STUDIES IN QUANTUM FIELD THEORY AT FINITE TEMPERATURE

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Shri V.C. Kuriakose, under my guidance in the Department of Physics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

Cochin - 22
September 15, 1982

K. Babu Joseph
Supervising teacher

DECLARATION

Certified that the work presented in this thesis is based on the original work done by me under the guidance of Dr.K. Babu Joseph in the Department of Physics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

Cochin - 22
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PREFACE

The work presented in this thesis has been carried out by the author in the Department of Physics, University of Cochin during 1978-82.

The thesis deals with certain quantum field systems exhibiting spontaneous symmetry breaking and their response to temperature. These models find application in diverse branches such as particle physics, solid state physics and non-linear optics. The nature of phase transition that these systems may undergo is also investigated.

The thesis contains seven chapters. The first chapter is introductory and gives a brief account of the various phenomena associated with spontaneous symmetry breaking. The chapter closes with a note on the effect of temperature on quantum field systems. In chapter 2, the spontaneous symmetry breaking phenomena are reviewed in more detail. Chapter 3, deals with the formulation of ordinary and generalised sine-Gordon field theories on a lattice and the study of the nature of phase transition occurring in these systems. In chapter 4, the effect of temperature on these models is studied, using the effective potential method. Chapter 5 is a continuation of this study for another model, viz, the φ^6 model. The nature of phase transition is also studied. Chapters 5 and 6 constitute a report of the investigations on the behaviour of coupling constants under thermal excitation in φ^4 theory, scalar electrodynamics, abelian and non-abelian gauge theories.

A part of these investigations has appeared in the form of the following publications:

1. Second order phase transition in two dimensional sine-Gordon field theory - Lattice model
   Pramana 11, 17 (1978)

2. Effective potential in sine-Gordon theory at finite temperature
   Phys. Lett. 88A, 447 (1982)

3. Finite temperature behaviour of a φ^6 field system
   J. Phys. A; Math. Gen. 15, 2231 (1982)

4. Temperature dependence of coupling constants
   Phys. Lett. B (In press)

5. Gauge couplings in hot environments
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Part of the work reported in this thesis has been carried out by me as a part-time research scholar while teaching at Union Christian College, Alwaye. But the award of a Teacher Fellowship by the University Grants Commission, New Delhi, enabled me to join this Department as a full-time research scholar. I am thankful to the University Grants Commission for providing me financial assistance in the form of a Teacher Fellowship and to the authorities of Union Christian College, Alwaye, and to the Government of Kerala for granting me deputation.

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SYNOPSIS

The thesis deals with certain quantum field systems exhibiting spontaneous symmetry breaking (SSB) and their response to temperature. These models are of interest in particle physics, solid state physics and nonlinear optics. The nature of phase transition that these systems may undergo is also investigated. The main theoretical tools employed to understand these properties are the lattice formalism, effective potential method and the renormalisation group.

Lattice gauge theory is a nonperturbative formalism and constitutes an attempt to explain various problems in quantum field theory, which were hitherto unanswered. There exists a close resemblance between this formulation and statistical physics. In fact many of the phenomena discovered independently in particle physics and condensed matter physics bear close resemblance as far as the theoretical questions are concerned. Phase transition is a process associated with the change of symmetry. Sine-Gordon field system is a nonlinear scalar field theory which exhibits SSB. In order to study the nature of phase transition associated with this symmetry breaking, the lattice formulation is used. The present investigation reveals that the sine-Gordon field system in 1+1 dimension undergoes a

second order phase transition from the disordered phase characterised by $\langle \varphi \rangle = 0$ to the ordered phase characterised by $\langle \varphi \rangle \neq 0$ where $\langle \varphi \rangle$ denotes the vacuum expectation value. The critical values of the parameters which characterise these two phases are also evaluated. The method is extended to generalised sine-Gordon field systems exhibiting a second order phase transition.

The fact that SSB in relativistic field theories disappears when the temperature of the system is increased above a critical value has significant consequences in particle physics. The most important physical consideration in statistical mechanics is the introduction of temperature. The relationship between SSB and temperature is investigated via the effective potential defined at finite temperature. As an application of this formalism, the effect of temperature on the 1+1 dimensional sine-Gordon field system discussed above is studied. The effective potential in the one-loop order at zero temperature is evaluated with a view to obtaining the renormalised Lagrangian for the system. The advantage of this procedure is that the quantum correction to soliton mass immediately follows. By evaluating the effective potential at finite temperature, the critical temperature above which the system regains the original symmetry is obtained. The soliton energy at finite temperature is found to be less than the classical soliton energy, and decreases smoothly with temperature and finally vanishes at the critical temperature, signalling a second order phase transition. The calculations are extended to the generalised sine-Gordon models. The study of generalised sine-Gordon models is motivated by the fact that for a particular case, the generalised sine-Gordon becomes the double sine-Gordon model which finds many applications in contemporary physics.

The formalism discussed above is applied to a φ^6 field system in 1+1 dimensions, which is a nonlinear field theory exhibiting SSB. This model is of importance in the theory of ferroelectric and structural phase transitions encountered in solid state physics. First, the model under consideration is shown to be renormalisable by evaluating the effective potential at zero temperature. Then the effective potential at finite temperature is evaluated. The calculations are done upto two-loop level. The critical temperature above which the system regains the symmetry property is calculated. Here the nature of phase transition of the model field system is investigated by evaluating the thermal average of the field variable and is shown to be one of first order.

With the advent of the Grand Unified era, studies on finite temperature behaviour of quantum field systems are of prime importance since these studies will help us to

speculate on the evolution of the universe at times as early as 10^{-35} sec after the big bang. In these studies the dependence of the coupling constant on temperature is to be taken into account as it can reduce drastically the amount of supercooling associated with the first order phase transition in grand unified theories. Using the vertex renormalisation procedure, the temperature behaviour of the coupling constant in two models is investigated. The massive φ^4 theory is considered first. The present investigation shows that when $m^2 > 0$, the scalar coupling constant decreases with temperature leading to a phase transition to a non-interacting phase. In a model with $m^2 < 0$, the coupling constant increases logarithmically. A renormalisation group study of the problem also supports these results. The temperature behaviour of the gauge coupling constant in scalar electrodynamics is also studied and it is found that the gauge coupling constant uniformly increases with temperature. This study is extended to include the temperature dependence of gauge coupling constants in abelian and non-abelian gauge theories. It is found that the gauge coupling constant in abelian gauge theory increases with temperature while that of non-abelian gauge theory decreases with temperature consistent with the idea of asymptotic freedom. This investigation is of relevance to Grand Unified Models of Cosmoligical evolution.

CONTENTS

One

INTRODUCTION

The success of Glashow, Weinberg, and Salam in unifying electromagnetic and weak interactions is a sign of progress towards understanding the intricacies of the fundamental interactions of nature. The formulation of unified gauge theories for strong, weak and electromagnetic interactions has led to new developments in the theories of elementary particles and quantum fields. Another development of considerable significance which has attracted great attention from the beginning of the last decade is the one relating to quark confinement. Various theories have been put forward to give a satisfactory explanation for the absence of free quarks. A new theory called quantum chromodynamics (QCD) along the line of quantum electrodynamics (QED) has been developed for strong interactions. QCD exhibits exact local colour symmetry.

Non-linear field systems are currently of considerable importance in various branches of physics. These systems are of special interest in the study of elementary particles and quantum field theory as they possess a particle spectrum which survives quantization. The most interesting of these objects, which are reminiscent of hadrons, arise in theories with spontaneous symmetry breaking (SSB). Therefore, it is natural to investigate the criteria for breaking symmetries spontaneously as well as the nature of the phase transition associated with symmetry breaking. Recently increasing attention has been paid to the study of the behaviour of quantum field systems at finite temperature, especially those systems with SSB. It is hoped that these studies will lead to a more transparent view of the various physical processes which may have occurred during the early stages of evolution of the universe. It is conjectured that the structure and properties of elementary particles in the early universe may be entirely different from those of the present time.

Spontaneously Broken Gauge Theories

The basis of a symmetry principle in physics is that some quantities remain invariant under certain transformations like translation, rotation etc. Gauge theories [1-5] are characterised by their invariance under a group (the gauge group) of symmetry transformations. Based on the type of gauge group involved in the symmetry transformation, gauge theories are classified into abelian and non-abelian types. The simplest gauge group is U(1) and the corresponding gauge theories are called abelian gauge theories. If higher symmetry groups such as SU(2), SU(3) etc. are involved, then the theory becomes non-abelian.

Consider the Lagrangian of a free fermion field $\psi(x)$:

$$\mathcal{L}_o = \bar{\psi}(x)\ (i\not{\partial} - m)\psi(x), \tag{1.1}$$

which is invariant under the phase transformation

$$\psi(x) \longrightarrow e^{i\alpha}\ \psi(x), \tag{1.2a}$$

where α is x-independent. This implies that the derivative of the field transforms like the field itself:

$$\partial_\mu \psi(x) \longrightarrow e^{i\alpha}\ \partial_\mu \psi(x). \tag{1.2b}$$

The group of transformation (1.2) is the abelian group U(1).

Yang and Mills [6] generalised the principle of gauge invariance to the case where the invariance is associated with a non-abelian internal symmetry group - SU(2). Their procedure can, indeed, be generalised to any internal symmetry group. Let $\varphi^i(x)$ (i=1,2,...n) be a set of fields. The Lagrangian describing the dynamics of the system will be invariant under a group G of transformations of the fields $\varphi^i(x)$, given by

$$\varphi^i(x) \longrightarrow \varphi^i(x) - i\alpha^a\ T^a_{ij}\ \varphi^j(x). \tag{1.3a}$$

Here T^a are the matrices of the representation to which the fields $\varphi(x)$ belong; α^a are C-numbers, infinitesimal x-independent parameters. Hence the derivatives of the fields transform like the fields themselves:

$$\partial_\mu \varphi^i(x) \longrightarrow \partial_\mu \varphi^i(x) - i\alpha^a \, T^a_{ij} \, \partial_\mu(\varphi^j(x)) . \qquad (1.3b)$$

The type of gauge invariance discussed above is known as global gauge invariance.

Now let α be x-dependent,

$$\Psi(x) \longrightarrow e^{i\alpha(x)} \Psi(x), \qquad (1.4a)$$

then, it can be seen that the derivative of the field no longer transforms like the field. This is true for abelian as well as non-abelian groups. But the local gauge invariance can be maintained by defining a covariant derivative D_μ:

$$D_\mu \Psi(x) \longrightarrow e^{i\alpha(x)} D_\mu \Psi(x). \qquad (1.4b)$$

To construct such an operator as D_μ, a new vector field $A_\mu(x)$ called a gauge field is introduced, which transforms according to

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{g} \partial_\mu \alpha(x), \qquad (1.5)$$

where g is a constant. Defining D_μ as

$$D_\mu = \partial_\mu - igA_\mu, \tag{1.6}$$

we can write the locally gauge invariant Lagrangian as

$$\mathcal{L}_0 \longrightarrow \mathcal{L}_1 = \bar{\psi}(x)(i\not{D} - m)\psi(x)$$

$$= \bar{\psi}(x)(i\not{\partial}-m)\psi(x) + g\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x) \tag{1.7}$$

If we wish to interpret A_μ as the field representing the photon, then a term corresponding to its kinetic energy must be added;

$$\mathcal{L}_1 \longrightarrow \mathcal{L}_2 = \mathcal{L}_1 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \tag{1.8}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{1.9}$$

Eq.(1.8) is nothing but the Lagrangian for QED. It can be seen that local gauge invariance will be spoiled if $\mathcal{L}_2$ contains a term proportional to $A_\mu A^\mu$. In other words, local gauge invariance demands the photon to be massless.

The same analysis can be extended to non-abelian gauge groups:

$$\varphi^i(x) \qquad \varphi^i(x) - i\alpha^a(x)\, T^a_{ij}\,\varphi^j(x). \tag{1.10}$$

Here N gauge fields $A_\mu^a(x)$ transforming according to

$$A_\mu^a(x) \longrightarrow A_\mu^a(x) - \frac{1}{g}\,\partial_\mu\alpha^a(x) + f_{abc}\alpha^b A_\mu^c, \qquad (1.11)$$

are introduced. g as before, is a constant and f_{abc} are the structure constants of the group,

$$[T_a, T_b] = i\, f_{abc} T_c. \qquad (1.12)$$

The covariant derivative

$$D_\mu \longrightarrow \partial_\mu - ig\, T_{ij}^a A_\mu^a(x), \qquad (1.13)$$

The Lagrangian which is locally gauge invariant can now be written:

$$\mathcal{L}_2 = \mathcal{L}_1 - \frac{1}{4} F_{\mu\nu}^a\, F^{\mu\nu a}, \qquad (1.14)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, f_{abc} A_\mu^b A_\nu^c \qquad (1.15)$$

In this case also the price that we have to pay for gauge invariance is that the gauge field shall be massless. Hence it is argued that local gauge invariance requires a set of massless vector fields, and the number of massless gauge vector bosons equals the number of generators of the gauge group.

It is found that, eventhough, the Lagrangian is invariant under some symmetry transformation, the vacuum of the theory may not be invariant. Then the symmetry is said to be spontaneously broken and there exist massless particles [7-9]. This result is known in the literature as the Goldstone theorem, and the corresponding massless particles are called Goldstone bosons. But, through the works of Higgs [10-12] and others [13-15], it was already known since 1964, that when a local gauge symmetry is spontaneously broken the vector particles acquire mass, and there would be no Goldstone bosons. This phenomenon is nowadays known as Higgs mechanism.

Unified Model of Weak and Electromagnetic Interactions

The classic work of Fermi [16] on β-decay can be treated as the starting point of several attempts made towards unifying weak and electromagnetic interactions. Though these two forces of interaction show some common features, they differ on various points. Both interactions are mediated by intermediate vector particles; nevertheless, there is a major difference between the two: weak interaction is mediated by massive vector mesons (W bosons) while electromagnetic interaction is mediated by a massless vector particle - the photon. The bosons being massive, the traditional theory of weak interaction is non-renormalisable, while the theory of electromagnetic interaction is renormalisable. The QED was, for years, the only consistent and successful theory we had in elementary particle physics.

It was Glashow [17] who first suggested the $SU(2)_L \otimes U(1)$ model for unified electromagnetic and weak interactions, and in this model the W-boson mass terms were put in by hand. The model in its present form was developed by Weinberg [18] and Salam [19]. They used the Higgs mechanism for generating W-boson masses. The advantage of using the Higgs mechanism is two fold. First, it explains the discrepancy between the photon and the intermediate vector boson masses; the photon remains massless as it corresponds to the unbroken symmetry subgroup associated with the conservation of charge, while the intermediate vector bosons get masses, as they correspond to symmetries which are broken. Secondly, the avoidance of an explicit vector boson mass term in the Lagrangian is necessary for the renormalisability of the model. This model is gauge invariant and renormalisable. The renormalisability of the 1967 model was proved by 't Hooft [20-22].

The most important criticism [3] of the existing gauge models is that none of these theories is sufficiently natural. That is, the parameters in these theories have to be carefully rigged so as to achieve even a qualitative agreement with experiment [23].

Quantum Chromodynamics

The salient features of strong interactions of hadrons may be understood using the quark model [24-28]. The factors which distinguish strong interaction from the others are: 1) at short distances (high momentum transfers) hadrons appear to consist of weakly or noninteracting quarks, 2) at long distances (low momentum transfers) the hadrons appear to consist of strongly interacting quarks. These facts suggest that the strength of the interaction between quarks is zero or very small when quarks are close together (asymptotic freedom) but becomes high when the quarks are far apart. It would therefore be impossible or very difficult for quarks to be separated from each other [29-30]. The index distinguishing different types of quarks is called flavour and each quark with a given flavour comes in three colours, with the same particle properties, except the colour quantum number. Physical hadrons are neutral with respect to the colour quantum number, or they are colour singlets.

A quantum field theoretic formulation, where all these aspects can be explained, is that of QCD [31-36], which is a non-abelian gauge theory based on the $SU(3)_c$

local gauge group. It has been formulated with the hope that it will provide a complete description of all strong interaction physics, just as QED has provided a highly accurate account of the interactions of photons and electrons. QCD is a renormalisable quantum field theory of strong interactions. The fundamental constituents of QCD are spin $\frac{1}{2}$ quarks and spin 1 gluons which interact with the quarks as well as among themselves [31,37]. In QCD the flavour index has no dynamical role while it is invariant under local $SU(3)_c$ symmetry. Hence a non-abelian gauge field $A_\mu^a(x)$, a=1,2,...8 which transforms as the adjoint representation of $SU(3)_c$ is needed. The quanta of the gauge field called gluons are coloured. A renormalisable locally $SU(3)_c$ gauge invariant Lagrangian can be written

$$\mathcal{L}_{SI} = i\, \bar{q}_\alpha^A\, \not{D}_{\alpha\beta} q_\beta^A - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \qquad (1.16)$$

$$(\alpha,\beta = 1,2,3$$
$$a = 1,2,\ldots 8$$
$$A = 1,2,\ldots N)\ ,$$

where

$$\not{D}_{\alpha\beta} = (\delta_{\alpha\beta}\partial_\mu - ig\, \lambda_{\alpha\beta}^a\, A_\mu^a)\gamma^\mu, \qquad (1.17)$$

and

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\, f_{abc} A_\mu^b A_\nu^c. \qquad (1.18)$$

The λ^a are $SU(3)_c$ matrices obeying the commutation relations:

$$[\lambda^a, \lambda^b] = i f^{abc} \lambda^c, \tag{1.19}$$

with f^{abc} the structure constants of $SU(3)_c$.

Asymptotic freedom. The failure of experiments hitherto performed to observe free quarks, requires that the nature of forces between quarks is essentially different from that of weak and electromagnetic interactions. The quark forces become weaker and weaker for short distances, while they grow stronger and stronger for longer distances. The property that the interactions between quarks is zero or vanishingly small, when the quarks are close together, is termed 'asymptotic freedom' - ie., the quarks behave as if they are free. This means that the $SU(3)_c$ gauge coupling constant g or the strong fine structure constant or running coupling constant $\alpha_s = g^2/4\pi$ becomes vanishingly small for short distances α_s is a function of q^2 where q_μ is a typical momentum relevant to the process considered. Only non-abelian gauge theories are asymptotically free [38-41]. QCD exhibits asymptotic freedom, and the observed scaling is good evidence that the strong interactions are described by non-abelian gauge fields. Using the renormalisation group technique, the strong fine structure constant is given by

$$\alpha_s(q^2) = \frac{g^2(q^2)}{4\pi} = \frac{12\pi}{(33-2N_f)\ \ln(q^2/\Lambda^2)} \qquad (1.20)$$

where N is the number of flavours. Λ being the only free parameter of QCD which has to be fixed by experiments.

Confinement of quarks. The success of quark model to explain properties of hadrons and resonances is beyond any doubt [42,43]. Though there are certain claims that fractionally charged particles have been observed [44,45] the present belief is that quarks are permanently hidden inside hadrons, and they cannot come out as free particles. There are various proposals to explain the confinement of quarks. A nice phenomenological approach to quark confinement is the method of semiclassical bag models; and in particular the MIT type models [46-48]. The bag models treat the hadron as a finite region in space to which almost free and light quarks are confined; the confinement is made possible by using certain boundary conditions. Though this method is less ambitious on theoretical grounds it has been impressively successful in reproducing known physical properties of several hadronic states.

Another method involves the construction of exact solutions of classical field equations in the interacting field theory that localize the energy in space, and that cannot be found by iterative procedures, starting from empty space vacua. Such solutions are called solitons [49-51],and they occur in field theories with degenerate vacua, in which case, they are frequently referred to as topological solitons [31,52-55].

The usual practice of removing the infinities hidden in quantum field theory is by introducing a cut-off, and working with a theory which is finite at each stage. However, Wilson [56] introduced the cut-off in a novel way, by formulating the theory on a finite lattice, with a finite number of degrees of freedom. He used the lattice version of the Feynman path integral formalism [1,57,58], and demonstrated the quark confinement mechanism. When a single abelian gauge field is coupled to massive quarks, there are two possible situations: 1) in the weak coupling limit (coupling limit $g \rightarrow 0$) the gauge field behaves like a normal free zero mass field and the quarks are unbound, while 2) in the strong coupling limit, the gauge field is massive, and the quarks are bound. This mechanism is similar to the one proposed by Schwinger [59,60] for two dimensional QED. Kogut and Susskind [61,62] gave a canonical Hamiltonian version of Wilson's lattice gauge theory. In their formalism the structure of the model is reduced to the interactions of an infinite collection of coupled rigid rotators. The gauge invariant configuration space consists of a collection of strings with quarks at their ends. The strings are lines of non-abelian electric flux. Quark confinement is a result of the inability of a string to break up without producing a pair of quarks.

Grand Unified Theories

A major breakthrough in the search for unity of all fundamental forces observed in nature, is the formulation of grand unified theories (GUTS) [63-67]. GUTS suggest that the strong and electroweak interactions originate from a single set of interactions, characterised by a single, basic, gauge coupling constant, while the 'matter fields' (quarks and leptons) are members of one multiplet. One of the spectacular aspects of this formulation is the prediction of decay of protons [68,69] resulting in baryon nonconservation. The high value of the lifetime of protons ($\simeq 10^{30}$ yrs.) entails that the grand unification scale is cosmologically high; hence, it is legitimate to assume that GUTS may have played an important role in the early stages of the evolution of the universe ($t < 10^{-35}$s) [68-73]. The basic hypothesis of grand unification is that there exists a simple group $G \supset G_1$, which is characterised by a single coupling constant g, and that all interactions are generated by G. In this representation quarks and leptons are members of the same multiplet of the group. The symmetry breakdown pattern is as shown below.

$$G \xrightarrow[(10^{15}\text{Gev})]{} G_1[\equiv SU(3)_c \otimes SU(2) \otimes U(1)] \xrightarrow[(10^{2}\text{Gev})]{} G_2[\equiv SU(3)_c \otimes U(1)]_{e.m}$$

Georgi and Glashow [65] showed that the SU(5) gauge group has the essential properties needed to represent G:

$$SU(5) \longrightarrow SU(3)_c \otimes SU(2) \otimes U(1) \longrightarrow SU(3)_c \otimes U(1)_{e.m.}$$

Using the renormalisation group approach [68-70] it can be seen that the U(1) coupling constant slightly increases with energy, while those of SU(2) and SU(3) decrease with energy.

Non-linear Scalar Field Theories

Non-linear scalar field theories are currently of considerable importance in various branches of physics like laser physics, solid state physics and particle physics [74-79]. Particle physicists are interested in certain non-linear fields as they possess a particle spectrum which survives quantisation [80-84]. Models such as φ^4, φ^6 and sine-Gordon are among the spontaneously broken non-linear field systems which are of considerable interest. Though perturbation theory is highly successful in lepton QED, there are phenomena where this formalism breaks down. In such cases one has to adopt some nonperturbative techniques, and the non-linear field theory is one where the non-perturbative approach has to be employed. These non-perturbative calculations have exposed in non-linear systems an unexpectedly rich particle structure in the quantal Hilbert space, resulting in novel effects like emergence of fermions from bosons, and also have provided new mechanisms for SSB without Goldstone bosons [83].

There are two main streams in the application of nonperturbative techniques in field theory; one is concerned with the finding of exact solutions of classical field eqns. [49-51] and the other mainly consists of approximation procedures such as semiclassical and lattice methods.

Semiclassical approximation methods. In this approach all field equations are treated as if they were describing classical field configurations, rather than quantum operator fields. The quantum property is, however, regained by quantising the classical solution through semiclassical methods [80-91]. In the case of spinless model, it is possible to find stable static solutions in one-space, one-time dimension, and higher dimensions are required for models with spin [83]. Usually two methods are employed; the first method gives time-independent classical solutions (static), and is relevant to quantum theory in the weak coupling limit [32]. The spirit of this procedure lies in expanding the Lagrangian about a time-independent solution in a harmonic oscillator approximation. This method is non-perturbative in the sense that it will yield states not accessible from free field theory by a perturbation expansion in the coupling constant. This method can be illustrated by the φ^4 model in 1+1 dimensions [82,86]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4} \varphi^4 \qquad (1.21)$$

The Lagrangian possesses $\varphi \longleftrightarrow -\varphi$ discrete symmetry. The potential function corresponding to this Lagrangian is:

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4. \qquad (1.22)$$

If $m^2 > 0$, the function V looks like fig.1.1 and the minimum occurs at $\varphi = 0$. On the other hand, if $m^2 < 0$, the potential looks like fig.1.2. In this case, V has two absolute minima at $\varphi = \pm m/\sqrt{\lambda}$, with a local maximum at $\varphi = 0$. This indicates that the symmetry is spontaneously broken by the vacuum state. The vacuum expectation value (vev) of the quantum field φ is

$$\langle 0|\varphi|0\rangle = m/\sqrt{\lambda} \ , \tag{1.23}$$

but the mass of the 'mesons' in the φ^4 theory is 2m. It can be seen that the Lagrangian (1.21) leads to a time-independent, but space-dependent solution of the form:

$$\varphi(x) = \frac{m}{\sqrt{\lambda}} \tanh \frac{m(x-a)}{\sqrt{2}} = \varphi^{(a)}_{kink} \ , \tag{1.24}$$

for any constant value of a. This solution is usually called the kink. It has a shape as shown in fig.1.3 and asymptotically approaches the two different values $\pm\, m/\sqrt{\lambda}$. The classical kink energy or mass has the value

$$E_c(\varphi_c) = \frac{2\sqrt{2}}{3}\,\frac{m^3}{\lambda} \tag{1.25}$$

The quantum-corrected mass or energy of the kink is [82]

$$E_k = \frac{2\sqrt{2}}{3\lambda}\,m^3 + \frac{m}{2\sqrt{6}} - \frac{3m}{\pi\sqrt{2}} \tag{1.26}$$

The second method [80,82] is a generalisation to field theory of the WKB approximation in quantum mechanics. In the weak coupling method, quantum fluctuations are considered around static classical solutions, while in this case, it is around certain periodic time-dependent classical solutions, chosen by imposing generalised Bohr-Sommerfeld quantisation conditions. As an application of this method, let us consider a two dimensional sine-Gordon field theory defined by the Lagrangian

$$\mathcal{L}(\varphi(x,t)) = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\varphi\right) - 1\right] \quad (1.27)$$

The equation of motion is

$$\Box \varphi + \frac{m^3}{\sqrt{\lambda}} \sin\left(\frac{\sqrt{\lambda}}{m}\varphi\right) = 0 \quad (1.28)$$

The Lagrangian (1.27) is invariant under the discrete symmetry, $\varphi \longleftrightarrow -\varphi$. The potential in this model has an infinite number of discrete degenerate minima at $\varphi = \frac{m}{\sqrt{\lambda}} 2\pi n$ $(n=0, \pm 1, \pm 2, \ldots)$, and this could lead to SSB. Analogous to the kink solution of the φ^4 theory, there exist in this case also static solutions connecting two different classical vacua at the two ends of space [fig.1.4]. The static solution is

$$\varphi_s(x) = \frac{4m}{\sqrt{\lambda}} \tan^{-1}(e^{xm}) , \quad (1.29)$$

with finite classical energy

$$E_c(\varphi_c) = \frac{8m^3}{\lambda} . \quad (1.30)$$

The quantum-corrected soliton mass is given by

$$M_{sol} = \frac{8m}{\gamma} \tag{1.31}$$

where

$$\gamma = \frac{\lambda/m^2}{(1 - \lambda/8\pi m^2)}$$

A solution of the form

$$-\varphi_s(x) = \frac{4m}{\sqrt{\lambda}} \tan^{-1}(e^{-xm}) = \varphi_A, \tag{1.32}$$

will also be soliton solution and is customarily called an antisoliton.

