into

$$\frac{t^{-n}}{\alpha} V_{,t} + u_2 V_{,x} + \frac{1}{2}u_{2,x} V + \frac{4\beta}{\alpha} t^{m-n} V_{,xxx} = 0 \quad (4.3.23)$$

Eliminating $V_{,t}$ from equations (4.3.22) and (4.2.23) we get

$$\frac{m-n}{2\alpha} t^{-n-1} - \frac{1}{2} u_{2,x} - \frac{3\beta}{\alpha} t^{m-n}\left(\frac{V_{,xx}}{V} \right)_{,x} = 0 \cdot \quad (4.3.24)$$

Integration of equation (4.3.24) with respect to x gives us

$$\frac{\beta}{\alpha} t^{m-n} \frac{V_{,xx}}{V} + \frac{1}{6} u_2 - \left(\frac{m-n}{6\alpha}\right) x t^{-n-1} = \lambda(t) \quad (4.3.25)$$

or

$$f(t) \left\{ \frac{\beta}{\alpha} t^{m-n} D^{*} + \frac{1}{6} u_2 - \left(\frac{m-n}{6\alpha}\right) x t^{-n-1} \right\} V$$

$$= f(t) \lambda(t) V \quad (4.3.26)$$

Thus we get the linear eigen value problem

$$LV = \mu V, \quad (4.3.27)$$

where $\mu = f(t) \lambda(t)$ and L is a linear operator defined by

$$L = f(t)\left\{\frac{\beta}{\alpha}\, t^{m-n}\, D^2 + \frac{1}{6}\, u_2 - \frac{m-n}{6\alpha} x t^{-n-1}\right\} . \qquad (4.3.28)$$

From equation (4.3.23) we get

$$V_{,t} = -\alpha t^n \left\{(4\beta/\alpha) t^{m-n}\, D^3 + u_2 D + \tfrac{1}{2} u_{2,x}\right\} V \qquad (4.3.29)$$

or

$$V_{,t} = -BV, \qquad (4.3.30)$$

where the operator B is defined by

$$B = \alpha t^n \left\{(4\beta/\alpha) t^{m-n}\, D^3 + u_2 D + \tfrac{1}{2} u_{2,x}\right\} . \qquad (4.3.31)$$

Equations (4.3.28) and (4.3.31) define the LPs, L and B. However, equation (4.3.30) implies that the eigen function V is in time evolution so that

$$L_{,t} = LB - BL. \qquad (4.3.32)$$

The $L_{,t}$ in (4.3.32) denotes the derivative with respect to both the explicit time dependence of L and the implicit dependence through $u_2(x,t)$.

From equations (4.3.27) and (4.3.30) we get the following results for which equation (4.3.32) holds:

$$\text{(i)} \quad m = n, \quad f(t) = C, \tag{4.3.33}$$

$$\text{(ii)} \quad m = 2n+1, f(t) = Ct^{n+1}, \tag{4.3.34}$$

where C is an arbitrary constant. For all other values of m and n the LPs are not consistent and hence the ABT exists only for the values of m and n defined in equations (4.3.33) and (4.3.34). Equation (4.3.33) implies that m and n can be both zero together and then the respective L and B are the well-known LPs of the constant coefficients KdV equation.

The above study shows that the variable coefficients KdV equation (3.4.1) is IST solvable and has PP whenever $m = n$ or $m = 2n+1$ and these properties are independent of the constant parameters α and β. For all other values of m and n, the system is non-integrable.

4.4. DISCUSSION

The variable coefficients KdV equation (3.4.1) that we have introduced is a new member in the families of integrable as well as non-integrable PDEs depending on the coefficients. The PP analysis leads to the ABT and LPs when it is integrable. The operator identity (4.3.32) of the LPs reveals that the system (3.4.1) can be integrable when $m = n$ and $m = 2n+1$ only, whereas for all other values of m and n, the system (3.4.1) is non-integrable. The soliton solutions are the products of IST solvable class of nonlinear PDEs [Helleman(1980); Bullough, Caudrey and Gibbs (1980)]. Above study shows that the variable coefficients KdV equation (3.4.1) has soliton not always, but in two special cases only. Hence, in general, a solitary wave solution of (3.4.1) need not be a soliton, and so, it need not be collisionally stable always.

