SOME PROBLEMS OF DISCRETE FUNCTION THEORY

DISCRETE
COMMUTATIVE DIFFERENCE OPERATOR THEORY

THESIS
SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
THRESIAMMA T. K.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF COCHIN
COCHIN 682 022
INDIA
1985

C E R T I F I C A T E

This is to certify that this thesis is a bona fide record of work by Thresiamma T.K., carried out in the Department of Mathematics and Statistics, University of Cochin, Cochin 682 022 under my supervision and guidance and that no part thereof has been submitted for a degree in any other University.

Wazir Hasan Abdi

DR. WAZIR HASAN ABDI

Former Professor and Head
of the Department of
Mathematics and Statistics
University of Cochin.

<u>STATEMENT</u>

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

Thresiamma T.K.
THRESIAMMA T.K.

ACKNOWLEDGEMENTS

It is my pleasant task and privilege to place on record, here, my sincere gratitude to Professor Wazir Hasan Abdi for having undertaken to guide me in the midst of his many commitments. His keen interest and availability have been a great help to me. My gratitude also goes to my co-supervisor Professor T. Thrivikraman for his valuable suggestions and timely help. I wish to thank Professor R.P.Agarwal, Professor W. Hahn and Dr. Arun Verma for their encouragements. I wish to acclaim the help I have had from the teaching and non-teaching staff and research scholars of the Department of Mathematics and Statistics, University of Cochin. The help and co-operation from Mr.A.Vijayakumar, Mr.Jacob K. Daniel, Miss.V.M.Sunandakumari and Mrs.Gracykutty Job during the preparation of this thesis is greatly appreciated.

I extend my sincere and heart-felt thanks to Revered Manager, Bursar, Fr.Jacob Kariatty. the Principal, my colleagues and non-teaching staff of Bharata Mata College, Thrikkakara, for their encouragement during the course of this work.

My thanks go to Mrs.Abdi and family, the A.C. sisters of Carmel Convent, Aftab and friends of NISTADS, librarians of the University of Lucknow for their hospitality, love and affection during my stay in Lucknow.

Finally, I express my gratitude to my superiors and my sisters of Congregation of Mother of Carmel, especially of Vimala Bhavan, Thrikkakara and all my near and dear ones, for their inspiration and help.

Joyfully I thank Mr. Joseph Kuttikal for his excellent typing and all sincere help.

S Y N O P S I S

This thesis is an attempt to study basic and bibasic commutative difference operators on the lines of J.L. Burchnall and T.W.Chaundy's " Commutative differential operators" [(1) Proc. London Math. Soc. (Sec.2) 21(1923) pp 420-440, (2) Proc. Royal Soc. London (A), 118(1928), pp 557-583, (3) Proc. Royal Soc. London (A), 134(1931), pp 471-485] on the discrete sets $\{p^m x_0\}$, $\{q^n y_0\}$, $\{q^m x_0, q^n y_0\}$, $\{p^m x_0, q^n y_0\}$ $m, n \in Z$, where p,q are positive real constants called bases, $p \neq q \neq 1$. Also bibasic pseudo-analytic functions are introduced using two bases p and q.

Though commutative differential operators play an important role in analysis, no basic or bibasic theory is available. This thesis is an attempt in this direction.

In the first chapter an outline of the theory of commutative differential operators done by Burchnall and Chaundy in the classical case is given. A historical survey of the study of q-difference equations, q-analytic function theory of C.J. Harman [A discrete analytic theory for geometric difference function, Ph.D. thesis of Adelaide (1972)], bibasic analytic functions of Khan M.A [" Contributions to the theory of generalized basic hypergeometric series, Ph.D. thesis, University of Lucknow], bianalytic functions of K.K.Velukutty [" Discrete bianalytic functions" Proc. Nat. Acad. Sci. India, 52(A),I, 1982], Discrete Pseudoanalytic functions of Mercy K Jacob [" Study of discrete Pseudo-analytic functions", Ph.D. thesis, University of Cochin, 1983] and the recent works already done by others have been stated. A list of the results established in the thesis is also given.

The second chapter deals with definition of basic difference operators, characteristic identity of two commutative basic difference operators and the specific nature of commutative difference operators. If

$$P_m = \sum_{k=0}^{m} a_k \theta^k, \quad Q_n = \sum_{k=0}^{n} b_k \theta^k$$

then we arrive at the results that if a_k and b_k are constants or q-periodic functions of x then P_m and Q_n are commutative. But if they are variable functions of x they are commutative. Hence we find the conditions for which these commute. We see that there are some relationships between the coefficients $a_k(x)$ and $b_k(x)$ which make the operators commute with each other. Some examples are constructed. Then we arrive at the result that the difference operators P and Q are commutative if and only if $F(P,Q) = 0$.

Some special commutative operators are developed in the third chapter. Taking $\delta = x\theta$, θ^n is factorised by means of δ. If two operators have a common factor, by transference of that factor we obtain new operators. And we show that if P' and Q' are the new operators obtained by transference of common factor of P and Q respectively then $F(P,Q) = F(P',Q')$. We get the result $f(\delta)x^a = f([a])x^a$, and find the inverse $f(\delta^{-1})x^a = f\left(\frac{1}{[a]}\right) x^a$. Using these results some q-difference equations are solved.

In the fourth chapter we define basic adjoint operators and their properties are listed. If P and Q are commutative, their adjoints also are commutative. The same results are obtained for transference also. Then it is shown that if a linear operator commutes with an operator P, it is a polynomial in that operator.

The bibasic commutative difference operators are studied by considering functions of x and y in R^2. Definitions of D_{px} and D_{qy} with bases p and q and their properties studied in the fifth chapter. Some special bibasic commutative difference operators are taken and some bibasic difference equations are solved.

The last chapter deals with bibasic pseudo-analytic functions and their properties. Pseudoanalytic functions of Mercy, bibasic analytic functions of Khan, bianalytic functions of K.K.Velukutty, and q-analytic functions of Harman are special cases of this. Bibasic analogues of $z^{(n)}$ and bex(z) of classical power and exponential functions etc are also given as examples of such functions.

Finally some lines for further study are suggested. A bibliography containing 75 references is also given.

C O N T E N T S

CHAPTER I

INTRODUCTION

The object of this thesis is to formulate a basic commutative difference operator theory for functions defined on a basic sequence, and a bibasic commutative difference operator theory for functions defined on a bibasic sequence of points, which can be applied to the solution of basic and bibasic difference equations. We give in this chapter a brief survey of the work done in this field in the classical case, as well as a review of the development of q-difference equations, q-analytic function theory, bibasic analytic function theory, bianalytic function theory, discrete pseudoanalytic function theory and finally a summary of results of this thesis.

1. A BRIEF SURVEY OF KNOWN RESULTS

(a) Theory of commutative differential operators

In this section we give an outline of the work done in the theory of commutative differential operators by J.L. Burchnall and T.W.Chaundy [1,2,3].

$$\text{Let } \varphi(D) = \alpha_o D^n + \alpha_1 D^{n-1} + \ldots + \alpha_n I \tag{1.1}$$

where $\alpha_o, \alpha_1, \ldots, \alpha_n$ are functions of x and $D = \frac{d}{dx}$, $D^m = \frac{d^m}{dx^m}$, be defined as a general polynomial differential operator of finite order n. Hence if f is any function of x, differentiable n times, then

$$\varphi(D)f = \alpha_o \frac{d^n f}{dx^n} + \alpha_1 \frac{d^{n-1} f}{dx^{n-1}} + \ldots + \alpha_n f \tag{1.2}$$

We denote such operators as $\varphi(D)$, $\psi(D)$ etc. by P,Q etc. (1.3)

If P and Q are two operators where

$$P = \alpha_o D^m + \alpha_1 D^{m-1} + \ldots + \alpha_m I \tag{1.4}$$

$$Q = \beta_o D^n + \beta_1 D^{n-1} + \ldots + \beta_n I \tag{1.5}$$

then in general they are not commutative.

Their alternant QP-PQ on expansion, is an operator of order not exceeding m+n-1. It vanishes identically if the coefficients of $f, f', f'', \ldots, f^{(m+n-1)}$ vanish under the condition that the m+n differential equations

$$\binom{n}{1} \beta_o \alpha_o' - \binom{m}{1} \alpha_o \beta_o' = 0$$

$$\binom{n}{2} \alpha_o'' \beta_o + \binom{n}{1} \beta_o \alpha_1' + \binom{n-1}{1} \beta_1 \alpha_o' - \binom{m}{2} \alpha_o \beta_o''$$

$$- \binom{m}{1} \alpha_o \beta_1' - \binom{m-1}{1} \alpha_1 \beta_o' = 0$$

....

$$\beta_o \alpha_m^{(n)} + \beta_1 \alpha_m^{(n-1)} + \dots + \beta_{n-1} \alpha_m' - \alpha_o \beta_n^{(m)}$$

$$- \alpha_1 \beta_n^{(m-1)} \quad \dots - \alpha_{m-1} \beta_n' = 0$$

are satisfied. However, all polynomial differential operators with constant coefficients are commutative.

Hence study of commutative differential operators results in a functional relation between the operators which is called the characteristic identity.

Eliminating $D^t y$ ($t = 0,1, \dots, m+n-1$) from

$$D^r (P-hI)y = 0 \ (r = 0,1, \dots, n-1) \quad (1.6)$$

$$D^s (Q-kI)y = 0 \ (s = 0,1, \dots, m-1) \quad (1.7)$$

an algebraic relation

$$F(h,k) = 0 \quad (1.8)$$

is obtained.

Hence if P and Q are commutative then

$$F(P,Q) = 0 \tag{1.9}$$

and conversely if a relation $F(P,Q) = 0$ can be found then P,Q commute.

Transformation of common factors of P and Q results in new operators P' and Q' and also

$$F(P,Q) = F(P', Q') \tag{1.10}$$

Thus the characteristic identity $F(P,Q) = 0$ remains invariant for the whole set of operators derived from a chosen P,Q. So restricted, the operators form a group.

Defining the adjoint of

$$P = \sum_{k=o}^{m} a_k(x)D^k \text{ as } P^* = \sum_{k=o}^{m} (-D)^k a_k(x) \tag{1.11}$$

T.W.Chaundy [1] proved that if P^*, Q^* are the adjoints of P and Q respectively, then

$$F(P,Q) = F(P^*, Q^*) \tag{1.12}$$

J.L. Burchnall and T.W. Chaundy [1] also defined another type of operators with $\delta = xD$ whose examples are

$$P = x^{-m}\, \delta(\delta-nI)\ (\delta-2nI)\ \ldots\ (\delta-(mn-n)I) \qquad (1.13)$$

$$Q = x^{-n}\, \delta(\delta-mI)\ (\delta-2mI)\ \ldots\ (\delta-(mn-m)I). \qquad (1.14)$$

Later T.W.Chaundy [2] solved many ordinary differential equations using δ. Analogously differential operators

$$\varphi(\partial_1, \partial_2, \ldots, \partial_n) = \Sigma\ \alpha\ \partial_1^a\ \partial_2^b \ldots\ \partial_n^k \qquad (1.15)$$

with $\partial_r = \dfrac{\partial}{\partial x_r}$ in the field of n independent variables $(x_1, x_2, \ldots, x_n)$ were studied.

H. Flanders [1], S.A.Amitsur [1] and others studied on commutative linear differential operators. V.P.Maslov [1] developed an operator theory considering the differential operator $P(x,D)$ and giving an order $P(x^2, D^1)$ showing D acts first.

Later Coddington [1,2,3] elaborated the theory of formal normal operators. Recently Hahn [5] has given an algebraic approach to commutative linear differential operators.

Eventhough much work has been done in this field, no basic or bibasic theory is available in literature.

(b) q-difference equations

A very extensive development of the theory of q-difference equations was carried by Jackson [1,2,3,4,5]. In 1908 Jackson used the difference operator

$$\Theta\ \varphi(x) = \frac{\varphi(qx) - \varphi(x)}{(q-1)x} \tag{1.16}$$

which gave rise to a series in which the coefficients follow the q-binomial form. In 1910 he introduced the concept of q-integration which he defined as the inverse q-difference operator Θ, as

$$\Theta_x^{-1} f(x) = \frac{1}{1-q} \int f(x)\ d(q,x) \tag{1.17}$$

Hahn [1] and Jackson [5] studied fundamental properties of the inverse operation, showing that under certain conditions, the q-integral tends to the Riemann integral as $q \to 1$. In fact the definite q-integrals are defined by

$$\int_0^x \Theta_x f(x)\ d(q,x) = f(\infty) - f(o) \tag{1.18}$$

$$\int_x^{\infty} \theta_x f(x)\, d(q,x) = f(\infty) - f(x) \tag{1.19}$$

whence $\int_a^b = \int_0^b - \int_0^a$

Correspondingly the q-integrals can be defined by

$$\int_0^x f(y)d(q,y) = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) \tag{1.20}$$

$$\int_x^{\infty} f(y)d(q,y) = (1-q)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x) \tag{1.21}$$

$$\int_0^{\infty} f(y)d(q,y) = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j) \tag{1.22}$$

Among other results Jackson deduced the formula for q-integration by parts

$$\int \left\{ \theta_x f(x) \right\} g(x)\, d(q,x) = (1-q)\; f(x)g(x) - \int f(qx) \left\{ \theta_x g(x) \right\} d(q,x) \tag{1.23}$$

In 1960, Abdi [1,2,3,4) revived interest in q-integration when he made a thorough study of q-Laplace transforms which were applied to the solution of certain q-difference and q-integral equations. Al-Salam [1] and Agarwal [2] in 1969 obtained q-analogues of Cauchy's multiple integral formula and fractional q-integrals.

Apart from these results in q-integration, research in q-difference theory has divided into two main streams, number theory and the general theory of q-difference equations.

Carmichael [1,2], Adams [5] and Tritzinsky [1], amongst others have evolved an extensive theory for linear q-difference equations. In 1943 Sawyer [1] studied the system $(F-\lambda G) = 0$ for second order and Chaundy [3,4] considered the general case. M.Upadhyay[1] solved q-difference equation of first order of Sawyer's type. Important contributions have been made by Hahn [2,3,4] and Abdi [2].

Abdi [5,6,7] introduced a 'bibasic' functional equation of the form $a(z)\ f(pz) + b(z)\ f(qz) + c(z)\ f(z) = 0$. He has also solved some bibasic functional equations.

(c). q-analytic function theory

In 1972 Harman [1] developed a discrete analytic theory for geometric difference functions.

He defined a lattice with geometric spacing, ie. points of the form

$$H = \{(\pm q^m x_o, \pm q^n y_o);\ m,n \in Z\} \tag{1.24}$$

where $o < q < 1$ and $x_o > o$, $y_o > o$ are fixed numbers.

Complex valued functions defined on the points of H are called discrete functions. Functions satisfying

$$\frac{f(x,y)-f(qx,y)}{(1-q)x} = \frac{f(x,y)-f(x,qy)}{(1-q)iy} = L[f(z)] \tag{1.25}$$

where $z = (x,y) \in H$, are called q-analytic functions. Hence the operator L is defined by

$$L[f(z)] = \bar{z}\, f(z) - x\, f(x,qy) + iy\, f(qx,y) \tag{1.26}$$

Defining a discrete domain D, he defined the q-analyticity of a discrete function f(z) in D. If $L[f(z)] = 0$ for every $z \in D$ such that $T(z) = \{(x,y), (qx,y), (x,qy)\} \subset D$, then f(z) is q-analytic in D. He also obtained the properties of q-analytic functions in D. As examples of q-analytic functions, he defined

$$z^{(n)} = \sum_{j=o}^{n} \binom{n}{j}_q x^{n-j}(iy)^j \tag{1.27}$$

$$e_q(z) = \sum_{j=o}^{\infty} \frac{(1-q)^j}{(1-q)_j} z^{(j)} \tag{1.28}$$

and discussed their properties.

(d). Bibasic analytic functions

Khan M.A [1] studied bibasic analytic functions choosing the lattice

$$\{ (p^m x_o, q^n y_o); m,n \in Z \}, p \neq q \neq 1 \tag{1.29}$$

and defining a discrete domain. He defined the bibasic difference operators D_{px}, D_{qy}, as

$$D_{px} f(z) = \frac{f(z) - f(px,y)}{(1-p)x} \tag{1.30}$$

$$D_{qy} f(z) = \frac{f(z) - f(x,qy)}{(1-q)iy} \tag{1.31}$$

where f(z) is a discrete function. Hence, if

$$T'(z) = \{(x,y),\ (px,y),\ (x,qy)\} \text{ and}$$

$$D_{px} = D_{qy}\ f(z) \qquad (1.32)$$

such that $T'(z) \subset D$, then f(z) is said to be bibasic analytic at $z \in D$. If (1.32) holds for every $z \in D$, then f(z) is said to be bibasic analytic in D.

Let $$L'[(f(z)] = \{\bar{z}-px+qiy\}\ f(z) - (1-p)x\ f(x,qy) + (1-q)iy\ f(px,y). \qquad (1.33)$$

Therefore f(z) is bibasic analytic in D if and only if

$$L'f(z) = 0, \qquad (1.34)$$

for every $z \in D$. He has given the properties of bibasic analytic functions and constructed examples of bibasic analytic functions, where

$$z^{(n)} = \sum_{j=o}^{n} \frac{(p)_n}{(p)_{n-j}(q)_j} \left\{\frac{(1-q)iy}{1-p}\right\}^{j} x^{n-j} \qquad (1.35)$$

$$ {}_0M_1 \ [-;p;z] = \sum_{n=0}^{\infty} \frac{z^{(n)}}{(p)_n} \tag{1.36} $$

$$ = e_p(x) \ eq\left(\frac{iy(1-q)}{1-p} \right) \tag{1.37} $$

According to Khan's definition

$$ D_{px} z^{(n)} = D_{qy} z^{(n)} = \left(\frac{1-p^n}{1-p} \right) z^{(n-1)}. \tag{1.38} $$

The theory of bibasic analytic functions is a two fold extension of the theory of q-analytic functions. The presence of two bases gives more freedom of the choice of the bases.

(e). Bianalytic functions

Velukutty [1] defined bianalytic functions taking the lattice

$$ H = \left\{ (q^m x_o, q^n y_o), \ m,n \in Z \right\}, \ o < q < 1, \ (x_o, y_o) \text{ are fixed, } x_o > o, \ y_o > o, \text{ and } p = q^{-1}. \tag{1.39} $$

He defined two operators R_q and R_p as

$$R_q f(z) = \bar{z} f(x,y) - xf(x,qy) + iyf(qx,y) \qquad (1.40)$$

$$R_p f(z) = \bar{z} f(x,y) - xf(x,py) + iyf(px,y) \qquad (1.41)$$

where $f: H \longrightarrow \emptyset$

$R_q f(z)$ and $R_p f(z)$ are respectively called the q- and p- residues of the function f at z. If the q-residue (p-residue) of f is zero at z, f is said to be q-analytic (p-analytic) at z. A function $f: H \longrightarrow \emptyset$ which is both q- and p- analytic in D (a discrete domain) is called bianalytic in D. In fact, it satisfies the equation (1.42)

$$R_q f(z) = R_p f(z) = 0 \text{ everywhere in D.} \qquad (1.43)$$

He has discussed the properties of bianalytic functions and how to construct bianalytic functions giving examples. $f(z) = \alpha z+\beta$, α, $\beta \in \emptyset$ is a trivial example of a bianalytic function in entire H.

(f). Pseudo-analytic functions

Mercy K. Jacob [1] introduced pseudo-analytic functions on the lattice,

$$\{(q^m x_0, q^n y_0), m,n \in Z\}, \quad 0 < q < 1, \; x_0 > 0, \; y_0 > 0$$

are fixed numbers. (1.44)

The theory of discrete pseudoanalytic functions is a generalisation of the theory of q-analytic functions. Discrete functions satisfying the inequality

$$|f(z) - f(z')| \leqslant k\ \sigma^{\mu} \tag{1.45}$$

where $z' = (x',y') \varepsilon D$, a discrete domain, $z \varepsilon A(z')$,

$$A(z') = \left\{ (qx',y'),\ (x',qy'),\ (q^{-1}x',y'),\ (x',q^{-1}y') \right\},$$

$$\sigma = (q^{-1} - 1) \max (x',y'),\ o < \mu \leqslant 1 \tag{1.46}$$

have been called discrete Hölder-type at $z' \varepsilon D$.

If the above inequality holds for all $z \varepsilon D$ then the function is called discrete Hölder-type in D, denoted as H(D). If $g_1, g_2 \varepsilon H(D)$ such that $\text{Im}(\bar{g}_1, g_2) > o$, then the row vector $g = [g_1, g_2]$ is called a generating vector. If $f = [f_1, f_2]'$ where f_1 and f_2 are real valued functions in D, we call the set of all such column vectors F(D).

Suppose $g = [g_1, g_2] \varepsilon G(D)$ and $W \varepsilon G.F(D)$, then

$$g^{\theta_x} w(z) = (g.\theta_x f)(z) \tag{1.47}$$

$$g^{\theta_y} w(z) = (g.\theta_y f)(z). \tag{1.48}$$

If w is a complex valued function defined in D, then w is called discrete g-pseudoanalytic of the first kind at $z \varepsilon D$, if

$$w \varepsilon G.F(D) \text{ and } g^{\theta_x} w(z) = {}_g\theta_y\, w(z). \tag{1.49}$$

If this relation holds for all $z \varepsilon D$, then w is called g-pseudoanalytic of the first kind in D, the class of such functions denoted by ${}_1P_D(g)$.

If $\quad w = (g.f),\ f \varepsilon F(D),\ g \varepsilon G(D),$ (1.50)

$w \varepsilon {}_1P_D(g)$, then $h = f_1 + i f_2$ is discrete g-pseudoanalytic of the second kind in D. The class of discrete g-pseudoanalytic functions of the second kind in D is denoted by ${}_2P_D(g)$.

2. SUMMARY OF RESULTS OF THE THESIS

Along with the above brief survey of work done in the field of differential operators, q-difference theory, q-analytic theory, bibasic analytic functions, bianalytic functions and pseudoanalytic functions, it is worth mentioning that no work has been done in basic commutative difference operators, bibasic commutative difference operators and bibasic pseudo-analytic functions. Hence an attempt is made in this thesis on these lines. Now we give the summary of results here.

Using the concept of differential operators defined and developed by J.L.Burchnall and T.W.Chaundy [1,2,3], a basic theory has been developed for functions defined on the set $\{q^m x_0, m \in Z\}$, $0 < q < 1$, $x_0 > 0$ are fixed and accordingly the conditions for basic difference operators to be commutative have been established. Some examples are also given. Then we try to establish some properties of these operators. Considering two operators P_m and Q_n, their characteristic identity is obtained. We see that two operators P_m and Q_n are commutative if and only if there is an algebraic relation of the form $F(P_m, Q_n)=0$. This is the content of Chapter Two.

