MULTIVARIATE TIME SERIES ANALYSIS OF INDUSTRIAL PHENOMENA

ARMA MODELLING OF TIME SERIES BASED ON RATIONAL APPROXIMATION OF SPECTRAL DENSITY FUNCTION

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

JESSY JOHN, C.

CERTIFICATE

Certified that the work reported in this thesis is based on the bona fide work done by Smt.Jessy John, C. under my guidance in the Department of Mathematics & Statistics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

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Chapter 1

INTRODUCTION

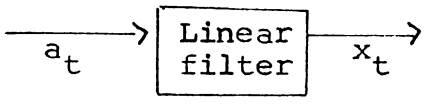
This study is concerned with Autoregressive-Moving Average (ARMA) models of time series. ARMA models form a subclass of the class of general linear models which represents stationary time series, a phenomenon encountered most often in practice by engineers, scientists and economists. It is always desirable to employ models which use parameters parsimoniously. Parsimony will be achieved by ARMA models because it has only finite number of parameters. Even though the discussion is primarily concerned with stationary time series, later we will take up the case of homogeneous nonstationary time series which can be transformed to stationary time series.

Time series models, obtained with the help of the present and past data is used for forecasting future values. Physical science as well as social science take benefits of forecasting models. The role of forecasting cuts across all fields of management—finance, marketing, production, business economics, as also in signal process, communication

engineering, chemical processes, electronics etc. This high applicability of time series is our motivation to this study.

1.1 Historical development

Beginnings were made in the mathematical approach of time series as early as 1809, when the French mathematician Fourier, introduced the idea that any series can be approximated as the sum of sine and cosine terms. In 1906 Schuster [32] applied Fourier's idea to analyse time series. Yule [38] in 1926 claimed that Schuster's approach is not adequate for prediction. Later in 1927, Yule [39] introduced linear filter model which claims to observe a time series as output from a linear filter with input as random shocks or white noise (defined in chapter 2). The linear filter gives an output



as a weighted sum of previous shocks, a present shock, and a level change μ,

$$x_t = \mu + a_t + \sum_{i=1}^{\infty} \psi_i a_{t-i}. \qquad (1.1)$$

He further showed that a series can be better described as a function of its past values, i.e.,he introduced the concept of autoregressive (AR) models. His study was restricted to autoregressive models of order four or less. Yule's work was extended by Walker [34]. He defined the general autoregressive scheme of equations, given by

$$x_t = \mu + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + a_t \tag{1.2}$$

Slutzky [32] introduced the moving-average (MA) scheme. The model suggested by Slutzky is given by

$$x_t = a_t - \sum_{i=1}^{q} \theta_i a_{t-i} \tag{1.3}$$

Wold [37] worked in the theoretical validity of the method and devised general representation of time-series. Wold [37] showed that any discrete, stationary time series, x_t, can be represented by an ARMA model. The new approach of time series analysis starts from Wold's work, who is the founder of ARMA models. Kolmogroff [22] did much work in the field of estimation of the parameters of a model. He was followed by Mann and Wald [25]. They developed the maximum likelihood estimation procedure for the solution of autoregressive process. Whittle [36] and Durbin [11], [12] obtained efficient methods for estimating AR and MA parameters.

Walker [35] extended the result to mixed ARMA models.

Kalman and Bucy [20] considered the following system:

$$Y_t = \sum_{i=1}^{m} \phi_i Y_{t-i} + \sum_{j=1}^{n} \gamma_j u_{t-j}$$

$$X_t = \sum_{k=1}^{p} \eta_k Y_{t-k} + v_t \qquad (1.4)$$

where

$t = 0,1,2....,$

Y_t is the true state of the system uncorrupted by noise,

ϕ_i is the i^{th} element of the transition vector which is constant for the purpose,

γ_i is the i^{th} element of the input vector,

u_t is the noise input to the system at time t,

X_t is the observed output or actual data measured at time t,

η_i is the i^{th} element of the observation vector, and

v_t is the measurement error at time t.

The sequences $\{u_t\}$ and $\{v_t\}$ are uncorrelated white noise, usually assumed Gaussian, with means and variances

$$E[u_t] = E[v_t] = 0$$

$$E[u_t u_{t'}] = E[v_t v_{t'}] = 0$$

$$E[u_t v_{t'}] = 0,$$

$$E[u_t u_t] = \sigma_u^2 < \infty, \text{ and}$$

$$E[v_t v_t] = \sigma_v^2 < \infty,$$

for all $t \neq t'$. This is also a linear filter model with the noise input separated into two components. Application of this model requires the knowledge of the physical system. This differs from the goal of allowing the data to determine the model, however it is a very general linear filter model and it is similar to the ARMA models. Brown [9] popularised exponential smoothing models given by

$$x_k(t) = \alpha x_{k-1}(t) + (1-\alpha)x_k(t-1) \qquad (1.5)$$

for time index $t = 0,1,2,...$and the index of smoothing $k = 1,2,...,x_o$ is defined to be x_t. Exponential smoothing is similar to curve fitting with polynomial regression. For polynomial trends of order s in the data, the forecast is a

linear combination of $x_1(t)$, $x_2(t),\ldots,x_s(t)$. For example, for stationary process the k^{th} step ahead forecast made at time t-1 is

$$\hat{x}(k;t-1) = x_1(t). \tag{1.6}$$

If a linear trend is suspected, the k-step ahead forecast from time t-1 is

$$\hat{x}(k;t-1) = 2x_1(t) - x_2(t) + \frac{\alpha}{1-\alpha}(x_1(t)-x_2(t))k; \tag{1.7}$$

The theory of exponential smoothing extends the forecasting to higher order polynomial trends as well as sinusoidal seasonality.

The exponential smoothing models, the autoregressive models and the moving average models are special cases of the ARMA model. ARMA models exhibit a simpler structure than Kalman-Bucy and linear filtering. Widespread acceptance of ARMA models is due to its simplicity and generality.

Akaike [1],[2],[3] developed a procedure for selecting an AR model for a given stationary time series. This procedure depends on the fact that any stationary process can be represented by an autoregressive model. Some loss of

parsimony is incurred through this assumption. For a given time series $\{x_t,\ t = 1,2,\ldots,T\}$ the Akaike's procedure known as final prediction error (FPE) scheme is as given below:

1. Determine an upper limit K for the order of autoregression to be considered,

2. Calculate $\bar{x}_T$ and $\{\gamma_k,\ k = 0,1,..K\}$ where

$$\bar{x}_T = \frac{1}{T} \sum_{t=1}^{T} x_t, \tag{1.8}$$

and

$$\hat{\gamma}_k = \frac{1}{T} \sum_{t=1}^{T-|k|} (x_t - \bar{x}_T)(x_{t+|k|} - \bar{x}_T), \tag{1.9}$$

3. A sequence of estimated autoregressive coefficients $\phi_{M,J};\ J = 1,2,3\ldots M$ is calculated for each order of autoregression to be considered, using the set of Yule-Walker equations (defined in chapter 2), $M = 1,2,\ldots K$,

4. Calculate the average sum of square of one-step ahead forecast errors S_M for each M where

$$S_M = \frac{1}{T} \sum_{t=1}^{T} \left\{ x_t - \sum_{m=1}^{M} \phi_{M,m} x_{t-m} (1 - \sum_{m=1}^{M} \phi_{M,m}) x_T \right\}^2 \tag{1.10}$$

with $x_o,\ x_{-1},\ldots x_{-K+1}$ defined as zeros.

5. Compute FPE(M), where

$$\text{FPE}(M) = \frac{T+M-1}{T-M-1} S_M; \quad M = 1,2,\ldots K, \qquad (1.11)$$

$$\text{and} \quad \text{FPE}(0) = \frac{T}{T-1} c(o)$$

6. The optimum order of autoregression is then chosen as the order P among orders 1 to K which achieves the minimum value of FPE.

There is a great deal of arbitrariness associated with the FPE criterion. Akaike himself exhibits this in a later paper, by presenting a class of FPE functions FPE_α defined by,

$$\text{FPE}_\alpha(M) = [1-T^{-\alpha}(M+1)]\,[1-T^{-1}(M+1)]^{-1} S_M,$$

which reduces to FPE when α equals unity. In 1974 Akaike developed a scheme for selection of a mixed model. Here also maximum orders of autoregressive and moving average operators are to be specified first. Then for each pair (p,q) of autoregressive and moving average orders a statistic AIC(p,q) is calculated. The expression given for AIC is

$$\text{AIC}(p,q) = T \log (\hat{\sigma}_a^2) + 2(p+q) \qquad (1.12)$$

where T is the record length and $\hat{\sigma}_a^2$ is the maximum likelihood

estimate of white noise. AIC may handle a broader class of models than FPE, it does exhibits a rapid increase in computation time when orders of moving average are increased, which is the main drawback of the criterion.

Anderson [4] has presented a multiple decision process for determining the order of an AR process with Gaussian noise. It requires to specify a minimum order m and maximum order k of autoregression. Then a series of probabilities $P_i (m \leq i \leq k)$ are selected such that P_i is the probability of deciding that the order of autoregression is i when it is actually less than i. This then describes k-m+1 regions R_m, $R_{m+1}, \ldots R_k$ of the original space of sample $\{x_t : t=1,2,\ldots T\}$. If a sample point falls in region R_p, then the order of autoregression is taken to be p. Another method selected by Anderson [5] is based upon the above procedure and follows the lines of backward elimination procedure of stepwise regression. The partial autocorrelations $\Phi_{k,k}$ are assumed to be distributed mutually independent and normally about zero when the true order is less than k. Then the partial autocorrelation function (PACF) (defined in chapter 2), is tested against zero successively beginning with k equal to L, then L-1 and so on until it is decided to be significant. The following relationships with ρ are used to

obtain the levels of significance and critical values

$$
\begin{aligned}
P_k &= \rho\left(\frac{L+m}{L-m}\right)/L+k-m,\ k=m+1,m+2,\ldots L \\
\beta_L &= P_L \\
\beta_K &= P_k \prod_{i=k+1}^{L} (1-\beta_i)^{-1}:\ k=L-1,L-2,\ldots,m+1 \\
\beta_k &= \text{Prob}\left\{ |\sqrt{T}\,\phi_{k,k}| > \delta_k \right\},
\end{aligned}
\tag{1.13}
$$

where $\sqrt{T}\,\phi_{k,k}$ has a limiting standard normal distribution. The arbitrary choice of $P_{m+1}, P_{m+2},\ldots P_L$ in the general procedure and ρ in the simpler formulation, the assumption of normal noise, and the difficulty of application of the general multiple decision problem are some drawbacks of the Anderson's approach.

Hannan [14] 1970, Cleveland [10] 1972, Jones [19] 1975 and Mclave [27] 1975, have also considered the problem of optimal choice of the order of AR processes.

Most of the procedures deal with either AR process or MA process. Even though the idea of mixed ARMA models were introduced earlier, it was Box and Jenkins who developed a model identification procedure for mixed ARMA process.

1.2 ARMA model identification procedure

In the analysis of time-series Box and Jenkins [8] developed their methods for stationary and homogeneous non-stationary series. Their approach can be divided into two parts namely model identification and forecasting. Model identification consists of different steps. In the identification procedure, the first step is to decide p and q and the second step is to find the estimates for parameters of the model. These are done with the help of autocorrelation function (ACF) and partial autocorrelation function (PACF). The k^{th} partial autocorrelation $\Phi_{k,k}$ is the partial correlation between x_t and x_{t+k} and is defined by the Yule-Walker equations on the autocorrelations

$$\rho_j = \sum_{i=1}^{k} \Phi_{k,i} \rho_{j-i} , \quad j=1,2,\ldots k \tag{1.14}$$

for k = 1,2,... The estimator of $\Phi_{k,k}$ is given in (2.43). The Box-Jenkins identification procedure for an ARMA(p,q) process is briefly given by the following:

(i) If p = 0, in which case the series follows a strict moving average process of order q, then the ACF obeys

$$\rho_k = 0 \text{ for all } k > q$$

and the PACF is dominated by damped exponentials or damped sine waves.

(ii) If $q = 0$, in which case the series is a strict autoregressive process of order p, then the ACF will damp out according to the difference equation

$$\rho_k = \sum_{i=1}^{p} \phi_i \rho_{k-i} \tag{1.15}$$

for all k. This appears as damped exponentials or damped sine waves or a mixture of both. Meanwhile, the PACF follows

$$\phi_{kk} = 0, \quad k > p.$$

(iii) If $p \neq 0$, $q \neq 0$, then the series represents an ARMA model of order (p,q), then the ACF follows the difference equation

$$\rho_k = \sum_{i=1}^{p} \phi_i \rho_{k-i} \qquad \text{for all } k > q, \tag{1.16}$$

which appears as damped exponentials or damped sine waves or mixture of both after q-p lags. The PACF is dominated by damped exponentials and damped sine waves after the first p-q lags.

There is vagueness in finding the values of p and q visually from the graphs even for AR and MA models, for which some definiteness is there in the procedure. But the procedure fails miserably when it comes to the mixed ARMA model.

Box and Jenkins [8] suggests different models for the same series. Again the highest order of the mixed ARMA model considered by them is (1,1) which is the simple case. This situation prompted us to look for an alternate procedure wherein:

(i) p and q can be determined by computation procedure rather than by reading graphs.

(ii) Higher order ARMA models are considered without much difficulty.

Mc Intire [28] presented a new ARMA model identification procedure based on the G-spectral estimates introduced by Morgan [29] and Houston [16]. The procedure due to Mc Intire can be summarised as:

1. Select a maximum order for autoregression, say L. Calculate the R and S arrays to column L+2 (for details of R and S arrays see Mc Intire [28]).

2. Find a column, say column n, of the S array with alternatively constant entries followed by a column with highly variable entries. Then consider n-1 as an estimator of p.

3. Determine wherein column n of the R array the zeros begin.

Then use

$$r_{p+1}(f_k) = 0 \text{ for } k \geq q-p+1,$$

where $f_k = \rho_k$. If the zeros start at $r_{\hat{p}+1}(f_m)$, then $\hat{q}-\hat{p}+1 = m$, so that $\hat{q} = m+\hat{p}-1$, where $\hat{q}$ is an estimator of q.

4. Check to see if the following hold:

$$s_{\hat{p}}(f_k) = c_1, \quad k \geq \hat{q} - \hat{p} + 1$$

$$s_{\hat{p}}(f_{\hat{q}-\hat{p}}) \neq c_1$$

$$s_{\hat{p}}(f_k) = c_2, \; k \leq -\hat{q}-\hat{p}$$

$$s_{\hat{p}}(f_{-\hat{q}-\hat{p}+1}) \neq c_2$$

$$r_{\hat{p}+1}(f_k) = 0, \; k \geq \hat{q}-\hat{p}+1$$

$$r_{\hat{p}+1}(f_{\hat{q}-\hat{p}}) \neq 0$$

$$r_{\hat{p}+1}(f_k) = 0 \quad k \leq -\hat{q}-\hat{p}-1$$

$$r_{\hat{p}+1}(f_{-\hat{q}-\hat{p}}) \neq 0.$$

Further investigate $s_{\hat{p}+1}(f_{\hat{p}-\hat{q}})$ and $s_{\hat{p}+1}(f_{-\hat{q}-\hat{p}-1})$ to see if these quantities are $-c_1$ and infinite, respectively.

The identification procedure suggested by McIntire is very complicated and it also fails to find a unique model

for a stationary series. For example, McIntire [28] suggests two models for series A of Box and Jenkins [8].

1.3 Author's work

Notations and definitions relevant of our study are given in the next chapter. Stationary time series, autocorrelations and partial autocorrelations of a stationary time series, difference operator, ARMA(p,q) models, ARIMA(p,d,q) models, etc. are defined. Different forms of the spectral density functions of a stationary time series are discussed.

The third chapter explains the new model identification technique. A unique ARMA(p,q) model representing a given stationary time series is obtained. Rational approximations of functions are discussed first. The theory of rational approximation due to Chebyshev (1962), is applied here. Applying this theory a unique rational approximation of the spectral density function is obtained which is in the form of

$$T_p^q(\cos\lambda) = \frac{a_o + a_1 \cos\lambda + \ldots + a_q \cos q\lambda}{1 + b_1 \cos\lambda + \ldots + b_p \cos p\lambda} \qquad (1.17)$$

where $\lambda \in [-\pi, \pi]$.

The rational form of the spectral density function of an ARMA(p,q) time series model is

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \quad \frac{1 + \sum_{j=1}^{q} \Theta_j^2 + 2 \sum_{j=1}^{q} (-\theta_j + \sum_{i=1}^{q-j} \Theta_i \Theta_{i+j}) \cos j\lambda}{1 + \sum_{j=1}^{p} \phi_j^2 + 2 \sum_{j=1}^{p} (-\phi_j + \sum_{i=1}^{p-j} \phi_i \phi_{i+j}) \cos j\lambda} \tag{1.18}$$

where $\lambda \in [-\pi, \pi]$, $\phi_1, \ldots \phi_p$ are autoregressive parameters and $\Theta_1, .., \Theta_q$ are moving average parameters, and σ_a^2 is the variance of the white noise a_t's and σ_x^2 is variance of the series. Equating (1.17) and (1.18) we obtain p+q+1 second degree equations in ϕ_i's, Θ_i's and σ_a^2.

An algorithm is developed to solve the nonlinear equations based on iteration. These p+q+1 parameters uniquely determine the ARMA(p,q) model for a given stationary time series.

In the first part of chapter four, the new model building procedure is tested using theoretical autocorrelations for nine different ARMA(p,q) models. It is found that the error between the theoretical values of the parameters and the estimated values of the parameters are very small proving the efficiency of the new technique for ARMA model estimation. In the second part of chapter four the new technique is applied to analyse simulated series and in the third part it is applied to original time series data. It is found that the new model identification procedure is highly suitable.

The fifth chapter gives some results in connection with the new ARMA model building technique. It is shown that the relation between the autocorrelations and the parameters in ARMA(p,q) model are same as that given by Box and Jenkins. Multivariate extension of the procedure is taken up for two variables.

Comparison of the new method for ARMA(p,q) model identification with some prominent methods like Box-Jenkin's method, McIntire's method are given in chapter six. It shows that this new technique gives a unique ARMA(p,q) model for a given stationary time series, whereas the other method gives different models for the same series.

In the concluding chapter, a brief discussion on the new technique for identifying ARMA(p,q) model representing a stationary time series is given. Also several areas are suggested for further research.

Chapter 2

NOTATIONS, DEFINITIONS AND PRELIMINARY NOTIONS

Some definitions and notations relevant for our study are discussed in this chapter.

2.1 Time series

A time series is a set of observations generated sequentially in time. If the time set is continuous, the time series is said to be continuous. If the set is discrete, the time series is said to be discrete. A time series is denoted by $\{x_t, t \in T\}$, T is called the index set. When apparent, the indication of the index set will be suppressed and the series will be denoted by $\{x_t\}$ or $\{x_t, t = \pm 1, \pm 2, \ldots\}$. In our studies only discrete time series, observed at equal intervals of time is considered. In the case of continuous time series it is very difficult to obtain observations. In such cases a discrete series is obtained by sampling the series at equal intervals of time and it is used for study. A time series is said to be deterministic if future values are exactly determined by some mathematical functions such as

$$x_t = \cos(2\pi t).$$

If the future values can be described only in terms of a probability distribution, then the series is said to be a statistical time series. We are confined to statistical time series in this thesis. Hereafter in this work, time series means statistical time series.

2.2 Stochastic process

A statistical phenomenon that evolves in time according to probabilistic laws is called a stochastic process. A stochastic process is represented by $\{X_t, t \in T\}$, where T is called the index set, which is a subset of the set of real numbers. As in the notation of time series if T is a continuous subset of real numbers then the stochastic process is known as continuous and if T is a discrete set then X_t is called a discrete stochastic process. We shall often refer to it simply as a process, omitting the word stochastic. An observed time series will be assumed to be a realization of a stochastic process.

2.3 Mean and product moments

The mean, μ_t, of the stochastic process $\{X_t, t \in T\}$ is defined as

$$\mu_t = E[X_t]. \tag{2.1}$$

The covariance between process elements at times t and t+k is defined by

$$\gamma_{k,t} = \mathrm{Cov}[X_t \; X_{t+k}]$$

$$\text{i.e.,} \gamma_{k,t} = E[(X_t - \mu_t)(X_{t+k} - \mu_{t+k})] \tag{2.2}$$

Since the covariance $\gamma_{k,t}$ defined in (2.2) is between the elements of the same process, it is called <u>autocovariance</u>. <u>Autocorrelations</u> of a process between process elements at times t and t+k is defined as

$$\rho_{k,t} = \frac{\gamma_{k,t}}{\gamma_{o,t}} \tag{2.3}$$

$$k = 0,1,2,....$$

2.4 Stationary process

Stationary stochastic process is a very important branch of stochastic processes. Stationarity of a process is based on the assumption that the process is in a particular state of statistical equilibrium. A stochastic process is said to be <u>strictly stationary</u> if its properties are unaffected by a shift in time origin, that is the joint probability distribution associated with n observations $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ made at any set of times $t_1, t_2, \ldots, t_n$ is

the same as that associated with n observations x_{t_1+k} $x_{t_2+k}, \ldots, x_{t_n+k}$ made at times t_1+k, $t_2+k, \ldots, t_n+k$. Stationarity of a stochastic process implies the following:

1. The mean and variance of the process are constants.

2. The autocovariance is a function of time only through the distance between the two time points involved.

That is μ_t and $\gamma_{k,t}$ do not depend on t. For stationary processes the t's are suppressed from the notation. If the properties (1) and (2) of a strictly stationary process is satisfied by a process, then it is called a <u>weakly stationary</u> process. It is clear that all strictly stationary processes are stationary in the weak sense also.

2.5 Estimates of mean and autocorrelations

The mean of a stationary time series $\{x_t\}$ can be estimated by $\hat{\mu} = \bar{x}$ given by

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \qquad (2.4)$$

where $\hat{\mu}$ is the estimate of μ and N is the number of

observations of the time series $\{x_t\}$. Estimate c_k of the k^{th} lag autocovariance, γ_k, of a stationary time series is obtained by

$$c_k = \frac{1}{N} \sum_{t=1}^{N-|k|} (x_t - \bar{x})(x_{t+|k|} - \bar{x}), \tag{2.5}$$

$k = 0, \pm 1, \pm 2 \ldots$, and the estimate r_k of the autocorrelation ρ_k is obtained by

$$r_k = \frac{c_k}{c_o} \tag{2.6}$$

The stationarity along with the property of ergodicity (Papoulis [30]) guarantees that both the estimates of the mean and autocovariances are consistent estimators. The matrix P_k defined by

$$P_k = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & 1 \end{bmatrix} \tag{2.7}$$

is called the <u>autocorrelation matrix</u> of order k. The autocorrelation matrix P_k of a stationary process must be positive definite (Box and Jenkins [8]) for all values of k. The set $\gamma = \{\gamma_k, k=0, \pm 1, \pm 2, \ldots\}$ is called the <u>autocovariance</u> function

and the set $\rho = \{ \rho_k, k=0,\pm1,\pm2,\ldots\}$ is called the auto-correlation function (ACF) of a stationary process. The graph of the ACF is called the correlogram.

2.6 Shift operators

The backward shift operator, B, is defined by

$$BX_t = x_{t-1} \tag{2.8}$$

and the difference operator, ∇, is defined by

$$\nabla x_t = x_t - x_{t-1}$$

or $$\nabla = 1 - B \tag{2.9}$$

2.7 White noise

A series of statistically independent, zero mean, finite variance random variables $\{a_t, t \in T\}$ are defined as white noise or series of random shocks.

$$\text{i.e., } E[a_t] = 0 \tag{2.10}$$

and

$$E[a_t a_{t+k}] = \begin{cases} \sigma_a^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{2.11}$$

2.8 Linear models

Most of the practical situations involving time series are covered by stationary time series or transformed stationary time series. Analysis of time series data consists of model identification and forecasting. The central theme of quantitative technique of forecasting is that the future can be predicted by discovering the patterns of events in the past. Model identification is the discovering of the patterns representing a given time series data. There are two equivalent forms for discrete linear models representing stationary time series. In the first form x_t is represented as a weighted sum of present and past values of the white noise process a_t. That is the model is

$$\tilde{x}_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \qquad (2.12)$$

where $\tilde{x}_t = x_t - \mu$ and $\psi_o = 1$. Using the backward shift operator (2.12) can be written as

$$\tilde{x}_t = \psi(B) a_t \qquad (2.13)$$

where

$$\psi(B) = 1 + \psi_1 B + \psi_2 B + \cdots \qquad (2.13a)$$

$\psi(B)$ is called the <u>transfer function</u> of the <u>linear filter</u>

relating $\tilde{x}_t$ to a_t. ψ_i, i = 0,1,2.... are called weights of the transfer function, where $\psi_o = 1$ always.

The second form of the model represents $\tilde{x}_t$, $\tilde{x}_t = x_t - \mu$, as a weighted sum of past values of $\tilde{x}_t$'s plus an added random shock a_t, that is

$$\tilde{x}_t = \pi_1 \tilde{x}_{t-1} + \pi_2 \tilde{x}_{t-2} + ... + a_t$$

$$= \sum_{i=1}^{\infty} \pi_i \tilde{x}_{t-i} + a_t \qquad (2.14)$$

Using the backward shift operator, equation (2.14) can be rewritten as

$$\pi(B)\, \tilde{x}_t = a_t \qquad (2.15)$$

where

$$\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - .... \qquad (2.15a)$$

The relationship between the ψ weights and π weights are given by Box and Jenkins [8].

$$\pi(B) = \psi^{-1}(B) \qquad (2.16)$$

The relationship (2.16) may be used to find the π weights, knowing the ψ weights and vice-versa.

The autocovariance γ_k at lag k of $\tilde{x}_t$ represented by (2.13) is given by

$$\gamma_k = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k}, \ldots . \tag{2.17}$$

$k = 0, \pm 1, \pm 2, \ldots .$ Substituting $k = 0$ we get $\gamma_o = \sigma_x^2$ as

$$\sigma_x^2 = \sigma_a^2 \sum_{i=1}^{\infty} \psi_i^2, \ldots \tag{2.18}$$

for σ_x^2 to be finite, the ψ weights must decrease fast. For a stationary time series variance is a constant. So it is a necessity that the convergence of (2.18) and thereby the fast decreasing of the ψ weights. The autocorrelations can be obtained as

$$\rho_k = \frac{\sigma_a^2}{\sigma_x^2} \sum_{j=0}^{\infty} \psi_j \psi_{j+k} \tag{2.19}$$

The autocovariances generating function is given by

$$\gamma(B) = \sum_{k=-\infty}^{\infty} \gamma_k B^k \tag{2.20}$$

and is shown that, (Box and Jenkins [8]) for $\tilde{x}_t$ given by (2.13), the autocorrelation generating function is of the form

$$\gamma(B) = \sigma_a^2 \psi(B) \psi(F), \tag{2.21}$$

where $F = B^{-1}$.

Considering the two equivalent forms given by (2.13) and (2.15), we can conclude that for a linear process to be stationary the generating function $\Psi(B)$ must converge for $|B| \leq 1$ and the condition for invertibility is that $\pi(B)$ must converge on or within the unit circle.

For practical purposes, it is difficult to estimate the parameters $\Psi_1, \Psi_2, \ldots$ defined in (2.13) and $\pi_1, \pi_2, \ldots$ defined in (2.15). Even if we estimate these parameters the precision of the estimated model will be less. Hence our aim is to obtain models which use parameters parsimoniously. Here we consider three different forms of linear models, which are very popular and which forms a subclass of linear models. In these models the number of parameters to be estimated is finite.

2.9 Moving-Average models

A moving average model of order q, abbreviated to MA(q), is defined as

$$\tilde{x}_t = a_t - \Theta_1 a_{t-1} - \ldots - \Theta_q a_{t-q}, \qquad (2.22)$$

and the stochastic process x_t, which can be represented by a moving-average model is called moving-average process.

The quantities $\Theta_1, \Theta_2, \ldots, \Theta_q$ are called moving-average parameters. Model defined in (2.22) can be rewritten as

$$\tilde{x}_t = \Theta(B)\, a_t, \tag{2.23}$$

where

$$\Theta(B) = 1 - \Theta_1 B - \Theta_2 B^2 - \ldots - \Theta_q B^q \tag{2.24}$$

and B is the backward shift operator. The operator $\Theta(B)$ is called the moving average operator. Comparison of the model (2.22) with the model defined by (2.13) shows that MA(q) model is a special case of the general linear model given by (2.13). Since $\Theta(B)$ is finite no condition is needed for stationarity or in other words an MA(q) model always represents a stationary process. Comparing the invertibility condition of the general models, an MA(q) model is invertible if the roots of the characteristic equation

$$\Theta(B) = 0 \tag{2.25}$$

lie outside the unit circle (Box and Jenkins [8]). The k^{th} lag autocovariance of an MA(q) process is given by

$$\gamma_k = E\left[(a_t - \Theta_1 a_{t-1} - \ldots - \Theta_q a_{t-q})(a_{t-k} - \Theta_1 a_{t-k-1} - \ldots - \Theta_q a_{t-k-q})\right] \tag{2.26}$$

On simplification we get

$$\gamma_k = \begin{cases} -\Theta_k + \Theta_1\Theta_{k+1} + \Theta_2\Theta_{k+2} + \ldots + \Theta_{q-k}\Theta_q, & k=1,2,\ldots,q. \\ 0, & k > q \end{cases} \tag{2.27}$$

and

$$\gamma_o = 1 + \Theta_1^2 + \Theta_2^2 + \ldots + \Theta_q^2 \tag{2.28}$$

From (2.27) and (2.28) we obtain the k^{th} lag autocorrelations using the relation $\rho_k = \frac{\gamma_k}{\gamma_o}$. Hence we get the theoretical autocorrelations as

$$\rho_k = \begin{cases} \dfrac{-\Theta_k + \Theta_1\Theta_{k+1} + \ldots + \Theta_{q-k}\Theta_q}{1 + \Theta_1^2 + \Theta_2^2 + \ldots + \Theta_q^2}, & k=1,2,\ldots q \\ 0, & k > q \end{cases} \tag{2.29}$$

By definition ρ_o is 1. The fact that the autocorrelations of MA(q) process is zero beyond the order q is very useful in the identification of an MA(q) model.

2.10 Autoregressive models

Autoregressive model of order p, abbreviated to AR(p), is defined as

$$\tilde{x}_t = \Phi_1\tilde{x}_{t-1} + \ldots + \Phi_p\tilde{x}_{t-p} + a_t, \tag{2.30}$$

where $\phi_1, \phi_2, \ldots, \phi_p$ are called the autoregressive parameters. Model defined in (2.30) is a special case of the general model defined by (2.15), with $\pi_i = \phi_i$, i=1,..,p and $\pi_i = 0$, $i > p$. The regression equation form of (2.30) is the reason for the name autoregressive model. Model (2.30) can be expressed using the backward shift operator as

$$\phi(B)\tilde{x}_t = a_t, \tag{2.31}$$

where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p \tag{2.32}$$

The operator $\phi(B)$ is called the autoregressive operator. As in the case of MA(q) model, comparison of an AR(p) model with general forms (2.13) and (2.15) gives AR(p) model is stationary if the roots of the characteristic equation

$$\phi(B) = 0 \tag{2.33}$$

lie outside the unit circle and since the series $\pi(B) = 1 - \phi_1 B - \ldots \phi_p B^p$ is finite and no restrictions are required on the parameters of the AR(p) model to ensure invertibility. The k^{th} lag autocovariance of an AR(p) process satisfy the following difference equation

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \ldots + \phi_p \gamma_{k-p}, \quad k > 0 \tag{2.34}$$

and

$$\gamma_o = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \ldots + \phi_p \gamma_p + \sigma_a^2 \tag{2.35}$$

On dividing each term in (2.34) by γ_o we get

$$\rho_k = \Phi_1\rho_{k-1} + \Phi_2\rho_{k-2} + \cdots + \Phi_p\rho_{k-p}, \qquad (2.36)$$

where $k > 0$.

i.e., $\Phi(B)\,\rho_k = 0, \quad k > 0 \qquad (2.37)$

where $\Phi(B)$ is the autoregressive operator defined in (2.32) and B operates on k. The solution of the difference equation (2.37) is givenbby

$$\rho_k = A_1G_1^k + A_2G_2^k + \ldots + A_pG_p^k \qquad (2.38)$$

where $G_1^{-1}, G_2^{-1}, \ldots, G_p^{-1}$ are the roots of the equation $\Phi(B) = 0$. The condition for stationarity implies that $|G_i^{-1}| > 1$, $i = 1,\ldots,p$. Using this condition we get that the autocorrelations of an AR(p) process will consist of a mixture of damped exponentials and damped sine waves. The set of p equations obtained by substituting $k = 1,2,..,p$ in (2.36) is called the Yule-Walker equations. Substituting the estimates of ρ_k and then solving the p equations, Yule-Walker estimates of the parameters can be obtained.

2.11 Autoregressive-Moving average models

Mixed autoregressive-moving average models of

order (p,q) are given by

$$\tilde{x}_t = \phi_1 \tilde{x}_{t-1} + \phi_2 \tilde{x}_{t-2} + \ldots + \phi_p \tilde{x}_{t-p} + a_t$$

$$- \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} \qquad (2.39)$$

The abbreviated form is ARMA(p,q). The process x_t represented by (2.39) is called an autoregressive-moving average process. Using the backward shift operator, the model given by (2.39) can be economically written as

$$\Phi(B)\tilde{x}_t = \Theta(B)a_t \qquad (2.40)$$

where $\Phi(B)$ and $\Theta(B)$ are as defined in (2.32) and (2.24) respectively. The stationarity of the process is ensured if the roots of the equation $\Phi(B) = 0$ lie outside the unit circle and invertibility is ensured if the roots of the equation $\Theta(B) = 0$ lie outside the unit circle. It is interesting to note that AR(p) and MA(q) models are special cases of mixed ARMA(p,q) models, which can be obtained by putting $q = 0$ and $p = 0$ respectively. That is if $q = 0$ in (2.39) then it reduces to an AR(p) model and if $p = 0$ then it reduces to a MA(q) model. Invertibility of a model guarantees a unique correspondence between the autocorrelation structure and the ARMA(p,q) model. That is, given an auto-correlation structure there exists at most one ARMA model

with that autocorrelation structure having an invertible moving average operator. The autocovariances, γ_k, of the ARMA(p,q) process satisfies the difference equation

$$\gamma_k = \phi_1\gamma_{k-1}+\ldots+\phi_p\gamma_{k-p}+\gamma_{xa}(k)-\theta_1\gamma_{xa}(k-1)\ldots.-\theta_q\gamma_{xa}(k-q) \tag{2.41}$$

where $\gamma_{xa}(k)$ is the cross covariance between $\tilde{x}_t$ and a_t at lag k and is given by

$$\gamma_{xa}(k) = 0 \quad k > 0$$
$$\gamma_{xa}(k) \neq 0 \quad k \leq 0 \tag{2.42}$$

Using (2.42) we get that the autocovariances and autocorrelations satisfy the following difference equations:

$$\gamma_k = \phi_1\gamma_{k-1}+\ldots+\phi_p\gamma_{k-p}, \quad k \geq q+1, \tag{2.43}$$

and

$$\rho_k = \phi_1\rho_{k-1}+\ldots+\phi_p\rho_{k-p}, \quad k \geq q+1 \tag{2.44}$$

In short, the autocorrelations for lags 1,2,...,q will be affected by the moving average part of the process and the autocorrelations ρ_i, $i \geq q+1$ will follow the pattern of an AR(p) process.

2.12 Partial autocorrelations

In the discussion of MA(q), AR(p) and ARMA(p,q) processes, we see that autocorrelations help to determine the order of an MA(q) process only. Partial autocorrelations defined below help to detect the order of the autoregressive part of the model. For an ARMA(p,q) process $\phi_j = 0$ for $j > p$. This idea is applied here. Partial autocorrelations, denoted by ϕ_{kk}, $k = 1,2,\ldots$ are defined as follows:

$$\phi_{11} = \rho_1$$

$$\phi_{kk} = \frac{|P_k^*|}{|P_k|} \qquad (2.45)$$

$k = 2,3,\ldots,P_k$ being the k^{th} order autocorrelation matrix and P_k^* is same as P_k except the last column, which is replaced by the vector $[\rho_1,\ldots,\rho_k]'$ and $|P_k|$, $|P_k^*|$ denote the determinants of the matrices P_k and P_k^* respectively. The partial autocorrelation ϕ_{kk} is the Yule-Walker estimate of the k^{th} autoregressive parameter, if the data represents an autoregressive process of order k. Hence for an autoregressive process of order p, the partial autocorrelations, ϕ_{kk}, $k=1,2,3\ldots$, will be nonzero for k less than or equal to p and zero for k greater than p. An MA(q) process may be written as an AR process of infinite order. So the partial autocorrelations ϕ_{kk}, $k = 1,2,3,\ldots$, of an MA(q) process should decline in

magnitude with increasing values of k and have no cut off after some lag. The partial autocorrelations of a mixed ARMA(p,q) process will follow the pattern of an AR(p) process upto the lag p and for $k > p$, it will follow the nature of the partial autocorrelations of an MA(q) process. The set $\{\phi_{kk}, k=1,2,\ldots\}$ is called the partial autocorrelation function (PACF). Estimates of ϕ_{kk}'s can be obtained by substituting the estimates of ρ_k in the matrices P_k and P_k^*.

2.13 Properties of ACF and PACF

The properties of the autocorrelations and partial autocorrelations discussed in the previous sections are summarised in table 2.1. Table 2.1 shows that the autocorrelations and partial autocorrelations are very useful tools in the analysis of time series.

2.14 Standard error of ACF and PACF

The standard error of ACF and PACF are needed in the identification procedure of a stationary time series. The variance of the estimated autocorrelations at lags k greater than some value q beyond which the theoretical ACF may be deemed to have 'died out', is given by Bartlett's approximation

$$\mathrm{Var}[r_k] \simeq \frac{1}{N}\left\{1 + 2\sum_{v=1}^{q}\rho_v^2\right\}, \quad k > q$$

$$\text{and} \quad \mathrm{S.E}[r_k] = \sqrt{\mathrm{Var}[r_k]}, \quad k > q \qquad (2.46)$$

Table 2.1 Characteristic behaviour of ACF and PACF of MA(q), AR(p) and ARMA(p,q) processes

Process	Autocorrelations (ACF)	Partial auto-correlations (PACF)
MA(q)	Spikes at lags 1 through q, then cut off	Tail off
AR(p)	Tail off according to $\rho_j = \Phi_1\rho_{j-1}+\ldots+\phi_p\rho_{j-p}$, $j > p$	Spikes at lags 1 through p and then cut off
ARMA(p,q)	Irregular pattern at lags 1 through q then tail off according to $\rho_j = \Phi_1\rho_{j-1}+\ldots+\phi_p\rho_{j-p}$	Tail off after lag p

where r_k is the estimate of ρ_k and S.E. stands for standard error. To test the hypothesis that the autocorrelations ρ_k are all essentially zero beyond some lag k = q, the standard error approximations defined in (2.46) can be used assuming the normality of the estimates. The covariance between the estimated autocorrelations r_k and r_{k+s} at two different lags k and k+s have been given by Bartlett [6] as

$$\text{Cov}\,[r_k, r_{k+s}] \simeq \frac{1}{N} \sum_{v=-\infty}^{\infty} \rho_v \rho_{v+s} \tag{2.47}$$

Standard errors of partial autocorrelations due to Quenoullie (Box and Jenkins [8]) are given by

$$\text{Var}\,[\hat{\phi}_{kk}] \simeq \frac{1}{n}, \quad k \geq p+1 \tag{2.48}$$

and $$\text{S.E}[\hat{\phi}_{kk}] \simeq \frac{1}{\sqrt{n}}, \quad k \geq p+1 \tag{2.49}$$

where $\hat{\phi}_{kk}$ is the estimate of ϕ_{kk}.