The existence of infinite number of conservation laws are considered as a necessary condition for the existence of soliton solutions of IST solvable equations [Bullough, Caudrey and Gibbs (1980)]. Here we are able to give two of these members for general 'm' and 'n'.

$$u_{,t} + ((\alpha/2)t^n u^2 + \beta t^m u_{,xx})_{,x} = 0 \qquad (4.4.1)$$

and

$$(\tfrac{1}{2}u^2)_{,t} + (\tfrac{\alpha}{3} t^n u^3 - \tfrac{\beta}{2}t^m u_{,x}^2 + \beta t^m u u_{,xx})_{,xx} = 0 \qquad (4.4.2)$$

The higher order conserved quantities are not so direct.

Chapter-V

SIMILARITY ANALYSIS AND EXACT SOLUTION OF A VARIABLE COEFFICIENTS KORETEWEG-DE VRIES EQUATION

5.1. INTRODUCTION

One of the most important methods for developing exact solutions of PDEs is that of reducing the number of variables exploiting continuous symmetries of the system. The solutions obtained by this procedure are generally called similarity solutions. This method has been widely used in the past for developing solutions as well as for the test of PP of various systems [Shen and Ames (1974), and Lakshmanan and Kaliappan (1983)].

5.2. LIE GROUPS, LIE ALGEBRAS AND SIMILARITY SOLUTIONS

Sophus Lie has widely investigated systems of PDEs that are invariant under transformation groups called Lie groups. A Lie group is a topological group in which there exists some neighbourhood N_0 of the identity that can be mapped homeomorphically onto an open bounded subset of the real Euclidean space E_n for some n. Knowing the

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Some results of this chaper find place in a paper published in the Journal of Mathematical Physics 27(11), November 1986 pp 2644-2646.

group of transformation the most general PDE invariant under the group can be constructed.

Given a Lie group G it is possible to construct a corresponding Lie algebra $\mathcal{L}$ [Sudarshan and Mukunda(1974) and Olver (1986)] in some neighbourhood of the identity. A Lie algebra $\mathcal{L}$ is a finite (n) dimensional real vector space in which a Lie bracket is defined which is linear, antisymmetric and satisfies Jacobi identity. For any m-parameter Lie group the infinitesimal operators form an m-dimensional Lie algebra.

A similarity solution is a solution obtained from group invariance. This integration procedure is based on the invariance of the differential equation under a continuous group of symmetries. The invariance of a first order differential equation under a group leads to the construction of an integrating factor and a reduction to quadrature. When a PDE is invariant under a transformation group, it is possible to find similarity solutions of the equation and its independent variables can be reduced by one. Knowing a symmetry group of a system of differential equations, we can construct new solutions of the system from known ones. Also new nonlinear PDEs reducible to the Painlevé equations can be derived through special transformations constructed by similarity variables of well-known one-dimensional soliton equations [Kawamoto (1983)]. Group invariant solutions have been used to describe the asymptotic behaviour of much more general solutions to systems of PDEs.

In chapter-IV we have analysed the existence of ABTs, LPs and the PP of the KdV equation with variable coefficients. In this chapter we are reporting some similarity solutions and in a particular case an exact solution of the equation using the standard similarity method.