There is another class of solution called 'doublet' or 'breather'. The forces between a soliton and an anti-soliton are attractive, and this could lead to classically bound soliton - antisoliton solutions. Such solutions are time-dependent [80,82]:

$$\varphi_B(x,t) = \frac{4m}{\sqrt{\lambda}} \tan^{-1}[\tan(\frac{N\gamma}{16}) \frac{\sin(mt \cos N\gamma/16)}{\cosh(mx \sin N\gamma/16)}] \tag{1.33}$$

where $N < 1,2,\ldots,8\pi/\gamma$. The shape of this solution is as shown in fig.1.5. The doublet masses are given by

$$M_N = \frac{16m}{\gamma} \sin(N\gamma/16) \tag{1.34}$$

This can be obtained by using the WKB quantisation procedure on the doublet solution.

Lattice theories. The programme initiated by Wilson [56] has attracted wide interest among physicists. Lattice formulation, which is a nonperturbative method, is of immense help in attacking problems associated with large distance behaviour, such as quark confinement, phase transitions, low lying spectra, coherent excitations and so on. The theory loses Lorentz invariance when formulated on a lattice. It is hoped that the continuum properties will reappear, when the lattice spacing goes to zero. Renormalisation group techniques [92,93] provide means to solve this problem.

There are two approaches to define a lattice theory. In the Hamiltonian formalism [61,62] space is replaced by a 3-dimensional lattice, while time remains continuous. The other approach is [56] the Euclidean lattice theory where the space and time are on equal footing, and are replaced by a four dimensional lattice. The quantisation is carried out [94,95], using the Feynman path integral technique. The starting point is the assumption of existence of an action integral $S[\varphi]$ depending upon classical fields, collectively denoted by $\varphi(x)$. The mean value of a physical quantity X is given by

$$\langle X \rangle = Z^{-1} \int \mathcal{D}\varphi X \exp(S[\varphi]) \tag{1.35}$$

where

$$Z = \int \mathcal{D}\varphi \exp(S[\varphi]) \tag{1.36}$$

The Euclidean Lagrangian lattice formalism is essentially identical with its statistical counterpart [94] in the following manner:

The field action	$\longleftrightarrow$	energy of a configuration
Vacuum functional integral	$\longleftrightarrow$	partition function
Coupling constant	$\longleftrightarrow$	temperature

The requirement of recovering a continuous theory in the limit of zero spacing is the same as finding a critical domain in statistical physics. The main difference between the two is that for most dynamical systems the relevant dimension of statistical physics is three, while Euclidean space-time has dimension four. This is fortunate in the sense that four seems to be the lowest critical dimension of systems with local symmetries. A derivative $\partial_\mu \varphi(x)$ is replaced by a sum over finite difference between different lattice sites. Thus

$$\partial_\mu \varphi(x) \longrightarrow \frac{1}{a} (\varphi_{x+\dot{a}\mu} - \varphi_x) \tag{1.37}$$

where a is the lattice spacing and $\hat{\mu}$ is the unit vector in the direction μ.

In the lattice formulation the inverse of the lattice spacing serves as the natural cut-off, the matter field is defined at the individual sites, while the gauge field, across the links joining the lattice sites.

Drell et al [96] have developed variational group techniques to study field theories on a discrete lattice. Their formulation differs from those of others on the definition of the gradient operator on the lattice. They defined the gradient operator as

$$\nabla f_j = \sum_k ik\, e^{ik\, j/\Lambda} f(k)$$

$$= \sum_{j'} f_{j'} \frac{1}{(2N+1)^2} \sum_k ike^{ik\,(j-j')/\Lambda} \qquad (1.38)$$

where Λ is the reciprocal of the lattice spacing, N is the number of lattice sites

and

$$f_j = \sum_{\underline{k}} e^{ik\, j/\Lambda} f(\underline{k}) \qquad (1.39)$$

The new definition of the gradient has the following advantages over the customary one (1.33): 1) for a free field, it leads to an energy spectrum in accord with the relativistic form of the energy - momentum relation for all $k \leq k_{max}$. 2) It automatically avoids doubling the fermion degree of freedom [61,62]. This method has been applied to the calculation of the ground state and low lying excitations of various quantum theories that are rendered finite in terms of a cut-off [97-100].

Phase Transition in Quantum Field Theory

It has been realised that there is a close relationship between particle physics and condensed matter physics; many of the theoretical puzzles of the two branches are closely related [101], and there exists a one-to-one correspondence between quantum field theory and statistical physics [102,103]. A number of phenomena discovered in these two branches exhibit close resemblance. For instance, the idea of SSB in quantum field theory - central to electroweak interaction and GUTS - was used long ago in solid state physics and in quantum statistics applied to theories of such phenomena as ferromagnetism, superfluidity and superconductivity. The Higgs mechanism of field theory resembles the Meissner effect observed in superconductivity [104-107].

The lattice gauge theory, put forward by Wilson, enhanced the use of the language of statistical physics in particle physics, and when the quantum field theory is formulated on a lattice, one can find the corresponding magnetic analogues, such as Ising, XY, Heisenberg models. In fact, any phase transition is a process associated with a change of symmetry. Phenomena which illustrate this, are the theories of Bose condensation, superconductivity, ferromagnetism etc. Statistical systems exhibit various types of phase transition as the temperature of the system is varied and distinct phases exhibit different qualitative properties. There exist a close analogy between gauge theories in d dimensions and nearest neighbour spin systems in d/2 dimensions. Thus four dimensional gauge models should exhibit a phase structure, similar to that of two dimensional statistical systems. The following table shows the systems that exhibit analogous behaviour [108].

4d-Gauge Theories	2d-Magnets
Z_2	Ising Model
Abelian Gauge	XY Model
Non-abelian	Heisenberg

Details of the analogy between statistical mechanics and equivalent Euclidean quantum systems are summarised below.

Quantum system	Statistical system
Ground state	Equilibrium state
Ground state expectation values of time ordered operators	Averages on the ensemble
Vacuum expectation value of field	Order parameter
Ground state energy	Free energy

Though the Monte-Carlo method of stimu'ating statistical system is an old one, only recently has it been applied to various gauge systems [109].

Quantum Field Theory at Finite Temperature

The discovery of Kirzhnits and Linde [110] that the spontaneous symmetry breaking in a relativistic field theory will disappear above a critical temperature led many physicists to study the behaviour of quantum field systems at finite temperature. We saw in the last section the correspondance between the quantum field systems and statistical systems. It is already an established fact that the broken symmetry of macroscopic systems ('ordered' systems) like ferromagnets, superconductors, will be restored above a critical temperature. Weinberg [111] showed how to calculate the critical temperature for general renormalisable field theories with symmetry broken at zero temperature.

The first step in the study of a quantum field theory at finite temperature, is to obtain the temperature Green's functions. The finite temperature Green's function is also given by a sum of Feynman diagrams, just as in zero temperature field theory, except that the energy integrals are replaced by sums over a discrete set of energy values [111]. Dolan and Jackiw [112] studied the finite temperature behaviour of quantum field systems by evaluating effective potential at finite temperature. They used functional diagrammatic methods for evaluating effective potential. The advantages of this method over Weinberg's method are that 1) it is less cumbersome and 2) it can be formulated purely in terms of an effective Lagrangian. Bernard [113] has also studied gauge fields at finite temperature using the functional integral method.

The finite temperature Green's function is defined by [114,115]

$$G_\beta(x_1 \ldots x_j) = \frac{\mathrm{Tr}(e^{-\beta H} T(\varphi(x_1) \ldots \varphi(x_j)))}{\mathrm{Tr}(e^{-\beta H})} , \qquad (1.40)$$

where H is the Hamiltonian governing the dynamics of the field $\varphi(x)$, and β^{-1} is proportional to temperature. T stands for the time ordering operator. Spontaneous symmetry violation can be studied with the help of the finite temperature effective action Γ^{β} [112]. Γ^{β} may be defined in the following way. The generating functional $W(J)$ for one particle irreducible Green's functions is defined as [1]

$$Z^{\beta}(J) = e^{i\omega^{\beta}(J)} = \frac{\mathrm{Tr}[e^{-\beta H}T(\exp(i\int d^4x\ \varphi(x)J(x)))]}{\mathrm{Tr}(e^{-\beta H})} \tag{1.41}$$

such that

$$\bar{\varphi}(x) = \frac{\delta\omega^{\beta}(J)}{\delta J(x)} \tag{1.42}$$

$\bar{\varphi}(x)$ evaluated at $J = 0$, is the thermodynamic average of the field $\varphi(x)$:

$$\bar{\varphi}(x)\Big|_{J=0} = \frac{\mathrm{Tr}(e^{-\beta H}\varphi(x))}{\mathrm{Tr}(e^{-\beta H})} \tag{1.43}$$

The effective action $\Gamma^{\beta}(\bar{\varphi})$ is given by the Legendre transform of $\omega^{\beta}(J)$. Thus,

$$\Gamma^{\beta}(\varphi) = \omega^{\beta}(J) - \int d^4x\ \bar{\varphi}(x)J(x). \tag{1.44}$$

It is assumed that H possesses a symmetry which in the normal course of events would imply that $\bar{\varphi} = 0$ at $J = 0$. In other words, symmetry violation is signalled by a nonvanishing value of $\bar{\varphi}$, for which $\delta\Gamma^{\beta}(\bar{\varphi})/\delta\bar{\varphi}(x) = 0$. It is assumed that the translation invariance is not violated and hence eq.(1.42) should be independent of x. Thus we can study $\Gamma^{\beta}(\bar{\varphi})$ for constant $\bar{\varphi}(x) = \sigma$. The effective potential $V^{\beta}(\sigma)$ is then defined by

$$V^{\beta} = - (\text{space-time vol})^{-1} \; \Gamma^{\beta}(\bar{\varphi})\Big|_{\bar{\varphi}=\sigma} \tag{1.45}$$

and symmetry breaking occurs when $\frac{\partial V^{\beta}}{\partial \sigma} = 0$ for $\sigma \neq 0$. The effective potential is the generating function for one particle irreducible Green's functions at zero momentum. The passage from zero temperature field theory to finite temperature field theory is effected by the following rule [112,113]:

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \frac{1}{-i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \tag{1.46}$$

$$(2\pi)^4\delta^4(k-k') \qquad (2\pi)^3 \frac{\beta}{i} \delta_{\omega,\omega'}\delta^3(k-k')$$

The summation over n is from $-\infty$ to $+\infty$. The four vector k has time component ω;

$$\begin{aligned} \omega_n &= \frac{2\pi n}{-i\beta} \qquad \text{(bosons)} \\ &= \pi(2n+1)/-i\beta \quad \text{(fermions)} \end{aligned} \tag{1.47}$$

The finite temperature 2-point function (Green's function) of a spinless field $\varphi(x)$ is defined by

$$D_\beta(x-x') = \frac{Tr(e^{-\beta H} T(\varphi(x)\ \varphi(x')))}{Tr(e^{-\beta H})} \tag{1.48}$$

Two diagonal representations for $D_\beta(x-x')$ can be given-- one in terms of imaginary time, and the other for real time [112].

Imaginary time

$$D_\beta(x) = \frac{1}{-i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} e^{-ikx} D_\beta(k) \tag{1.49}$$

For non-interacting fields

$$D_\beta(k) = \frac{i}{k^2 - m^2} = \frac{-i}{\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^2 + m^2} \tag{1.50}$$

Real time

$$D_\beta(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \bar{D}_\beta(k) \tag{1.51}$$

Here k is a real Minkowski four-vector. In the absence of interactions

$$\bar{D}_\beta(k) = \frac{i}{k^2-m^2+i\varepsilon} + \frac{2\pi}{(e^{\beta(k^2+m^2)^{\frac{1}{2}}} - 1)} \delta(k^2-m^2) \tag{1.52}$$

For Fermi fields, the 2-point function is

$$S_\beta(x-x') = \frac{\mathrm{Tr}(e^{-\beta H} T(\psi(x)\bar{\psi}(x')))}{\mathrm{Tr}(e^{-\beta H})} \qquad (1.53)$$

As before, in the imaginary time formulation

$$S_\beta(x) = \frac{1}{-i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} S_\beta(k) \qquad (1.54)$$

Here $\omega_n = \pi(2n+1)/-i\beta$. For free fermions

$$S_\beta(k) = \frac{i}{\not{k}-m} \qquad (1.55)$$

In the real time formulation

$$S_\beta(x) = \int \frac{d^4k}{(2\pi)^4} \bar{S}_\beta(k) \qquad (1.56)$$

In the absence of interactions

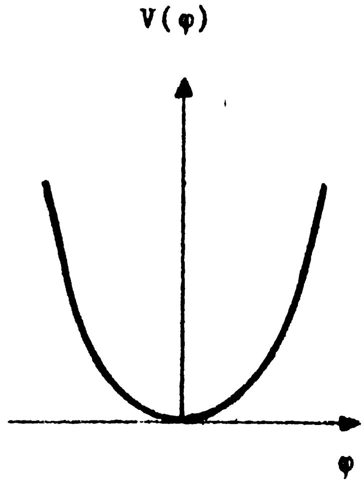
$$\bar{S}_\beta(k) = \frac{i}{\not{k}-m} - \frac{2\pi}{(e^{\beta(\vec{k}^2+m^2)^{1/2}}+1)} (\not{k}+m)\,\delta(k^2-m^2) \qquad (1.57)$$

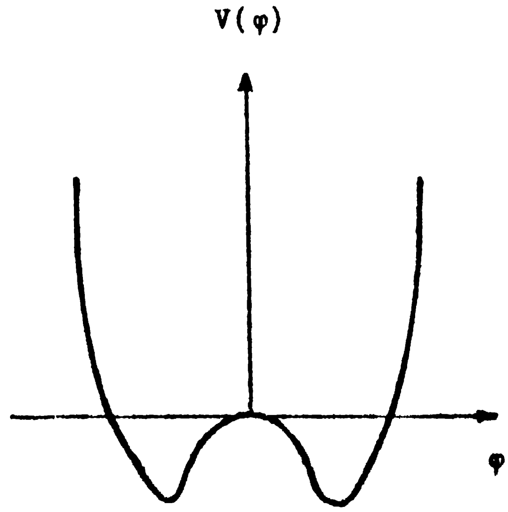
Hadronic matter in unusual environments. We have presented above the effect of temperature on spontaneously-symmetry-broken theories. Now we shall explore the properties of hadronic matter in unusual environments, in particular at high temperature. One of the first attempts to explore the properties of hadrons at high temperatures came from Hagedorn [116,117], who found that the density of hadronic states increases exponentially with energy and argued that this implies the existence of a limiting temperature, above which hadronic matter cannot exist. Cabbibo and Parisi [118] argued that this implies the liberation of quarks above a critical temperature. Several authors [119-125] studied the properties of QCD at finite temperature. All of them have stressed the fact that the zero temperature renormalisation prescriptions are sufficient enough to eliminate the various ultraviolet divergences. Weinberg [111] has explicitly shown this in the one-loop order.

Polyakov [126] and Susskind [127] studied the temperature dependence of lattice gauge theories and shown that as the temperature is increased these theories undergo a phase transition to an unconfined phase. Gross et al [128] have given a detailed discussion of the properties of QCD at finite temperature from the point of view of perturbation theory, semiclassical methods (instantons and effective lattice gauge theories. Their calculations are also in agreement with the previous results.

The phase transition from hadronic phase to quark phase can also be understood by studying finite temperature behaviour of the true vacuum of QCD. At zero temperature, the perturbative vacuum has an energy density B, which is higher than that of the true vacuum. This difference vanishes at high temperature, and the confining phase is destroyed [126]. Muller and Rafelski [129] treated B as a temperature-dependent quantity, and found that the phase transition to the perturbative ground state occurs at the critical temperature $T_{c.r.} = 1.3\ B^{1/4} \sim 190$ MeV.

The most interesting consequences of the phase transitions in gauge theories are connected with cosmology [70,72,130-132]. The advent of GUTS has opened exciting new possibilities for exploring the history of the universe at the earliest times. Indeed, we may now even speculate on the evolution of the universe at times as early as 10^{-35} sec after the big bang. The full GUT symmetry can be restored above 10^{15} GeV, and such temperatures could be achieved only at cosmic time $t \sim 10^{-35}$ sec [71,133]: the early universe was the ultimate high energy accelerator. As the universe expands and cools, it undergoes a series of phase transitions. These phase changes occur as the high degree of symmetry at temperatures close to the Planck mass $G^{-1/2} = 1.22 \times 10^{19}$ GeV, is reduced in stages to the present "zero temperature" symmetry $SU(3)_c \otimes U(1)_{e.m.}$ The precise nature of the phase transition depends on the surviving symmetry at a given temperature [132].





φ^4 potential

Fig. 1.1 ($m^2 > 0$)

Fig. 1.2 ($m^2 < 0$)

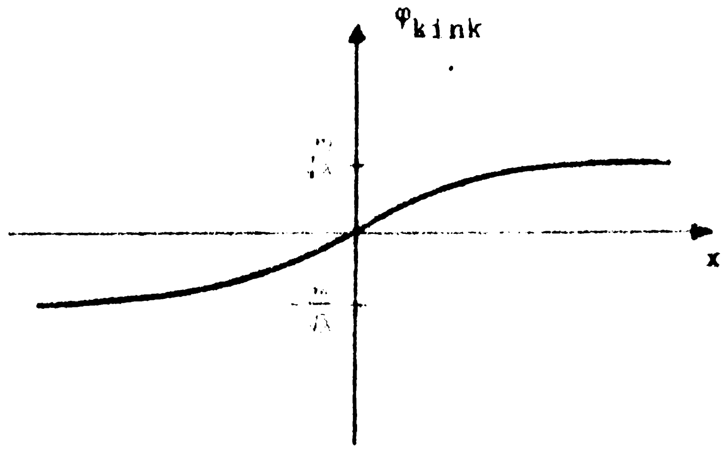


Fig. 1.3 Static solution in φ^4 theory

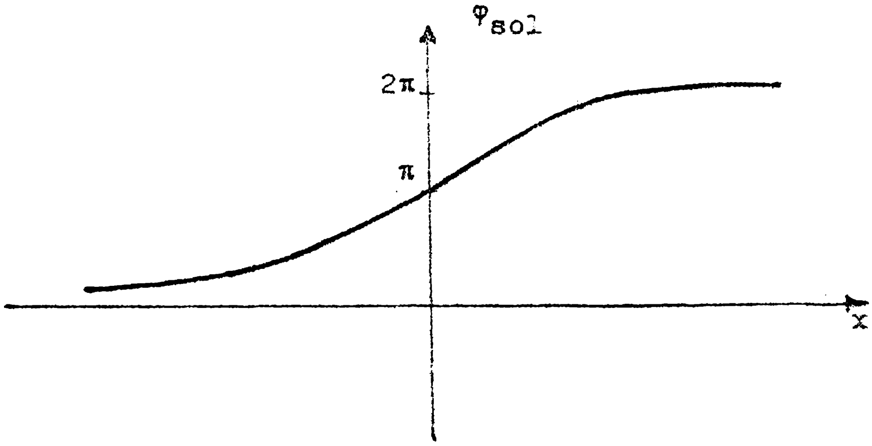


Fig. 1.4 The sine-Gordon solution

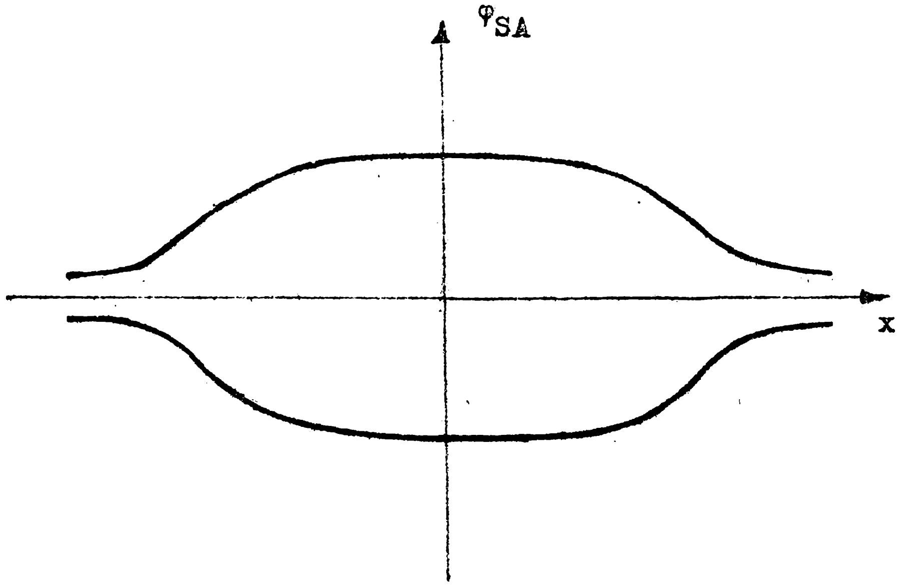


Fig. 1.5 The General shape of the doublet solution.

Two

THEORIES WITH SPONTANEOUS SYMMETRY BREAKING

Introduction

The role played by symmetry and invariance principles in the development of physics, is beyond easy estimation. While symmetry principles were useful in classical mechanics, it was in quantum mechanics that symmetry principles assumed a pivotal role in the very formalism employed. The symmetry properties, as they exhibit the group property, can be discussed using the representation theory of groups which will lead to elegant and far reaching mathematical and physical results. As one proceeds from atomic to nuclear, and then to elementary particle physics, ie, from quantum mechanics to quantum field theory, the representation theory of groups becomes a powerful and indispensible tool in the description of various kinds of invariance.

In the context of Lagrangian field theory, the symmetry of the system implies that the Lagrangian $\mathcal{L}(\varphi, \partial_\mu \varphi)$ of the system is invariant under certain symmetry groups of transformations. Then it follows from Noether's theorem [134] that there exists a set of conserved current density operators $j_\mu^a(x)$, such that

$$\partial_\mu j_\mu^a(x) = 0. \tag{2.1}$$

The charge density associated with the current density $j_\mu^a(x)$ is defined

$$Q^a = \int d^3x \; j_o^a(x) , \tag{2.2}$$

which is the generator of a symmetry transformation of the field. The vacuum state, however, may or may not remain invariant under this transformation, namely

$$Q^a|0\rangle = 0 \tag{2.3}$$

or

$$Q^a|0\rangle \neq 0. \tag{2.4}$$

The second mechanism, ie., eventhough, the Lagrangian is invariant under the symmetry group of transformations, the vacuum is not, is known as spontaneous symmetry breaking (SSB).

Now some model field theories which exhibit SSB will be considered. One of the simplest models is the one with a single scalar field, described by the Lagrangian (c.f. eq (1.21)):

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} m^2\varphi^2 - \frac{\lambda}{4} \varphi^4 \tag{2.5}$$

The only symmetry this Lagrangian possesses, is the reflection symmetry: $\varphi \longleftrightarrow -\varphi$. The potential function is given by,

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4}\varphi^4 \quad \lambda > 0. \tag{2.6}$$

For $m^2 > 0$, the potential function is sketched in fig.1.1, and the minimum occurs at $\varphi = 0$. On the other hand, for $m^2 < 0$, the potential is of the form given in fig.1.2. Now $\varphi = 0$ is not a minimum, and there are two symmetric absolute minima at $\varphi = \pm (-m^2/\lambda)^{1/2}$.

In quantum field theory the ground state corresponds to the minimum of the potential. Thus, for $m^2 < 0$, the vev of the field is not zero and has the value $\pm (-m^2/\lambda)^{1/2}$ to zeroth order. Let σ denote the vev of the field:

$$\langle\varphi\rangle_0 = \sigma = \pm \left[-\frac{m^2}{\lambda}\right]^{\frac{1}{2}} \tag{2.7}$$

Either value of σ may be chosen, and by convention the plus sign is chosen. Clearly the vacuum is not an eigenstate of the symmetry operation since $\sigma \neq -\sigma$, and the symmetry is said to be 'spontaneously broken'. In order to carry out a perturbation calculation, the field φ is shifted to a new value φ' defined by

$$\varphi' = \varphi - \sigma , \tag{2.8}$$

so that $\langle\varphi'\rangle_0 = 0$.