2.15 Illustrative examples

For illustration let us consider some simple examples.

Example 1. MA(1) process

The MA(1) model is given by

$$\tilde{x}_t = a_t - \theta_1 a_{t-1}$$

and the moving average operator is

$$\Theta(B) = 1 - \Theta_1 B.$$

Condition for invertibility is given as roots of $1 - \Theta_1 B = 0$ must lie outside the unit circle, which gives

$$|\Theta_1| < 1.$$

Autocorrelations are given by

$$\gamma_o = (1 + \Theta_1^2)\sigma_a^2$$

and

$$\rho_k = \begin{cases} \dfrac{-\Theta_1}{1+\Theta_1^2} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

The partial autocorrelations given by

$$\Phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_2 \\ \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdots & \\ \rho_{k-1} & \rho_{k-2} & \cdot & \cdots & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \cdot & \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdot & \cdots & \\ \rho_{k-1} & \rho_{k-2} & \cdot & \cdots & 1 \end{vmatrix}}$$

On simplification the PACF of MA(1) process satisfy the equation

$$\phi_{11} = \frac{-\theta_1}{1+\theta_1^2}$$

and

$$\phi_{kk} = -\theta_1^k \frac{1-\theta_1^2}{1-\theta_1^{2(k+1)}}$$

$k = 2,3,\ldots$

Example 2. AR(1) process

The AR(1) model is given by

$$\tilde{x}_t = \phi_1 \tilde{x}_{t-1} + a_t$$

and the AR operator is

$$\phi(B) = 1 - \phi_1 B.$$

The condition for stationarity of an AR(1) process is given by

$$-1 < \phi_1 < 1.$$

Using (2.36) we get the autocorrelations of the AR(1) process satisfy the equation

$$\rho_k = \phi_1 \rho_{k-1}, \quad k = 1,2,\ldots$$

which gives

$$\rho_1 = \phi_1$$

and

$$\rho_k = \phi_1^k$$

The partial autocorrelations are given by

$$\phi_{11} = \phi_1$$

and

$$\phi_{kk} = 0, \quad k \geq 2.$$

Example 3. ARMA(1,1) process

ARMA(1,1) model is given by

$$\tilde{x}_t = \phi_1 \tilde{x}_{t-1} + a_t - \theta_1 a_{t-1}$$

The condition for stationarity of an ARMA(1,1) process is given by $-1 < \phi_1 < 1$ and that for invertibility is given

by $-1 < \Theta_1 < 1$. Using (2.41) we get

$$\gamma_o = \Phi_1 \gamma_1 + \sigma_a^2 - \Theta_1 \gamma_{xa}(-1)$$

$$\gamma_1 = \Phi_1 \gamma_o - \Theta_1 \sigma_a^2$$

$$\gamma_k = \Phi_1 \gamma_{k-1}, \quad k \geq 2$$

but $\gamma_{xa}(-1) = (\Phi_1 - \Theta_1)\sigma_a^2$

which implies

$$\gamma_o = \frac{1+\Theta_1^2 - 2\Phi_1\Theta_1}{1-\Phi_1^2}\sigma_a^2$$

$$\gamma_1 = \frac{(1-\Phi_1\Theta_1)(\Phi_1-\Theta_1)}{1-\Theta_1^2}\sigma_a^2$$

and

$$\gamma_k = \Phi_1 \gamma_{k-1}, \quad k \geq 2.$$

Further we obtain the autocorrelations using γ_o, γ_1 and γ_k, $k \geq 2$ defined above as

$$\rho_1 = \frac{(1-\Phi_1\Theta_1)(\Phi_1-\Theta_1)}{1+\Theta_1^2-2\Phi_1\Theta_1}$$

and

$$\rho_k = \Phi_1 \rho_{k-1}, \quad k \geq 2.$$

The partial autocorrelation $\Phi_{11} = \rho_1$. After lag 1, the partial autocorrelations behave like that of MA(1).

2.16 ARIMA models

So far the discussion was concentrated on stationary time series. There are many series which are not stationary, but most of the nonstationary series exhibit homogeneity, apart from the local level and trend, namely, one part of the series behaves much like any other part. Such series are called Linear nonstationary process or homogeneous nonstationary process. These type of series can be made stationary by differencing the series suitable number of times. The condition for a series to be stationary is that the roots of the equation $\Phi(B) = 0$ must lie outside the unit circle. So if the roots lie inside or on the unit circle then the process will be non-stationa If the roots lie inside the unit circle then the process is explosive non-stationary and if the roots lie on the circle, then it corresponds to a homogeneous nonstationary process. The class of models representing the homogeneous nonstationary models are known as autoregressive integrated moving average (ARIMA) models and it is given by

$$\Phi(B)\tilde{x}_t = \Theta(B)a_t$$

where $\bar{\Phi}(B)$ is a nonstationary autoregressive operator, such that d of the roots of $\bar{\Phi}(B) = 0$ are unity and the remaining roots lie outside the unit circle. Hence it can be written as

$$\bar{\Phi}(B)\tilde{x}_t = \phi(B)(1-B)^d\tilde{x}_t = \Theta(B)a_t, \tag{2.50}$$

where $\phi(B)$ is a stationary autoregressive operator. Again (2.50) can be rewritten as

$$\phi(B)\nabla^d\tilde{x}_t = \Theta(B)a_t, \tag{2.51}$$

since $\nabla^d\tilde{x}_t = (1-B)^d\tilde{x}_t$. Equivalently, substituting $\nabla^d\tilde{x}_t = w_t$, we get

$$\phi(B)w_t = \Theta(B)a_t. \tag{2.52}$$

It is very interesting to note that the d^{th} difference of the series can be represented by a stationary invertible ARMA process. It can be seen that $\tilde{x}_t$ can be obtained by

$$\tilde{x}_t = s^d w_t \tag{2.53}$$

where, d = 1 gives

$$^{s}w_t = \sum_{j=-\infty}^{t} \tilde{x}_j$$

$$= (1+B+B^2+\ldots)\tilde{x}_t$$

$$= (1-B)^{-1}\tilde{x}_t$$

$$= \nabla^{-1}\tilde{x}_t$$

repeating the process $\tilde{x}_t$ will be represented by an infinite sum, that is, the process is called an autoregressive integrated moving average process usually denoted by ARIMA(p,d,q) where p is the order of the stationary autoregressive operator, d is the order of differencing to obtain stationarity and q is the order of the moving average part. The three different sub-classes of ARIMA(p,d,q) are ARI(p,d), IMA(d,q) and ARIMA(p,d,q). Estimates of autocorrelations of an ARIMA(p,d,q) process are as defined in section 2.5, but the pattern of the correlogram will be different from that of a stationary process. If the series represent a nonstationary process, then the autocorrelations will remain large even at long lag. This property can be used to detect the nonstationarity as well as the order, d, of differencing to remove nonstationarity of a series.

2.17 Spectral density function

The previous discussions reveal that a time series may be described by its autocorrelation structure in the time domain. Equivalently a series may be depicted by its spectral density, that is the Fourier transform of the ACF in the frequency domain.

Let $\{\gamma_k,\ k = 0, \pm1, \pm2, \ldots\}$ be the autocovariance function of a discrete stationary time series, then the power spectrum of the series is defined as the Fourier transform of the autocovariance function, that is the power spectrum, $P(\lambda)$, is defined by

$$P(\lambda) = \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\lambda k}, \tag{2.54}$$

where $\lambda \in [-\pi, \pi]$. The spectral density function $s(\lambda)$, is defined as the Fourier transform of the ACF, $\{\rho_k;\ k=0,\pm1,\pm2,\ldots\}$ and is given by

$$s(\lambda) = \sum_{k=-\infty}^{\infty} \rho_k e^{-i\lambda k}, \tag{2.55}$$

where $\lambda \in [-\pi, \pi]$. Since ρ_k is an even function, $\rho_k = \rho_{-k}$, (2.55) becomes

$$s(\lambda) = 1 + 2 \sum_{k=1}^{\infty} \rho_k \cos(\lambda k), \tag{2.56}$$

$\lambda \in [-\pi, \pi]$

The property that the autocorrelations of a stationary time series decrease rapidly implies that

$$|s(\lambda)| \leq \sum_{k=-\infty}^{\infty} |\rho_k| < \infty \tag{2.57}$$

and (2.57) implies that $s(\lambda)$, the spectral density function of a stationary time series converges uniformly. Using the Fourier inverse transformation the autocorrelations can be obtained from the spectral density function by

$$\rho_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\lambda k) s(\lambda) d\lambda \tag{2.58}$$

The spectral distribution function is defined as

$$S(\lambda) = \int_{-\pi}^{\lambda} s(v) dv \tag{2.59}$$

Further $s(\lambda) \geq 0$, Anderson [5], when $S(\lambda)$ is absolutely continuous and

$$\int_{-\pi}^{\pi} S(\lambda) d\lambda = 1. \tag{2.60}$$

With this knowledge of the spectral density function of a stationary time series, the spectral density function of a linear process can be investigated.

Consider the autocovariance generating function

$$\gamma(B) = \sum_{k=-\infty}^{\infty} \gamma_k B^k \tag{2.61}$$

(For details see Box and Jenkins [8] and Anderson [5]). Substituting $B = e^{-i\lambda}$ in (2.61) the equation becomes

$$\gamma(e^{-i\lambda}) = \sum_{k=-\infty}^{\infty} \gamma_k e^{-ik\lambda} \tag{2.62}$$

The right hand side of equation (2.62) is the expression of power spectrum i.e.,

$$\gamma(e^{-i\lambda}) = P(\lambda) \tag{2.63}$$

But it is shown that, Box and Jenkins [8], for a linear process defined by (2.13)

$$\gamma(B) = \sigma_a^2 \psi(B) \psi(B^{-1}) \tag{2.64}$$

Hence using (2.62), (2.63) and (2.64), we get

$$P(\lambda) = \sigma_a^2 \psi(e^{-i\lambda}) \psi(e^{i\lambda})$$

$$\text{i.e, } P(\lambda) = \sigma_a^2 \left|\psi(e^{-i\lambda})\right|^2. \tag{2.65}$$

From the definitions of the power spectrum and the spectral

density function we get

$$s(\lambda) = \frac{P(\lambda)}{\sigma_x^2} \tag{2.66}$$

hence the spectral density function of a linear process represented by (2.13) is

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \left| \psi(e^{-i\lambda}) \right|^2 \tag{2.67}$$

The spectral density function of a linear process obtained in (2.67) is of great importance. For an MA(q) model, by definition

$$\psi(B) = \theta(B),$$

which along with (2.67) gives the spectral density function of an MA(q) process as

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \left| \theta(e^{-i\lambda}) \right|^2 \tag{2.68}$$

On simplification (2.68) becomes

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \left[1 + \sum_{i=1}^{q} \theta_i^2 + 2 \sum_{i=1}^{q} \left(-\theta_i + \sum_{j=1}^{q-i} \theta_j \theta_{j+i} \right) \cos(i\lambda) \right] \tag{2.69}$$

where $\lambda \in [-\pi, \pi]$. For an AR(p) process $\psi(B) = \phi^{-1}(B)$, which

along with (2.67) gives the spectral density function of an AR(p) process as

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \frac{1}{|\phi(e^{i\lambda})|^2}, \tag{2.70}$$

which on simplification becomes

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \frac{1}{1+\sum_{i=1}^{p} \phi_i^2 + 2\sum_{i=1}^{p} (-\phi_i + \sum_{j=1}^{p-i} \phi_j \phi_{j+i}) \cos(i\lambda)} \tag{2.71}$$

$\lambda \in [-\pi, \pi]$.

Mixed ARMA(p,q) model provides

$$\psi(B) = \frac{\Theta(B)}{\phi(B)}$$

Substituting this $\psi(B)$ in (2.67) we get the spectral density function of the ARMA(p,q) process as

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \frac{|\Theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \tag{2.72}$$

and the simplified form of $s(\lambda)$ is

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \frac{1+\sum_{i=1}^{q} \Theta_i^2 + 2\sum_{i=1}^{q} (-\Theta_i + \sum_{j=1}^{q-i} \Theta_j \Theta_{j+i}) \cos(i\lambda)}{1+\sum_{i=1}^{p} \phi_i^2 + 2\sum_{i=1}^{q} (-\phi_i + \sum_{j=1}^{p-i} \phi_j \phi_{j+1}) \cos(i\lambda)} \tag{2.73}$$

where $\lambda \in [-\pi, \pi]$.

The different rational forms of the spectral density function defined by (2.69), (2.71) and (2.73) of MA(q), AR(p) and ARMA(p,q) processes respectively, are used to develop the new model identification procedure explained in the next chapter.

Chapter 3

THEORETICAL DEVELOPMENT OF R-SPEC PROCEDURE FOR ARMA(p,q) MODEL IDENTIFICATION*

3.1 Introduction to R-spec procedure

A time series may be described by its autocorrelation structure. In chapter 1 and chapter 2 we have discussed several techniques of model identification. In the last part of chapter 2, we see that there is one-one correspondence between spectral density function and its ARMA(p,q) model. In this chapter a new method is developed to estimate the rational form of the spectral density function of a given stationary time series. As a result of it, a new method is developed to estimate the parameters along with the order (p,q) of an ARMA(p,q) model. The spectral density function of a given time series is estimated using its autocorrelation function.

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* Results of this chapter were presented as papers by the author, entitled (1) "ARMA models from spectral density function--An alternative to Box-Jenkins" in the Third Annual Conference of ISTPA at Delhi during August 1981, and (2) "Rational approximation of the spectral density function of stationary time series--Unique rational approximation", in the Sixth Annual Conference of ISTPA at Trivandrum during December 1984.

The Chebyshev minimax approximation as a ratio of polynomials is effected for the spectral density function. Using this rational approximation of the spectral density function, the R-spec method for ARMA(p,q) model identification procedure is developed. The uniqueness of the Chebyshev approximation gives a unique ARMA(p,q) model.

3.2 Spectral density function and its estimate

The spectral density function, $s(\lambda)$, of a stationary time series is defined in (2.56) as the Fourier transform of the autocorrelation function and it is given by

$$s(\lambda) = 1 + 2 \sum_{i=1}^{\infty} \rho_i \cos(i\lambda) , \qquad (3.1)$$

$$\lambda \in [-\pi, \pi].$$

The estimate $\hat{s}(\lambda)$ of $s(\lambda)$ can be obtained by substituting the estimated autocorrelation function, $\{r_k,\ k=0,1,2,..\}$.

$$\text{i.e., } \hat{s}(\lambda) = 1 + 2 \sum_{k=1}^{N} r_k \cos(k\lambda), \qquad (3.2)$$

where r_k, $k=1,...,N$ are the estimates of ρ_k, $k=1,2,...N$.

3.3 Rational approximation of continuous functions

The notion of rational approximation of continuous functions is considered here. According to Pade' approximation theory a continuous function f(x) in an interval [a,b] and having the Maclaurin series representation,

$$f(x) = \sum_{j=0}^{\infty} c_j x^j \tag{3.3}$$

in the neighbourhood of zero can be approximated to a rational function of the form,

$$\frac{P_m(x)}{Q_k(x)},$$

where $P_m(x) = \sum_{j=0}^{m} a_j x^j$ (3.4)

and $Q_k(x) = \sum_{j=0}^{k} b_j x^j$ (3.5)

The expansion of f(x) in a series of Chebyshev polynomials is given by

$$f(x) = \frac{1}{2} c_o + \sum_{j=1}^{\infty} c_j T_j(x) \tag{3.6}$$

where

$$T_j(x) = \cos(r \cos^{-1} x) \tag{3.7}$$

Now Chebyshev expansion of f(x) is used to generate a rational approximation in a manner analogous to that used to generate Pade' approximation from Maclaurin series expansion and the rational form is

$$T_k^m(x) = \frac{\sum_{j=0}^{m} a_j T_j(x)}{\sum_{j=0}^{k} b_j T_j(x)}, \tag{3.8}$$

where a_j, $j=1,\ldots,m$ and $b_j, j=1,\ldots,k$ are to be determined. The equations (3.7) and (3.8) takes elegant forms

$$f(\cos\lambda) = \frac{1}{2} c_o + \sum_{j=1}^{\infty} c_j \cos(j\lambda) \tag{3.9}$$

and

$$T_k^m(\cos\lambda) = \frac{\sum_{j=0}^{m} a_j \cos(j\lambda)}{\sum_{j=0}^{k} b_j \cos(j\lambda)}, \quad \lambda \in [-\pi,\pi], \tag{3.10}$$

by substituting $x = \cos\lambda$ in (3.7) and (3.8) and using the fact that $T_j(\cos\lambda) = \cos(j\lambda)$.

Substituting $c_i = \rho_i$, $i=1,2,\ldots$, in (3.9) we get

$$f(\cos\lambda) = \frac{1}{2} \rho_o + \sum_{i=1}^{\infty} \rho_i \cos(i\lambda) \tag{3.11}$$

Equations (3.1) and (3.11) gives

$$s(\lambda) = 2\, f(\cos\lambda) \tag{3.12}$$

3.4 Main theorem

Given a set of observations $x_1, x_2, \ldots, x_N$, at equal intervals of time, from a stationary time series, there exists a unique best fit ARMA(p,q) model obtained from the estimated rational form of the spectral density function.

For the proof of the main theorem we need the following lemmas (See Blum [7]) and the proof of them are given in the appendix.

Lemma 3.1

For every function $f \in C[a,b]$ there is a best uniform rational approximation in the class $R_m^n[a,b]$, where $R_m^n[a,b]$ is defined as the set $R_m^n[a,b] = \{P/Q:$ deg $P \leq n$, deg $Q \leq m$ and $Q(x) > 0\}$ and by a best approximating rational function for f on [a,b] we mean a rational function $R^* \in R_m^n[a,b]$ such that

$$\sup_{x \in [a,b]} |f(x) - R^*(x)| \leq \sup_{x \in [a,b]} |f(x) - R(x)|$$

holds for all rational functions $R \in R_m^n[a,b]$. Again by best uniform rational approximation we mean, the best approximating rational function for all $x \in [a,b]$.

Lemma 3.2

Let $R^* = \frac{P^*}{Q^*} \in R_m^n[a,b]$ and let $f \in C[a,b]$.

The residue $r^* = f - R^*$ assumes its extreme values $\pm \|f-R^*\|_\infty$ with successively alternating signs on at least $2 + \max\{n+\deg Q^*, m + \deg P^*\}$ points in $[a,b]$ if and only if R^* is the best approximation to f in $R_m^n[a,b]$, where f is a real measurable function of x and the essential supremum of $|f|$ is denoted by $\|f\|_\infty$.

Lemma 3.3

The best rational approximation in $R_m^n[a,b]$ of a continuous function is unique.

Proof of the Main theorem

For a given stationary time series let the autocorrelation function be $\{\rho_k, k=0, \pm1, \pm2, \ldots\}$. The Fourier transform of the autocorrelation function given by (3.1) is known as the spectral density function of the series. In section 2.17 it is shown that the spectral density function

of an ARMA(p,q) process can be expressed as a rational function given by

$$s(\lambda) = \frac{\sigma_a^2\left[1+\sum_{i=1}^{q}\theta_i^2+2\sum_{i=1}^{q}(-\theta_i+\sum_{j=1}^{q-i}\theta_j\theta_{j+1})\cos(i\lambda)\right]}{\sigma_x^2\left[1+\sum_{i=1}^{p}\phi_i^2+2\sum_{i=1}^{p}(-\phi_i+\sum_{j=1}^{p-i}\phi_j\phi_{j+i})\cos(i\lambda)\right]} \qquad (3.13)$$

The right hand side of (3.13) is apparently a rational function of $\cos\lambda$. The estimate of $s(\lambda)$ in terms of autocorrelation function is a polynomial in $\cos\lambda$. This suggests that we can find the best uniform and unique rational approximation of $s(\lambda)$ to obtain the corresponding ARMA(p,q) model. Substituting the estimate r_k of ρ_k, k=1,2,3,...and ρ_o=1 in the expression of the spectral density function $s(\lambda)$ given in (3.1) we obtain the estimate of $s(\lambda)$ and $f(\cos\lambda)$.

Let $T_k^l(\cos\lambda)$ be the initial rational approximation of $f(\cos\lambda)$, obtained, using the minimax approximation procedure due to Pade'. (Ralston [31]). (The algorithm to obtain T_k^l is given in section 3.4). $T_k^l(\cos\lambda)$ is given by

$$T_k^l(\cos\lambda) = \frac{\sum_{i=0}^{l} a_i\ (\cos i\lambda)}{\sum_{i=0}^{k} b_i\ \cos(i\lambda)}, \qquad (3.14)$$

b_o= 1. Let N = 1+k and let $\{\lambda_i^{(0)};\ i=1,\ldots,N+1\}$ be a sequence of points in $[-\pi,\pi]$ such that

$$-\pi \le \lambda_o^{(0)} < \lambda_1^{(0)} < \ldots < \lambda_{N+1}^{(0)} \le \pi$$

Determine the coefficients $a_j^{(0)}$, j=0,1,...,l, $b_j^{(0)}$, j=1,2,...,k

and a quantity $e^{(0)}$ such that

$$f(\cos \lambda_i^{(0)} - T_k^l(\cos \lambda_i^{(0)}) = (-1)^i e^{(0)}, \qquad (3.15)$$

$0 \leq i \leq N+1$, $f(\cos \lambda_i^{(0)})$ is defined in (3.11) and $T_k^l(\cos \lambda) \in R_m^n [-\pi, \pi]$. If

$$r_1(\lambda) = f(\cos \lambda) - T_k^l(\cos \lambda) \qquad (3.16)$$

then $e^{(0)}$ serves as local extremum of $r_1(\lambda)$ with alternate signs. This leads to a system of nonlinear equations given by

$$\sum_{j=0}^{l} a_j \cos(j \lambda_i^{(0)}) - \left[f(\cos \lambda_i^{(0)}) + (-1)^i e^{(0)}\right]\left[\sum_{j=0}^{k} b_j \cos(j \lambda_i^{(0)})\right] = 0 \qquad (3.17)$$

where $0 \leq i \leq N+1$ and $b_o = 1$. When these N+2 equations in N+2 unknowns $a_o^{(0)}, a_1^{(0)}, \ldots, a_l^{(0)}; b_1^{(0)}, b_2^{(0)}, b_3^{(0)}, \ldots, b_k^{(0)}$ and $e^{(0)}$ are solved, we compute $r_1(\lambda)$. Since $r_1(\lambda)$ is continuous with at least N+2 alternating signs, using multiple exchange algorithm or second algorithm of Remés, (Blum [7], Ralston [31]), we get another set of points $\lambda_o^{(1)}, \lambda_1^{(1)}, \ldots \lambda_{N+1}^{(1)}$ on which $r_1(\lambda)$ alternates in sign with greater extreme (absolute) value. The second set of λ values can be used to

generate $a_j^{(1)}$'s and $b_j^{(1)}$'s, which will then determine $e^{(1)}$. Continue this procedure until $e^{(n)}$ appears to have converged. This gives the converged coefficients $a_o^{(n)}$, $a_1^{(n)}$,...., $a_l^{(n)}$; $b_1^{(n)}$, $b_2^{(n)}$,...., $b_k^{(n)}$ and $e^{(n)}$, which determine the best uniform and unique rational approximation of $f(\cos\lambda)$ using lemma 3.1, lemma 3.2 and lemma 3.3.

Let $T_k^{l*}(\cos\lambda)$ be the best uniform and unique rational approximation of $f(\cos\lambda)$. Then $T_k^{l*}(\cos\lambda)$ is given by

$$T_k^{l*}(\cos\lambda) = \frac{\sum_{i=0}^{l} a_i^{(n)} \cos(i\lambda)}{\sum_{i=0}^{k} b_i^{(n)} \cos(i\lambda)} \qquad (3.18)$$

$b_o = 1$. Using (3.12) and (3.18) we get

$$\hat{s}(\lambda) = 2T_k^{l*}(\cos\lambda)$$

$$\text{i.e., } \hat{s}(\lambda) = 2\,\frac{\sum_{i=0}^{l} a_i^{(n)} \cos(i\lambda)}{\sum_{i=0}^{k} b_i^{(n)} \cos(i\lambda)} \qquad (3.19)$$

Using (3.13) and (3.19) we get the order of the ARMA(p,q) model as

$$p = k$$

and $$q = l \qquad (3.20)$$

also we get the following equation

$$\frac{\sigma_a^2 \left[1+\sum_{i=1}^{q}\theta_i^2+2\sum_{i=1}^{q}\left(-\theta_i+\sum_{j=1}^{q-i}\theta_j\theta_{j+i}\right)\cos(i\lambda)\right]}{\sigma_x^2 \left[1+\sum_{i=1}^{p}\phi_i^2+2\sum_{i=1}^{p}\left(-\phi_i+\sum_{j=1}^{p-i}\phi_j\phi_{j+i}\right)\cos(i\lambda)\right]} = 2\,\frac{\sum_{i=1}^{q} a_i^{(n)}\cos(i\lambda)}{\sum_{i=1}^{p} b_i^{(n)}\cos(i\lambda)} \tag{3.21}$$

where $b_o = 1$ and $\lambda \in [-\pi,\pi]$

Equation (3.21) gives rise to a set of equations, by equating the coefficients of $\cos(j\lambda)$ on both sides in the numerator and denominator. The equations thus obtained are given by

$$2a_o^{(n)} = \frac{\sigma_a^2}{\sigma_x^2}\,\frac{\left[1+\sum_{i=1}^{q}\theta_i^2\right]}{\left[1+\sum_{i=1}^{p}\phi_i^2\right]} \tag{3.22}$$

$$\frac{a_i^{(n)}}{2a_o^{(n)}} = \frac{-\theta_i+\sum_{j=1}^{q-i}\theta_j\theta_{j+i}}{1+\sum_{i=1}^{q}\theta_i^2} \tag{3.23}$$

$i = 1,2,\ldots,q$ and

$$\frac{b_i^{(n)}}{2} = \frac{-\phi_i+\sum_{j=1}^{p-i}\phi_j\phi_{j+i}}{1+\sum_{i=1}^{p}\phi_i^2} \tag{3.24}$$

$i = 1,2,\ldots,p.$

Rearranging equations (3.23) and (3.24) we get

$$\Theta_i = -\frac{a_i^{(n)}}{2a_o^{(n)}}\left(1+\sum_{i=1}^{q}\Theta_i^2\right)+\sum_{j=1}^{q-i}\Theta_j\Theta_{j+i}\,, \qquad (3.25)$$

$$i=1,2,\ldots q$$

and

$$\Phi_i = -\frac{b_i^{(n)}}{2}\left(1+\sum_{i=1}^{p}\Phi_i^2\right)+\sum_{j=1}^{p-i}\Phi_j\Phi_{j+i} \qquad (3.26)$$

$$i=1,2,\ldots p.$$

Let $\bar{\Phi} = [\Phi_1, \Phi_2, \ldots, \Phi_p]'$. (3.27)

Equate (3.27) and (3.26) which implies that

$$\bar{\Phi} = \bar{f}(\bar{\Phi}) \qquad (3.28)$$

The following iteration procedure is developed to obtain $\hat{\bar{\Phi}}$, the estimate of the autoregressive parameters of the ARMA(p,q) model. Substitute $\Phi_i = 0$, i=1,2,..p, in the right hand side of (3.26). Then we get the initial value of $\bar{\Phi}$ as

$$\bar{\Phi}_o = \left[-\frac{b_1^{(n)}}{2}, \frac{-b_2^{(n)}}{2}, \ldots, \frac{-b_p^{(n)}}{2}\right]' \qquad (3.29)$$

Then using the initial value $\bar{\Phi}_o$ defined in (2.29), apply the iteration successively so that

$$\bar{\Phi}_i = \bar{f}(\bar{\Phi}_{i-1}), \qquad (3.30)$$

where i stands for the number of iteration, $\bar{\Phi}_o$ is as defined in (2.29) and i=1,2,..,n+1. Continue the iteration until we get $\bar{\Phi}_{n+1} = \bar{\Phi}_n$.

Hence we obtain,

$$\hat{\bar{\Phi}} = \bar{\Phi}_n \tag{3.31}$$

where $\bar{\Phi}_n$ is the converged value of the AR parameters in the iteration procedure.

Similarly, let

$$\bar{\Theta} = [\theta_1, \theta_2, \ldots, \theta_q]' \tag{3.32}$$

using (3.25) we get

$$\bar{\Theta} = \bar{g}(\bar{\Theta}) \tag{3.33}$$

Substitute $\Theta_i = 0$, i=1,2,..,q in the right hand side of (3.25), then we get the initial value of $\bar{\Theta}$ as

$$\bar{\Theta}_o = \left[\frac{-a_1^{(n)}}{2a_o^{(n)}}, \frac{-a_2^{(n)}}{2a_o^{(n)}} \ldots, \frac{-a_q^{(n)}}{2a_o^{(n)}}\right]' \tag{3.34}$$

The iteration procedure explained in the case of the estimation

of AR parameters is repeated to obtain the estimates of the moving-average parameter. Starting the iteration with $\bar{\Theta}_o$ defined in (3.34) as initial value we obtain

$$\bar{\Theta}_i = \bar{g}(\bar{\Theta}_{i-1}),$$

where i stands for the number of iteration and m is the number of iteration such that,

$$\bar{\Theta}_{m+1} = \bar{g}(\bar{\Theta}_m) \quad \text{and} \quad \bar{\Theta}_{m+1} = \bar{\Theta}_m$$

Hence we obtain the estimate $\hat{\bar{\Theta}}$ as

$$\hat{\bar{\Theta}} = \bar{\Theta}_m$$

After obtaining $\hat{\bar{\Phi}}$ and $\hat{\bar{\Theta}}$, σ_a^2 is obtained from (3.22) by substituting $\hat{\bar{\Phi}}$ and $\hat{\bar{\Theta}}$ we get $\hat{\bar{\Phi}} = \bar{\Phi}_n$. Then using equation (3.27) we get

$$\hat{\bar{\Phi}} = [\Phi_1^{(n)}, \Phi_2^{(n)}, \ldots, \Phi_p^{(n)}]' \tag{3.35}$$

and

$$\hat{\bar{\Theta}} = \bar{\Theta}_m$$

implies

$$\hat{\bar{\Theta}} = [\Theta_1^{(m)}, \Theta_2^{(m)}, \ldots, \Theta_q^{(m)}]' \tag{3.36}$$

Equations (3.35) and (3.36) determines the unique ARMA(p,q) model, namely,

$$\tilde{x}_t = \phi_1^{(n)} \tilde{x}_{t-1} + \phi_2^{(n)} \tilde{x}_{t-2} + \ldots + \phi_p^{(n)} \tilde{x}_{t-p} + a_t - \theta_1^{(m)} a_{t-1} \ldots - \theta_q^{(m)} a_{t-q} \tag{3.37}$$

which proves the theorem.

3.5 R-Spec procedure for ARMA(p,q) model identification

The R-spec procedure for identifying the ARMA(p,q) model representing a given time series has the following steps.

Step 1

Given a time series data find the estimates r_k's of the autocorrelations. Then check the stationarity of the data using the S.E. of the estimated r_k's. If the data is not stationary, obtain the transformed stationary series (See note

Step 2

After finding the estimates of the ACF of a stationary time series, find the estimates of the PACF.

Step 3

Find the cut off points of both ACF and PACF. Graphs of ACF and PACF can be used for this purpose.

Step 4

To start the algorithm to obtain the rational approximation of the spectral density function, it is required to obtain m and n, the upper bounds for the degrees of denominator and numerator respectively. Behaviour of both the graphs of the ACF and the PACF exhibit the nature of damped exponentials and damped sine waves after a finite number of points, if the data represents a mixture of autoregressive and moving average phenomenon. The PACF will have a cut off after a finite number of terms in the case of an AR model while its ACF will exhibit the nature of damped exponentials. If the data represents an MA model then its ACF will have a cut off after a finite number of terms and the PACF resembles the nature of the ACF of an AR process. Hence a reasonably good cut off point, say Q, in the graph of the ACF gives an upper bound for q, the order of the moving average part and a similar point, say P, in the graph of the PACF gives an upper bound for p, the order of the autoregressive part. Now take m = P and n = Q. With these values of m and n

Minimax-approximation procedure due to Padé is applied to obtain the initial rational approximation of the spectral density function. The initial rational approximations, which are of the form $T_k^l(\cos\lambda)$, $0 \le l \le Q$ and $0 \le k \le P$ given by

$$T_k^l(\cos\lambda) = \frac{\sum_{j=0}^{l} a_j \cos(j\lambda)}{\sum_{j=0}^{k} b_j \cos(j\lambda)} \tag{3.38}$$

where $b_o = 1$ and $\lambda \in [-\pi, \pi]$, are found with the help of the following equations (Ralston [31]).

$$a_o = \frac{1}{2} \sum_{j=0}^{k} \rho_j b_j \tag{3.39}$$

$$a_r = \frac{1}{2} \sum_{j=0}^{k} b_j (\rho_{|r-j|} + \rho_{r+j}), \quad r=1,\ldots l \tag{3.40}$$

and

$$0 = \frac{1}{2} \sum_{j=0}^{k} b_j (\rho_{|r-j|} + \rho_{r+j}), \quad r=l+1,\ldots,l+k \tag{3.41}$$

Estimate the coefficients of $T_k^l(\cos\lambda)$ for all orders $0 \le l \le Q$ and $0 \le k \le P$, using the set of equations (3.39), (3.40) and (3.41). Then find the error of each $T_k^l(\cos\lambda)$ from $f(\cos\lambda)$ at sufficiently large number of points, so that we obtain the extreme values of the error for each

$T_k^1(\cos\lambda)$. Select that $T_k^1(\cos\lambda)$ with minimum of maximum error, as the initial rational approximation.

Step 5

After obtaining the initial rational approximation of the spectral density function the next step is to obtain the improved form of the rational approximation applying the main theorem.

Let $N = 1+k$ and let $\{\lambda_i^{(0)},\ i=0,1,2,\ldots n+1\}$ be a sequence of points in $[-\pi,\pi]$ such that

$$-\pi \le \lambda_0^{(0)} < \lambda_1^{(0)} \ldots < \lambda_{n+1}^{(0)} \le \pi.$$

Determine the coefficients $a_j^{(0)}$, $j=0,1,\ldots,l$; $b_j^{(0)}$, $j=1,2,\ldots$ and a quantity $e_o^{(0)}$ such that

$$f(\cos\lambda_i^{(0)}) - T_k^1(\cos\lambda_i^{(0)}) = (-1)^i e^{(0)} \qquad (3.42)$$

$0 \le i \le n+1$ and $f(\cos\lambda) = \frac{s(\lambda)}{2}$ and $T_k^1(\cos\lambda) \in R_P^Q[-\pi,\pi]$

If $r_1(\lambda) = f(\cos\lambda) - T_k^1(\cos\lambda)$, then $e_o^{(0)}$ serves as local extrema with alternate signs. This leads to a system of nonlinear equations given by

$$\sum_{j=0}^{l} a_j^{(0)} \cos(j\lambda_i^{(0)}) - [f(\cos\lambda_i^{(0)}) + (-1)^i e^{(0)}] \sum_{j=0}^{k} b_j^{(0)} \cos(j\lambda \qquad (3.43)$$

$0 \le i \le n+1$, $b_o = 1$. To start the computation procedure

substitute the extremum value of the error of $T_k^l(\cos\lambda)$ from $f(\cos\lambda)$ equal to $e_o^{(0)}$ and take $\lambda_i^{(0)}$, i=0,1,...,n+1, the corresponding extremum points, then solve the system of nonlinear equations given by (3.43) to obtain $a_o^{(1)}$, $a_1^{(1)},\ldots,a_l^{(1)}$, $b_1^{(1)}$, $b_2^{(1)},\ldots,b_k^{(1)}$. Using these estimated $a_i^{(1)}$'s and $b_j^{(1)}$'s determine $e^{(1)}$. Then applying the Reme's second algorithm (see appendix) we obtain another set of points $\lambda_o^{(1)}$, $\lambda_1^{(1)},\ldots,\lambda_{n+1}^{(1)}$, on which $r_1(\lambda)$ alternates sign with greater extreme (absolute) value. Then using the second set of values determine

$$a_o^{(2)}, a_1^{(2)},\ldots, a_l^{(2)}; \quad b_1^{(2)}, b_2^{(2)},\ldots, b_k^{(2)}$$

This procedure is to be continued until $e^{(n)}$ gets converged. Let $T_k^{l*}(\cos\lambda)$ be the rational approximation, determined by the converged coefficients.

$$\text{i.e., } T_k^{l*}(\cos\lambda) = \frac{a_o^{(n)} + a_1^{(n)}\cos\lambda + \ldots + a_l^{(n)}\cos(l\lambda)}{1 + b_1^{(n)}\cos\lambda + \ldots + b_k^{(n)}\cos(k\lambda)} \tag{3.44}$$

Thus we obtain the unique rational approximation of the spectral density function as

$$\hat{s}(\lambda) = 2T_k^{l*}(\cos\lambda) \tag{3.45}$$

Application of the main theorem gives that the order of the autoregressive part is p = k and the order of the moving average part is q = 1. Hence we get

$$T_p^q(\cos\lambda) = T_k^{1*}(\cos\lambda) \text{ with } q = 1 \text{ and } p = k.$$

and

$$T_p^q(\cos\lambda) = \frac{\sum_{j=0}^{q} a_j^{(n)} \cos(j\lambda)}{\sum_{j=0}^{p} b_j^{(n)} \cos(j\lambda)} \tag{3.46}$$

$b_o = 1$ and n being the number of the last iteration. Hence we obtain the estimate of the best uniform and unique rational approximation of the spectral density function as

$$\hat{s}(\lambda) = 2T_p^q(\cos\lambda) \tag{3.47}$$

Step 6 Estimation of the ARMA(p,q) parameters

Using the rational form of the spectral density function given in (3.12) and the estimated form (3.44) we get the following system of equations, by equating the coefficients of $\cos(j\lambda)$ on both sides in the numerator and denominator.

$$2a_o^{(n)} = \frac{\sigma_a^2\left[1+\sum_{i=1}^{q}\Theta_i^2\right]}{\sigma_x^2\left[1+\sum_{j=1}^{p}\Phi_j^2\right]} \tag{3.48}$$

$$\frac{a_i^{(n)}}{2a_o^{(n)}} = \frac{-\theta_i + \sum_{j=1}^{q-i} \theta_j \theta_{j+i}}{1 + \sum_{j=1}^{q} \theta_j^2} \qquad (3.49)$$

$$i=1,\ldots,q$$

and

$$\frac{b_i^{(n)}}{2} = \frac{-\phi_i + \sum_{j=1}^{p-i} \phi_j \phi_{j+i}}{1 + \sum_{j=1}^{p} \phi_j^2}, \quad i=1,\ldots,p \qquad (3.50)$$

On rearranging (3.48), (3.49) and (3.50) we get

$$\sigma_a^2 = \frac{2a_o^{(n)} \sigma_x^2 \left[1 + \sum_{j=1}^{p} \phi_j^2\right]}{\left[1 + \sum_{j=1}^{q} \theta_j^2\right]} \qquad (3.51)$$

$$\theta_i = \frac{-a_i^{(n)}}{2a_o^{(n)}} \left[1 + \sum_{j=1}^{q} \theta_j^2\right] + \sum_{j=1}^{q-i} \theta_j \theta_{j+i} \qquad (3.52)$$

and

$$\phi_i = \frac{-b_i^{(n)}}{2} \left[1 + \sum_{j=1}^{p} \phi_j^2\right] + \sum_{j=1}^{p-i} \phi_j \phi_{j+i} \qquad (3.53)$$

To obtain the estimates of θ_i, $i=1,\ldots,q$ and ϕ_i, $i=1,\ldots,p$, we apply the iteration procedure explained in the proof of the theorem.