The Third Chapter deals with some special basic commutative difference operators $\delta = x\theta$, and their properties. We are able to obtain commutative difference operators easily using these δ. Given a pair of commutative difference operators, we shall form new pairs by transference of common factors if any, and hence we can find that a set of q-difference equations is generated by the same characteristic identity. If f(δ) is a polynomial operator in δ, we define f(δ^{-1}) and prove some theorems connected to these polynomial operators. Applying these operators and results related to them, we solve some basic difference equations.

In the Fourth Chapter we use concepts of J.L.Burchnal and T.W.Chaundy [1] and Coddington [1,2,3] to develop basic adjoint operators, normal operators, and self-adjoint operators. Here we prove that if the coefficients are constants or q-periodic functions of x, then the basic operator is normal. When the co-efficients are not constants or q-periodic functions, then we see that an operator P in general is not a normal operator. But we could not obtain a complete characterisation. We also see that if a first order basic difference operator Q commutes with an operator P of order m, then P is a polynomial in that operator Q. We conclude this chapter by giving some applications of adjoint operators.

Abdi [5,6,7] introduced the concept of bibasic functional equations. The Fifth Chapter is an attempt to define bibasic difference operators D_{px} and D_{qy}. We study the effect of action of these operators on functions f(x,y) defined on R^2. Also we discuss the nature of these operators and conditions under which they commute. Considering bibasic difference operators of the form

$$P_m = \sum_{k,j=o}^{m} a_{k,j} D_{px}^{k} D_{qy}^{j} (.), \quad k+j \leqslant m.$$

Some special bibasic operators also are considered and their properties established. We see that the alternant of two linear operators is also a linear operator and hence we are able to form repeated alternants of such operators. Finally, we solve some bibasic difference equations in R^2.

The last chapter starts with the definition of functions which are pseudoanalytic in certain domain in the discrete geometric space with two unconnected bases p and q. We try to establish some of their properties. We see that this theory reduces to that of basic pseudo-analytic functions, bibasic analytic functions, bi-analytic functions and q-analytic functions under certain conditions. We also define $z^{(n)}$ and bex(z) in the bibasic case. However, these are examples of bibasic analytic functions which are different from those of Khan [1], where $D_{px}z^{(n)}$ and $D_{qy}z^{(n)}$ are given in terms of p alone. But in the new definition of $z^{(n)}$ we get these in terms of both p and q. We also show that continuation from both the axes is possible by taking suitable continuation operators. Finally, we establish an analogue of Maclaurin series.

There are some interesting problems related to these theories like study of integral curves and second degree partial difference equations which we have not attempted here.

CHAPTER-II

BASIC COMMUTATIVE DIFFERENCE OPERATORS

In this chapter we study different types of basic commutative difference operators (ie. commutative q-difference operators) and their behaviour when the coefficients are constants, q-periodic functions of x or variable functions of x which satisfy certain conditions. In general, all operators are not commutative. Hence an attempt is made here to determine the operators which commute with each other. Also we show that two operators P_m and Q_n are commutative if and only if they satisfy a functional equation $F(P_m, Q_n)=0$.

1. BASIC DIFFERENCE OPERATORS

Here we consider the difference operator θ on real valued functions of a variable x, where

$$\theta\varphi(x) = \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, \text{ Jackson [1] and} \tag{2.1}$$

$$\theta^r[\varphi(x)g(x)] = \sum_{j=o}^{r} \binom{r}{j}_q q^{j(j-r)} \theta^j\varphi(q^{r-j})\ \theta^{r-j}g(x), \tag{2.2}$$

Hahn [4]

$$\left.\begin{aligned} \binom{r}{j}_q &= \frac{(1-q)_r}{(1-q)_j\ (1-q)_{r-j}} \\ (1-q)_r &= (1-q)(1-q^2)\ \dots\ (1-q^r)\ . \end{aligned}\right\} \tag{2.3}$$

Accordingly we define the general basic polynomial difference operator of order m.

$$P_m = \sum_{k=o}^{m} a_k \theta^k \tag{2.4}$$

where a_k are constants or functions of x, $x = q^\alpha x_o$, $q \in (o,1)$, $x_o > o$ fixed, $\alpha \in Z$ and $a_m \neq o$.

We also define

$$P = \sum_{k=o}^{\infty} a_k \theta^k = \lim_{m \to \infty} \sum_{k=o}^{m} a_k \theta^k$$

$$= \lim_{m \to \infty} P_m \text{ , in the usual sense.}$$

$$f(x) = \sum_{k=o}^{m} a_k x^k, \quad |x| < x_o, \quad x = q^\alpha x_o, \quad o < q < 1,$$

$\alpha \in Z$, $x_o > o$, a_k constants or variable functions of x is called the associated polynomial.

If m is infinite, question of convergence will arise involving both the operand and the coefficients. We cannot discuss the convergence of the operator by itself without a knowledge of the operand. If the operand is merely a polynomial in x of n^{th} degree, the infinite operators terminate at θ^n. In other cases, if we consider infinite operators we have to make restrictions.

It is easily seen that such operators can be combined by the algebraic operations of addition, subtraction and multiplication.

If P_m and Q_n are two basic difference operators, then we define the addition and multiplication as;

$$\left\{ P_m \pm Q_n \right\} f = P_m f \pm Q_n f \tag{2.5}$$

$$\left\{ P_m \, Q_n \right\} f = P_m \left\{ Q_n f \right\} \tag{2.6}$$

We see from (2.5) and (2.6) that the basic difference operators follow the fundamental laws of arithmetical combination except possibly the commutative law of multiplication.

2. BASIC COMMUTATIVE DIFFERENCE OPERATORS

Two operators are said to be commutative if they satisfy the commutative law of multiplication. In other words, if

$$P_m = \sum_{k=o}^{m} a_k \theta^k \text{ and } Q_n = \sum_{k=o}^{n} b_k \theta^k$$

$$a_m \neq o, \quad b_n \neq o$$

are two basic difference operators of order m and n respectively, they are commutative if

$$P_m Q_n = Q_n P_m . \tag{2.7}$$

We now inquire what sort of basic difference operators are commutative

(a) Operators P_m and Q_n commute if a_k and b_k are constants:

(i) Let m,n be finite

$$P_m Q_n = \left(\sum_{k=o}^{m} a_k\theta^k\right) \left(\sum_{k=o}^{n} b_k\theta^k\right)$$

$$= \sum_{r=o}^{m+n} C_r\theta^r, \text{ where } C_r = \sum_{j=o}^{r} a_j b_{r-j}$$

$$= Q_n P_m$$

Hence the result.

(ii) Let m,n be infinite,

$$PQ = \left(\sum_{k=o}^{\infty} a_k\theta^k\right) \left(\sum_{k=o}^{\infty} b_k\theta^k\right)$$

$$= \sum_{r=o}^{\infty} C_r\theta^r \text{ where } C_r = \sum_{j=o}^{r} a_j b_{r-j}$$

$$= QP .$$

Both sides will exist if for

$\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ the radii of convergence are P_1, P_2 respectively and $\theta^{(k)} f(x) = O(x^k)$. Then the radius of convergence of the product series will be min (P_1, P_2). Therefore for infinite m,n and also when the associated series converges and the operand is bounded, the operators are commutative.

(b) Operators P_m and Q_n commute if a_k and b_j are q-periodic functions of x.

Let

$$P_m = \sum_{k=0}^{m} a_k(x)\theta^k, \quad Q_n = \sum_{j=0}^{n} b_j(x)\theta^j$$

Then

$$(P_m Q_n)f = \sum_{k=0}^{m} a_k(x) \left(\sum_{j=0}^{n} \sum_{i=0}^{k} \binom{k}{i} q^{i(i-k)} \theta^i \, b_j(q^{k-i}x) \, \theta^{k+j-i} f \right) \tag{2.8}$$

$$(Q_n P_m)f = \sum_{j=0}^{n} b_j(x) \left(\sum_{k=0}^{m} \sum_{i=0}^{j} \binom{j}{i}_q q^{i(i-j)} \theta^i a_k(q^{j-i}x) \, \theta^{j+k-i} f \right). \tag{2.9}$$

Since $a_k(x)$, $b_j(x)$ are q-periodic functions of x, we have

$$a_k(x) = a_k(qx) \dots = a_k(q^n x), \ \forall\ k=o, \dots m$$

$$b_j(x) = b_j(qx) \dots = b_j(q^m x), \ \forall\ j=o, \dots n$$

$$\theta\ a_k(x) = \theta\ b_j(x) = o$$

Therefore (2.8) reduces to

$$P_m\ Q_n(f) = \sum_{k=o}^{m} a_k(x) \sum_{j=o}^{n} b_j(x)\theta^{k+j}\ (f)$$

$$= \sum_{k=o}^{m} a_k(x)\theta^k \sum_{j=o}^{n} b_j(x)\theta^j\ (f)$$

$$= \sum_{r=o}^{m+n} C_r\theta^r f$$

where $$C_r = \sum_{j=o}^{r} a_j(x)\ b_{r-j}(x)$$

$$= Q_n\ P_m(f), \text{ from (2.9)}.$$

Hence P_m and Q_n commute.

A similar argument applies when m and n are infinite, since q-periodic functions $a_k(x)$, $b_j(x)$ remain constant at all points of the set $\{q^\alpha x_o,\ \alpha \in Z\}$, $x_o > o$ and $o < q < 1$ fixed.

(c) Basic difference operators P_m and Q_n with coefficients $a_k(x)$, $b_j(x)$ not q-periodic functions of x and m,n finite are in general, not commutative.

They commute if the condition that m+n+1 basic difference equations are satisfied.

Let $P_m = \sum_{k=0}^{m} a_k(x)\theta^k, \; Q_n = \sum_{j=0}^{n} b_j(x)\theta^j.$

Therefore,

$$P_mQ_n(f) = \sum_{k=0}^{m} a_k(x)\Big[\sum_{j=0}^{n}\sum_{i=0}^{k} \binom{k}{i}_q q^{i(i-k)} \theta^i b_j(q^{k-i}x)\theta^{k+j-i}(f)\Big]$$

$$Q_nP_m(f) = \sum_{j=0}^{n} b_j(x)\Big[\sum_{k=0}^{m}\sum_{i=0}^{j} \binom{j}{i}_q q^{i(i-j)} \theta^i a_k(q^{j-i}x)\theta^{j+k-i}(f)\Big].$$

Therefore

$$Q_nP_m - P_mQ_n \neq 0 \text{ in general.} \tag{2.10}$$

Now $Q_nP_m - P_mQ_n = 0$, if

$$A(x,\theta)F = 0, \tag{2.11}$$

where $F = \text{Transpose}\,(f_o, f_1, f_2, \ldots, f_{m+n})$ (2.12)

$f_o = f,\; f_1 = \theta f \ldots,\; f_{m+n} = \theta^{m+n} f$

and $A(x,\theta)$ is the square matrix of order $(m+n+1)$ given by

$$
\begin{bmatrix}
S(o,b,a_o) & S(o,b,a_1) & \cdot\cdot & S(o,b,a_m) & 0 & 0 & \cdot\cdot & & 0 \\
0 & S(1,b,a_o(qx)) & \cdot\cdot & \cdot\cdot & S(1,b,a_m(qx)) & 0 & \cdot\cdot & & 0 \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot \\
0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & S(n-1,b,a_m(q^{n-1}x)) & 0 \\
0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & (S(n,b,a_m(q^n x)) - S(m,a,b_n(q^m x))) \\
0 & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & -S(m-1,a,b_n(q^{m-1}x)) & 0 \\
\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot \\
0 & -S(1,a,b_o)qx)) & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & -S(1,a,b_n(qx)) & & 0 \\
-S(o,a,b_o) & -S(o,a,b_1) & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot\ -S(o,a,b_n) & 0 & & 0
\end{bmatrix}
\quad \cdot\cdot \ (2.13)
$$

where

$$S(j,b,a_i(q^j x)) = \sum_{k=j}^{n} b_k(x)\theta^{k-j} a_i(q^j x) \binom{k}{k-j}_q q^{j(j-k)} \quad (2.14)$$

.. ..

$$- S(j,a,b_i(q^j x) = \sum_{k=j}^{m} a_k(x)\theta^{k-j} b_i(q^j x) \binom{k}{k-j}_q q^{j(j-k)}.$$

From these m+n+1 basic difference equations, we get the relationship between the coefficients if they commute.

3. EXAMPLE WHICH GIVES THE RELATIONSHIP BETWEEN THE COEFFICIENTS OF THE FORM (2.11), (2.12) AND (2.13)

Consider the basic difference operators L and M with variable coefficients, where

$$L = \theta^2 + a(x)\theta + b(x)I \quad (2.15)$$

$$M = \theta + c(x)I. \quad (2.16)$$

Let $c(x)$ be a chosen function on the set $\{q^\alpha x_0\}$, $\alpha \in Z$ such that $c(q^\alpha x_0) \to 0$ as $\alpha \to \infty$. We try to determine $a(x)$ and $b(x)$ in terms of $c(x)$ so that L and M may commute.

Now formally,

$$(LM)f = [\theta^2 + a(x)\theta + b(x)I]\,[\theta f + c(x)f]$$

$$= \theta^3 f + \left\{ c(q^2x) + a(x) \right\} \theta^2 f$$

$$+ \left\{ (1+q)q^{-1}\theta c(qx) + a(x)\ c(qx) + b(x) \right\} \theta f$$

$$+ \left\{ \theta^2\ c(x) + a(x)\ \theta c(x) + b(x)\ c(x) \right\} f \qquad (2.17)$$

Also,

$$(ML)f = [\theta + c(x)I]\,[\theta^2 f + a(x)\theta\ f + b(x)f\ ]$$

$$= \theta^3 f + \left\{ a(qx) + c(x) \right\} \theta^2 f$$

$$+ \left\{ \theta a(x) + c(x)\ a(x) + b(qx) \right\} \theta f$$

$$+ \left\{ \theta b(x) + c(x)\ b(x) \right\} f \cdot \qquad (2.18)$$

From (2.17) and (2.18)

$$(LM-ML)f = \left\{ c(q^2x) + a(x) - a(qx) - c(x) \right\} \theta^2 f$$

$$+ \left\{ (1+q)q^{-1}\ \theta c(qx) + a(x)\ c(qx) + b(x) - \theta a(x) \right.$$

$$\left. - c(x)\ a(x) - b(qx) \right\} \theta f$$

$$+ \left\{ \theta^2\ c(x) + a(x)\theta\ c(x) + b(x)\ c(x) \right.$$

$$\left. - \theta b(x) - c(x)\ b(x) \right\} f\,. \qquad (2.19)$$

Hence L and M are not commutative in general, but L and M will commute if LM - ML = 0, ie. if the following basic difference equations are satisfied.

$$c(q^2x) + a(x) - a(qx) - c(x) = 0 \tag{2.20}$$

$$(1+q)q^{-1}\ \theta c(qx) + a(x)\ c(qx) + b(x)$$
$$- \theta a(x) - c(x)\ a(x) - b(qx) = 0 \tag{2.21}$$

$$\theta^2 c(x) + a(x)\ \theta c(x) - \theta b(x) = 0. \tag{2.22}$$

We see that (2.20), (2.21), (2.22) follow from (2.19).
From (2.20)

$$c(q^2x) - c(x) = a(qx) - a(x).$$

Therefore,

$$\frac{a(x) - a(qx)}{(1-q)x} = \frac{c(x) - c(q^2x)}{(1-q)x}$$

ie. $$\theta\ a(x) = \frac{c(x) - c(q^2x)}{(1-q)x}. \tag{2.23}$$

Integrating both sides of (2.23) we get,

$$\int \theta a(x)\ d(q,x) = \int \frac{c(x) - c(q^2x)}{(1-q)x}\ d(q,x)$$

ie. $a(x) = (1-q)x \sum_{i=o}^{\infty} \frac{q^i \{ c(q^i x) - c(q^{i+2}x) \}}{(1-q)q^i x}$,

by(1.20)

$= c(x) + c(qx)$ since $c(q^{\alpha}x_o) \longrightarrow o$ as $\alpha \longrightarrow \infty$.

Therefore

$$a(x) = c(x) + c(qx) \qquad (2.24)$$

From (2.22)

$$\theta b(x) = \theta^2 c(x) + a(x)\,\theta c(x)$$

$$= \theta^2 c(x) + \{ c(x) + c(qx) \}\,\theta c(x)$$

Integrating by parts using

$$\int \{ \theta_x f(x) \} g(x)\, d(q,x) = (1-q)\, f(x)\, g(x) - \int f(qx) \{ \theta_x g(x) \}\, d(q,x),$$

Jackson [4,5]

We get

$$\int \theta b(x)\, d(q,x) = \int \theta^2 c(x)\, d(q,x) + \int \theta c(x) \{ c(x) + c(qx) \}\, d(q,x)$$

Therefore

$$b(x) = \theta\, c(x) + (1-q)\, c(x) \left\{ c(x) + c(qx) \right\}$$

$$- \int c(qx)\, \theta \left\{ c(x) + c(qx) \right\} d(q,x)$$

ie. $$b(x) = \theta c(x) + (1-q) \left\{ c(x) \right\}^2 + (1-q)\, c(x)\, c(qx)$$

$$- \int \left\{ \frac{c(x) - c(q^2x)}{(1-q)x} \right\} c(qx)\, d(q,x)$$

$$= \theta c(x) + (1-q) \left\{ c(x) \right\}^2 + (1-q)\, c(x)\, c(qx)$$

$$-(1-q)x \sum_{i=0}^{\infty} \left[q^i \left\{ \frac{c(q^ix)c(q^{i+1}x) - c(q^{i+2}x)c(q^{i+1}x)}{(1-q)q^ix} \right\} \right]$$

by (1.20)

$$= \theta c(x) + (1-q) \left\{ c(x) \right\}^2 + (1-q)\, c(x)\, c(qx)$$

$$- c(x)\, c(qx)$$

Therefore

$$b(x) = \theta\, c(x) - q\, c(x)\, c(qx) + (1-q) \left\{ c(x) \right\}^2 \qquad (2.25)$$

Thus from (2.24) and (2.25) it is clear that we can find the coefficients a(x) and b(x) in terms of c(x).

This example illustrates that, if two basic difference operators with variable coefficients, which are not q-periodic, commute, then some relationship exists between the coefficients.

In the general case however we get m+n+1 basic difference equations, from which we get the relationships between the coefficients.

We now establish some properties of these operators.

4. PROPERTIES OF BASIC DIFFERENCE OPERATORS

Theorem 1

If f is a solution of the equation

$\sum_{k=o}^{m} a_k(x)\theta^k f = o$, then cf is also a solution where c is any arbitrary constant.

Proof

As $\theta\,(cf(x)) = c\,\theta\, f(x)$

and $\theta^r\,(cf\,(x)) = c\,\theta^r f(x)$ (2.26)

We get $P_m(cf(x)) = \sum_{k=0}^{m} a_k(x)\theta^k cf(x)$

$$= c \sum_{k=0}^{m} a_k(x)\theta^k f(x)$$

$$= c\, P_m f(x).$$

Theorem 2

If $f_1, f_2, \ldots, f_m$ are m distinct solutions of the homogeneous equation $P_m f = 0$, $c_1 f_1 + c_2 f_2 + \ldots + c_m f_m$ is a solution where $c_1, c_2, \ldots, c_m$ are arbitrary constants.

Proof

$$P_m(c_1 f_1 + c_2 f_2 + \ldots + c_m f_m)$$

$$= c_1 P_m f_1 + c_2 P_m f_2 + \ldots + c_m P_m f_m$$

$$= 0, \text{ by theorem 1}.$$

Theorem 3

If $g = g_0(x)$ be any solution of the non-homogeneous equation $P_m g = r(x)$, then if $f(x)$ is the complete primitive of $P_m f = 0$, then $g = g_0(x) + f(x)$ will be the most general solution of $P_m g = r(x)$.

Proof

θ^r is distributive, Jackson [1].

Therefore P_m is distributive, where

$$P_m = \sum_{k=o}^{m} a_k(x)\theta^k$$

Therefore

$$P_m[g_o(x) + f(x)] = P_m g_o(x) + P_m f(x)$$
$$= r(x)$$

since $P_m g_o(x) = r(x)$.

Therefore $P_m f(x) = 0$

and $g = g_o(x) + f(x)$

involves m arbitrary constants.

It is, therefore, the most general solution of $P_m g = r(x)$.

Theorem 4

P_m and Q_n are commutative difference operators if and only if, given a constant g, we can find an h such that the equations $(P_m - gI)\ \varphi = 0$, $(Q_n - hI)\ \varphi = 0$ have a common solution $Y(g,h) = 0$.

Proof

Let $P_m = \sum_{k=o}^{m} a_k \theta^k$, $Q_n = \sum_{k=o}^{n} b_k \theta^k$

where a_k, b_k are constants or variables. Let Y_1, Y_2, ... Y_m be a linearly independent set of solutions of the basic difference equation

$$(P_m - gI)\ \varphi = 0 . \tag{2.27}$$

We assume P_m and Q_n are commutative. Then

$$(P_m - gI)\ Q_n\ Y_1 = Q_n(P_m - gI)\ Y_1 = 0 . \tag{2.28}$$

Thus $Q_n\ Y_1$ is a solution of the equation (2.27). Similarly the solutions $Q_n\ Y_2$, $Q_n\ Y_3$, ..., $Q_n\ Y_m$ c n be obtained. Then we have

$$\begin{aligned} Q_nY_1 &= \alpha_{11}Y_1 + \alpha_{12}\ Y_2 + \dots + \alpha_{1m}Y_m \\ Q_nY_2 &= \alpha_{21}Y_1 + \alpha_{22}Y_2 + \dots + \alpha_{2m}Y_m \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Q_nY_m &= \alpha_{m1}Y_1 + \alpha_{m2}Y_2 + \dots + \alpha_{mm}Y_m \end{aligned} \tag{2.29}$$

Now let $Y = c_1Y_1 + c_2Y_2 + \dots + c_mY_m$

Then $QY = hY$. (2.30)

provided that h and the constants c satisfy the equations $hc_r = \alpha_{r1}c_1 + \alpha_{r2}c_2 + \dots + \alpha_{rm}c_m$ $(r = 1, \dots, m)$. In order that these equations may be consistent it is necessary that h be determined by the relation

$$\begin{vmatrix} \alpha_{11}-h & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22}-h & \cdots & \alpha_{2m} \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mm}-h \end{vmatrix} = 0$$

Thus corresponding to each g there exist m values of the constant h such that the equation

$$(P_m - gI)\,\varphi = 0$$

$$(Q_n - hI)\,\varphi = 0$$

have a common solution.

Thus Y is a common solution of

$$(P_m - gI)\ \varphi = 0 \text{ and } (Q_n - hI)\ \varphi = 0 . \qquad (2.31)$$

The constants g and h are therefore connected by some functional relation $Y(g,h) = 0$.