In terms of φ', the Lagrangian (2.5) becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi')^2 + \mu^2\varphi'^2 - \lambda\sigma\varphi'^3 - \frac{1}{4}\lambda\varphi'^4 \tag{2.9}$$

From this, it can be seen that boson states have mass $-2m^2 > 0$, and the new Lagrangian does not exhibit the symmetry of the original one.

A slightly more complicated, but at the same time more interesting, model of a charged scalar field is given by the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi^*)(\partial_\mu\varphi) - \frac{m^2}{2}(\varphi^*\varphi) - \frac{\lambda}{4!}(\varphi^*\varphi)^2, \tag{2.10}$$

where

$$\varphi = \varphi_1 + i\varphi_2 \tag{2.11}$$

The Lagrangian (2.10) is invariant under U(1), and the corresponding symmetry operation is

$$\varphi \longrightarrow \varphi e^{i\theta} , \tag{2.12}$$

where θ is a real constant. The potential function is

$$V(\varphi, \varphi^*) = \frac{m^2}{2} (\varphi^* \varphi) + \frac{\lambda}{4!} (\varphi^* \varphi)^2$$

$$= \frac{m^2}{2} (\varphi_1^2+\varphi_2^2) + \frac{\lambda}{4!} (\varphi_1^2+\varphi_2^2)^2 . \qquad (2.13)$$

When $m^2 < 0$, the absolute minima occur not at the origin, but on the circle

$$\sqrt{\varphi_1^2 + \varphi_2^2} = (- \frac{m^2}{\lambda})^{\frac{1}{2}}$$

It is possible to define the axis in the $\varphi_1 - \varphi_2$ plane so that

$$\langle \varphi_1 \rangle_o = \sigma = (- \frac{m^2}{\lambda})^{\frac{1}{2}}, \quad \langle \varphi_2 \rangle_o = 0. \qquad (2.14)$$

Shifting the fields in the following way:

$$\varphi_1 = \varphi_1' + \sigma$$

$$\varphi_2 = \varphi_2' ; \qquad (2.15)$$

then, in terms of φ', the Lagrangian (2.10) takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1' \partial_\mu \varphi_1) + m \varphi_1'^2 + \frac{1}{2} (\partial_\mu \varphi_2' \partial_\mu \varphi_2')$$

$$- \frac{\lambda \sigma}{12} \varphi_1'(\varphi_1'^2 + 4\varphi_1'^2) - \frac{\lambda}{4!} (\varphi_1'^2+\varphi_2'^2) + \frac{11}{24} \frac{m^2}{\lambda}$$

(2.16)

This shows that the field φ_1' has $(\text{mass})^2$, $-2m^2 > 0$, and the field φ_2' has zero mass. Another way of looking at the symmetry breaking of the Lagrangian (2.10) leads to the famous linear σ-model [135-137].

A much more general example is furnished by an n-component real scalar field φ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^i)^2 - \frac{m^2}{2} \varphi^i \varphi^i - \frac{\lambda}{4!} (\varphi^i \varphi^i)^2 \; ; \tag{2.17}$$

is invariant under the orthogonal group O(n) in n-dimensions. If $m^2 < 0$, the potential has a ring of minima at $\sigma = (-m/\lambda)^{1/2}$. Let the n^{th} component of φ be the one which develops a nonvanishing vev, namely

$$\langle \varphi \rangle_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \sigma \end{pmatrix} \tag{2.18}$$

The new feature that has appeared is that there is still a non-trivial group, which leaves the vacuum invariant. This subgroup is O(n-1) with $\frac{1}{2}(n-1)(n-2)$ generators. Proceeding as above, it can be seen that the Lagrangian (2.17) contains a massive field with bare mass $-2m^2 > 0$ and (n-1) massless fields. Thus to each generator of the original group, which leaves the vacuum invariant, there corresponds a massless boson.

The Goldstone Theorem

Goldstone [7] conjectured that when there is spontaneous breaking of a continuous symmetry in a quantum field theory, there must exist massless spin zero particles, which have come to be known as Goldstone particles. We saw earlier that the Lagrangian (2.10) transforms into a new form (2.16), when the SSB is taken into account. It contains a field φ_1' with positive mass while the field φ_2' has zero mass, exhibiting the Goldstone mode. This can be intuitively understood in the following way [138]. In the vacuum all the state vectors are lined up with the same phase and magnitude, $|\varphi| = \sigma$. Oscillations are then of two types: one in magnitude giving rise to the massive quanta of type φ_1'; while little energy is required to rotate all $\varphi(x)$ by a common phase giving rise to the Goldstone mode represented in this example by φ_2'. In other words, if the symmetry is represented by a continuous variable (like a rotation of phase), the "direction" of breaking can be slightly different in different places, and waves due to perturbations in this direction must always be possible; little energy is associated with long wave disturbances and we have particles of zero mass [139].

If there is more than one way to vary φ, while keeping the energy a minimum, then there will be more than one Goldstone boson. Thus the Lagrangian (2.17), when

SSB is taken into account, contains (n-1) Goldstone bosons. In general if the Lagrangian is invariant under a group G, but the vacuum has a lower symmetry, ie., it is invariant under a group G' where $G' \subset G$, then the number of massless bosons is given by, n = dim G - dim G'. Physical illustration of Goldstone bosons is given by excitations of zero frequency mode in solid state physics. Phonons in crystals and liquid helium [140] and magnons in ferromagnets [141] are some examples of excitations with zero frequency.

Higgs Mechanism

In conventional gauge theory, all the gauge fields are massless, whereas if gauge theories are to be applied to weak interaction, the gauge fields should be massive. The possibility that SSB might provide a solution, was first suggested by Nambu [142] and Anderson [143], who were inspired by the appearance of SSB in the theory of superconductivity [144]. The implementation of this idea was bottled by the Goldstone conjecture, as no massless scalar particles are observed. Towards mid-sixties, it was realised that [10] the Goldstone theorem applies to global symmetries only, while it fails for local gauge theories, through a mechanism, now known as Higgs mechanism. In the presence of gauge fields, the Goldstone theorem is no longer applicable because of the fact that the gauge fields can absorb the Goldstone fields, and in doing so, the gauge fields become massive. Thus a single mechanism can account for the disappearance of the Goldstone bosons and the emergence of massive gauge fields.

The simplest model, which illustrates how gauge theories evade Goldstone bosons, was presented by Higgs [10]. Consider the Lagrangian

$$\mathcal{L} = (\partial_\mu \varphi^*)(\partial_\mu \varphi) - m^2(\varphi^* \varphi) - \lambda(\varphi^* \varphi)^2 \tag{2.19}$$

This Lagrangian is invariant under a U(1) group of transformations and exhibits Goldstone bosons. Now a locally gauge invariant Lagrangian is given by [1]

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\varphi^*(\partial^\mu - ieA^\mu)\varphi - m^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2$$
$$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.20}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Under local gauge transformations,

$$\varphi(x) \longrightarrow \varphi'(x) = e^{-i\Theta(x)}\varphi(x)$$

$$\varphi^*(x) \longrightarrow \varphi^{*\prime}(x) = e^{i\Theta(x)}\varphi^*(x)$$

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu - \frac{1}{e}\partial_\mu \Theta(x) \tag{2.21}$$

Lagrangian (2.20) is invariant. If $m^2 > 0$, (2.21) gives the Lagrangian for scalar electrodynamics. When $m^2 < 0$, the minima of the potential occur at

$$|\varphi|^2 = -\frac{m^2}{2\lambda} = \frac{\sigma^2}{2}, \qquad (2.22)$$

so that

$$\langle\varphi\rangle_0 = \frac{\sigma}{\sqrt{2}}. \qquad (2.23)$$

Instead of shifting φ as we have done earlier, here the field φ will be reparametrized exponentially as

$$\begin{aligned}\varphi &= e^{i\xi/\sigma}(\sigma + \eta)/\sqrt{2} \\ &= \frac{1}{\sqrt{2}}(\sigma + \eta + i\xi) + \text{quadratic and higher order terms} \qquad (2.24)\end{aligned}$$

The fields ξ and η will be real and the field ξ is associated with the spontaneously broken U(1) symmetry, as it can be seen that in the absence of the gauge field A_μ, the ξ field will be massless. Substituting (2.24) in (2.20), we find,

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{2}\partial_\mu\xi\,\partial^\mu\xi + \\ &\frac{1}{2}e^2\sigma^2 A_\mu A^\mu - e\sigma A_\mu\partial^\mu\xi + m^2\eta^2. \qquad (2.25)\end{aligned}$$

This relation shows that the η field has mass $-2m^2$. The fields A_μ and ξ are mixed up in such a way that an interpretation is not immediately apparent. However, the interpretation will become easy if the gauge function is chosen to be $\xi(x)/\sigma$ (unitary gauge); then,

$$\varphi \longrightarrow \varphi' = \frac{\sigma + \eta}{\sqrt{2}} \tag{2.26}$$

$$A_\mu \longrightarrow A'_\mu = A_\mu - \frac{1}{e\sigma} \partial_\mu \xi ,$$

so that

$$\mathcal{L} = \frac{1}{2} [(\partial_\mu + ieA'_\mu)(\sigma + \eta)][\partial_\mu - ieA'_\mu)(\sigma + \eta)] - \frac{1}{2}m^2(\sigma + \eta)^2 - \frac{1}{4} \lambda(\sigma + \eta)^4 - \frac{1}{4}F'_{\mu\nu} F'^{\mu\nu} ; \tag{2.27}$$

which yields:

$$\mathcal{L} = - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2} e^2\sigma^2A'_\mu A'^\mu + \frac{1}{2}e^2 A'^2_\mu \eta(2\sigma + \eta) - \frac{1}{2}\eta^2(3\lambda\sigma^2 + \mu^2) - \lambda\sigma \eta^3 - \frac{1}{4}\lambda\eta^4 \tag{2.28}$$

This Lagrangian contains no massless particles. There is a scalar η-meson, with mass $3 \lambda \sigma^2 + m^2$, and a massive vector meson A'_μ with mass 2σ. The ξ-field, known as a would be Goldstone boson, has been 'gauged away' to become the longitudinal mode of the vector A'_μ.

We shall extend the above formulation to non-abelian gauge fields. Let the symmetry (the simplest) be SU(2). The scalar mesons are put in the triplet representation; the fields transform as

$$\delta\varphi_i = -i\varepsilon^j L^j_{ik}\varphi_k = \varepsilon^j \varepsilon^{ijk}\varphi_k. \qquad (2.29)$$

The scalar part of the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi_i + g\varepsilon^{ijk} A^j_\mu \varphi_k)(\partial^\mu\varphi_i + g\varepsilon^{ij'k'} A^{\mu j'}\varphi_{k'}) - V(\varphi^2) \qquad (2.30)$$

When the symmetry is broken spontaneously, V has a non-zero minimum; let the third component acquire a vev:

$$\langle\varphi\rangle = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix} \qquad (2.31)$$

Though, now the vacuum is no longer invariant under T_1 and T_2, T_3 remains a good symmetry.

Parametrizing φ as before

$$\varphi = \exp\left\{\frac{1}{\sigma}(\xi_1 L^1 + \xi_2 L^2)\right\}\begin{pmatrix} 0 \\ 0 \\ \sigma+\eta \end{pmatrix}, \qquad (2.32)$$

we can see that the fields ξ_1 and ξ_2 are the would be Goldstone bosons associated with the two broken degrees of freedom. But a redefinition of the gauge functions, as done earlier, leads to the disappearance of the massless fields. Also the vector mesons, corresponding to the broken symmetry generators, acquire a mass $M = g\sigma$. Since the T_3 symmetry survives, there remains one massless vector meson. The Lagrangian (2.30) thus becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}g^2\sigma^2\varepsilon^{ij3} A_\mu^j A^{\mu j'} - V[(\sigma+\eta)^2] \qquad (2.33)$$

A physical example of the Higgs phenomenon is superconductivity, which has the same properties as described above, except that it is nonrelativistic. The Higgs Lagrangian (2.20) in the static case, is identical to the Ginzburg-Landau free energy in the theory of type II superconductors [145,146].

GWS Model for Electroweak Interactions

In the model for electroweak interaction known as the Glashow-Weinberg-Salam (GWS) model, the vector bosons acquire mass via the Higgs mechanism. The original model describes only the weak and electromagnetic interactions of leptons. A naive extension of this model to include quarks, will lead to the existence of strangeness changing neutral currents, contrary to experimental observations. But Weinberg [147,148] showed that hadrons could be incorporated in the model, by implementing a mechanism due to Glashow, Iliopoulos and Maiani [149]. The GIM mechanism necessitated the introduction of a fourth quark called the charmed quark. The GWS model supplemented with the GIM mechanism predicted both the existence of strangeness conserving neutral currents and of the charmed quark, both of which were subsequently discovered [150–154].

The GWS model is based on the gauge group $SU(2) \otimes U(1)$. SU(2) group has three generators T^i, $(i=1,2,3)$ and therefore three gauge bosons A^i_μ, while U(1) has only one generator Y and one gauge boson B_μ. The coupling constant for SU(2) gauge group is denoted by g and that for U(1) by g'. The GWS model is a chiral model, in which parity violation is incorporated by assigning left and right handed fermions to different representations. All left handed fermions transform according to the doublet representation of SU(2), while right handed fermions are singlets, and for this reason, the GWS gauge group is often written as $SU(2)_L \otimes U(1)$. The Gell-Mann-Nishijima relation $Q = T^3_L + \frac{1}{2}Y$ defines the electric charge operator for the model (T^i and Y are sometimes referred to as weak isospin and weak hypercharge, respectively).

The Lagrangian can now be assumed to be the sum:

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_l \tag{2.34}$$

where

$$\mathcal{L}_g = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} ; \tag{2.35}$$

the field strength tensors being defined by

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^i_\mu A^k_\nu \tag{2.36}$$

and

$$f_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.37}$$

$$\mathcal{L}_l = \bar{R} i\gamma^\mu(\partial_\mu + ig'B_\mu)R + \bar{L} i\gamma^\mu(\partial_\mu + \frac{i}{2} g'B_\mu - ig \frac{\tau^i}{2} A^i_\mu)L \tag{2.38}$$

Here L represents the left-handed $SU(2)_L$ doublet:

$$L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \tag{2.39}$$

where

$$e_L = \frac{1}{2}(1 - \gamma_5)e \tag{2.40}$$

and R is the right-handed SU(2) singlet such that

$$R = e_R = \frac{1}{2}(1 + \gamma_5)e \qquad (2.41)$$

The theory has the following defects: 1) it contains four massless weak gauge bosons (A^1, A^2, A^3, B), whereas nature admits only one massless boson (photon); 2) the local $SU(2)_L$ invariance forbids an electron mass term. But Higgs mechanism is at our disposal to get rid of these defects. Introducing a complex doublet of scalar particles

$$\varphi = \begin{pmatrix} \varphi^* \\ \varphi^o \end{pmatrix} \qquad (2.42)$$

which transforms like an SU(2) doublet, we can add a new piece to the Lagrangian (2.34)

$$\mathcal{L}_H = (D^\mu \varphi)^* (D_\mu \varphi) - V(\varphi^* \varphi) , \qquad (2.43)$$

where

$$V(\varphi^* \varphi) = m^2(\varphi^* \varphi) + \lambda(\varphi^* \varphi)^2 , \qquad (2.44)$$

and

$$D_\mu \varphi = (\partial_\mu - \frac{ig'}{2} B_\mu - \frac{ig}{2} \tau^i A^i_\mu) \varphi \qquad (2.45)$$

We may also add an interaction piece representing the

Yukawa couplings of the scalars to the fermions

$$\mathcal{L}_I = - G_e[\bar{R}\, \varphi^+ L + \bar{L}\, \varphi R]. \qquad (2.46)$$

Now let us imagine that $m^2 < 0$ in (2.37), and consider the consequence of SSB. The vev of the field is given by (2.23). Symmetries associated with the generators T^1, T^2, T^3 & Y are spontaneously broken. However the subgroup U(1) generated by the electric charge operator $Q = T_L^3 + \frac{1}{2}Y$ is unbroken. Hence $SU(2)_L \otimes U(1)$ is broken down to the U(1), for $m^2 < 0$. This corresponds to the case of the existence of one massless gauge boson (photon) and three massive bosons.

The Lagrangian can be expanded about the minimum of the potential, by parametrising φ in the exponential form

$$\varphi = e^{i \frac{\xi^i \tau^i}{2\sigma}} \begin{pmatrix} 0 \\ \frac{\sigma+\eta}{\sqrt{2}} \end{pmatrix} \qquad (2.47)$$

By redefining the gauge function (unitary gauge), we can put

$$\varphi \longrightarrow \varphi' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sigma+\eta \end{pmatrix} \qquad (2.48)$$

Substituting this value of φ in the Lagrangian, we can see that there is only one physical Higgs particle with mass $-2m^2 > 0$. The would be Goldstone bosons, which are three in number, have been 'eaten' to give masses to the three of the four gauge bosons.

The kinetic energy term may be rewritten as

$$(D^{\mu}\varphi)^{*}(D_{\mu}\varphi) = \tfrac{1}{2}(0\sigma)\left[\tfrac{g}{2}\vec{\tau}\vec{A}_{\mu} + \tfrac{g'}{2}B_{\mu}\right]^{2}\binom{0}{\sigma} + \eta \text{ terms}$$

$$= \frac{\sigma^{2}}{4}\left((g'B_{\mu}-gA^{3}_{\mu})^{2} + g^{2}((A^{1}_{\mu})^{2} + (A^{2}_{\mu})^{2})\right)$$

$$= M^{2}_{W}\, W^{+\mu}\, W^{-}_{\mu} + \frac{M^{2}_{Z}}{2} Z^{\mu}Z_{\mu} \,, \qquad (2.49)$$

where

$$W^{\pm}_{\mu} = \frac{A^{1}_{\mu} \mp iA^{2}_{\mu}}{\sqrt{2}} \,, \qquad (2.50)$$

$$Z_{\mu} = \frac{+\, g'B_{\mu} - gA^{3}_{\mu}}{\sqrt{g^{2} + g'^{2}}}$$

$$= \sin\theta_{W}\, B_{\mu} - \cos\theta_{W}\, A^{3}_{\mu} \qquad (2.51)$$

are charged and neutral massive gauge fields with mass

$$M^{2}_{W} = g^{2}\sigma^{2}/4 \qquad (2.52)$$

$$M^{2}_{Z} = (g^{2} + g'^{2})\,\frac{\sigma^{2}}{4} = \frac{M^{2}_{W}}{\cos^{2}\theta_{W}} \qquad (2.53)$$

Here $\tan \theta_W = g'/g$ defines the weak (or Weinberg's) angle. The fourth gauge boson

$$A_\mu = \frac{gB_\mu + g'A_\mu^3}{\sqrt{g^2 + g'^2}} = \cos \theta_W B_\mu + \sin \theta_W A_\mu^3 \qquad (2.54)$$

is massless - the photon associated with the unbroken U(1) group.

The interaction part of the Lagrangian may be rewritten:

$$\begin{aligned} \mathcal{L}_I &= - G_e \sigma \, [\bar{e}_R e_L + \bar{e}_L e_R] \; + \\ &= - G_e \sigma \, \bar{e} e \; , \end{aligned} \qquad (2.55)$$

so that the electron has acquired a mass

$$m_e = G_e \sigma \; . \qquad (2.56)$$

Since right-handed helicity neutrinos do not exist, neutrinos remain massless. The $SU(2)_L \otimes U(1)$ gauge fields A_μ, Z_μ and the electroweak gauge fields B_μ, A_μ^3 can be related as

$$\begin{pmatrix} B_\mu \\ A_\mu^3 \end{pmatrix} \quad \begin{pmatrix} \cos \theta_W & \sin \theta_W \\ \sin \theta_W & -\cos \theta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \qquad (2.57)$$

Spontaneous Symmetry Breaking by Radiative Corrections

Coleman and Weinberg [155,156] have shown that radiative corrections can produce SSB in theories for which the semiclassical (tree) approximation does not indicate such breakdown. The radiative corrections are determined using the method of generating functionals and effective potentials. The functional methods were first introduced into quantum field theory by Schwinger [157,158], and extended to the study of spontaneous symmetry breakdown by Jona-Lasinio [159]. The study of SSB by finding the minima of the potential is a semiclassical approximation. But the method of Jona-Lasinio is based on a new function called the effective potential [160], such that the minima of the effective potential give, without any approximation, the true vacuum states of the theory.

The calculation of effective potential is not an easy job; it requires an infinite summation of Feynman diagrams. But we can do the calculations through a method called loop expansions [155]; first summing all diagrams with no closed loops (tree graph), then those with one closed loop etc. At each stage of this expansion, an infinite summation is required, but this summation is trivial. Jackiw [161] uses the Feynman path integral method to obtain a simple formula for the effective potential. The formula has the advantage of summing all the relevant Feynman graphs to a given order of the loop expansion. It generates all orders of the loop expansion representing each order by a finite number of graphs. The formula for the effective potential $V(\sigma)$ is [161]

$$V(\sigma) = V_0(\sigma) - \frac{1}{2} i\hbar \int \frac{d^4k}{(2\pi)^4} \ln \det i \mathcal{D}^{-1}(ab)\{\sigma, k\} + i\hbar \langle \exp(i/\hbar \int d^4x \hat{\mathcal{L}}_I \{\sigma, \varphi_a(x)\})\rangle \quad (2.58)$$

The first term is the classical (tree) approximation. The second term is the contribution of all graphs with one closed loop, and so on. The loop expansion corresponds to the semiclassical expansion in powers of $\hbar$.

In order to study the effect of loop corrections, we shall consider a simple model - the theory of a massless, quantically self interacting field, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}A(\partial_\mu \varphi)^2 - \frac{1}{2}B\varphi^2 - \frac{1}{4!}C\varphi^4 \quad (2.59)$$

Where A, B and C are the wave function, mass and coupling constant renormalisation counter terms, to be determined by imposing the definitions of the scale of the renormalised field, the renormalised mass and the renormalised coupling constant [155].

Contributions to the lowest order (tree approximation) of the effective potential come from the graph shown in fig.2.1. Thus

$$V = \frac{\lambda}{4!}\varphi_c^4 \cdot \tag{2.60}$$

To next order (one-loop approximation), we have an infinite series of polygon graphs shown in fig.2.2 as well as contributions from the mass and the coupling constant counter terms. Thus we may write the renormalised effective potential to one-loop order,

$$V = \frac{\lambda}{4!}\varphi_c^4 - \frac{1}{2}B\,\varphi_c^2 - \frac{1}{4!}\,C\,\varphi_c^4 + i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{\frac{1}{2}\lambda\varphi_c^2}{k^2+i\varepsilon}\right)^n \tag{2.61}$$

The last term in this expression can be rewritten as

$$\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln(1+\frac{\lambda\varphi_c^2}{2k^2}) \; .$$

This integral is still u.v. divergent and may be evaluated by introducing a cut off parameter Λ. The constant B is determined by fixing the renormalised mass to vanish:

$$\left.\frac{d^2V}{d\varphi_c^2}\right|_{\varphi_c=0} = 0 \; ; \tag{2.62}$$

the constant c is determined by defining the coupling constant at some off-mass-shell position in momentum space. Thus,

$$\left.\frac{d^4V}{d\varphi_c^4}\right|_{\varphi_c=M} = \lambda \tag{2.63}$$

where M is some number with the dimension of a mass and is completely arbitrary. Thus the effective potential upto one-loop order is given by

$$V = \frac{\lambda}{4!}\varphi_c^4 + \frac{\lambda^2 \varphi_c^4}{256\pi^2}\left[\ln\left(\frac{\varphi_c^2}{M^2}\right) - \frac{25}{26}\right]. \tag{2.64}$$

This shows that the one-loop corrections have turned the minimum at the origin into a maximum, and caused a new minimum to appear away from the origin, thus causing the onset of SSB, which was absent in the original form of the potential.

The above formulation can be extended to the case of massless scalar electrodynamics, described by the

Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi_1 - eA_\mu\varphi_2)^2 + \frac{1}{2}(\partial_\mu\varphi_2 + eA_\mu\varphi_1)^2$$

$$-\frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2 + \text{counter terms} \qquad (2.65)$$

The effective potential calculated as described above, is

$$V = \frac{\lambda}{4!}\varphi_c^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\varphi_c^4\left(\ln\frac{\varphi_c^2}{M^2} - \frac{25}{6}\right). \qquad (2.66)$$

This has a minimum away from origin, signalling the breakdown of symmetry. The masses of the scalar meson and vector meson [155] are given by

$$m^2(s) = \frac{3e^4}{8\pi^2}\langle\varphi\rangle^2 \qquad (2.67)$$

and

$$m^2(v) = e^2\langle\varphi\rangle^2$$

respectively. Thus massless scalar electrodynamics, with radiative corrections, resembles the abelian Higgs theory.

Dynamical Symmetry Breaking

We shall close this chapter by mentioning a new mechanism by which symmetry can be broken known as dynamical symmetry breaking [162-165]. Dynamical symmetry breaking is meant for any spontaneous breakdown of symmetry for which the associated Goldstone bosons are composite rather than elementary. In this approach we consider the electroweak theory without scalars, but some dynamical mechanism by which fermion - antifermion condensates could acquire vevs, thereby providing breakdown of electroweak symmetry.

Weinberg [166] and Susskind [167] have proposed a mechanism for the formation of a fermion-antifermion condensate, where in addition to the normal electroweak gauge symmetry and fermions, the existence of heavy fermions (called technifermions F_T) and are associated heavy colour (called technicolour) are also being postulated. The technigauge symmetry, which is not only asymptotically free but also increases fast with lower q^2 so that the force between F_T and $\bar{F}_T$ becomes strong at $E \simeq 1$ Tev , so that $\langle \bar{F}_T F_T \rangle \simeq (1 \text{ Tev})^3$. If the F_T's also transform non-trivially under the usual electroweak group G_W, this will provide a breaking of the G_W symmetry. The G_W - gauge bosons will then acquire mass $m \simeq 1$ Tev which is roughly the expected mass for W and Z bosons by the standard electroweak theory.