Let $\bar{\Phi} = [\Phi_1, \Phi_2, \ldots, \Phi_p]'$, then from equation (3.53) we get

$$\bar{\Phi} = \bar{f}(\bar{\Phi}) \tag{3.54}$$

Put $\Phi_i = 0, i=1,\ldots,p$ in the right hand side of (3.53), then the initial value Φ_o of $\bar{\Phi}$ is obtained as

$$\bar{\Phi}_o = \left[\frac{-b_1^{(n)}}{2}, \frac{-b_2^{(n)}}{2}, \ldots, \frac{-b_p^{(n)}}{2} \right]' \tag{3.56}$$

Then substitute these values of Φ_i's in (3.53) to obtain $\bar{\Phi}_1$. i.e., we get

$$\bar{\Phi}_1 = \bar{f}(\bar{\Phi}_o)$$

Then find $\bar{\Phi}_2 = \bar{f}(\bar{\Phi}_1)$, $\bar{\Phi}_3 = \bar{f}(\bar{\Phi}_2), \ldots, \bar{\Phi}_{n+1} = \bar{f}(\bar{\Phi}_n)$.

Continue the iteration procedure until we get $\bar{\Phi}_{n+1} = \bar{\Phi}_n$. This converged value $\bar{\Phi}_n = [\Phi_1^{(n)}, \Phi_2^{(n)}, \ldots, \Phi_p^{(n)}]'$ is the estimates of the parameters of the AR part of the model.

Next, we obtain the estimates of the moving average parameters. Let $\bar{\Theta} = [\Theta_1, \Theta_2, \ldots, \Theta_q]'$, then (3.52) implies

$$\bar{\Theta} = \bar{g}(\bar{\Theta})$$

We obtain the initial value $\bar{\theta}_o$ as

$$\bar{\Theta}_o = \left[\frac{-a_1^{(n)}}{2a_o^{(n)}}, \frac{-a_2^{(n)}}{2a_o^{(n)}}, \ldots, \frac{-a_q^{(n)}}{2a_o^{(n)}}\right]'$$

by substituting $\Theta_i = 0$, i=1,...,q, in the right hand side of (3.52). Repeat the iteration procedure explained in the case of AR parameters and obtain the converged value $\bar{\theta}_m = [\Theta_1^{(m)}, \Theta_2^{(m)}, \ldots, \Theta_q^{(m)}]'$. Using the estimated parameters $\bar{\Phi}_n$ and $\bar{\theta}_m$ in (3.51), the estimate of σ_a^2 is obtained. Thus the estimated parameters are

$$\hat{\bar{\Phi}} = [\Phi_1^{(n)}, \Phi_2^{(n)}, \ldots, \Phi_p^{(n)}]'$$

$$\hat{\bar{\Theta}} = [\Theta_1^{(m)}, \Theta_2^{(m)}, \ldots, \Theta_q^{(m)}]'$$

i.e.,

$$\begin{aligned} \hat{\phi}_1 &= \phi_1^{(n)} \\ \hat{\phi}_2 &= \phi_2^{(n)} \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ \hat{\phi}_p &= \phi_p^{(n)} \end{aligned} \qquad (3.60)$$

and

$$\begin{aligned}
\hat{\Theta}_1 &= \Theta_1^{(m)} \\
\hat{\Theta}_2 &= \Theta_2^{(m)} \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
\hat{\Theta}_q &= \Theta_q^{(m)}
\end{aligned} \tag{3.61}$$

Further using $\hat{\Theta}_i$'s and $\hat{\Phi}_i$'s we obtain

$$\hat{\sigma}_a^2 = \frac{2a_o^{(n)} \hat{\sigma}_x^2 \left[1 + \sum_{i=1}^{p} \hat{\Phi}_i^2\right]}{1 + \sum_{i=1}^{q} \hat{\Theta}_i^2} \tag{3.62}$$

Equation (3.60),(3.61) and (3.62) determines the estimated ARMA(p,q) model and the estimated unique ARMA(p,q) model is given by

$$\tilde{x}_t = \hat{\Phi}_1 \tilde{x}_{t-1} + \hat{\Phi}_2 \tilde{x}_{t-2} + \cdots + \hat{\Phi}_p \tilde{x}_{t-p} + a_t - \hat{\theta}_1 a_{t-1} \cdots - \hat{\Theta}_q a_{t-q} \tag{3.63}$$

Note 1.

If the given time series is not stationary, but homogeneous non-stationary we transform the series to

stationary series by repeated differencing. If d is the number differencing, d = 0,1,2,..., then the transformed stationary time series will be $w_t = \nabla^d x_t$. Then we apply the identification procedure to the differenced series.

Chapter 4

DATA ANALYSIS WITH R-SPEC MODEL IDENTIFICATION PROCEDURE

In chapter 3 the R-spec model identification procedure using the rational form of the spectral density function is presented. In this chapter the R-spec technique is tested using theoretical autocorrelations of MA, AR and ARMA models, autocorrelations of simulated series and using some observed time series data. The R-spec procedure is developed for the model identification of stationary time series. In the case of homogeneous nonstationary time series, the series will be transformed to stationary series first and then the analysis will be carried out. After obtaining the stationary time series data, the R-spec procedure has the following steps. (1) Obtain the estimates of the ACF and the PACF and then determine P and Q, the upper bounds for p and q respectively. (2) Find the estimate of the rational form of the spectral density function using the procedure developed in chapter 3. (3) Using the estimated rational form of the spectral density function (obtained in step 2) estimate the parameters of the ARMA model.

In chapter 2, AR(p), MA(q) and ARMA(p,q) models as well as their autocorrelations are defined. An MA(q)

model is given by

$$\tilde{x}_t = a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q} \tag{4.1}$$

and the k^{th} lag autocorrelation of an MA(q) model is defined as

$$\left.\begin{aligned} \rho_k &= \frac{-\theta_k + \theta_1\theta_{k+1} + \ldots + \theta_{q-k}\theta_q}{1 + \sum_{i=1}^{q} \theta_i^2}, \quad k=1,2,\ldots q \\ \rho_k &= 0, \quad k > q \end{aligned}\right\} \tag{4.2}$$

The partial autocorrelations are defined as

$$\phi_{kk} = \frac{|P_k^*|}{|P_k|}, \quad k = 1,2,\ldots \tag{4.3}$$

where P_k and P_k^* are as defined in chapter 2. For a given MA(q) model we obtain the theoretical ACF and PACF using (4.2) and (4.3) respectively.

An AR(p) model is given by

$$\tilde{x}_t = \phi_1 \tilde{x}_{t-1} + \ldots + \phi_p \tilde{x}_{t-p} + a_t, \tag{4.4}$$

and its k^{th} lag autocorrelations are given by

$$\rho_k = \phi_1 \rho_{k-1} + \ldots + \phi_p \rho_{k-p}, \quad k > 0 \tag{4.5}$$

Hence for a given AR(p) model the ACF and PACF are determined using (4.5) and (4.3) respectively.

An ARMA(p,q) model is defined by

$$\tilde{x}_t = \phi_1 \tilde{x}_{t-1} + \ldots + \phi_p \tilde{x}_{t-p} + a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q} \tag{4.6}$$

and its ACF is given by

$$\rho_k = \phi_1 \rho_{k-1} + \ldots + \phi_p \rho_{k-p} + \frac{\gamma_{xa}(k)}{\gamma_o} - \frac{\theta_1 \gamma_{xa}(k-1)}{\gamma_o} - \ldots - \frac{\theta_q \gamma_{xa}(k-q)}{\gamma_o}, \tag{4.7}$$

where

$$\gamma_o = \phi_1 \gamma_1 + \ldots + \phi_p \gamma_p + \sigma_a^2 - \theta_1 \gamma_{xa}(-1) - \ldots - \theta_q \gamma_{xa}(-q), \tag{4.8}$$

and

$$\gamma_{xa}^{(k)} = \begin{cases} 0, & k > 0 \\ \neq 0 & k \leq 0 \end{cases}, \tag{4.9}$$

$$\gamma_{xa}(k) = E\left[\tilde{x}_{t-k}\, a_t\right]$$

From equation (4.7) ρ_k can be simplified as

$$\rho_k = \phi_1 \rho_{k-1} + \ldots + \phi_p \rho_{k-p}, \quad k \geq p+1 \tag{4.10}$$

Therefore using (4.7), (4.8), (4.9) and (4.10) the ACF of an ARMA(p,q) process can be determined. The expression for PACF is same as in (4.3).

4.1 Examples

In this section the R-spec technique is tested using theoretical autocorrelations of nine models.

Example 4.1

The model used in the first example is an MA(1) model given by

$$\tilde{x}_t = a_t - 0.5a_{t-1} \tag{4.11}$$

The ACF and PACF of (4.11) are given in table 4.1 and table 4.2 respectively. From the correlogram (fig.4.1(a)) and the graph of the PACF (fig.4.1(b)) we see that the ACF has a cut off after the first lag and the PACF tapers off. Hence P = 0 and Q = 1. In this case there is only one order namely (0,1) for the initial rational form of the spectral density function. Using (3.39) and (3.40) we get the initial rational form of the spectral density function as

$$T_o^1(\cos\lambda) = 0.5 - 0.4\cos(\lambda), \tag{4.12}$$

$\lambda \in [-\pi, \pi]$

But the spectral density function is given by

$$s(\lambda) = 2\left[0.5 - 0.4\cos(\lambda)\right] \tag{4.13}$$

From equations (4.12) and (4.13), the difference between $s(\lambda)$ and $2T_o^1(\cos\lambda)$ is obtained as

$$f(\cos\lambda) - T_o^1(\cos\lambda) = 0 \tag{4.14}$$

for all λ and $f(\cos\lambda) = \frac{s(\lambda)}{2}$. Since $T_o^1(\cos\lambda)$, the initial rational approximation of the spectral density function itself is identically equal to $f(\cos\lambda)$ no improvement of $T_o^1(\cos\lambda)$ is needed and hence the estimate of the rational form of the spectral density function is

$$T_o^{1*}(\cos\lambda) = T_o^1(\cos\lambda)$$

$$\text{i.e., } T_o^{1*}(\cos\lambda) = 0.5 - 0.4\cos\lambda \tag{4.15}$$

Using (4.15) the following equations are obtained:

$$a_o = 0.5$$
$$a_1 = -0.4 \tag{4.16}$$

Using (4.16), (3.51) and (3.52) the following equations are

obtained:

$$\sigma_a^2 = \sigma_x^2(1 + \theta_1^2)2a_o \tag{4.17}$$

and

$$\theta_1 = 0.4(1 + \theta_1^2) \tag{4.18}$$

Applying the method of iteration to equation (4.18) the estimate of θ_1 is obtained. Table 4.3 gives the iteration for θ_1. From table 4.3 the estimated value of θ_1 is

$$\hat{\theta}_1 = 0.4999999$$

Then using $\hat{\theta}_1$ and (4.17) the estimate of σ_a^2 is obtained as

$$\hat{\sigma_a^2} = 1.0$$

Substituting $\hat{\theta}_1$, the estimated model is obtained as

$$\hat{\tilde{x}}_t = a_t - 0.4999999a_{t-1} \tag{4.19}$$

Table 4.4 displays the difference between the original values of θ_1 and σ_a^2 and their estimates respectively.

Example 4.2

An MA(2) model is taken as the second example. The model used is

$$\tilde{x}_t = a_t - 0.7a_{t-1} + 0.12a_{t-2} \tag{4.20}$$

Tables 4.5 and 4.6 give the ACF and PACF of x_t represented by (4.20). Fig.4.2(a) is the correlogram and fig.4.2(b) is the graph of the PACF. Fig.4.2(a) shows that the ACF has a cut off after the second term and fig.4.2(b) shows that its PACF tapers off. So the upper bounds for p and q are taken as P = 0 and Q = 2. Using (3.39) and (3.40) the initial estimates of the rational forms of the spectral density function are obtained as

$$T_o^1(\cos\lambda) = 0.5 - 0.5211379\cos(\lambda) \qquad (4.21)$$

and

$$T_o^2(\cos\lambda) = 0.5 - 0.5211379\cos(\lambda) + 0.079768\cos(2\lambda) \qquad (4.22)$$

Substituting the values of $\hat{\rho}_k$'s in the expression of $s(\lambda)$, the expected spectral density function, and using (4.22) it is seen that $f(\cos\lambda) - T_o^2(\cos\lambda) = 0$ for all values of λ. The differences of $T_o^1(\cos\lambda)$ and $T_o^2(\cos\lambda)$ from $f(\cos\lambda)$ are given in table 4.7. Since $f(\cos\lambda) - T_o^2(\cos\lambda) = 0$ for all values of λ, $2T_o^2(\cos\lambda)$ is selected as the initial rational approximation of $s(\lambda)$ and further we see that no improvement is needed for $T_o^2(\cos\lambda)$, since the maximum error is zero, i.e., $2T_o^{2*}(\cos\lambda) = 2T_o^2(\cos\lambda)$. Equation (4.21) determines p = 0 and q = 2 and the following equations:

$$\sigma_a^2 = \frac{2a_o\sigma_x^2}{1+\theta_1^2+\theta_2^2}, \tag{4.23}$$

$$\Theta_1 = 0.5211379(1 + \Theta_1^2+ \Theta_2^2)+ \Theta_1\Theta_2 \tag{4.24}$$

and

$$\Theta_2 = -0.079766(1 + \Theta_1^2+ \Theta_2^2) \tag{4.25}$$

Table 4.8 is the iteration table for Θ_1 and Θ_2. The estimated values of Θ_1 and Θ_2 are

$$\hat{\Theta}_1 = 0.6999997$$

$$\hat{\Theta}_2 = -0.1199999$$

From (4.23) estimate of σ_a^2 is obtained by substituting $\hat{\Theta}_1$ and $\hat{\Theta}_2$ and it is given by

$$\hat{\sigma}_a^2 = 1.0000002$$

Hence the estimated model is given by

$$\hat{\hat{x}}_t = a_t - 0.6999997a_{t-1} + 0.1199999a_{t-2} \tag{4.26}$$

Table 4.9 is the error table of the estimated MA(2) parameters from their respective original values.

Example 4.3

An MA(3) model is considered as the third example and the model used is given by

$$\tilde{x}_t = a_t - 0.4a_{t-1} + 0.2a_{t-2} - 0.15a_{t-3} \qquad (4.27)$$

Tables 4.10 and 4.11 give the ACF and PACF of the model given by (4.27). From the correlogram (fig.4.4(a)) and the graph of the PACF (fig.4.4(b)) the upper bounds of p and q are obtained as P = 0 and Q = 3. Hence $T_o^1(\cos\lambda)$, $T_o^2(\cos\lambda)$ and $T_o^3(\cos\lambda)$ are considered for initial rational approximation of the spectral density function. The initial forms are given by

$$T_o^1(\cos\lambda) = 0.5 - 0.4171779\cos(\lambda) \qquad (4.28)$$

$$T_o^2(\cos\lambda) = 0.5 - 0.4171779\cos(\lambda) + 0.2126789\cos(2\lambda) \qquad (4.29)$$

and

$$T_o^3(\cos\lambda) = 0.5 - 0.4171779\cos(\lambda) + 0.2126789\cos(2\lambda) - 0.1226993\cos(3\lambda) \qquad (4.30)$$

Now $\hat{s}(\lambda)$, the expected spectral density function is given by

$$\hat{s}(\lambda) = 1 + 2\left[-0.4171779\cos(\lambda) + 0.2126789\cos(2\lambda) - 0.1226993\cos(3\lambda)\right] \qquad (4.31)$$

Table 4.12 gives the differences of $T_o^1(\cos\lambda)$, $T_o^2(\cos\lambda)$ and $T_o^3(\cos\lambda)$ from $f(\cos\lambda)$ and it gives that $T_o^3(\cos\lambda)$ has got the minimum maximum error. Since the difference $f(\cos\lambda) - T_o^3(\cos\lambda) = 0$ for all λ, we get $T_o^{3*}(\cos\lambda) = T_o^3(\cos\lambda)$ i.e., the estimated rational form is $T_o^3(\cos\lambda)$ itself. Equation (4.30) determines the order of the model as $p = 0$ and $q = 3$. Further (4.30) determines the following equations:

$$\sigma_a^2 = \frac{\sigma_x^2}{1+\theta_1^2+\theta_2^2+\theta_3^2} \tag{4.32}$$

$$\theta_1 = 0.4171779(1+\theta_1^2+\theta_2^2+\theta_3^2)+\theta_1\theta_2+\theta_2\theta_3 \tag{4.33}$$

$$\theta_2 = -0.2126789(1+\theta_1^2+\theta_2^2+\theta_3^2)+\theta_1\theta_3 \tag{4.34}$$

$$\theta_3 = 0.1226993(1+\theta_1^2+\theta_2^2+\theta_3^2) \tag{4.35}$$

The method of iteration is applied to equations (4.33), (4.34) and (4.35). Table 4.13 gives the iteration table for θ_1, θ_2 and θ_3. The estimated values of θ_1, θ_2 and θ_3 obtained from table 4.13 are:

$$\hat{\theta}_1 = 0.3999999$$

$$\hat{\theta}_2 = -0.2000000$$

$$\hat{\theta}_3 = 0.1499998$$

Substituting $\hat{\Theta}_1$, $\hat{\Theta}_2$ and $\hat{\Theta}_3$ in (4.32) $\hat{\sigma_a^2}$ is obtained as

$$\hat{\sigma_a^2} = 1.0000001$$

Table 4.14 gives the differences of $\hat{\Theta}_1$, $\hat{\Theta}_2$, $\hat{\Theta}_3$ and $\hat{\sigma_a^2}$ from their original values.

Example 4.4

The model used in this example is given by

$$\tilde{x}_t = a_t + 0.7a_{t-1} + 0.25a_{t-2} - 0.08a_{t-3} + 0.05a_{t-4} \qquad (4.36)$$

Table 4.15 gives the ACF and table 4.16 gives the PACF of the series represented by (4.36). From the correlogram (fig.4.6(a)) and the graph of the PACF (fig.4.6(b)) the upper bound Q = 4 and the upper bound P = 0. The initial rational approximations of the spectral density function are given by

$$T_o^1(\cos\lambda) = 0.5 + 0.5501473\cos(\lambda) \qquad (4.37)$$

$$T_o^2(\cos\lambda) = 0.5 + 0.5501473\cos(\lambda) + 0.1162418\cos(2\lambda) \qquad (4.38)$$

$$T_o^3(\cos\lambda) = 0.5 + 0.5501473\cos(\lambda) + 0.1162418\cos(2\lambda) - 0.07365\cos(3\lambda) \qquad (4.39)$$

and $T_o^4(\cos\lambda) = 0.5 + 0.5501473\cos(\lambda) + 0.1162418\cos(2\lambda)$

$$- 0.07365\cos(3\lambda) + 0.032025\cos(4\lambda) \qquad (4.40)$$

Now substituting ρ_k, k=1,2,...in the expression of $s(\lambda)$, the spectral density function, we get $\hat{s}(\lambda) = 2T_o^4(\cos\lambda)$ Table 4.17 represents the differences of $T_o^1(\cos\lambda)$, $T_o^2(\cos\lambda)$ $T_o^3(\cos\lambda)$ and $T_o^4(\cos\lambda)$ from $f(\cos\lambda)$. Fig.4.7 is the graph of the differences given in table 4.17. From table 4.17 we get the maximum error corresponding to $T_o^4(\cos\lambda)$ is zero and it is minimum when compared to the maximum errors due to $T_o^1(\cos\lambda)$, $T_o^2(\cos\lambda)$ and $T_o^3(\cos\lambda)$. In this case we get $\hat{s}(\lambda) = 2T_o^4(\cos\lambda)$ which implies no improvement is needed for $T_o^4(\cos\lambda)$. i.e., the rational approximation of $s(\lambda)$ is $T_o^4(\cos\lambda)$given in (4.40). From (4.40) we get the order of the ARMA(p,q) model as p = 0 and q = 4 and also the following equations.

$$\sigma_a^2 = \frac{\sigma_x^2}{1 + \sum_{i=1}^{4} \theta_i^2} \qquad (4.41)$$

$$\theta_1 = -0.5501473\left(1 + \sum_{i=1}^{4} \theta_i^2\right) + \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_4 \qquad (4.42)$$

$$\theta_2 = -0.1162418\left(1 + \sum_{i=1}^{4} \theta_i^2\right) + \theta_1\theta_3 + \theta_2\theta_4 \qquad (4.43)$$

$$\Theta_3 = -0.07365(1 + \sum_{i=1}^{4} \Theta_i^2) + \Theta_1\Theta_4 \qquad (4.44)$$

and

$$\Theta_4 = -0.032025(1 + \sum_{i=1}^{4} \Theta_i^2) \qquad (4.45)$$

Table 4.18 is the iteration table, obtained by applying the method of iteration to the equations (4.42), (4.43), (4.44) and (4.45). The estimates of the MA(4) parameters obtained from table 4.18 are

$$\hat{\Theta}_1 = -0.6999992$$

$$\hat{\Theta}_2 = -0.2499962$$

$$\hat{\Theta}_3 = 0.079943$$

$$\hat{\Theta}_4 = -0.0500037$$

and the estimated value of σ_a^2 using equation (4.41) is

$$\hat{\sigma}_a^2 = 1.0000002.$$

Table 4.19 gives the errors between the original values and the estimated values of Θ_1, Θ_2, Θ_3, Θ_4 and σ_a^2 respectively.

Example 4.5

An AR(1) model is considered here. The model

used is given by

$$\tilde{x}_t = -0.35\tilde{x}_{t-1} + a_t \qquad (4.46)$$

The ACF and PACF of the model given in (4.46) is given in table 4.20 and table 4.21 respectively. Fig.4.8(a) is the correlogram and fig.4.8(b) is the graph of the PACF. From these two figures we take P = 1 and Q = 2. Various initial rational forms for the spectral density function for $p \leq 1$ and $q \leq 2$ are given as

$$T_o^1(\cos\lambda) = 0.5 - 0.35\cos(\lambda) \qquad (4.47)$$

$$T_o^2(\cos\lambda) = 0.5 - 0.35\cos(\lambda) + 0.1225\cos(2\lambda) \qquad (4.48)$$

$$T_1^o(\cos\lambda) = \frac{0.3908686}{1 + 0.623608\cos(\lambda)} \qquad (4.49)$$

$$T_1^1(\cos\lambda) = \frac{0.441667 - 0.3258334\cos(\lambda)}{1 + 0.3333333\cos(\lambda)} \qquad (4.50)$$

$$T_1^2(\cos\lambda) = \frac{0.3908701 - 4.38E{-}06\cos(\lambda) + 3.75E{-}06\cos(2\lambda)}{1 + 0.6235991\cos(\lambda)} \qquad (4.51)$$

Equations (4.49) and (4.51) show that $T_1^o(\cos\lambda) \simeq T_1^2(\cos\lambda)$. Hence we need not find the errors of $T_1^2(\cos\lambda)$ from $f(\cos\lambda)$. Table 4.22 gives the differences of various initial rational forms from $f(\cos\lambda)$ and column 3 of the table shows the

differences $f(\cos\lambda) - T_1^o(\cos\lambda) = 0$, for all values of λ, which gives that $2T_1^o(\cos\lambda)$ is the initial rational approximation of $s(\lambda)$, the spectral density function. Since the error of $2T_1^o(\cos\lambda)$ from $s(\lambda)$ is zero for all values of λ in $[-\pi,\pi]$ we get $T_1^{o*}(\cos\lambda)$ is equal to $T_1^o(\cos\lambda)$. Hence the estimated rational form of the spectral density function is

$$2T_1^{o*}(\cos\lambda) = \frac{2 \times 0.3908686}{1 + 0.623608 \cos(\lambda)} \qquad (4.52)$$

From (4.52) the order of the ARMA(p,q) model is obtained as $p = 1$ and $q = 0$. Further (4.52) determines the following equations:

$$\sigma_a^2 = 2 \times 0.390866\, \sigma_x^2 (1 + \phi_1^2) \qquad (4.53)$$

and

$$\phi_1 = -\frac{0.623608}{2}(1 + \phi_1^2) \qquad (4.54)$$

Table 4.23 is the iteration table for ϕ_1, obtained using equation (4.54). The estimated value of ϕ_1 obtained from table 4.23 is

$$\hat{\phi}_1 = -0.3499999.$$

Substituting $\hat{\phi}_1$ in (4.53) the estimate of σ_a^2 is obtained as

$$\hat{\sigma_a^2} = 0.9999993.$$

Hence the estimated model is given by

$$\hat{\tilde{x}}_t = -0.3499999\tilde{x}_{t-1} + a_t \qquad (4.55)$$

Table 4.24 gives the errors of $\hat{\phi}_1$ and $\hat{\sigma}_a^2$ from ϕ_1 and σ_a^2 respectively.

Example 4.6

The model considered is given by

$$\tilde{x}_t = 0.5\tilde{x}_{t-1} + a_t \qquad (4.56)$$

The ACF and PACF of (4.56) is given in table 4.26 and table 4.27 respectively. From the correlogram (fig.4.10(a)) and the graph of the PACF (fig.4.10(b)) of (4.56) the upper bounds for p and q are obtained as P = 1 and Q = 2. Here we have to find $T_o^1(\cos\lambda)$, $T_1^o(\cos\lambda)$, $T_1^1(\cos\lambda)$, $T_o^2(\cos\lambda)$ and $T_1^2(\cos\lambda)$

$$T_o^1(\cos\lambda) = 0.5 + 0.5\cos(\lambda) \qquad (4.57)$$

$$T_1^o(\cos\lambda) = \frac{0.3}{1 - 0.8\cos(\lambda)} \qquad (4.58)$$

$$T_1^1(\cos\lambda) = \frac{0.3 + 0 \times \cos(\lambda)}{1 - 0.8\cos(\lambda)}$$

$$\text{i.e., } T_1^1(\cos\lambda) = \frac{0.3}{1 - 0.8\cos(\lambda)}$$

$$\text{i.e., } T_1^1(\cos\lambda) = T_1^o(\cos\lambda) \tag{4.59}$$

$$T_o^2(\cos\lambda) = 0.5 + 0.5\cos(\lambda) + 0.25\cos(2\lambda) \tag{4.60}$$

and

$$T_1^2(\cos\lambda) = \frac{0.3 + 0x\cos(\lambda) + 0x\cos(2\lambda)}{1 - 0.8\cos(\lambda)}$$

$$\text{i.e., } T_1^2(\cos\lambda) = \frac{0.3}{1 - 0.8\cos(\lambda)}$$

$$\text{i.e., } T_1^2(\cos\lambda) = T_1^o(\cos\lambda) \tag{4.61}$$

Table 4.27 gives the differences of $T_o^1(\cos\lambda)$, $T_1^o(\cos\lambda)$ and $T_o^2(\cos\lambda)$ from $f(\cos\lambda)$ and it shows that column 2, the error column corresponding to $T_1^o(\cos\lambda)$ has got minimum of maximum errors. From table 4.27 we obtain the maximum value of $f(\cos\lambda) - T_1^o(\cos\lambda)$ is equal to 0.3E-5 and the points at which the error attains the extremum values are $-\pi$, 0 and π. Hence substituting $E^{(0)}$ = 0.3E-5 in the equation

$$f^{(0)}(\cos\lambda_i) - \left[T_1^o(\cos\lambda_i)\right]^{(0)} = E^{(0)} \tag{4.62}$$

(the suffice '(0)' stands for the initial iteration and $\lambda_i = -\pi$, 0 and π) and solving (4.62) we get a_o = 0.3,

$b_1 = -0.8$ and $E^{(1)} = 0.3E\text{-}5$. This solution gives that

$$\left[T_1^o(\cos\lambda)\right]^{(1)} = \frac{0.3}{1 - 0.8\cos(\lambda)} \tag{4.63}$$

Now from (4.58) and (4.63) we see that the iteration converges at the first step itself. Table (4.28) gives the iteration values for $T_1^{o*}(\cos\lambda)$ and we get the estimated rational form of the spectral density function is given by

$$2T_1^{o*}(\cos\lambda) = 2\left[T_1^o(\cos\lambda)\right]^{(1)} \tag{4.64}$$

which implies

$$\hat{s}(\lambda) = 2 \times \frac{0.3}{1 - 0.8\cos(\lambda)} \tag{4.65}$$

Using (4.65) the order of the ARMA(p,q) model is determined as $p = 1$ and $q = 0$. Also (4.65) gives the following equations

$$\sigma_a^2 = 2 \times 0.3\, \sigma_x^2(1 + \phi_1^2) \tag{4.66}$$

and

$$\phi_1 = 0.4(1 + \phi_1^2) \tag{4.67}$$

Table 4.29 is the iteration table for ϕ_1 obtained using (4.67) and the estimated value of ϕ_1 is given by

$$\hat{\phi}_1 = 0.4999999.$$

Then from (4.66) $\hat{\sigma}_a^2$ is obtained as

$$\hat{\sigma}_a^2 = 1.0001249.$$

Table 4.30 gives the error between the estimated and original values of the parameters.

Example 4.7

An AR(2) model is considered here. The model considered is given by

$$\tilde{x}_t = 0.75\tilde{x}_{t-1} - 0.14\tilde{x}_{t-2} + a_t \tag{4.68}$$

The ACF and PACF of (4.68) is given in table 4.31 and table 4.32 respectively. From the correlogram (fig.4.12(a)) and the graph of the PACF (fig.4.12(b)) the upper bounds of p and q are obtained as P = 2 and Q = 1. So the initial rational approximations $T_k^l(\cos\lambda)$, $0 \leq l \leq 1$ and $0 \leq k \leq 2$ are considered. The initial rational forms are given by

$$T_o^1(\cos\lambda) = 0.5 + 0.6578947\cos(\lambda) \tag{4.69}$$

$$T_1^o(\cos\lambda) = \frac{0.1801986}{1 - 0.9721956\cos(\lambda)} \tag{4.70}$$

$$T_1^1(\cos\lambda) = \frac{0.2201512 + 0.082198\cos(\lambda)}{1 - 0.8507403\cos(\lambda)} \tag{4.71}$$

$$T_2^o(\cos\lambda) = \frac{0.1757341}{1 - 1.080842\cos(\lambda) + 0.17698\cos(2\lambda)} \tag{4.72}$$

$$T_2^1(\cos\lambda) = \frac{0.1757336 - 0.00051858\cos(\lambda)}{1 - 1.080842\cos(\lambda) + 0.1769795\cos(2\lambda)} \tag{4.73}$$

Table 4.33 gives the differences of $T_k^1(\cos\lambda)$'s given by (4.69), (4.70), (4.71), (4.72) and (4.73) from $f(\cos\lambda)$ and it shows that the maximum error corresponding to $T_2^o(\cos\lambda)$ is minimum when compared to the maximum error due to other rational forms. Hence $2T_2^o(\cos\lambda)$ is taken as the initial rational approximation of the spectral density function. Here we get the maximum value of $|f(\cos\lambda) - T_2^o(\cos\lambda)|$ at $\lambda = 0$ and the maximum value is equal to 0.000599. From the graph of the errors (fig.4.13) of the rational forms from $f(\cos\lambda)$, we find that the errors have alternative signs at $-\pi$, -0.9π, -0.8π,...,-0.1π, 0, 0.1π,...,π. The algorithm for obtaining the improved form of the R-spec is applied here. $E^{(0)}$ is taken as 0.0002 which is less than the maximum value obtained in table 4.33. To solve for a_o, b_1 and b_2 we need three equations. Here we select the points 0, 0.5π and π and $E^{(0)} = 0.0002$ to start the algorithm. Table 4.34 gives the values obtained. The maximum error is now at 0.1π and the value of the maximum error is 0.000586. Table 4.34

also shows that the coefficients b_1, b_2 and a_o are converged upto four decimal places. Hence the converged rational form is given by

$$T_2^{o*}(\cos\lambda) = \frac{0.1757}{1 - 1.0809\cos(\lambda) + 0.1770\cos(2\lambda)} \quad (4.74)$$

The estimated rational form of the spectral density function given in (4.74) determines p = 2 and q = 0, the order of the model. Also (4.74) determines the following equations:

$$\Phi_1 = \frac{1.0809}{2}(1 + \Phi_1^2 + \Phi_2^2) + \Phi_1\Phi_2 \quad (4.75)$$

$$\Phi_2 = -\frac{0.1770}{2}(1 + \Phi_1^2 + \Phi_2^2) \quad (4.76)$$

and

$$\sigma_a^2 = 2 \times 0.1757 \times \sigma_x^2(1 + \Phi_1^2 + \Phi_2^2) \quad (4.77)$$

Table 4.35 gives the iteration for Φ_1 and Φ_2 using the equations (4.75) and (4.76). The estimated values of Φ_1 and Φ_2 are given by

$$\hat{\Phi}_1 = 0.7500838$$

and $$\hat{\Phi}_2 = -0.1400271$$

Then using $\hat{\Phi}_1$ and $\hat{\Phi}_2$ in (4.77), $\hat{\sigma}_a^2$ is obtained as

$$\hat{\sigma}_a^2 = 0.9998062$$

and the model estimated is

$$\hat{\tilde{x}}_t = 0.75008\tilde{x}_{t-1} - 0.14002\tilde{x}_{t-2} + a_t \qquad (4.78)$$

The differences of $\hat{\Phi}_1$, $\hat{\Phi}_2$ and $\hat{\sigma}_a^2$ from Φ_1, Φ_2 and σ_a^2 are given in table 4.35.

Example 4.8

An ARMA(1,1) model as given by

$$\tilde{x}_t = 0.43\tilde{x}_{t-1} + a_t - 0.05a_{t-1} \qquad (4.79)$$

is considered here. Table 4.36 contains the ACF and table 4.37 contains the PACF of (4.79). The correlogram (fig.4.14(a)) and the graph of the PACF (fig.4.14(b)) show that P = 1 and Q = 2. The various initial rational forms are

$$T_o^1(\cos\lambda) = 0.5 + 0.3875247\cos(\lambda) \qquad (4.80)$$

$$T_o^2(\cos\lambda) = 0.5 + 0.3875247\cos(\lambda) + 0.1666356\cos(2\lambda) \qquad (4.81)$$

$$T_1^o(\cos\lambda) = \frac{0.469916}{1 - 0.5712608\cos(\lambda)} \qquad (4.82)$$

$$T_1^1(\cos\lambda) = \frac{0.3591675 - 0.035847\cos(\lambda)}{1 - 0.7257995\cos(\lambda)} \qquad (4.83)$$

$$T_1^2(\cos\lambda) = \frac{0.359366 - 0.03585\cos(\lambda) + 1.6E{-}06\cos(2\lambda)}{1 - 0.7258066\cos(\lambda)} \qquad (4.84)$$

Expressions of $T_1^1(\cos\lambda)$ in (4.83) and $T_1^2(\cos\lambda)$ in (4.84) show that $T_1^2(\cos\lambda) \simeq T_1^1(\cos\lambda)$. So $T_1^2(\cos\lambda)$ need not be considered. Table 4.38 gives the differences of $T_0^1(\cos\lambda)$, $T_0^2(\cos\lambda)$, $T_1^0(\cos\lambda)$ and $T_1^1(\cos\lambda)$ from $f(\cos\lambda)$. The graph of these errors are given in fig.4.15. Table 4.38 shows that the maximum error corresponding to $T_1^1(\cos\lambda)$ is minimum when compared to the maximum errors of other columns. Hence $T_1^1(\cos\lambda)$ is taken as the initial rational approximation. Further the maximum error corresponding to $T_1^1(\cos\lambda)$ is equal to 0.000001 which is very small. So $T_1^1(\cos\lambda)$ is taken as $T_1^{1*}(\cos\lambda)$. Hence the estimated rational form of the spectral density function is given by

$$\hat{s}(\lambda) = 2T_1^1(\cos\lambda)$$

$$\text{i.e., } \hat{s}(\lambda) = \frac{2\left[0.3591675 - 0.035847\cos(\lambda)\right]}{1 - 0.7257995\cos(\lambda)} \qquad (4.85)$$

Equation (4.85) determines that the order of the ARMA model is p = 1 and q = 1. Comparing the coefficients in (4.85) with the corresponding coefficients of the theoretical

rational form of the spectral density function of an ARMA(1,1) model, the following equations are obtained.

$$\sigma_a^2 = \frac{2a_o \sigma_x^2 (1 + \Phi_1^2)}{(1 + \Theta_1^2)} \tag{4.86}$$

$$\Phi_1 = \frac{0.7257995}{2}(1 + \Phi_1^2) \tag{4.87}$$

and

$$\Theta_1 = \frac{0.035847}{2x0.3591675}(1 + \Theta_1^2) \tag{4.88}$$

The method of iteration is applied to (4.87) and (4.88). The iteration table of Φ_1 is table 4.39 and that of Θ_1 is table 4.40. The estimated values of Φ_1 and Θ_1 are

$$\hat{\Phi}_1 = 0.4299997$$

$$\hat{\Theta}_1 = 0.0499997.$$

Substituting $\hat{\Phi}_1$ and $\hat{\Theta}_1$ in (4.86) the estimate of σ_a^2 is obtained as

$$\hat{\sigma}_a^2 = 1.0000027$$

Now the estimated model using R-spec procedure is given by

$$\hat{\tilde{x}}_t = 0.4299997\tilde{x}_{t-1} + a_t - 0.04999997a_{t-1} \tag{4.89}$$

Table 4.41 shows the error between the estimated values and the original values of the ARMA(1,1) parameters.