The form of $Y(g,h) = 0$ can be obtained directly from the eliminant of the basic difference equations

$$(P_m - gI)\ \varphi = 0,\ (Q_n - hI)\ \varphi = 0$$

Their common solution Y will be more generally a common solution of m+n basic difference equations

$$\theta^r\ (P_m - gI)\ \varphi = 0\ ,\ r = o,1,\ \ldots,\ n-1$$

$$\theta^s\ (Q_n - hI)\ \varphi = 0\ ,\ s = o,1,\ \ldots,\ m-1\ .$$

Eliminating φ, $\theta\varphi$, $\theta^2\varphi$, ..., $\theta^{m+n-1}\varphi$ from these m+n equations, we get

$$\begin{array}{llllll}
a_o(x)-g & a_1(x) & \cdots & a_m(x) \; 0 \; 0 & \cdots & 0 \\
\theta a_o(x) & \{a_o(qx)-g+\theta a_1(x)\} & \cdots & & \cdots & 0 \\
\theta^2 a_o(x) & \{\binom{2}{1}_q q^{-1}\theta a_o(qx)+\theta^2 a_1(x)\} & \cdots & & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\theta^{n-1} a_k(x) & \{\binom{n-1}{1}_q q^{2-n}\theta^{n-2} a_o(qx)+\theta^{n-1} a_1(x)\} & \cdots & & \cdots & 0 \\
b_o(x)-h & b_1(x) & \cdots & b_n(x) & \cdots & 0 \\
\theta b_o(x) & \{b_o(qx)-h+\theta b_1(x)\} & \cdots & & \cdots & 0 \\
\theta^2 b_o(x) & \{\binom{2}{1}_q q^{-1}\theta b_o(qx)+\theta^2 b_1(x)\} & \cdots & & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\theta^{m-1} b_o(x) & \{\binom{m-1}{1}_q q^{2-m}\theta^{m-2} b_o(qx)+\theta^{m-1} b_1(x)\} & \cdot & & \cdots & b_n(q^{n-2}x
\end{array}$$

$$= 0 \ . \qquad (2.32)$$

This gives us $Y(g,h) = 0$.

Hence if P_m and Q_n are commutative given g, we get a common solution Y such that $Y(g,h) = 0$.

Conversely

If for every g, we can find h such that $(P_m - gI)\ \varphi = 0$, $(Q_n - hI)\ \varphi = 0$ have a common solution $Y(g,h)$, then P_m and Q_n are commutative operators.

Proof

Operating on the common solution Y with $P_m\, Q_n - Q_n\, P_m$, we get

$$
\begin{aligned}
(P_m\, Q_n - Q_n\, P_m)\, Y &= P_m\, h\, Y - Q_n\, g\, Y \\
&= ghY - hgY \\
&= 0\ .
\end{aligned}
$$

Therefore $(P_m\, Q_n - Q_n\, P_m)\ \varphi = 0$ has infinitely many solutions $Y(g,h)$, which we have seen, are linearly independent.

It is thus an identity and hence

$$P_m\, Q_n - Q_n\, P_m = 0\ . \tag{2.33}$$

Therefore P_m and Q_n are commutative. Hence the result.

Theorem 5

Any two operators P_m and Q_n are commutative if and only if they are connected by an algebraic identity $F(P_m, Q_n) = 0$ with constant coefficients.

Proof (necessity)

Assume P_m and Q_n are commutative. Then

$$Q_n P_m - P_m Q_n = 0, \text{ and hence}$$

$$A(x, \theta) F = 0 \quad , \text{ by}(2.11, 2.12, 2.13) .$$

Also we get

$$(P_m - gI) \varphi = 0 = (Q_n - hI)\varphi \quad , \text{ by } (2.31) .$$

ie $\quad F(P_m, Q_n)\varphi = F(g,h) \varphi = 0$, by theorem (4) . (2.34)

Hence Y is a solution of $F(P_m, Q_n)\varphi = 0$. (2.35)

Thus the basic difference equation

$$F(P_m, Q_n)\varphi = 0 \tag{2.36}$$

is satisfied by every Y. Since g is arbitrary, there are infinitely many Y's and hence (2.36) is an identity unless the Y's are linearly dependent. Then

$$F(P_m, Q_n) = 0 .$$

Now if Y's are linearly dependent we have

$$\alpha_1 Y_1 + \alpha_2 Y_2 + \ldots + \alpha_s Y_s = 0 , \qquad (2.37)$$

where each Y_s is a solution of $(P_m - g_s I)\varphi = 0$ and the g's are all distinct. Here we take one solution each of every distinct equation $(P_m - g_s I)\varphi = 0$; since the sum $\alpha_1 Y_1 + \alpha_2 Y_2 + \ldots \alpha_s Y_s$, of solutions of the same equation is itself a solution of the equation

$$P_m = \sum_{k=o}^{m} a_k(x)\theta^k, \quad P_m^2 f = \sum_{k=o}^{m} a_k(x) \left[\sum_{j=0}^{m} \sum_{i=0}^{k} \binom{k}{i}_q q^{i(i-k)} \theta^i a_j(q^{k-i}x)\, \theta^{k+j-i}(f) \right] .$$

Therefore

$$P_m{}^2 = \sum_{k=o}^{m} a_k(x) \left[\sum_{j=o}^{m} \sum_{i=o}^{k} \binom{k}{i}_q q^{i(i-k)} \theta^i a_j(q^{k-i}x)\theta^{k+j-i}(.) \right] \text{etc}.$$

Operation with $P_m, P_m{}^2, \ldots, P_m{}^{s-1}$ on (2.37), gives s-1 relations

$$\alpha_1 g_1 Y_1 + \alpha_2 g_2 Y_2 + \ldots + \alpha_s g_s Y_s = 0$$

$$\cdots \quad \cdots \quad \cdots$$

$$\alpha_1 g_1^{s-1} Y_1 + \alpha_2 g_2^{s-1} Y_2 + \ldots + \alpha_s g_s^{s-1} Y_s = 0 .$$

Since no $\alpha_s Y_s$ is zero, the determinant

$$\begin{vmatrix} g_1 & g_2 & \cdots & g_s \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ g_1^{s-1} & g_2^{s-1} & \cdots & g_s^{s-1} \end{vmatrix}$$ must vanish.

But since all g's are distinct this is not zero. Hence $F(P_m, Q_n) = 0$ identically by (2.36).

Sufficiency

Proof

Suppose if possible P_m and Q_n are not commutative.

ie. $\quad Q_n P_m - P_m Q_n \neq 0$

ie. $\quad A(x,\theta)F \neq 0$, from (2.11),(2.12),(2.13)

This implies that there is no relationship between the coefficients of P_m and Q_n. Hence there is no common solution Y such that (2.34) is satisfied.

Hence $F(P_m, Q_n) \neq 0$. Thus sufficiency is established.

<u>Example</u>

$$\text{Consider } P_2 = qx^2\theta^2 + x\theta - I$$

$$Q_3 = q^3x^3\theta^3 + qx^2\theta^2 - (1+q)x\theta + (1+q)I.$$

Here it is seen that

$$P_2Q_3 = Q_3P_2 .$$

ie. P_2 and Q_3 are commutative. Now we find F(g,h) for these operators.

$$\theta^r \, (P_2 - gI)\, \varphi = 0, \quad r = o,1,2$$

$$\theta^s \, (Q_3 - hI)\, \varphi = 0, \quad s = o,1$$

ie. $$qx^2\theta^2\varphi + x\theta\varphi - (1+g)\,\varphi = 0 \tag{2.38}$$

$$q^3x^2\theta^3\varphi + (2q+q^2)\ x\theta^2\varphi - g\theta\varphi = 0 \qquad (2.39)$$

$$q^5x^2\theta^4\varphi + (2q^2+2q^3+q^4)x\theta^3\varphi + (2q+q^2-g)\theta^2\varphi = 0 \qquad (2.40)$$

$$q^3x^3\theta^3\varphi + qx^2\theta^2\varphi - (1+q)x\theta\varphi + (1+q-h)\ \varphi = 0 \qquad (2.41)$$

$$q^6x^3\theta^4\varphi + (2q^3+q^4+q^5)\ x^2\theta^3\varphi - h\theta\varphi = 0 \qquad (2.42)$$

Eliminating $\theta^4\varphi$, $\theta^3\varphi$, $\theta^2\varphi$, $\theta\varphi$, φ from (2.38), (2.39), (2.40), (2.41), (2.42), we get,

$$\begin{vmatrix} 0 & 0 & qx^2 & x & -(1+g) \\ 0 & q^3x^2 & (2q+q^2)x & -g & 0 \\ q^5x^2 & (2q^2+2q^3+q^4)x & (2q+q^2-g) & 0 & 0 \\ 0 & q^3x^3 & qx^2 & -(1+q)x & (1+q-h) \\ q^6x^3 & (2q^3+q^4+q^5)x^2 & 0 & -h & 0 \end{vmatrix} = 0$$

ie. $g^3-(2q+q^2)g^2-h^2-2(1+q)gh = 0$. (2.43)

Thus $F(g,h) = g^3-(2q+q^2)g^2-h^2 - 2(1+q)gh = 0$.

Thus $F(P_2,Q_3) = P_2{}^3-(2q+q^2)P_2{}^2-Q_3{}^2 - 2(1+q)P_2Q_3 = 0$.

Thus we have verified the theorem for this example.

CHAPTER III

SOME SPECIAL BASIC COMMUTATIVE DIFFERENCE OPERATORS AND SOLUTION OF BASIC DIFFERENCE EQUATIONS

In this chapter we define some special basic commutative difference operators P_m and Q_n of order m and n respectively. We obtain new operators $P_m{}'$ and $Q_n{}'$ by transference of the common factors of P_m and Q_n respectively and show that the characteristic relations $F(P_m, Q_n)$ and $F(P_m{}', Q_n{}')$ are primarily the same. Taking polynomial operators $f(\delta)$ in terms of $\delta = x\theta$, we obtain their inverse f). We prove some related theorems and hence solve some basic difference equations.

1. SPECIAL BASIC COMMUTATIVE DIFFERENCE OPERATORS

Let
$$P_m = \theta^m = q^{\frac{-m(m-1)}{2}} x^{-m} \prod_{k=o}^{m-1} (\delta-[k]I) \tag{3.1}$$

and
$$Q_n = \theta^n = q^{\frac{-n(n-1)}{2}} x^{-n} \prod_{k=o}^{n-1} (\delta-[k]I) , \tag{3.2}$$

where $\delta = x\theta$ and $[k] = \frac{1-q^k}{1-q}$.

<u>Lemma</u> $P_m Q_n = Q_n P_m = \theta^{m+n}$

Proof

Let the lemma be true for $Q_r P_m$.

Hence $$Q_r P_m f = q^{-\frac{(m+r)(m+r-1)}{2}} x^{-(m+r)} \prod_{k=o}^{m+r-1} (\delta-[k]I)f$$

$$= \theta^{m+r} f$$

Now $$Q_{r+1} P_m f = \theta \, \theta^{m+r} f$$

$$= q^{-\frac{(m+r)(m+r-1)}{2}} \theta \left[x^{-(m+r)} \prod_{k=o}^{m+r-1} (\delta-[k]I)f \right]$$

$$= q^{-\frac{(m+r)(m+r-1)}{2}} \left[q^{-(m+r)} x^{-(m+r)} \theta \prod_{k=o}^{m+r-1} (\delta-[k]I)f \right.$$

$$\left. - [m+r] \, q^{-(m+r)} x^{-(m+r+1)} \prod_{k=o}^{m+r-1} (\delta-[k]I)f \right]$$

$$= q^{-\frac{(m+r)(m+r-1)}{2}} q^{-(m+r)} \left[x^{-(m+r+1)} \delta \prod_{k=o}^{m+r-1} (\delta-[k]I)f \right.$$

$$\left. - [m+r] \, x^{-(m+r+1)} \prod_{k=o}^{m+r-1} (\delta-[k]I)f \right]$$

$$= q^{-\frac{(m+r)(m+r+1)}{2}} x^{-(m+r+1)} \prod_{k=o}^{m+r-1} (\delta-[k])(\delta-[m+r])f$$

$$= q^{-\frac{(m+r)(m+r+1)}{2}} x^{-(m+r+1)} \prod_{k=o}^{m+r} (\delta-[k]I)f$$

$$= \Theta^{m+r+1} f \ .$$

Hence if the lemma is true for $Q_r P_m$ it is true for $Q_{r+1} P_m$.

Now $Q_1 P_m = \Theta\Theta^m f = \Theta [q^{-\frac{m(m-1)}{2}} x^{-m} \prod_{k=o}^{m-1} (\delta-[k]I)]f$

$$= q^{-\frac{m(m-1)}{2}} \Theta [x^{-m} \prod_{k=o}^{m-1} (\delta-[k]I)f$$

$$= q^{-\frac{m(m-1)}{2}} [q^{-m} x^{-m} \Theta \prod_{k=o}^{m-1} (\delta-[k]I)f$$

$$- [m]q^{-m} x^{-(m+1)} \prod_{k=o}^{m-1} (\delta-[k]I)f]$$

$$= q^{-\frac{m(m-1)}{2}} q^{-m}[x^{-(m+1)} \delta \prod_{k=o}^{m-1} (\delta-[k]I)f$$

$$-[m] x^{-(m+1)} \prod_{k=o}^{m-1} (\delta-[k]I)f]$$

$$= q^{-\frac{m(m+1)}{2}} x^{-(m+1)} \prod_{k=o}^{m-1} (\delta-[k]I)(\delta-[m]I)f$$

$$= q^{\frac{-m(m+1)}{2}} x^{-(m+1)} \prod_{k=o}^{m} (\delta-[k]I)f$$

$$= \theta^{m+1}f .$$

Hence the lemma is true for all n

ie. $Q_n P_m = \theta^{m+n}$ is true for all m and n .

Similarly we can prove that

$P_m Q_n = \theta^{m+n}$ for all m and n.

Hence the result,

Also $\theta\ \theta^2 f = x^{-1}\delta q^{-1}x^{-2}\delta(\delta-1)f$

$$= x^{-1}x\theta q^{-1}x^{-2}x\theta(x\theta f-f)$$

$$= \theta q^{-1}x^{-1}\ [qx\theta^2 f+\theta f-\theta f]$$

$$= \theta\ \theta^2 f$$

$$= \theta^3 f$$

The result follows by induction.

<u>Characteristic identity of P_m and Q_n</u>

We have $P_m = \theta^m$, $Q_n = \theta^n$.

Consider the basic difference equations

$$\theta^r(P_m - gI)f = 0, \quad r = 0,1,\ldots, n-1$$

$$\theta^s(Q_n - hI)f = 0, \quad s = 0,1,\ldots, m-1.$$

Hence

$$\left.\begin{array}{l}
\theta^m f - gf = 0 \\
\theta^{m+1} f - g\theta f = 0 \\
\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots \\
\theta^{m+n-1} f - g\theta^{n-1} f = 0 \\
\theta^n f - hf = 0 \\
\theta^{n+1} f - h\theta f = 0 \\
\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots \\
\theta^{m+n-1} - h\theta^{m-1} f = 0
\end{array}\right\} \qquad (3.3)$$

Eliminating f, θf, $\theta^2 f$ ... $\theta^{m+n-1}f$ from equations (3.3) we get that the determinant

$$\begin{vmatrix} -g & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -g & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -g & 0 \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \cdots & & & -g & 0 & \cdots & 1 \\ -h & 0 & 0 & \cdots & \cdots & & & 0 & 1 & 0 \cdots & 0 \\ 0 & -h & 0 & \cdots & \cdots & & & 0 & 0 & 1 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & \cdots & \cdots & & & \cdot & -h & 0 \cdots & 1 \end{vmatrix} =$$

Expanding we get

$$g^n - h^m = 0 .$$

ie. $$F(g,h) = g^n - h^m . \qquad (3.4)$$

Therefore $$F(P_m, Q_n) = P_m^n - Q_n^m,$$ by (2.34)

2. DERIVATION OF NEW OPERATORS BY TRANSFERENCE OF COMMON FACTORS

$$P_m = \theta^m = q^{\frac{-m(m-1)}{2}} x^{-m} \prod_{k=o}^{m-1} (\delta-[k]I)$$

$$Q_n = \theta^n = q^{\frac{-n(n-1)}{2}} x^{-n} \prod_{k=o}^{n-1} (\delta-[k]I)$$

Let $m < n$.

Then we see that the operators δ, $(\delta-1)$, $(\delta-[2])$, ..., $(\delta-[m-1])$ are common for both.

Hence we can transfer any one of these operators to the left end of P_m and Q_n. This transformation leads us to new operators. Hence given a pair P_m, Q_n; we get m pair of new operators. Let them be $P_m{}'$, $Q_n{}'$; $P_m{}^2$, $Q_n{}^2$, ..., $P_m{}^m$, $Q_n{}^m$.

Now we transfer δ to the left and find $P_m{}'$, $Q_n{}'$.

$$P_m = q^{\frac{-m(m-1)}{2}} x^{-m} \delta(\delta-1)(\delta-[2]I)\ldots(\delta-[m-1]I)$$

Therefore

$$P_m' = q^{\frac{-m(m-1)}{2}} \delta x^{-m}(\delta-1)(\delta-[2]I) \ldots (\delta-[m-1]I)$$

$$= q^{\frac{-m(m-1)}{2}} x\theta x^{-m}(\delta-1)(\delta-[2]I)\ldots (\delta-[m-1]I)$$

$$P_m'f = q^{\frac{-m(m-1)}{2}} x\theta[x^{-m} \prod_{k=1}^{m-1} (\delta-[k]I)]f$$

$$= q^{\frac{-m(m-1)}{2}} x [q^{-m} x^{-m} \theta \prod_{k=1}^{m-1} (\delta-[k]I)f$$

$$- [m] q^{-m} x^{-(m+1)} \prod_{k=1}^{m-1} (\delta-[k]I)f]$$

$$= q^{\frac{-m(m-1)}{2}} q^{-m} x^{-m} \prod_{k=1}^{m-1} (\delta-[k]I)(\delta-[m]I) f$$

$$= q^{\frac{-m(m+1)}{2}} x^{-m} \prod_{k=1}^{m} (\delta-[k]I)f ,$$

Hence

$$P_m' = q^{\frac{-m(m+1)}{2}} x^{-m} \prod_{k=1}^{m}(\delta-[k] I)$$

$$Q_n' = q^{\frac{-n(n+1)}{2}} x^{-n} \prod_{k=1}^{n}(\delta-[k]I).$$

The other pairs also can be obtained similarly.

Now we find the characteristic identity of P_m', Q_n'.

Consider the basic difference equations

$$\theta^r(P_m' - g'I)f = 0, \quad r = 0, 1, \ldots, n-1$$

$$\theta^s(Q_n' - h'I)f = 0, \quad s = 0, 1, \ldots, m-1 .$$

Eliminating $f, \theta f, \ldots, \theta^{m+n-1}f$ from these $m+n$ basic difference equations, we get the same form as (3.4).

$$\text{Hence} \quad F(g', h') = g'^n - h'^m.$$

$$\text{Therefore } F(P_m, Q_n) = F(P_m', Q_n')$$

$$\text{Similarly } F(P_m', Q_n') = F(P_m'', Q_n'') \ldots.$$

$$= F(P_m^m, Q_n^m) = F(P_m, Q_n)$$

$$= P_m^n - Q_n^m.$$

Hence a set of basic difference equations is generated by the same characteristic identity.

Example

Let $m = 2,\ n = 3$

$$P_2 = \theta^2,\ Q_3 = \theta^3$$

Hence $\theta^r(P_2-gI)\ f = 0,\ r = 0,1,2,$

$\theta^s(Q_3-hI)\ f = 0,\ s = 0,1$ give

$$\theta^2 f - gf = 0 \qquad (3.5)$$

$$\theta^3 f - g\theta f = 0 \qquad (3.6)$$

$$\theta^4 f - g\theta^2 f = 0 \qquad (3.7)$$

$$\theta^3 f - hf = 0 \qquad (3.8)$$

$$\theta^4 f - h\theta f = 0 \ . \qquad (3.9)$$

Eliminating f, θf, $\theta^2 f$, $\theta^3 f$, $\theta^4 f$ from (3.5), (3.6), (3.7), (3.8), (3.9) we have

$$\begin{vmatrix} -g & 0 & 1 & 0 & 0 \\ 0 & -g & 0 & 1 & 0 \\ 0 & 0 & -g & 0 & 1 \\ -h & 0 & 0 & 1 & 0 \\ 0 & -h & 0 & 0 & 1 \end{vmatrix} = 0$$

ie $$g^3 - h^2 = 0 \ . \qquad (3.10)$$

Now $P_2 = \theta^2 \quad = q^{-1}\ x^{-2}\ \delta(\delta-1)$

$Q_3 = \theta^3 \quad = q^{-3}\ x^{-3}\ \delta(\delta-1)(\delta-[2])$

Transferring δ-1, we get

$$P_2' = q^{-1}(\delta-1)x^{-2}\,\delta = q^{-2}\theta^2-(q^{-2}+q^{-1})x^{-1}\theta$$

$$Q_3' = q^{-3}(\delta-1)x^{-3}\,\delta(\delta-[2])$$

$$= q^{-3}\theta^3-(q^{-2}+q^{-3}+q^{-4})x^{-1}\theta^2+(q^{-2}+q^{-3}+q^{-4})x^{-2}\theta$$

Now $\theta^r(P_2'-g'I)f = 0, \quad r = 0,1,2$

$\theta^s(Q_3'-h'I)f = 0, \quad s = 0,1$

give $q^{-2}\theta^2 f - (q^{-2}+q^{-1})x^{-1}\theta f - g'f = 0$ (3.11)

$q^{-2}\theta^3 f - (q^{-2}+q^{-3})x^{-1}\theta^2 f + \left\{(q^{-2}+q^{-3})x^{-2}-g^1\right\}\theta f = 0$ (3.12)

$q^{-2}\theta^4 f - (q^{-3}+q^{-4})x^{-1}\theta^3 f + \left\{(q^{-3}+q^{-4})x^{-3}\right.$

$\left.(q^{-4}+q^{-5})x^{-2}-g^1\right\}\theta^2 f - (q^{-3}+2q^{-4}+q^{-5})x^{-3}\theta f = 0$ (3.13)

$q^{-3}\theta^3 f - (q^{-2}+q^{-3}+q^{-4})x^{-1}\theta^2 f$

$+ (q^{-2}+q^{-3}+q^{-4})x^{-2}\theta f - h'f = 0$ (3.14)

$q^{-3}\theta^4 f - (q^{-3}+q^{-4}+q^{-5})x^{-1}\theta^3 f$

$+ (q^{-3}+2q^{-4}+2q^{-5}+q^{-6})x^{-2}\theta^2 f$

$- (q^{-4}+2q^{-5}+2q^{-6}+q^{-7})x^{-3}\theta f - h'\theta f = 0$. (3.15)

Eliminating f, θf, $\theta^2 f$, $\theta^3 f$, $\theta^4 f$ from (3.11), (3.12), (3.13), (3.14), (3.15) we get that the determinant

$$\begin{vmatrix} -g' & -(q^{-2}+q^{-1})x^{-1} & q^{-2} & 0 \\ 0 & \left\{(q^{-2}+q^{-3})x^{-2}-g'\right\} & -(q^{-2}+q^{-3})x^{-1} & q^{-2} \\ 0 & -(q^{-3}+2q^{-4}+q^{-5})x^{-3} & \left\{(q^{-3}+q^{-4})x^{-3}+(q^{-4}+q^{-5})x^{-2}-g'\right\} & -(q^{-3}+q^{-4})x^{-1} \\ -h' & (q^{-2}+q^{-3}+q^{-4})x^{-2} & -(q^{-2}+q^{-3}+q^{-4})x^{-1} & q^{-3} \\ 0 & \left\{(q^{-4}+2q^{-5}+2q^{-6}+2q^{-7})x^{-3}-h'\right\} & \left\{q^{-3}+2q^{-4}+2q^{-5}+q^{-6}\right\}x^{-2} & -(q^{-3}+q^{-4}+q^{-5}) \end{vmatrix}$$

vanishes.