5.3. SIMILARITY TRANSFORMATIONS OF A PDE

We shall give the essential details [Bluman and Cole (1974)] of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a PDE,

$$F(x,\ t,\ u,\ u_t,\ u_x,\ u_{xx},\ \ldots) = 0 \tag{5.3.1}$$

under a family of one-parameter infinitesimal continuous point group transformations

$$x = x + \varepsilon\, X(x,t,u) + O(\varepsilon^2), \tag{5.3.2}$$

$$t = t + \varepsilon\, T(x,t,u) + O(\varepsilon^2), \tag{5.3.3}$$

$$u = u + \varepsilon\, U(x,t,u) + O(\varepsilon^2). \tag{5.3.4}$$

Here X, T and U are the infinitesimals of the variables x, t and u, respectively, and ε is an infinitesimal parameter. The derivatives of u are also transformed according to

$$u_x = u_x + \varepsilon\,[U_x] + O(\varepsilon^2), \qquad (5.3.5)$$

$$u_t = u_t + \varepsilon\,[U_t] + O(\varepsilon^2), \qquad (5.3.6)$$

$$u_{xxx} = u_{xxx} + \varepsilon\,[U_{xxx}] + O(\varepsilon^2), \qquad (5.3.7)$$

where $[U_x]$, $[U_t]$ and $[U_{xxx}]$ are the infinitesimals of the transformations of derivatives u_x, u_t and u_{xxx}. These are called the first and third extensions depending on the order of the derivative term. These " extensions" [Bluman and Cole (1974)] are given by

$$[U_x] = U_x + (U_u - X_x)u_x - T_x\,u_t - X_u\,u_x^2 - T_u\,u_x\,u_t, \qquad (5.3.8)$$

$$[U_t] = U_t + (U_u - T_t)u_t - X_t\,u_x - T_u\,u_t^2 - X_u\,u_x\,u_t, \qquad (5.3.9)$$

and

$$[U_{xxx}] = U_{xxx} + (3U_{xxx} - X_{xxx})\,u_x - T_{xxx}\,u_t + 3(U_{xuu} - X_{xxu})\,u_x^2 - 3T_{xxu}\,u_x\,u_t + (U_{uuu} - 3X_{xuu})u_x^3 + 3(U_{xu} - X_{xx})u_{xx}$$

$$- 3T_{xx}\, u_{xt} - 3T_{xuu}\, u_x^2 u_t + 3(U_{uu} - 3X_{xu}) u_x\, u_{xx}$$

$$- 3T_{xu}\, u_t\, u_{xx} - 6T_{xu}\, u_{xt}\, u_x - 3T_x\, u_{xxt}$$

$$+ (U_u - 3X_x) u_{xxx} - X_{uuu}\, u_x^4 - 6X_{uu}\, u_x^2\, u_{xx}$$

$$- 3T_{uu}\, u_x^2\, u_{xt} - T_{uuu}\, u_x^3\, u_t - 3X_u\, u_{xx}^2$$

$$- 3T_u\, u_x\, u_{xxt} - 3T_u\, u_{xx}\, u_{xt} - 3T_{uu}\, u_x\, u_t\, u_{xx}$$

$$- 4X_u\, u_x\, u_{xxx} - T_u\, u_t\, u_{xxx}. \qquad (5.3.10)$$

The invariance requirement of (5.3.1) under the set of transformations (5.3.2)-(5.3.10) leads to the invariant surface condition

$$T \frac{\partial F}{\partial t} + X \frac{\partial F}{\partial x} + U \frac{\partial F}{\partial u} + [U_x] \frac{\partial F}{\partial u_x}$$

$$+ [U_t] \frac{\partial F}{\partial u_t} + [U_{xxx}] \frac{\partial F}{\partial u_{xxx}} = 0 \qquad (5.3.11)$$

On solving (5.3.11), the infinitesimals X, T and U can be uniquely determined, which give the similarity group under which the system (5.3.1) is invariant. By the infinitesimal transformations (5.3.2)-(5.3.4) we have

$$u(x+\varepsilon X + O(\varepsilon^2),\ t + \varepsilon T + O(\varepsilon^2))$$

$$= u + \varepsilon U + O(\varepsilon^2) \qquad (5.3.12)$$

On expanding and equating the $O(\varepsilon)$ terms on either side of (5.3.12) we get