Example 4.9

The model used is ARMA(2,1) given by

$$\tilde{x}_t = -0.8\tilde{x}_{t-1} - 0.12\tilde{x}_{t-2} + a_t - 0.02a_{t-1} \qquad (4.90)$$

The ACF and the PACF of (4.90) are given in table 4.42 and table 4.43 respectively. The correlogram (fig.4.16(a)) and the graph of the PACF (fig.4.16(b)) show that P = 2 and Q = 2. The initial rational forms obtained are given by

$$T_o^1(\cos\lambda) = 0.5 - 0.725941\cos(\lambda) \qquad (4.91)$$

$$T_o^2(\cos\lambda) = 0.5 - 0.725941\cos(\lambda) + 0.4607528\cos(2\lambda) \qquad (4.92)$$

$$T_1^o(\cos\lambda) = \frac{0.1392337}{1+0.9939272\cos(\lambda)} \qquad (4.93)$$

$$T_1^1(\cos\lambda) = \frac{0.1679876 - 0.0578591\cos(\lambda)}{1 + 0.914709\cos(\lambda)} \qquad (4.94)$$

$$T_2^o(\cos\lambda) = \frac{0.1358823}{1 + 1.1097737\cos(\lambda) + 0.1679748\cos(2\lambda)} \qquad (4.95)$$

$$T_2^1(\cos\lambda) = \frac{0.1402517 - 0.0078743\cos(\lambda)}{1 + 1.0832257\cos(\lambda) + 0.1451134\cos(2\lambda)} \tag{4.96}$$

$$T_2^2(\cos\lambda) = \frac{0.1402997 - 0.0079631\cos(\lambda) + 0.000015\cos(2\lambda)}{1+1.0829709\cos(\lambda) + 0.1449202\cos(2\lambda)} \tag{4.97}$$

Considering $T_2^1(\cos\lambda)$ and $T_2^2(\cos\lambda)$ we see that $a_2 \simeq 0$ and all other corresponding coefficients in $T_2^2(\cos\lambda)$ is equal to $T_2^1(\cos\lambda)$ upto three decimal places, which implies that $T_2^2(\cos\lambda) \simeq T_2^1(\cos\lambda)$. Table 4.44 gives the differences of $T_o^2(\cos\lambda)$, $T_1^1(\cos\lambda)$, $T_1^o(\cos\lambda)$, $T_2^o(\cos\lambda)$ and $T_2^1(\cos\lambda)$ from $f(\cos\lambda)$. Since $(f(\cos\lambda) - T_o^1(\cos\lambda))$ and $(f(\cos\lambda) - T_o^2(\cos\lambda))$ are considerably large for all points, $f(\cos\lambda) - T_o^1(\cos\lambda)$ and $f(\cos\lambda) - T_o^2(\cos\lambda)$ are not included in table 4.44. From table 4.44, it is obtained that the maximum error corresponding to $T_2^1(\cos\lambda)$ is minimum and that maximum is 0.5E-5, which we take to be equal to zero. Hence we obtain the rational approximation as the initial rational form and no further improvement is needed. The following equations as well as the order p = 2 and q = 1 are obtained from (4.96).

$$\sigma_a^2 = \frac{2\text{x}0.1402517\,\sigma_x^2(1 + \phi_1^2 + \phi_2^2)}{(1 + \theta_1^2)} \tag{4.98}$$

$$\Phi_1 = -1.0832257(1 + \Phi_1^2 + \Phi_2^2) + \Phi_1\Phi_2 \qquad (4.99)$$

$$\Phi_2 = \frac{-0.1451134(1 + \Phi_1^2 + \Phi_2^2)}{2} \qquad (4.100)$$

and $$\Theta_1 = \frac{0.0078743}{2 \times 0.1402517}(1 + \theta_1^2) \qquad (4.101)$$

The method of iteration is applied to obtain Φ_1, Φ_2 and Θ_1. Table 4.45 gives the iteration values of Φ_1 and Φ_2 and table 4.46 gives the iteration values of Θ_1. From these two tables we obtain the estimated values of Φ_1, Φ_2 and Θ_1 as

$$\hat{\Phi}_1 = -0.7999845$$

$$\hat{\Phi}_2 = -0.1199973$$

$$\hat{\Theta}_1 = 0.0278286$$

Then σ_a^2 is estimated as

$$\hat{\sigma}_a^2 = 1.0365793$$

The errors of the estimated values of the parameters from their original values are shown in table 4.47.

Note 4.1

Examples 4.1 through 4.9 show that the order of the ARMA model determined using R-spec technique is same as that of the original model used in each example. Further, the differences between the estimated values and the original values of the parameters are very small.

4.2 Analysis of Simulated Series

In this section, the R-spec technique is applied to three simulated series. Standard normal deviates are used as a_t's, the white noise. The random numbers used for simulation are given in the appendix. Certainly, distributions other than N(0,1) could have been applied, but these would only affect the estimates of the ACF. This thesis is concerned only with model selection once the ACF have been estimated. Thus for simplicity only N(0,1) deviates have been used.

Example 4.10

The analysis of a simulated series using an MA(2) model is given in this example. The model used for simulation is,

$$\tilde{x}_t = a_t - 0.5a_{t-1} + 0.2a_{t-2} \qquad (4.102)$$

The estimated ACF and PACF of the simulated series are given

in table 4.48 and table 4.49 respectively. The correlogram (fig.4.18(a)) and the graph of the PACF (fig.4.18(b)) show that P = 0 and Q = 2. Hence $T_o^1(\cos\lambda)$ and $T_o^2(\cos\lambda)$ are considered for initial rational approximation of the spectral density function. The initial rational forms are given by

$$T_o^1(\cos\lambda) = 0.5 - 0.42597\cos(\lambda) \tag{4.103}$$

and

$$T_o^2(\cos\lambda) = 0.5 - 0.42597\cos(\lambda) + 0.12842\cos(2\lambda) \tag{4.104}$$

Table 4.50 shows the differences of $T_o^1(\cos\lambda)$ and $T_o^2(\cos\lambda)$ from $f(\cos\lambda)$ and it shows that the maximum absolute error corresponding to $T_o^2(\cos\lambda)$ is minimum when compared to that of $T_o^1(\cos\lambda)$. So $T_o^2(\cos\lambda)$ is taken as the initial rational approximation of $f(\cos\lambda)$. The maximum value of $|f(\cos\lambda) - T_o^2(\cos\lambda)|$ is equal to 0.0031 and the extremum is attained at the point 0. The improved rational approximation is obtained by taking $T_o^2(\cos\lambda)$ as the initial form. Table 4.51 gives the maximum absolute error as well as the coefficients in each iteration. The improved rational approximation is

$$T_o^{2*}(\cos\lambda) = 0.49833 - 0.42351\cos(\lambda) + 0.12757\cos(2\lambda) \tag{4.105}$$

Hence the estimated rational form of the spectral density function is

$$\hat{s}(\lambda) = 2T_o^{2*}(\cos\lambda) \tag{4.106}$$

Equations (4.105) and (4.106) determine the order of the model as p = 0, q = 2 and the following equations

$$\Theta_1 = 0.42493(1+\Theta_1^2+\Theta_2^2)+\Theta_1\Theta_2 \tag{4.107}$$

$$\Theta_2 = -0.12799(1+\Theta_1^2+\Theta_2^2) \tag{4.108}$$

and $$\sigma_a^2 = \frac{2a_o^{(2)}\sigma_x^2}{1+\Theta_1^2+\Theta_2^2} \tag{4.109}$$

The method of iteration is applied to equations (4.107) and (4.108) to obtain the estimates of Θ_1 and Θ_2. Table 4.52 is the iteration table for Θ_1 and Θ_2. From table 4.52 we obtain

$$\hat{\Theta}_1 = 0.45096$$

$$\hat{\Theta}_2 = -0.15719$$

Substituting $\hat{\Theta}_1$ and $\hat{\Theta}_2$ in (4.109), the estimate of σ_a^2 is obtained as

$$\hat{\sigma}_a^2 = 0.99483.$$

The differences of the estimated values of the parameters from their corresponding original values are given in table 4.54. The estimated model, using the R-spec procedure, for the simulated series is,

$$\tilde{x}_t = a_t - 0.45096a_{t-1} + 0.15719a_{t-2} \qquad (4.110)$$

and its residual variance is $\hat{\sigma}_a^2 = 0.99483$.

Example 4.11

Analysis of AR(1) series is given in this example. The model used for simulation is

$$\tilde{x}_t = 0.74\tilde{x}_{t-1} + a_t. \qquad (4.111)$$

The estimated ACF and PACF of the simulated series are given in table 4.54 and table 4.55 respectively. From the correlogram (fig.4.20(a)) and the graph of the PACF (fig.4.20(b), P = 1 and Q = 1. The initial rational forms are

$$T_o^1(\cos\lambda) = 0.5 + 0.7256\cos(\lambda), \qquad (4.112)$$

$$T_1^o(\cos\lambda) = \frac{0.15329}{1-0.95172\cos(\lambda)} \qquad (4.113)$$

and $$T_1^1(\cos\lambda) = \frac{0.1549-0.0025\cos(\lambda)}{1-0.95111\cos(\lambda)} \qquad (4.114)$$

From (4.113) and (4.114) we get $T_1^1(\cos\lambda) \simeq T_1^o(\cos\lambda)$. The

differences of $T_o^1(\cos\lambda)$ and $T_1^o(\cos\lambda)$ from $f(\cos\lambda)$ are given in table 4.56. The maximum value of $|f(\cos\lambda) - T_1^o(\cos\lambda)|$ is minimum when compared to that of $T_o^1(\cos\lambda)$. Table 4.56 shows that $|f(\cos\lambda) - T_1^o(\cos\lambda)|$ attains its maximum value at $\lambda = 0$ and the maximum value is 0.08102. Taking $T_1^o(\cos\lambda)$ as the initial rational approximation, the converged rational approximation $T_1^{o*}(\cos\lambda)$ is obtained. Table 4.57 gives the differences $(f(\cos\lambda) - T_1^{o(i)}(\cos\lambda))$, i being the number of iteration and $\lambda = -\pi, -0.9\pi \ldots, 0, \ldots 0.9\pi, \pi$. Table 4.58 gives the maximum absolute error, $|E_2^{(i)}|$, the coefficients $a_o^{(i)}$ and $b_1^{(i)}$ in each iteration. The rational approximation obtained is

$$T_1^{o*}(\cos\lambda) = \frac{0.15513}{1-0.95056\cos(\lambda)} \tag{4.115}$$

Hence the estimated rational form of the spectral density function is

$$\hat{s}(\lambda) = 2T_1^{o*}(\cos\lambda). \tag{4.116}$$

Equations (4.115) and (4.116) determine p = 1, q = 0 and the following equations

$$\phi_1 = 0.47528(1 + \phi_1^2) \tag{4.117}$$

and

$$\sigma_a^2 = 2 \times 0.15513(1 + \phi_1^2)\sigma_x^2 \tag{4.118}$$

Applying the iteration procedure to (4.117), the estimate of ϕ_1 is obtained in table 4.59 as

$$\hat{\phi}_1 = 0.72532.$$

Then from (4.118) $\hat{\sigma}_a^2$ is obtained as

$$\hat{\sigma}_a^2 = 1.00963.$$

Hence the estimated model is

$$\hat{\tilde{x}}_t = 0.72532\tilde{x}_{t-1} + a_t \qquad (4.119)$$

with residual variance $\hat{\sigma}_a^2 = 1.00963$. Table 4.60 gives the differences between the original values and their estimated values of the parameters.

Example 4.12

An ARMA(1,1) series simulated using the model

$$\tilde{x}_t = 0.68\tilde{x}_{t-1} + a_t - 0.09a_{t-1} \qquad (4.120)$$

is considered in this example. The estimated ACF and PACF of the simulated series are given in table 4.61 and table 4.62 respectively. From the correlogram (fig.4.22(a)) and the graph of the PACF (fig.4.22(b)) we take P = 1 and Q = 2. The initial

rational forms $T_k^l(\cos\lambda)$, $0 \leq l \leq 2$ and $0 \leq k \leq 1$ are

$$T_o^1(\cos\lambda) = 0.5 + 0.62537\cos(\lambda) \tag{4.121}$$

$$T_o^2(\cos\lambda) = 0.5+0.62537\cos(\lambda)+0.42323\cos(2\lambda) \tag{4.122}$$

$$T_1^o(\cos\lambda) = \frac{0.21374}{1-0.90012\cos(\lambda)} \tag{4.123}$$

$$T_1^1(\cos\lambda) = \frac{0.21059-0.03327\cos(\lambda)}{1 - 0.92556\cos(\lambda)} \tag{4.124}$$

$$T_1^2(\cos\lambda) = \frac{0.20888-0.03717\cos(\lambda)-0.0025\cos(2\lambda)}{1 - 0.93103\cos(\lambda)} \tag{4.125}$$

The differences of $T_k^l(\cos\lambda)$, $0 \leq l \leq 2$, $0 \leq k \leq 1$ are calculated. Table 4.63 gives the differences of $T_1^o(\cos\lambda)$, $T_1^1(\cos\lambda)$ and $T_1^2(\cos\lambda)$ from $f(\cos\lambda)$. The differences of $T_o^1(\cos\lambda)$ and $T_o^2(\cos\lambda)$ are considerably large, and therefore $(f(\cos\lambda) - T_o^1(\cos\lambda))$ and $(f(\cos\lambda) - T_o^2(\cos\lambda))$ are not included in table 4.63. Further, table 4.63 shows that the maximum value of $|f(\cos\lambda) - T_1^1(\cos\lambda)|$ is minimum. Hence $T_1^1(\cos\lambda)$ is selected as the initial rational approximation of $f(\cos\lambda)$. Table 4.64 gives the differences $f(\cos\lambda) - T_1^{1(i)}(\cos\lambda)$, where i stands for the number of iteration. Table 4.65 shows the convergence of the maximum

absolute error as well as the coefficients $a_o^{(i)}$, $a_1^{(i)}$ and $b_1^{(i)}$. The converged rational approximation is given as

$$T_1^{1*}(\cos\lambda) = \frac{0.21115-0.03401\cos(\lambda)}{1-0.925687\cos(\lambda)} \tag{4.126}$$

Hence we get

$$\hat{s}(\lambda) = 2T_1^{1*}(\cos\lambda) \tag{4.127}$$

Equations (4.126) and (4.127) determine p = 1, q = 1 and the following equations

$$\Phi_1 = 0.46284(1 + \Phi_1^2) \tag{4.128}$$

$$\Theta_1 = 0.08053(1 + \Theta_1^2) \tag{4.129}$$

and $$\sigma_a^2 = \frac{2 \times 0.21116(1+\Phi_1^2)\sigma_x^2}{1 + \Theta_1^2} \tag{4.130}$$

Applying the method of iteration to equations (4.128) and (4.129), the estimates of Φ_1 and Θ_1 are obtained in table 4.66 and table 4.67 respectively. The estimated values of Φ_1 and Θ_1 are

$$\hat{\Phi}_1 = 0.67162$$

$$\hat{\Theta}_1 = 0.0809$$

Then $\hat{\sigma}_a^2$ is obtained using (4.130) as

$$\hat{\sigma}_a^2 = 0.9959.$$

Hence the estimated model is given by

$$\hat{\tilde{x}}_t = 0.67162\tilde{x}_{t-1} + a_t - 0.0809a_{t-1} \qquad (4.131)$$

The differences between the original values and the estimated values of the parameters are shown in table 4.68.

4.3 Analysis of Observed Time Series Data

The R-spec technique is applied to three observed time series data. Example 4.13 is the analysis of series C, from Box and Jenkins [8]. Example 4.14 is the analysis of series D from Box-Jenkins [8]. The analysis of Indian population data is given in example 4.15. The data are given in the appendix.

Example 4.13 (Analysis of series C)

The estimated ACF of the series is given in table 4.69 Figure 4.24 (the correlogram) shows that series C is non-stationary. So the ACF of the differenced series (table 4.70) is estimated. The correlogram (fig.4.25(a)) of the differenced

series shows that the ACF die out and hence the differenced series is stationary. The PACF of the differenced series is given in table 4.71. From the correlogram and the graph of the PACF (fig.4.25(b)) we take P = 1 and Q = 0. In this case there is only one initial rational form given by

$$T_1^{o}(\cos\lambda) = \frac{0.112121}{1-0.9697\cos(\lambda)} \tag{4.132}$$

Differences of $T_1^{o(i)}(\cos\lambda)$, i=0,1,2,3 from $f(\cos\lambda)$ are given in table 4.72. Table 4.73 shows the convergence of the rational approximation and we obtain,

$$T_1^{o*}(\cos\lambda) = \frac{0.11668}{1-0.96626\cos(\lambda)} \tag{4.133}$$

Equation (4.133) implies

$$\hat{s}(\lambda) = 2T_1^{o*}(\cos\lambda) \tag{4.134}$$

Equation (4.133) determines p = 1, q = 0 and the following equations

$$\phi_1 = 0.48313(1+\phi_1^2) \tag{4.135}$$

and $$\sigma_a^2 = 2 \times a_o \times \sigma_x^2(1+\phi_1^2) \tag{4.136}$$

Applying the method of iteration to equation (4.135) the

estimate of ϕ_1 is obtained in table 4.74. The estimated value of ϕ_1 is

$$\hat{\phi}_1 = 0.78$$

Using (4.136) and $\hat{\phi}_1$, $\hat{\sigma}_a^2$ is obtained as

$$\hat{\sigma}_a^2 = 0.0178$$

The estimated model for series C is

$$\hat{w}_t = 0.78 w_{t-1} + a_t \qquad (4.137)$$

where $w_t = \nabla \tilde{x}_t$. Substituting $w_t = \tilde{x}_t - \tilde{x}_{t-1}$ in (4.137) the model becomes

$$\hat{\tilde{x}}_t = 1.78\tilde{x}_{t-1} - 0.78\tilde{x}_{t-2} + a_t \qquad (4.138)$$

and the residual variance is

$$\hat{\sigma}_a^2 = 0.0178.$$

Example 4.14 (Analysis of series D)

The estimated ACF and PACFof series D are given in table 4.75 and table 4.76 respectively. The correlogram (fig.4.27(a)) shows that the series is stationary. The correlogram and the graph of the PACF (fig.4.27(b)) show

that P = 1 and Q = 0. In this example also there is only one initial rational form given by

$$T_1^{o}(\cos\lambda) = \frac{0.07494}{1-0.98850\cos(\lambda)} \tag{4.139}$$

Table 4.77 gives the differences $f(\cos\lambda) - T_1^{o(i)}(\cos\lambda)$ i=0,1 and 2 and table 4.78 shows the convergence of the rational approximation. From table 4.78 we obtain $T_1^{o*}(\cos\lambda) = T_1^{o(2)}(\cos\lambda)$.

$$\text{i.e., } T_1^{o*}(\cos\lambda) = \frac{0.61100}{1-0.98972\cos(\lambda)} \tag{4.140}$$

So the estimated rational form of the spectral density function is

$$\hat{s}(\lambda) = T_1^{o*}(\cos\lambda) \tag{4.141}$$

Equation (4.140) determines p = 1 and q = 0. Also we get the following equations

$$\phi_1 = 0.49486(1+\phi_1^2) \tag{4.142}$$

$$\text{and } \sigma_a^2 = 2 \times 0.0611(1+\phi_1^2)\sigma_x^2 \tag{4.143}$$

Applying the method of iteration to (4.142) the estimated ϕ_1

is obtained in table 4.79 and the estimated value is

$$\hat{\phi}_1 = 0.8611$$

Then using (4.143) σ_a^2 is estimated as

$$\hat{\sigma}_a^2 = 0.07695$$

Hence the estimated model for series D is

$$\hat{\tilde{x}}_t = 0.8611\tilde{x}_{t-1} + a_t \qquad (4.144)$$

with residual variance equal to 0.07695.

Example 4.15 (Analysis of Indian Population Data)

The population data from 1947 to 1980 is taken as the series and the R-spec technique is used to identify the model.

The estimated ACF of P_t, P_t being the population at time t, is given in table 4.80. The correlogram (fig.4.29) shows that the series $\{P_t\}$ is non-stationary. Table 4.81 and Table 4.82 are the ACF of ∇P_t and $\nabla^2 P_t$ respectively. Correlogram of ∇P_t (fig.4.30) shows that ∇P_t is also non-stationary while figure 4.31(a), the correlogram of $\nabla^2 P_t$ shows that $\nabla^2 P_t$ is stationary. The estimated PACF of $\nabla^2 P_t$

is given in table 4.83. The correlogram and the graph of the PACF (fig.4.31(b)) show P = 1 and Q = 1. The initial rational forms are

$$T_o^1(\cos\lambda) = 0.5-0.5266\cos(\lambda) \tag{4.145}$$

$$T_1^o(\cos\lambda) = \frac{0.23427}{1+1.00922\cos(\lambda)} \tag{4.146}$$

$$T_1^1(\cos\lambda) = \frac{0.456896-0.44119\cos(\lambda)}{1+0.163702\cos(\lambda)} \tag{4.147}$$

The differences of $T_o^1(\cos\lambda)$ and $T_1^1(\cos\lambda)$ from $f(\cos\lambda)$ are given in table 4.84. Since the difference $f(\cos\lambda) - T_1^o(\cos\lambda)$ for $\lambda \in [-\pi,\pi]$ is very large, these differences are not included in this table. From 4.84, $T_1^1(\cos\lambda)$, having the minimum of maximum absolute error, is selected as the initial rational approximation of the spectral density function. Table 4.85 shows the convergence of $|E_2^{(i)}|$, $a_o^{(i)}$, $a_1^{(i)}$ and $b_1^{(i)}$, i being the number of iteration. The converged rational approximation is

$$T_1^{1*}(\cos\lambda) = \frac{0.45631-0.43995\cos(\lambda)}{1+0.16736\cos(\lambda)} \tag{4.148}$$

Hence we get

$$\hat{s}(\lambda) = 2T_1^{1*}(\cos\lambda). \tag{4.149}$$

Equation (4.148) determines p = 1, q = 1 and the following equations

$$\phi_1 = -0.08382(1 + \phi_1^2) \tag{4.150}$$

$$\theta_1 = 0.482066(1 + \theta_1^2) \tag{4.151}$$

and

$$\sigma_a^2 = \frac{2 \times 0.45631 \times (1+ \phi_1^2)\ \sigma^2_{\nabla^2 P_t}}{1 + \theta_1^2} \tag{4.152}$$

The estimate of ϕ_1 is obtained in table 4.86, applying the method of iteration to (4.150) and the estimate of θ_1 is obtained in table 4.87 using the equation (4.151) and then $\hat{\sigma}_a^2$ is obtained using (4.152). The estimates are

$$\hat{\phi}_1 = -0.084$$

$$\hat{\theta}_1 = 0.762$$

and $\hat{\sigma}_a^2 = 0.4514$

The estimated model for the Indian population is given as

$$\hat{w}_t = -0.084 w_{t-1} + a_t - 0.762 a_{t-1} \tag{4.153}$$

where $w_t = \nabla^2 P_t$. Substituting $w_t = \nabla^2 P_t$ and rearranging

the terms, the model (4.153) becomes

$$\hat{P}_t = 1.916P_{t-1}-0.832P_{t-2}-0.084P_{t-3}+a_t-0.762a_{t-1} \quad (4.154)$$

and the residual variance $\hat{\sigma}_a^2 = 0.4514$.

Note 4.2

The R-spec technique is applied to three observed time series, series C, series D and the Indian population data. Series C gives observations on chemical process temperature readings made at every minute and series D gives observations on chemical process viscocity readings made at every hour. The census figures published by the Central Statistical Organisation is taken as one possible realization of the process namely, human population growth of the nation.

Note 4.3

The model estimated for the Indian population data is given by (4.154). Taking a_o, a_1 and a_2 equal to zero and using (4.154), we can calculate a_3, a_4,...,a_{33}. Now the estimate of the population for 1981 can be obtained from (4.154) using the data of 1980, 1979, 1978 and a_{33}. The estimated value is

$$\hat{p}_{34} = 67.663,$$

which means the estimated population for 1981 using the identified ARMA(1,1) model is 0.677×10^{9}.

Table 4.1

Theoretical ACF of the MA(1) model

$\tilde{x}_t = a_t - 0.5a_{t-1}$

ρ_1	$= -0.4$
ρ_k	$= 0, \quad k > 1$

Table 4.2

PACF of the MA(1) model

$\tilde{x}_t = a_t - 0.5a_{t-1}$

Lags	1	2	3	4	5
1-5	-0.4	-0.1904761	-0.0941176	-0.04692079	-0.02344321
6-10	-0.01171946	-0.00585946	-0.002929697	-0.001464844	-0.000732422
11-15	-0.000366211	-0.000183105	-0.00009155	-0.00004577	-0.000022888

Table 4.3

Iteration table for MA(1) Parameter θ_1

Number of Iteration	$\hat{\theta}_1$
0	0.2
1	0.464
2	0.4861184
3	0.49455244
4	0.4978217
5	0.4991305
6	0.4996524
7	0.499861
8	0.4999441
9	0.4999777
10	0.499991
11	0.4999964
12	0.4999985
13	0.49999985
14	0.4999997
15	0.4999998
16	0.4999999
17	0.4999999

Table 4.4

Errors of the estimated parameters of MA(1) model

Parameter (1)	Original value of the parameter (2)	Estimated value (3)	Difference (2) - (3)
θ_1	0.5	0.4999999	0.0000001
σ_a^2	1.0	1.0	0.0

Table 4.5

Theoretical ACF of the MA(2) model

$\tilde{x}_t = a_t - 0.7a_{t-1} + 0.12a_{t-2}$

$\rho_1 = -0.5211379;$	$\rho_2 = 0.079766;$	$\rho_k = 0, \quad k > 2$

Table 4.6

PACF of the MA(2) model

$\tilde{x}_t = a_t - 0.7a_{t-1} + 0.12a_{t-2}$

Lags	1	2	3	4	5
1-5	-0.5211379	-0.2633367	-0.1249722	-0.0562843	-0.02444346
6-10	-0.01035988	-0.004319	-0.001780122	-0.0007278	-0.0002959
11-15	-0.0001198	-0.00004833	-0.00001946	-0.000007222	-0.000003140

Table 4.7*

Errors in various Initial Rational Approximations to $f(\cos\lambda)$

k	E_1	E_2
-1.0	-0.079766	0
-o.9	-0.0758619	0
-0.8	-0.064532	0
-o.7	-0.0468852	0
-0.6	-0.024649	0
-0.5	0.00000	0
-0.4	0.024649	0
-0.3	0.0468852	0
-0.2	0.064532	0
-0.1	0.0758619	0
0.0	0.079766	0
0.1	0.0758619	0
0.2	0.06532	0
0.3	0.0468852	0
0.4	0.024649	0
0.5	0	0
0.6	-0.024644	0
0.7	-0.0468852	0
0.8	-0.06532	0
0.9	-0.0758619	0
1.0	-0.079766	0

$\lambda = k\pi$

$E_1 = f(\cos\lambda) - \left[0.5 - 0.5211379 \cos(\lambda)\right]$

$E_2 = f(\cos\lambda) - \left[0.5 - 0.5211379 \cos(\lambda) + 0.079766 \cos(2\lambda)\right]$

* The errors E_1 and E_2, are calculated for 201 points in $[-\pi,\pi]$ and twentyone values are given in this table.

Table 4.8

Iteration table for MA(2) Parameters Θ_1 and Θ_2

Number of Iteration	$\hat{\Theta}_1$	$\hat{\Theta}_2$
0	0.5211379	-0.079766
1	0.6244177	-0.1019367
2	0.6660923	-0.1116954
3	0.684458	-0.1161516
4	0.6928119	-0.1182111
5	0.6966622	-0.1191673
6	0.6984473	-0.1196122
7	0.6992769	-0.1198193
8	0.699663	-0.1199157
9	0.6998429	-0.1199816
10	0.699912	-0.1199897
11	0.6999568	-0.1199949
12	0.6999758	-0.1199971
13	0.699988	-0.1199985
14	0.6999932	-0.1199991
15	0.6999965	-0.1199995
16	0.699998	-0.1199997
17	0.6999987	-0.1199998
18	0.6999992	-0.1199998
19	0.6999994	-0.1199999
20	0.6999995	-0.1199999
21	0.6999996	-0.1199999
22	0.6999997	-0.1199999
23	0.6999997	-0.1199999

Table 4.9

Errors of the estimated parameters of MA(2) model

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
θ_1	0.7	0.6999997	0.0000003
θ_2	-0.12	-0.1199999	-0.0000001
σ_a^2	1.0	1.0000002	-0.0000002

Table 4.10

Theoretical ACF of the MA(3) model

$$\tilde{x}_t = a_t - 0.4a_{t-1} + 0.2a_{t-2} - 0.15a_{t-3}$$

$\rho_1 = -0.4171779$; $\rho_2 = 0.2126789$; $\rho_3 = -0.1226993$; $\rho_k = 0$, $k > 3$

Table 4.11

PACF of the MA(3) model

$$\tilde{x}_t = a_t - 0.4a_{t-1} + 0.2a_{t-2} - 0.15a_{t-3}$$

Lags	1	2	3	4	5
1-5	-0.4171779	0.04678359	-0.027784	-0.0844273	-0.02112544
6-10	0.0039883	-0.00733322	-0.007009	-0.000735	-0.00004577
11-15	-0.00096203	-0.0004903	-0.00001295	-0.00005759	-0.00009669

Table 4.12*

Errors in Various Initial Rational Approximations to $f(\cos\lambda)$

k	E_1	E_2	E_3
-1.0	0.3353782	0.1226993	0.0
-0.9	0.2441816	0.0721208	0.0
-0.8	0.0278052	-0.0379161	0.0
-0.7	-0.1824153	-0.1166939	0.0
-0.6	-0.2713266	-0.0992658	0.0
-0.5	-0.2126789	0	0.0
-0.4	-0.072795	0.0992658	0.0
-0.3	0.0509725	0.1166939	0.0
-0.2	0.1036375	0.037916	0.0
-0.1	0.09994	-0.0721208	0.0
0.0	0.0899796	-0.1226993	0.0
0.1	0.09994	-0.0721208	0.0
0.2	0.1036375	0.0379161	0.0
0.3	0.0509723	0.1166939	0.0
0.4	-0.072795	0.0992658	0.0
0.5	-0.2126789	0	0.0
0.6	-0.2713266	-0.0992658	0.0
0.7	-0.1824153	-0.1166939	0.0
0.8	0.0278052	-0.0379161	0.0
0.9	0.2441816	0.0721208	0.0
1.0	0.3353782	0.1226993	0.0

$\lambda = k\times\pi$

$E_1 = f(\cos\lambda) - [0.5-0.4171779\cos(\lambda)]$

$E_2 = f(\cos\lambda) - [0.5-0.4171779\cos(\lambda) + 0.2126789\cos(2\lambda)]$

$E_3 = f(\cos\lambda) - [0.5-0.4171779\cos(\lambda) + 0.2126789\cos(2\lambda) - 0.1226993\cos(3\lambda)]$

* The errors E_1, E_2 and E_3 are calculated for 201 points in $[-\pi,\pi]$ and twentyone values are given in this table.

Table 4.13

Iteration of the MA(3) Parameters Θ_1, Θ_2 and Θ_3

Iteration Number	$\hat{\Theta}_1$	$\hat{\Theta}_2$	$\hat{\Theta}_3$
0	0.4171779	-0.2126789	0.1226993
1	0.4001125	-0.2113274	0.1514504
2	0.3956033	-0.2005058	0.1506362
3	0.3991807	-0.1997475	0.149619
4	0.4000157	-0.2000899	0.1498931
5	0.3999756	-0.2000438	0.1500019
6	0.3999797	-0.2000025	0.1499997
7	0.3999963	-0.1999998	0.149998
8	0.399999	-0.2000005	0.1499994
9	0.3999997	-0.2000001	0.1499998
10	0.3999999	-0.2000000	0.1499998
11	0.3999999	-0.2000000	0.1499998

Table 4.14

Errors of the estimated parameters of the MA(3) model

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
Θ_1	0.4	0.3999999	0.0000001
Θ_2	-0.2	-0.200000	0
Θ_3	0.15	0.1499998	0.0000002
σ_a^2	1.0	1.0000001	0.0000001

Table 4.15

Theoretical ACF of MA(4) model

$$\tilde{x}_t = a_t + 0.7a_{t-1} + 0.25a_{t-2} - 0.08a_{t-3} + 0.05\,a_{t-4}$$

$\rho_1 = 0.5501473$; $\rho_2 = 0.1162418$; $\rho_3 = -0.07365$; $\rho_4 = 0.032025$; $\rho_k = 0$, $k > 4$

Table 4.16

PACF of MA(4) model

$$\tilde{x}_t = a_t + 0.7a_{t-1} + 0.25a_{t-2} - 0.08a_{t-3} + 0.05a_{t-4}$$

Lags	1	2	3	4	5
1-5	0.5501473	-0.2673308	-0.1177555	0.0860266	-0.0532226
6-10	0.001687411	0.018411	-0.013278	0.002168471	0.00335856
11-15	-0.0030349	0.0007944	0.0005797	-0.0006792	0.00024236

Table 4.17*

Errors in Various Initial Rational Approximations to $f(\cos\lambda)$

k	E_1	E_2	E_3	E_4
-1.0	0.1578693	0.0416275	-0.0320225	0.0
-0.9	0.1274364	0.0333948	-0.0098954	0.0
-0.8	0.0390683	0.00314764	0.0259067	0.0
-0.7	-0.0800592	-0.00441385	0.0259068	0.0
-0.6	-0.1635211	-0.0694795	-0.0098954	0.0
-0.5	-0.1482643	-0.0320225	-0.0320225	0.0
-0.4	-0.0443526	0.0496887	-0.0098954	0.0
-0.3	0.0600313	0.095952	0.0259067	0.0
-0.2	0.0845865	0.0486658	0.0259067	0.0
-0.1	0.0606467	-0.0333948	0.0098954	0.0
0.0	0.0105693	-0.1056725	-0.0320225	0.0
0.1	0.0606467	-0.0333948	0.0098954	0.0
0.2	0.0845865	0.0486658	0.0259067	0.0
0.3	0.0600313	0.095952	0.0259067	0.0
0.4	-0.0443526	0.0496887	-0.0098954	0.0
0.5	-0.1482643	-0.0320225	-0.0320225	0.0
0.6	-0.1635211	-0.0694795	-0.0098954	0.0
0.7	-0.0800592	0.00441385	0.0259068	0.0
0.8	0.0390683	0.00314764	0.0259067	0.0
0.9	0.1274364	0.0333948	-0.0098954	0.0
1.0	0.1578693	0.0416275	-0.0320225	0.0

$\lambda = kx\pi$

$E_1 = f(\cos\lambda) - [0.5+0.5501473\cos(\lambda)]$

$E_2 = f(\cos\lambda) - [0.5+0.5501473\cos(\lambda)+0.1162418\cos(2\lambda)]$

$E_3 = f(\cos\lambda) - [0.5+0.5501473\cos(\lambda)+0.1162418\cos(2\lambda) -0.07365\cos(3\lambda)]$

$E_4 = f(\cos\lambda) - [0.5+0.5501473\cos(\lambda)+0.1162418\cos(2\lambda) -0.07365\cos(3\lambda)+0.032025\cos(4\lambda)]$

* The errors E_1, E_2, E_3 and E_4 are calculated for 201 points in $[-\pi,\pi]$ and twentyone values are given in this table.

Table 4.18

Iteration Table for MA(4) Parameters

Θ_1, Θ_2, Θ_3 and Θ_4

Iteration Number	$\hat{\Theta}_1$	$\hat{\Theta}_2$	$\hat{\Theta}_3$	$\hat{\Theta}_4$
0	-0.5501473	-0.1162418	0.07365	-0.032025
1	-0.6698905	-0.1979851	0.0797927	-0.042357
2	-0.7028718	-0.2357332	0.0818141	-0.0479129
3	-0.7071291	-0.2499725	0.0811133	-0.0499137
4	-0.7040698	-0.2510077	0.0794945	-0.0499137
5	-0.7015573	-0.2507108	0.0803058	-0.0502001
6	-0.7005041	-0.2507278	0.0799709	-0.0500873
7	-0.7000006	-0.250203	0.0799898	-0.0500382
8	-0.6999264	-0.2500247	0.079978	-0.0500071
9	-0.6999494	-0.2499708	0.0799889	-0.0500008
10	-0.6999804	-0.2499767	0.0799926	-0.050001
11	-0.6999957	-0.2499876	0.0799944	-0.0500025
12	-0.7000003	-0.2499941	0.0799946	-0.0500034
13	-0.7000005	-0.2499963	0.0799944	-0.0500037
14	-0.6999998	-0.2499966	0.0799943	-0.0500037
15	-0.6999992	-0.2499962	0.07999943	-0.0500037
16	-0.6999992	-0.2499962	0.0799943	-0.0500037

Table 4.19

Errors of the estimated parameters of the MA(4) model

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
Θ_1	-0.7	-0.6999992	-0.0000008
Θ_2	-0.25	-0.2499962	-0.0000038
Θ_3	0.08	0.0799943	0.0000057
Θ_4	-0.05	-0.0500037	0.0000037
σ_a^2	1.0	1.0000002	-0.0000002

Table 4.20

Theoretical ACF of the AR(1) model.I

$$\tilde{x}_t = -0.35\tilde{x}_{t-1} + a_t$$

Lags	1	2	3	4	5
1-5	-0.35	0.122502	-0.042875	0.0150062	-0.0052521
6-10	0.0018382	-0.0006433	0.0002251	-0.0000787	0.0000275
11-15	-0.0000096	0.33E-5	-0.11E-5	0.3E-5	-0.1E-6

Table 4.21

Partial Auto Correlations of the AR(1) model.I

$$\tilde{x}_t = -0.35\tilde{x}_{t-1} + a_t$$

Lags	1	2	3	4	5
1-5	-0.35	-0.596E-7	-0.22E-7	-0.19E-7	-0.74E-7
6-10	-0.13E-7	-0.67E-7	-0.35E-7	0.75E-7	-0.174E-7
11-15	0.105E-7	-0.58E-7	-0.38E-7	-0.75E-7	-0.28E-7

Table 4.22*

Errors in various rational** approximations to $f(\cos\lambda)$

k	E_1	E_2	E_3	E_4
-1.0	-0.096169	0.023540	0	-0.026438
-0.9	-0.074248	0.032865	0	0.005785
-0.8	-0.049262	0.036138	0	0.035821
-0.7	-0.022981	0.033437	0	00.063005
-0.6	00.003019	0.025654	0	0.086920
-0.5	0.027365	0.014292	0	0.107318
-0.4	00.048890	0.001221	0	0.124067
-0.3	0.066634	-0.011575	0	0.137108
-0.2	0.079847	-0.022246	0	0.146421
-0.1	0.087990	-0.029297	0	0.152005
0	0.097403	-0.031759	0	0.086875
0.1	0.087990	-0.029297	0	0.152005
0.2	0.079847	-0.022246	0	0.146421
0.3	0.066634	-0.011575	0	0.137108
0.4	0.048890	0.001221	0	0.124067
0.5	0.027365	0.014292	0	0.107318
0.6	0.003019	0.025654	0	0.086920
0.7	-0.022981	0.033437	0	0.063005
0.8	-0.049262	0.036138	0	0.035821
0.9	-0.074248	0.032865	0	0.005785
1.0	-0.096169	0.023540	0	-0.026438

$\lambda = k\times\pi$

$E_1 = f(\cos\lambda) - (0.5-0.35\cos(\lambda))$, $E_2 = f(\cos\lambda) - (0.5-0.35\cos(\lambda) + 0.1225\cos(2\lambda))$

$E_3 = f(\cos\lambda) - (0.3908686 \,/\, (1+0.623608 \times \cos(\lambda))$

$E_4 = f(\cos\lambda) - (0.441667-0.3258334\cos(\lambda)) \,/\, (1+0.333333\cos(\lambda))$

* Errors E_1, E_2, E_3 and E_4 are calculated for 201 points in $[-\pi,\pi]$ and twentyone values are given in this table.