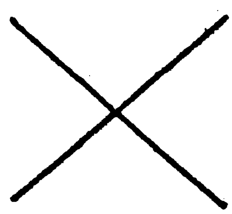


Fig.2.1 The zero-loop approximation for the effective potential

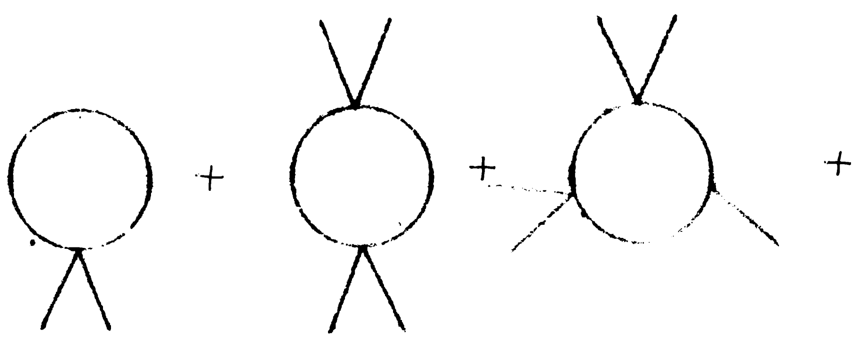


Fig. 2.2 The one-loop approximation for the effective potential

Three

SECOND ORDER PHASE TRANSITION IN SINE-GORDON FIELD SYSTEMS

Introduction

The term soliton coined by Zabusky and Kruskal [168] applies to solutions of certain non-linear evolution equations [74,80-84]. Solitons now find many applications in almost all branches of science and technology [75-79,169-171]. Recently the existence of solitons with fermion number $\frac{1}{2}$ in condensed matter and relativistic field theories, has been conjectured [172-174]. One of the non-linear equations, which have been thoroughly investigated, is the sine-Gordon [SG] equation in 1+1 dimension. It possesses soliton solutions [80-84]. Physicists as well as mathematicians are interested in these equations, because of their wide range of applications. The particle physicists are interested in these equations as they possess a particle spectrum, which survives quantisation.

It was Skyrme [175-177] who first suggested that the solitons of two-dimensional SG equation, when quantised, exhibit particle-like behaviour. He further identified these particles with fermions. Later Coleman [178], using an entirely independent procedure, discussed a mapping between SG and massive Thirring models, the latter being described by the Lagrangian

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m_F\bar{\psi}\psi - \tfrac{1}{2}g(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma_{\mu}\psi). \qquad (3.1)$$

The relationship between the SG coupling constant λ and the four fermion coupling constant g of the Thirring model, is

$$\frac{\lambda}{4\pi m^2} \longleftrightarrow \frac{1}{1 + \frac{g}{8\pi}} . \qquad (3.2)$$

The equivalence is valid only for the range $\frac{\lambda}{m^2} < 8\pi$. Coleman [178] further conjectured that the quantum soliton of the SG theory is the fermion of the massive Thirring model, and that the fundamental boson of the former theory is a bound state of the latter theory.

Simon and Griffiths [179] have obtained several rigorous results for the two dimensional φ^4 theory, by considering the φ^4 field theory as a proper limit of a generalised Ising model. For a scalar field theory in two-dimensions,

$$\mathcal{L} = \tfrac{1}{2}(\partial_{\mu}\varphi)^2 + b\varphi^2 + \tfrac{1}{4}g\,\varphi^4 - J\varphi \;, \; g > 0 \qquad (3.3)$$

Simon-Girffiths Theorem implies the non-existence of any phase transition for $J \neq 0$, and also rules out the possibility of a first-order phase transition at $J = 0$. Chang [180] has shown that a two dimensional φ^4 field does exhibit a second order phase transition at $J = 0$. The ultraviolet (u.v) divergence is controlled by normal-ordering the Hamiltonian according to a fixed mass. He has also shown that there is no contradiction between Simon-Girffiths Theorem and the existence of a second order phase transition. Drell et al [96] have developed nonperturbative variational techniques to study field theories on a discrete lattice, introduced by Wilson [56] for studying the phenomena for which the usual weak coupling perturbation theory is of no use. Drell et al [96] considered a two dimensional scalar field theory exhibiting SSB, of the form

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \lambda(\varphi^2 - f^2)^2, \qquad (3.4)$$

and showed that the phase transition from the ground state with $\langle\varphi\rangle_0 = 0$ and those exhibiting an SSB, such that $\langle\varphi\rangle_0 \neq 0$, is a second order one. Here we study a two-dimensional SG field theory on a lattice, using the variational technique of Drell et al. We find that the nature of the phase transition between the ground states corresponding to $\langle \varphi \rangle_0 = 0$ and $\langle \varphi \rangle_0 \neq 0$, is a second order one, and that it is in accord with the Simon-Griffiths Theorem. We have also extended these calculations to a generalised SG type field theory in two-dimensions with a potential of the form $[\cos^{\ell}(\frac{\sqrt{\lambda}}{m}\varphi) - 1]$.

Two Dimensional Sine-Gordon Field Theory on a Lattice

The SG field system is described by a single scalar field $\varphi(x,t)$ with a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^4}{\lambda}(\cos(\frac{\sqrt{\lambda}}{m}\varphi) - 1). \tag{3.5}$$

This yields a Hamiltonian

$$H = \int dx[\frac{1}{2}\pi^2(x,t) + \frac{1}{2}(\frac{\partial \varphi}{\partial x})^2 + \frac{m^4}{\lambda}(\cos(\frac{\sqrt{\lambda}}{m}\varphi) - 1)], \tag{3.6}$$

where

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x,t)}.$$

The salient features of this Lagrangian have been described in Chapter 1. The Lagrangian (3.5) possesses the discrete symmetries, $\varphi \longleftrightarrow -\varphi \pm 2n\pi$, where $n = 0,1,2,3\ldots$ The potential function corresponding to Lagrangian (3.5),

$$V(\varphi) = \frac{m^4}{\lambda}[1 - \cos(\frac{\sqrt{\lambda}}{m}\varphi)], \tag{3.7}$$

has an infinite number of discrete degenerate absolute minima at $\varphi = \pm \frac{m}{\sqrt{\lambda}} 2\pi n$ (n = 0,1,2,...) and these could lead to SSB. The particle spectrum of the SG Hamiltonian (3.6) gives the soliton a mass (or energy) with a classical value

$$E_c(\varphi_c) = 8m^3/\lambda \quad (3.8)$$

We shall formulate the SG theory on a linear lattice [96]. The passage from the continum field theory to the lattice model is carried out in the following way. The continuous space is replaced by a discrete lattice of linear dimension L, with a lattice spacing $1/\Lambda$, defined so that there are 2N+1 points on a side. This necessitates that the continuous variable x be replaced by a discrete variable j, for each site on the lattice, such that $j = 0, \pm 1, \pm 2, \ldots \pm N$. The advantage of this programme is that Λ serves as a natural cut-off for the momentum k, which can remove the u.v. divergences in the renormalisation procedure. The one dimensional 'volume integral' is replaced by a summation; accordingly,

$$\int dx \longrightarrow \frac{1}{\Lambda} \sum_j . \quad (3.9)$$

The fields will be defined at the lattice sites, and can be expanded in terms of their Fourier components:

$$\varphi(x) \longrightarrow \varphi_j = \sum_{k=-k_{max}}^{k_{max}} \varphi(k)\, e^{ikj/\Lambda}$$

$$\varphi(k) = \frac{1}{2N+1} \sum_j \varphi_j\, e^{-ikj/\Lambda} \tag{3.10}$$

$$\pi(x) \longrightarrow \pi_j = \sum_{k=-k_{max}}^{k_{max}} \pi(k) e^{ikj/\Lambda}$$

$$\pi(k) = \frac{1}{2N+1} \sum_j \pi_j e^{-ikj/\Lambda} ,$$

where $k = \frac{2\pi n}{L}$ ($n = 0, \pm 1, \pm 2, \ldots$) such that $k_{max} = \frac{2\pi N}{L}$. The canonical commutation relation on the lattice between the fields φ_i and its canonically conjugate variable π_i, can be written as

$$[\pi_j , \varphi_i] = - i \Lambda\, \delta_{ji} \tag{3.11}$$

The gradient operator in the lattice formulation is usually written as a difference operator,

$$\nabla_\ell \varphi = \Lambda\, (\varphi_{\ell+1} - \varphi) \tag{3.12}$$

But this leads to a doubling of the fermionic degrees of freedom. Though this definition of the gradient operator does not cause any difficulty in the present calculation, we will follow the method of Drell et al [96]. Consider an arbitrary function f_j with its Fourier expansion,

$$f_j = \sum_k f(k)\, e^{ikj/\Lambda} \tag{3.13}$$

Then

$$\partial_j f_j = \Lambda\left[\sum_k f(k)\, e^{ik(j+1)/\Lambda} - \sum_k f(k)\, e^{ikj/\Lambda}\right]$$

$$= \sum_k f(k)\, e^{ikj/\Lambda}$$

$$= \sum_{j'} f_{j'}\left[\frac{1}{2N+1} \sum_k ik e^{ik(j-j')/\Lambda}\right] \tag{3.14}$$

This definition of the gradient enables us to write

$$\int dx\, \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 \longrightarrow \frac{1}{\Lambda} \sum_j \frac{1}{2} (\partial_j \varphi_j)^2$$

$$= \frac{1}{\Lambda} \sum_{jj'} \frac{1}{2} \Lambda^2 \varphi_j \varphi_{j'} D(j-j') ,$$

where

$$D(j-j') = \frac{1}{2N+1} \sum_k \frac{k^2}{\Lambda^2}\, e^{ik(j-j')/\Lambda} , \tag{3.15}$$

is a sum of correlation terms.

The value of D(j-j') can be found as

$$D(j) = \frac{4N(N+1)}{(2N+1)^2}\frac{\pi^2}{3} \xrightarrow[N\to\infty]{} \frac{\pi^2}{3} \text{ for } j = 0 \tag{3.16}$$

$$= \frac{(2\pi)^3 (-1)^j \cos(\pi j/2N+1)}{2(2N+1)^3 \sin(\pi j/2N+1)} \xrightarrow[N\to\infty]{} \frac{2(-1)^j}{j^2} \text{ for } j \neq 0.$$

Thus the Hamiltonian (3.6) now reads

$$H = \frac{1}{\Lambda}\sum_j \pi_j^2 + \frac{1}{\Lambda}\sum_{jj'} \frac{1}{2}\Lambda^2 \varphi_j \varphi_{j'} D(j-j') +$$

$$\frac{1}{\Lambda}\sum_j \frac{m^4}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}\varphi_j - 1\right) \tag{3.17}$$

Introducing two dimensionless variables as

$$\varphi_j = x_j$$
$$\pi_j = \Lambda p_j, \tag{3.18}$$

such that

$$[p_i, x_{j'}] = -i\delta_{ij'}, \tag{3.19}$$

and rescaling m and λ as

$$m^2 = m_o^2 \Lambda^2$$
$$\lambda = \lambda_o \Lambda^2 \tag{3.20}$$

the Hamiltonian (3.17) can be cast into the form

$$H = \Lambda \sum_j [\tfrac{1}{2}p_j^2 + \tfrac{1}{2} D(0)x_j^2 + \gamma(\cos(\beta x_j) - 1)] +$$

$$\Lambda \sum_{j_1 \neq j_2} [\tfrac{1}{2} D(j_1-j_2)\, x_{j_1} x_{j_2}], \qquad (3.21)$$

where

$$\beta = \frac{\sqrt{\lambda_0}}{m_0} \quad \text{and} \quad \gamma = \frac{m_0^4}{\lambda_0} \qquad (3.22)$$

Variational Calculation with a Single-Site Basis

The energy of the field system in one of the vacua corresponding to SSB, satisfies the eigenvalue equation

$$H|\psi\rangle = E|\psi\rangle \qquad (3.23)$$

This state, $|\psi\rangle$, will hereafter be referred to as the ground state of the system. In order to perform a variational calculation, we construct $|\psi\rangle$ as the product of eigenstates at each site j:

$$|\psi\rangle = \prod_j |\psi_j\rangle \qquad (3.24)$$

where the correlation between different sites is ignored. This approximation yields a 'single-site basis'. The trial states $|\psi_j\rangle$ are constructed by introducing creation and annihilation operators through the field variables x_j and p_j:

$$x_j = \frac{1}{2\omega_j}(a_j + a_j^+)$$

$$ip_j = (\frac{\omega_j}{2})(a_j - a_j^+) \ , \tag{3.25}$$

where ω_j is some parameter. The annihilation operator destroys the vacuum $|0_j\rangle$ at the site j in the sense,

$$a_j|0_j\rangle = 0. \tag{3.26}$$

From (3.19), it can be seen that the creation-annihilation operators satisfy the commutation relation

$$[a_j, a_{j'}^+] = \delta_{jj'}. \tag{3.27}$$

Repeated application of a_j^+'s on the vacuum state yields

$$|n_j\rangle = \frac{1}{(n_j!)^{\frac{1}{2}}}(a_j^+)^{n_j}|0_j\rangle \tag{3.28}$$

This enables us to cast $|\psi_j\rangle$ in the form

$$|\psi_j\rangle = \sum_{n_j=0}^{\infty} C_{n_j}^j |n_j\rangle \tag{3.29}$$

satisfying the relation

$$\langle \psi_j | \psi_{j'} \rangle = \delta_{jj'}. \tag{3.30}$$

The ground state energy in this basis,

$$\langle \Psi | H | \Psi \rangle = E_o(\Psi) =$$

$$\Lambda [\sum_j \langle \psi_j | \tfrac{1}{2}p^2 + \tfrac{1}{2}D(0)x_j^2 + \gamma(\cos(\beta x_j)-1) | \psi_j \rangle +$$

$$\tfrac{1}{2} \sum_{j_1 \neq j_2} D(j_1-j_2) \langle \psi_{j_1} | x_{j_1} | \psi_{j_1} \rangle \langle \psi_{j_2} | x_{j_2} | \psi_{j_2} \rangle] \tag{3.31}$$

Now we make the assumption that the ground state is translationally invariant so that the same $|\psi_j\rangle$ can be adopted for each site j, ie, $C^j_{n_j} = C_{n_j}$. This enables us to write

$$\begin{aligned} \langle \psi_j | x_j^n | \psi_j \rangle &\longrightarrow \langle \psi | x^n | \psi \rangle \\ \langle \psi_j | p_j^2 | \psi_j \rangle &\longrightarrow \langle \psi | p^2 | \psi \rangle \end{aligned} \tag{3.32}$$

It is apparent from (3.15) that

$$\sum_{j_1} D(j_1-j_2) = 0,$$

so that

$$\sum_{j_1=j_2} D(j_1-j_2) = -\sum_{j} D(0)$$

$$= -(2N+1)D(0). \qquad (3.33)$$

Then (3.31) can be put in the form

$$E_0(\Psi) = (2N+1)\wedge[\langle\Psi|\tfrac{1}{2}p^2 + \tfrac{1}{2}D(0)x^2+\gamma(\cos(\beta x)-1)|\Psi\rangle$$

$$- \tfrac{1}{2}D(0)\langle\Psi|x|\Psi\rangle^2]. \qquad (3.34)$$

Our aim here is to vary the trial state $|\Psi\rangle$ in order to minimise the value $E_0(\Psi)$. It can be seen that, except for the last term, (3.34) would have given the expectation value of a positive-definite Schrodinger Hamiltonian. The last term corresponds to the square of the expectation value of the mean field strength $\langle\Psi|x|\Psi\rangle$. This suggests that $\langle\Psi|x|\Psi\rangle$ can be chosen as one of the variational parameters in any trial wave function. The minimisation is carried in two steps: first, a Lagrange multiplier is introduced and the variation is carried out with $\langle\Psi|x|\Psi\rangle$ held fixed; the lowest value of $E_0(\Psi)$ is found out by varying over all values of $\langle\Psi|x|\Psi\rangle$

Let us define

$$\bar{H}(J) = \frac{1}{2}p^2 + \frac{1}{2}D(0)x^2 + \gamma(\cos(\beta x)-1) - J(x)\mathbf{x}, \quad (3.35)$$

and let

$$\Gamma(J) = \langle \Psi_0 | \bar{H}(J) | \Psi_0 \rangle \quad (3.36)$$

be its ground-state eigenvalue. It then follows that

$$\frac{\partial \Gamma(J)}{\partial J(x)} = -\langle \Psi_0 | x | \Psi_0 \rangle = -x(J). \quad (3.37)$$

For small J, invoking parity considerations to yield a nonvanishing vev $\langle x \rangle$, we try an expansion for x(J):

$$x(J) = \sum_{r=0}^{\infty} C_{2r+1} J^{2r+1} \quad (3.38)$$

Eqs.(3.37) and (3.38) suggest that

$$\Gamma(J) = \Gamma(0) - \frac{1}{2}C_1 J^2 - \frac{1}{4}C_3 J^4 + \ldots \quad (3.39)$$

Since $J \ll 1$, (3.38) can be inverted to obtain

$$J(x) = \frac{1}{C_1}x + \frac{C_3}{C_1^4}x^3 + \ldots \quad (3.40)$$

From (3.34) and (3.35) we find,

$$\frac{E(J)}{(2N+1)\Lambda} \equiv \mathcal{E}(J) = \Gamma(J) + Jx(J) - \frac{1}{2}D(0)x^2(J). \quad (3.41)$$

Using (3.39) and (3.40), this can be cast into the form

$$\mathcal{E}(J) = \Gamma(0) - \frac{\eta x^2}{2C_1} + \frac{C_3}{C_1^4}\left(\frac{1}{4} + \eta\right)x^4 + \ldots \quad (3.42)$$

where $\eta = C_1 D(0) - 1.$ (3.43)

Eqs.(3.35) and (3.38) fix C_1 to be always positive, and $\mathcal{E}(J)$ will be a minimum for small x, if

$$0 < \eta << 1 \quad \text{and} \quad C_3 < 0 \quad (3.44)$$

The value of x corresponding to the minimum of $\mathcal{E}(J)$ denoted by x_c is given by

$$x_c^2 = \frac{\eta C_1^3}{|C_3|}(1-4\eta). \quad (3.45)$$

To show that there exists a range of β and γ, for which these conditions on C_1 and C_3 are satisfied, we introduce a two-parameter trial state $|\psi\rangle$ having the form of a displaced Gaussian form as

$$|\psi_o\rangle = e^{-i\langle x\rangle p}|0\rangle, \quad \langle\psi_o|x|\psi_o\rangle = \langle x\rangle. \quad (3.46)$$

Taking the expectation value of (3.35) in this state, we find,

$$\Gamma(J) = \langle \Psi_0 | \bar{H}(J) | \Psi_0 \rangle =$$

$$\frac{\omega^2}{8} + \frac{D(0)}{8\omega^2} + \frac{D(0)}{2}\langle x\rangle^2 +$$

$$\gamma(\cos(\beta\langle x\rangle))K(\omega) - \gamma - J\langle x\rangle, \qquad (3.47)$$

where

$$K(\omega) = 1 - \frac{\beta^2}{2!\,(2\omega)^2} + \frac{\beta^4}{4!}\,\frac{3}{(2\omega)^4} + \ldots$$

The variational conditions are

$$\frac{\partial \Gamma(J)}{\partial \omega} = 0 , \qquad (3.48)$$

and

$$\frac{\partial \Gamma(J)}{\partial \langle x\rangle} = 0. \qquad (3.49)$$

Expanding $\omega(J)$ as a power series in J, we write

$$\omega(J) = \sum_{s=0}^{\infty} \omega_{2s} J^{2s}. \qquad (3.50)$$

Equating the coefficient of each power of J to zero, we obtain the following relations.

$$\frac{\omega_0^4 - D(0)}{\gamma} = -\beta^2 \qquad (3.51)$$

$$C_1 = \frac{1}{(\omega_c)^4} > 0 \tag{3.52}$$

$$\omega_2 = \frac{\gamma\beta^4}{8(\omega_o)^{11}} \tag{3.53}$$

$$C_3 = \frac{\gamma\beta^4}{2(\omega_o)^{16}} \left[\frac{\gamma\beta^4}{16(\omega_o)^6} - \frac{1}{3}\right]. \tag{3.54}$$

Eqs.(3.43) and (3.54) yield

$$\omega_o = [D(O)]^{1/4} \left[1 - \frac{\eta}{4} + O(\eta^2)\right]. \tag{3.55}$$

This equation together with (3.51) gives

$$\gamma\beta^2 = m_o^2 = (1+2\eta)\eta D(O) + O(\eta^3). \tag{3.56}$$

The condition $C_3 < 0$ is guaranteed, if

$$\gamma\beta^4 = \lambda_o < \frac{16}{3}(\omega_o)^6. \tag{3.57}$$

It can be seen that (3.55) and (3.57) can be satisfied simultaneously – which implies that the condition for $0<x_c<< 1$ has been satisfied. This may be stated as

$$m_o^2 > m_{cr}^2 = 0,$$

$$\lambda_o < \lambda_{cr} \underset{N\to\infty}{\longrightarrow} \frac{16}{3}(\pi^2/3)^{6/4}. \tag{3.58}$$

Choosing the parameters m_0 and λ_0 corresponding to the critical values given in the last equation, it follows that

$$\langle x \rangle = x_c \propto \left[\frac{2\eta D(0)}{\lambda_0} \right]^{1/2} \propto \left[\frac{(m_0^2 - m_{cr}^2)}{\lambda_0} \right]^{1/2} \qquad (3.59)$$

Thus we can show that x_c can be made arbitrarily small by appropriate choice of m_0 and λ_0, without encountering a false minimum at $\langle x \rangle = 0$. This implies that a transition from the phase $\langle x \rangle = 0$ to the phase $\langle x \rangle \neq 0$ can occur by a smooth variation of the constants, and the critical point corresponds to $m_{cr} = 0$ and $\lambda_{cr} = \frac{16}{3}(\pi^2/3)^{6/4}$. This is not a first order phase transition, which is distinguished by a discontinuous change from one phase to another.

Calculation of an Upper Bound on the Soliton Mass

In order to compute an upper bound on the classical soliton mass, we shall work again in a single-site basis [96]. In this case the calculation can be carried out in a simple way, by adding and subtracting the diagonal term $j_1 = j_2$ in the double sum in (3.31). Thus

$$E_s = \langle \Psi_s | H | \Psi_s \rangle =$$

$$\Lambda \Big[\sum_j \langle \Psi_j | \tfrac{1}{2} p_j^2 + \gamma(\cos(\beta x) - 1) + \tfrac{1}{2} D(0)(x_j^2 - x_j \langle \Psi_j | x_j | \Psi_j \rangle) | \Psi_j \rangle$$

$$+ \frac{1}{2} \sum_{j_1 j_2} D(j_1 - j_2) \langle \Psi_{j_1} | x_{j_1} | \Psi_{j_1} \rangle \langle \Psi_{j_2} | x_{j_2} | \Psi_{j_2} \rangle \qquad (3.60)$$

The last term in this equation represents the gradient term in (3.6), with the matrix element replacing its classical strength. This is the only term which couples different lattice sites; hence the soliton energy may be minimised by repeating the procedure for the vacuum states. In this case the Lagrange multiplier $J(j)$ is introduced for each site, and the soliton energy is minimised by keeping

$$\langle x_j \rangle = \langle \Psi_j | x_j | \Psi_j \rangle$$

fixed. Thus, in analogy with (3.35) and (3.41), we introduce

$$\bar{H}(J(j)) = \frac{1}{2N+1} \sum_j \Big[\frac{p_j^2}{2} + \frac{D(0) x_j^2}{2} + \gamma(\cos(\beta x_j) - 1) - J(j) x_j \qquad (3.61)$$

Defining

$$\Gamma(J(j)) = \langle \Psi_s | \bar{H} | \Psi_s \rangle$$

$$= \frac{1}{2N+1} \sum_j \Gamma_j(J(j)),$$

where $\Gamma_j(J(j))$ is the same function calculated in (3.39), we find

$$\frac{1}{\Lambda(2N+1)} E_s = \frac{1}{2N+1} \sum_j \left[\Gamma_j(J(j)) + J(j))\langle x_j \rangle - \tfrac{1}{2}D(0)\langle x_j \rangle^2 \right]$$

$$+ \frac{1}{2N+1} \sum_{j_1,j_2} \tfrac{1}{2} D(j_1-j_2) \langle x_{j_1} \rangle \langle x_{j_2} \rangle \qquad (3.62)$$

It is apparent that, except for an interval of finite length on the lattice, the value of $\langle x(j) \rangle$ must be arbitrarily close to x_c and 0 where x_c is given by (3.45). Let us introduce D as a parameter in the variational calculation (fig.3.1), such that

$$D = \frac{2j_o + 1}{\Lambda}, \qquad (3.63)$$

where

$$\langle x_j \rangle = x_c \quad \text{for } j > j_o$$

$$= 0 \quad \text{for } j < -j_o$$

Now the difference between the energy of the soliton and vacuum states can be put in the form as

$$E_s(J) - E_o(J) = \Lambda \sum_{j=-j_o}^{j_o} \left[\Gamma_j(J(j)) + J(j) \langle x_j \rangle - \tfrac{1}{2}D(0)\langle x_j \rangle^2 \right] - \frac{2j_o+1}{2N+1} E_o(J) + \Lambda \sum_{j_1 j_2} \tfrac{1}{2}D(j_1-j_2) \langle x_{j_1} \rangle \langle x_{j_2} \rangle$$

(3.64)

The last term represents the 'kinetic' energy due to the change of $\langle x_j \rangle$ in the transition region, and may be evaluated by a linear approximation to $\langle x_j \rangle$ between $-j_o$ and j_o. Thus,

$$\Lambda \sum_{j_1 j_2} \tfrac{1}{2} D(j_1 - j_2) \langle x_{j_1} \rangle \langle x_{j_2} \rangle = \Lambda (2j_o + 1) \frac{x_c^2}{(2j_o+1)^2}$$

$$= \frac{\Lambda x_c^2}{D\Lambda} \qquad (3.65)$$

For calculating the difference of the first two terms in (3.64), we set $J(j) = 0$. Using (3.42) and (3.63) this difference gives, to leading order in $x_c^2 << 1$,

$$(2j_o+1)\Lambda \Gamma(0) - \frac{2j_o+1}{2N+1} E_o(J) = D\Lambda^2 \frac{\eta}{2c_1} x_c^2 \qquad (3.66)$$

Combining (3.65) and (3.66), we get

$$E_s(J) - E_o(J) = \Lambda \left(D\Lambda \frac{\eta}{2c_1} x_c^2 + \frac{x_c^2}{D\Lambda} \right), \qquad (3.67)$$

which is minimised at

$$(D\Lambda) = (2c_1/\eta)^{1/2} \qquad (3.68)$$

to give

$$E_s(J) - E_o(J) = 2x_c^2 (\eta/2c_1)^{1/2} \Lambda \qquad (3.69)$$

From (3.68) we find

$$\eta = \frac{2c_1}{(D\Lambda)^2}$$

This enables us to rewrite x_c in (3.59) as

$$x_c = \frac{2}{(D\Lambda)\lambda_0^{1/2}} .$$

Or equivalently,

$$D \sim 2(\lambda^{1/2} \cdot x_c)^{-1} \qquad (3.70)$$

Hence (3.69) can be put in the form

$$E_s(J) - E_0(J) \simeq \lambda^{1/2} x_c^3 \qquad (3.71)$$

Thus, the one soliton energy is, apart from numerical factors, a rescaled version of the semiclassical result for the soliton mass $\frac{m^3}{\lambda}$ in (3.8). Since we are free to choose x_c small enough, the soliton mass is finite and small, and no matter how large the cut-off Λ is made.