$$T\frac{du}{dt} + X\frac{du}{dx} - U = 0 \qquad (5.3.13)$$

The solutions of (5.3.13) are obtained by Lagrange's condition

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U} \qquad (5.3.14)$$

Equations (5.3.14) give the solution

$$x = x(t,\ c_1,\ c_2), \qquad (5.3.15)$$

$$u = u(t,\ c_1,\ c_2), \qquad (5.3.16)$$

where c_1, c_2 are arbitrary integration constants. The constant c_1 plays the role of an independent variable called the similarity variable σ and c_2 that of a dependent variable called the similarity solution $f(\sigma)$ such that

$$u(x,t) = f(\sigma) \tag{5.3.17}$$

Substituting (5.3.17) in the original equation (5.3.1) the resultant equation is an ODE involving only the derivatives with respect to the similarity variable σ.

5.4. SIMILARITY TRANSFORMATION AND LIE ALGEBRA OF VARIABLE COEFFICIENTS KdV EQUATION

Under the family of infinitesimal transformations (5.3.2)-(5.3.4) the variable coefficients KdV equation (3.4.1) yields

$$[U_t] + \alpha t^n (u_x U + u[U_x]) + \alpha n t^{n-1} u u_x T$$
$$+ \beta t^m [U_{xxx}] + \beta m t^{m-1} u_{xxx} T = 0 \tag{5.4.1}$$

On substituting the expressions for the extensions from (5.3.8)-(5.3.10) and solving for the infinitesimals X, T and U we get the constraint equations

$$- X_t + \alpha t^n (U + u(U_u - X_x)) + n\alpha t^{n-1} uT = 0, \tag{5.4.2}$$

$$U_t + \alpha t^n u U_x + \beta t^m U_{xxx} = 0, \tag{5.4.3}$$

$$tU_u - 3t\, X_x + m\, T = 0, \tag{5.4.4}$$

$$U_u - T_t = 0,\ U_{xu} - X_{xx} = 0,\ U_{uu} - 3X_{ux} = 0, \tag{5.4.5}$$

$$T_x = T_u = X_u = 0 \tag{5.4.6}$$

The constraints (5.4.2)-(5.4.6) can be uniquely solved. Then we get the following solutions for X,T and U.

(i) When m and n are arbitrary,

$$T = 0, \tag{5.4.7}$$

$$X = a[\alpha t^{n+1}/(n+1)] + b, \tag{5.4.8}$$

$$U = a \tag{5.4.9}$$

For the Lie algebra,

$$G_1 = \frac{\alpha t^{n+1}}{n+1} \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{5.4.10}$$

$$G_2 = \frac{\partial}{\partial x}, \tag{5.4.11}$$

$$[G_1, G_2] = 0 \tag{5.4.12}$$

(ii) When $m = 3n + 5$,

$$T = t, \tag{5.4.13}$$

$$X = (2+n)x + a[\alpha t^{n+1}/(n+1)] + b, \tag{5.4.14}$$

$$U = u + a. \tag{5.4.15}$$

The Lie algebra is the same as in the last case [(5.4.10)-(5.4.12)].

(iii) when $m = -2$ and $n = -\frac{3}{2}$,

$$T = t^{\frac{1}{2}}, \tag{5.4.16}$$

$$X = -(xt^{-\frac{1}{2}}/2) - 2a\alpha t^{-\frac{1}{2}} + b, \tag{5.4.17}$$

$$U = (ut^{-\frac{1}{2}}/2) + (x/4\alpha) + a. \tag{5.4.18}$$

The Lie algebra is same as in (5.4.10)-(5.4.12) with $n = \frac{-3}{2}$.

In all the above cases [(5.4.7)-(5.4.18)], a and b are arbitrary integration constants.

5.5. SIMILARITY SOLUTIONS

Using (5.3.14) and (5.3.17) we can find the similarity variables, similarity reduced equations, and similarity solutions for the above three cases [(5.4.7)-(5.4.18)].