Expanding this determinant we get

$$g'^3 - h'^2 = 0 .$$

Hence $\qquad F(P_2', Q_3') = F(P_2, Q_3)$ from (3.10) .

3. POLYNOMIAL OPERATORS AND THE INVERSE OPERATOR

We see that $\qquad \delta = x\theta,\ \delta^2 = \delta\delta = (x\theta)(x\theta)$.

Therefore, $\qquad \delta^2 \neq x^2\theta^2,\ x^n\theta^n \neq \delta^n$

Now we consider the polynomial operator

$$f(\delta) = \sum_{k=o}^{m} a_k\delta^k, \ a_k \text{ constants} .$$

<u>Theorem 1</u>

$$f(\delta)x^a = f([a])\, x^a \text{ for every } a \qquad (3.16)$$

<u>Proof</u>

$$\delta x^a = x\theta x^a$$

$$= x[a]x^{a-1}$$

$$= [a]x^a . \qquad (3.17)$$

We assume $\delta^m x^a = [a]^m x^a$.

Therefore $\delta^{m+1}x^a = \delta[[a]^m x^a]$

$$= [a]^m x\theta x^a$$

$$= [a]^{m+1}x^a. \qquad (3.18)$$

By induction the result is true for all m. Therefore,

$$f(\delta)x^a = [a_oI+a_1\delta+a_2\delta^2+ \ldots + a_m\delta^m]x^a$$

$$= a_ox^a + a_1[a]x^a+\ldots + a_m[a]^m x^a,$$

from (3.17) and (3.18)

$$= [a_oI+a_1[a]+ a_2[a]^2+ \ldots + a_m[a]^m]x^a$$

$$= f([a])x^a.$$ Hence the result.

Theorem 2

$$f(\delta^{-1})x^a = f(1/[a])x^a, \quad a > o . \qquad (3.19)$$

Proof

$$\delta^{-1}x^a = \theta^{-1} x^{-1} x^a$$

$$= \theta^{-1} x^{a-1}$$

$$= \int x^{a-1} d(q,x)$$

$$= (1-q)x \sum_{k=o}^{\infty} q^k(q^k x)^{a-1} \quad \text{by } (1.20)$$

$$= (1-q)x^a \sum_{k=o}^{\infty} (q^a)^k .$$

Hence

$$\delta^{-1} x^a = \frac{(1-q)x^a}{1-q^a}$$

$$= \frac{x^a}{[a]} . \qquad (3.20)$$

Assume

$$\delta^{-n} x^a = \frac{x^a}{[a]^n} .$$

Therefore

$$\delta^{-(n+1)}x^a = \delta^{-1}(\delta^{-n} x^a)$$

$$= \delta^{-1} (x^a/([a]^n))$$

$$= 1/([a]^n) \theta^{-1}x^{-1}x^a$$

$$= 1/([a]^n)\theta^{-1}x^{a-1}$$

$$= 1/([a]^n)(x^a/([a]))$$

$$= x^a/([a]^{n+1}) \qquad (3.21)$$

By induction, we get the result. Therefore if

$$f(\delta^{-1}) = (a_o I + a_1\delta^{-1} + \ldots + a_n\delta^{-n})$$

Then $$f(\delta^{-1})x^a = (a_o I + a_1\delta^{-1} + a_2\delta^{-2} + \ldots + a_n\delta^{-n})x^a$$

$$= \left\{(a_o I + a_1(1 / [a]) + a_2(1 / [a]^2) + \ldots + a_n(1 / [a]^n)\right\} x^a$$

$$= f(1 / [a])x^a \text{ from (3.20) and (3.21)}$$

<u>Theorem 3</u>

$$f(\delta)(x^a y) = x^a f(q^a\delta + [a])y$$

<u>Proof</u>

$$\delta(x^a y) = x\Theta(x^a y)$$

$$= x[q^a x^a \Theta y + [a]\ x^{a-1} y]$$

$$= x^a[q^a\delta + [a]\ ]y$$

. . . .

$$\delta^n(x^a y) = x^a[q^a\delta + [a]]^n\ y$$

$[q^a\delta + [a]]^n$ means $[q^a\delta+[a]]\ [q^a\delta+[a]]\ \ldots$ operating n times on y. Therefore

$$f(\delta)\ (x^a y) = [a_o I + a_1\delta + a_2\delta^2 + \ldots + a_n\delta^n]\ (x^a y)$$

$$= \left\{ a_o x^a + a_1 x^a[q^a\delta+[a]] + a_2 x^a[q^a\delta+[a]]^2 + \ldots + a_n x^a[q^a\delta + [a]]^n \right\} y$$

$$= x^a f(q^a\delta + [a])y.$$

Hence the result.

<u>Theorem 4</u>

If $f_1, f_2, \ldots, f_n$ are polynomials with constant coefficients, then

$$x^{a_n} f_n(\delta) x^{a_{n-1}} f_{n-1}(\delta) \ \ldots \ x^{a_2} f_2(\delta) x^{a_1} f_1(\delta)$$

$$= x^{a_1+a_2+\ldots+a_n} f_1(\delta)\ f_2(q^{a_1}\delta+[a_1])$$

$$f_3(q^{a_2+a_1}\delta+[a_1+a_2]) \ \ldots\ldots\ldots$$

$$f_n(q^{a_1+a_2+\ldots+a_{n-1}}\delta + [a_1+a_2+\ldots+a_{n-1}]) \qquad (3.22)$$

Proof

In theorem 3 we have proved that

$$f(\delta)\ (x^a y) = x^a f(q^a\delta+[a])y$$

Now we apply this result from right to left on the left side of the required equality. Therefore

$$x^{a_n} f_n(\delta)\ x^{a_{n-1}} f_{n-1}(\delta) x^{a_{n-2}} f_{n-2}(\delta)\ \dots\ x^{a_2} f_2(\delta)\ x^{a_1} f_1(\delta)$$

$$= x^{a_n} f_n(\delta) x^{a_{n-1}} f_{n-1}(\delta) x^{a_{n-2}} f_{n-2}(\delta)\ \dots\ x^{a_1+a_2} f_1(\delta) f_2(q^{a_1}\delta+[a_1])$$

$$= x^{a_n} f_n(\delta) x^{a_{n-1}} f_{n-1}(\delta)\ \dots\ x^{a_1+a_2+a_3} f_1(\delta) f_2(q^{a_1}\delta+[a_1])$$

$$f_3(q^{a_1+a_2}\delta+[a_1+a_2])$$

$$\dots\dots\dots\dots\dots\dots$$

$$= x^{a_1+a_2+a_3+\dots+a_n} f_1(\delta) f_2(q^{a_1}\delta+[a_1]) f_3(q^{a_1+a_2}\delta+[a_1+a_2])$$

$$\dots\dots\dots\ f_n(q^{a_1+a_2+\dots+a_{n-1}}\delta+[a_1+a_2+\dots+a_{n-1}])$$

Hence the result.

Corollary (1)

Put $a_r = a,\ r = 1,2,\ \ldots,\ n$

$f_r(\delta) = f(\delta)$ then (3.22) reduces to

$$[x^a f(\delta)]^n = x^{na} f(\delta)\ f(q^a\delta+[a])\ \ldots\ldots\ldots$$
$$f(q^{(n-1)a}\delta + [(n-1)]) \tag{3.23}$$

Corollary (2)

Put $a = -1$ and $f(\delta) = \delta$ in (3.23)

Then (3.23) reduces to

$$(x^{-1}\delta)^n = x^{-n}\delta(q^{-1}\delta+[-1])(q^{-2}\delta+[-2]\ \ldots$$
$$(q^{-(n-1)}\delta+\ [-n(n-1)])$$

$$= x^{-n}\ q^{\frac{-n(n-1)}{2}}\ \delta(\delta-1)\ \ldots\ (\delta-[n-1])$$

Since

$$q^{-1}\delta + [-1] = q^{-1}(\delta-1)$$

$$q^{-2}\delta + [-2] = q^{-2}\delta - q^{-2}[2]$$
$$= q^{-2}(\delta-[2])$$ and so on.

We see that

$$x^{-1}\delta \;=\; x^{-1}\,x\Theta = 0$$

Therefore (3.24) reduces to

$$\Theta^{n} \;=\; q^{\frac{-n(n-1)}{2}}\,x^{-n}\,\delta(\delta-1)\,..\,(\delta-[n-1]I)$$

$$=\; q^{\frac{-n(n-1)}{2}}\,x^{-n}\prod_{k=o}^{n-1}(\delta-[k]I)$$

This is the same as (3.1) and (3.2). Hence by using theorem (4) we can form basic commutative difference operators easily.

<u>A commutative triad</u>

$$P_3 \;=\; q^{-6}x^{-3}\,\delta(\delta-1)(\delta-[5]I)$$

$$Q_4 \;=\; q^{-10}x^{-4}\delta(\delta-1)(\delta-[3]I)(\delta-[6]I)$$

$$R_5 \;=\; q^{-5}x^{-5}\delta(\delta-1)(\delta-[3]I)(\delta-[4]I)(\delta-[7]I)$$

We find that P_3, Q_4, R_5 form a commutative triad.

Also

$$P_3^4 = Q_4^3$$

$$Q_4^5 = R_5^4$$

$$R_5^3 = P_3^5$$

$$P_3R_5 = Q_4^2$$

$$Q_4R_5 = P_3^3$$

Example

Let $P = q^{-3}x^{-2}\delta(\delta-[3])$

$Q = q^{-6}x^{-3}\delta(\delta-[2])(\delta-[4])$

Then $PQ = q^{-3}x^{-2}\delta(\delta-[3])q^{-6}x^{-3}\delta(\delta-[2])(\delta-[4])$

$= q^{-9}x^{-2}\delta x^{-3}(q^{-3}\delta-[3]+[-3])\delta(\delta-[2])(\delta-[4])$

Now $(q^{-3}\delta-[3]+[-3]) = q^{-3}\delta-[3] + \left(\frac{1-q^{-3}}{1-q}\right)$

$= q^{-3}\delta - \left(\frac{1-q^3}{1-q}\right)-q^{-3}\left(\frac{1-q^3}{1-q}\right)$

$= q^{-3}\delta - \left(\frac{1-q^3}{1-q}\right)(1+q^{-3})$

$= q^{-3}(\delta-[6])$

Therefore

$$PQ = q^{-9}x^{-5}(q^{-3}\delta+[-3])q^{-3}(\delta-[6])\delta(\delta-[2])(\delta-[4])$$

$$= q^{-9}x^{-5}q^{-3}(\delta-[3])(\delta-[6])\delta(\delta-[2])(\delta-[4])$$

$$= q^{15}x^{-5}\delta(\delta-[2])(\delta-[3])(\delta-[4])(\delta-[6])$$

$$= QP$$

Also we see that

$$P^3 = q^{-21}x^{-6}\delta(\delta-[2])(\delta-[3])(\delta-[4])(\delta-[5])(\delta-[7])$$

$$= Q^2$$

4. SOLUTION OF BASIC DIFFERENCE EQUATIONS

a) Consider the difference equation

$$\theta^m f = 0$$

ie. $$q^{\frac{-m(m-1)}{2}} x^{-m} \prod_{k=o}^{m-1} (\delta-[k]I)f = 0$$

ie. $$\prod_{k=o}^{m-1} (\delta-[k]I)f(x) = 0$$

Let $\sum_{k=o}^{m-1} f_k(x)$ be the solution of $\Theta^m f = 0$

Therefore

$$(\delta-[k])f_k(x) = 0$$

ie. $$x\Theta f_k(x) = \left(\frac{1-q^k}{1-q}\right)f_k(x)$$

ie. $$x\left[\frac{f_k(x)-f_k(qx)}{(1-q)x}\right] = \left(\frac{1-q^k}{1-q}\right)f_k(x)$$

or $$f_k(x)-f_k(qx) = (1-q^k)f_k(x)$$

or $$f_k(qx) = \left\{1-(1-q^k)\right\} f_k(x)$$

or $$f_k(qx) = q^k f_k(x)$$

Hence $$f_k(x) = c_k x^k$$

Hence in this case polynomial solutions are possible upto any degree.

b) Consider $(\delta-\lambda x)\ f(x) = 0$

ie $$f(x)-f(qx)-(1-q)\lambda x f(x) = 0$$

or $$\left\{1-(1-q)x\lambda\right\} f(x) = f(qx)$$

Therefore $\quad f(x) = \dfrac{f(qx)}{\{1-x(1-q)\lambda\}}$

or $\quad f(x) = \dfrac{C}{\prod_{r=o}^{\infty}[1-x(1-q)^{-1}\lambda q^{r}]}$

$$= e_q\{(1-q)\lambda x\}$$

converges for all x since $o < q < 1$.

c) Consider the basic difference equation

$$qx^2\theta^2 f(x) + x\theta f(x) - f(x) = x^3$$

Now by (3.1)

$$qx^2\theta^2 = qx^2q^{-1}x^{-2}\delta(\delta-I)$$

$$= \delta^2-\delta$$

So we have $\quad (\delta^2-I)f(x) = x^3$

Therefore $\quad f(x) = (\delta^2-1)^{-1}x^3$

$$= \frac{x^3}{[3]^2-1} \quad \text{by (3.16)}$$

Hence $\quad f(x) = \dfrac{x^3}{(1+q+q^2)^2-1}$

d) Consider the basic difference equation

$$q\theta^2 f(x) = (qx^2\theta^2 + x\theta - m^2)\, f(x)$$

Multiplying both sides by x^2 we have

$$qx^2\theta^2 f = x^2(qx^2\theta^2 + x\theta - m^2) f$$

or $$\delta(\delta-1)f = x^2 \left\{ \delta(\delta-1) + \delta - m^2 \right\} f$$

or $$\delta(\delta-1)f = x^2\, (\delta^2 - m^2) f \qquad (3.25)$$

Let $f = f_o + f_1 + f_2 + \ldots$ be a solution. Then (3.25) gives

$$\delta(\delta-1)f_o = 0 \qquad (3.26)$$

$$\delta(\delta-1)f_1 = x^2(\delta^2 - m^2) f_o \qquad (3.27)$$

$$\delta(\delta-1)f_2 = x^2(\delta^2 - m^2) f_1 \qquad (3.28)$$

$$\cdots\cdots\cdots\cdots$$

$$\delta(\delta-1)f_{r+1} = x^2(\delta^2 - m^2) f_r$$

Now we solve $\delta(\delta-1)f_o = 0$ by the power series method and consider the particular case

$$f_o = 1 \text{ or } x \text{ (say)}$$

We substitute $f_o = x$ in (3.27)

ie. $\delta(\delta-1)f_1 = x^2(\delta^2-m^2)x$

$= x^3(1-m^2)$ by (3.16)

$= (1-m^2)x^3$

Therefore $f_1 = \delta^{-1}(\delta-1)^{-1}(1-m^2)x^3$

$= (1-m^2)\delta^{-1}(\delta-1)^{-1}x^3$

$$= \frac{(1-m^2)x^3}{[3]\ [2]} \text{ by (3.16)} \qquad (3.29)$$

Substituting (3.29) in (3.28) we have

$$\delta(\delta-1)f_2 = \frac{x^2(\delta^2-m^2)(1-m^2)x^3}{[3]\ [2]}$$

$$= \frac{x^5([3]^2-m^2)(1-m^2)}{[3]\ [2]} \text{ by (3.16)}$$

$$= \frac{(1-m^2)([3]^2-m^2)x^5}{[3]\ [2]}$$

Therefore $$f_2 = \frac{(1-m^2)([3]^2-m^2)}{[3]\ [2]}\delta^{-1}(\delta-1)^{-1}x^5$$

$$= \frac{(1-m^2)([3]^2-m^2)x^5}{[2]\,[3]\,[4]\,[5]} \quad \text{by (3.16)}$$

And so on

$$f_r = \frac{(1-m^2)([3]^2-m^2)\ \ldots\ ([2r-1]^2-m^2)\ x^{2r+1}}{[\ 2r+1\ ]!}$$

Therefore

$$f = f_o + f_1 + f_2 + \ldots + f_r + \ldots \text{ gives}$$

$$f(x) = x + \frac{(1-m^2)x^3}{[3]!} + \frac{(1-m^2)([3]^2-m^2)x^5}{[5]!} + \ldots$$

$$+ \ldots + \frac{(1-m^2)([3]^2-m^2)\ \ldots([2r-1]^2-m^2)x^{2r+1}}{[2r+1]!}$$

$$+ \ldots$$

If we take $f_o = 1$

Then $f_1 = \dfrac{-m^2x^2}{[2]!}$

$$f_2 = \frac{-m^2([2]^2-m^2)x^4}{[4]!}$$

. . . .

$$f_r = \frac{-m^2([2]^2-m^2) \ldots ([2r-2]^2-m^2)x^{2r}}{[2r]!}$$

Therefore $$f(x) = 1 - \frac{m^2x^2}{[2]!} - \frac{m^2([2]^2-m^2)x^4}{[4]!} \quad \ldots\ldots$$

$$- \frac{m^2([2]^2-m^2) \ldots ([2r-2]^2-m^2)x^{2r}}{[2r]!}$$

$$\ldots\ldots\ldots$$

5. GENERAL HYPERGEOMETRIC EQUATION

Consider the equation

$$f(\delta)\varphi = x^h g(\delta)\varphi$$

Suppose $$f(\delta)\varphi = f([a])x^a + x^h g(\delta)\varphi$$

The equations in the solution by successive approximation are now

$$f(\delta)\varphi_o = f([a])x^a$$

$$f(\delta)\varphi_1 = x^h g(\delta)\varphi_o$$

$$f(\delta)\varphi_2 = x^h g(\delta)\varphi_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(\delta)\varphi_n = x^h g(\delta)\varphi_{n-1}$$

Solving these we have

$$\varphi_o = x^a$$

which gives $f(\delta)\varphi_1 = x^h g(\delta)x^a$

$$= x^h g([a])x^a \quad \text{from (3.16)}$$

$$= x^{a+h}g([a])$$

Therefore $\varphi_1 = f^{-1}(\delta)x^{a+h}g([a])$

$$= g([a])f^{-1}(\delta)x^{a+h}$$

$$\varphi_1 = \frac{g([a]x^{a+h}}{f([a+h])} \quad \text{by (3.16)}$$

Then $f(\delta)\varphi_2 = x^h g(\delta)\varphi_1$

$$= \frac{x^h g(\delta)x^{a+h}g([a])}{f([a+h])}$$

$$= \frac{x^{a+2h}g([a])g([a+h])}{f([a+h])} \quad \text{by (3.16)}$$

Therefore $\varphi_2 = \dfrac{x^{a+2h}g([a])g([a+h])}{f([a+h])f([a+2h])}$ by (3.16)

And so on

$$\varphi_n = \frac{g([a])g([a+h)\ldots g([a+(n-1)h])x^{a+nh}}{f([a+h])f([a+2h])\ldots f([a+nh])}$$

or

$$\varphi_n = \prod_{r=1}^{n} \frac{g([a+(r-1)h]x^{a+nh}}{f([a+rh])}$$

write

$$s_n(a) = \varphi_o + \varphi_1 + \ldots + \varphi_n$$

$$= x^a + \frac{g([a])x^{a+h}}{f([a+h])} + \frac{g([a])g([a+h])x^{a+2h}}{f([a+h])f([a+2h])}$$

$$+ \ldots + \frac{g([a])g([a+h])\ldots g([a+(n-1)h]x^{a+nh}}{f([a+h])\ \ldots f([a+nh])}$$

Therefore

$$f(\delta)s_n(a) = f([a])x^a + x^h g(\delta)s_{n-1}(a)$$

Therefore

$$\left\{f(\delta)-x^h g(\delta)\right\} s_n(a) = f([a])x^a - x^h g(\delta)\varphi_n$$

If $f([a]) = 0$

Then $\left\{ f(\delta)-x^h g(\delta) \right\} s_n(a)$

$$= \frac{-g([a])\ldots g([a+nh])x^{a+(n+1)h}}{f([a+h])\ldots f([a+nh])}$$

If $f([a]) = 0$ we get $g([a]) = 0$

$\ldots = g([a+nh]) = 0$ by (3.69)

Then $\left\{ f(\delta)-x^h g(\delta) \right\} s_n(a) = 0$

or $\varphi = s_n(a)$ is a solution of

$$f(\delta)\varphi = x^h g(\delta)\varphi$$

If $m \longrightarrow \infty$, then we get

$$s(a) = x^a + \frac{g([a])x^{a+h}}{f([a+h])} + \ldots \qquad (3.30)$$

or $\varphi = s(a)$ is a solution.

If $f([a]) = 0$ has distinct roots, we may get many solutions.

Convergence

To find the convergence of the sequence of these solutions we should know $f(\delta)$ and $g(\delta)$.

Suppose $$f(\delta) = f_o\delta^k + f_1\delta^{k-1} + \dots$$

$$g(\delta) = g_o\delta^j + g_1\delta^{j-1} + \dots$$

Therefore $$\frac{\varphi_n}{\varphi_{n-1}} = \frac{g([a+(n-1)h])x^h}{f([a+nh])}$$

or $$\frac{\varphi_n}{\varphi_{n-1}} = \left(\frac{g_o}{f_o}\right)x^h[nh]^{j-k}$$

which is convergent when

$$\left|\frac{\varphi_n}{\varphi_{n-1}}\right| < 1 \quad \text{ie. if } k > j$$

Hence $$\frac{\varphi_n}{\varphi_{n-1}} \longrightarrow 0 \quad \text{in this case}$$

Therefore the solution (3.30) is convergent for all finite x^h.