Sine-Gordon Type Theories

The foregoing consideration can be extended to a generalised SG potential of the form

$$V(\varphi) = \frac{m^4}{\lambda}[1 - \cos^{\ell}(\frac{\sqrt{\lambda}}{m}\varphi)], \qquad (3.72)$$

where ℓ is a positive integer. This potential is invariant under the symmetry operation $\varphi \longleftrightarrow -\varphi \pm 2n\pi$ and has an infinite number of discrete absolute minima at $\varphi = \pm \frac{m}{\sqrt{\lambda}} 2n\pi$, and this could lead to SSB. Proceeding as in the case of the ordinary SG model, we can find

$$\frac{\omega_o^4 - D(0)}{\gamma} = -\beta^2 \ell \qquad (3.73)$$

$$C_1 = \frac{1}{(\omega_o)^4} \qquad (3.74)$$

$$\omega_2 = \frac{\gamma\beta^4}{8(\omega_o)^{11}} \ell(3\ell-2) \qquad (3.75)$$

$$C_3 = \frac{\gamma\beta^4}{2(\omega_o)^{16}}\left[\frac{\gamma\beta^4}{16(\omega_o)^6} \ell(3\ell-2) - \frac{1}{3}\right]\ell(3\ell-2) \qquad (3.76)$$

We can also arrive at, as above, the following results

$$m_o^2 == \frac{1}{\ell}\left[D(0)\eta + D(0)2\eta^2 + O(\eta^3)\right] > m_{cr}^2 = 0 \qquad (3.77)$$

$$\lambda_o < \lambda_{cr} = \frac{1}{\ell(3\ell - 2)} \frac{16}{3} [D(0)]^{6/4} \qquad (3.78)$$

Though it has not yet been established that the generalised SG model based on a potential of the form

$[1 - \cos^{\ell}(\frac{\sqrt{\lambda}}{m}\varphi)]$ contains a soliton sector, except for the case $\ell = 4$ [181], we can conclude that the generalised SG theory also undergoes a second-order phase transition. From the above calculations, it can be seen that we cannot get the results in the continuum limit by a naive approximation of taking the limit $\frac{1}{\Lambda} \longrightarrow 0$. To get the results for the case $\frac{1}{\Lambda} \longrightarrow 0$ we have to employ renormalisation group ideas [93].

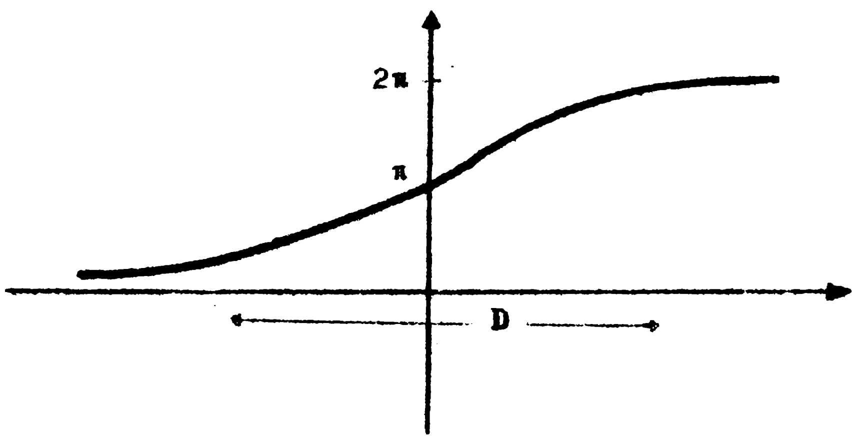


Fig. 3.1 The soliton configuration

Four

EFFECTIVE POTENTIAL IN SINE-GORDON THEORY AT FINITE TEMPERATURE

Introduction

In the previous chapter we presented a formulation of the two dimensional SG field theory on a lattice. It is also shown there that the SG field system exhibits a second order phase transition at zero temperature; ie, by appropriate choice of the parameters, the system can be made to undergo a transition from the ordered phase $\langle \varphi \rangle \neq 0$ to the disordered phase $\langle \varphi \rangle = 0$. In this chapter we study the effect of temperature on the SG field system, since the most important physical consideration in statistical mechanics, is the introduction of temperature into the formalism.

In recent years, there has been growing interest in field theoretical models for condensed matter physics. Since the pioneering work by Krumhausl and Schrieffer [182] on the study of statistical mechanics of the φ^4 field system, many have studied the classical statistical mechanics of kinks of a variety of one dimensional systems [183-188]. The classical statistical mechanics of the static and dynamic properties of the SG system have been calculated [184-186], and stability of the system under thermal fluctuations has also been studied [188].

Though classical statistical mechanics is eminently successful in describing many of the observed phenomena in condensed matter physics, the method fails to account for renormalisation of the boson and soliton energy and bound states of fluctuations (eg "breathers" in SG system), while such cases can be handled within the framework of quantum statistical mechanics. In parallel to the development of classical statistical mechanics of 1-dimensional non-linear systems, there has been remarkable advance in the quantum field theory of 1+1 dimensional non-linear systems [80-84]. Recently Maki and Takayama [189] studied the quantum statistical mechanics of the SG system, but the very detailed discussion of the quantum statistical mechanics of the SG system given by them does not, however, clarify the nature of the phase transition. They have used the functional integral method to study the thermodynamic properties of the SG system.

In the present work, we first calculate the quantum correction to soliton energy by evaluating the effective potential at one-loop level [155,190]. It is interesting to find that our result agrees with the other well-known results [80,81]. Temperature is introduced into the formalism by calculating the effective potential at finite temperature [112]. This is possible, since the differential equations satisfied by finite temperature Green's functions, are identical with those of the zero temperature theory [114,115]. The difference lies in the boundary conditions. Whereas the familiar causal boundary conditions at $\pm\infty$ are appropriate at zero temperature, periodic boundary conditions for imaginary time are relevant at finite temperature, as described in chapter 1. It is found that the symmetry breaking of this model can be removed by increasing the temperature above a critical value [110]. The phase transition, that this model may undergo, is clarified to be one of second order. The calculations are extended to generalised SG type theories defined by a potential of the form $V(\varphi) = \cos^{l}(\frac{\sqrt{\lambda}}{m}\varphi)$.

Effective Potential Approach to Find Quantum Corrected Mass

The Lagrangian for the two dimensional SG theory is written

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^4}{\lambda}(\cos(\frac{\sqrt{\lambda}}{m}\varphi) - 1). \quad (4.1)$$

The properties of this type of Lagrangian have already been discussed in Chapters 1 and 3. The potential function

$$V(\varphi) = \frac{m^4}{\lambda}[1 - \cos(\frac{\sqrt{\lambda}}{m}\varphi)], \tag{4.2}$$

corresponding to this Lagrangian has an infinite number of discrete degenerate minima at $\varphi = \pm \frac{m}{\sqrt{\lambda}} 2\pi n = \sigma$ (n=0,1,2,...) which signal the existence of SSB. The soliton solution of the equation of motion (see (1.28)) corresponding to the Lagrangian (4.1) is

$$\varphi_s = \frac{4m}{\sqrt{\lambda}} \arctan(e^{mx}), \tag{4.3}$$

with classical soliton energy as

$$E_c = \frac{8m^3}{\lambda} . \tag{4.4}$$

We shall evaluate the effective potential to one-loop order at zero temperature. The presence of SSB requires that the field be shifted from φ to $\varphi+\sigma$ where σ is a classical constant scalar field, such that after the shift

$$\langle 0|\varphi|0\rangle = 0. \tag{4.5}$$

Hence the Lagrangian (4.1), after the shift, becomes

$$\mathcal{L}(\varphi+\sigma) = \frac{1}{2}(\partial_\mu(\varphi+\sigma))^2 + \frac{m^4}{\lambda}(\cos(\frac{\sqrt{\lambda}}{m}(\varphi+\sigma))-1). \tag{4.6}$$

From this the new propagator for the field can be found out:

$$i\Delta(\sigma, k) = \frac{-1}{k^2 - M^2} \tag{4.7}$$

where

$$M^2 = m^2 - \cos\left(\frac{\sqrt{\lambda}}{m}\sigma\right). \tag{4.8}$$

The effective potential [155,161] can be expressed in the form

$$V(\sigma) = V_o(\sigma) + V_1(\sigma) + V_2(\sigma) + \ldots \tag{4.9}$$

The first term in this series gives the classical tree approximation; the second term represents the contribution of all graphs with one closed loop; and so on. Here the calculations are done only upto one-loop order. The zero-loop (tree approximation) part of the effective potential is, thus, given by

$$V_o(\sigma) = \frac{m^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m}\sigma\right)\right]. \tag{4.10}$$

The one-loop contribution to the effective potential [161]

$$V_1(\sigma) = \frac{-i}{2}\int \frac{d^2k}{(2\pi)^2}\ln(k^2-M^2) \tag{4.11}$$

The integration can be performed by rotating this integral into Euclidean space; thus,

$$V_1(\sigma) = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2} \ln(k^2 - M^2) \qquad (4.12)$$

The u.v. divergence in this integral can be removed by introducing a cut off at $k^2 = \Lambda^2$; so that,

$$V_1(\sigma) = \frac{M^2}{8\pi} \ln(\frac{\Lambda^2}{M^2}). \qquad (4.13)$$

Here the expressions that vanish as $\Lambda^2 \longrightarrow \infty$, are thrown out. Hence the effective potential at zero-temperature upto one-loop order, can be written as

$$V(\sigma) = \frac{m^4}{\lambda} [1 - \cos(\frac{\sqrt{\lambda}}{m} \sigma)] + \frac{M^2}{8\pi} \ln(\frac{\Lambda^2}{M^2}). \qquad (4.14)$$

The cut-off parameter can be removed by adding a counter term $A \cos(\frac{\sqrt{\lambda}}{m} \sigma)$. The value of A can be determined using the renormalisation condition

$$\left.\frac{d^2V}{d\sigma^2}\right|_{\sigma=0} = m^2, \qquad (4.15)$$

which yields

$$A = -\frac{m^2}{8\pi}\ln(\frac{\Lambda^2}{m^2}) + \frac{m^2}{8\pi} . \qquad (4.16)$$

Hence (4.14) now takes the form

$$V(\sigma) = \frac{m^4}{\lambda}[1 - \cos(\frac{\sqrt{\lambda}}{m}\sigma)] - \frac{M^2}{8\pi}\ln(\cos(\frac{\sqrt{\lambda}}{m}\sigma)) + \frac{m^2}{8\pi}\cos(\frac{\sqrt{\lambda}}{m}\sigma). \quad (4.17)$$

For $\frac{\sqrt{\lambda}}{m}$ small, it can be seen that quantum effects manifest themselves through the shift $\frac{m^4}{\lambda} \longrightarrow \frac{m^4}{\lambda} - \frac{m^2}{8\pi}$. We thus obtain the quantum corrected soliton mass [190] in the form

$$(E_c = \frac{8m^3}{\lambda}) \longrightarrow (E_s = \frac{8m^3}{\lambda} - \frac{m}{\pi}) \quad (4.18)$$

This result agrees with that of earlier works [80,82]. In the above calculation we have used only one-loop correction term, and if we go for higher loops, though the calculations may become very tedious, better values for the soliton mass can be obtained.

Sine-Gordon Theory at Finite Temperature

We shall now study the behaviour of the SG theory at finite temperature, by evaluating the temperature-dependent effective potential, $V^T(\sigma)$ [112]. In this calculation the Boltzmann constant k is set equal to unity. The one-particle irreducible vacuum graphs are the same for both zero and finite temperature formulations. Thus the effective potential at finite temperature to all loops, can be written

$$V^T(\sigma) = V_o(\sigma)+V_1^T(\sigma)+i \langle \exp(i\int d^3x \mathcal{L}_I(\sigma,\varphi))\rangle \quad (4.19)$$

Here $V_o(\sigma)$ is the classical potential - the zero-loop contribution to the effective potential. The zero-loop contribution is temperature-independent. $V_1^T(\sigma)$ gives the one-loop contribution. Higher loop contributions are given by $\langle \exp(i\int d^3x \mathcal{L}_I(\sigma,\varphi))\rangle$ - the sum of all the one-particle irreducible vacuum graphs. Thus in the present case, the zero-loop contribution to the effective potential is,

$$V_o(\sigma) = \frac{m^4}{\lambda}(1 - \cos(\frac{\sqrt{\lambda}}{m}\sigma)) \quad (4.20)$$

Using the rules given in (1.46) and (1.47), the one-loop contribution to the effective potential at finite temperature can be written as

$$V_1^T(\sigma) = \frac{T}{2}\sum_n\int \frac{dk}{2\pi} \ln(-4\pi^2n^2T^2 - E_M^2) \quad (4.21)$$

where $E_M^2 = k^2 + M^2$. Though the sum over n diverges, it may be evaluated in the following way. Define

$$X(E) = \sum_n \ln(4\pi^2n^2T^2 + E^2).$$

Then

$$\frac{dX(E)}{dE} = \sum_n \left(\frac{2E}{4\pi^2 n^2 T^2 + E^2} \right).$$

Invoking the identity [191]

$$\sum_{n=-\infty}^{\infty} \frac{1}{a^2+n^2} = \frac{\pi}{a}\coth(\pi a), \tag{4.22}$$

we find

$$\frac{dX(E)}{dE} = \frac{2}{T} \left(\frac{1}{2} + \frac{1}{e^{E/T} - 1} \right).$$

This leads to the result

$$X(E) = \frac{2}{T} \left(\frac{E}{T} + T \ln(1 - e^{-E/T}) \right). \tag{4.23}$$

Hence (4.21) may be rewritten as,

$$V_1^T(\sigma) = \int \frac{dk}{2\pi} \left[\frac{E_M}{2} + T \ln(1 - e^{-E_M/T}) \right]$$

$$= V_1^o(\sigma) + \bar{V}_1^T(\sigma), \tag{4.24}$$

where

$$V_1^o(\sigma) = \int \frac{dk}{2\pi} \left(\frac{E_M}{2} \right) \tag{4.25}$$

gives the usual zero-temperature one-loop approximation to the effective potential which may be compared with (4.12), and

$$\bar{V}_1^T(\sigma) = \frac{T}{\pi}\int_0^\infty dk\, \ln(1 - e^{-E_M/T}) \qquad (4.26)$$

gives the temperature-dependent part at the one-loop level. It is evident that $\bar{V}_1^T$ vanishes, as it should, at zero temperature, $T \rightarrow 0$. Introducing x^2 as

$$x^2T^2 = E_M^2 - M^2$$

we have

$$\bar{V}_1^T(\sigma) = \frac{T^2}{\pi}\int_0^\infty dx\, \ln(1 - e^{-(x^2+M^2/T^2)^{1/2}}) \cdot \qquad (4.27)$$

This integral may be evaluated in the high temperature limit by giving a Taylor expansion for $\bar{V}_1^T(\sigma)$. Thus we find

$$\bar{V}_1^T(\sigma) = -\frac{\pi T^2}{6} + \frac{MT}{4} \qquad (4.28)$$

The critical temperature above which the SSB may be removed can be calculated in the following way. At zero temperature, the effective potential $V^T(\sigma) = V^0(\sigma)$ possesses a symmetry breaking solution, ie,

$$\frac{\partial V^0(\sigma)}{\partial\sigma} = 0 \quad \text{for } \sigma \neq 0.$$

If the finite temperature contribution can eliminate symmetry breaking, then

$$\frac{\partial V^T(\sigma)}{\partial \sigma} = 2\sigma \frac{\partial V^T(\sigma)}{\partial \sigma^2} = 0 \quad \text{only if } \sigma = 0. \qquad (4.29)$$

$\frac{\partial V^T(\sigma)}{\partial \sigma^2}$ is assumed to be positive for large σ^2. Writing

$$V^T(\sigma) = V^o(\sigma) + \bar{V}^T(\sigma)$$

then (4.29) becomes

$$\left.\frac{\partial V^o(\sigma)}{\partial \sigma^2}\right|_{\sigma=0} + \left.\frac{\partial \bar{V}^T(\sigma)}{\partial \sigma^2}\right|_{\sigma=0} \geq 0 \qquad (4.30)$$

Since

$$\left.\frac{\partial V^o(\sigma)}{\partial \sigma^2}\right|_{\sigma=0} = \frac{m^2}{2}$$

the critical temperature is defined by the relation

$$\left.\frac{\partial \bar{V}^{T_c}}{\partial \sigma^2}\right|_{\sigma=0} = -\frac{m^2}{2} \qquad (4.31)$$

Using (4.28) in this equation, we can see that the critical temperature above which the SG system does not exhibit SSB is

$$T_c = \frac{8m^3}{\lambda} \qquad (4.32)$$

This is large in the weak coupling limit. Thus we find that, when the temperature is raised above a value given by $T_c = 8m^3/\lambda$, the symmetry is restored in the case of 1+1 dimensional SG field system.

Let us now calculate the temperature-dependent boson mass of the field system. In the lowest order, the correction to the mass square term comes from  [80,82]. Hence the temperature-dependent mass in the one-loop order is given by

$$m_T^2 = m^2 - \delta m^2 - \frac{\lambda}{2} \int \frac{i}{k^2 - m^2}$$

$$= m^2 - \delta m^2 - \frac{\lambda}{2} \int \frac{dk}{2\pi} \left[\frac{1}{2E_M} + \frac{1}{E_M(e^{E_M/T} - 1)} \right] \tag{4.33}$$

The mass counter term δm^2 cancels the zero-temperature contribution - the first term in the integral. Thus in the high temperature limit, ie $T \gg m$, we are left with

$$m_T^2 = m^2 - \frac{\lambda T}{8 m} \tag{4.34}$$

The same result can be obtained from the temperature-dependent effective potential, the temperature-dependent mass being defined as

$$m_T^2 = 2 \left. \frac{\partial V^T(\sigma)}{\partial \sigma^2} \right|_{\sigma=0} \tag{4.35}$$

The quantum corrected soliton energy at zero temperature is given by

$$E_s = 8m/\gamma \tag{4.36}$$

where

$$\gamma = \frac{\lambda}{m^2\left(1 - \frac{\lambda}{8\pi m^2}\right)} .$$

Hence, using (4.34), we can find the soliton energy at finite temperature:

$$E_s^T = \frac{8m}{\gamma}\,(1 - T/T_c)^{\frac{1}{2}}; \tag{4.37}$$

or equivalently, we have

$$\frac{E_s^T}{E_s} = (1 - T/T_c)^{\frac{1}{2}} . \tag{4.38}$$

This relation tells us that the soliton energy E_s^T continuously decreases with temperature and vanishes at the critical temperature T_c, contrary to the observation made by Maki and Takayama [189]. In the weak coupling limit we can see from (4.32) that $T_c >> m$, and (4.38) rests on the assumption $T >> m$; hence, we argue that the decrease of E_s^T to zero is smooth and monotone. Hence it follows that the phase transition in the SG field system is one of second order.

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A large number of experiments have been performed to measure soliton energy [192,193]. It has been found that the experimentally observed value for soliton energy is less than the classical value [194]. Maki and Takayama [189] have shown that their calculation can account for the observed value of soliton energy. Recently Maki [195] has also shown that, in the region $T \gg m$, the discrepancies between the experimental and classical values of the energy of magnetic solitons can be accounted for in terms of the quantum effect. Since in the present calculation the soliton energy at finite temperature is less than the classical value, by appropriate choice of the parameters, the present result can account for the observed value of soliton energy.

Breathers in the Sine-Gordon System at Finite Temperature

The SG equation possesses another class of exact solutions called doublets or breathers [80,82]. The force between a soliton and an antisoliton is attractive, and hence, they may form a doublet, described by

$$\varphi_B(x,t) = \frac{4m}{\sqrt{\lambda}} \text{arc tan}\left[\tan\left(\frac{N\gamma}{16}\right) \frac{\sin(mt\cos(N\gamma/16))}{\cosh(mx\sin(N\gamma/16))}\right], \tag{4.39}$$

with an energy

$$E_N^B = \frac{16m}{\gamma} \sin(\frac{N\gamma}{16}). \qquad (4.40)$$

where $N = 1,2,3.... < 8\pi/\gamma$. It is seen from (4.40) that there is a finite number of doublet states, and as the coupling γ increases, these states disappear one by one, decaying into soliton-antisoliton pairs. Proceeding as before, we can find the energy of the breather at finite temperature [196]:

$$E_N^{BT} = 2E_s(1 - T/T_c)^{\frac{1}{2}} \sin(\frac{N\gamma}{16}) \qquad (4.41)$$

The energy of the breather also vanishes at the critical temperature, and the finite temperature behaviour of the energy of breather is similar to that of the energy of the ordinary soliton.

Sine-Gordon Type Theories

In this section we will consider a generalised SG field defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^4}{\lambda} \cos^{\ell}(\frac{\sqrt{\lambda}}{m}\varphi), \qquad (4.42)$$

where ℓ is a positive integer. The potential function corresponding to this Lagrangian is characterised by an infinite set of discrete degenerate minima at

$\sigma = \pm \frac{(2n+1)}{2} \frac{\pi m}{\sqrt{\lambda}}$ ($n = 0,1,2...$). This generalisation is motivated by the fact that for $\ell = 4$, (4.42) leads to the double SG equation [181]. Double SG equations are of current interest as some physical phenomena appear to be governed by these equations [197]. It is known that the double SG equation is not integrable [197] and the one-loop corrections are no longer all of the quantum corrections; nevertheless, methods for the one-loop correction can still be applied in the present case.

The soliton solution equation of motion corresponding to the Lagrangian (4.42) for $\ell = 4$, is given by

$$\varphi_s = \frac{m}{\sqrt{\lambda}} \text{arc tan}(mx), \tag{4.43}$$

with classical soliton energy

$$E_c = \frac{\pi m^3}{\lambda} . \tag{4.44}$$

A procedure similar to the one described above helps us to find the effective potential corresponding to the Lagrangian (4.42) in the one-loop approximation:

$$V(\sigma) = \frac{m^4}{\lambda} \cos^{\ell}\left(\frac{\sqrt{\lambda}}{m}\sigma\right) - \frac{M^2}{8\pi} \ln \cos\left(\frac{\sqrt{\lambda}}{m}\sigma\right) + \frac{\ell m^2}{8\pi} \cos\left(\frac{\sqrt{\lambda}}{m}\sigma\right), \tag{4.45}$$

where

$$M^2 = \ell m^2 \cos^{\ell}(\frac{\sqrt{\lambda}}{m}\sigma).$$

The temperature-dependent part of the effective potential in the present case is:

$$\bar{V}^T(\sigma) = -\frac{\pi}{6}T^2 + \frac{M}{4}T \qquad (4.46)$$

Invoking (4.31), we can find the value of the critical temperature also:

$$T_c = \sqrt{\ell}\,\frac{8m^3}{\lambda}. \qquad (4.47)$$

When $\ell = 4$, from (4.45) we can find the quantum-corrected soliton mass (see (4.44)) as,

$$(E_c = \frac{\pi m^3}{\lambda}) \longrightarrow (E_s = \frac{\pi m^3}{\lambda} - \frac{m}{8}). \qquad (4.48)$$

In this case the critical temperature is

$$T_c = 16\,\frac{m^3}{\lambda}\,, \qquad (4.49)$$

and it is seen that the soliton energy at finite temperature exhibits features similar to those of the ordinary SG theory.

To conclude, we find that using the method of effective potential, the salient features of the finite temperature behaviour of SG and generalized SG field systems can be analysed in a satisfactory manner.

Five

FINITE TEMPERATURE BEHAVIOUR OF A φ^6 FIELD SYSTEM

Introduction

The significance of studying the effect of temperature on quantum field system has been discussed in the first chapter. In the preceding chapter we considered the response of a particular quantum field system to temperature. Here we discuss another scalar field model in 1+1 dimensions and study the effect of temperature on this system. The functional diagrammatic method employed earlier, has been adopted in the present case too. We will consider a φ^6 self interacting field system which exhibits SSB. It has been shown that [198,199] such a field system in 1+1 dimensions possesses soliton solutions. This system enjoys $\varphi \longleftrightarrow -\varphi$ symmetry, and has a positive mass square term in the Lagrangian. Using lattice approximation and block-spin renormalisation group method Boyanovsky and Masperi [200] have shown that the nature of phase transition associated with such a field system may be second order or first order, depending on the relative depths of the wells and intersite coupling.