The set of infinitesimals (5.4.7)-(5.4.9) gives the similarity variable

$$\sigma_1 = t \tag{5.5.1}$$

and the similarity reduced equation

$$\frac{df_1}{d\sigma_1} + \frac{(n+1)\ a\ \alpha\ \sigma_1{}^n}{a\alpha t^{n+1} + (n+1)b} f_1 = 0 \tag{5.5.2}$$

The corresponding similarity solution is

$$u(x,t) = [(n+1)ax/a\alpha t^{n+1} + (n+1)b]+f_1 \tag{5.5.3}$$

Equations (5.5.2) and (5.5.3) give an exact solution of the variable coefficients KdV equation (3.4.1)

$$u(x,t) = [a(n+1)x+c]\ [a\alpha t^{n+1} + b(n+1)] \tag{5.5.4}$$

The solution (5.5.4) is not so useful as the third derivative with respect to the variable x vanishes.

The set of infinitesimals (5.4.13)-(5.4.15) yields the similarity variable

$$\sigma_2 = \frac{x}{t^{n+2}} + \frac{a\alpha}{(n+1)t} + \frac{b}{(n+2)t^{n+2}} \tag{5.5.5}$$

The corresponding similarity reduced equation is

$$\frac{\beta d^3 f_2}{d\sigma_2{}^3} + \alpha f_2 \quad \frac{df_2}{d\sigma_2} + f_2 - (n+2)\sigma_2 \frac{df_2}{d\sigma_2} = 0 \tag{5.5.6}$$

and the similarity solution is

$$u(x,t) = t\ f_2(\sigma_2) - a. \tag{5.5.7}$$

When $n = -3$, equation (5.5.6) can be reduced to a second-order equation by integration with respect to σ_2. This yields

$$\beta \frac{d^2 f_2}{d\sigma_2{}^2} + \frac{\alpha}{2} f_2{}^2 + \sigma_2 f_2 = \text{const.} \tag{5.5.8}$$

Equation (5.5.8) is not easily solvable. From equation (5.4.16)-(5.4.18) we get the similarity variable

$$\sigma_3 = xt^{\frac{1}{2}} + 4a\alpha t^{\frac{1}{2}} - bt \tag{5.5.9}$$

The corresponding similarity reduced equation is

$$\beta \frac{d^3 f_3}{d\sigma_3^{\,3}} + a\,\alpha\, f_3 \frac{df_3}{d\sigma_3} + \frac{b}{2\alpha} = 0 \qquad (5.5.10)$$

and the similarity solution is

$$u(x,t) = -\sigma_3/2\alpha + bt/2\alpha + t^{\frac{1}{2}} f_3(\sigma_3) \qquad (5.5.11)$$

Equation (5.5.10) can be exactly solved for the case $b = 0$. This gives the following solution of the variable coefficients KdV equation (3.4.1) for $m = -2$, $n = \frac{-3}{2}$:

$$u(x,t) = \frac{-(4\alpha+x)t^{\frac{1}{2}}}{2\alpha} + \frac{4t^{\frac{1}{2}}}{[(\sqrt{-\alpha/3\beta})(x+4a\alpha)t^{\frac{1}{2}} + c]^2} \qquad (5.5.12)$$

The exact solution (5.5.12) is real valued only when $\alpha < 0$ or $\beta < 0$ and not both simultaneously negative. The solution (5.5.12) has no characteristics of a stable configuration like " soliton" [Scott, Chu and Mc Laughlin (1973)].

5.6. SELF-SIMILAR SOLUTION

The self-similar solution can be developed for the variable coefficients KdV equation (3.4.1) using the dimensional analysis. The self-similar transformation is very much identical to the similarity transformations; nevertheless self-similar solutions are not always obtainable by similarity procedure.