And $$\frac{\varphi_n}{\varphi_{n-1}} \longrightarrow \left(\frac{g_o}{f_o}\right) x^h \text{ if } k = j$$

Therefore it is convergent if

$$|x^h| < \left| \frac{f_o}{g_o} \right|$$

and if $k < j$ it is divergent except at $x^h = 0$

Singularities

When $k = j$, the exceptional points are given by

$$|x^h| = \left| \frac{f_o}{g_o} \right|$$

Let $h = 1$ and $x = Mx'$ where

$$M = \left(\frac{f_o}{g_o}\right)^{\frac{1}{h}}$$

If we take f_o, $g_o = 1$, then f, g are of the forms

$$f(\delta) = \delta^k + f_1\delta^{k-1} + f_2\delta^{k-2} + \ldots$$

$$g(\delta) = \delta^k + g_1\delta^{k-1} + \ldots \text{ since } k = j$$

The limits of convergence of the series are now $|x| = 1$.

Theorem 5:

If z is a solution of the basic difference equation

$$q^{-a}(\delta-[a])f(\delta)z = x^m g(\delta)z, \tag{3.31}$$

then
$$\varphi = \prod_{s=0}^{r-1} q^{\frac{-r[2a+(r-1)m]}{2}} \left\{\delta-[a+sm]\right\} \tag{3.32}$$

is a solution of the basic difference equation

$$q^{-(a+rm)}\left\{\delta-[a+rm]\right\} f(\delta)\varphi = x^m g(\delta)\varphi \tag{3.33}$$

Proof
$$q^{-(a+rm)}\left\{\delta-[a+rm]\right\} f(\delta)\varphi$$

$$= {}^{-(a+rm)}\left\{\delta-[a+rm]\right\} f(\delta)$$

$$\prod_{s=0}^{r-1} q^{\frac{-r\left\{2a+(r-1)m\right\}}{2}}\left\{(\delta-[a+sm])\right\} z \text{ by (3.32)}$$

$$= q^{-(a+rm) - \frac{r\left\{2a+(r-1)m\right\}}{2}} \left\{\delta-[a+rm]\right\}$$

$$\prod_{s=0}^{r-1} \left\{\delta-[a+sm]\right\} \quad f(\delta)z.$$

$$= q^{\frac{-(r+1)\left\{2a+rm\right\}}{2}} \prod_{s=0}^{r} \left\{\delta-[a+sm]\right\} f(\delta)z$$

$$= q^{\frac{-r\left\{2a+(r+1)m\right\}}{2}} q^{-a}(\delta-[a]) \prod_{s=1}^{r} \left\{\delta-[a+sm]\right\} f(\delta)z$$

$$= q^{\frac{-r\left\{2a+(r+1)m\right\}}{2}} \prod_{s=1}^{r} \left\{\delta-[a+sm]\right\} q^{-a}(\delta-[a])f(\delta)z$$

$$= q^{\frac{-r\left\{2a+(r+1)m\right\}}{2}} \prod_{s=1}^{r} \left\{\delta-[a+sm]\right\} x^m g(\delta)z \text{ from } (3.31)$$

$$= x^m q^{\frac{-r\left\{2a+(r-1)m\right\}}{2}} \prod_{s=0}^{r-1} \left\{\delta-[a+sm]\right\} g(\delta)z$$

$$= x^m g(\delta)\varphi \text{ from } (3.32)$$

Hence the theorem.

CHAPTER IV

BASIC ADJOINT DIFFERENCE OPERATORS

Associated with the theory of basic commutative difference operators is that of basic adjoint operators which are also of importance in the theory of basic difference equations. Hence we define the basic adjoint difference operator P_m^* of P_m and establish some of their properties analogous to Chaundy [1]. We also define basic normal difference operators and basic self adjoint operators on the lines of Coddington [1] and construct some examples. And we derive the result that if an operator P_m commutes with a first order operator Q_1 it is a polynomial in terms of Q_1.

1. Definitions

a) Basic adjoint difference operator

If $P_m = \sum_{k=o}^{m} a_k(x)\theta^k(..)$, then its basic adjoint P_m^* is defined as

$$P_m^* = \sum_{k=o}^{m} (-1)^k \theta^k a_k(x)\ (..) \tag{4.1}$$

b) Basic normal operator

An operator P_m is called basic normal if

$$P_m P_m^* \quad = \quad P_m^* P_m \quad \text{in the sense that}$$

$$P_m P_m^* f \quad = \quad P_m^* P_m f$$

c) Basic self-adjoint operator

An operator P_m is called basic self-adjoint if $P_m = P_m^*$.

2. BASIC NORMAL OPERATORS

Theorem 1

In general $P_m P_m^* \neq P_m^* P_m$

Proof

$$P_m \quad = \quad \sum_{k=o}^{m} a_k(x)\theta^k(..)$$

$$P_m^* \quad = \quad \sum_{k=o}^{m} (-1)^k \theta^k[a_k(x)(..)]$$

$$P_m P_m^* f = [\sum_{k=o}^{m} (-1)^k a_o(x)\theta^k a_k(x) + \sum_{k=o}^{m} (-1)^k a_1(x).$$

$$\theta^{k+1} a_k(x) + \ldots + (-1)^m a_m(x)\theta^m a_m(x)]f$$

$$+ \sum_{k=1}^{m} (-1)^k \binom{k}{k-1}_q q^{-(k-1)} a_o(x)\theta^{k-1} a_k(qx) + \ldots$$

$$+ \binom{m}{m-1}_q q^{-(m-1)} a_m(x)\theta^{m-1} a_o(qx)]\theta f$$

$$+ \ldots + [(-1)^m a_o(x) a_m(q^m x) + \ldots + (-1)^m a_m(x).$$

$$\theta^m a_m(q^m x)]\theta^m f + \ldots + (-1)^m a_m(x) a_m(q^{2m} x)\theta^{2m} f \quad (4.2)$$

$$P_m^* P_m f = [a_o^2(x) - a_1(qx)\theta a_o(x) - \theta a_1(x) a_o(x)$$

$$+ a_2(q^2 x)\theta^2 a_o(x) + (1+q) q^{-1} \theta a_2(qx)\theta a_o(x)$$

$$+ \ldots + (-1)^m \sum_{j=o}^{m} \binom{m}{j}_q q^{j(j-m)} \theta^j a_m(q^{m-j} x)\theta^{m-j} a_o(x)]f$$

$$+ \ldots + (-1)^m a_m(q^m x) a_m(q^m x)\theta^{2m} f \quad (4.3)$$

From (4.2) and (4.3) we see that the corresponding coefficients are not the same. Hence

$$P_m P_m^* \neq P_m^* P_m$$

Theorem 2

If the coefficients of the operator P_m are constants, it is normal

Proof

Let $\quad P_m = \sum_{k=o}^{m} a_k \theta^k(..)$

Hence $\quad P_m^* = \sum_{k=o}^{m} (-1)^k \theta^k a_k(..)$ where a_k are constants

Therefore $P_m^* = \sum_{k=o}^{m} (-1)^k a_k \theta^k(..)$

And $\quad P_m P_m^* f = a_o^2 f + (2a_o a_2 - a_1^2)\theta^2 f + (2a_o a_4 - 2a_1 a_3 + a_2^2)\theta^4 f + \ldots$

$$+[(-1)^m a_o a_m + (-1)^{m-1} a_1 a_{m-1} \ldots - a_{m-1} a_1 + a_m a_o]\theta^m f$$

$$+ \ldots + (-1)^m a_m^2 \theta^{2m} f \qquad (4.4)$$

$$P_m{}^*P_mf = a_o^2f + (2a_oa_2-a_1^2)\theta^2f + \ldots + (-1)^m a_m^2\theta^{2m}f \qquad (4.5)$$

Hence from (4.4) and (4.5)

$$P_m P_m^* = P_m^*P_m$$

Hence the result.

Theorem 3

If P_m is an operator whose coefficients are q-periodic functions of x, then

$$P_mP_m{}^* = P_m{}^* P_m$$

Proof

In this case

$$a_k(x) = a_k(qx) = \ldots a_k(q^mx)$$

And $$\theta a_k(x) = 0 \text{ for all } a_k(x)$$

Hence $$P_mP_m{}^*f = P_m{}^*P_mf$$

$$= a_o^2(x)f + [2a_o(x)a_2(x)-a_1^2(x)]\theta^2f$$

$$+ \ldots + (-1)^m a_m^2(x)\theta^{2m}f$$

Hence the result.

Remark

From theorems (1), (2) and (3) we see that when the coefficients are variable functions of x which are not q-periodic, only some operators commute with their basic adjoints. However, we give examples of normal operators with variable coefficients which are not q-periodic.

3. EXAMPLES

(i) Basic normal operators

(a) Consider

$$P_2 = qx^2\theta^2 + x\theta - I$$

$$P_2^* = q\theta^2x^2 - \theta x - I$$

$$P_2P_2^*f = (qx^2\theta^2 + x\theta - I)(q\theta^2x^2 - \theta x - I)f$$

$$= q^{10}x^4\theta^4f + \{q^9+2q^8+3q^7+2q^6+q^5-q^4\}x^3\theta^3f$$

$$+ \{q^7+3q^6+4q^5+5q^4+2q^3-q^2-2q\}x^2\theta^2f$$

$$+ (q+q^2-2)x\theta f - (q+q^2-2)f \qquad (4.6)$$

$$P_2^*P_2 f = (q\theta^2x^2-\theta x-I)(qx^2\theta^2f+x\theta f-f)$$

$$= q^{10}x^4\theta^4 f + (q^9+2q^8+3q^7+2q^6+q^5-q^4)\ x^3\theta^3 f$$

$$+ (q^7+3q^6+4q^5+5q^4+2q^3-q^2-2q)\ x^2\theta^2 f$$

$$+ (q+q^2-2)x\theta f - (q+q^2-2)f \qquad (4.7)$$

From (4.6) and (4.7) $P_2P_2^* f = P_2^* P_2 f$.

Hence P_2 is basic normal.

(b) Consider

$$P_1 = x\theta + I$$

$$P_1^* = -\theta x + I$$

Hence $P_1P_1^* f = (x\theta + I)\ (-\theta xf+f)$

$$= (x\theta + I)\ [-qx\theta f]$$

$$= -q^2x^2\theta^2 f - 2qx\theta f$$

$$P_1^* P_1 f = (-\theta x + I)\ (x\theta f + f)$$

$$= -q^2x^2\theta^2 f - 2qx\theta f$$

Hence $P_1P_1^* f = P_1^* P_1 f$

Hence P_1 is normal.

(ii) Basic self adjoint operators

a) Consider $P_2 = \theta^2 - (1+q)x^{-2}I$

$P_2^* = \theta^2 - (1+q)x^{-2}I$

Hence ∴ $P_2 = P_2^*$

Hence P_2 is self adjoint

b) Consider $P_2 = (1+q)\theta^2 + x^2I$

$P_2^* = (1+q)\theta^2 + x^2I$

4. RESULTS

(1) If P_m is a basic difference operator, then its basic adjoint is unique.

(2) If P_m^* is the adjoint of P_m and Q_n^* is the adjoint of Q_n then, $P_m^* \pm Q_n^*$ is the adjoint of $P_m \pm Q_n$; $Q_n^* P_m^*$ is the adjoint of P_mQ_n and $Q_n^* P_m^* - P_m^* Q_n^*$ is the adjoint of $P_mQ_n - Q_nP_m$.

(3) $\alpha\theta^n$ and $(-\theta)^n \alpha$ are adjoints, α constant or variable.

(4) $\alpha_oI + \alpha_1\theta + \alpha_2\theta^2 + \ldots + \alpha_n\theta^n$, and $\alpha_oI - \theta\alpha_1 + \theta^2\alpha_2 + \ldots + (-\theta)^n\alpha_n$ are adjoints, α_i constants or variables.

(5) $\alpha_o\theta\alpha_1\theta\alpha_2 \dots\ \theta\alpha_n$ and $(-1)^n\alpha_n\theta\ \dots\ \alpha_2\theta\alpha_1\theta\alpha_o$ are adjoints, α_i constants or variables.

(6) $(\theta-\alpha_1)(\theta-\alpha_2)\dots(\theta-\alpha_n)$ and $(-1)^n\ (\theta+\alpha_n)\dots(\theta+\alpha_2)(\theta+\alpha_1)$ are adjoints, α_i constants or variables.

(7) $\alpha_o\theta\alpha_1\ \theta\alpha_2\ \theta\alpha_1\ \theta\alpha_o$ is identical with its adjoint and is therefore self-adjoint. α_i constants or variables. Proof of the above statements, being easy, are omitted.

5. THE CHARACTERISTIC IDENTITY $F(P_m{}^*, Q_n{}^*) = 0$

<u>Theorem 4</u>

If P_m and Q_n are commutative then $P_m{}^*$ and $Q_n{}^*$, their adjoints, are also commutative and $F(P_m{}^*, Q_n{}^*) = 0$

<u>Proof</u>

If P_m and Q_n are commutative then

$$Q_nP_m - P_mQ_n = 0 \text{ and } F(P_m, Q_n) = 0 \text{ by}(2.36)$$

Now by result (2)

$$P_m^* Q_n^* - Q_n^* P_m^* \text{ is adjoint to } Q_n P_m - P_n Q_m$$

Let Y be a common solution of

$$(P_m - gI)f = 0 \text{ and } (Q_n - hI)f = 0$$

Then Y is a common solution of

$$(P_m^* - gI)f = 0 \text{ and } (Q_n^* - hI)f = 0$$

Operating on Y with the operator $P_m^* Q_n^* - Q_n^* P_m^*$

We have

$$\begin{aligned}(P_m^* Q_n^* - Q_n^* P_m^*)Y &= P_m^* hY - Q_n^* gY \\ &= ghY - hgY \\ &= 0\end{aligned}$$

Hence $(P_m^* Q_n^* - Q_n^* P_m^*)Y = 0$ has infinitely many distinct solutions. Hence $P_m^* Q_n^* - Q_n^* P_m^* = 0$. Hence P_m^* and Q_n^* commute with each other.

So $F(P_m^*, Q_n^*) = 0$ by (2.36)

Hence the result.

Example

Consider $P_2 = \theta^2$, $Q_3 = \theta^3$

Hence $F(g,h) = g^3 - h^2$ by (3.10)

Therefore $F(P_2, Q_3) = P_2^3 - Q_3^2 = (\theta^2)^3 - (\theta^3)^2$

$= 0$

$P_2^* = \theta^2$

$Q_3^* = -\theta^3$

$$F(g,h) = \begin{vmatrix} 0 & 0 & 1 & 0 & -g \\ 0 & 1 & 0 & -g & 0 \\ 1 & 0 & -g & 0 & 0 \\ 0 & -1 & 0 & 0 & -h \\ -1 & 0 & 0 & -h & 0 \end{vmatrix} = g^3 - h^3$$

Hence

$F(P_2^*, Q_3^*) = (P_3^*)^3 - (Q_2^*)^2$

$= (\theta^2)^3 - (-\theta^3)^2$

$= 0$

6. OPERATOR P_m AS POLYNOMIAL IN Q_1 A FIRST ORDER BASIC DIFFERENCE OPERATOR

Theorem 5

If a first order basic difference operator Q_1 commutes with

$$P_m = \sum_{k=o}^{m} a_k(x)\theta^k(..)$$

Then

$$P_m = \sum_{k=o}^{m} A_k(x)Q_1^k$$

Proof

Suppose P_m and Q_1 commute where

$$Q_1 = x\theta + I$$

Then

$$P_mQ_1 = Q_1P_m = a_o(x)I + [xa_o(x) + 2a_1(x)]\theta$$

$$+ [qxa_1(x) + (2+q)a_2(x)]\theta^2 + ...$$

$$+ ... + [\binom{m}{1}_q a_m(x) + a_m(x)]\theta^m$$

$$+ q^m xa_m(x)\theta^{m+1} \qquad (4.8)$$

Now let

$$P_m = \sum_{k=o}^{m} A_k(x) Q_1^k$$

$$= A_o(x)I + A_1(x)\ (x\theta+I)$$

$$+ A_2(x)[qx^2\theta^2 + 3x\theta + I] + \ldots$$

$$+ \ldots + A_m(x)\ (x\theta+I)^m . \qquad (4.9)$$

Comparing coefficients of $I, \theta, \theta^2 \ldots \theta^m$ of (4.8) and (4.9) we get

$A_o(x), A_1(x) \ldots A_m(x)$ in terms of $a_o(x), a_1(x) \ldots a_m(x)$.

Hence the result.

Example

$$\text{Let } P_2 = qx^2\theta^2 - (1+q)x\theta + (1+q)I$$

$$Q_1 = x\theta + I .$$

$$\text{Then} \quad P_2Q_1 = Q_1P_2 = q^3x^3\theta^3 + qx^2\theta^2 - (1+q)x\theta + (1+q)I . \qquad (4.10)$$

Let $P_2 = a(x)Q_1^2 + b(x)Q_1 + c(x)I$

$$= a(x)qx^2\theta^2 + \left\{3a(x)+b(x)\right\} x\theta$$

$$+ \left\{a(x) + b(x) + c(x)\right\} I . \qquad (4.11)$$

Comparing coefficients of I, θ, θ^2 from (4.10) and (4.11) we get

$$a(x) = 1$$

$$b(x) = -(q+4)$$

$$c(x) = 2q+4$$

Hence $P_2 = Q_1^2-(q+4)Q_1+ (2q+4)I$.

Therefore P_2 is a polynomial in Q_1. But this is not true for Q_1^*. Since

$$Q_1^* = -\theta x + I$$

$$P_2Q_1^* = Q_1^* P_2 = q^4x^3\theta^3f .$$

Hence from (4.11) and (4.12) we see that a(x), b(x), c(x) are zeros. Therefore we cannot write P_2 as a polynomial in Q_1^*. Now we consider P_2^* and see that P_2^* cannot be written as a polynomial in Q_1^*.

$$P_2{}^* = q\theta^2x^2 + (1+q)\theta x + (1+q)I$$

$$Q_1{}^* = -\theta x + I$$

$$P_2^* Q_1^* = Q_1{}^* P_2^*$$

$$= -q^8x^3\theta^3-(q^3+2q^4+2q^5+2q^6+q^7)x^2\theta^2$$

$$-(4q^2+3q^3+2q^4+q^5+2q)x\theta \,. \qquad (4.13)$$

If we write

$$P_2{}^* = a(x)Q_1{}^{*2}+b(x)Q_1{}^*+ c(x)I$$

$$= q^3a(x)x^2\theta^2 + \{(q^2a(x)-qb(x)\} x\theta$$

$$+ c(x)I \,, \qquad (4.14)$$

Comparing (4.13) and (4.14) we get

$$c(x) = 0$$

$$a(x) = -(1+2q+2q^2+2q^3+q^4)$$

$$b(x) = 3q + q^2 - q^4 + 2-q^5 \,.$$

But from (4.14) it is clear that this is not true. Hence eventhough $P_2{}^*Q_1{}^* = Q_1{}^*P_2{}^*$, $P_2{}^*$ is not a polynomial in Q_1^*, since Q_1 is not symmetric.

7. APPLICATION OF ADJOINT OPERATORS

Theorem 6

If the complete solution of the basic difference equation $P_m\varphi = 0$ is given then the adjoint equation $P_m^*\varphi = 0$ can be solved. Further the solutions can be expressed explicitly in terms of those of the other.

Proof

In results (5) and (6) of section 4 we see that $\alpha_o\theta\alpha_1\ \theta\alpha_2\ \ldots\ \theta\alpha_n$ and $(-1)^n\ \alpha_n\theta\ \ldots\ \alpha_2\theta\alpha_1\theta\alpha_o$ are adjoints and $(\theta-\alpha_1)(\theta-\alpha_2)\ \ldots\ (\theta-\alpha_n)$ and $(-1)^n\ (\theta+\alpha_n)\ \ldots\ (\theta+\alpha_2)$ $(\theta+\alpha_1)$ are adjoints.

$$P_m\varphi \quad = \quad 0 \text{ gives}$$

$$\sum_{k=o}^{m} a_k\theta^k\varphi = 0\ .$$

If $Y_1, Y_2, \ldots, Y_m$ are a linearly independent solutions, then

$$Y = c_1Y_1 + c_2Y_2 + \ldots + c_mY_m\ .$$

If these m functions are not linearly independent, then constants $c_1, c_2, \ldots, c_m$ may be determined so that

$$c_1Y_1 + c_2Y_2 + \ldots + c_mY_m = 0 \text{ identically.}$$

Differentiating the above equation (m-1) times

$$\begin{aligned} c_1\,\theta Y_1 + c_2\theta Y_2 + \ldots + c_m\theta Y_m &= 0 \\ c_1\,\theta^2 Y_1 + c_2\theta^2 Y_2 + \ldots + c_m\theta^2 Y_m &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_1\theta^{m-1}Y_1 + c_2\theta^{m-1}Y_2 + \ldots + c_m\theta^{m-1}Y_m &= 0 \end{aligned}$$

Hence

$$\Delta = \begin{vmatrix} Y_1 & Y_2 & \cdots & Y_m \\ \theta Y_1 & \theta Y_2 & \cdots & \theta Y_m \\ \cdots & \cdots & \cdots & \cdot \\ \cdots & \cdots & \cdot & \cdot \\ \theta^{m-1}Y_1 & \theta^{m-1}Y_2 & \cdot & \theta^{m-1}Y_m \end{vmatrix} \tag{4.15}$$

$$\Delta_r = \begin{vmatrix} Y_1 & Y_2 & \cdots & Y_{r-1} & Y_{r+1} & \cdots & Y_m \\ \theta Y_1 & \theta Y_2 & \cdots & \theta Y_{r-1} & \theta Y_{r+1} & \cdots & \theta Y_m \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \theta^{r-1} Y_1 & \theta^{r-1} Y_2 & \cdots & \theta^{r-1} Y_{r-1} & \theta^{r-1} Y_{r+1} & \cdots & \theta^{r-1} Y_m \\ \theta^{r+1} Y_1 & \theta^{r+1} Y_2 & \cdots & \theta^{r+1} Y_{r-1} & \theta^{r+1} Y_{r+1} & \cdots & \theta^{r+1} Y_m \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \theta^{m-1} Y_1 & \theta^{m-1} Y_2 & \cdots & \theta^{m-1} Y_{r+1} & \theta^{m-1} Y_{r+1} & \cdots & \theta^{m-1} Y_m \end{vmatrix},$$

$$r = 1, \ldots, m .$$

Then we can show that Δ_1/Δ, $\Delta_2/\Delta \ldots \Delta_m/\Delta$ form a set of linearly independent solutions of the adjoint equation $P_m^* \varphi = 0$.