Besides its importance in particle physics as a model scalar field theory, the φ^6 self interacting model finds applications in solid state physics also, where it has been used to explain the first order phase transition from the ferroelectric to paraelectric state and the structural phase transitions observed in crystals [201-203].

We saw in Chapter 2, a new mechanism for SSB due to Coleman and Weinberg [155]. In this mechanism, the vacuum of the original field system is shifted from the symmetric point to an asymmetric point, when quantum corrections are added to the original Lagrangian, which will give rise to SSB. Recently Rajaraman and Rajalakshmy [204] have shown that a symmetry spontaneously broken, can be restored due to radiative corrections. By taking a self interacting scalar field of the type:

$$\mathcal{L}(x,t) = \frac{1}{\lambda}\left[\frac{1}{2}(\partial_\mu\varphi)^2 - (\varphi^2 + a^2)(\varphi^2-1)^2 + a^2\right] \quad (5.1)$$

they showed that for a certain range of the parameters a and λ, the symmetry can be restored via radiative corrections.

We take a φ^6 coupled scalar field in 1+1 dimensions, described by the Lagrangian

$$\mathcal{L}\left\{\varphi(x)\right\} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}\lambda^2\varphi^2\left(\varphi^2 - \frac{m}{\lambda}\right)^2, \quad (5.2)$$

where $m, \lambda > 0$. The potential function

$$V(\varphi) = \frac{1}{2}\lambda^2\varphi^2 (\varphi^2 - \frac{m}{\lambda})^2, \tag{5.3}$$

has the shape shown in fig.5.1. There are three absolute minima: one at $\varphi = 0$ and the other two at $\varphi = \pm(m/\lambda)^{1/2}$. The static solution is given by [198-199]

$$\varphi_c(x) = [\frac{m}{2\lambda}(\tanh(mx) + 1)]^{1/2}, \tag{5.4}$$

and the soliton mass comes out to be

$$M = \frac{m^2}{4\lambda} . \tag{5.5}$$

In the present study we first show that this model is renormalisable at zero temperature. It is also shown that the spontaneously broken symmetry can be restored by raising the temperature of the field system. The value of the critical temperature is calculated quantitatively and the nature of phase transition associated with this system is shown to be one of first order.

Effective Potential at T = 0

We shall now proceed to evaluate the effect of quantum corrections on the potential function (5.3) at zero temperature. Since the potential function has three absolute minima (one at $\varphi = 0$ and the other two at $\varphi = \pm (m/\lambda)^{1/2} = \sigma$), the vacuum around $\varphi = \pm (m/\lambda)^{1/2}$ would lead to SSB. Hence in order to have vanishing vevs the field φ should be given a shift: $\varphi \longrightarrow \varphi+\sigma$, where σ is a classical constant scalar:

$$\sigma = (m/\lambda)^{1/2} , \tag{5.6}$$

such that after the shift we have

$$\langle 0 | \varphi | 0 \rangle = 0 .$$

The Lagrangian (5.2), after the shifting, is given by

$$\mathcal{L}(\varphi+\sigma) = \frac{1}{2}(\partial_\mu(\varphi+\sigma))^2 - \frac{1}{2}\lambda^2(\varphi+\sigma)^2((\varphi+\sigma)^2 - \frac{m}{\lambda})^2, \tag{5.7}$$

or

$$\mathcal{L}(\varphi+\sigma) = \frac{1}{2}(\partial\mu\,\varphi)^2 - \frac{1}{2}(m^2 - 12\lambda m\sigma^2 + 15\lambda^2\sigma^4)\varphi^2 + \ldots . \tag{5.8}$$

Taking (5.6) into account, it can be seen that the terms which contain odd powers of φ vanish, and the mass of the particles is given by

$$m_\varphi^2 = 4m^2 . \tag{5.9}$$

The quadratic term in (5.8) defines a new propagator for the fields while higher powers comprise an interaction

Lagrangian $\mathcal{L}_I$. The inverse propagator is

$$i\Delta^{-1}(\sigma, k) = k^2 - M^2, \tag{5.10}$$

where

$$M^2 = m^2 - 12\lambda m\sigma^2 + 15\lambda^2\sigma^4.$$

The effective potential [155,161] can be expressed in the form

$$V(\sigma) = V_o(\sigma) + V_1(\sigma) + V_2(\sigma) + \ldots \tag{5.11}$$

Here the first term represents the classical (tree) approximation, the second term gives the contribution of all graphs with one closed loop; the third term gives the contribution of all graphs with two closed loops, and so on. The graphs corresponding to these contributions are shown in fig.5.2. In the present calculation, we consider only the first two terms in (5.11). The zero-loop (tree approximation) part of the effective potential (the dot in fig.5.2) is

$$V_o(\sigma) = \frac{1}{2}\lambda^2\sigma^2(\sigma^2 - \frac{m}{\lambda})^2 \tag{5.12}$$

The one-loop contribution (fig.5.2b) to the effective potential [161] is

$$V_1(\sigma) = -\frac{i}{2}\int \frac{d^2k}{(2\pi)^2}\ln(k^2 - M^2) \tag{5.13}$$

On rotating this integral into Euclidean space, we find

$$V_1(\sigma) = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2} \ln(k^2 - M^2). \qquad (5.14)$$

This integral is u.v. divergent, and the divergence can be removed by introducing a cut off at $k^2 = \Lambda^2$ so that

$$V_1(\sigma) = \frac{M^2}{8\pi} \ln\left(\frac{\Lambda^2}{M^2}\right). \qquad (5.15)$$

Hence the effective potential upto one-loop level can be written,

$$V(\sigma) = \frac{1}{2}\lambda^2\sigma^2\left(\sigma^2 - \frac{m}{\lambda}\right)^2 + \frac{M^2}{8\pi}\ln\left(\frac{\Lambda^2}{M^2}\right). \qquad (5.16)$$

Since the parameters in the original Lagrangian (5.2) are not renormalised, (5.16) shows divergence when $\Lambda^2 \longrightarrow 0$. The divergence in (5.16) in the lowest order perturbation theory, is caused by the diagrams shown in fig.5.3, while there are no infinities associated with the σ^6 term; for instance, the graph  is finite. Hence the effective potential with the necessary counter terms is

$$V(\sigma) = \frac{\lambda^2\sigma^2}{2}\left(\sigma^2 - \frac{m}{\lambda}\right) + \frac{M^2}{8\pi}\ln\left(\frac{\Lambda^2}{M^2}\right) + C_1 + C_2\sigma^2 + C_3\sigma^4. \qquad (5.17)$$

The constants C_1, C_2 and C_3 are determined by using the following normalisation conditions on $V(\sigma)$:

$$V(\sigma)\Big|_{\sigma=(m/\lambda)^{1/2}} = 0,$$

$$\frac{d^2V}{d\sigma^2}\Big|_{\sigma=(m/\lambda)^{1/2}} = 4m^2, \tag{5.18}$$

$$\frac{d^4V}{d\sigma^4}\Big|_{\sigma=(m/\lambda)^{1/2}} = 156\,\lambda m.$$

Imposing these conditions on (5.17), we find

$$C_1 = -\frac{m^2}{8\pi}\ln\left(\frac{\Lambda^2}{4m^2}\right) - \frac{240}{8\pi}m^2$$

$$C_2 = \frac{12\lambda m}{8\pi}\ln\left(\frac{\Lambda^2}{4m^2}\right) - \frac{777}{16\pi}\lambda m \tag{5.19}$$

$$C_3 = -\frac{15}{8\pi}\lambda^2\ln\left(\frac{\Lambda^2}{4m^2}\right) + \frac{1257}{16\pi}\lambda^2$$

Thus the renormalised effective potential at zero temperature in the one-loop approximation becomes

$$V(\sigma) = \frac{1}{2}\lambda^2\sigma^2\left(\sigma^2 - \frac{m}{\lambda}\right)^2 + \frac{M^2}{8\pi}\ln\left(\frac{4m^2}{M^2}\right). \tag{5.20}$$

It is evident from this expression that $V(\sigma)$ has no cut-off dependence. Since this procedure may be extended straightforwardly to higher loops, the model field system is seen to be renormalisable.

Effective Potential at Finite Temperature

In this section we shall evaluate the effective potential at finite temperature and show that the symmetry breaking present in the model can be removed by raising the temperature above a certain value called the critical temperature. The temperature-dependent effective potential is represented by $V^T(\sigma)$. In this calculation, the Boltzmann constant k is set equal to unity. Following the arguments given in the preceding chapter, we find that SSB in this model can be removed by raising the temperature above a critical value, defined by (see (4.31))

$$\left.\frac{\partial \bar{V}^{T_c}}{\partial \sigma^2}\right|_{\sigma=0} = -\frac{m^2}{2} . \qquad (5.21)$$

Using (4.19), we can calculate the effective potential at finite temperature, and in this case the calculations are done upto two-loop level.

The zero-loop contribution to the effective potential is given by

$$V_o(\sigma) = \frac{1}{2}\lambda^2\sigma^2(\sigma^2 - \frac{m}{\lambda})^2 \qquad (5.22)$$

The one-loop contribution (fig.5.2b) to the effective potential can be found [112]:

$$V_1^T(\sigma) = -\frac{i}{2}\frac{T}{-i}\sum_n \int \frac{dk}{2\pi} \ln(-4\pi^2n^2T^2 - k^2 - M^2)$$

$$= \frac{T}{2}\sum_n \int \frac{dk}{2\pi} \ln(-4\pi^2n^2T^2 - E_M^2), \qquad (5.23)$$

where $E_M^2 = k^2 + M^2$. This sum integration has been performed in Chapter Four (see 4.21). Hence

$$V_1^T(\sigma) = \int \frac{dk}{2\pi}[\frac{E_M}{2} + T \ln(1 - e^{-E_M/T}]$$

$$= V_1^o(\sigma) + \bar{V}_1^T(\sigma). \qquad (5.24)$$

Here

$$V_1^o(\sigma) = \int \frac{dk}{2\pi} \frac{E_M}{2}, \qquad (5.25)$$

gives the usual zero-temperature one-loop approximation to the effective potential, which may be compared with (5.14). $\bar{V}_1^T(\sigma)$ gives the temperature-dependent part of the effective potential,

$$\bar{V}_1^T(\sigma) = \frac{T}{\pi}\int_0^\infty dk\ \ln(1 - e^{-E_M/T}) \qquad (5.26)$$

By the equation (4.26)

$$\bar{V}_1^T(\sigma) = -\frac{\pi T^2}{6} + \frac{MT}{4}\ . \qquad (5.27)$$

Invoking (5.21), we find that the critical temperature correct to one-loop order;

$$T_c = \frac{1}{3}\frac{m^2}{\lambda}. \qquad (5.28)$$

We shall now proceed to evaluate the two-loop corrections to $V^T(\sigma)$. Our motivation for doing this is to investigate the effect of higher order loops in determining the critical temperature in a more precise manner. The two-loop contributions come from two graphs (fig.5.2c and fig.5.2d). The contribution from fig.5.2c with proper combinatorial factor can be put as [113]

$$V_{2a}^T(\sigma) = \frac{(-24\lambda m)}{8}\left(\frac{1}{-i\beta}\sum_n\int\frac{dk}{2\pi}\ \frac{i}{k^2 - M^2}\right)^2. \qquad (5.29)$$

Since we are interested only in the temperature-dependent terms, we find

$$\bar{V}_{2a}^T(\sigma) = \frac{(-24\lambda m)}{2}\left[\frac{\partial}{\partial M^2}\ (\bar{V}_1^T(\sigma))\right]^2 \qquad (5.30)$$

The mass dependent term in $V_1^T(\sigma)$, according to (5.27), is written separately:

$$\bar{V}_1^T(\sigma,M) = \frac{MT}{4}$$

Therefore

$$\bar{V}_{2a}^T(\sigma) = -\frac{3}{16}\frac{\lambda m T^2}{M^2} . \tag{5.31}$$

The contribution from fig.5.2d can be put as

$$V_{2b}^T(\sigma) = \frac{(24\lambda m)^2}{12}\sigma^2(-i)\iiint \frac{\delta^2(k_1+k_2+k_3)}{(k_1^2+M^2)(k_2^2+M^2)(k_3^2+M^2)}$$

(5.32)

Invoking the rules given in (1.46), we find

$$V_{2b}^T(\sigma) = -48\lambda^2 m^2\sigma^2 T^2 \sum_{n_1}\sum_{n_2}\sum_{n_3}\iiint\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}$$

$$\frac{\delta(k_1+k_2+k_3)\ \delta_{n_1+n_2+n_3}}{(4\pi^2n_1^2T^2+E_{M_1}^2)(4\pi^2n_2^2T^2+E_{M_2}^2)(4\pi^2n_3^2T^2+E_{M_3}^2)}$$

(5.33)

The summation is done first. Using the Kronecker delta, the n_3 summation can be eliminated. By the method of partial fractions, the summation can be carried out. Thus we find, in the high temperature limit (T >> m),

$$\bar{V}^T_{2b}(\sigma) = - \frac{96\lambda^2 m^2 \sigma^2 T^3}{\pi^2} \iint \frac{dk_1 dk_2}{(k_1^2+M^2)(k_2^2+M^2)((k_1+k_2)^2+M^2)} \tag{5.34}$$

This integral may be evaluated as follows. Let

$$I = \iint \frac{dk_1 dk_2}{(k_1^2+M^2)(k_2^2+M^2)((k_1+k_2)^2+M^2)}$$

We define

$$f(x) = \int \frac{dk\ e^{ikx}}{k^2+M^2}$$

as the one-dimensional Fourier transform. This integral, when evaluated, gives

$$f(x) = \frac{\pi}{M} e^{-M|x|} \tag{5.35}$$

Thus we find

$$I = \frac{1}{2\pi} \int dx\ f^3(x)$$

$$= \frac{1}{\pi} \int_0^\infty dx \left(\frac{\pi}{M} e^{-Mx} \right)^3 .$$

Thus

$$\bar{V}^{T}_{2b}(\sigma) = - \frac{32\lambda^2 m^2 \sigma^2}{M^4} T^3 . \tag{5.37}$$

Hence the temperature-dependent part of the effective potential to the two-loop level is

$$\bar{V}^{T}(\sigma) = - \frac{\pi}{6} T^2 + \frac{M}{4} T - \frac{3}{16} \frac{\lambda m T^2}{M^2} - \frac{32\lambda^2 m^2 \sigma^2 T^3}{M^4} . \tag{5.38}$$

The critical temperature is evaluated using the relation (5.21). Thus,

$$- \frac{3}{2} \lambda T_c - \frac{9}{4} \frac{\lambda^2}{m^2} T_c^2 - \frac{32\lambda^2}{m^2} T_c^3 = - \frac{m^2}{2} \tag{5.39}$$

This is a cubic equation in T_c. Introducing y as $T_c = y - \frac{3}{128}$, the eq.(5.39) can be cast into the standard form

$$y^3 + py + q = 0$$

where

$$p = \frac{3}{64} \frac{m^2}{\lambda} - 1.156 \times 10^{-4} \tag{5.40}$$

$$q = - \frac{1}{64} \frac{m^4}{\lambda^2} - 1.098 \times 10^{-3} \frac{m^2}{\lambda} + 2.574 \times 10^{-5} \tag{5.41}$$

From these relations it is seen that

$$\frac{q^2}{4} + \frac{p^3}{27} > 0,$$

which is the condition to be satisfied by (5.39) that it has one real root, so that the critical temperature is determined uniquely. The critical temperature correct to two-loops is then

$$T_c = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \frac{3}{128} \qquad (5.42)$$

Nature of Phase Transition

The above calculations reveal that the SSB present in the φ^6 model can be removed by raising the temperature above a critical value. In the language of superconductivity, we may restate this in terms of a phase transition from the ordered phase characterised by $\langle\varphi\rangle \neq 0$ to a disordered phase characterised by $\langle\varphi\rangle = 0$, as the temperature of the system is increased. We may study the nature of the phase transition in the following way [131]. The vev $\langle 0|\varphi|0\rangle = \sigma$ is replaced by its thermal average $\langle\varphi\rangle_T = \sigma_T$, taken with respect to a Gibbs ensemble, so that the order parameter of the theory becomes explicitly temperature-dependent.

Finite temperature Greens functions are defined by the relation

$$\langle - - - - \rangle = \frac{Tr(e^{-H/T} \ldots)}{Tr(e^{-H/T})} ,$$

where H is the Hamiltonian governing the system. The parameter characterising the thermodynamic equilibrium state of the φ particles of the system is given by the density of the particles in momentum space:

$$n_k = \frac{1}{(e^{\omega_k/T} - 1)}$$

where $n_k = \langle a_k^+ a_k \rangle$, $\omega_k = (k^2+m^2)^{1/2}$. a_k and a_k^+ are the usual annihilation and creation operators.

The equation of motion corresponding to the Lagrangian (5.2) is

$$\Box\varphi - m^2\varphi + 4\lambda m\varphi^3 - 3\lambda^2\varphi^5 = 0 \qquad (5.43)$$

On shifting the field from φ to $\varphi+\sigma$, and taking the Gibbs average of the corresponding equation, we find

$$\langle \Box(\varphi+\sigma)\rangle - m^2\langle\varphi+\sigma\rangle + 4\lambda m \langle(\varphi+\sigma)^3\rangle - 3\lambda^2\langle (\varphi+\sigma)^5\rangle = 0 \qquad (5.44)$$

Using the standard finite temperature Greens function method [205] we find, in the high temperature limit,

$$\langle\varphi^2\rangle = \frac{1}{2\pi}\int^{\infty} dk \frac{1}{(k^2+m^2)^{1/2}\left(e^{\frac{(k^2+m^2)^{1/2}}{T}} - 1\right)}$$

$$= \frac{T}{m} \tag{5.45}$$

By a similar calculation,

$$\langle\varphi^4\rangle = \frac{3}{4}\frac{T^2}{m^2}, \tag{5.46}$$

$$\langle\varphi\rangle = \langle\varphi^3\rangle = \langle\varphi^5\rangle = 0. \tag{5.47}$$

Then (5.44) becomes

$$\Box\sigma_T - m^2\sigma_T + 4\lambda m\sigma_T^3 + 6\lambda\sigma_T T - 3\lambda^2\sigma_T^5 -$$

$$15\lambda^2\sigma_T^3\frac{T}{m} - \frac{45}{4}\lambda^2\frac{T^2}{m^2}\sigma_T = 0 \tag{5.48}$$

Assuming σ_T to be constant, we obtain

$$\sigma_T(-m^2 + 4\lambda m\sigma_T^2 + 6\lambda T - 3\lambda^2\sigma_T^4 -$$

$$15\lambda^2\frac{T}{m}\sigma_T^2 - \frac{45}{4}\lambda^2\frac{T^2}{m^2} = 0 \tag{5.49}$$

This equation has three solutions:

$$\sigma_T = 0 \qquad (5.50)$$

$$\sigma_T^2 = \frac{(4\lambda m - 15\lambda^2 \frac{T}{m}) \pm \sqrt{(4\lambda m - 15\lambda^2 \frac{T}{m})^2 - 12\lambda^2 (m^2 - 6\lambda T + \frac{45}{4}\lambda^2 \frac{T^2}{m^2})}}{6\lambda^2} \qquad (5.51)$$

Each solution of these equations defines a possible phase of the field system with its characteristic excitations. On heating the field system from absolute zero, the two branches of σ_T^2 given by (5.51) can coincide at a temperature $T_1 = 0.10 \frac{m^2}{\lambda}$ yielding a common value for σ_T, viz., $\sigma_{T_1}^2 = \frac{5}{12} \frac{m}{\lambda}$. Nevertheless, this is not a phase transition; as temperature increases further, the two branches of σ_T^2 will again separate. The existence of the separate branches of σ_T^2 implies that the phase transition at the critical temperature T_c is one of first order.

The mass of the excitations may be found by making the shift $\sigma \longrightarrow \sigma + \delta\sigma$ in (5.48). Retaining only terms linear in $\delta\sigma$:

$$\Box\delta\sigma - [m^2 + 15\lambda^2 \sigma_T^4 + 45\lambda^2 \sigma_T^2 \frac{T}{m} + \frac{45}{4}\lambda^2 \frac{T^2}{m^2} - 6\lambda T - 12\lambda m \sigma_T^2]\,\delta\sigma = 0, \qquad (5.52)$$

from which the excitation mass is found:

$$M_\varphi^2 = m -6\lambda T + \frac{45}{4}\lambda^2\frac{T^2}{m^2} - 12\lambda m\sigma_T^2 + 45\lambda^2\frac{T}{m}\sigma_T^2 + 15\lambda^2\sigma_T^4. \qquad (5.53)$$

The disordered phase is associated with excitations of mass

$$M_\varphi^2 = m^2 -6\lambda T + \frac{45}{4}\frac{\lambda^2 T^2}{m^2} \qquad (5.54)$$

The existence of distinct solutions for σ_T, as given by (5.50) and (5.51), may be indicative of a domain structure of the vacuum. In the case of the Higgs model such a domain structure has been speculated upon [131], wherein adjacent domains are associated with opposite signs of σ_T. The domains are separated by kinks, but this is not a stable configuration because they define degenerate minima of the effective potential. The situation is different in the φ^6 model. There is a five-fold multiplicity of values of σ_T which can be associated with different domains in the vacuum. Eventhough domains carrying condensate values, σ_T, which differ only in sign may join together due to collapse of kink walls, there still may be some domains with different absolute values of σ_T. These latter configurations may be assumed to be stable.

It is worth mentioning in this context that the existence of a domain wall structure has been very well established experimentally in the case of ferroelectrics which are described by a φ^6 coupled phenomenological model defined in terms of polarisation as the order parameter [201-203].

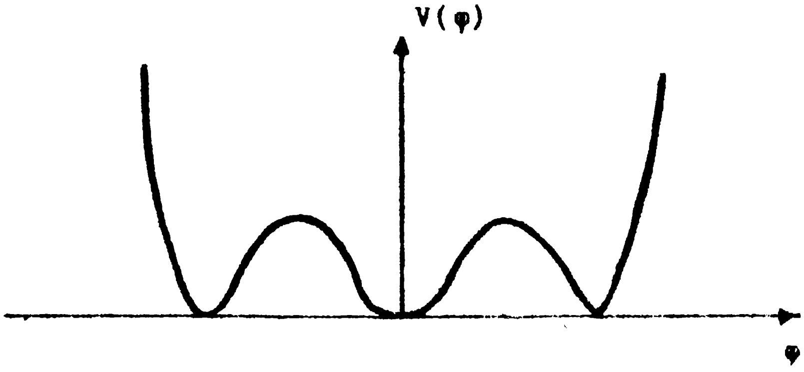


Fig. 5.1 The φ^6 potential

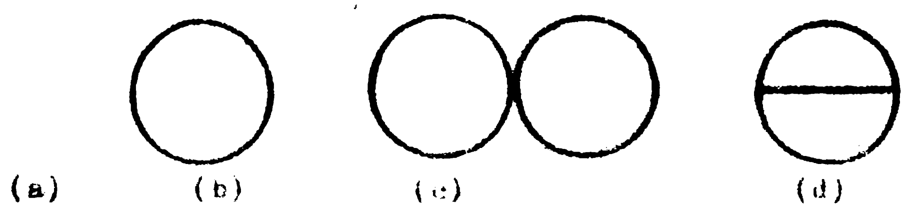


Fig. 5.2

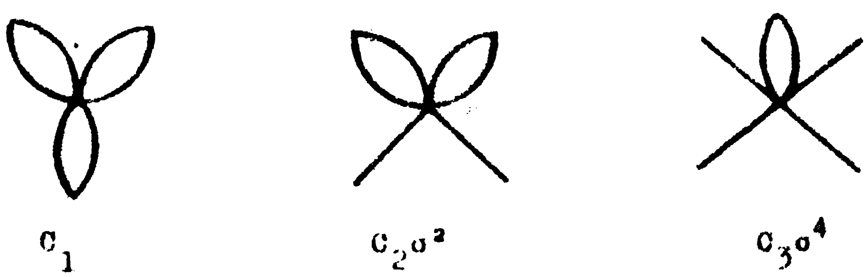


Fig. 5.3

Six

TEMPERATURE DEPENDENCE OF COUPLING CONSTANTS

Introduction

Finite temperature behaviour of quantum field systems has become a very relevant field of investigation with the advent of Grand Unified Theories (GUTS) [69,72,131,132]. Since the grand unification scale is cosmologically high [$T > 10^{15}$ GeV or $t < 10^{-35}$s], it should not come as a surprise that grand unified interactions may have played an important role in the 'earliest' stages of the evolution of the universe. In fact the unification of strong, weak and electromagnetic interactions into a unified field theory, can only occur at very high energies which cannot be reached, at present, in laboratories [66,67].

In the early universe, characterised by very high temperatures, the gauge symmetry of electroweak and grand unified theories were unbroken. As the universe cooled, a series of first order phase transitions may have taken place, which brought these symmetries into their present spontaneously broken form. The most popular mechanism [206-208] for the breaking of these symmetries is based on the Coleman-Weinberg model [158]. The phase transition in

Coleman-Weinberg type models is first order, but the rate of transition is very low [208-210]. For electroweak theory, Witten [210] has shown that the phase transition is driven by unexpected sources. He used a clever method by which he made the φ^4 coupling term negative and temperature-dependent, which is the driving force for the phase transition. The same mechanism is used for GUTS too [206-8,211]. Recent studies of the SU(5) model have shown that the first order phase transition occurs at a temperature of approximately 1 GeV, indicating extreme super cooling. In their calculations the temperature dependence of the coupling constant has not been taken into account. If the temperature dependence is taken into consideration, the amount of supercooling may be drastically reduced and the transition temperature may be raised to 2×10^{10} GeV [211].