** Initial

Table 4.23

Iteration table for AR(1) parameter ϕ_1

Number of iteration	$\hat{\phi}_1$
0	-0.3421181
1	-0.348299
2	-0.3496296
3	-0.3499191
4	-0.3499823
5	-0.3499961
6	-0.3499991
7	-0.3499997
8	-0.3499999
9	-0.3499999

Table 4.24

Errors of the estimated parameters of the AR(1) model.1

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
ϕ_1	-0.35	-0.3499999	-0.0000001
σ_a^2	1.0	0.9999993	0.7E-6

Table 4.25

Theoretical ACF of the AR(1) model · II

$$\tilde{x}_t = 0.5\tilde{x}_{t-1} + a_t$$

Lags	1	2	3	4	5
1-5	0.5	0.25	0.125	0.0625	0.03125
6-10	0.01525	0.0078125	0.0039062	0.0019531	0.0009765
10-15	0.0004882	0.0002441	0.0001221	0.000061	0.0000305
16-20	0.0000152	0.0000076	0.0000038	0.0000019	0.0000004

Table 4.26

PACF of the AR(1) model · II

$$\tilde{x}_t = 0.5\tilde{x}_{t-1} + a_t$$

Lags	1	2	3	4	5
1-5	0.5	0	-0.89E-7	-0.89E-7	-0.819E-7
6-10	-0.81956E-7	-0.80E-7	-1.48E-7	0.917E-7	-0.5704E-7
11-15	-0.3911E-7	0.3539E-7	0.6495E-7	-0.996E-7	0.3337E-8

Table 4.27*

Errors in various rational approximations to $f(\cos\lambda)$

k	E_1	E_2	E_3
-1.0	-0.225358	0.3E-5	0.018948
-0.9	-0.250077	0	-0.031481
-0.8	-0.258960	0.1E-5	-0.084676
-0.7	-0.246435	0	-0.131296
-0.6	-0.206033	0.1E-5	-0.159838
-0.5	-0.131008	-0.1E-5	-0.157688
-0.4	-0.016542	-0.2E-5	-0.113828
-0.3	0.134743	0	-0.024867
-0.2	0.303093	-0.1E-5	0.094741
-0.1	0.443806	0	0.204446
0	0.499995	-0.3E-5	0.249995
0.1	0.443806	0	0.204446
0.2	0.303093	-0.1E-5	0.094741
0.3	0.134743	0	-0.024867
0.4	-0.016542	-0.2E-5	-0.113828
0.5	-0.131008	-0.1E-5	-0.157688
0.6	-0.206033	0.1E-5	-0.159838
0.7	-0.246435	0	-0.131296
0.8	-0.258960	0.1E-5	-0.084676
0.9	-0.250077	0	-0.031481
1.0	-0.225358	0.3E-5	0.018948

$\lambda = kx\,\pi$

$E_1 = f(\cos\lambda) - [0.5+0.5\cos(\lambda)]$

$E_2 = f(\cos\lambda) - (0.3/(1-0.8\cos\lambda))$

$E_3 = f(\cos\lambda) - (0.5+0.5\cos(\lambda) + 0.25\cos(2\lambda))$

* Errors E_1, E_2, E_3 are calculated for 201 values of $\lambda \in [-\pi, \pi]$ and twentyone values are included in this table.

Table 4.28

Iteration table for the convergence of the coefficients of the R-spec estimate as well as the error

Iteration Number (i)	Error $E^{(i)}$	$a_o^{(i)}$	$b_1^{(i)}$
0	0.5E-5	0.3	0.8
1	0.5E-5	0.3	0.8

Table 4.29

Iteration table for the AR(1) parameter ϕ_1

Number of Iteration	$\hat{\phi}_1$
0	0.464
1	0.4861184
2	0.4945244
3	0.4978217
4	0.4991305
5	0.499861
6	0.4999444
7	0.4999777
8	0.499991
9	0.499994
10	0.4999985
11	0.4999997
12	0.4999998
13	0.4999999
14	0.4999999

Table 4.30

Errors of the estimated parameters of of the AR(1) model.II

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
ϕ_1	0.5	0.4999999	0.0000001
σ_a^2	1.0	1.0001249	-0.0001249

Table 4.31

Theoretical ACF of the AR(2) model

$$\tilde{x}_t = 0.75\tilde{x}_{t-1} - 0.14\tilde{x}_{t-2} + a_t$$

Lag	1	2	3	4	5
1-5	0.657895	0.353421	0.172961	0.080241	0.035967
6-10	0.015741	0.006771	0.002874	0.001208	0.000737
11-15	0.000384	0.000185	0.000085	0.000038	0.000016
16-20	0.000007	0.000003	0.000001	0.0	0

Table 4.32

PACF of the AR(2) model

$$\tilde{x}_t = 0.75\tilde{x}_{t-1} - 0.14\tilde{x}_{t-2} + a_t$$

Lag	1	2	3	4	5
1-5	0.657895	-0.139999	-0.298023E-6	0.745058E-7	-0.249594E-6
6-10	0.242144E-6	-0.819564E-7	-0.628643E-7	-0.6419624E-7	0.419581E-7
11-15	0.586908E-7	-0.336819E-7	0.351705E-7	-0.669625E-7	0

Table 4.33*

Errors in various initial rational approximations to $f(\cos\lambda)$

k	E_1	E_2	E_3	E_4
-1.0	0.076406	-0.005068	0.0001980	-0.00013
-0.9	0.067190	0.000016	-0.0001241	0.000276
-0.8	0.037156	0.008722	0.0001265	0.000717
-0.7	-0.014302	0.021148	-0.0001297	0.000757
-0.6	-0.069975	0.036766	0.0001393	0.000842
-0.5	-0.108099	0.051666	-0.0001544	0.001741
-0.4	-0.107925	0.053971	0.0001838	0.003013
-0.3	-0.059861	0.023906	-0.0002398	0.003554
-0.2	0.02461	-0.051870	0.0003517	-0.003846
-0.1	0.10858	-0.149289	-0.0005288	0.005064
0	0.14425	-0.197065	0.00059911	0.005968
0.1	0.10858	-0.149289	-0.0005288	0.005064
0.2	0.02461	-0.051870	0.0003517	-0.003846
0.3	-0.059861	0.023906	-0.0002398	0.003554
0.4	-0.107925	0.053971	0.0001838	0.003013
0.5	-0.108099	0.051666	-0.0001544	0.001741
0.6	-0.069975	0.036766	0.0001393	0.000842
0.7	-0.014302	0.021148	-0.0001297	0.000757
0.8	0.037156	0.008722	0.0001265	0.000717
0.9	0.067190	0.000016	-0.0001241	0.000276
1.0	0.076406	-0.005068	0.0001980	-0.000130

$\lambda = k\pi$

$$E_1 = f(\cos\lambda) - \left[0.5+0.6578947\cos(\lambda)\right]$$

$$E_2 = f(\cos\lambda) - \left[\frac{0.2201512+0.082198\cos(\lambda)}{1-0.8507403\cos(\lambda)}\right]$$

$$E_3 = f(\cos\lambda) - \left[\frac{0.1757341}{1-1.080842\cos(\lambda)+0.17698\cos(2\lambda)}\right]$$

$$E_4 = f(\cos\lambda) - \left[\frac{0.1757336-0.00051858\cos(\lambda)}{1-1.080842\cos(\lambda)+0.1769795\cos(2\lambda)}\right]$$

* The errors E_1, E_2, E_3 and E_4 are calculated for 201 values of $\lambda \in [-\pi,\pi]$ and twentyone values are given in this table.

Table 4.34

Iteration table for the convergence of the coefficients of the R-spec estimate as well as the error

Iteration Number	Maximum of $f(\cos\lambda) - T_2^o(\cos\lambda)$	a_o	b_1	b_2
0	0.000599	0.1757341	1.080842	0.17698
1	0.000586	0.1757	1.0809	0.1770

Table 4.35.a

Iteration table for the AR(2) parameters ϕ_1 and ϕ_2

Iteration Number	$\hat{\phi}_1$	$\hat{\phi}_2$
0	0.54045	-0.0885
1	0.654711	-0.1150427
2	0.703945	-0.1276065
3	0.7272363	-0.1337962
4	0.7386525	-0.1368895
5	0.7443372	-0.1384446
6	0.747189	-0.1392386
7	0.7486249	-0.1396243
8	0.7493492	-0.1398241
9	0.7497149	-0.1399251
10	0.7498996	-0.1399761
11	0.7499929	-0.1400019
12	0.7500401	-0.1400149
13	0.7500639	-0.1400215
14	0.7500759	-0.1400264
15	0.7500801	-0.1400264
16	0.7500838	-0.1400271

Table 4.35.b

Errors of the estimated parameters of the AR(2) model

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
ϕ_1	0.75	0.7500838	-0.0000838
ϕ_2	-0.14	-0.1400271	0.0000271
σ_a^2	1.0	0.9998062	0.0001938

Table 4.36

Theoretical ACF of the ARMA(1,1) model

$$\tilde{x}_t = 0.43\tilde{x}_{t-1} + a_t - 0.05a_{t-1}$$

Lags	1	2	3	4
1-4	0.3875247	0.1666356	0.0716533	0.030809
5-8	0.0132486	0.0056968	0.0024496	0.0010533
9-12	0.0004529	0.0001947	0.0000837	0.0000359
13-16	0.0000154	0.0000066	0.0000028	0.0000012
17-18	0.0000005	0.0000002		

Table 4.37

PACF of the ARMA(1,1) model

$$\tilde{x}_t = 0.43\tilde{x}_{t-1} + a_t - 0.05a_{t-1}$$

Lags	1	2	3	4
1-4	0.3875247	0.03801841	0.004453	0.00051972
5-8	0.60683E-4	0.703E-5	0.83E-6	0.32E-6
9-12	-0.2E-7	-0.5E-9	-0.4E-7	-0.2E-7
13-15	-0.3E-7	-0.5E-7		

Table 4.38

Errors in various initial rational approximations to f(cos λ)

k	E_1	E_2	E_3	E_4
-1.0	-0.131051	0.005840	-0.029716	0.00001
-0.9	-0.141864	-0.019378	-0.035253	0.00002
-0.8	-0.142363	-0.044706	-0.038977	0.00001
-0.7	-0.129705	-0.065189	-0.039267	-0.000001
-0.6	-0.101146	-0.075262	-0.033901	-0.000001
-0.5	-0.054745	-0.069695	-0.02027	-0.00001
-0.4	0.009204	-0.045308	0.003741	0.000001
-0.3	0.085851	-0.003584	0.03785	0
-0.2	0.163339	0.046594	0.076766	0
-0.1	0.222976	0.088855	0.109182	0
0	0.245643	0.105561	0.121991	0
0.1	0.222976	0.088855	0.10918	0
0.2	0.163339	0.046594	0.076766	0
0.3	0.085851	-0.003584	0.03785	0
0.4	0.099204	-0.04508	0.003741	0.000001
0.5	-0.054745	-0.069695	-0.02027	-0.000001
0.6	-0.101146	-0.075262	-0.033901	-0.000001
0.7	-0.129705	-0.065189	-0.039267	-0.000001
0.8	-0.142363	-0.044706	-0.038977	0
0.9	-0.141864	-0.019378	-0.035253	0
1.0	-0.131051	-0.005840	-0.029716	0

$\lambda = k\pi$

$$E_1 = f(\cos\lambda) - [0.5+0.3875247\cos(\lambda)]$$

$$E_2 = f(\cos\lambda) - [0.5+0.3875247\cos(\lambda)+0.1666356\cos(2\lambda)]$$

$$E_3 = f(\cos\lambda) - \left[\frac{0.4069919}{1-0.5712608\cos(\lambda)}\right]$$

$$E_4 = f(\cos\lambda) - \left[\frac{0.3591675-0.035847\cos(\lambda)}{1-0.7257995\cos(\lambda)}\right]$$

Table 4.39

Iteration table for the parameter ϕ_1

of the ARMA(1,1) model

Number of Iteration	$\hat{\phi}_1$
0	0.3628997
1	0.4106921
2	0.4241092
3	0.4241092
4	0.4294311
5	0.4298223
6	0.4299444
7	0.4299824
8	0.4299943
9	0.429998
10	0.4299992
11	0.4299996
12	0.4299997
13	0.4299997

Table 4.40

Iteration table for the parameter θ_1

of the ARMA(1,1) model

Number of Iteration	$\hat{\theta}_1$
0	0.0498751
1	0.0499991
2	0.0499997
3	0.0499997

Table 4.41

Errors of the estimated parameters of the ARMA(1,1) model

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
Φ_1	0.43	0.4299997	0.0000001
Θ_1	0.05	0.0499997	0.0000003
σ_a^2	1.0	1.0000027	-0.0000027

Table 4.42

Theoretical ACF of the ARMA(2,1) model

$\tilde{x}_t = -0.8\tilde{x}_{t-1} - 0.12\tilde{x}_{t-2} + a_t - 0.02a_{t-1}$

Lags	1	2	3	4	5
1-5	-0.725941	0.4607528	-0.2814893	0.1699011	-0.1021421
6-10	0.0613255	-0.0368034	0.0220837	-0.0132505	0.0079504
11-15	-0.0047703	0.0028622	-0.0017173	0.0010304	-0.0006183
16-20	0.000371	-0.0002227	0.0001336	-0.0000801	0.000048
21-25	-0.0000288	0.0000173	-0.0000105	0.0000064	-0.39E-5

Table 4.43

PACF of the ARMA(2,1) model

$\tilde{x}_t = -0.8\tilde{x}_{t-1} - 0.12\tilde{x}_{t-2} + a_t - 0.02a_{t-1}$

Lags	1	2	3	4	5
1-5	-0.725941	-0.01400337	-0.003941655	-0.000111103	-0.00000316
6-10	-0.00000015	-0.00000018	-0.000000022	0.000000152	0.00000022
11-15	-0.000000027	-0.000000058	0.000000098	-0.000000069	0.0

Table 4.44*

Errors in various initial rational approximation to $f(\cos \lambda)$

k	E_1	E_2	E_3	E_4
-1.0	0.017281	-0.004259	0.579387E-3	-0.23E-5
-0.9	0.009345	-0.004050	0.37536E-3	0.11E-5
-0.8	0.0030098	-0.003382	0.181213E-3	0
-0.7	-0.007976	-0.002481	0.52E-5	0
-0.6	-0.003952	-0.0015034	-0.146821E-3	0.16E-5
-0.5	-0.006259	-0.5644E-3	-0.2803E-3	0.15E-5
-0.4	-0.007917	0.276432E-3	-0.383243E-3	0.12E-5
-0.3	-0.009081	0.009630	-0.46584E-3	0
-0.2	-0.0098474	0.001472	-0.523046E-3	-0.5E-5
-0.1	-0.010281	0.001785	-0.555083E-3	0.14E-5
0	-0.010425	0.001887	-0.569031E-3	-0.13E-5
0.1	-0.010281	0.001785	-0.555083E-3	0.14E-5
0.2	-0.0098474	0.001472	-0.52304E-3	-0.5E-5
0.3	-0.009081	0.009630	-0.46584E-3	0
0.4	-0.007917	0.276432E-3	-0.383243E-3	0.12E-5
0.5	-0.006249	-0.5644E-3	-0.2803E-3	0.15E-5
0.6	-0.003952	-0.0015034	-0.146821E-3	0.16E-5
0.7	-0.007976	-0.002481	0.52E-5	0
0.8	0.0030098	-0.003382	0.181213E-3	0
0.9	0.009345	-0.004050	0.375316E-3	0.11E-5
1.0	0.017281	-0.004259	0.579387E-3	-0.23E-5

$\lambda = k\pi$

$$E_1 = f(\cos\lambda) - \frac{0.1392337}{1+0.9939272\cos(\lambda)}$$

$$E_2 = f(\cos\lambda) - \frac{0.1679876-0.0578591\cos(\lambda)}{1+0.914709\cos(\lambda)}$$

$$E_3 = f(\cos\lambda) - \frac{0.1358823}{1+1.1097737\cos(\lambda)+0.167974\cos(2\lambda)}$$

$$E_4 = f(\cos\lambda) - \frac{0.1402517-0.0078743\cos(\lambda)}{1+1.0832257\cos(\lambda)+0.1451134\cos(2\lambda)}$$

* The errors E_1, E_2, E_3 and E_4 are calculated for 201 values of $\lambda \in [-\pi, \pi]$ and twentyone values are given in this table.

Table 4.45

Iteration table for the parameters ϕ_1 and ϕ_2 of the ARMA(2,1) model

Number of Iteration	$\hat{\phi}_1$	$\hat{\phi}_2$
0	-0.5415856	-0.0725339
1	-0.6640067	-0.0941907
2	-0.7226349	-0.1051579
3	-0.7544004	-0.1112132
4	-0.772612	-0.1147115
5	-0.783373	-0.1167859
6	-0.7898419	-0.1180352
7	-0.7937703	-0.1187947
8	-0.7961702	-0.119259
9	-0.797642	-0.1195438
10	-0.7985466	-0.1197188
11	-0.7991033	-0.1198266
12	-0.7994462	-0.119893
13	-0.7996575	-0.1199339
14	-0.7997878	-0.1199591
15	-0.7998681	-0.1199747
16	-0.7999176	-0.1199843
17	-0.7999482	-0.1199902
18	-0.7999671	-0.1199938
19	-0.7999769	-0.1199961
20	-0.7999845	-0.1199973

Table 4.46

Iteration table for the parameter θ_1 of the ARMA(2,1) model

Number of Iteration	$\hat{\theta}_1$
0	0.0281297
1	0.0281519
2	0.0281519

Table 4.47

Errors of the estimated parameters of of the ARMA(2,1) model

Parameter (1)	Original value (2)	Estimated value (3)	Difference (2) - (3)
ϕ_1	0.8	0.7999845	0.0000155
ϕ_2	-0.12	-0.1199973	-0.0000027
θ_1	0.02	0.0281519	-0.0081519
σ_a^2	1.0	1.0365793	-0.0365793

Table 4.48

Estimated ACF of simulated MA(2) Series using the model $\tilde{x}_t = a_t - 0.5a_{t-1} + 0.2a_{t-2}$

Lags	1	2	3	4	5
1-5	-0.4260	0.1284	-0.0001	-0.0030	-0.0441
6-10	0.0551	-0.0575	-0.0129	0.0756	-0.0017
11-15	-0.0153	0.0813	0.0223	0.0879	0.0344

Table 4.49

Estimated PACF of simulated MA(2) Series using the model $\tilde{x}_t = a_t - 0.5a_{t-1} + 0.2a_{t-2}$

Lags	1	2	3	4	5
1-5	-0.4260	-0.0648	0.0375	0.0225	0.0520
6-10	0.0284	0.1259	0.2534	0.2877	0.1835
11-15	0.1518	-0.1272	0.0275	0.1821	0.1267

Table 4.50*

Errors in various initial rational approximations to $f(\cos\lambda)$

k	E_1	E_2
-1.0	0.1463719	-0.002896
-0.9	0.1030306	-0.0008629
-0.8	0.0420845	0.002400
-0.7	-0.0373564	0.002327
-0.6	-0.1049159	-0.001023
-0.5	-0.1314295	-0.003011
-0.4	-0.104727	0.0008342
-0.3	-0.0371476	0.002536
-0.2	0.042454	0.002463
-0.1	0.1028974	-0.0009945
0	0.1252918	-0.003126
0.1	0.102894	-0.0009945
0.2	0.042454	0.002463
0.3	-0.0371476	0.002536
0.4	-0.104727	0.0008342
0.5	-0.1314295	-0.003011
0.6	-0.1049159	-0.001023
0.7	-0.0373564	0.002327
0.8	0.0420845	0.002400
0.9	0.1030306	-0.0008629
1.0	0.1463719	-0.0028961

$\lambda = k\pi$

$E_1 = f(\cos\lambda) - [0.5-0.425975\cos(\lambda)]$

$E_2 = f(\cos\lambda) - [0.5-0.425975\cos(\lambda)+0.1284185\cos(2\lambda)]$

* The errors E_1 and E_2 are calculated for 201 values of $\lambda \in [-\pi,\pi]$ and twentyone values are included in this table.

Table 4.51

Iteration table for the rational approximation $T_o^{2*}(\cos\lambda)$

i	$E^{(i)}$	$a_o^{(i)}$	$a_1^{(i)}$	$a_2^{(i)}$
0	0.0031	0.5	-0.4259758	0.1284185
1	0.0026	0.4982572	-0.4236072	0.1271667
2	0.00258	0.498328	-0.4235113	0.1275661

Table 4.52

Iteration table for Θ_1 and Θ_2

Number of Iteration	Θ_1	Θ_2
0	0.4249322	-0.1279941
1	0.4542337	-0.1532025
2	0.4529915	-0.157407
3	0.4513533	-0.1574299
4	0.4509743	-0.1572412
5	0.4509486	-0.1571898
6	0.4509591	-0.1571848
7	0.450963	-0.1571858
8	0.4509636	-0.1571863
9	0.4509636	-0.1571864
10	0.4509635	-0.1571864

Table 4.53

Errors of the estimated values of the parameters from their original values

Parameters (1)	Original value (2)	Estimated value (3)	Error (2) - (3)
Θ_1	0.50	0.45096	0.04904
Θ_2	0.20	0.15719	0.04281
σ_a^2	1.00	0.9948319	0.00517

Table 4.54

Estimated ACF of the simulated AR(1) Series using the model $\tilde{x}_t = 0.74\tilde{x}_{t-1} + a_t$

Lags	1	2	3	4	5
1-5	0.7256	0.5311	0.3912	0.2714	0.1890
6-10	0.1552	0.1004	0.0682	0.0570	0.0490
11-15	0.0341	0.0218	-0.0350	0.0099	0.0131

Table 4.55

Estimated PACF of simulated AR(1) Series using the model $\tilde{x}_t = 0.74\tilde{x}_{t-1} + a_t$

Lags	1	2	3	4	5
1-5	0.7256	-0.1143	-0.0328	0.0064	-0.05598
6-10	-0.1140	-0.1099	0.0021	0.0591	-0.0959
11-15	0.0321	0.0252	0.0380	-0.0878	-0.0136

Table 4.56*

Errors in various initial rational approximations to $f(\cos\lambda)$

k	E_1	E_2
-1.0	0.3251	0.02086
-0.9	0.22969	-0.04066
-0.8	0.0349	-0.01061
-0.7	0.0507992	0.02600
-0.6	-0.14109	0.01623
-0.5	-0.3739	-0.02719
-0.4	-0.50222	0.00485
-0.3	-0.55227	-0.030850
-0.2	-0.37141	0.0492594
-0.1	0.36716	-0.008844
0	0.38400	-0.08102
0.1	0.36716	-0.008844
0.2	-0.37141	0.0492594
0.3	-0.55227	-0.030850
0.4	-0.50222	0.00485
0.5	-0.3739	-0.02719
0.6	-0.14109	0.01623
0.7	0.050799	0.02600
0.8	0.0349	-0.01061
0.9	0.22969	-0.04066
1.0	0.3251	0.02086

$\lambda = k\pi$

$$E_1 = f(\cos\lambda) - \left[0.5+0.7256\cos(\lambda)\right]$$

$$E_2 = f(\cos\lambda) - \left[\frac{0.15329}{1-0.95172\cos(\lambda)}\right]$$

* The errors E_1 and E_2 are calculated for 201 values of $\lambda \in [-\pi,\pi]$ and twentyone values are included in this table.

Table 4.57

Errors of $T_1^{o(i)}(\cos\lambda)$ from $f(\cos\lambda)$,

i being the number of iteration

k	$E_2^{(o)}$	$E_2^{(1)}$	$E_2^{(2)}$
-1.0	0.02086	0.02002	0.01987
-0.9	-0.04066	-0.04151	-0.04167
-0.8	-0.01061	-0.01153	-0.01169
-0.7	0.02600	0.02497	0.02478
-0.6	0.01623	0.01501	0.01478
-0.5	-0.02719	-0.02874	-0.02903
-0.4	0.00485	0.00277	0.00236
-0.3	-0.03085	-0.03387	-0.03448
-0.2	0.04926	0.04499	0.04399
-0.1	-0.00884	0.00948	-0.00819
0	-0.08102	-0.04499	-0.04399
0.1	-0.00884	0.00948	-0.00819
0.2	0.04926	0.04499	0.04399
0.3	-0.03085	-0.03387	-0.03448
0.4	0.00485	0.00277	0.00236
0.5	-0.02719	-0.02874	-0.02903
0.6	0.01623	0.01501	0.01478
0.7	0.02600	0.02497	0.02478
0.8	-0.01061	-0.01153	-0.01169
0.9	-0.04066	-0.04151	-0.04167
1.0	0.02086	0.02002	0.01987

$\lambda = kx\pi$

$$E_2^{(o)} = f(\cos\lambda) - \frac{0.15329}{1-0.95172\cos(\lambda)}$$

$$E_2^{(1)} = f(\cos\lambda) - \frac{0.15484}{1-0.95067\cos(\lambda)}$$

$$E_2^{(2)} = f(\cos\lambda) - \frac{0.15513}{1-0.95056\cos(\lambda)}$$

Table 4.58

Iteration table for the rational approximation $T_1^{o*}(\cos\lambda)$

i	$\lvert E_2^{(i)} \rvert$	$a_o^{(i)}$	$b_1^{(i)}$
0	0.08102	0.15329	-0.95172
1	0.04499	0.15484	-0.95067
2	0.04399	0.15513	-0.95056

Table 4.59

Iteration table for Φ_1

i	$\Phi_1^{(i)}$
0	0.4752824
1	0.5826455
2	0.6366292
3	0.6679128
4	0.6873094
5	0.6998031
6	0.7080397
7	0.7135511
8	0.7172749
9	0.7198072
10	0.7215369
11	0.7227217
12	0.7235351
13	0.7240942
14	0.7244788
15	0.7247436
16	0.7249261
17	0.7250517
18	0.7251384
19	0.7251981
20	0.7252392
21	0.7252676
22	0.7252871
23	0.7253006
24	0.7253099
25	0.7253163

Table 4.60

Errors of the estimated values of the parameters from their original values

Parameter (1)	Original value (2)	Estimated value (3)	Error (2) - (3)
Φ_1	0.74	0.7253	0.0147
σ_a^2	1.00	1.00963	-0.00963

Table 4.61

Estimated ACF of the simulated ARMA(1,1) series using the model $\tilde{x}_t = 0.68\tilde{x}_{t-1}+a_t-0.09a_{t-1}$

Lags	1	2	3	4	5
1-5	0.62537	0.42323	0.28917	0.19795	0.13035
6-10	0.09061	0.05982	0.04210	0.01987	0.01638
11-15	-0.09233	0.05863	0.01525	0.018976	0.02977

Table 4.62

Estimated PACF of the simulated ARMA(1,1) series using the model $\tilde{x}_t = 0.68\tilde{x}_{t-1}+a_t -0.09a_{t-1}$

Lags	1	2	3	4	5
1-5	0.62537	0.01458	-0.08115	-0.07035	-0.00849
6-10	-0.02119	-0.06611	0.06957	-0.00805	-0.03455
11-15	0.02708	0.04077	-0.06091	-0.01867	-0.01199

Table 4.63*

Errors in various initial rational approximations to $f(\cos\lambda)$

k	E_1	E_2	E_3
-1.0	0.03669	0.02253	0.02306
-0.9	-0.00322	-0.01689	-0.01652
-0.8	0.01892	0.00679	0.00676
-0.7	0.00614	-0.00313	-0.00369
-0.6	0.00422	-0.00030	-0.00125
-0.5	-0.10064	0.01562	0.01483
-0.4	-0.04213	-0.02658	0.02599
-0.3	-0.00903	0.02588	0.03003
-0.2	-0.10834	-0.05308	-0.04962
-0.1	0.08610	0.07662	0.07366
0	0.21863	-0.02340	-0.09493
0.1	0.08610	0.07662	0.07366
0.2	-0.10834	-0.05308	-0.04962
0.3	-0.00903	0.02588	0.03003
0.4	-0.04213	-0.02658	-0.02599
0.5	-0.10064	0.01562	0.01483
0.6	0.00422	-0.00030	-0.00125
0.7	0.00614	-0.00313	-0.00369
0.8	0.01892	0.00679	0.00676
0.9	-0.00322	-0.01699	-0.01652
1.0	0.03669	0.02253	0.02306

$\lambda = k\pi$

$$E_1 = f(\cos\lambda) - \frac{0.21374}{1-0.90012\cos(\lambda)}$$

$$E_2 = f(\cos\lambda) - \frac{0.21059-0.03327\cos(\lambda)}{1-0.92556\cos(\lambda)}$$

$$E_3 = f(\cos\lambda) - \frac{0.20888-0.03717\cos(\lambda)-0.0025\cos(2\lambda)}{1-0.92558\cos(\lambda)}$$

* The errors E_1, E_2 and E_3 are calculated for 201 values of $\lambda \in [-\pi,\pi]$ and twentyone values are included in this table.

Table 4.64

Errors of $T_1^{1(i)}(\cos\lambda)$ from $f(\cos\lambda)$,

i being the number of iteration

k	$E_2^{(o)}$	$E_2^{(1)}$
-1.0	0.02253	0.02165
-0.9	-0.01689	-0.01778
-0.8	0.00679	0.00589
-0.7	-0.00313	-0.00405
-0.6	-0.00030	-0.00124
-0.5	0.01562	0.01463
-0.4	-0.02658	-0.02766
-0.3	0.02588	0.02460
-0.2	-0.05308	-0.05494
-0.1	0.07662	0.07271
0	-0.02340	-0.03087
0.1	0.07662	0.07271
0.2	-0.05308	-0.05494
0.3	0.02588	0.02460
0.4	-0.02658	-0.02766
0.5	0.01562	0.01463
0.6	-0.00030	-0.00124
0.7	-0.00314	-0.00405
0.8	0.00679	0.00589
0.9	-0.01689	-0.01778
1.0	0.02253	0.02165

$\lambda = k\pi$

$$E_2^{(o)} = f(\cos\lambda) - \frac{0.21059-0.03327\cos(\lambda)}{1-0.92556}$$

$$E_2^{(1)} = f(\cos\lambda) - \frac{0.21158-0.03401\cos(\lambda)}{1-0.925687\cos(\lambda)}$$

Table 4.65

Iteration table for the rational approximation $T_1^{1*}(\cos\lambda)$

i	$\lvert E_1^{(i)} \rvert$	$a_o^{(i)}$	$a_1^{(i)}$	$b_1^{(i)}$
0	0.07662	0.21059	-0.03327	-0.92556
1	0.07271	0.21158	-0.03401	-0.925687

Table 4.66

Iteration table for ϕ_1

i	$\phi_1^{(i)}$
0	0.4628435
1	0.5619957
2	0.6090276
3	0.6345189
4	0.6491909
5	0.6579083
6	0.6631822
7	0.666407
8	0.6683915
9	0.6696175
10	0.6703768
11	0.6708477
12	0.67114
13	0.6713216
14	0.6714344
15	0.6715045
16	0.6715481
17	0.6715752
18	0.671592
19	0.6716025
20	0.671609
21	0.671613
22	0.6716156
23	0.6716171
24	0.6716181
25	0.6716187
26	0.6716191
27	0.6716193
28	0.6716195
29	0.6716195

Table 4.67

Iteration table for Θ_1

i	$\Theta_1^{(i)}$
1	0.0803714
2	0.0808906
3	0.0808973
4	0.0808974
5	0.0808974

Table 4.68

Errors of the estimated values of the parameters from their original values

Parameters (1)	Original value (2)	Estimated value (3)	Error (2)-(3)
Φ_1	0.68	0.6716	0.0084
Θ_1	0.09	0.0809	0.0091
σ_a^2	1.000	0.9959	0.0041

Table 4.69

Estimated ACF of Series C

Lags	1	2	3	4	5
1-5	0.98	0.94	0.90	0.85	0.80
6-10	0.75	0.69	0.64	0.58	0.52
11-15	0.47	0.41	0.36	0.30	0.25

Table 4.70

Estimated ACF of the differenced Series ∇x_t, where $\{x_t\}$ is Series C

Lags	1	2	3	4	5
1-5	0.80	0.65	0.53	0.44	0.38
6-10	0.32	0.26	0.19	0.14	0.14
11-15	0.10	0.09	0.07	0.07	0.07

Table 4.71

Estimated PACF of the differenced series ∇x_t, where $\{x_t\}$ is Series C

Lags	1	2	3	4	5
1-5	0.80	-0.01	-0.01	0.06	0.03
6-10	-0.03	-0.01	-0.08	0.00	0.10
11-15	-0.14	0.10	-0.05	0.05	0.02

Table 4.72

Errors of $T_1^{o(i)}(\cos\lambda)$ from $f(\cos\lambda)$, i being the number of iteration

k	$E_1^{(0)}$	$E_1^{(1)}$	$E_1^{(2)}$
-1.0	0.024477	0.0225291	0.0220586
-0.9	0.0095755	0.00758103	0.007099
-0.8	0.0297664	0.0276275	0.0271093
-0.7	0.0110529	0.00864012	0.00805289
-0.6	0.0320509	0.0291747	0.0284691
-0.5	0.0105787	0.0069276	0.0060191
-0.4	0.0519093	0.0469252	0.0456505
-0.3	0.0773336	0.0699969	0.0679969
-0.2	0.0541914	0.0448903	0.0399194
-0.1	0.0391157	0.0473802	0.0409147
0	-0.3091911	-0.0699933	-0.06799
0.1	0.0391157	0.0473802	0.0409147
0.2	0.0541919	0.0448903	0.0399194
0.3	0.0773336	0.0699969	0.0679969
0.4	0.0519093	0.0469252	0.0456505
0.5	0.0105787	0.0069276	0.0060191
0.6	0.0320509	0.0291747	0.0284691
0.7	0.0110529	0.00864012	0.00805289
0.8	0.0297664	0.0276275	0.0271093
0.9	0.0095755	0.00758103	0.007899
1.0	0.024477	0.0225291	0.0220586

$$\lambda = k\pi$$

$$E_1^{(0)} = f(\cos\lambda) - \frac{0.1121212}{1-0.9696969\cos(\lambda)}$$

$$E_1^{(1)} = f(\cos\lambda) - \frac{0.11577}{1-0.96655\cos(\lambda)}$$

$$E_1^{(2)} = f(\cos\lambda) - \frac{0.11668}{1-0.96627\cos(\lambda)}$$

Table 4.73

Iteration table for the rational approximation $T_1^{1*}(\cos\lambda)$

Iteration Number	$\lvert E_1^{(i)} \rvert$	$a_o^{(i)}$	$b_1^{(i)}$
0	0.3091911	0.1121212	-0.9696969
1	0.0699933	0.1157724	-0.9665475
2	0.0679969	0.1166809	-0.9662654

Table 4.74

Iteration table for Φ_1

Iteration Number	$\Phi_1^{(i)}$
0	-0.485
1	-0.5990841
2	-0.6590673
3	-0.6956693
4	-0.7197185
5	-0.7362275
6	-0.747885
7	-0.756276
8	-0.7623974
9	-0.7669061
10	-0.7702503
11	-0.7727435
12	-0.7746092
13	-0.7760094
14	-0.7770624
15	-0.7778556
16	-0.7784538
17	-0.7789053
18	-0.7792463
19	-0.779504
20	-0.7796988
21	-0.7798462
22	-0.7799576
23	-0.7800419
24	-0.7801057
25	-0.780154
26	-0.7801905
27	-0.7802181
28	-0.7802391
29	-0.7802549
30	-0.7802669
31	-0.7802759
32	-0.7802828
33	-0.780288
34	-0.7802919
35	-0.7802949

Table 4.75

The estimated ACF of Series D

Lags	1	2	3	4	5
1-5	0.86	0.74	0.62	0.53	0.46
6-10	0.41	0.35	0.31	0.27	0.24
11-15	0.22	0.20	0.18	0.15	0.14
16-20	0.13	0.16	0.19	0.21	0.23

Table 4.76

The estimated PACF of Series D

Lags	1	2	3	4	5
1-5	0.86	-0.02	0.00	0.01	0.03
6-10	0.03	-0.02	0.01	0.00	0.01
11-15	0.05	0.01	-0.04	-0.03	0.07

Table 4.77

Errors of $T_1^{o(i)}(\cos\lambda)$ from $f(\cos\lambda)$, i being the number of iteration

k	$E_1^{(0)}$	$E_1^{(1)}$	$E_1^{(2)}$
-1.0	-0.0576878	0.050147	0.050708
-0.9	-0.006616	0.001112	0.005374
-0.8	0.0659981	0.074327	0.073708
-0.7	0.033901	0.043386	0.042683
-0.6	-0.0786528	-0.067188	0.068036
-0.5	0.0650575	0.079363	0.0789
-0.4	0.1194575	0.1409047	0.139341
-0.3	0.1662093	0.201523	0.199004
-0.2	0.00706135	0.079542	0.074697
-0.1	-0.4100499	-0.189056	-0.198593
0	-0.3599708	0.005712	0.18795
0.1	-0.4100499	-0.189056	-0.198593
0.2	0.00706135	0.079542	0.074697
0.3	0.1662093	0.201523	0.199004
0.4	0.1194575	0.1409047	0.139341
0.5	0.0650575	0.079363	0.0789
0.6	-0.0786528	0.067188	0.068036
0.7	0.0339101	0.043386	0.042683
0.8	0.0659981	0.074327	0.073708
0.9	-0.006616	0.001112	0.005374
1.0	-0.0576878	0.050147	0.050708

$\lambda = k\pi$

$$E^{(0)} = f(\cos\lambda) - \frac{0.0749425}{1-0.9885057\cos(\lambda)}$$

$$E^{(1)} = f(\cos\lambda) - \frac{0.060637}{1-0.9902507\cos(\lambda)}$$

$$E^{(2)} = f(\cos\lambda) - \frac{0.0611002}{1-0.9897238\cos(\lambda)}$$

Table 4.78

Iteration table for the rational approximation

$T_1^{o*}(\cos \lambda)$

Iteration Number	$\lvert E^{(i)} \rvert$	$a_o^{(i)}$	$b_1^{(i)}$
0	0.41005	0.0749425	-0.9885057
1	0.201523	0.060637	-0.990250
2	0.199004	0.0611002	-0.989724

Table 4.79

Iteration table for Φ_1

Iteration Number	Φ_1
0	0.494862
1	0.6160479
2	0.6826696
3	0.7254863
4	0.7553229
5	0.777187
6	0.7937684
7	0.8086588
8	0.8168679
9	0.8250702
10	0.8317347
11	0.8371989
12	0.8417118
13	0.8454612
14	0.8485916
15	0.8534235
16	0.851216
17	0.8552856
18	0.8568602
19	0.8581944
20	0.8593266
21	0.860289
22	0.861108

Table 4.80

Estimated ACF of the population data

Lags	1	2	3	4	5
1-5	0.9150	0.8287	0.7450	0.6604	0.5762
6-10	0.4920	0.4096	0.3280	0.2475	0.1678
11-15	0.0893	0.0132	-0.0518	-0.1133	0.1700

Table 4.81

Estimated ACF of the first differenced population data

Lags	1	2	3	4	5
1-5	0.7425	0.6296	0.5690	0.52190	0.4819
6-10	0.4111	0.1519	0.0672	0.0490	0.1740
11-15	0.1503	0.0298	-0.0410	-0.0302	-0.0555

Table 4.82

Estimated ACF of $\nabla^2 P_t$, $\{P_t\}$ being the population data

Lags	1	2	3	4	5
1-5	-0.52661	0.0436	-0.0064	-0.0084	0.0012
6-10	0.0069	-0.0276	0.0370	-0.1273	0.1389
11-15	0.0199	0.0409	-0.0439	0.0273	-0.0763

Table 4.83

Estimated PACF of $\nabla^2 P_t$

Lags	1	2	3	4	5
1-5	-0.5266	-0.3234	-0.2261	-0.1815	-0.1497
6-10	-0.1135	-0.1337	-0.0851	-0.0431	-0.0205

Table 4.84*

Errors in various initial rational approximations to $f(\cos\lambda)$

k	E_1	E_2
-1.0	-0.0436	-0.00364
-0.9	-0.0415	-0.00198
-0.8	-0.03527	0.00144
-0.7	-0.02563	0.00358
-0.6	-0.01347	0.00260
-0.5	0	-0.00053
-0.4	0.01347	-0.00315
-0.3	0.02563	-0.00325
-0.2	0.03527	-0.00083
-0.1	0.0415	0.00218
0	0.0436	0.00352
0.1	0.0415	0.00218
0.2	0.03527	-0.00083
0.3	0.02563	-0.00325
0.4	0.01347	-0.00315
0.5	0	-0.00053
0.6	-0.01347	0.00260
0.7	-0.02563	0.00358
0.8	-0.03527	0.00144
0.9	-0.0415	-0.00198
1.0	-0.0436	-0.00364

$\lambda = k\pi$

$E_1 = f(\cos\lambda) - [0.5-0.52661\cos(\lambda)]$

$E_2 = f(\cos\lambda) - \dfrac{0.4569-0.44119\cos(\lambda)}{1+0.16370\cos(\lambda)}$

* The errors E_1 and E_2 are calculated for 201 values $\lambda \in [-\pi,\pi]$ and twentyone values are included in this table.