Let $\Delta_1/\Delta = X_1, (\Delta_2/\Delta) = X_2, \ldots \Delta_m/\Delta = X_m$.

Then the Wronskian of these m functions is

$$\begin{vmatrix} X_1 & X_2 & \cdots & X_m \\ \theta X_1 & \theta X_2 & \cdots & \theta X_m \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \theta^{m-1}X_1 & \theta^{m-1}X_2 & \cdots & \theta^{m-1}X_m \end{vmatrix}$$

$= \triangle^{-1}$, where $\triangle$ is given in (4.15) .

nd $\triangle^{-1} \neq 0$.

Hence the m functions $X_1, X_2, \ldots, X_m$ are linearly independent. Now we prove that $X_1, X_2, \ldots, X_m$ are the solutions of $P_m{}^*\varphi = 0$ or the solution of the equation

$$\sum_{k=0}^{m} (-1)^k \theta^k a_k \varphi = 0 .$$

Now the equation whose solutions are $Y_1, Y_2, \ldots, Y_m$ is say

$$R_1 Y = \triangle_1^{-1} \begin{vmatrix} Y & Y_2 & \cdots & Y_m \\ \theta Y & \theta Y_2 & \cdots & \theta Y_m \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \theta^{m-1}Y & \theta^{m-1}Y_2 & \cdots & \theta^{m-1}Y_m \end{vmatrix}$$

where R_1 is the operator with leading term θ^{m-1}.
We write this with the coefficient of $\theta^{m-1}Y$ as unity.
Then

$$(P_m - \theta R_1)\varphi = 0$$

is also an equation of order m-1 whose roots are $Y_2 \dots Y_m$.

Hence $\quad P_m - \theta R_1 == \alpha R_1$ for some α.

Therefore $\quad P_m = (\theta + \alpha I)R_1$.

Hence $\quad (\theta+\alpha I)R_1Y_1 = P_mY_1 = 0$.

$$\text{Now } R_1Y_1 = \Delta_1^{-1} \begin{vmatrix} Y_1 & Y_2 & \dots & Y_m \\ \theta Y_1 & \theta Y_2 & \dots & \theta Y_m \\ \cdot & \cdot\ \cdot & \dots & \cdot \\ \theta^{m-1}Y_1 & \theta^{m-1}Y_2 & \dots & \theta^{m-1}Y_m \end{vmatrix} = \Delta/\Delta_1 .$$

Thus $\quad (\theta+\alpha I)(\Delta/\Delta_1) = 0$.

Therefore $\quad (\theta-\alpha I)(\Delta_1/\Delta) = 0$. $\quad$ (4.16)

But $\quad P_m^* = -\theta R_1(\theta-\alpha I)$

$$= -\theta R_1(\theta-\alpha I) .$$

-G 3498-

Hence $P_m^*(\triangle_1/\triangle) = 0$.

Therefore in a similar manner

$$P_m^*(\triangle_r/\triangle) = 0 \text{ for every } r .$$

Hence the result .

Generalisation of theorem 6

If P_m and Q_n are commutative and have leading terms θ^m, θ^n respectively and if Y_r is the solution of

$$(P_m-gI)\varphi = 0, \ (Q_n-h_rI)\varphi = 0 \ (r=1,\ldots,m),$$

then $\triangle_r/\triangle$ is a solution common to the adjoint equations

$$(P_m^* - g\,I)\varphi = 0, \ (Q_n^* - h_r\,I)\varphi = 0$$
$$r = 1, \ldots, m$$

Proof

We have seen that

$$(\theta-\alpha I)(\triangle_1/\triangle) = 0 \text{ from } (4.16) .$$

Now $R_1Q_nY_r = R_1h_rY_r = 0 \ (r = 2, \ldots, m)$.

Hence $R_1 Q_n \quad = \quad Q_1 R_1$, where Q_1 is some operator.

Hence $(\theta+\alpha I)Q_1 R_1 = (\theta+\alpha I)R_1 Q_n = (P_m - gI)Q_n$

$$= Q_n(P_m - gI)$$

$$= Q_n(\theta+\alpha I)R_1 . \qquad (4.17)$$

Hence $(\theta+\alpha I)Q_1 \quad = \quad Q_n(\theta+\alpha I)$.

But $(Q_1 - h_1 I)R_1 Y_1 = R_1(Q_n - h_1 I)Y_1 = 0$.

Therefore $(Q_1 - h_1 I)(\triangle / \triangle_1) \quad = \quad 0$.

Hence for some operator s_1

$$Q_1 - h_1 I \quad = \quad s_1(\theta+\alpha I) .$$

From (4.17)

$$(Q_n - h_1 I)(\theta+\alpha I) \quad = \quad (\theta+\alpha I)(Q_1 - h_1 I)$$

$$= \quad (\theta+\alpha I)s_1(\theta+\alpha I) .$$

And so $Q_n - h_1 I \quad = \quad (\theta+\alpha I)s_1$.

By taking adjoints we have

$$Q_n^* - h_1 I \quad = \quad -\theta \; s_1(\theta-\alpha I)$$

Therefore

$$(Q_n^* - h_1 I)(\triangle_1/\triangle) \quad = \quad 0 .$$

Hence the result.

CHAPTER V

BIBASIC COMMUTATIVE DIFFERENCE OPERATORS

We can extend the theory of basic commutative difference operators to bibasic commutative difference operators. If $f(x,y)$ is a function of two variables in R^2 and if D_{px} and D_{qy} denote the difference operators we see that these difference operators are commutative. But all bibasic polynomial difference operators are not commutative. Hence in this chapter an attempt is made to study the conditions under which the bibasic difference operators are commutative. Also we define some special bibasic commutative difference operators and study their properties. Using these operators we solve some bibasic-difference equations.

1. BIBASIC DIFFERENCE OPERATORS AND THEIR PROPERTIES

Let $x = p^m x_o$, $y = q^n y_o$, $p \neq q$ and $p,q \neq 1$ are fixed; where $m \in Z$, $n \in Z$; $x_o > o$, $y_o > o$ fixed.

$$\text{Let } f(x,y) = \sum_{j,k=o}^{m} \alpha_{j,k} x^k y^j, \; k+j \leq m, \text{ where}$$

$|x| < x_o$, $|y| < y_o$.

We define the bibasic difference operators as :

$$D_{px}f(x,y) = \frac{f(x,y)-f(px,y)}{(1-p)x} \tag{5.1}$$

$$D_{qy}f(x,y) = \frac{f(x,y)-f(x,qy)}{(1-q)y} \tag{5.2}$$

We see that D_{px} and D_{qy} are commutative. Consider the bibasic polynomial difference operators

$$P_m = \sum_{k,j=0}^{m} a_{k,j} D_{px}^{k} D_{qy}^{j}, \quad k+j \leq m$$

$$Q_n = \sum_{k,j=0}^{n} b_{k,j} D_{px}^{k} D_{qy}^{j}, \quad k+j \leq n .$$

Convergence of

$$\sum_{k,j=0}^{m} a_{k,j} D_{px}^{k} D_{qy}^{j} f(x,y) \text{ as } m \to \infty \tag{5.3}$$

depends on the nature of f(x,y). We cannot therefore discuss the convergence of the operator by itself apart from knowledge of the operand. We can see that (5.3) converges if f(x,y) converges. Hence we can consider bibasic difference operators of finite or infinite order.

If the operand is a polynomial of n^{th} degree in x and y these infinite operators terminate at n. In other cases we treat infinite operators as **purely** symbolic.

Now if we consider three bibasic polynomial difference operators P,Q,R we see that

$$P + Q = Q + P$$
$$P + (Q+R) = (P+Q) + R$$
$$P(QR) = (PQ)R$$
$$P(Q+R) = PQ+PR ,$$

Hence these operators obey the fundamental laws of arithmetic combination except the commutative law. In general PQ $\neq$ QP. For example, consider

$$P = a_o(x,y)I + a_1(x,y)D_{px} + a_2(x,y)D_{qy}$$
$$Q = b_o(x,y)I + b_1(x,y)D_{px} + b_2(x,y)D_{qy} .$$

Hence $$PQ = [a_o(x,y)b_o(x,y) + a_1(x,y)D_{px}b_o(x,y) + a_2(x,y)D_{qy}\, b_o(x,y)]I$$
$$+ [a_o(x,y)\; b_1(x,y) + a_1(x,y)\; b_o(px,y) + a_1(x,y)D_{px}b_1(x,y) + a_2(x,y)D_{qy}b_1(x,y)]D_{px}$$

$$+ [a_1(x,y)b_1(px,y)]D_{px}^2 + a_2(x,y)b_2(x,qy)D_{qy}^2$$

$$+ [a_1(x,y)b_2(px,y) + a_2(x,y)b_1(x,qy)]D_{px}D_{qy}$$

$$+ [a_o(x,y)b_2(x,y) + a_1(x,y)D_{px}b_2(x,y)$$

$$+ a_2(x,y)b_o(x,qy)+a_2(x,y)D_{qy}b_2(x,y)]D_{qy} \qquad (5.4)$$

$$QP = [b_o(x,y)\ a_o(x,y) + b_1(x,y)D_{px}\ a_o(x,y) +$$

$$+ b_2(x,y)D_{qy}\ a_o(x,y)]I$$

$$+ [b_o(x,y)\ a_1(x,y) + b_1(x,y)\ a_o(px,y)$$

$$+ b_1(x,y)D_{px}\ a_1(x,y) + b_2(x,y)D_{qy}\ a_1(x,y)]D_{px}$$

$$+ [b_o(x,y)\ a_2(x,y) + b_1(x,y)D_{px}\ a_2(x,y)$$

$$+ b_2(x,y)\ a_o(x,qy) + b_2(x,y)D_{qy}\ a_2(x,y)]D_{qy}$$

$$+ [b_1(x,y)\ a_2(px,y) + b_2(x,y)\ a_1(x,qy)]D_{px}D_{qy}$$

$$+ [b_1(x,y)\ a_1(px,y)D_{px}^2 + b_2(x,y)a_2(x,qy)D_{qy}^2 . \qquad (5.5)$$

From (5.4) and (5.5) we see that

$$PQ \neq QP$$

2. BIBASIC COMMUTATIVE DIFFERENCE OPERATORS

a) Operators P_m and Q_n are commutative if their coefficients are constants

$$(P_m\, Q_n)f = \left(\sum_{k,j=o}^{m} a_{k,j} D_{px}^{k} D_{qy}^{j}\right)\left(\sum_{k,j=o}^{n} b_{k,j} D_{px}^{k} D_{qy}^{j}\right)f,$$

where $a_{k,j}$ and $b_{k,j}$ are constants,

$$\begin{aligned} &= [a_{oo}\, b_{oo} I + (a_{oo}\, b_{1o} + a_{1o}\, b_{oo}) D_{px} \\ &\quad + (a_{oo}\, b_{o1} + a_{o1}\, b_{oo}) D_{qy} \\ &\quad + (a_{oo}\, b_{11} + a_{1o}\, b_{o1} + a_{o1}\, b_{1o} + a_{11} b_{oo}) D_{px} D_{qy} \\ &\quad + \dots + a_{m,o}\, b_{n,o}\, D_{px}^{m+n} + a_{o,m}\, b_{o,n} D_{qy}^{m+n}] f \\ &= (Q_n\, P_m) f\, . \end{aligned}$$

Hence the result.

b) Operators P_m and Q_n are commutative if their coefficients are bi-periodic functions, ie. p-periodic in x and q-periodic in y.

In this case $a_{oo}(px,y) = a_{oo}(x,y)$

$$a_{oo}(x,qy) = a_{oo}(x,y)$$

and so on.

Also $D_{px}a_{oo}(x,y) = 0 = D_{qy}a_{oo}(x,y)$ and so on.

Therefore these coefficients act as constants. Hence we get the same result as (5.6).

c) Operators P_m and Q_n with variable coefficients are commutative if they satisfy $\frac{(m+1)(m+2)}{2} + \frac{(n+1)(n+2)}{2} + 1$ bibasic difference equations. That is if the coefficients of the two operators are related,

$$(P_m Q_n)f = \left(\sum_{k,j=o}^{m} a_{k,j}(x,y)D_{px}^{k}D_{qy}^{j}\right)\left(\sum_{k,j=o}^{n} b_{k,j}(x,y)D_{px}^{k}D_{qy}^{j}f\right)$$

and

$$(Q_n P_m)f = \left(\sum_{k,j=o}^{n} b_{k,j}(x,y)D_{px}^{k}D_{qy}^{j}\right)\left(\sum_{k,j=o}^{m} a_{k,j}(x,y)D_{px}^{k}D_{qy}^{j}f\right)$$

Hence

$$(P_mQ_n-Q_nP_m)f = [\sum_{k=o}^{m} a_{k,o}(x,y)D_{px}^{k}b_{oo}(x,y)$$

$$- \sum_{k=o}^{n} b_{k,o}(x,y)D_{px}^{k}a_{oo}(x,y)]f$$

$$+ \ldots + [a_{m,o}(x,y)b_{n,o}(p^m x,y)$$

$$- b_{n,o}(x,y)a_{m,o}(p^m x,y)]D_{px}^{m+n}f$$

$$+ [a_{o,m}(x,y)b_{o,n}(x,q^m y) -$$

$$- b_{o,n}(x,y)a_{o,m}(x,q^n y)]D_{qy}^{m+n}f .$$

These operators are commutative if and only if

$$(P_mQ_n - Q_nP_m)f = 0 .$$

ie. we get $\frac{(m+1)(m+2)}{2} + \frac{(n+1)(n+2)}{2} + 1$

bibasic difference equations to be satisfied, which are

$$\sum_{k=o}^{m} a_{k,o}(x,y)D_{px}^{k}b_{oo}(x,y) - \sum_{k=o}^{n} b_{k,o}(x,y)D_{px}^{k}a_{oo}(x,y) = 0$$

. . . .

$$a_{m,o}(x,y)b_{n,o}(p^{m}x,y) - b_{n,o}(x,y)a_{m,o}(p^{n}x,y) = 0$$

$$a_{o,m}(x,y)b_{o,n}(x,q^{m}y) - b_{o,n}(x,y)a_{o,m}(x,q^{n}y) = 0$$

Hence the result.

3. ALTERNANTS OF BIBASIC DIFFERENCE LINEAR OPERATORS

If two bibasic difference operators P and Q are non-commutative, then PQ-QP is called their alternant.

Result 1

The alternant of two linear bibasic difference operators is also a linear bibasic difference operator.

Proof

Consider

$$P = \alpha_1(x,y)D_{px} + \alpha_2(x,y)D_{qy}$$

$$Q = \beta_1(x,y)D_{px} + \beta_2(x,y)D_{qy} \text{ .}$$

Then PQ-QP $= [\alpha_1(x,y)D_{px}\beta_1(x,y)+\alpha_2(x,y)D_{qy}\beta_1(x,y)$

$- \beta_1(x,y)D_{px}\alpha_1(x,y)-\beta_2(x,y)D_{qy}\alpha_1(x,y)]D_{px}$

$+[\alpha_1(x,y)D_{px}\beta_2(x,y)+\alpha_2(x,y)D_{qy}\beta_2(x,y)$

$- \beta_1(x,y)D_{px}\alpha_2(x,y)-\beta_2(x,y)D_{qy}\alpha_2(x,y)]D_{qy}$

$+[\alpha_1(x,y)\beta_1(px,y)-\beta_1(x,y)\alpha_1(px,y)]D_{px}^2$

$+[\alpha_1(x,y)\beta_2(px,y)+\alpha_2(x,y)\beta_1(x,qy)-\beta_1(x,y)\alpha_2(px,y)$

$- \beta_2(x,y)\alpha_1(x,qy)]D_{qy}\,D_{px}$

$+[\alpha_2(x,y)\beta_2(x,qy)-\beta_2(x,y)\alpha_2(x,qy)]D_{qy}^2$.

This is also a linear operator. Hence the result. We denote the alternant PQ-QP by (P,Q).

<u>Note</u> Since (P,Q) is linear, we can form its alternant with P and with Q respectively. Hence we get $\{P,(P,Q)\}$ and $\{Q,(P,Q)\}$ etc. These are again linear. Hence we get a succession of alternants in this way.

Result 2

If P,Q,R are three linear bibasic operators, then

$$\left\{P,(Q,R)\right\} + \left\{Q,(R,P)\right\} + \left\{R,(P,Q)\right\} = 0 .$$

Proof being easy, is not given.

Result 3

If P and Q are commutative, then

$$\left\{P,(Q,R)\right\} = \left\{Q,(P,R)\right\} .$$

Proof

$$\left\{P,(Q,R)\right\} = P(QR-RQ) - (QR-RQ)P \tag{5.7}$$

$$\begin{aligned} \left\{Q,(P,R)\right\} &= QPR-QRP - PRQ+RPQ \\ &= (PQR-PRQ) - (QRP-RQP) \\ &= P(QR-RQ) - (QR-RQ)P \\ &= \left\{P,(Q,R)\right\} \end{aligned} \tag{5.8}$$

From (5.7) and (5.8) we get the result.

Result 4

The sequence of repeated alternants that can be constructed from a given pair of linear bibasic difference operators in general does not terminate.

Proof

Consider two linear bibasic difference operators

$$P = D_{px} + D_{qy}$$

$$Q = xD_{px} + yD_{qy}$$

Then $PQ \neq QP$.

Hence $(Q,P) = (1-p)xD_{px}^{2} - D_{px} - D_{qy} + (1-q)y\,D_{qy}^{2}$.

This is a linear operator.

Hence $$\{(Q,P),P\} = (1-p)^{2}xD_{px}^{3} - (1-p)D_{px}^{2} - (1-q)D_{qy}^{2} + (1-q)^{2}yD_{qy}^{3}.$$

This is again linear and so on. We get a succession of linear operators of the form

$$(1-p)^{n}x\,D_{px}^{n+1} - (1-p)^{n-1}D_{px}^{n} - (1-q)^{n-1}D_{qy}^{n} + (1-q)^{n}yD_{qy}^{n+1}. \qquad (5.9)$$

Hence the result, since (5.9) does not terminate unless $p < 1$, $q < 1$.

Result 5

If Y is a solution of Pf=0, Qf=0, where P and Q are two bibasic linear operators, then Y is a constant or is bi-periodic.

Proof

Since Y is a solution of Pf=0 and Qf=0, we get PY=0, QY=0 and hence (Q,P)Y=0. Hence Y is a solution of (Q,P)f=0. Hence by results (3) and (4), Y is a solution of all repeated alternants. But if Y is a solution of n linearly independent bibasic linear operators in a field of n variables, we get n independent equations of the form

$$\alpha_{11} P_1 Y + \alpha_{12} P_2 Y + \ldots + \alpha_{1n} P_n Y = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{n1} P_1 Y + \alpha_{n2} P_2 Y + \ldots + \alpha_{nn} P_n Y = 0 .$$

This is possible only if $P_1Y=0$, $P_2Y=0$ ... $P_nY=0$ which gives Y = a constant or Y is bi-periodic. Hence the result.

4. SPECIAL BIBASIC COMMUTATIVE DIFFERENCE OPERATORS

Let
$$\delta_{px} = x D_{px}$$
$$\delta_{qy} = y D_{qy} .$$

Then we see that

$$\delta_{px}\,\delta_{qy} = \delta_{qy}\,\delta_{px}$$

Hence these operators are commutative. Now we consider the operators

$$D^m_{px} = p^{\frac{-m(m-1)}{2}}\,x^{-m}\,\prod_{k=o}^{m-1}[\delta_{px}-[k]_p I\,]$$

$$D^n_{qy} = q^{\frac{-n(n-1)}{2}}\,y^{-n}\,\prod_{k=o}^{n-1}[\delta_{qy}-[k]_q I\,]$$

Then by the same argument as in (3.3) we see that

$$D^m_{px}\,D^n_{qy} = D^n_{qy}\,D^m_{px}$$

But here there are no common factors as in (3.1) and (3.2) and hence derivation of new operators by transference of common factors is not possible in this case.

$$\delta^{-1}_{px} = D^{-1}_{px}\,cx^{-1}$$

$$\delta^{-1}_{qy} = D^{-1}_{qy}\,y^{-1}$$

Hence if $f(\delta)_q = a\,\delta_{px} + b\,\delta_{qy} + c\,I.$

Then $f(\delta_{pq})(xy)^m = (axD_{px}+byD_{qy}+cI)(xy)^m$

$= (a[m]_p+b[m]_q+c)(xy)^m$

$= f([m])\ (xy)^m$

If $f(\delta_{pq}^{-1}) = (a\delta_{px}^{-1} + b\delta_{qy}^{-1} + cI)$

Then $f(\delta_{pq}^{-1})(xy)^m = (a\ \delta_{px}^{-1}+ b\ \delta_{qy}+ cI)(xy)^m$

$= aD_{px}^{-1}x^{-1}(xy)^m+bD_{qy}^{-1}y^{-1}(xy)^m+c(xy)^m$

$= [a\frac{1}{[m]_p} + b\frac{1}{[m]_q} + c](xy)^m$

The arguments being same as (3.16) and (3.19) we leave the detailed proof.

5. SOLUTION OF BIBASIC DIFFERENCE EQUATIONS

Theorem

Any linear combination of two solutions of a linear homogeneous bibasic difference equation if they exist is again a solution.

Proof

We prove the theorem for the general equation of the second order. Consider the bibasic difference equation

$$[D_{px}^2 + C_4 D_{qy}^2 + C_3 D_{px} D_{qy} + C_2 D_{px} + C_1 D_{qy} + C_o I] f(x,y) = 0$$

where c_i, $i = 0, \ldots, 4$ are constants.

If possible let

$$[D_{px}^2 + C_4 D_{qy}^2 + C_3 D_{px} D_{qy} + C_2 D_{px} + C_1 D_{qy} + C_o I] f$$

$$= (D_{px} + K_1 D_{qy} + K_2 I)\ (D_{px} + K_1' D_{qy} + K_2' I) f \qquad (5.10)$$

Hence $\qquad (D_{px} + K_1 D_{qy} + K_2 I) f = 0$

and $\qquad (D_{px} + K_1' D_{qy} + K_2' I) f = 0$

Then $\qquad (K_1 - K_1') D_{qy} f + (K_2 - K_2') f = 0$

ie $\qquad D_{qy} f = \left\{ \dfrac{K_2' - K_2}{K_1 - K_1'} \right\} f$

ie $\qquad \dfrac{f(x,y) - f(x,qy)}{(1-q)y} = \left\{ \dfrac{K_2' - K_2}{K_1 - K_1'} \right\} f(x,y)$

ie $\qquad \left[1 - \left\{ \dfrac{K_2' - K_2}{K_1 - K_1'} \right\} (1-q)y \right] f(x,y) = f(x,qy)$

Hence $f(x,y) = \prod_{n=0}^{\infty} \dfrac{f(x,q^n y)}{\left[1-\left\{\dfrac{K_2'-K_2}{K_1-K_1'}\right\}(1-q)q^n y\right]}$

exists when $q < 1$.