Renormalisation group studies reveal that the coupling constants vary with q^2 (the momentum transfer square), at which they are probed. The strong interaction coupling constant decreases as q^2 increases, while U(1) coupling constant increases slightly with q^2 (fig.6.1). Collins and Perry [212] showed, using renormalisation group arguments, that the non-abelian gauge coupling constant of $SU(3)_c$ field theory of quarks and gluons decreases with density of the quark soup, and that perturbation theory becomes more reliable under such conditions. In the present study we make an attempt to investigate the temperature behaviour of coupling constants. We have chosen two models i) φ^4 theory ii) scalar electrodynamics (SED). In the case of φ^4 theory we find that when the mass square is positive, the coupling constant decreases with temperature, while when m^2 is negative the coupling constant increases with temperature. The calculations are verified using the renormalisation group approach. In the case of SED, the gauge coupling constant is found to increase with temperature.

φ^4 Theory

We consider a massive scalar particle theory defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4 \quad , \quad \lambda > 0 \qquad (6.1)$$

The parameters appearing in the Lagrangian are bare or unrenormalised. The renormalisation of these quantities at zero temperature has been discussed elsewhere [213]. In the present study we are interested only on the renormalisation of the coupling constant λ . The renormalised coupling constant can be obtained by the vertex renormalisation procedure. The theory contains one kind of vertex only (fig.6.2) which is of order λ. This single vertex can be represented by

$$\Gamma^1 = -i\lambda(2\pi)^4\ \delta^4(k_1+k_2-k_3-k_4). \tag{6.2}$$

The vertex correction, in the lowest order, comes from three graphs (fig.6.3), and the correction is of order λ^2. The contributions from these graphs are identical, and hence it is sufficient to consider any one of them. Using the Feynman rules (Appendix-B), the contribution from the first graph is

$$\Gamma^2 = \frac{(-i\lambda)^2}{2}\int \frac{d^4k_5}{(2\pi)^4}\ \frac{d^4k_6}{(2\pi)^4}\ \frac{i}{(k_5^2-m^2)}\ \frac{i}{(k_6^2-m^2)}\ (2\pi)^8.$$

$$\delta^4(k_1+k_2-k_5-k_6).\ \delta^4(k_5+k_6-k_3-k_4). \tag{6.3}$$

The aim of the present study is to obtain an expression for the temperature-dependent coupling constant. This is achieved by translating the zero temperature formalism to finite temperature formalism by using the prescriptions stated in chapter 1. Thus we can find that Γ^1 is temperature independent while Γ^2 is temperature dependent.

Rewriting Γ^2 in the form

$$\Gamma_\beta^2 = \frac{(-i\lambda)^2}{2}\frac{1}{(-i\beta)} \sum_{n_1}\sum_{n_2} \int \frac{d^3k_5}{(2\pi)^3}\frac{d^3k_6}{(2\pi)^3} \frac{i}{(\omega_{n_1}^2-k_5^2-m^2)} \frac{i}{(\omega_{n_2}^2-k_6^2-m^2)}$$

$$(2\pi)^6(-i\beta)^2\,\delta^3(k_1+k_2-k_5-k_6)\;\delta(\omega_{k_1}+\omega_{k_2}-\omega_{k_5}-\omega_{k_6})\,.$$

$$\delta^3(k_5+k_6-k_3-k_4)\;\delta(\omega_{k_5}+\omega_{k_6}-\omega_{k_3}-\omega_{k_4})\,. \tag{6.4}$$

Here $\beta \sim \frac{1}{T}$ (the inverse of temperature). The calculations can be simplified by putting the external momenta to zero. Thus we find

$$\Gamma_\beta^2 = \frac{\lambda^2}{2}\frac{1}{-i\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\omega_n^2-k^2-m^2)^2}\cdot(2\pi)^3(-i\beta)$$

$$\delta^3(k_1+k_2-k_3-k_4)\;\delta(\omega_{n_1}+\omega_{n_2}-\omega_{n_3}-\omega_{n_4}) \tag{6.5}$$

Expressing Γ_β as

$$\Gamma_\beta = \Gamma^1 + \Gamma_\beta^2$$

where

$$\Gamma_\beta = -i\lambda_\beta (2\pi)^4 \, \delta^4(k_1 + k_2 - k_3 - k_4).$$

λ_β is identified to be the temperature-dependent coupling constant. Thus the temperature-dependent coupling constant λ_β in the one-loop approximation, taking into account the contributions from the remaining graphs also,

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{2} \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\omega_n^2 - k^2 - m^2)^2} \tag{6.6}$$

Setting $E_k^2 = k^2 + m^2$, and substituting for ω_n as $\omega_n = \frac{2\pi n i}{\beta}$ $(n = 0, \pm 1, \pm 2, \ldots)$, we obtain

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{2\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + E_k^2\right)^2} \tag{6.7}$$

The summation is done first in the following way

$$\sum_n \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + E_k^2\right)^2} = - \frac{1}{2E_k} \frac{d}{dE_k} \left[\sum_n \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + E_k^2\right)} \right] \tag{6.8}$$

The identity [191]

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + a^2)} = - \frac{1}{2a^2} + \frac{\pi}{2a} \coth(\pi a)$$

gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\frac{4\pi^2 n^2}{\beta^2} + E_k^2)} = \beta\left(\frac{1}{2E_k} + \frac{1}{E_k(e^{\beta E_k} - 1)} \right)$$

This enables us to write (6.8) in the following form

$$\sum_n \frac{1}{(\frac{4\pi^2 n^2}{\beta^2} + E_k^2)^2} = \beta\left[\frac{1}{4E_k^3} + \frac{1}{2E_k^3(e^{\beta E_k} - 1)} + \frac{\beta e^{\beta E_k}}{2E_k^2(e^{\beta E_k} - 1)^2}\right]$$

(6.9)

Thus (6.6) can be cast into the form

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{2} \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{4E_k^3} + \frac{1}{2E_k^3(e^{\beta E_k} - 1)} + \frac{\beta e^{\beta E_k}}{2E_k^2(e^{\beta E_k} - 1)^2}\right]$$

(6.10)

The temperature-independent term cancels with the renormalisation counter term at zero temperature. Hence the λ appearing on the r.h.s of (6.10) will be a renormalised one. Thus

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{4} \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{E_k^3(e^{\beta E_k} - 1)} + \frac{\beta e^{\beta E_k}}{E_k^2(e^{\beta E_k} - 1)^2} \right]$$

(6.11)

$$= \lambda - \frac{3\lambda^2}{4} \frac{1}{2\pi^2} \int_0^\infty k^2 dk \left[\frac{1}{(k^2+m^2)^{3/2} (e^{\beta(k^2+m^2)^{1/2}} - 1)} + \frac{\beta e^{\beta(k^2+m^2)^{1/2}}}{(k^2+m^2)(e^{\beta(k^2+m^2)^{1/2}} - 1)^2} \right] \tag{6.12}$$

$$= \lambda + \frac{3\lambda^2}{4\pi^2} \int_0^\infty k^2 dk \frac{d}{dk^2} \left[\frac{1}{(k^2+m^2)^{1/2} (e^{\beta(k^2+m^2)^{1/2}} - 1} \right]$$

Putting $x^2 = \beta^2 k^2$, $\beta m = a$, we find

$$\lambda_\beta = \lambda + \frac{3\lambda^2}{4\pi^2} \int_0^\infty x^2 dx \frac{d}{dx^2} \left[\frac{1}{(x^2+a^2)^{1/2} (e^{(x^2+a^2)^{1/2}} - 1)} \right] \tag{6.13}$$

Integrating by parts, we get

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{8\pi^2} \int_0^\infty dx \left[\frac{1}{(x^2+a^2)^{1/2} (e^{(x^2+a^2)^{1/2}} - 1)} \right]$$

This integral can be evaluated in the high temperature limit ($T \gg m$), yielding

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{8\pi^2} \left[\frac{\pi}{2m\beta} + \frac{1}{2} \ln\left(\frac{\beta m}{4\pi}\right) \right]$$

This may be rewritten as

$$\lambda_\beta = \lambda - \frac{3\lambda^2}{8\pi^2} \left[\frac{\pi T}{2m} + \frac{1}{2}\ln\left(\frac{m}{4\pi T}\right)\right] \tag{6.14}$$

$$= \lambda\left[1 - \frac{3\lambda}{8\pi^2}\left(\frac{\pi T}{2m}\right) + \frac{1}{2}\ln\left(\frac{m}{4\pi T}\right)\right].$$

This shows that in the absence of SSB (ie $m^2 > 0$ in (6.1)), the scalar coupling constant is seen to decrease with temperature. If we neglect the ln T term - which is justified at the high temperature - then, a critical temperature T_a may be defined corresponding to the vanishing of the coupling constant:

$$T_a = \frac{16\pi m}{3\lambda} - \frac{m}{\pi}\ln\left(\frac{m}{4\pi}\right) \tag{6.15}$$

The temperature T_a signals the emergence of a non-interacting phase for the field system. It is known that at zero temperature the model can possess asymptotic freedom only for a negative value of λ [214]. The present result must be contrasted with this because it holds for the physically interesting case of positive λ. If the onset of 'no-interaction' is a genuine phase transition, then it should also work in the reverse. It is not difficult to glean a few illustrations from physics where forces get weakened with rise of temperature. The rupturing of chemical bonds and disappearance of the phonon field picture in solids under thermal agitation are examples that bring out this mechanism at least in a qualitative manner. However, it is hoped that a clear and specific quantitative comparison regarding the behaviour of scalar coupling constants can be made by studying the anharmonic vibration in the context of a self-coupled phonon field at finite temperature.

We encounter an imaginary mass ($m^2 < 0$) in (6.1) for a theory with SSB. However the imaginary terms may be separated, and it is hoped that they will disappear when higher order effects are taken into account [112], the resulting expression for λ_β now reads

$$\lambda_\beta = \lambda + \frac{3\lambda^2}{16\pi^2} \ln(4\pi T). \qquad (6.16)$$

It is seen that with SSB, λ increases uniformly with temperature. This provides a justification for Witten's wellknown recipe [210]. The present result may be applied to the Ginzburg-Landau model for superconductivity which predicts a variation of the penetration depth δ with the quadratic coupling constant λ : $\delta \sim \sqrt{\lambda}$ [215]. At high temperatures, since λ increases, the penetration depth δ increases and as a result, superconductivity is inevitably lost.

Renormalisation Group Approach

The above calculations can be checked using the renormalisation group [216]. We shall use a temperature-dependent, mass-independent renormalisation programme [217]. The RG equation for the present problem can be written in the form

$$[T \frac{\partial}{\partial T} + \beta \frac{\partial}{\partial \lambda} + (1+\gamma_m) m \frac{\partial}{\partial m} + \gamma \int d^4x \; \varphi_c(x) \frac{\delta}{\delta \varphi_c(x)}] \Gamma = 0 \tag{6.17}$$

where β, γ and γ_m are the RG functions. Following Coleman and Weinberg [155], the above equation can be recast into the following forms:

$$(T \frac{\partial}{\partial T} + \beta \frac{\partial}{\partial \lambda} + (1+\gamma_m) m \frac{\partial}{\partial m} + \gamma \varphi_c \frac{\partial}{\partial \varphi_c}) V^T(\varphi_c) = 0, \tag{6.18}$$

$$(T \frac{\partial}{\partial T} + \beta \frac{\partial}{\partial \lambda} + (1+\gamma_m) m \frac{\partial}{\partial m} + \gamma \varphi_c \frac{\partial}{\partial \varphi_c} + 2\gamma) Z = 0, \tag{6.19}$$

Where $V^T(\varphi_c)$ is the effective potential at finite temperature and Z is the scale of the field. The temperature-dependent effective potential at the one-loop level evaluated in the high temperature limit is [112]

$$V^T(\varphi_c) = \frac{1}{2} m^2 \varphi_c^2 + \frac{\lambda}{4!} \varphi_c^4 + \frac{M^4}{64\pi^2} \ln\left(\frac{T^2}{4\pi^2 m^2} \right) + \frac{M^2 T^2}{24} - \frac{M^3 T}{12\pi} \tag{6.20}$$

where

$$M^2 = m^2 + \frac{1}{2}\lambda\,\varphi_c^2$$

It is convenient to use

$$V^{(4)} = \frac{\partial^4 V}{\partial \varphi_c^4}$$

in place of V^T in (6.18). Introducing a scale parameter t,

$$t = \ln(T/m) \,, \tag{6.21}$$

equations (6.15) and (6.16) can be rewritten as

$$\left(-\frac{\partial}{\partial t} + \bar{\beta}\,\frac{\partial}{\partial \lambda} + \bar{\gamma}\,\varphi_c\,\frac{\partial}{\partial \varphi_c} + 4\bar{\gamma}\right)V^{(4)} = 0 \,, \tag{6.22}$$

$$\left(-\frac{\partial}{\partial t} + \bar{\beta}\,\frac{\partial}{\partial \lambda} + \bar{\gamma}\,\varphi_c\,\frac{\partial}{\partial \varphi_c} + 2\bar{\gamma}\right)Z = 0 \,, \tag{6.23}$$

where

$$\bar{\beta} = \beta/\gamma_m$$

$$\bar{\gamma} = \gamma/\gamma_m .$$

Imposing the renormalisation conditions [155], we can find the zero-loop values for $V^{(4)}$ and Z:

$$V^{(4)}(0,\lambda) = \lambda \tag{6.24}$$

$$Z(0,\lambda) = 1 \tag{6.25}$$

Combining these equations with (6.22) and (6.23) we obtain

$$\bar{\gamma} = \frac{1}{2} \frac{\partial}{\partial t} Z(0,\lambda) \tag{6.26}$$

$$\bar{\beta} = \frac{\partial}{\partial t} V^4(0,\lambda) - 4\bar{\gamma}\lambda \tag{6.27}$$

Thus, we can find $\bar{\beta}$ and $\bar{\gamma}$ exactly, if we know the derivatives that occur on the right side of the above equations. The required result can be obtained if we go for the one-loop calculations. We will denote the temperature dependent coupling constant by λ_β such that the RG function $\bar{\beta}$ satisfies the ordinary differential equation [218]

$$\frac{d\lambda_\beta}{dt} = \bar{\beta}(\lambda_\beta) \tag{6.28}$$

with the boundary condition $\lambda_\beta(t=0) = \lambda$. In the one-loop approximation the value of $V^{(4)}$ can be obtained from (6.20):

$$V^{(4)}(t,\lambda) = \lambda - \frac{3\lambda^2}{8\pi^2} (\frac{\pi}{2} \frac{T}{m} - \frac{1}{2} \ln(T/m)] \tag{6.29}$$

The one-loop correction to Z vanishes; hence

$$Z = 1. \tag{6.30}$$

These results yield

$$\bar{\gamma} = 0 , \tag{6.31}$$

and

$$\bar{\beta} = - \frac{3\lambda^2}{8\pi^2} \left(\frac{\pi}{2} e^t - \frac{1}{2} t\right). \tag{6.32}$$

The differential equation (6.28) now reads

$$\frac{d\lambda_\beta}{dt} = - \frac{3\lambda_\beta^2}{8\pi^2} \left(\frac{\pi}{2} e^t - \frac{1}{2} t\right) \tag{6.33}$$

On integrating this equation we find

$$\lambda_\beta = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2} \ln(T/m) + \frac{3\lambda}{16\pi}(T/m)} \tag{6.34}$$

Thus the RG calculation gives an improved expression for λ_β - the temperature-dependent scalar coupling constant. But it is remarkable to note that both perturbation and RG calculations predict the same type of thermal behaviour for the coupling constant.

Scalar Electrodynamics

We shall now extend the above calculations to the case of gauge coupling constant in SED described by the Lagrangian

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\varphi^*(\partial_\mu - ieA_\mu)\varphi - m^2(\varphi^*\varphi) -$$

$$\lambda(\varphi^*\varphi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (6.35)$$

In this case two vertices are to be considered with respect to the gauge coupling constant e. The vertices are as shown in figs.6.4 and 6.5 which are of orders e and e^2 respectively. But it can be seen that the lowest order vertex correction at finite temperature to the vertex fig.6.4 vanishes. The vertex represented in fig.6.5 is expressed as

$$\Gamma^1 = 2ie^2 g_{\mu\nu}(2\pi)^4 \delta^4(k_1 + p_1 - p_2 - k_2). \qquad (6.36)$$

The lowest order correction to this vertex is given by fig.6.6, and using the Feynman rules (Appendix-B) the contribution from this graph is

$$\Gamma^2 = 2e^4 g_{\mu\nu}\int \frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{(k_1+k_3)(k_2+k_4)}{(k_3^2-m^2)\,q^2(k_4^2-m^2)}$$

$$(2\pi)^8\,\delta^4(k_2-k_4-q)\quad\delta^4(k_3-p_1-p_2-k_4). \qquad (6.37)$$

As before, Γ^2 is temperature-dependent and hence

$$\Gamma^2_\beta = 2e^4 \, g_{\mu\nu} \frac{1}{-i\beta} \sum_n \int \frac{d^4k}{(2\pi)^3} \cdot \frac{1}{(\omega_n^2 - k^2 - m^2)^2} \cdot$$

$$(2\pi)^3(-i\beta) \quad \delta^3(k_1+p_1-p_2-k_2) \quad \delta(\omega_{n_1}+\omega_{n_2}-\omega_{n_3}-\omega_{n_4}). \tag{6.38}$$

Expressing Γ_β as

$$\Gamma_\beta = \Gamma^1 + \Gamma^2_\beta ,$$

where

$$\Gamma_\beta = 2ie^2_\beta \, g_{\mu\nu}(2\pi)^4\delta^4(k_1+p_1-p_2-k_2). \tag{6.39}$$

Here e_β is defined as the temperature-dependent gauge coupling constant. Thus we obtain

$$e^2_\beta = e^2[1 + \frac{e^2}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \quad \frac{1}{(\omega_n^2-k^2-m^2)^2}] \tag{6.40}$$

It is now straightforward to see that (see (6.7)), in the high temperature limit, the temperature-dependent gauge coupling constant

$$e^2_\beta = e2[1 + \frac{e^2}{4\pi^2} \quad \frac{\pi T}{2m} - \frac{1}{2} \ln (\frac{4\pi T}{m}))] \tag{6.41}$$

Thus we can see that the gauge coupling constant in SED increases with temperature.

We have studied two models wherein the scalar and gauge coupling constants are temperature-dependent, SSB is the critical factor that determined the nature of the temperature variation. In the early universe when there was thermal equilibrium, the temperature of relativistic particles must have varied with time approximately as $T \sim (t)^{-1/2}$ [72]. This implies that the coupling constants could vary with time under conditions of thermal equilibrium, thus realising Dirac's [219,220] hypothesis of time variation of constants of nature.

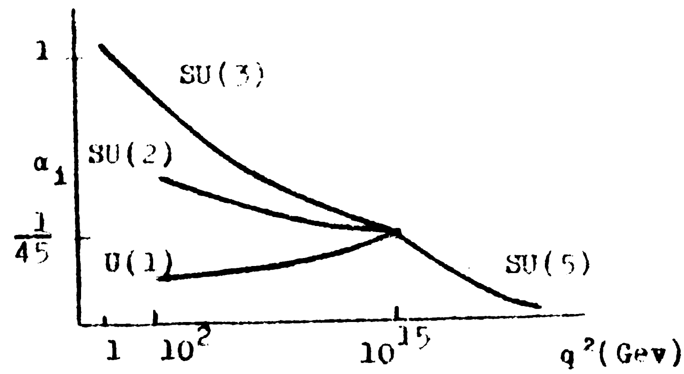


Fig. 6.1 The qualitative behaviour of coupling constants with energy

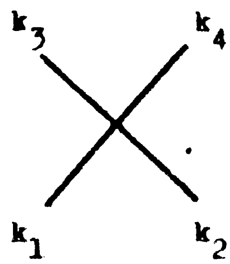


Fig. 6.2 The bare vertex in φ^4 theory

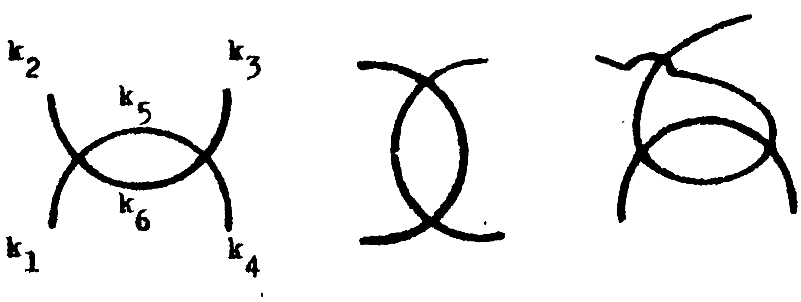
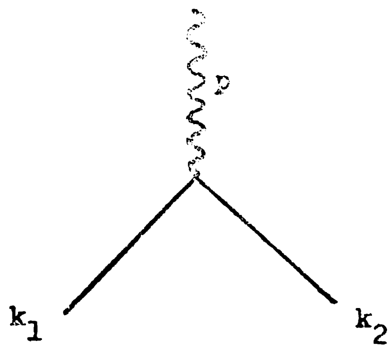
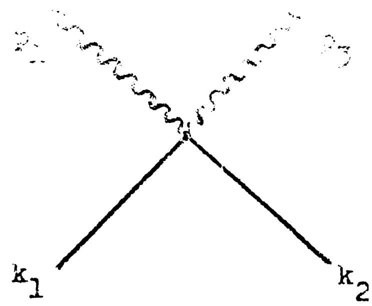


Fig. 6.3 Vertex correction in φ^4 theory at the one loop level





Bare vertices in SED

Fig. 6.4

Fig. 6.5

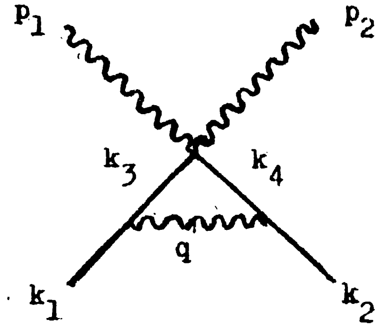


Fig. 6.6 Vertex correction of the order of e^2 in SED

Seven

GAUGE COUPLINGS IN HOT ENVIRONMENTS

In the preceding chapter we have discussed the significance of studying the temperature dependence of coupling constants. There we considered the behaviour of the coupling constants in φ^4 theory and scalar electrodynamics (SED) as a function of temperature. This investigation is extended in the present chapter to include the behaviour of gauge coupling constants in hot environments. We discuss the case of abelian and non-abelian gauge theories separately.

Abelian Gauge Theory

Here we will restrict ourselves to the case of quantum electrodynamics (QED), described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_{\mu}A^{\mu})^2 + \bar{\psi}(i\gamma_{\mu}\partial^{\mu} - m - e\gamma_{\mu}A^{\mu})\psi \,. \tag{7.1}$$

The expression for the temperature-dependent coupling constant in this model, is obtained through the vertex renormalisation technique. The calculation is limited to the lowest order. The bare vertex (fig.7.1) of this model is represented by

$$\Gamma^1 = -ie\,\gamma^{\mu}(2\pi)^4\delta^4(p_1-p_2-k_1). \qquad (7.2)$$

The renormalisation of this vertex to the lowest order is achieved through the Feynman diagrams sketched in fig.7.2. Let the contribution from this diagram be denoted by Γ^2. Using Feynman rules for QED (Appendix-B), Γ^2 can be evaluated in the Feynman gauge

$$\Gamma^2 = (-ie)^3\int\frac{d^4p_4}{(2\pi)^4}\,\frac{d^4p_3}{(2\pi)^4}\,\frac{d^4k_2}{(2\pi)^4}\cdot\gamma^{\mu}\,\frac{i}{\not{p}_4-m}\,\gamma^{\lambda}\,\cdot$$

$$\frac{-ig_{\lambda\nu}}{k_2^2}\,\gamma^{\nu}\,\frac{i}{\not{p}_3-m}\,\cdot$$

$$(2\pi)^{12}\,\delta^4(p_4-p_3-k_1)\,\delta^4(p_1-p_4-k_2)\,\delta^4(p_3+k_2-p_2). \qquad (7.3)$$

With the help of Dirac delta functions, two of the integrations can be performed; thus

$$\Gamma^2 = -e^3\Big[\int\frac{d^4k}{(2\pi)^4}\,\gamma^{\mu}\,\frac{(-k+m)}{(k^2-m^2)}\,\gamma^{\nu}\,\frac{1}{k^2}\,\gamma^{\mu}\frac{(-k+m)}{(k^2-m^2)}$$

$$(2\pi)^4\,\delta^4(p_1-p_2-k_1), \qquad (7.4)$$

where we have set the external momenta zero. The above equation can be further simplified as

$$\Gamma^{(2)} = -4e^3\gamma^\mu(2\pi)^4\delta^4(p_1-p_2-k_1)\int\frac{d^4k}{(2\pi)^4}\left[\frac{k^2}{k^2(k^2-m^2)^2} - \frac{2\not{k}m}{k^2(k^2-m^2)^2} + \frac{m^2}{k^2(k^2-m^2)^2}\right] \qquad (7.5)$$

The second term in the integral is linear in k, and hence can be dropped. The bare vertex Γ^1 is temperature-independent while Γ^2 is temperature dependent, and the latter will henceforth be denoted by Γ^2_β. It can be seen that the third term in (7.5) vanishes in the high temperature limit (T >> m). Using the rules given in (1.46), Γ^2_β can be expressed as

$$\Gamma^2_\beta = -4e^3\gamma^\mu(2\pi)^4\delta^4(p_1-p_2-k_1)\frac{1}{-i\beta}\Sigma\int\frac{d^3k}{(2\pi)^3}\cdot\left[\frac{1}{\left(\frac{4\pi^2n^2}{\beta^2}+k^2+m^2\right)^2}\right] \qquad (7.6)$$

This sum-integral has been already evaluated in the last chapter (see (6.7)). Thus we find

$$\Gamma^2_\beta = \frac{-ie^3\gamma^\mu}{\pi^2}(2\pi)^4\delta^4(p_1-p_2-k_1)\left[\frac{\pi T}{2m}+\frac{1}{2}\ln\left(\frac{m}{4\pi T}\right)\right], \qquad (7.7)$$

where we have kept only the temperature-dependent terms. Writing

$$\Gamma_\beta = \Gamma^1 + \Gamma_\beta^2 \quad (7.8)$$

where Γ_β is defined as

$$\Gamma_\beta = -ie_\beta\gamma^\mu(2\pi)^4\delta^4(p_1-p_2-k_1) \quad (7.9)$$

with e_β being the temperature-dependent gauge coupling constant, we find

$$e_\beta = e[1 + \frac{e^2}{\pi^2}(\frac{\pi T}{2m} + \frac{1}{2}\ln(\frac{m}{4\pi T}))] \quad (7.10)$$

This shows that the U(1) gauge coupling constant increases uniformly with temperature.