Table 4.85

Iteration table for the rational approximation $T_1^{1*}(\cos \lambda)$

Iteration Number	$\lvert E^{(i)} \rvert$	$a_o^{(i)}$	$a_1^{(i)}$	$b_1^{(i)}$
0	0.00358	0.456896	-0.441190	0.163702
1	0.003	0.456314	-0.4399473	0.1676337
2	0.003	0.456314	-0.4399473	0.1676337

Table 4.86

Iteration table for Φ_1

Iteration Number	$\Phi_1^{(i)}$
0	-0.0838168
1	-0.0844056
2	-0.0844139
3	-0.0844141
4	-0.0844141

Table 4.87

Iteration table for Θ_1

Iteration Number	$\Theta_1^{(i)}$
0	0.4820664
1	0.5940928
2	0.6522099
3	0.6871267
4	0.7096707
5	0.7248507
6	0.7353482
7	0.7427375
8	0.7480027
9	0.7517864
10	0.754522
11	0.7565085
12	0.7579555
13	0.7590118
14	0.7597844
15	0.76035
16	0.7607644
17	0.761084
18	0.7612913
19	0.761455
20	0.7615751
21	0.761728
22	0.7617756
23	0.7618105
24	0.7616633
25	0.761728

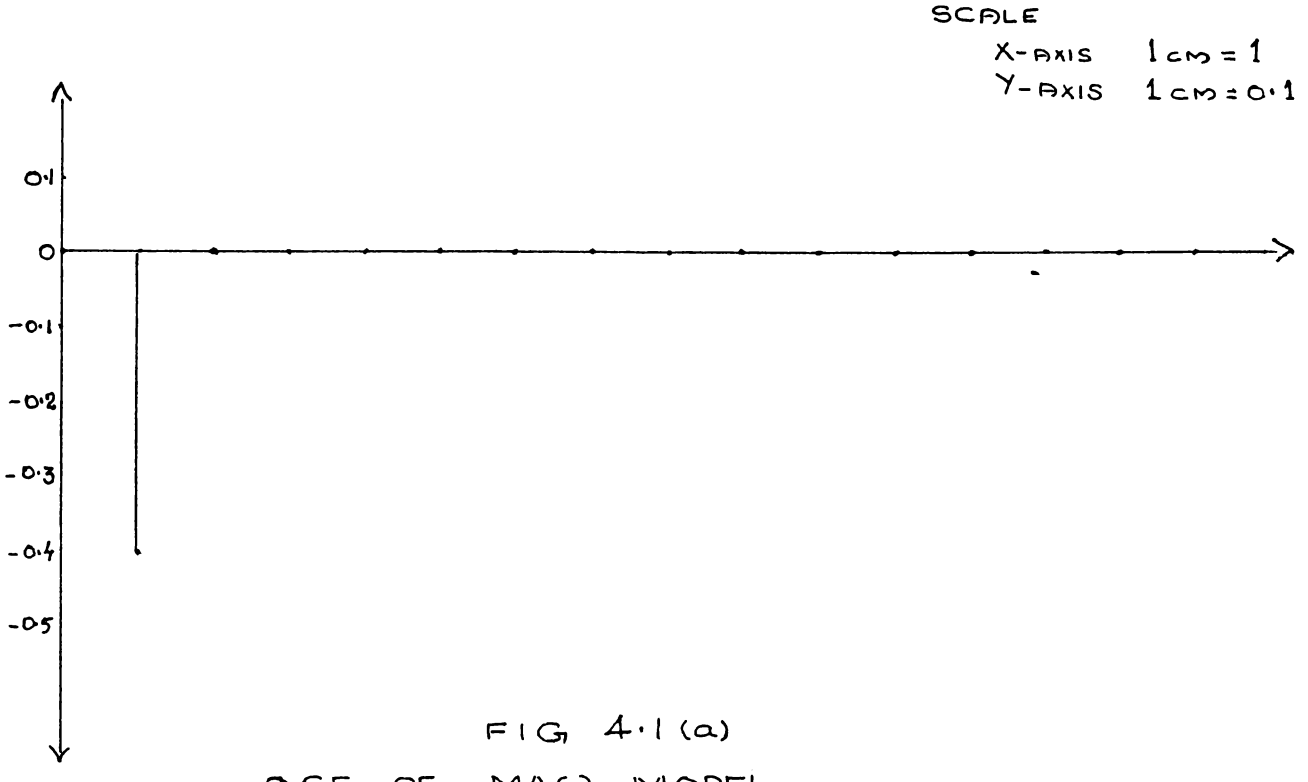


FIG 4·1 (a)

ACF OF MA(1) MODEL

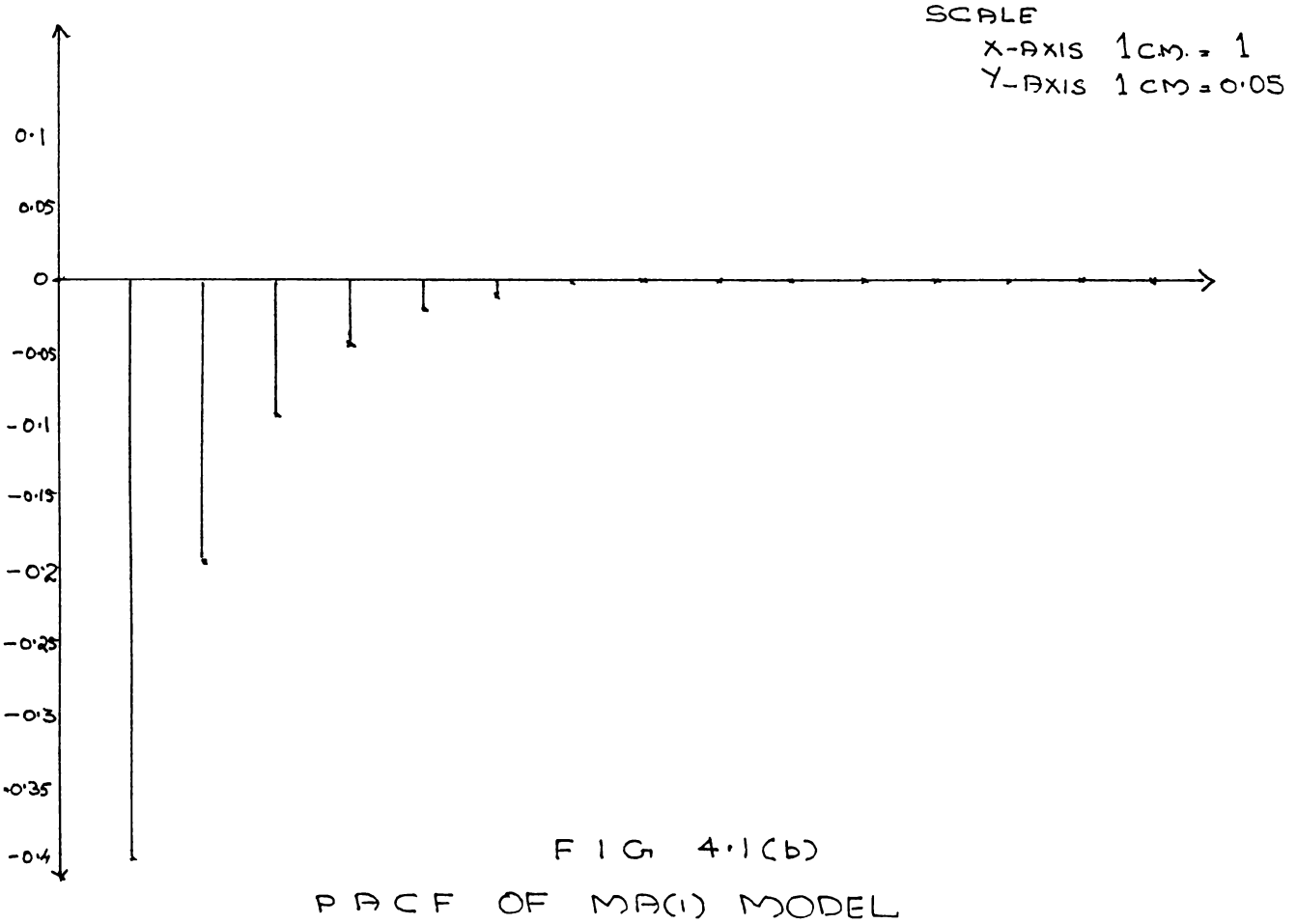


FIG 4·1 (b)