For the variable coefficients KdV equation (3.4.1) we get the self-similar transformation

$$u(x,t) = t^{(m-3n-2)/3} F(\eta) \qquad (5.6.1)$$

where $\eta(x,t)$ is the self-similar variable

$$\eta(x,t) = xt^{-(m+1)/3} \qquad (5.6.2)$$

Equation (5.6.1) yields the following self-similarity reduced ODE, on substituting in (3.4.1):

$$\beta \frac{d^3F}{d\eta^3} + \alpha F \frac{dF}{d\eta} - \left(\frac{m+1}{3} \right)\eta \frac{dF}{d\eta} + \frac{m-3n-2}{3} F = 0 \qquad (5.6.3)$$

Unfortunately equation (5.6.3) cannot be easily solved for any values of m and n.

5.7. DISCUSSION

Using the well-known Ablowitz-Ramani-Segur (ARS) conjecture [Ablowitz, Ramani and Segur (1978, 1980a,b), Ablowitz and Segur (1981)] one can study the PP of a PDE by reducing it to an ODE, using similarity or self similar transformations. Equation (5.5.2) is linear and so it is clearly Painlevé-type. For n = -3, the equation (5.5.8) is not a Painlevé-type equation whereas (5.5.10) can be integrated once and reduced to Painlevé-type. This equation (5.5.15) can be reduced to a second-order equation for n = -1, but it is not Painlevé-type.

The exact solution (5.5.12) that we developed has no smooth property of a soliton solution, which indicates that the system has decaying solutions other than soliton solutions when coefficients of KdV equation are variables.

Chapter-VI

COMPLETELY INTEGRABLE KORTEWEG-DE VRIES EQUATION WITH VARIABLE COEFFICIENTS

6.1. INTRODUCTION

Ward (1984) has pointed out the apparent defect in the Painlevé test for PDEs suggested by Weiss et al. (1983). For example the expansion (4.2.4) could, a priori, miss some essential singularities which may lead to erroneous conclusions. Even for ODEs it often requires a great deal of work to show that the expansion gives only poles.

The Painlevé analysis of PDEs by means of the expansion (4.2.4) is similar to that for ODEs. In the case of ODEs the coefficients u_j in (4.2.4) are constants and it is required that the recursion relations do not determine 'k' constants where 'k' is the order of the ODE. Then the expansion (4.2.4) can be considered as a general solution. In the case of PDEs u_j in (4.2.4) are functions of 'n' variables and according to Cauchy-Kovalewskaya theorem, expansion (4.2.4) determines the general solution if we introduce two arbitrary functions φ and u_p of n-1 variables. But it is impossible to guarantee that the arbitrariness will be left after the process of summation and the expansion (4.2.4) will remain the general solution.

Doktorov and Sakovich (1985) while studying the nonlinear Klein-Fock-Gordon PDE observed that the Painlevé analysis of Weiss, Tabor and Carnevale (1983) holds good in the case of no resonances at all (φ is an arbitrary function of n variables) or with one resonance (it could even be fixed) and the compatibility condition for φ be an equation of order $k \geq 2$, rather than an identity, (its solution φ will contain 'k' arbitrary functions of n-1 variables). Thus both the 'correct number of resonances' in the expansion (4.2.4) and the requirement for compatibility conditions to be satisfied identically are additional postulates in the case of PDEs.

In this chapter we check the integrability of (3.4.1) by showing the equivalence of (3.4.1) to KdV and cylindrical KdV (cKdV) equations whose integrabilities are known.

6.2. EXCEPTIONALITY AND EQUIVALENCE

Existence of infinite number of nontrivial conserved densities is associated with the complete integrability of a dynamical system. This property is called exceptionality by Abellanas and Galindo (1981).

Two evolution equations

$$u_t = k(x,t,u,u_x, \ldots) \qquad (6.2.1)$$

and

$$u'_{t'} = k'(x',t',u',u'_{x'}, \ldots) \qquad (6.2.2)$$

are called equivalent if there exists an invertible transformation

$$x' = s_1(t)x + s_2(t), \qquad (6.2.3)$$

$$t' = s_3(t), \qquad (6.2.4)$$

$$u'(x',t') = s_4(t)\, u(x,t) + r(x,t), \qquad (6.2.5)$$

which takes a solution u(x,t) of equation (6.2.1) into a solution u'(x', t') of equation (6.2.2.)