Substituting for $D_{qy}f$ in the equation (5.10) we have

$$D_{px}f = \left\{\frac{-K_1K_2' + K_2K_1'}{K_1 - K_1'}\right\} f(x,y)$$

Hence $\dfrac{f(x,y)-f(px,y)}{(1-p)x} = \left\{\dfrac{-K_1K_2' + K_2K_1'}{K_1-K_1'}\right\} f(x,y)$

Then $f(x,y) = \prod_{n=o}^{\infty} \dfrac{f(p^n x,y)}{\left[1-\left\{\dfrac{K_2K_1'-K_2'K_1}{K_1-K_1'}\right\}(1-p)p^n x\right]}$

exists when $p < 1$. Let these be denoted by $f_1(x,y)$ and $f_2(x,y)$. Comparing coefficients of (5.10) we have

$$K_1 K_1' = c_4 \implies K_1 = c_4/K_1'$$

$$K_1+K_1' = c_3 \implies K_1'^2 - c_3K_1' + c_4 = 0$$

ie. $$K_1 = \frac{c_3 \pm \sqrt{c_3^2 - 4\,c_4}}{2}$$

exists when $c_3^2 - 4\,c_4 \nless 0$

Similarly $K_2K_2' = c_o \Longrightarrow K_2 = c_o/K_2'$

$$K_2 + K_2' = c_2 \Longrightarrow K_2'^2 - c_2K_2' + c_o = 0$$

ie. $$K_2' = \frac{c_2 \pm c_2^2 - 4c_o}{2}$$

exists when $c_2^2 - 4c_o \nless 0$

Then $$K_1 = \frac{2c_4}{c_3 \pm \sqrt{c_3^2 - 4c_4}}$$ and

$$K_2 = \frac{2c_o}{c_2 \pm \sqrt{c_2^2 - 4c_o}}$$

$$K_1K_2' + K_2K_1' = c_1$$

Then $$D_{px}^2f_1 + c_4D_{qy}^2f_1 + c_3D_{px}D_{qy}f_1 + c_2D_{px}f_1 + c_1D_{qy}f_1 + c_of_1 = 0 \qquad (5.11)$$

and $$D_{px}{}^2 f_2 + c_4 D_{qy}{}^2 f_2 + c_3 D_{px} D_{qy} f_2$$

$$+ c_2 D_{px} f_2 + c_1 D_{qy} f_2 + c_o f_2 = 0 \qquad (5.12)$$

Multiplying (5.11) by A_1 and (5.12) by A_2 and adding we have

$$D_{px}{}^2(A_1f_1+A_2f_2)+c_4D_{qy}{}^2(A_1f_1+A_2f_2)$$

$$+ c_3D_{px}D_{qy}(A_1f_1+A_2f_2)+c_2D_{px}(A_1f_1+A_2f_2)$$

$$+ c_1D_{qy}(A_1f_1+A_2f_2)+c_o(A_1f_1+A_2f_2) = 0$$

Hence $A_1f_1 + A_2f_2$ is a solution of (5.46)

Consider the bibasic difference equation

$$[pq\delta_{px}{}^2\delta_{qy}{}^2 + p\delta_{px}{}^2\delta_{qy} - p\delta_{px}{}^2$$

$$+ q\delta_{px}\delta_{qy}{}^2 + \delta_{px}\delta_{qy} - \delta_{px} - q\delta_{qy}{}^2$$

$$- \delta_{qy} + I]\, f(x,y) = (xy)^3$$

ie. $(p\delta_{px}{}^{2}+\delta_{px}-I)(q\delta_{qy}{}^{2}+\delta_{qy}-I)f(x,y) = (xy)^{3}$

ie. $(\delta_{px}{}^{2}-I)\ (\delta_{qy}{}^{2}-I)\ f(x,y) = (xy)^{3}$

Hence $f(x,y) = (\delta_{px}{}^{2}-I)^{-1}\ (\delta_{qy}{}^{2}-I)^{-1}(xy)^{3}$

$$= \frac{(xy)^{3}}{([3]_{p}{}^{2}-1)\ ([3]_{q}{}^{2}-1)} \quad \text{by (5.45)}$$

CHAPTER VI

BIBASIC PSEUDOANALYTIC FUNCTIONS

Bibasic pseudoanalytic functions

Discrete analytic function theory is concerned with complex valued functions defined at certain lattice points in the complex plane. Harman [1] used the lattice $\{(\pm q^m x_0, \pm q^n y_0)\}$ suitable for q-function theory and developed the concept of q-analytic functions. Using Harman's lattice, Mercy Jacob [1] introduced discrete pseudoanalytic functions. Khan [1] extended Harman's q-analytic functions to bibasic analytic functions using the lattice $\{(\pm p^m x_0, \pm q^n y_0)\}$ where p and q are not related. Here an attempt is made to establish bibasic pseudoanalytic functions using the bibasic lattice of Khan.

1. THE LATTICE

For convenience we consider only the first quadrant of the complex plane. We define the discrete plane B' with respect to some fixed point $z_0 = (x_0, y_0)$, as the set of lattice points

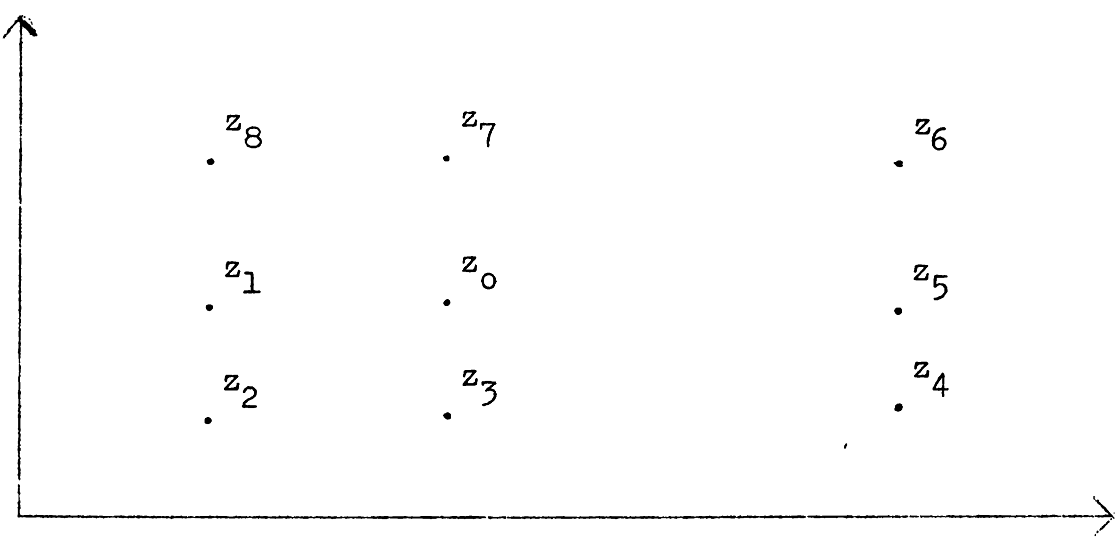
$$B' = \{ (p^m x_o, q^n y_o),\ m,n \in Z \}$$

$$x_o > o,\ y_o > o,\ p \neq q \text{ and}$$

$$p,q \neq 1 \text{ fixed} \qquad (6.1)$$

z_o will be called the origin of B'.

Two lattice points z_i, $z_{i+1} \in B'$ are said to be " adjacent" if z_{i+1} is one of (px_i, y_i), $(p^{-1}x_i, y_i)$, (x_i, qy_i) or $(x_i, q^{-1}y_i)$ where $z_i = (x_i, y_i)$



Let

$$A(z) = \{ (px,y),\ (x,qy),\ (p^{-1}x,y),\ (x,q^{-1}y) \} \qquad (6.2)$$

A 'discrete curve C' in B' joining z_o and z_n is denoted by the sequence

$$C = \langle z_o, z_1, \ldots, z_n \rangle \qquad (6.3)$$

where z_i, z_{i+1}, $i = 0, 1, \ldots, n-1$ are adjacent points of B'.

If points are distinct ($z_i \neq z_j$, if $i \neq j$), then the discrete curve C is said to be simple.

'A discrete closed curve C' in B' is a discrete curve $\langle z_o, z_1, \ldots, z_n \rangle$ where $\langle z_o, z_1, \ldots, z_{n-1} \rangle$ is simple and $z_o = z_n$.

$T(z) = \{ (x,y), (px,y), (x,qy) \}$ is called the triad of z. (6.4)

$$S(z) = \{ (x,y), (px,y), (px,qy), (x,qy) \} \qquad (6.5)$$

is called the " bibasic set" with respect to $z \in B'$. The discrete closed curve around S(z) is $\langle (x,y), (px,y), (px,qy), (x,qy), (x,y) \rangle$ and this order of points is taken as the positive direction. A discrete domain D is composed of a union of bibasic sets.

Therefore

$$D = \bigcup_{i=1}^{n} S(z_i) \tag{6.6}$$

If $\overline{C}$ is the closed curve formed by joining adjacent points of the discrete closed curve C, then $\overline{C}$ encloses certain points of B', denoted by Int(C).

A " finite discrete domain" D is defined as

$$D = \{ z \in B';\ z \in C \bigcup Int(C) \} \tag{6.7}$$

∂D = D-Int(D) denotes the discrete closed curve around the finite discrete domain D.

Let f: D $\longrightarrow \emptyset$. Then f is called a discrete function. The bibasic operators defined by Khan [1] are

$$D_{px}f(z) = \frac{f(z) - f(px,y)}{(1-p)x} \tag{6.8}$$

$$D_{qy}f(z) = \frac{f(z) - f(x,qy)}{(1-q)iy} \tag{6.9}$$

Now we define the operators B^+ and B^- as follows.

$$B^+f(z) = \tfrac{1}{2}[D_{px}f(z) + D_{qy}f(z)] \tag{6.10}$$

$$B^-f(z) = \tfrac{1}{2}[D_{px}f(z) - D_{qy}f(z)] \tag{6.11}$$

2. DISCRETE BIBASIC PSEUDOANALYTIC FUNCTION

(a) Hölder type discrete bibasic functions

Let D be a discrete domain and $f:D \rightarrow \emptyset$. Suppose $z' = (x',y') \in D$ and $|f(z) - f(z')| \leq k\sigma^{\mu}$ (6.12) where $\sigma = \max \{ (p^{-1}-1)x', (q^{-1}-1)y' \}$ for every $z \in A(z')$, μ and k are real constants $0 < \mu \leq 1$, then we say that the function f is Hölder-type discrete bibasic at z'.

If f is Hölder-type discrete bibasic at any $z \in D$ such that $A(z) \not\subset D$, then we say that f is Hölder-type discrete bibasic in D. The class of such functions on D will be denoted by HB(D).

If D is a discrete domain and if $f,g \in HB(D)$ then $fg \in HB(D)$

(b) Generating vector space

Let us take two discrete bibasic functions $g_1(z)$ and $g_2(z) \in HB(D)$ such that $Im(\bar{g}_1 \, g_2) > 0$ throughout the discrete domain D. Then the row vector $g = [g_1 g_2]$ is called a generating vector and the set of all generating vectors $\{g\}$ is the generating space over D denoted by GB(D).

The components of g cannot be equal and also cannot be equal and opposite in sign.

Now if $f_1(z)$ and $f_2(z)$ are real valued functions in D, consider $f = [f_1 f_2]'$. The set of all such column vectors will be denoted by FB(D).

Now if g is a generating vector and W a discrete function defined on D, then for any W, a unique element $f \in FB(D)$ can be found such that,

$$W(z) = (g.f)(z)$$

$$= g_1(z)\, f_1(z) + g_2(z)\, f_2(z),$$

for all $z \in D$.

If possible let

$$W(z) = (g.f)(z) \text{ and}$$

$$W(z) = (g.F)(z)$$

Therefore $(g.f)(z) = (g.F)(z)$

ie. $g_1f_1 + g_2f_2 - [g_1F_1 + g_2F_2] = 0$

ie. $g_1(f_1-F_1) + g_2(f_2-F_2) = 0.$

Hence $f_1 = F_1,\ f_2 = F_2$

ie. f is unique.

(c) Discrete bibasic pseudoanalytic functions

Here instead of 1 and i we assign two arbitrary functions $g_1(z)$ and $g_2(z)$. Hence W(z) = GB.FB(D) where (.) means multiplication of a row and a column vector.

Thus GB.FB(D) form a vector space over R.

We define

$$_gD_{px}W(z) = (g.D_{px}f)(z) \quad (6.13)$$

$$_gD_{qy}W(z) = (g.D_{qy}f)(z) \quad (6.14)$$

where D_{px} and D_{qy} are given in (6.8) and (6.9).

Definition

If W is a discrete function defined over D, then W is called discrete bibasic g-pseudoanalytic of the first kind at $z \in D$ if

$$W \in GB.FB(D) \text{ and } {}_gD_{px}W(z) = {}_gD_{qy}W(z) \tag{6.15}$$

If this relation holds for all $z \in D$ such that $T(z) \subset D$, then W is called bibasic g-pseudoanalytic of the first kind in D.

The class of all discrete bibasic g-pseudo-analytic functions of the first kind in D is denoted by ${}_1B_D(g)$. Then ${}_1B_D(g)$ forms a vector space over R.

Definition

If $W = (g.f)$; $f \in FB(D)$, $g \in GB(D)$, and $W \in {}_1B_D(g)$, then $h = f_1 + if_2$ is called discrete bibasic g-pseudoanalytic of the second kind in D.

The class of all such functions of the second kind in D is denoted by ${}_2B_D\,(g)$.

Note

Each component g_1, g_2 of the generating vector is itself an element of ${}_1B_D(g)$.

Theorem (1)

A complex valued function W will be discrete bibasic g-pseudoanalytic of the first kind in a discrete domain D if and only if an $f \in FB(D)$ is found such that $\bar{B}f$ is orthogonal to g throughout D.

Proof

(a) Necessary

Suppose that $W \in {}_1B_D(g)$, then

$W = g.f$, $f \in FB(D)$, $g \in GB(D)$.

Hence $\quad {}_gD_{px}\, W(z) = {}_gD_{qy}\, W(z)$, by (6.15)

ie. $\quad (g.D_{px}f)(z) = (g.D_{qy}f)(z)$, by (6.13) and (6.14)

ie. $\quad [g.(D_{px}-D_{qy})f](z) = 0$

ie. $\quad \frac{1}{2}[g.(D_{px}-D_{qy})f](z) = 0$

ie. $\quad (g.\bar{B}f)(z) = 0$ by (6.11) $\qquad$ (6.16)

ie. $\quad \bar{B}f$ is orthogonal to g.

(b) Sufficient

Suppose that $W = g.f$, $f \in FB(D)$, $g \in GB(D)$ and $\bar{B}f$ is orthogonal to g, then $(g.\bar{B}f)(z) = 0$.

ie. $\frac{1}{2}[g.(D_{px}-D_{qy})f](z) = 0$

ie. $(g.D_{px}f)(z) = (g.D_{qy}f)(z)$

Hence by (6.15), $W \varepsilon {}_1B_D(g)$. Thus the theorem.

Definition

A discrete function $f(x,y)$ is said to be biperiodic if it is p-periodic in x and q-periodic in y, $p \neq q$, $p,q \neq 1$.

Theorem 2

${}_gDW(z) = 0$ if and only if $W = (g.f)(z)$ where f is bi-periodic.

Proof

Let $W = g.f$, $g \varepsilon GB(D)$, $f \varepsilon FB(D)$ is an element of ${}_1B_D(g)$. Then by (6.16)

$$(g.B^{-}f)(z) = 0$$

and $$(g.B^{+}f)(z) = {}_gDW(z).$$

But $$\overline{B^{+}f(z)} = B^{-}\overline{f(z)}$$

and $$\overline{B^{-}f(z)} = B^{-}\overline{f(z)}. \qquad (6.17)$$

Hence $[\bar{g}.\ (\overline{B^+f})](z) = 0$.

ie. $(\bar{g}.\ B^+\bar{f})(z) = 0$ (by 6.17)

ie. $(\bar{g}.\ B^+f)(z) = 0$, since f is real valued.

Thus we get,

$$g_1(z)\ B^+f_1(z) + g_2(z)\ B^+f_2(z) = {}_gDW(z)$$

and $\overline{g_1(z)}\ B^+f_1(z) + \overline{g_2(z)}\ B^+f_2(z) = 0$.

Solving

$$B^+f_1(z) = \frac{\overline{g_2(z)}\ {}_gDW(z)}{g_1(z)\ \overline{g_2(z)} - \overline{g_1(z)}\ g_2(z)} \tag{6.18}$$

and

$$B^+f_2(z) = \frac{-\overline{g_1(z)}\ {}_gDW(z)}{g_1(z)\ \overline{g_2(z)} - \overline{g_1(z)}\ g_2(z)} . \tag{6.19}$$

If ${}_gDW(z) = 0$, then (6.18) and (6.19) we have

$$B^+f_1(z) = 0 = B^+f_2(z)$$

ie. $(D_{px} + D_{qy})\ f_1(z) = 0$

and $(D_{px} + D_{qy})\ f_2(z) = 0$.

Hence

$$\frac{f_1(z)-f_1(px,y)}{(1-p)x} + \frac{f_1(z)-f_1(x,qy)}{(1-q)iy} = 0$$

and

$$\frac{f_2(z)-f_2(px,y)}{(1-p)x} + \frac{f_2(z)-f_1(x,qy)}{(1-q)iy} = 0$$

since f_1 and f_2 are real valued, equating real and imaginary parts to zero, we have

$$f_1(z) - f_1(px,y) = 0$$

$$f_1(z) - f_1(x,qy) = 0$$

$$f_2(z) - f_2(px,y) = 0$$

$$f_2(z) - f_2(x,qy) = 0$$

Hence f is bi-periodic, ie. p-periodic in x and q-periodic in y.

(b) <u>Sufficiency</u>

Suppose that $W = (g.f)$, $g \in GB(D)$, $f \in FB(D)$ is an element of ${}_1B_D(g)$ and f is bi-periodic.

$$
\begin{aligned}
{}_{g}DW(z) &= (g.B^{+}f)(z) \\
&= g.\tfrac{1}{2}(D_{px}f + D_{qy}f)(z) \\
&= 0 ,
\end{aligned}
$$

since f is p-periodic in x and q-periodic in y.
Hence the theorem.

Remark

Solutions of the equation, ${}_{g}DW(z)=0$ are called g-pseudo constants. We can represent a g-pseudoconstant by g.f where f is p-periodic in x and q-periodic in y.

3. DEDUCTIONS

(1) The discrete bibasic pseudo-analytic function theory developed here is a two fold extension of the theory of discrete basic pseudo analytic functions in the following sense.

The two variables of the function run on two different bases as against one in the discrete basic pseudoanalytic function theory. Thus for $p = q < 1$, q-pseudoanalytic function theory becomes a particular case of this.

(2) When $B^{-}f(z)$ of (6.11) becomes zero, the discrete pseudoanalytic function theory reduces to Khans [1] bibasic analytic function theory.

(3) When $p = q^{-1}$ in the second case, we get Velukutty's [1] bianalytic functions where $o < q < 1$.

(4) If further $p = q < 1$ and $B^{-} f(z) = 0$, then this reduces to Harmans [1] q-analytic function theory.

4. EXAMPLES OF BIBASIC ANALYTIC FUNCTIONS

Harman [1] defined $z^{(n)}$ with base q by using continuation operator as,

$$z^{(n)} = \sum_{j=o}^{n} \frac{(1-q)_n}{(1-q)_{n-j}(1-q)_j} (iy)^j \, x^{n-j} \tag{6.20}$$

This is q-analytic. Khan [1] defined $z^{(n)}$ with bases p and q, as the following

$$z^{(n)} = \sum_{j=o}^{n} \frac{(p)_n}{(p)_{n-j}(q)_j} \left\{ \frac{(1-q)iy}{1-p} \right\}^j x^{n-j} , \tag{6.21}$$

where

$$D_{px}z^{(n)} = D_{qy}z^{(n)} = \left(\frac{1-p^n}{1-p}\right)z^{(n-1)}. \qquad (6.22)$$

This analogue of $z^{(n)}$ suffers from a disadvantage that $D_{px}z^{(n)}$ and $D_{qy}z^{(n)}$ are both given in terms of p. We define $z^{(n)}$ using the same bibasic lattice $\{(p^m x_o, q^n y_o)\}$ given in (6.1) as

$$z^{(n)} = \sqrt{[n]_p!\,[n]_q!}\ \sum_{r=0}^{n} \frac{x^{n-r}(iy)^r}{[n-r]_p!\,[r]_q!} \qquad (6.23)$$

$$= \sqrt{[n]_p!\,[n]_q!}\ \sum_{r=o}^{n} \frac{x^{r}(iy)^{n-r}}{[r]_p!\,[n-r]_q!}$$

which removes this difficulty. We now prove that $z^{(n)}$ is bibasic analytic.

<u>Theorem 3</u>

$$z^{(n)} = \sqrt{[n]_p!\,[n]_q!}\ \sum_{r=o}^{n} \frac{x^{n-r}(iy)^r}{[n-r]_p!\,[r]_q!}$$

is bibasic analytic in D.

Proof

$z^{(n)}$ will be bibasic analytic in D if $D_{px}z^{(n)} = D_{qy}z^{(n)}$.

Consider

$$D_{px}z^{(n)} = D_{px}\sqrt{[n]_p!\,[n]_q!}\sum_{r=o}^{n}\frac{x^{n-r}(iy)^r}{[n-r]_p!\,[r]_q!}$$

$$= \sqrt{[n]_p!\,[n]_q!}\sum_{r=o}^{n}\frac{(iy)^r}{[n-r]_p!\,[r]_q!}\left\{\frac{x^{n-r}-p^{n-r}x^{n-r}}{(1-p)x}\right\}$$

$$= \sqrt{[n]_p[n]_q}\sqrt{[n-1]_p!\,[n-1]_q!}\sum_{r=o}^{n-1}\frac{x^{n-r-1}(iy)^r}{[n-r-1]_p!\,[r]_q!}$$

$$= \sqrt{[n]_p\,[n]_q}\;z^{(n-1)}. \tag{6.24}$$

Now,

$$D_{qy}z^{(n)} = \sqrt{[n]_p!\,[n]_q!}\sum_{r=o}^{n}\frac{x^{n-r}}{[n-r]_p!\,[r]_q!}\left\{\frac{(iy)^r-q^r(iy)^r}{(1-q)iy}\right\}$$

$$=\sqrt{[n]_p[n]_q}\sqrt{[n-1]_p![n-1]_q!}\sum_{r=o}^{n}\frac{x^{n-r}(iy)^{r-1}}{[n-r]_p![r-1]_q!}$$

$$=\sqrt{[n]_p[n]_q}\sqrt{[n-1]_p!\,[n-1]_q!}\sum_{r=o}^{n-1}\frac{x^{n-1-r}(iy)^{r}}{[n-1-r]_p![r]_q!}$$

$$=\sqrt{[n]_p\,[n]_q}\;z^{(n-1)}. \qquad (6.25)$$

From (6.24) and (6.25) we get,

$$D_{pz}z^{(n)} = D_{qy}z^{(n)} = \sqrt{[n]_p\,[n]_q}\;z^{(n-1)}.$$

Hence $z^{(n)}$ is bibasic analytic in D. Also $z^{(o)} = 1$.