PACF OF MA(1) MODEL

SCALE
X-AXIS 1cm = 1
Y-AXIS 1cm = 0.2

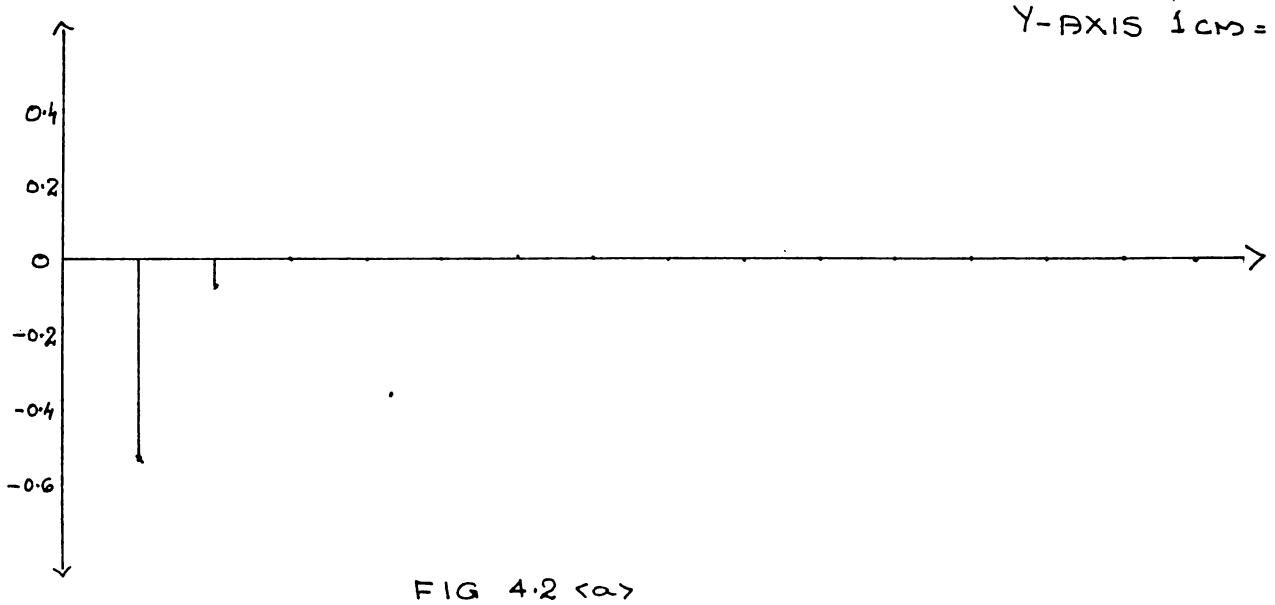


FIG 4.2 (a)
ACF OF MA(2) MODEL

SCALE
X-AXIS 1cm = 1
Y-AXIS 1cm = 0.05

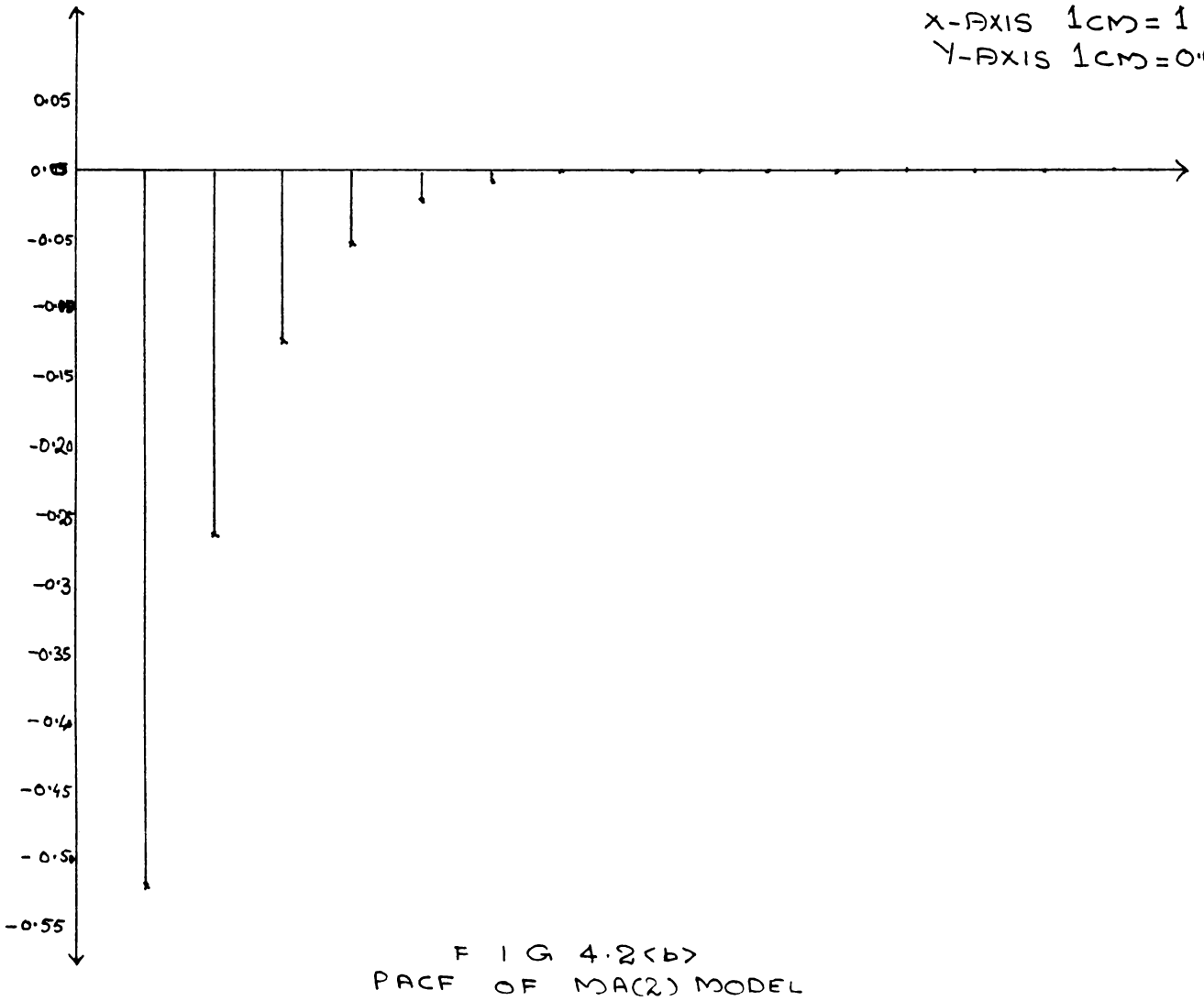


FIG 4.2(b)
PACF OF MA(2) MODEL

SCALE

X-AXIS 1 CM = 0·1

Y-AXIS 1 CM = 0·05

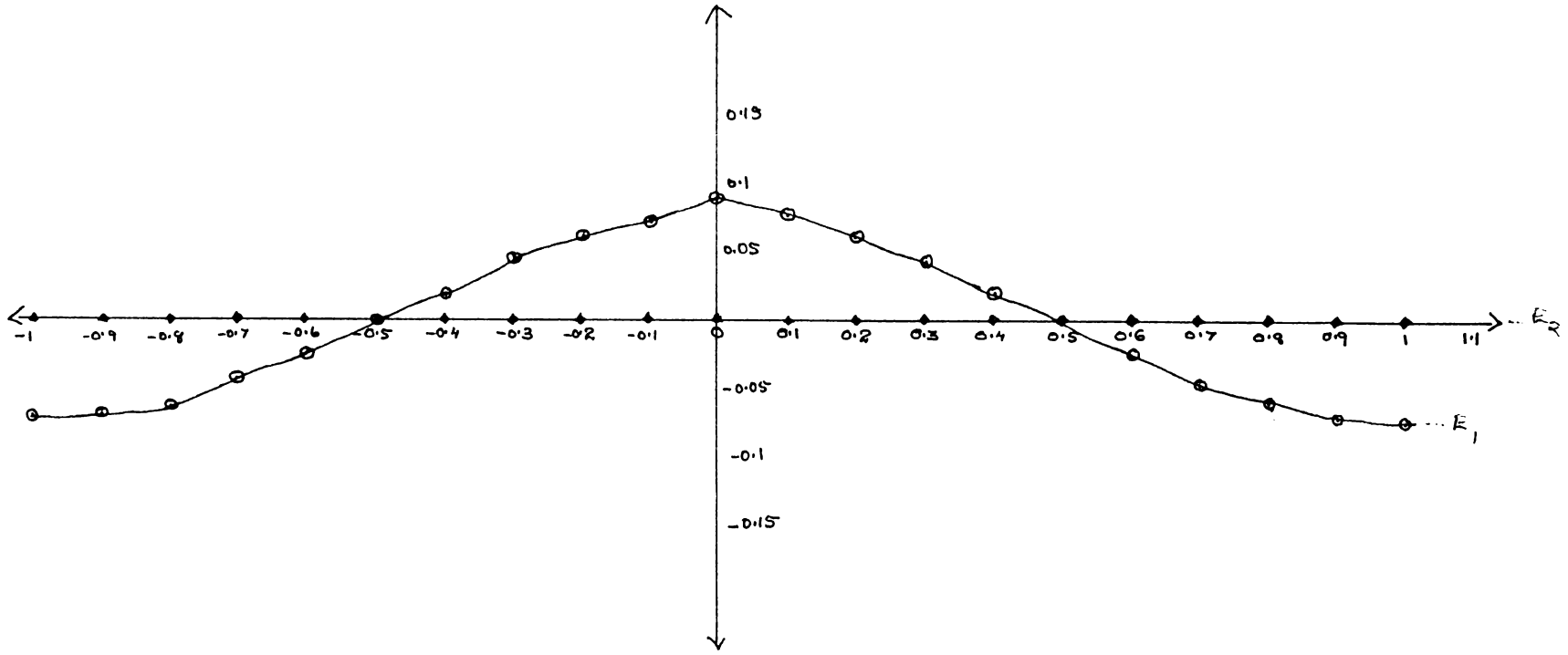


FIG 4·3

ERRORS OF INITIAL R-SPEC ; TABLE - 4·7

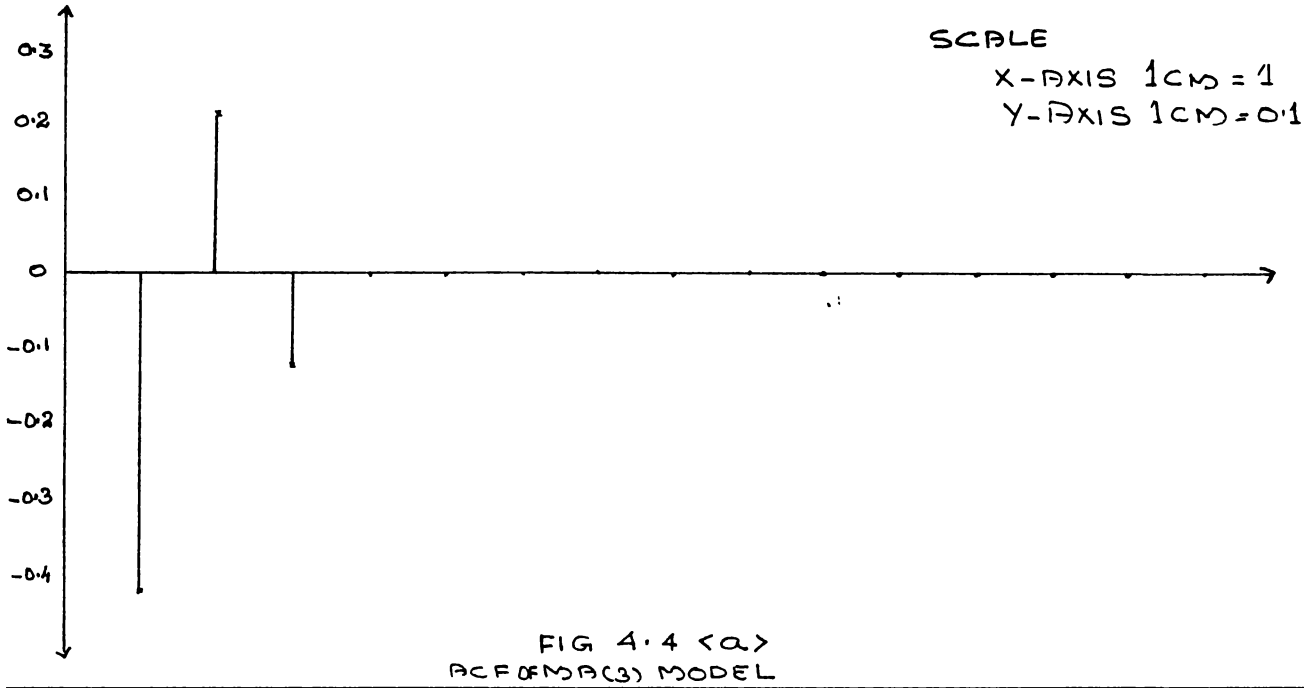


FIG 4·4 <a>
ACF OF MA(3) MODEL

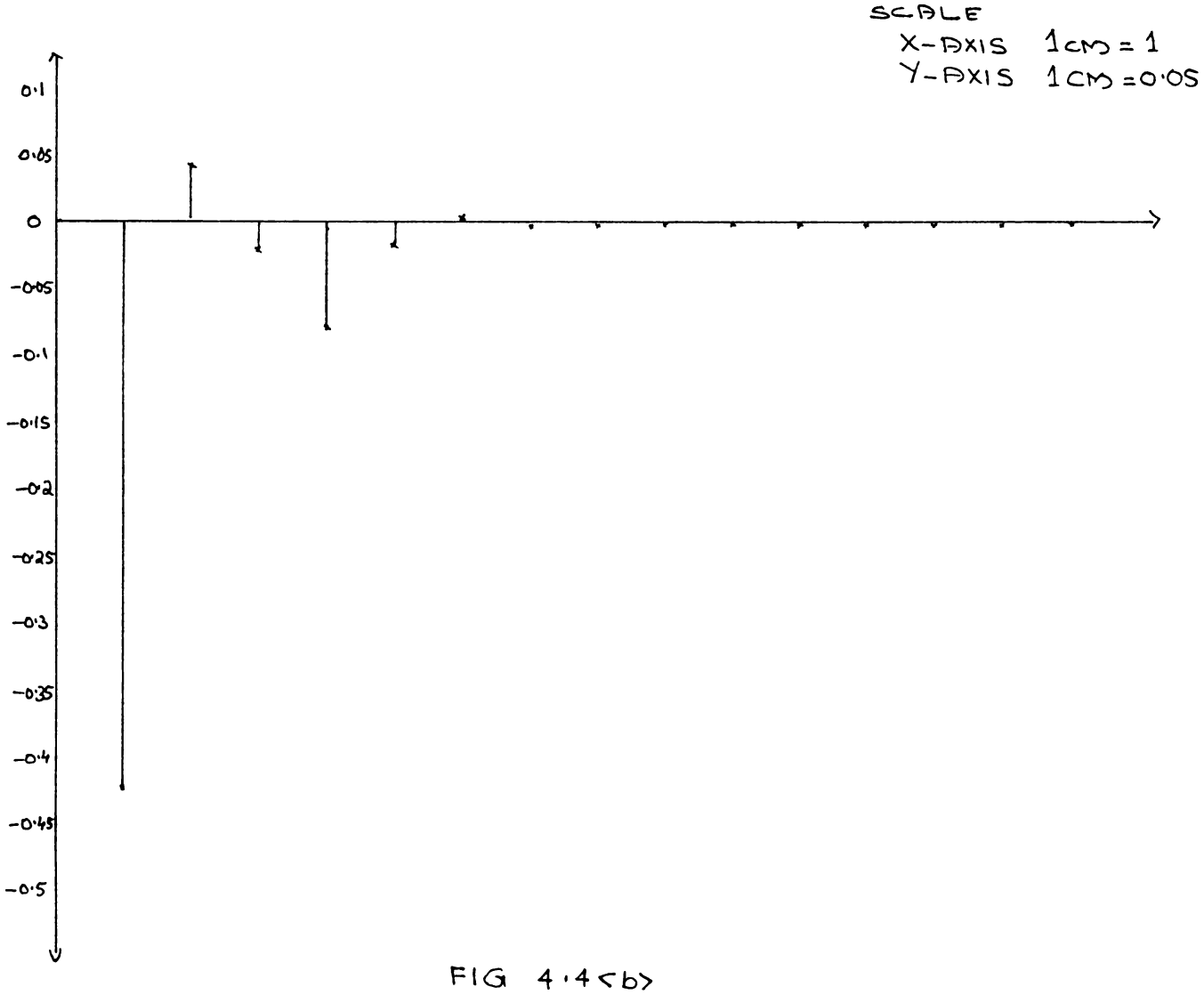


FIG 4·4 <b>
PACF OF MA(3) MODEL

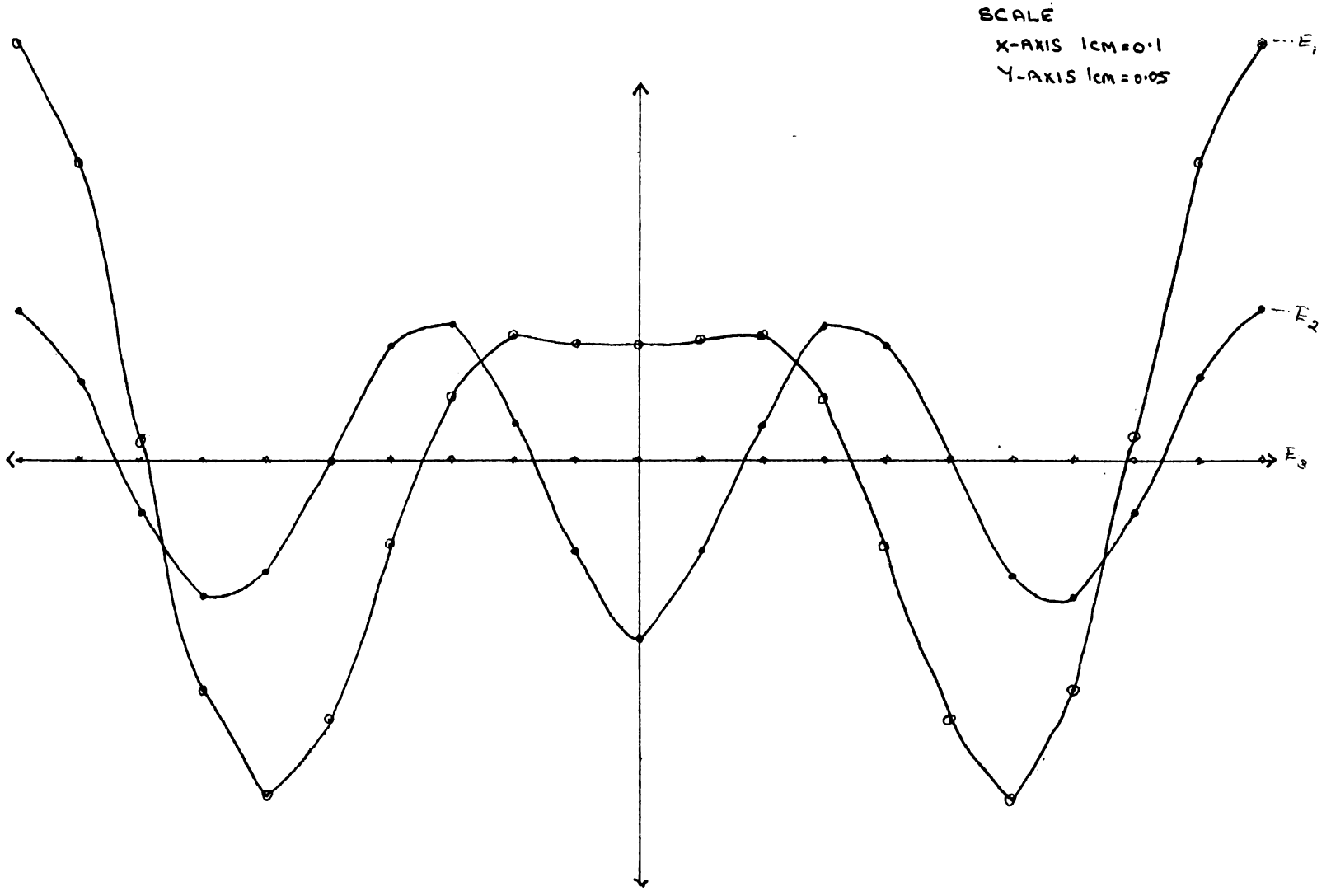


FIG 4.5
ERRORS OF INITIAL R-SPEC ; TABLE 4.12

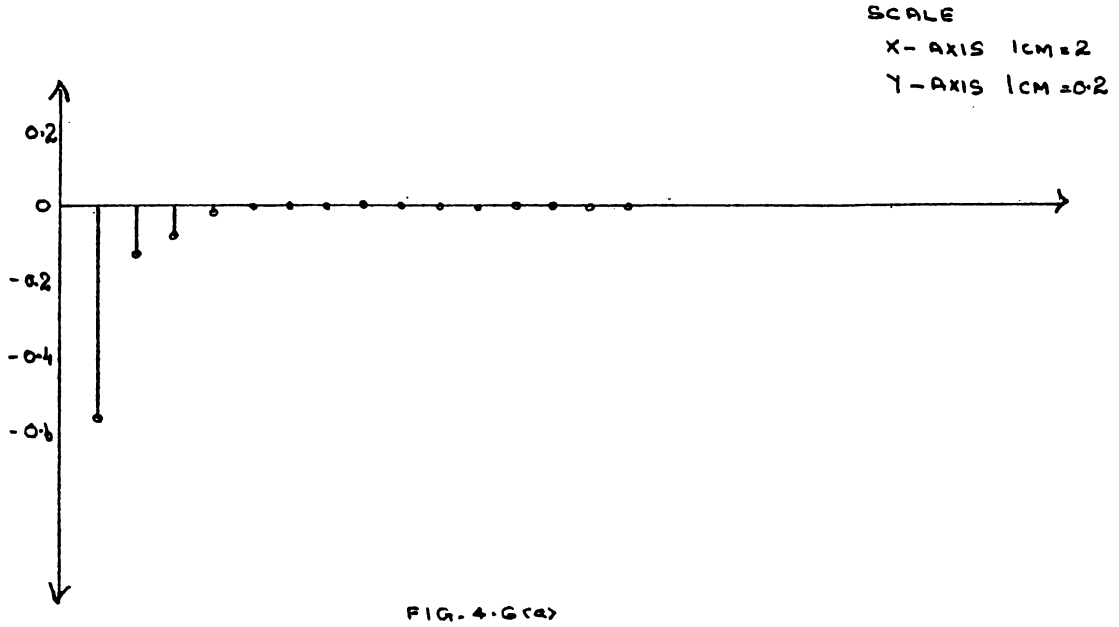


FIG. 4·6(a)
ACF OF MA(4) MODEL

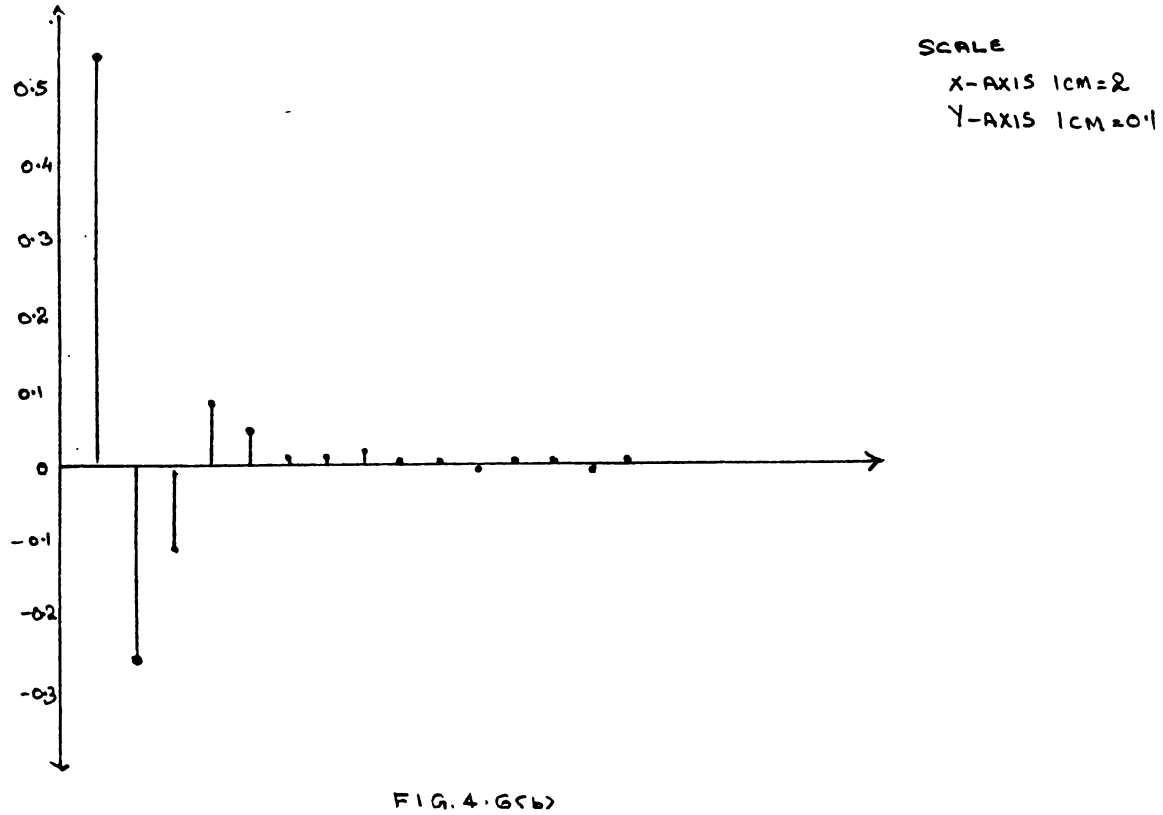


FIG. 4·6(b)
PACF OF MA(4) MODEL

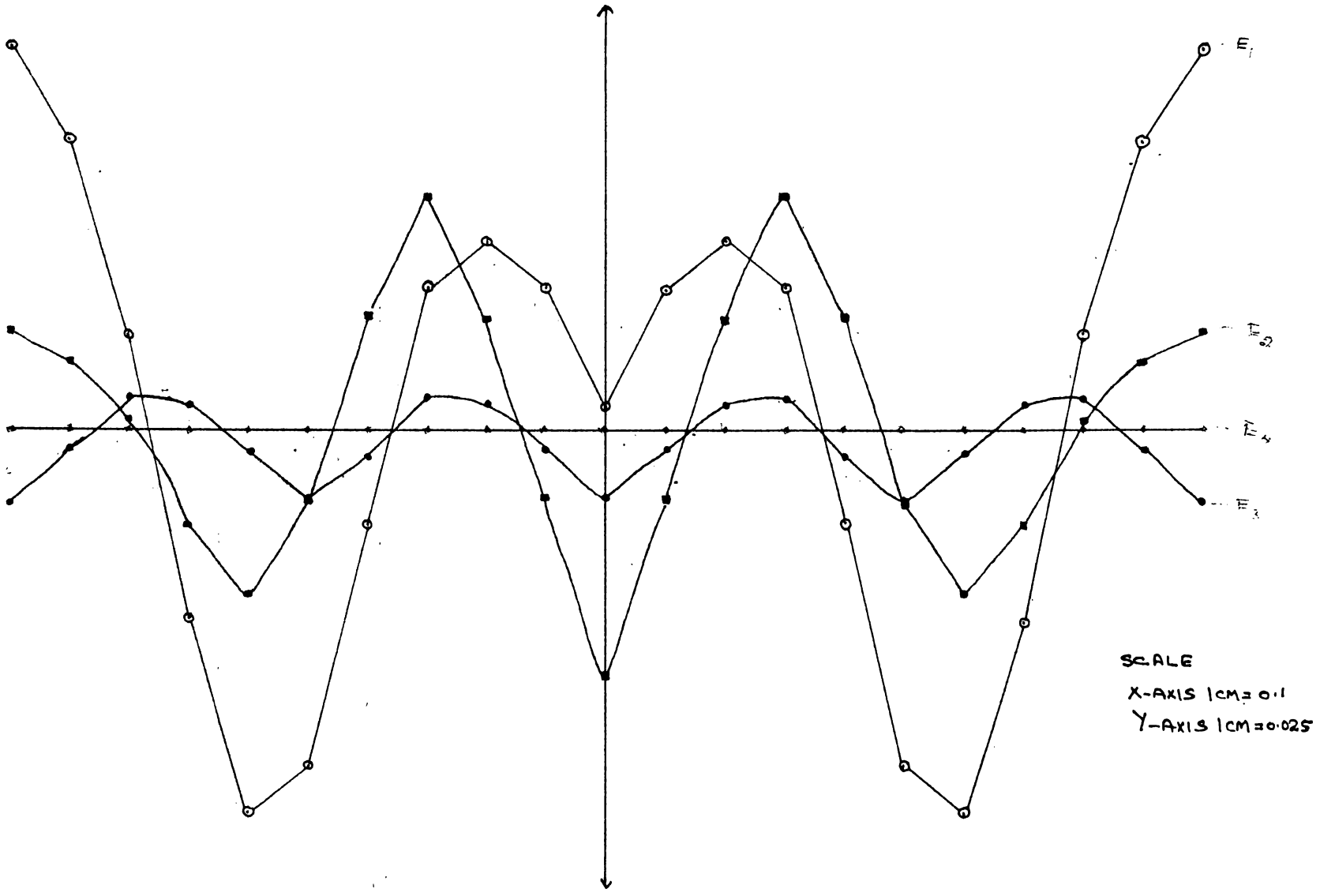


FIG 4.7

ERRORS OF INITIAL R-SPEC : TABLE 4.17

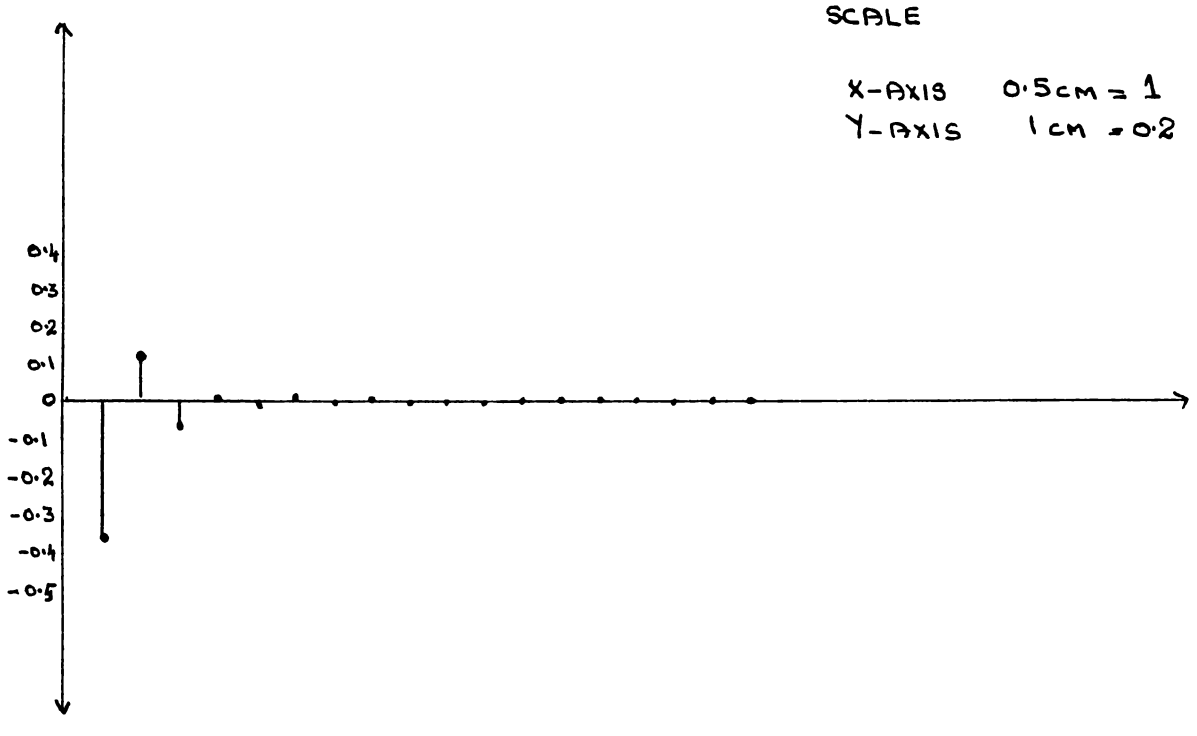


FIG 4.8(a)
ACF OF ARCI) MODEL I

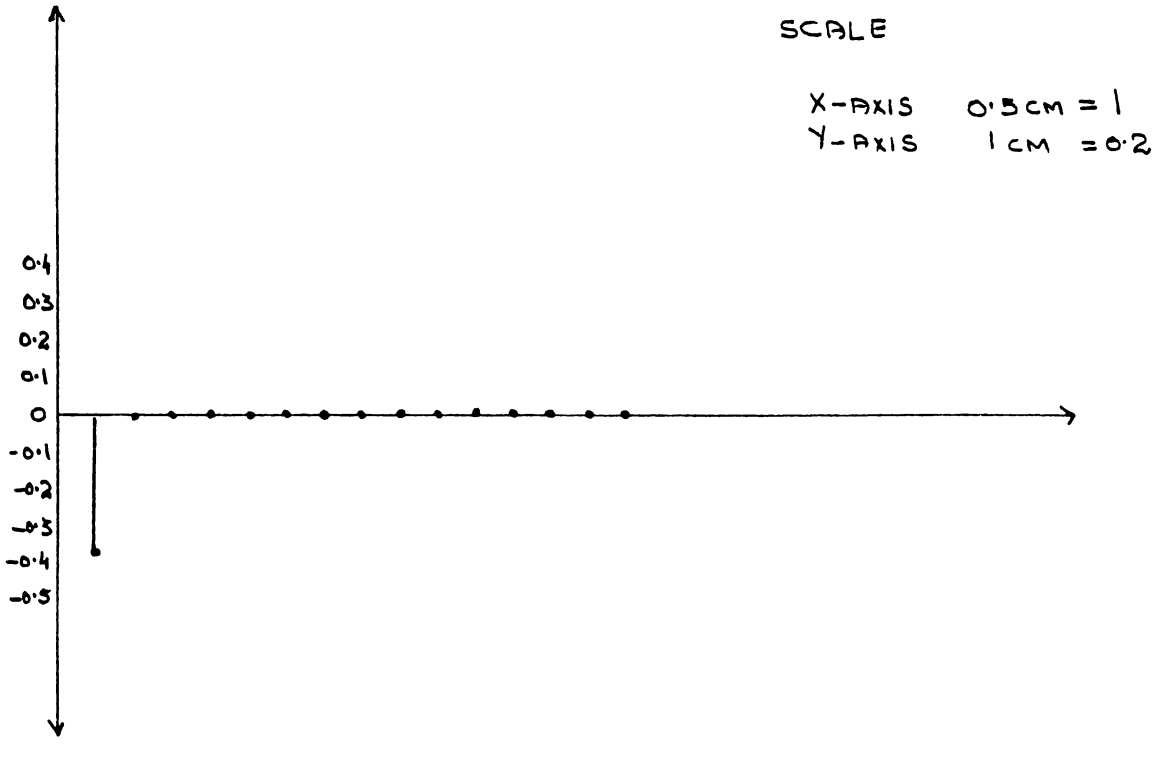


FIG 4.8(b)
PACF OF ARCI) MODEL I

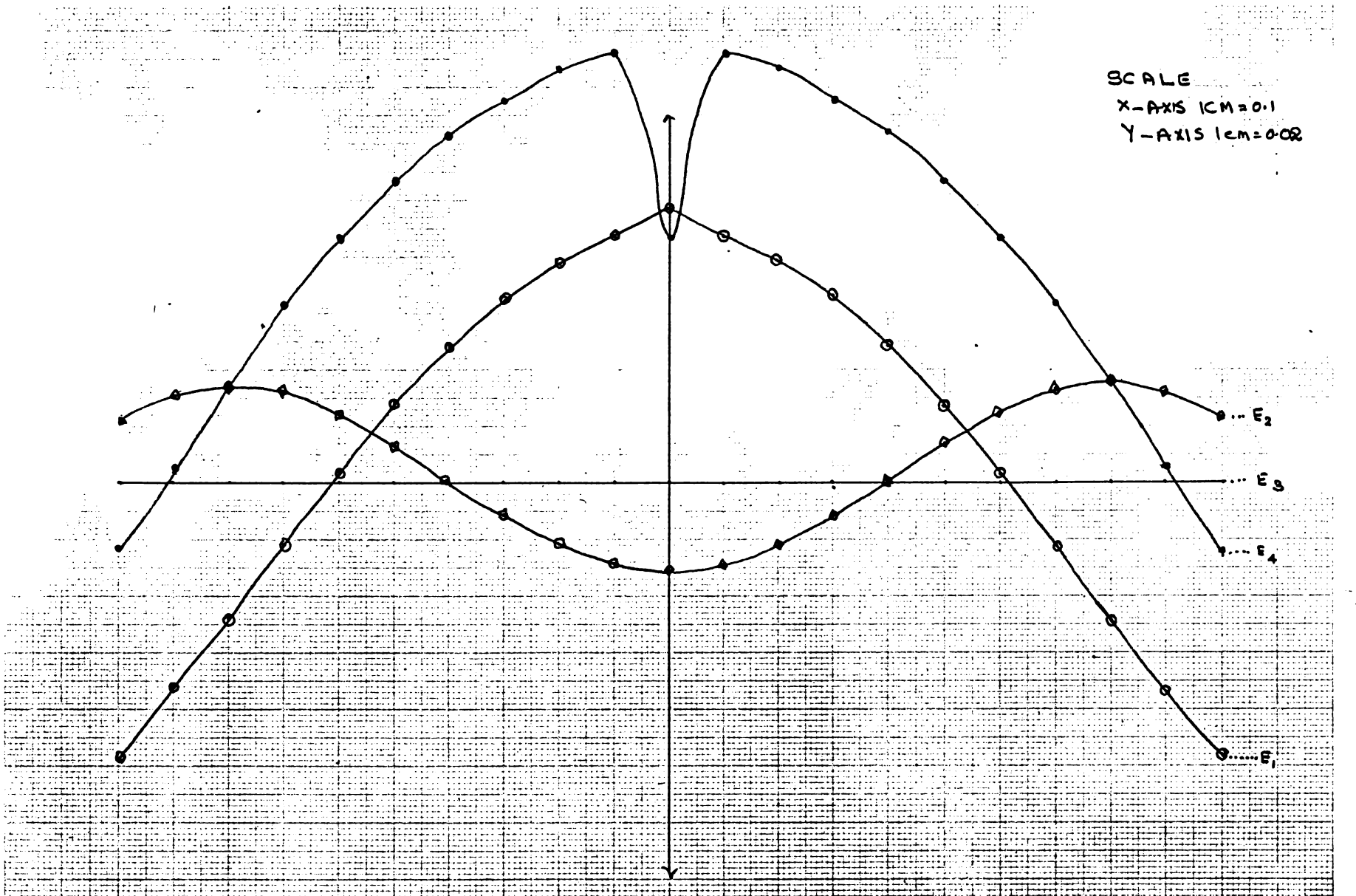


FIG 4.9

ERRORS OF INITIAL R-SPEC ; TABLE 4.22

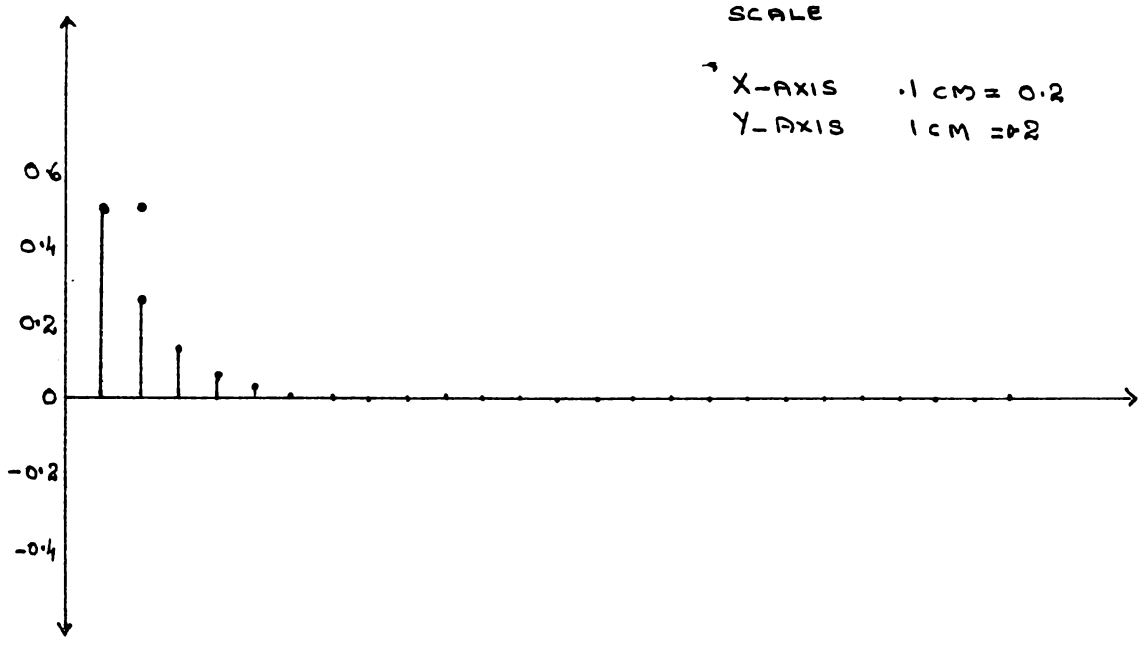


FIG 4.10(a)
ACF OF ARCI MODEL II

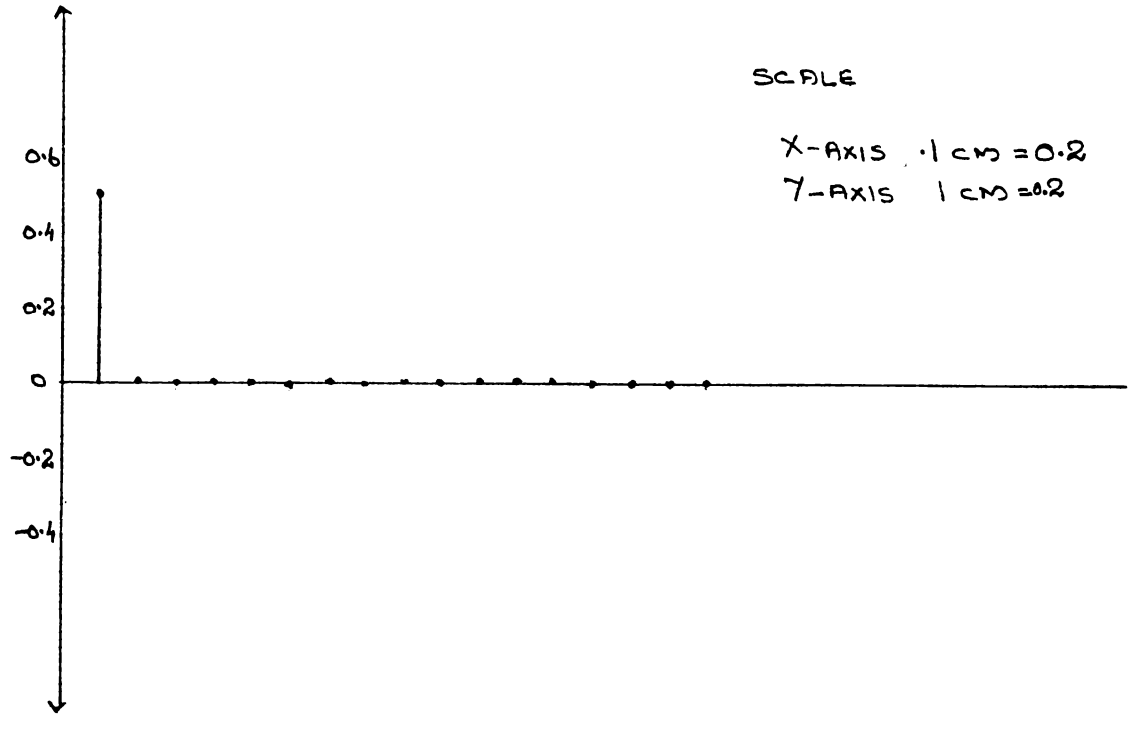


FIG 4.10(b)
PACF OF ARCI MODEL II

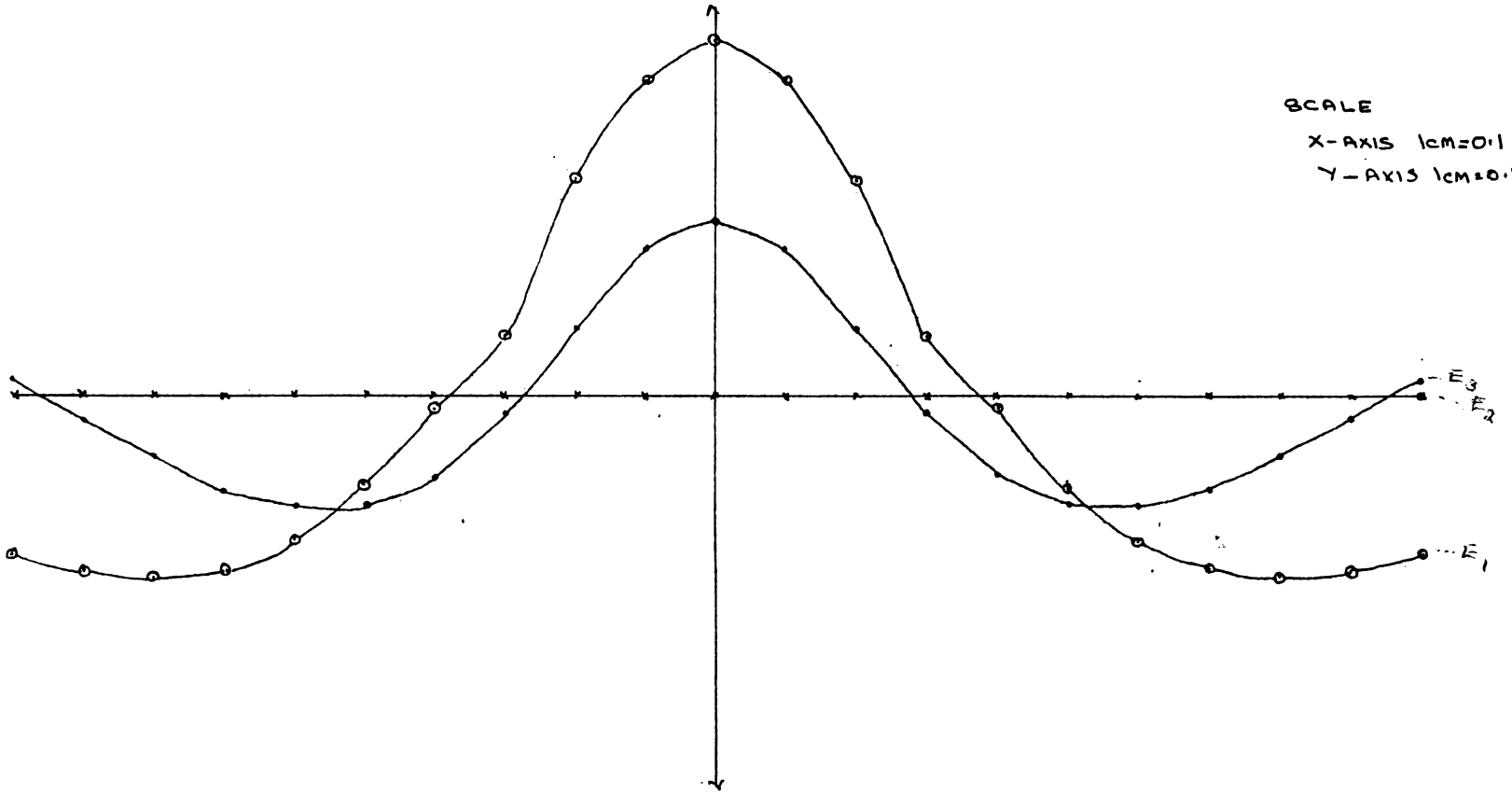


FIG 4·11.

ERRORS OF INITIAL R-SPEC : TABLE 4·27

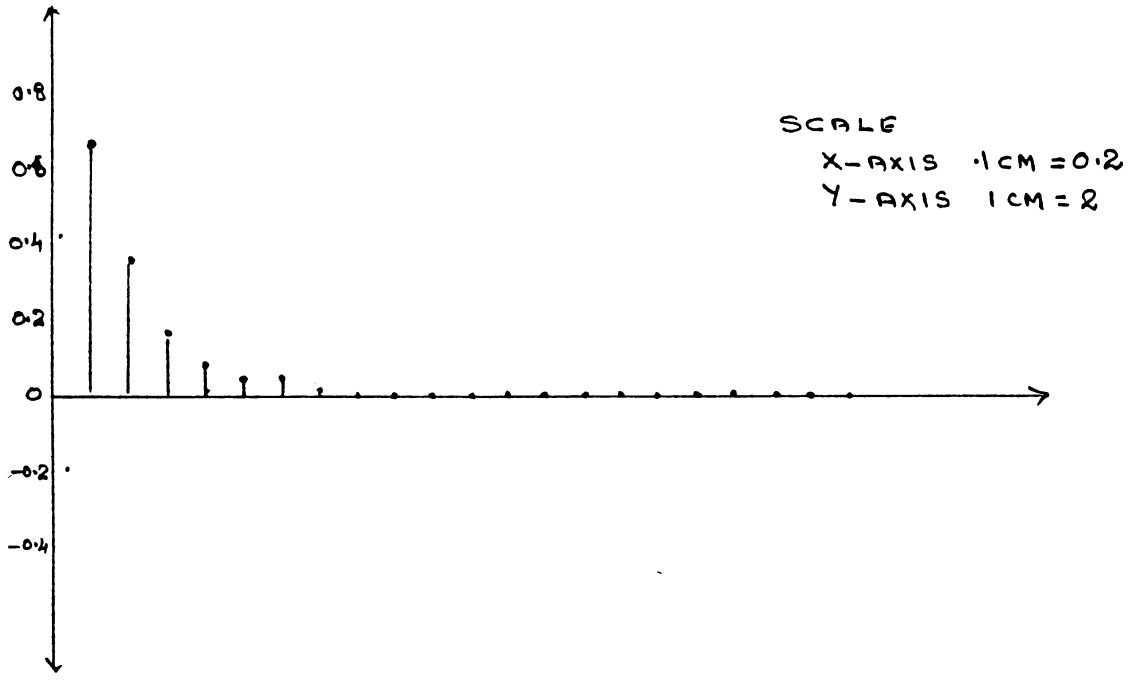


FIG 4.12 (a)
ACF OF AR(2) MODEL

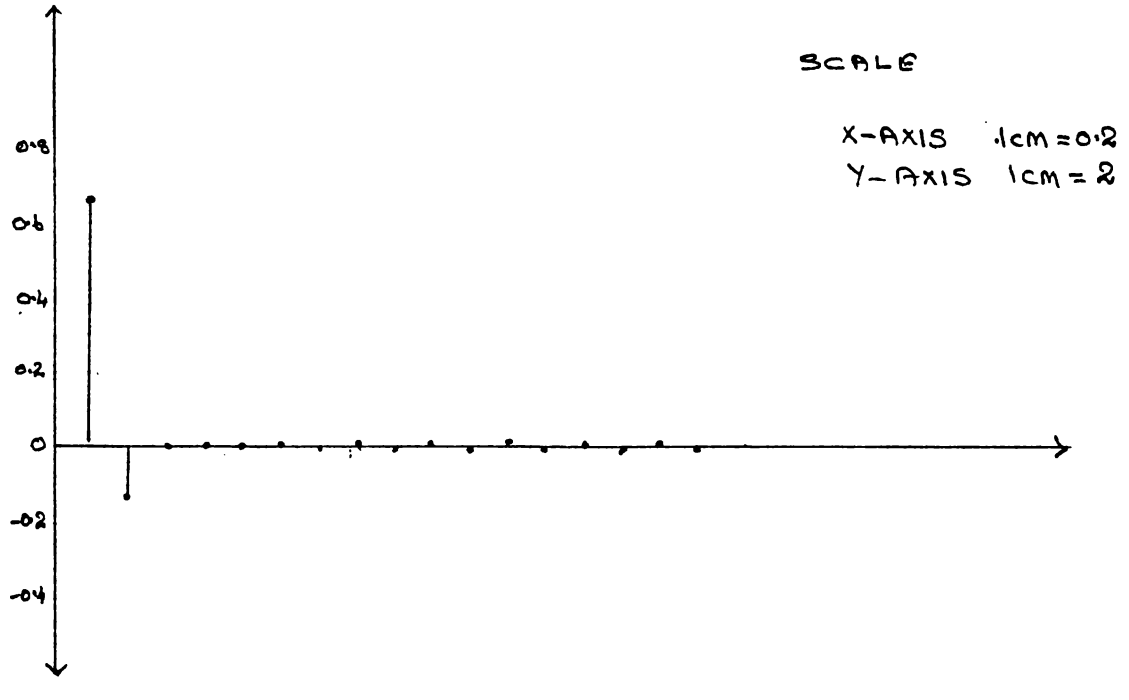


FIG 4.12 (b)
PACF OF AR(2) MODEL

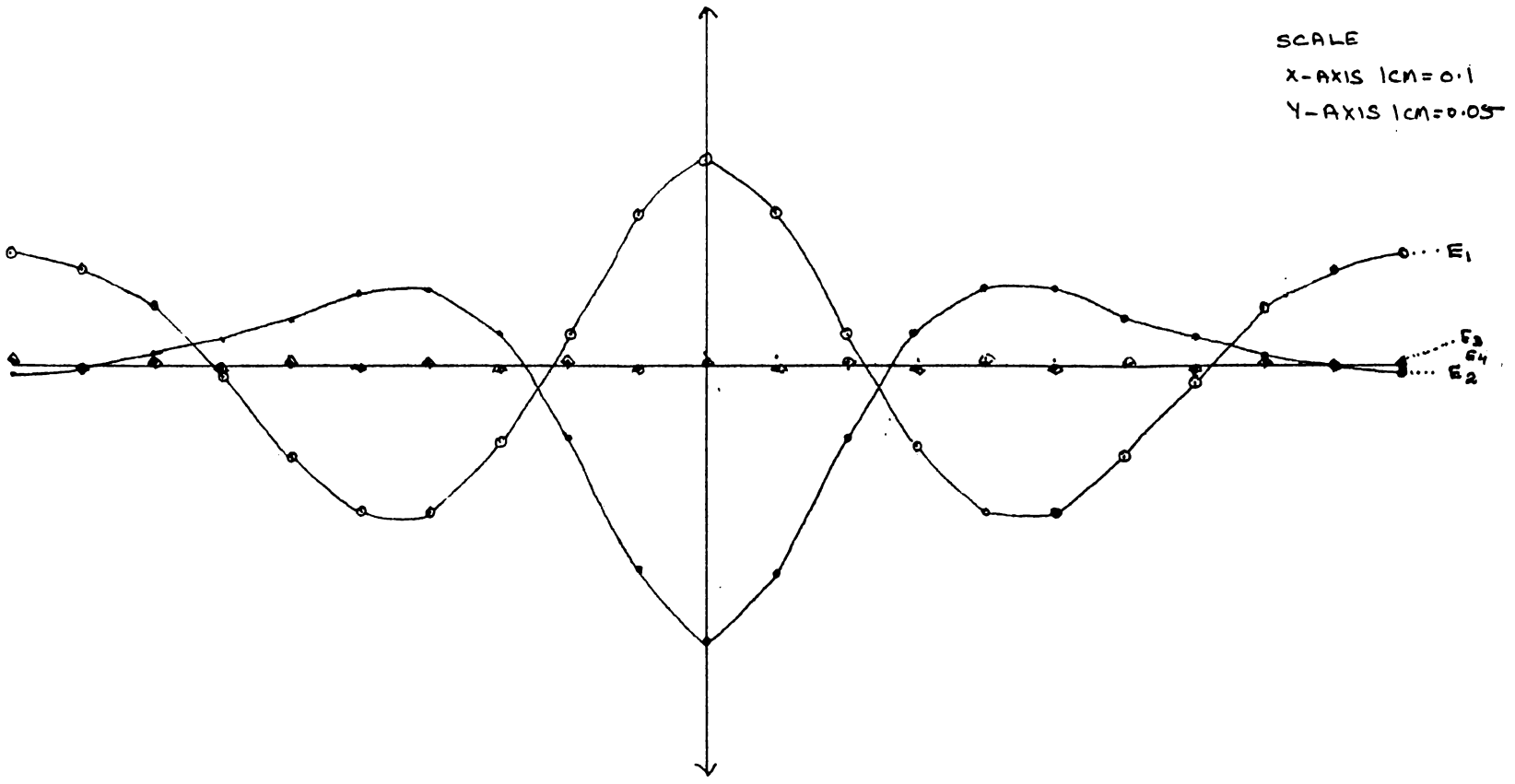


FIG 4·13

ERRORS OF INITIAL R-SPEC: TABLE 4·33

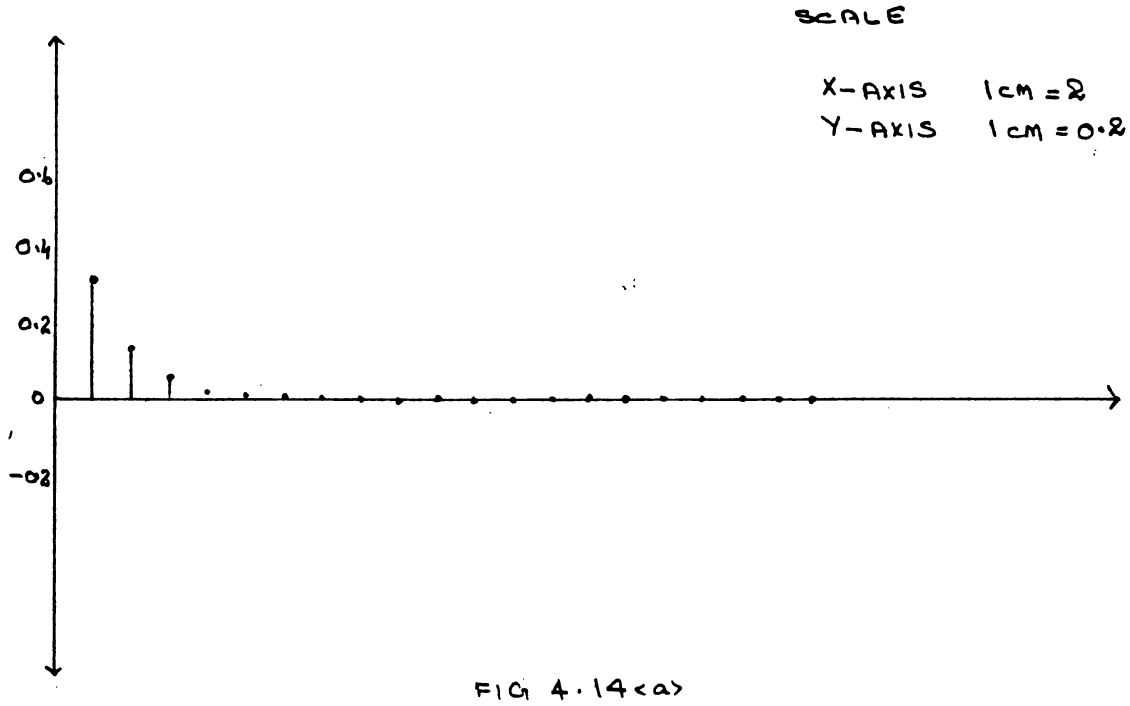


FIG 4·14(a)
ACF OF ARMA(1,1) MODEL

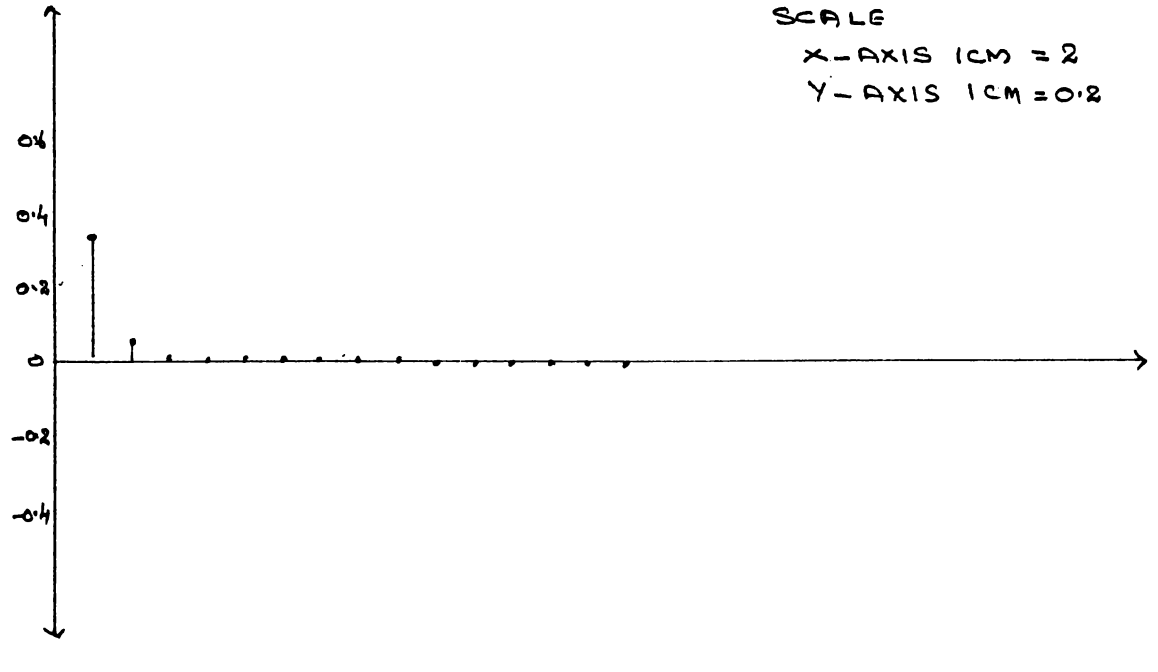
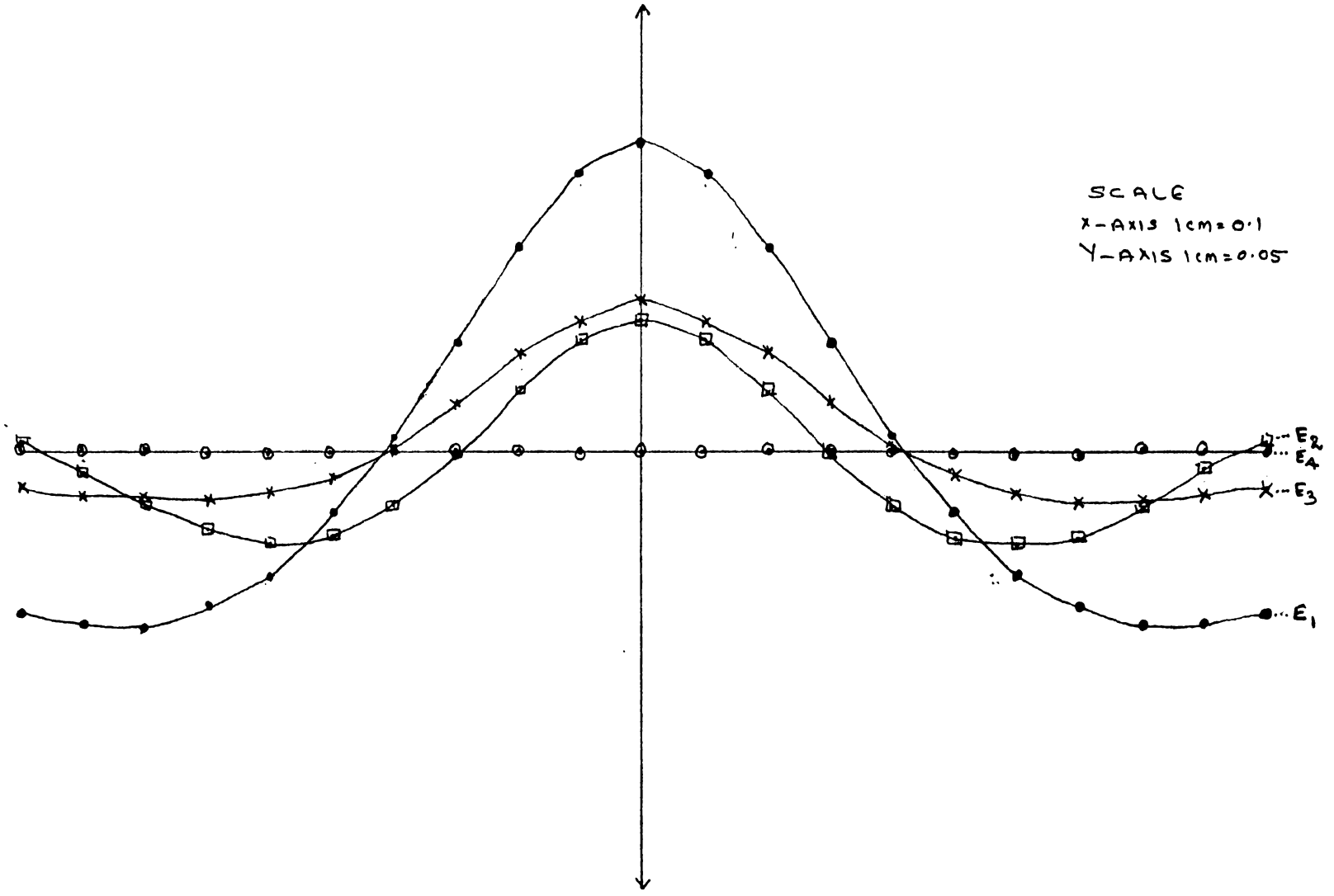