Non-abelian Gauge Theories

A non-abelian gauge theory with fermions included, is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\bar{\Psi}_i\gamma_\mu D^\mu_{ij}\Psi_j \quad (7.11)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc}A_\mu A^c_\nu ,$$

and

$$D^\mu_{ij} = \partial^\mu\delta_{ij} - ig A^{a\mu}T^a_{ij}.$$

Here the f^{abc} are the structure constants of the group, and T^a are the matrices representing the group generators. The Feynman rules for non-abelian gauge theory are given in Appendix-B. To understand the nature of variation of the gauge coupling constant in (7.11), it is sufficient to consider the quark-quark-gluon vertex (fig.7.3). The bare vertex Γ^1 is

$$\Gamma^1 = -ig\gamma^\mu T^a_{ij}(2\pi)^4\,\delta^4(p_1-p_2-k_1) \tag{7.12}$$

Now we will consider the vertex diagrams at the one-loop level. In the case of the q - q - g vertex, there are two graphs at the one-loop level (fig.7.4). Let the contributions from these graphs be denoted by Γ^{2a} and Γ^{2b}. It can be seen that the first one is identical to the graph appropriate to QED. The contribution corresponding to this graph (fig.7.4a) in the Feynman gauge can be written as

$$\Gamma^{2a} = -4g^3\gamma^\mu T^a_{mn}T^c_{im}T^c_{nj}\left[\int\frac{d^4k}{(2\pi)^4}\quad\frac{k^2}{k^2(k^2-m^2)^2} - \frac{2\not{k}m}{k^2(k^2-m^2)^2} + \frac{m^2}{k^2(k^2-m^2)^2}\right](2\pi)^4\,\delta^4(p_1-p_2-k_1) \tag{7.13}$$

We use the relation [221], taking the gauge group as SU(n),

$$T^a_{ij}T^a_{k\ell} = \frac{1}{2}\left[\delta_{i\ell}\delta_{jk} - \frac{\delta_{ij}}{n}\delta_{k\ell}\right] \tag{7.14}$$

where n is the dimensionality of the group representation. For SU(3) in the fundamental representation n = 3; then, (7.14) can be rewritten as

$$\Gamma^{2a} = \frac{4g^3\gamma^\mu T^a_{ij}}{6} \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{(k^2-m^2)^2} + \frac{m^2}{k^2(k^2-m^2)^2} \right] \quad (7.15)$$

The term linear in k vanishes at high temperatures, and hence dropped. The delta function is left out for convenience. Since the second term vanishes in the high temperature limit, the temperature-dependent part of Γ^{2a} can be written as

$$\Gamma^{2a}_\beta = \frac{4g^3\gamma^\mu T^a_{ij}}{2n} \frac{1}{-i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + k^2 + m^2\right)^2} \quad (7.16)$$

This equation is the same as (7.6); hence

$$\Gamma^{2a}_\beta = \frac{ig^3\gamma^\mu T^a_{ij}}{n\pi^2} \left[\frac{\pi T}{2m} + \frac{1}{2} \ln\left(\frac{m}{4\pi T}\right) \right] \quad (7.17)$$

Now we will consider the contribution Γ^{2b} from the graph fig.7.4b.

$$\Gamma^{2b} = -ig^3 f^{abc} T^b_{i\ell} T^c_{\ell j} \int \frac{d^4k_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}$$

$$[(k_1-k_2)_\nu \frac{\delta_{\mu\rho}}{k_2^2} \gamma^\rho \frac{1}{(\not p_3 - m)} \gamma^\sigma \frac{g_{\nu\sigma}}{k_3^2} + (k_2-k_3)_\mu \frac{\delta_{\nu\rho}}{k_2^2} \gamma^\rho \frac{1}{(\not p_3 - m)} \gamma^\sigma \frac{g_{\nu\sigma}}{k_3^2} +$$

$$(k_3-k_1)_\lambda \delta_{\mu\sigma} \frac{g_{\lambda\rho}}{k_2^2} \gamma^\rho \frac{1}{(\not p_3 - m)} \gamma^\sigma \frac{1}{k_3^2}].$$

$$(2\pi)^{12} \delta^4(k_1+k_2+k_3)\ \delta^4(p_1+k_2-p_3)\ \delta^4(p_3+k_3-p_2) \tag{7,18}$$

$$= -ig^3 f^{abc} T^b_{i\ell} T^c_{\ell j} \int \frac{d^4k_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \left\{ (k_1-k_2)_\nu \frac{\gamma^\mu}{k_2^2} \frac{(\not p_3+m)}{(p_3^2-m^2)} \cdot \right.$$

$$\left. \frac{\gamma^\nu}{k_3^2} + (k_2-k_3)_\mu \frac{\gamma^\nu}{k_2^2} \frac{(\not p_3+m)}{(p_3^2-m^2)} \frac{\gamma^\nu}{k_3^2} + (k_3-k_1)_\lambda \frac{\gamma^\lambda}{k_2^2} \frac{(\not p_3+m)}{(p_3^2-m^2)} \frac{\gamma^\mu}{k_3^2} \right\}$$

$$(2\pi)^{12} \delta^4(k_1+k_2+k_3)\ \delta^4(p_1+k_2-p_3)\ \delta^4(p_3+k_3-p_2). \tag{7.19}$$

Using the Dirac delta function two of the integrations can be carried out, and neglecting the external momenta, we find

$$\Gamma^{2b} = -ig^3 f^{abc} T^b_{i\ell} T^c_{\ell j} \int \frac{d^4k}{(2\pi)^4} [-k_\nu \frac{\gamma^\mu}{k^2} \frac{(\not k+m)}{(k^2-m^2)} \frac{\gamma^\nu}{k^2} +$$

$$2k_\mu \frac{\gamma^\nu}{k^2} \frac{(\not k+m)}{(k^2-m^2)} \frac{\gamma^\nu}{k^2} - k_\lambda \frac{\gamma^\lambda}{k^2} \frac{(\not k+m)}{(k^2-m^2)} \frac{\gamma^\mu}{k^2} \quad (2\pi)^4\ \delta^4(p_1-p_2-k_1). \tag{7.20}$$

Dropping the terms which are linear in k, we find

$$\Gamma^{2b} = -ig^3 f^{abc} T^b_{i\ell} T^c_{\ell j} \int \frac{d^4k}{(2\pi)^4} \left[- \frac{2\gamma^\mu}{k^2(k^2-m^2)} - \frac{4k_\mu \not{k}}{k^4(k^2-m^2)} \right] \tag{7.21}$$

(The delta factor has been omitted). The temperature dependent part Γ^{2b}_β, in the high temperature limit, is of the form

$$\Gamma^{2b}_\beta = -ig^3 f^{abc} T^b_{i\ell} T^c_{\ell j} \frac{1}{-i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \Big[-2\gamma^\mu \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + k^2\right)^2}$$

$$- \frac{4\left(\gamma_o \frac{4\pi^2 n^2}{\beta^2}, (\vec{\gamma}\cdot\vec{k})\vec{k}\right)}{\left(\frac{4\pi^2 n^2}{\beta^2} + k^2\right)^3} \Big] \tag{7.22}$$

Note that

$$f^{abc} T^c_{\ell j} = -i[T^a, T^b]_{\ell j}$$

$$= -i[T^a_{\ell k} T^b_{kj} - T^b_{\ell k} T^a_{kj}].$$

Further, by invoking (7.14), we find

$$f^{abc} T^c_{\ell j} T^c_{i\ell} = \frac{in}{2} T^a_{ij} \tag{7.23}$$

The first term in (7.22), viz;

$$- \frac{2\gamma^{\mu}}{\beta} \sum_{n} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + k^2\right)^2}$$

is analogous to (7.16); when the summation is carried out, we find

$$- \frac{2\gamma^{\mu}}{\beta} \sum_{n} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left(\frac{4\pi^2 n^2}{\beta^2} + k^2\right)^2} = \frac{-2\gamma^{\mu}}{4\pi^2} \int_0^{\infty} \frac{dk}{k(e^{\beta k} - 1)} \tag{7.24}$$

The first part of the second term yields

$$- \frac{\gamma^{o}}{4\pi^2} \int_0^{\infty} \frac{dk}{k(e^{\beta k} - 1)} + \frac{\gamma^{o}}{4\pi^2} \int_0^{\infty} dk \frac{\beta e^{\beta k}}{(e^{\beta k} - 1)^2} \tag{7.25}$$

Similarly the second part of the second term now becomes

$$\frac{\vec{\gamma}}{4\pi^2} \int_0^{\infty} \frac{dk}{k(e^{\beta k} - 1)} - \frac{\vec{\gamma}}{4\pi^2} \int_0^{\infty} dk \frac{\beta e^{\beta k}}{(e^{\beta k} - 1)^2} \tag{7.26}$$

Combining (7.25) and (7.26), we find

$$- \frac{\gamma^{\mu}}{4\pi^2} \int_0^{\infty} \frac{dk}{k(e^{\beta k} - 1)} + \frac{\gamma^{\mu}}{4\pi^2} \int_0^{\infty} dk \frac{\beta e^{\beta k}}{(e^{\beta k} - 1)^2} \tag{7.27}$$

Substituting (7.23), (7.24) and (7.27) in (7.22), we obtain

$$\Gamma_{\beta}^{2b} = ig^3\gamma^{\mu}T^a_{ij} \; \frac{n}{2} \left[-\frac{3}{4\pi^2}\int_0^{\infty} \frac{dk}{k(e^{\beta k}-1)} - \frac{1}{4\pi^2}\int_0^{\infty} dk \, \frac{d}{dk}\left(\frac{1}{e^{\beta k}-1} \right) \right] \tag{7.28}$$

The above integrals exhibit the infrared divergences, characteristic of Yang-Mills theories [128,131,222]. Accordingly, we introduce an infrared cut-off $\Lambda(T)$ which is assumed to depend on temperature T; $\Lambda(T)$ may be assumed to be small [131,222]. Hence

$$\Gamma_{\beta}^{2b} = ig^3\gamma^{\mu}T^a_{ij} \; \frac{n}{2} \left\{ \frac{-3}{4\pi^2} \int_0^{\infty} \frac{dk}{[k^2+\Lambda^2(T)]^{1/2}\,(e^{\beta(k^2+\Lambda^2(T))^{1/2}}-1)} + \frac{1}{4\pi^2}\left(\frac{1}{e^{\beta\Lambda(T)}-1} \right) \right. \tag{7.29}$$

$$= ig^3\gamma^{\mu}T^a_{ij} \; \frac{n}{2} \left[\frac{-3}{4\pi^2} \, \frac{\pi T}{2\Lambda(T)} + \frac{1}{2}\ln\left(\frac{\Lambda(T)}{4\pi T}\right) + \frac{1}{4\pi^2} \, \frac{1}{(e^{\Lambda(T)/T}-1)} \right] \tag{7.30}$$

Combining (7.17) and (7.30),

$$\Gamma_\beta^2 = \frac{ig^3\gamma^\mu T^a_{ij}}{n\pi^2}\left[\frac{\pi T}{2m} + \frac{1}{2}\ln\left(\frac{m}{4\pi T}\right)\right] +$$

$$ig^3\gamma^\mu T^a_{ij}\ \frac{n}{2}\left[-\frac{3}{4\pi^2}\left(\frac{\pi T}{2\Lambda(T)} + \frac{1}{2}\ln\left(\frac{\Lambda(T)}{4\pi T}\right)\right) +\right.$$

$$\left.\frac{1}{4\pi^2}\ \frac{1}{(e^{\Lambda(T)/T} - 1)}\right] \qquad (7.31)$$

Defining Γ_β, as before, as

$$\Gamma_\beta = \Gamma^1 + \Gamma_\beta^2$$

where

$$\Gamma_\beta = -ig_\beta\gamma^\mu T^a_{ij}\ ,$$

the temperature-dependent coupling constant g_β is, therefore, given by

$$g_\beta = g\left[1 - \frac{1}{n\pi^2}\left(\frac{\pi T}{2m} + \frac{1}{2}\ln\left(\frac{m}{4\pi T}\right)\right) + \frac{3n}{4\pi^2}\left(\frac{\pi T}{2\Lambda(T)} +\right.\right.$$

$$\left.\left.\frac{1}{2}\ln\left(\frac{\Lambda(T)}{4\pi T}\right)\right) - \frac{n}{4\pi^2}\ \frac{1}{(e^{\Lambda(T)/T} - 1)}\right]. \qquad (7.32)$$

The temperature behaviour of the coupling constant in QCD can be obtained by putting $n = 3$ in the above equation. This relation shows that the gauge coupling constant in non-abelian gauge theories decreases with temperature.

Discussion

The work of Collins and Perry [212] showed that, under conditions of high density, QCD perturbation theory becomes more reliable because of two factors, viz, asymptotic freedom and mass generation for gluons. Asymptotic freedom implies that quarks and gluons become free as a result of the vanishing of the gauge coupling constant. The mass generation mechanism for gluons ensures a screening length and an infrared cut-off [128]. The present investigation shows that the QCD coupling constant decreases with temperature, to one loop level accuracy in perturbation theory. At the qualitative level this confirms the renormalisation - group inspired conjecture [125]:

$$\bar{g}^2 = \frac{24\pi^2}{(33-2N_f)\ \ln\ (T/\Lambda)}$$

where $\bar{g}$ is an effective running coupling constant. Kajantie and Kapusta [125], on the basis of renormalisation group and linear response theories, argue that a unique effective coupling constant as a function of temperature or density does not exist but depends on the physical circumstances, for instance whether one is speaking of static electric or static magnetic interactions. Nevertheless, for the static limit, they define an approximate running coupling constant in the form

$$\bar{g}^2(k,M,T\ g(M)) = \frac{g^2(M)}{1 + \frac{11}{48\pi^2} g^2(M)N \ln(\frac{k^2}{M^2}) + \frac{m^2(g^2(M),T)}{k^2}}$$

where M denotes the renormalisation point, and m^2 the electric or magnetic screening mass of gluons. Our derivation of the temperature-dependent coupling constant g_β based on the quark-quark-gluon vertex, involves a straightforward application of perturbation theory at high temperature and has, besides the expected logarithmic terms, exposed a linear dependence on T. Even if we substitute the gluon screening mass, $\Lambda(T) \sim gT$ or g^2T, depending on the nature of the screening [128] into the expression for g_β (7.32), the linear dependence persists along with the logarithmic term. It appears that, at the one-loop level, these two types of terms arise in all gauge theories at finite temperature (see preceding chapter). We believe that g_β is a physically significant parameter, which can be defined for QCD at any temperature irrespective of the details of the screening mechanism.

The existence of a temperature-dependent coupling constant g_β in a grand unified theory such as SU(5) has important implications for the calculation of various processes leading to the currently observed matter-antimatter asymmetry. For SU(5), g_β is obtained by setting n = 5 in (7.32). Estimates of the decoupling temperature, T_d, at which a given particle species goes out of thermal equilibrium, made in the literature [223], ignore the temperature dependence of the coupling constants. It must be admitted that, for a true, picture, we should know the temperature dependence of the gravitation constant G besides that of α $(=g^2/4\pi)$. This problem cannot be solved until a satisfactory quantum theory of gravity is developed, and possibly, unified with the formalism of grand unified theories familiar today.

The monotone decrease of g_β with temperature is in agreement with ideas of asymptotic freedom as well as deconfinement transition investigated in lattice QCD [224]. At the deconfinement transition, colour screening decouples the constituents and chiral symmetry is restored.

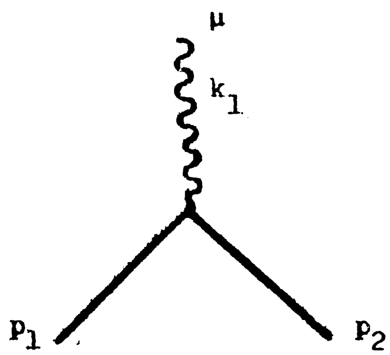


Fig. 7.1 Bare vertex in QED

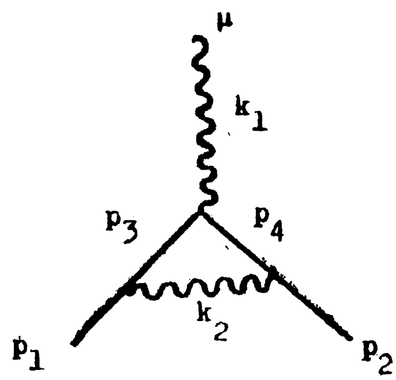


Fig. 7.2 Lowest order vertex correction in QED

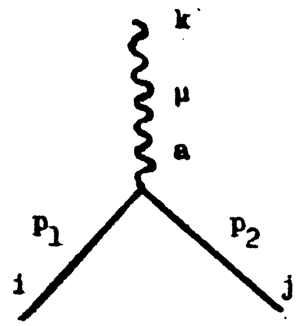


Fig. 7.3 Quark-quark-gluon vertex

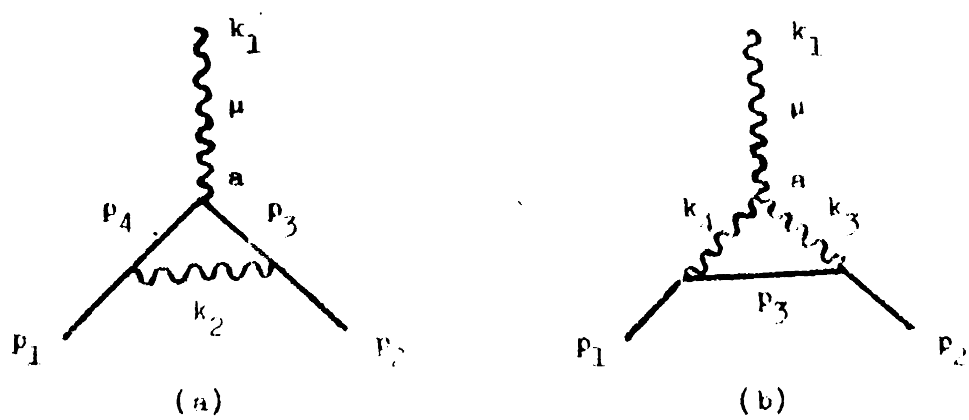


Fig. 7.4 Lowest order correction to q-q-g vertex

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APPENDIX - A

NOTATION AND CONVENTIONS

The natural unit is used throughout (ie, c and $\hbar$ are set equal to 1). The space-time coordinates $(t,x,y,z = (t,\vec{x})$ are denoted by the contravariant four-vector:

$$x^{\mu} \quad (x^0, x^1, x^2, x^3) \quad (t,\ x,\ y,\ z)$$

The covariant four-vector x_{μ} is defined as

$$x_{\mu} = (x_0, x_1, x_2, x_3) \quad (t, -x, -y, -z) \quad (t\ , -\vec{x}) = g_{\mu\nu} x^{\mu}$$

where

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Summation over repeated indices will always be implied. The inner product implies:

$$x^2 = x_{\mu} x^{\mu} = t^2 - \vec{x}^2.$$

Momentum vectors are defined

$$p^{\mu} = (E,\ p_x, p_y, p_z)$$

and

$$p^2 = p_{\mu} p^{\mu} = E^2 - \vec{p}^2.$$

Dirac γ matrices satisfy the following relation:

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g_{\mu\nu}$$

with

$$\gamma^{0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\gamma^{i} = \begin{bmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{bmatrix}$$

where

$$\sigma^{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^{2} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \quad \sigma^{3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are the Pauli 2x2 spin matrices.

The inner product of a γ-matrix with an ordinary four-vector is denoted by

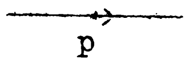
$$\gamma^{\mu}p_{\mu} = \not{p} = \gamma^{0}E - \vec{\gamma}\cdot\vec{p}$$

APPENDIX - B

RULES FOR FEYNMAN DIAGRAMS

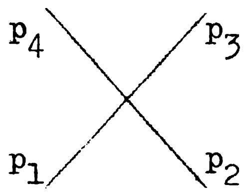
<u>φ^4 theory</u>

Internal boson line



$$\frac{i}{p^2-m^2}$$

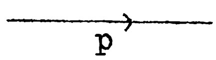
Vertex



$$-i\lambda(2\pi)^4\ \delta^4(p_1+p_2-p_3-p_4)$$

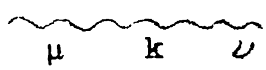
<u>Scalar electrodynamics (SED)</u>

Internal boson line



$$\frac{i}{p^2-m^2}$$

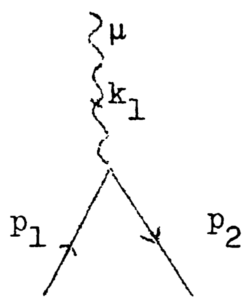
Internal photon line



$$\frac{-ig_{\mu\nu}}{k^2}$$

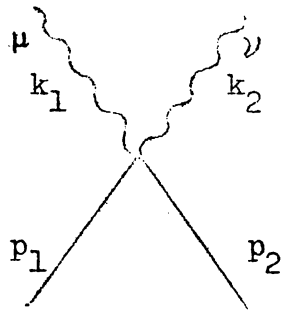
Vertex

1.



$$-ie(p_1+p_2)_\mu(2\pi)^4\delta^4(p_1+k_1-p_2)$$

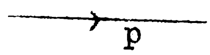
2.



$$2ie^2 g_{\mu\nu}(2\pi)^4\delta^4(p_1+k_1-p_2-k_2)$$

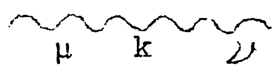
Quantum electrodynamics (QED)

Internal electron line



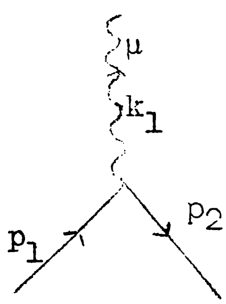
$$\frac{i}{\not{p}-m}$$

Internal photon line



$$\frac{-ig_{\mu\nu}}{k^2}$$

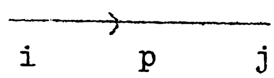
Vertex



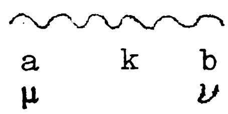
$$-ie\gamma^{\mu}(2\pi)^4\delta^4(p_1-k_1-p_2)$$

Non-abelian gauge theory

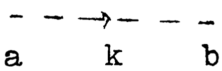
Internal fermion line



$$\frac{i\delta_{ij}}{\not{p}-m}$$

Internal gluon line 

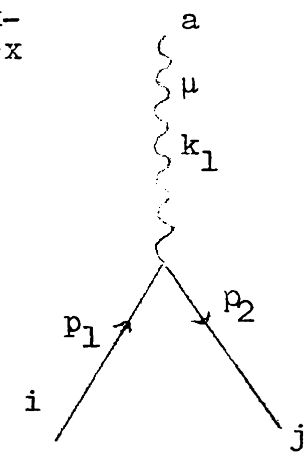
$-i\delta_{ab}\left[\frac{g_{\mu\nu}}{k^2} \frac{-(1-a)k_\mu k}{k^4} \right]$

Internal ghost line 

$-i\delta_{ab} \frac{1}{k^2}$

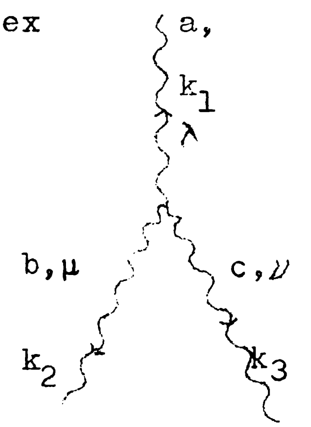
Vertex

1. Quark-quark-gluon vertex



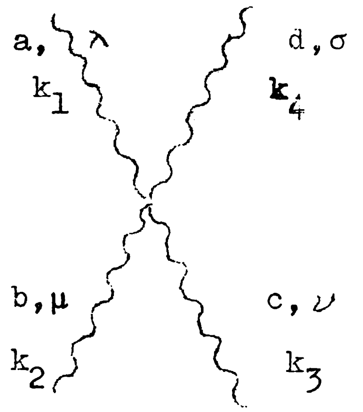
$$-ig\gamma^\mu T_{ij}{}^a (2\pi)^4 \delta^4 (p_1-k_1-p_2)$$

2. 3-Gluon vertex



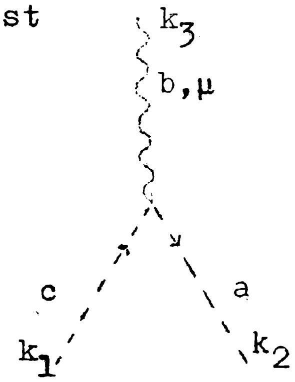
$$gf^{abc}[(k_1-k_2)_\nu \; g_{\lambda\mu} + (k_3-k_1)_\mu g_{\nu\lambda} + (k_2-k_3)_\lambda \; g_{\mu\nu}] (2\pi)^4\delta^4(k_1+k_2+k_3)$$

3. 4-Gluon vertex



$$\{-ig^2 f^{abc} f^{cde}(g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) - ig^2 f^{ace} f^{bde}(g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) - ig^2 f^{ade} f^{cbe}(g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\mu} g_{\sigma\nu})\}$$
$$(2\pi)^4 \delta^4(k_1 - k_2 + k_3 - k_4)$$

4. Gluon-ghost-ghost vertex



$$g f^{abc} k_2^{\mu} (2\pi)^4 \delta^4(k_1 - k_3 - k_2)$$