<u>Exponential function</u>

Here we define the bibasic analogue of the exponential function as

$$\text{bex}(z) = \sum_{r=o}^{\infty}\frac{z^{(r)}}{\sqrt{[r]_p![r]_q!}}, \qquad (6.26)$$

where z is given by (6.23).

Theorem 4

bex(z) is bibasic analytic in D.

Proof

Let $z \in D$.

Then

$$D_{px}\ \text{bex}(z) = D_{px} \sum_{r=o}^{\infty} \frac{z^{(r)}}{\sqrt{[r]_p! [r]_q!}}$$

$$= \sum_{r=1}^{\infty} \frac{1}{\sqrt{[r]_p! [r]_q!}} \sqrt{[r]_p\ [r]_q}\ z^{(r-1)}$$

$$= \sum_{r=1}^{\infty} \frac{z^{(r-1)}}{\sqrt{[r-1]_p!\ [r-1]_q!}}$$

$$= \sum_{r=o}^{\infty} \frac{z^{(r)}}{\sqrt{[r]_p!\ [r]_q!}}$$

$$D_{qy}\ \text{bex}(z) = \sum_{r=1}^{\infty} \frac{z^{(r-1)}}{\sqrt{[r-1]_p!\ [r-1]_q!}}$$

$$= \sum_{r=0}^{\infty} \frac{z^{(r)}}{\sqrt{[r]_p!\ [r]_q!}} \ .$$

Hence bex(z) is bibasic analytic in D.

Now we show that $z^{(n)}$ satisfy Laplace's equation.

$$D_{px} z^{(n)} = \sqrt{[n]_p\ [n]_q}\ z^{(n-1)}, \text{ by } (6.24)$$

$$D_{px}{}^2 z^{(n)} = \sqrt{[n]_p\ [n]_q}\sqrt{[n-1]_p\ [n-1]_q}\ z^{(n-2)}. \quad (6.27)$$

Similarly,

$$D_{qy}{}^2 z^{(n)} = \sqrt{[n]_p\ [n]_q}\ \sqrt{[n-1]_p [n-1]_q}\ z^{(n-2)}. \quad (6.28)$$

From (6.27) and (6.28) we get,

$$(D_{px}{}^2 - D_{qy}{}^2)\ z^{(n)} = 0 \ .$$

In a similar manner we get,

$$(D_{px}{}^2 - D_{qy}{}^2)\ \text{bex}(z) = 0 \ .$$

5. DISCRETE BIBASIC ANALYTIC CONTINUATION

Harman [1] has extended the discrete plane to include points on the positive half axes, and has shown that continuation into Q' from both the axes is possible. Here taking a similar approach we consider continuation operators $\mathcal{B}_{px}$ and $\mathcal{B}_{qy}$.

(a) The bibasic operator $Lf(z) = \{[z-px+iqy]f(z) -(1-p)x\ f(x,qy)+(1-q)iy\ f(px,y)\}$, defined by Khan[1] involves a bibasic triad of points

$$T(z) = \{(x,y),\ (px,y),\ (x,qy)\}.$$

From these it follows that given the value of a bibasic analytic function f at any two points of T(z), then it is uniquely determined at the third point.

(b) Hence, if a bibasic analytic function f is defined at the set of points $\{(p^m x,y);\ m \in Z\}$, then it can be uniquely continued as a bibasic analytic function to all points of B' lying below this set of points.

(c) Similarly continuation is possible from $\{(x,q^n y);n \in Z\}$.

(d) If f is defined on the sets $\{(p^m x,y); m \in Z\}$ and $\{(x,q^n y); n \in Z\}$, then it has a unique continuation as a bibasic analytic function to all points of B'.

(e) If f is defined on the sets $\{(p^m x,y); m = 0,1,2,\ldots\}$ and $\{(x,q^n y); n = 0,1,2,\ldots\}$, then by repeated application of (a), the function has unique continuation into the rectangular region $\{(p^m x,q^n y); m = 0,1,\ldots; n = 0,1,\ldots\}$.

(f) Let

$$X = \{(p^m x_o,0);\ m \in Z\}$$

$$Y = \{(0,q^n y_o);\ n \in Z\},$$

where (x_o,y_o) is the fixed point of definition of B', by (6.1). Then we define the extended discrete plane

$$\bar{B} = B' \cup X \cup Y .$$

The discrete rectangular domain R' is defined by

$$R' = \{(p^m x_o,q^n y_o); m=0,1,\ldots;n=0,1,\ldots\} .$$

If X^+, Y^+ are defined by

$$X^+ = \{(p^m x_o,0); m = 0,1,\ldots\}$$

$$Y^+ = \{(0;q^n y_o); n = 0,1,\ldots\},$$

then the extended rectangular domain R is defined as

$$\bar{R} = R' \cup X^+ \cup Y^+. \quad (6.29)$$

The values on the axes, of a discrete function f defined on R' are defined to be

$$f(x,o) = \lim_{n \to \infty} f(x, q^n y_o) = \lim_{y \to \infty} f(x,y)$$

$$f(o,y) = \lim_{m \to \infty} f(p^m x_o, y), (x,y) \varepsilon R'$$

$$= \lim_{x \to o} f(x,y).$$

Now we define the continuation operators as follows,

$$\mathcal{E}_{px}[f(o,y)] = \sum_{j=o}^{\infty} \frac{1}{[j]_p} x^j D_{qy}{}^j [f(o,y)]$$

$$\mathcal{E}_{qy}[f(x,o)] = \sum_{j=o}^{\infty} \frac{1}{[j]_q!} (iy)^j D_{px}{}^j [f(x,o)]$$

To verify these definitions we consider $z^{(n)}$ by continuation.

$$\mathcal{E}_{qy} f[(x,o)] = \sum_{j=o}^{\infty} \frac{1}{[j]_q!}(iy)^j D_{px}{}^j [\sqrt{[n]_p![n]_q!}\ \frac{x^n}{[n]_p!}\ ]$$

$$= \sqrt{\frac{[n]_q!}{[n]_p!}} \sum_{j=o}^{\infty} \frac{1}{[j]_q!} (iy)^j D_{px}{}^j (x^n)$$

$$= \sqrt{\frac{[n]_q!}{[n]_p!}} \sum_{j=o}^{n} \frac{1}{[j]_q!} (iy)^j \frac{[n]_p!}{[n-j]_p!} x^{n-j} ,$$

since $D_{px}{}^j = 0$ when $j > n$.

$$= \sqrt{[n]_p![n]_q!} \sum_{j=o}^{n} \frac{x^{n-j}(iy)^j}{[j]_q![n-j]_p!} \qquad (6.30)$$

Now again,

$$\mathcal{E}_{px}[f(o,y)] = \sum_{j=o}^{\infty} \frac{1}{[j]_p!} x^j D_{qy}^j [f(o,y)]$$

$$= \sum_{j=o}^{n} \frac{1}{[j]_p!} x^j D_{qy}{}^j [\sqrt{[n]_p![n]_q!}\ \frac{(iy)^n}{[n]_q!}\ ]$$

$$= \sqrt{\frac{[n]_p!}{[n]_q!}} \sum_{j=o}^{n} \frac{1}{[j]_p!} x^j \frac{[n]_q}{[n-j]_q!} (iy)^{n-j} ,$$

since $D_{qy}{}^j (iy)^n = 0$ for $j > n$.

$$= \sqrt{[n]_p! \, [n]_q!} \sum_{j=o}^{n} \frac{x^j (iy)^{n-j}}{[n-j_q!][j]_p!} . \qquad (6.31)$$

From (6.30) amd (6.31) we see that continuation operators can be used to derive f(z) since (6.30) and (6.31) are equal to $z^{(n)}$ of (6.23). In a similar manner we can extend the exponential function using continuation operators.

6. DISCRETE BIBASIC MACLAURIN SERIES

We can find an analogue for the Maclaurin series about the point $z_o = 0$. To include the point (o,o), we extend the definition of $\bar{R}$ of (6.29) as follows,

$$\overline{R_o} = \bar{R} \bigcup (o,o) . \qquad (6.32)$$

A discrete function f is said to be bibasic analytic on

$\overline{R_o}$ if it is bibasic analytic on $\overline{R}$ and in addition

$$\lim_{(x,y) \to (o,o)} D^j_{pq}[f(x,y)]$$

exists and the limit is denoted by $D^j_{pq}f(o,o)$.

Under certain conditions the discrete Maclaurin series can be shown to represent a bibasic-analytic function, provided, the series converges. We consider the following.

<u>Theorem 5</u>

Let f be bibasic analytic in $\overline{R_o}$. If $f(z) = \mathcal{L}_{qy}f(x,o) = \mathcal{L}_{px} f(o,y)$, the series representations of $\mathcal{L}_{qy}$, $\mathcal{L}_{px}$, being absolutely convergent in $\overline{R_o}$, then

$$\mathbf{f(z)} = \sum_{j=o}^{\infty} \frac{D_{pq}{}^{j}f(o,o)\ \mathbf{z}^{(j)}}{\sqrt{[j]_p![j]_q!}}, \tag{6.33}$$

the series being absolutely convergent for all $z \in \overline{R_o}$.

<u>Proof</u>

$$f(z) = \mathcal{L}_{px}\ [f(o,y)]$$

$$= \sum_{j=o}^{\infty} \frac{1}{[j]_p!} x^j D^j_{qy} [f(o,y)] .$$

Hence,

$$f(x,o) = \lim_{y \to o} \sum_{j=o}^{\infty} \frac{1}{[j]_p!} x^j D^j_{qy} [f(o,y)]$$

$$= \sum_{j=o}^{\infty} \frac{1}{[j]_p!} x^j D^j_{pq} [f(o,o)] .$$

Also, $f(z) = \mathcal{E}_{qy} f(x,o)$

$$= \sum_{j=o}^{\infty} \frac{1}{[j]_q!} (iy)^j D_{px}{}^j f(x,o) .$$

hence,

$$f(z) = \sum_{j=o}^{\infty} \frac{1}{[j]_q!} (iy)^j D^j_{px} \sum_{k=o}^{\infty} \frac{1}{[k]_p!} x^k D^k_{pq} f(o,o)$$

$$= \sum_{j=o}^{\infty} \frac{1}{[j]_p! [j]_q!} D^j_{pq} f(o,o) \sum_{k=o}^{j} \frac{[j]_p!}{[j-k]_p!} x^{j-k} (iy)^j$$

$$= \sum_{j=o}^{\infty} \frac{D^j_{pq} f(o,o)}{\sqrt{[j]_p! [j]_q!}} \sum_{k=o}^{j} \frac{\sqrt{[j]_p! [j]_q!}}{[j-k]_p! [j]_q!} (iy)^j x^{j-k}$$

$$= \sum_{j=o}^{\infty} \frac{D^j_{pq} f(o,o) z^{(j)}}{\sqrt{[j]_p! \, [j]_q!}} .$$

Hence the result.

If $\lim\sup \left\{ \left| D^j_{pq} \; f(o,o) \right|^{1/j} \right\} = a$, then the series (6.33) converges absolutely for all z such that

$$||z|| < \frac{1}{a} \left[\frac{1}{1-q} + \frac{1}{1-p} \right] ,$$

using the similar approach of Harman [1]. We can prove that the series representations $\mathcal{L}_{qy}$, $\mathcal{L}_{px}$ are uniformly and absolutely convergent in $\overline{R_o}$.

Only a limited treatment of power series has been carried out in the discrete analytic function theory. The form of $z^{(n)}$ and bex(z) of (6.23) and (6.26) suggests that further extensions should be possible. But a suitable convolution operator is to be defined. Of fundamental importance to such a study would be the determination of suitable bounds for the function $D^j_{pq}[f(z_o)]$. This may then lead to general conditions for the convergence of discrete power series.

CONCLUDING REMARKS

In this thesis an attempt has been made to establish a theory of basic and bibasic commutative difference operators, solution of basic and bibasic difference equations, and bibasic pseudoanalytic functions. Classical analysis, q-theory, bibasic theory, q-analytic function theory, bibasic analytic function theory, discrete pseudoanalytic function theory, have been utilised to develop the above concepts.

We have proved that two operators P_m and Q_n are commutative if and only if the characteristic equation $F(P_m, Q_n)=0$ is satisfied. But this can be proved by using integral curves on the lines of J.L. Burchnall and T,W. Chaundy [2,3] where Abelian coefficients occur.

We have used some commutative operators to solve basic and bibasic difference equations. There are some interesting problems related to these theories like study of integral curves and second degree partial difference equations which we have not been attempted here.

Only some properties of bibasic analytic functions have been established here. But the technique can be used for deeper study of such functions.

BIBLIOGRAPHY

ABDI, WAZIR HASAN

[1] On q-Laplace transforms, Proc. Nat. Acad. Sci. India 29, A(1960), pp 389-408.

[2] A study of q-Laplace transforms, Ph.D. Thesis, University of Lucknow (1961).

[3] Application of q-Laplace transforms to the solution of certain q-integral equations, Rend. Circ. Mat. Palermo, 11(1962), pp 1-13.

[4] On certain q-difference equations and q-Laplace transform, Proc. Nat. Inst. Sci. India 28, A(1962), pp 1-15.

[5] A bibasic functional equation, Japan-United States Seminar on ordinary differential and functional equations, Lecture notes in Mathematics 243(1971), pp 324-327.

[6] Complex index solutions of a linear bibasic functional equation, Mathematica Balkanica 4.1(1974), pp 1-7.

[7] Singular solutions of a bibasic functional equation, Proc. Nat. Acad. Sci. (India) (in print).

ADAMS C.R.

[1] The general theory of a class of linear partial q-difference equations, Trans. Amer. Math. Soc. 26(1924), pp. 283-312.

[2] Existence theorems for a linear partial difference equation of the intermediate type, Trans. Amer. Math. Soc. 28(1926), pp 119-128.

[3] On the linear ordinary q-difference equations,
- Annals of Math. 30(1929), pp 195-205.

[4] On the linear partial q-difference equation of
- general type, Trans. Amer. Math. Soc. 31(1929), pp 360-371.

[5] Linear q-difference equations, Bull. Amer. Math
- Soc. 37(1931), pp 361-400.

AGARWAL R.P.

[1] Some basic hypergeometric identities, Annals de
- la sc soc, Brussels (1953).

[2] Certain fractional q-integrals and q-deri vatives,
- Proc. Cambridge Philos. Soc. 66(1969), pp 370.

[3] Certain basic hypergeometric identities associated
- with moch theta functions, Quart. J. Math. Oxford(2) 20(1969), pp 121-128.

AGARWAL R.P and VERMA A

[1] A general transformation theory for basic hyper-
- geometric series with unconnected bases, Proc. Cambridge Phil. Soc. 63(1967).

AL-SALAM W.A.

[1] Some fractional q-integrals and q-derivatives,
- Proc. of the Edinburgh Mathematical Soc. 15(1966), pp 135-140.

AL-SALAM W.A and VERMA A

[1] Some orthogonality Preserving operators, Proc. Amer. Math. Soc. 23, No.1(1969), pp 136-139.

AMITSUR S.A.

[1] 'Commutative linear differential operators', Pacific J. Math. 8(1958), pp 1-10.

BALSAM R.E.

[1] 'Normal ordinary differential operators', Dissertation, University of California, Los Angeles (1965).

BERGMAN S

[1] Linear operators in the theory of partial differential equations, Trans. Amer. Math. Soc. Vol.53(1943), pp 130-155.

BERS L

[1] Theory of pseudoanalytic functions, Lecture notes (mimeographed) New York University (1953).

[2] An outline of the theory of pseudoanalytic functions, Bull. Amer. Math. Soc. Vol.62(1956), pp 291-331.

[3] Formal powers and power series, Comm. Pure. Appl. Math. Vol.9(1956).

BURCHNALL J.L and CHAUNDY T.W.

[1] 'Commutative ordinary differential operators', Proc. London Math. Soc. (Sec.2) 21(1923), pp 420-440.

[2] 'Commutative ordinary differential operators', Proc. Royal Soc. London (A), 118(1928), pp 557-583.

[3] 'Commutative ordinary differential operators II - the identity $P^n=Q^m$', Proc. Royal Soc. London (A), pp 471-485.

CARMICHAEL R.D.

[1] Linear difference equations and their analytic solutions, Trans. Amer. Math. Soc. 12(1911), pp 243-284.

[2] The general theory of linear q-difference equations, Amer. J. Math. 34(1912), pp 147-168.

CHAUNDY T.W.

[1] The differential calculus, Oxford(1935).

[2] Elementary differential equations, Oxford(1969).

[3] Differential equations with polynomial solutions, Quart J. of Math., Vol.20(1949), pp 105-120.

[4] Second-order linear differential equations with polynomial solutions, Quart. J. Math. 4(1953), pp 81-95.

CODDINGTON E.A.

[1] 'Generalized resolutions of the identity for symmetric ordinary differential operators', Ann. of Math., 68(1958), pp 378-392.

[2] 'Normal extensions of formally normal operators', Pacific J. Math, 10(1960), pp 1203-1209.

[3] 'Formally normal operators having no normal extensions', Can. J. Math, 17(1965), pp 1030-1040.

DAVIS H.T.

[1] 'The Theory of Linear Operators', Principia Press, Indiana (1936).

DAVIS R.H.

[1] 'Singular normal differential operators', Technical Report No.10, Department of Mathematics, University of California, Berkelay, California (1955).

DEETER C.R and LORD M.E.

[1] 'Further theory of operational calculus on discrete analytic functions', J. Math. Anal. Appl. 26(1969), pp 92-113.

DUNFORD N and SCHWARTZ J.T

[1] 'Linear Operators, Part II', Interscience, New York (1963).

DUFFIN R.J.

[1] 'Basic properties of discrete analytic functions', Duke Math.J. 23(1956).pp 335-364.

DUFFIN R.J and DURIS C.S.

[1] 'Discrete analytic continuation of solutions of difference equations.' J. Math. Anal. Appl. 9(1964), pp 252-267.

EULER L

[1] 'Introduction in Analysin Infinitorum', Lausanne, Vol.I (1748).

FLANDERS H

[1] 'Commutative linear differential operators' Technical Report No.1, Department of Mathematics, University of California, Berkely California(1955).

HAHN W

[1] 'Beiträge zur theorie der Heineschen Reihen', Math. Nachr. 2(1949), pp 340-379.

[2] 'Uber die höheren Heineschen Reihen und eine einheitliche Theorie der sogenannten speziellen Funktionen', Math. Nachr. 3(1950), pp,257-294.

[3] 'On linear geometric difference equations with
- accessory parameters', Funk. Ekv. 14(1971), pp 73-78.

[4] 'Lineare geometrische differenzen gleichungen',
- Bericht Nr. 169(1981).

[5] 'Commutative linear differential operators',
- Technische Universitat graz Institut für
Mathematik I Graz, Austria (1981).

HARMAN C.J.

[1] 'A discrete analytic theory for geometric difference
- function', Ph.D. Thesis in Mathematics, University
of Adelaide (1972).

HEAVISIDE O.

[1] 'On operators in Physical Mathematics', Proc. Royal
- Socl. A(1893), pp 504-529.

HEINE E

[1] 'Handbuch der theorie der kugel funktionen',
- Berlin 1878.

ISAACS R.P.

[1] 'A finite difference function theory', Universidad
- Nacional Tucuman Revista 2(1941), pp 177-201.

JACKSON F.H.

[1] 'On q-functions and a certain difference operator', Trans. Roy. Soc. Edinburgh 46(1908), pp 253-281.

[2] 'q-form of Taylor's series', Messenger of Math. 38(1909), pp 57-61.

[3] 'q-difference equations,' Amer. J. Math. 32(1910), pp 305-314.

[4] 'On q-definite integrals', Quart. J. Pure and Appl. Math. 41(1910), pp 193-203.

[5] 'Basic integration', Quart. J. Math. 2(1951), pp 1-16.

KHAN M.A.

[1] 'Contributions to the theory of generalised basic hypergeometric series', Ph.D. Thesis, University of Lucknow (1977).

LORD M.E.

[1] 'A discrete algebraic derivative', J. Math. Anal. Appl. 25(1969), pp.701-709.

MASLOV V.P.

[1] 'Operational methods', Moscow (1976).

MERCY K JACOB

[1] 'A study of discrete pseudoanalytic functions,' Ph.D. thesis, University of Cochin (1983).

MILLER W. Jr.

[1] 'Lie Theory and q-difference equations', Siam J. Math. Anal 1(1970), pp 171-188.

MILNE-THOMSON L.M.

[1] 'The calculus of finite differences', Macmillan(1965).

PINCHERLE S

[1] 'Ricerche Sopra una classe important die funzioni monodrome', Giornale die matematiche 18(1880), pp 92-136.

STARCHER G.W.

[1] 'On identities arising from solutions of q-difference equations and interpretations in number theory', Bull. Amer.Math. Soc. 36(1930), pp 577-581.

SAWYER W.W.

[1] 'Differential equations with polynomial solutions,'
- Quart.J. of Math. Vol.20(1949), pp.20-30.

SELMO TAUBER and DONALD DEAN

[1] 'On difference equations equations containing step
- and delta functions', J. Soc. Indust. Appl. Math.
Vol.8, No.1(1960).

TRIZINSKY W.J.

[1] 'Analytic theory of q-difference equations,' Acta
- Math. 61(1931), pp 1-38.

UPADHYAY M

[1] 'Q-differential equations with polynomial solutions',
- Proc. Amer. Math. Soc. 18(1967), pp 57-62.

[2] 'Fractional q-integration and integral representations
- of basic double hypergeometric series of higher order,'
Ganita, Vol.23 No.1(1972).

VEKUA I.N.

[1] 'Generalised analytic functions,' Addison Wesley
- Pub. Co. London (1962).

VELUKUTTY K.K.

[1] 'Discrete bianalytic functions', Proc. Nat. Acad.
- Sci. India, 52(A), I (1982), pp 69-74.

VERMA A and PRASAD J

[1] 'Characterisation of some orthogonal polynomials', publications De L'institut mathematique Nouvelle serie, tome 9(23), 1969, pp 177-180.

WILLIAMS W.E.

[1] 'Properties of the solution of a certain functional equation', Proc. Camb. Math. Soc. 57(1961), pp 439-441.