Abellanas and Galindo (1985) have shown that corresponding to an exceptional non-autonomous " flow" of the type

$$u_t = u_{xxx} + f(t,u,u_x), \quad f(t,o,o) = 0 \tag{6.2.3}$$

there exists an autonomous exceptional " flow" equivalent to it.

6.3. VARIABLE COEFFICIENTS KdV EQUATION

Let us make the transformation

$(t,x,u) \longrightarrow (\tau,x,v)$ by

$$\left.\begin{aligned} t &= t(\tau), \\ x &= x\,, \\ u(x,t) &= v(x,\tau)\; a(\tau), \end{aligned}\right| \tag{6.3.1}$$

where $a(\tau)$ and $t(\tau)$ are to be determined so that the transformed equation is exceptional. Equation (3.4.1) is transformed into

$$v_\tau + \frac{a_\tau}{a}.v + t^n t_\tau a.\alpha v v_x + t^m t_\tau.\beta v_{xxx} = 0. \tag{6.3.2}$$

Let us choose the transformation (6.3.1) requiring that

$$\left.\begin{aligned} &t_\tau\, t^n\, a = 1 \\ \text{and}\quad & \\ &t_\tau\, t^m \quad = 1 \end{aligned}\right| \tag{6.3.3}$$

Then the following cases arise.

(a) When $m \neq -1$

$$t = [(m+1)(\tau+c)]^{\frac{1}{m+1}},$$

$$a = [(m+1)(\tau+c)]^{\frac{m-n}{m+1}},$$

where c = constant; let c = o.

(b) When $m = -1$

$$t = c\, e^{\tau},$$

$$a = (c\, e^{\tau})^{-(n+1)}$$

where c = constant $\neq$ o; let, c = 1.

Then for $m = n \neq -1$ we make use of the transformation (a) and obtain

$$v_{\tau} + \alpha v v_{x} + \beta v_{xxx} = 0, \qquad (6.3.4)$$

which is the KdV equation.

When $m = n = 2n+1 = -1$ making use of the transformation (b) we again obtain the KdV equation. When $m=2n+1 \neq -1$ by making use of (a) we get the cKdV equation

$$v_\tau + \frac{v}{2\tau} + \alpha v v_x + \beta v_{xxx} = 0 \qquad (6.3.5)$$

It is a simple exercise to show that the transformations are equivalence transformations.

6.4. DISCUSSION

We have shown that in the cases $m = n$ and $m = 2n+1$ the KdV equation (3.4.1) with variable coefficients is equivalent to the pKdV or cKdV equations which are known to be completely integrable. Therefore in these cases the equation is integrable confirming the results already obtained in chapter-IV.

6.5. CONCLUSION

In this thesis we have presented qualitative studies of certain KdV equations with variable coefficients. The well-known KdV equation is a model for waves propagating on the surface of shallow water of constant depth. This model is considered as fitting into waves reaching the shore.

Renewed attempts have led to the derivation of KdV type equations in which the coefficients are not constants. Johnson's equation is one such equation. We have used this model to study the interaction of waves. It has been found that three-wave interaction is possible, there is transfer of energy between the waves and the energy is not conserved during interaction.

As has been pointed out in chapter-III, the study of KdV equations with variable coefficients is relevant in the context of water waves. In order to study such equations from the point of view of integrability, we have introduced a model equation in chapter-III. This model is studied in chapters IV, V and VI. In chapters IV and V, we have used the concept of PP in the analysis. We have been able to find the cases in which the equation represents integrable systems. The conclusions are confirmed in the last chapter by showing that in the case of integrability the equations are closely related to the well-known pKdV and cKdV equations.

We have not obtained general solution of the model equation. It will be interesting to find the soliton solutions of the equation when it is IST solvable. Another interesting problem is that of finding the solution of the model equation for general m and n and then studying the time evolution for various values of m and n. Such a study may shed some light on the possible connection between movable singularity, the PP and the soliton stability of particular solutions of a nonlinear PDE.

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