FIG 4·14(b)
PACF OF ARMA(1,1) MODEL


SCALE
X-AXIS 1cm=0·1
Y-AXIS 1cm=0·05
E2
E4
E3
E1

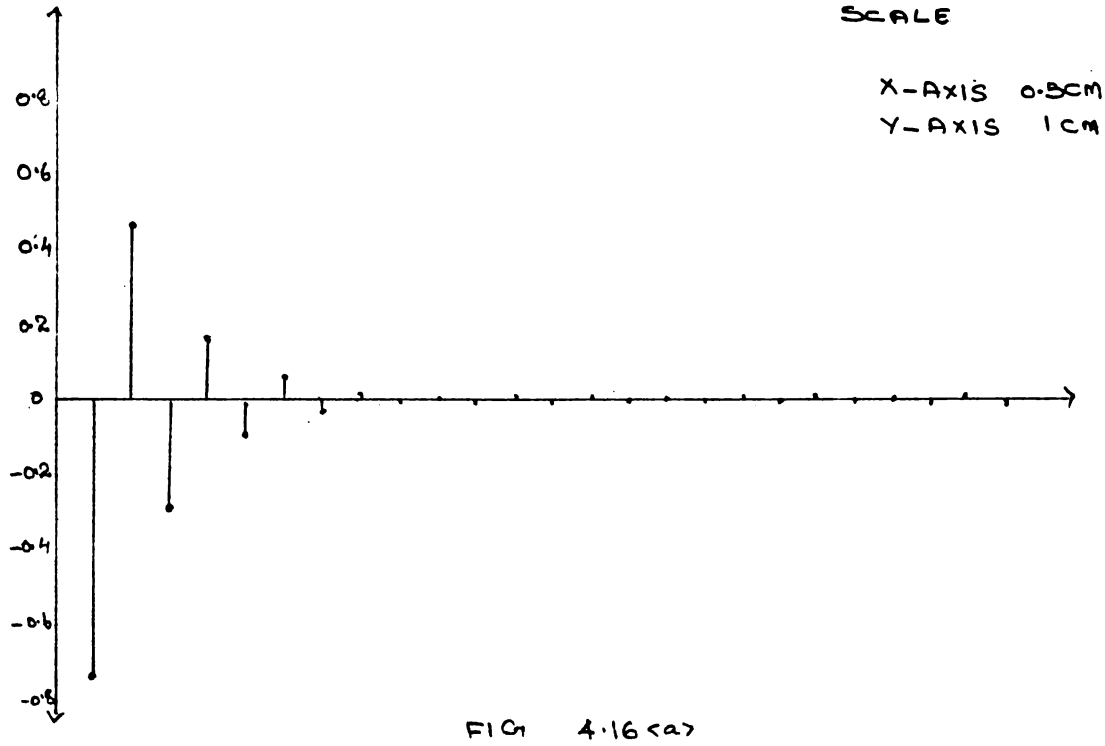


FIG 4.16 (a)
ACF OF ARMA(2,1) MODEL

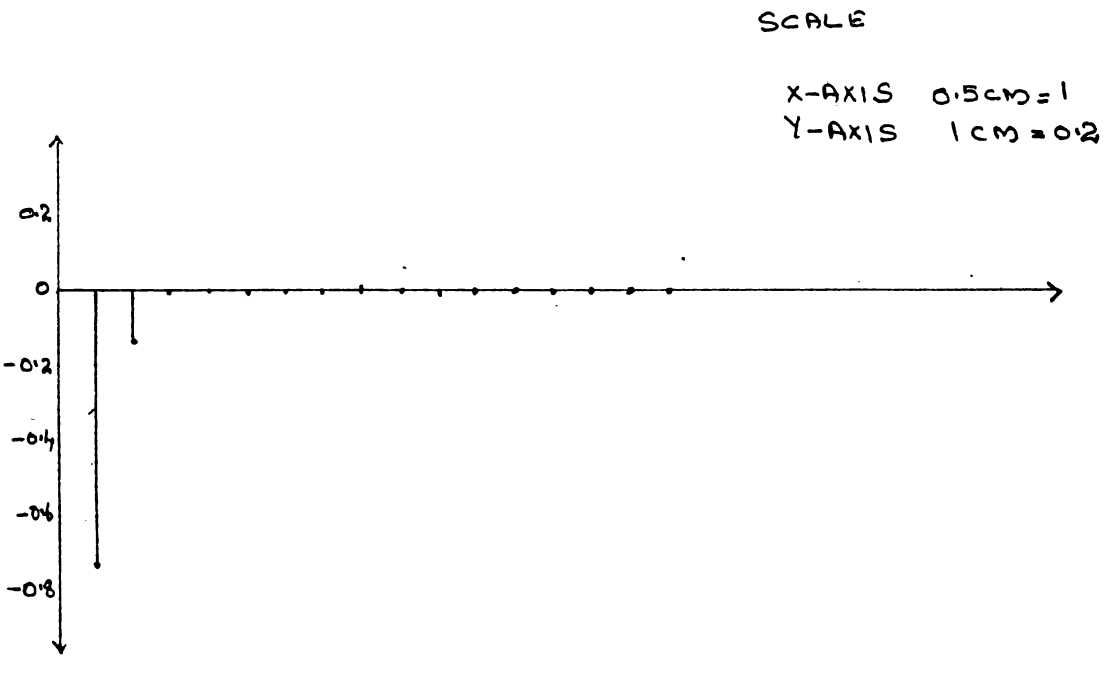


FIG 4.16 (b)
PACF OF ARMA(2,1) MODEL

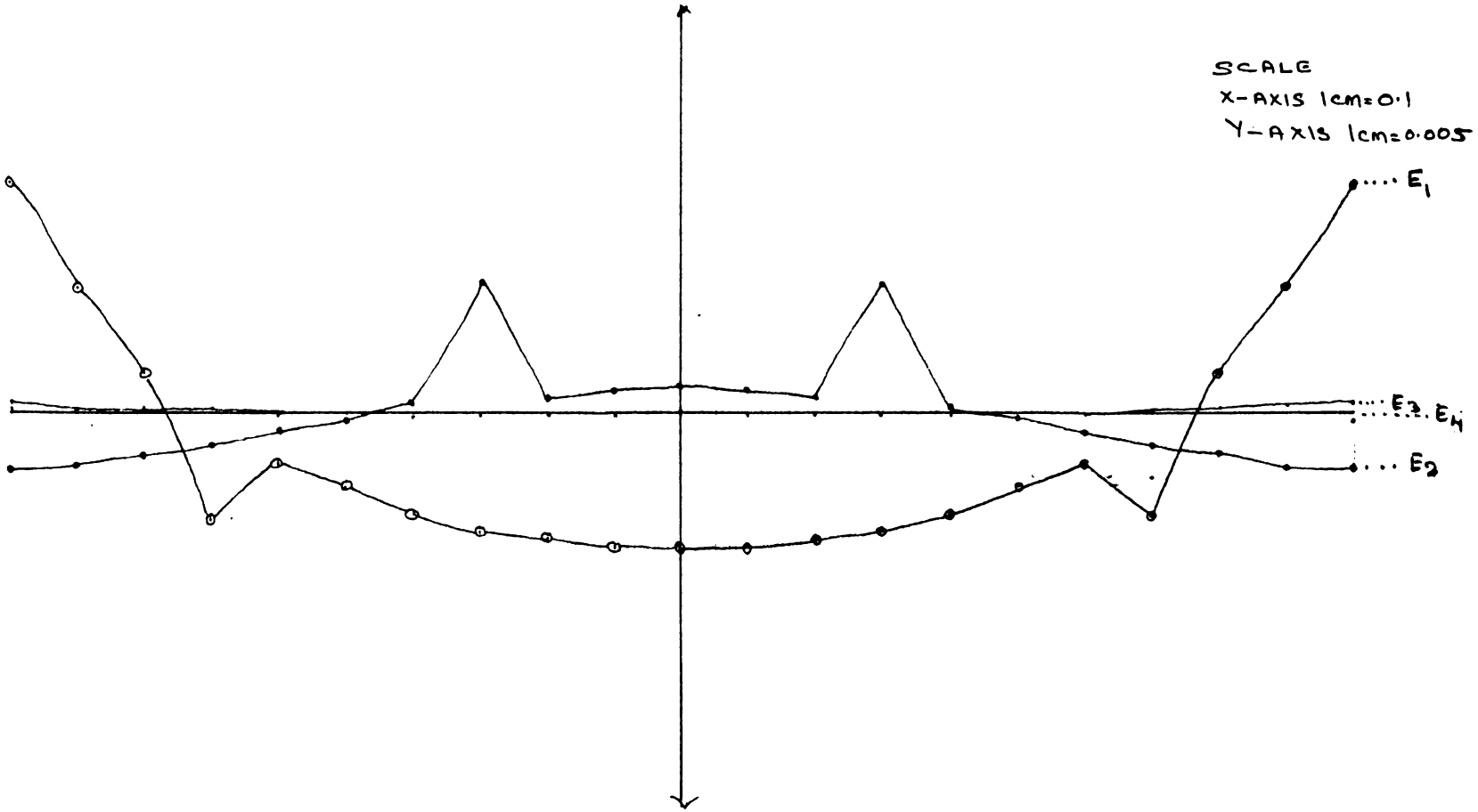


FIG 4.17

ERRORS OF INITIAL R-SPEC : TABLE 4.44

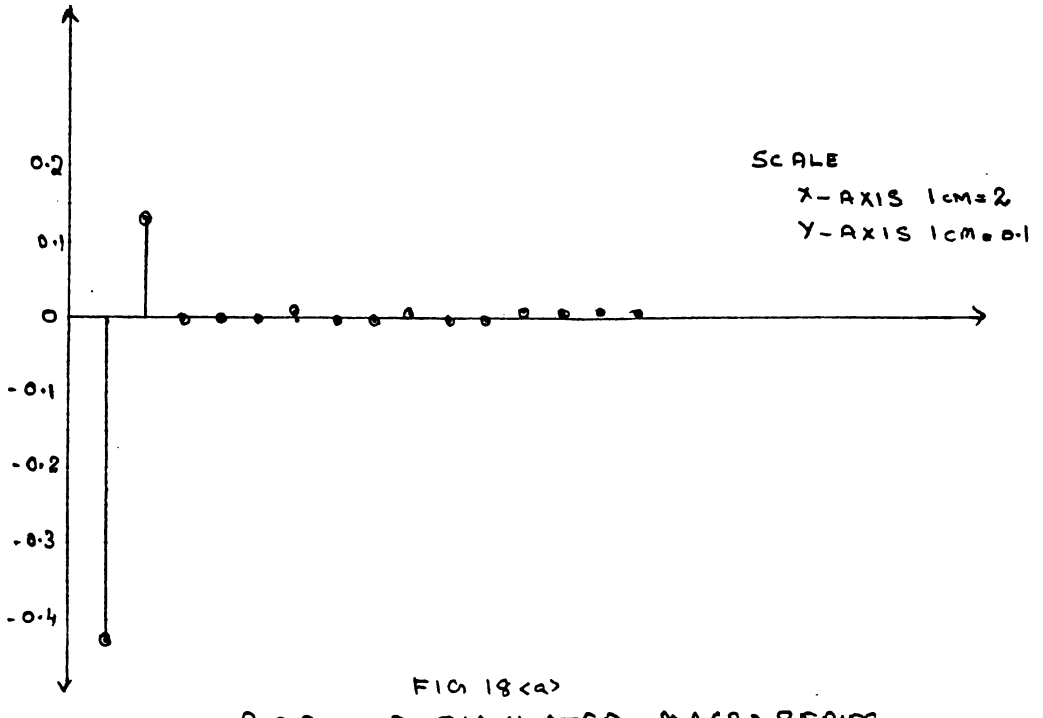


FIG 18(a)
ACF OF SIMULATED MA(2) SERIES

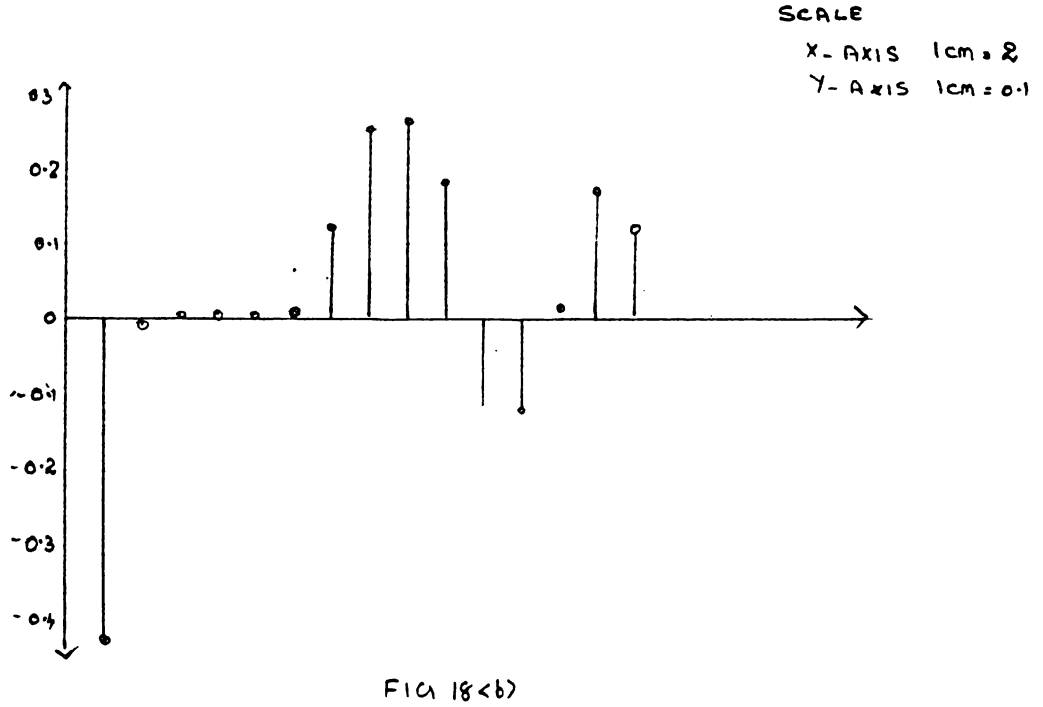


FIG 18(b)
PACF OF SIMULATED MA(2) SERIES

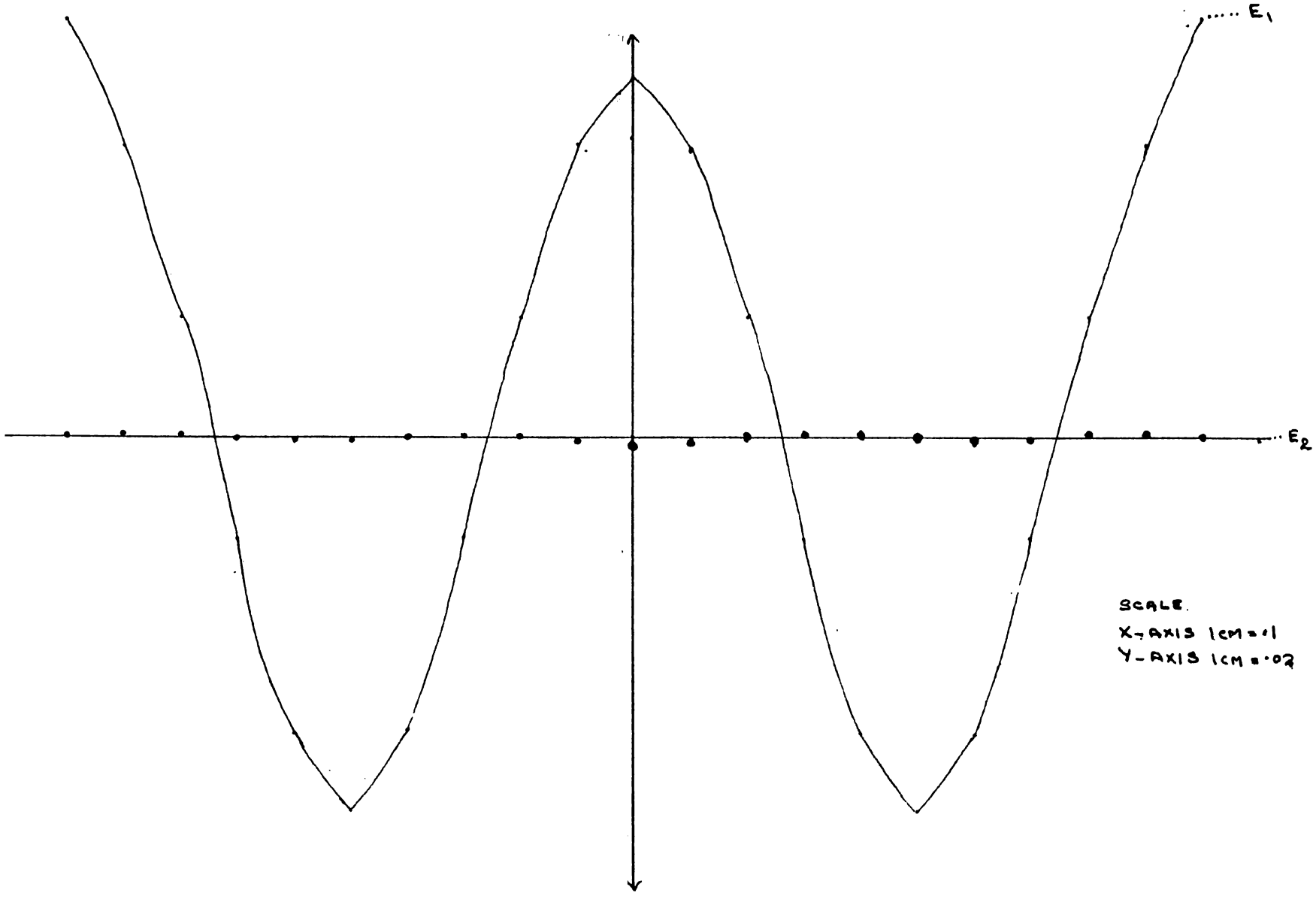


FIG 4·19
ERRORS OF INITIAL R-SPEC : TABLE 4·5.0

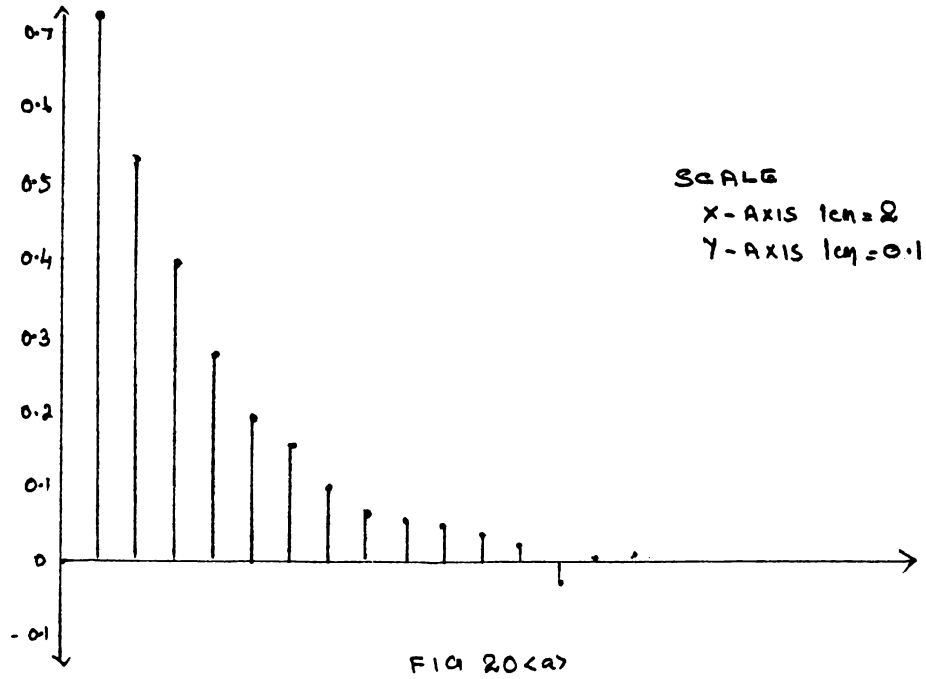


FIG 20(a)
ACF OF SIMULATED ARCD SERIES

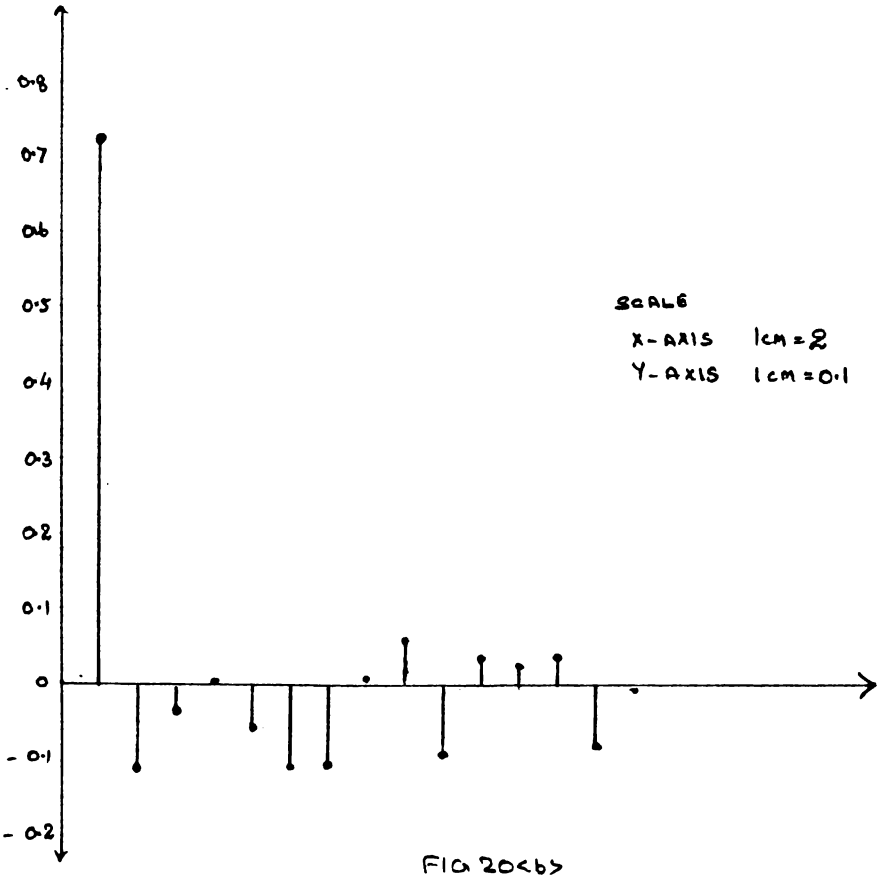


FIG 20(b)
PACF OF SIMULATED ARCD SERIES

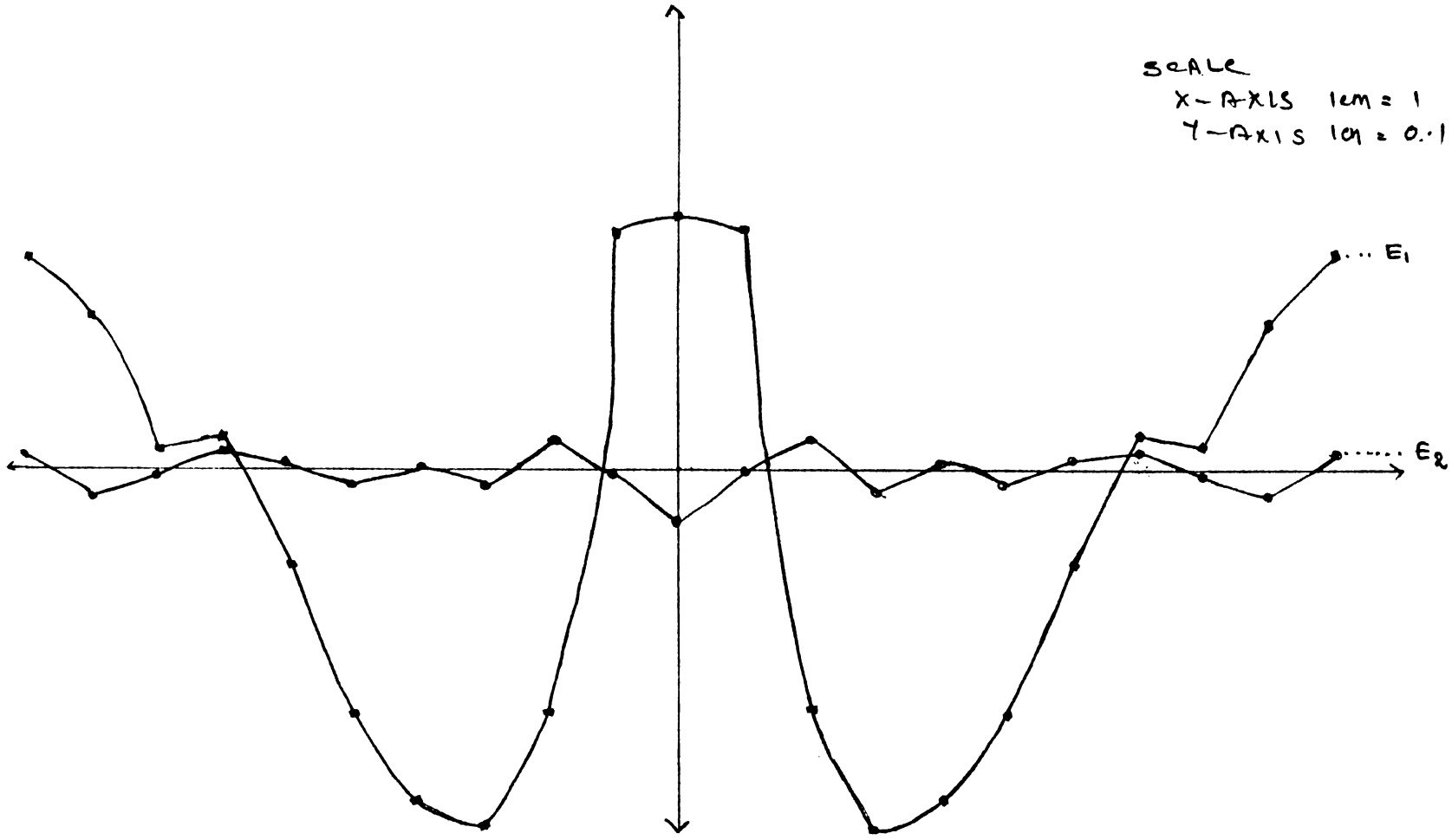


FIG 4.21
ERRORS OF INITIAL R-SPEC : TABLE 4.56

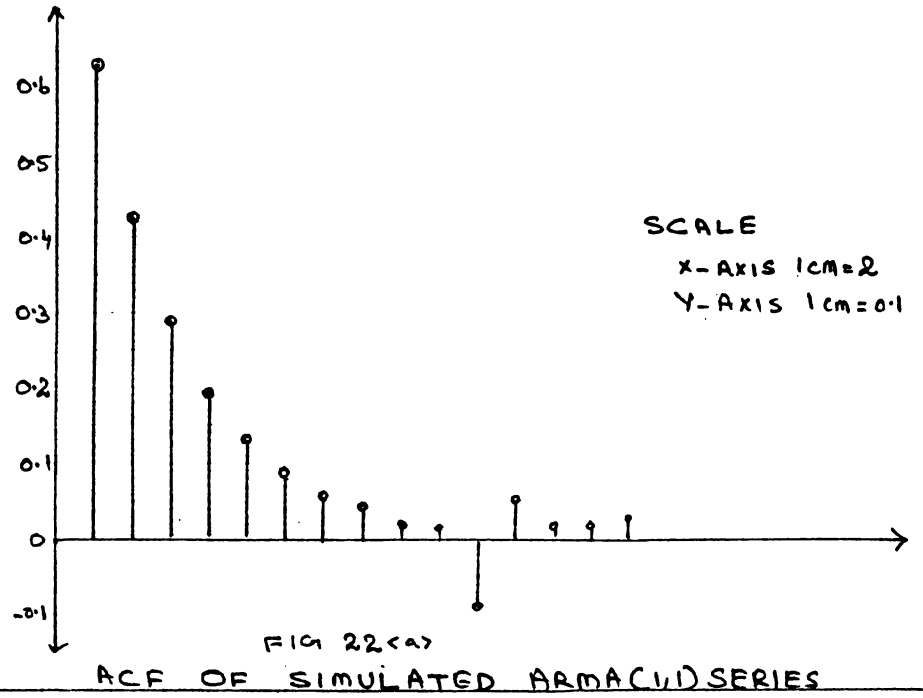


FIG 22(a)

ACF OF SIMULATED ARMA(1,1) SERIES

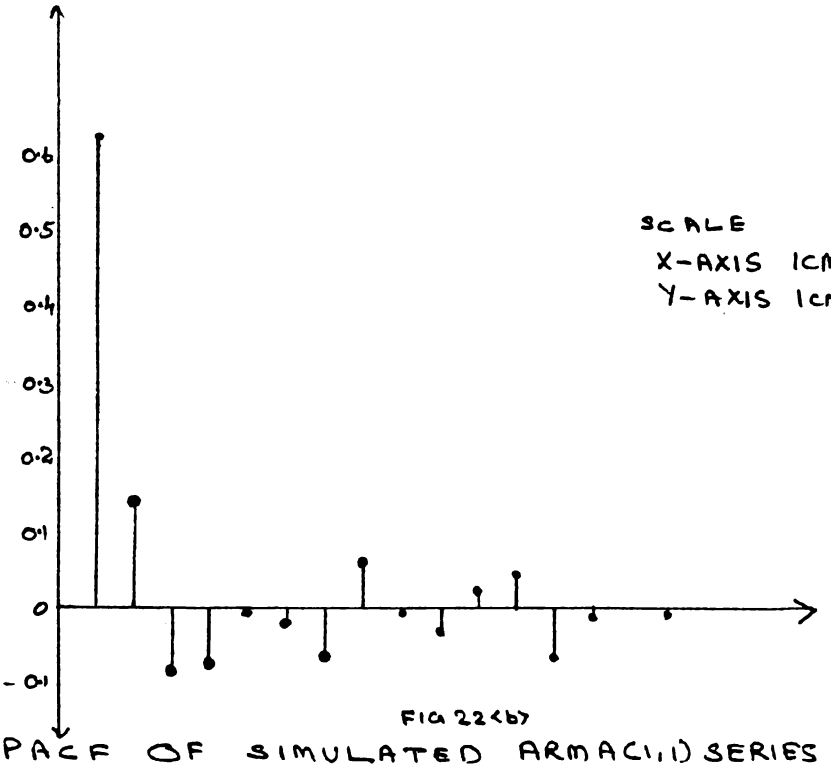


FIG 22(b)

PACF OF SIMULATED ARMA(1,1) SERIES

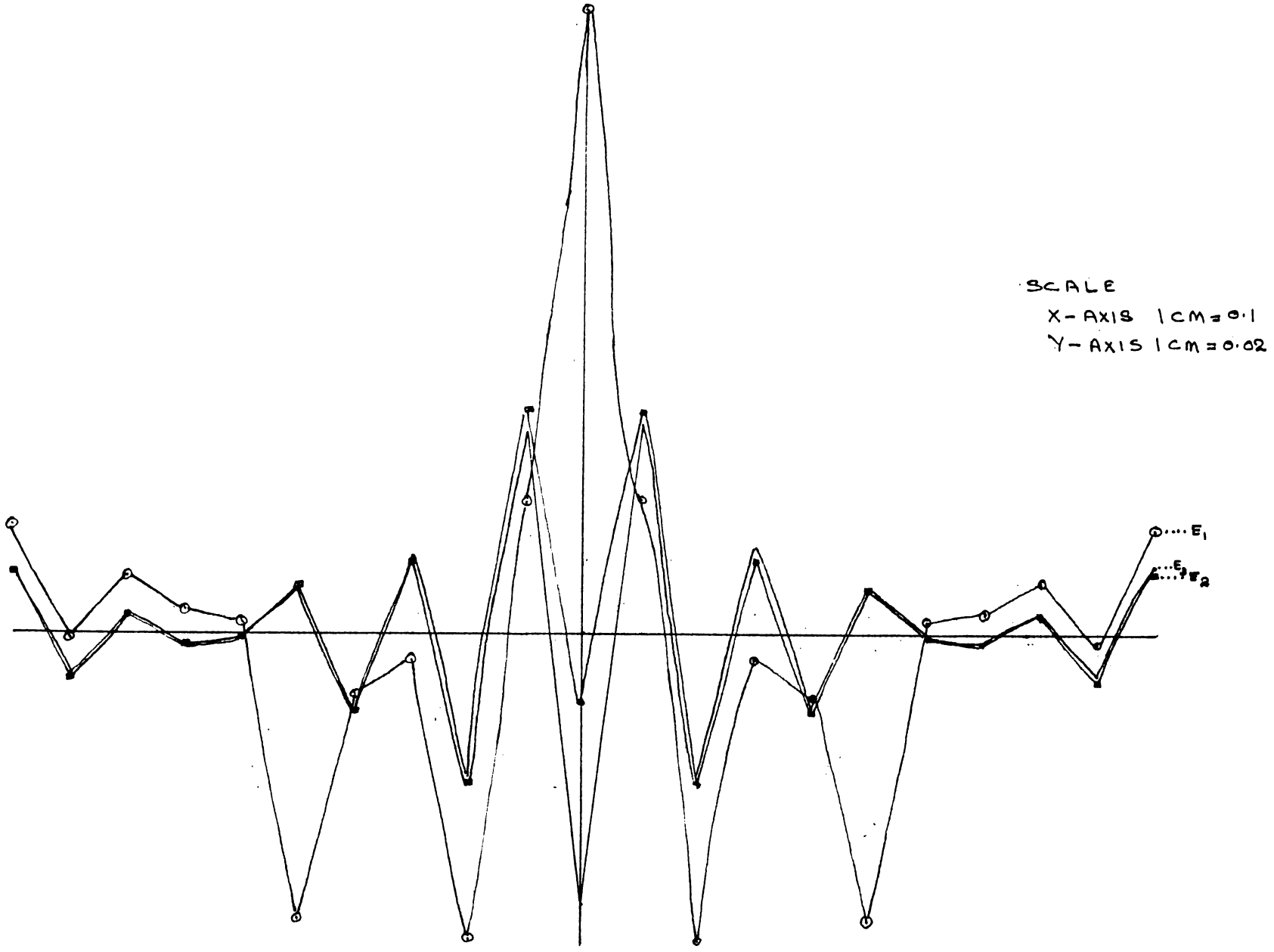


FIG 4·23

ERRORS OF INITIAL R-SPEC : TABLE 4·63

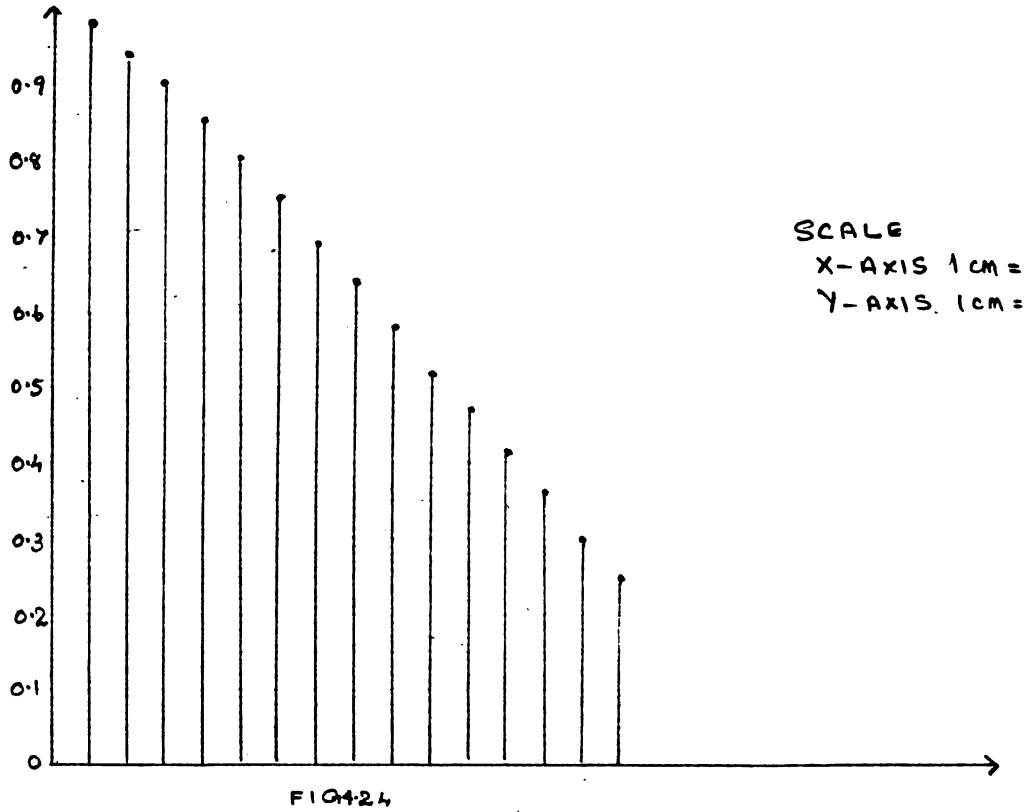


FIG 4.24
ACF OF SERIES C

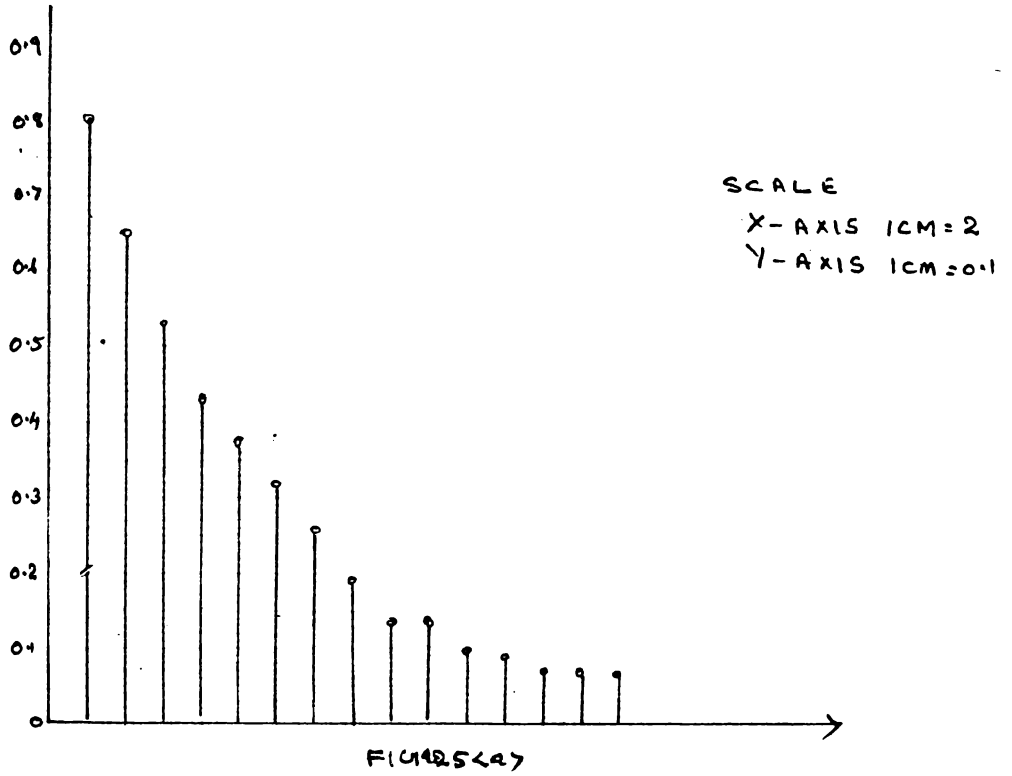


FIG 4.25(a)
ACF OF DIFFERENCED SERIES C

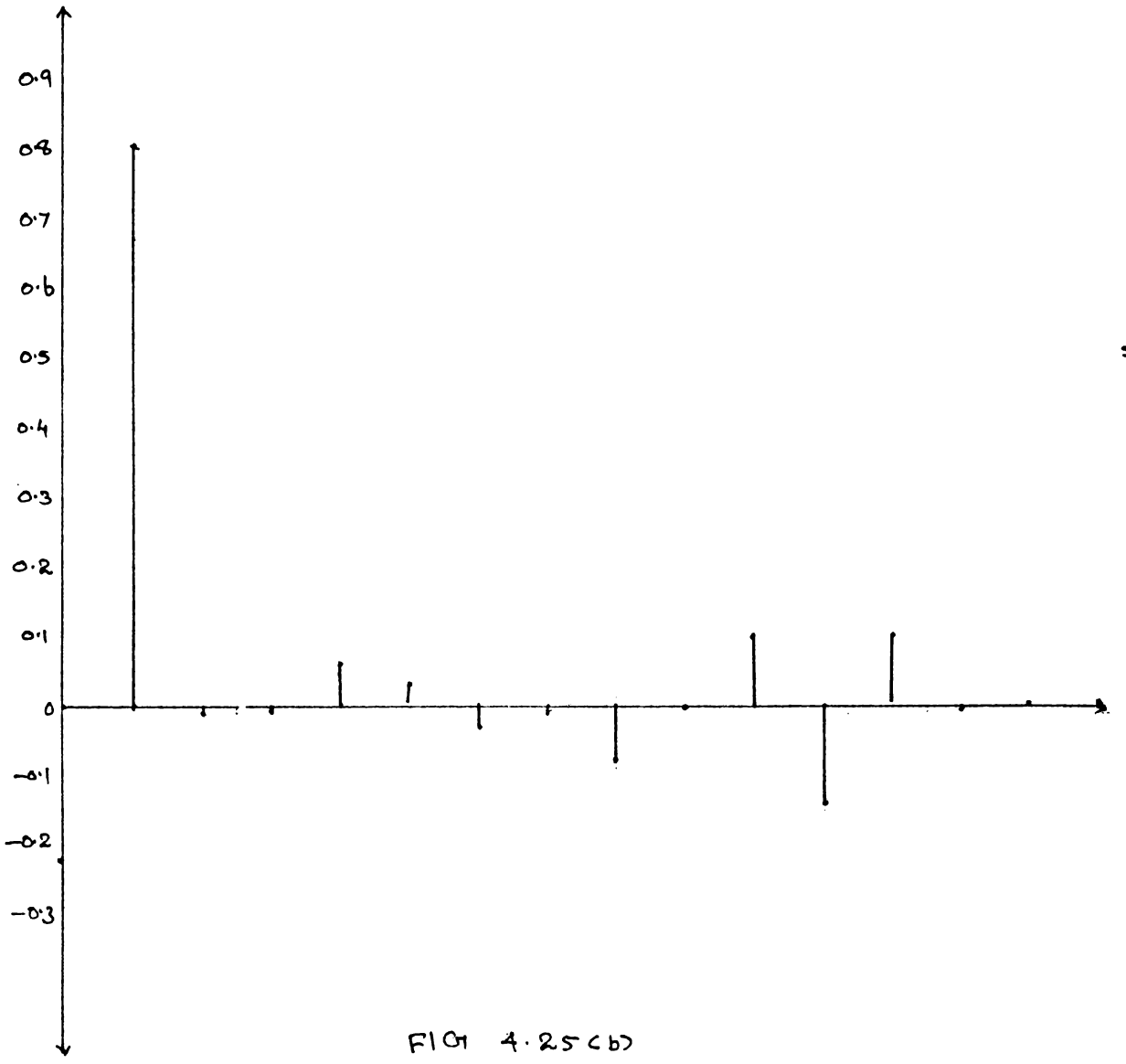


FIG 4·25(b)

PACF OF DIFFERENCED SERIES C

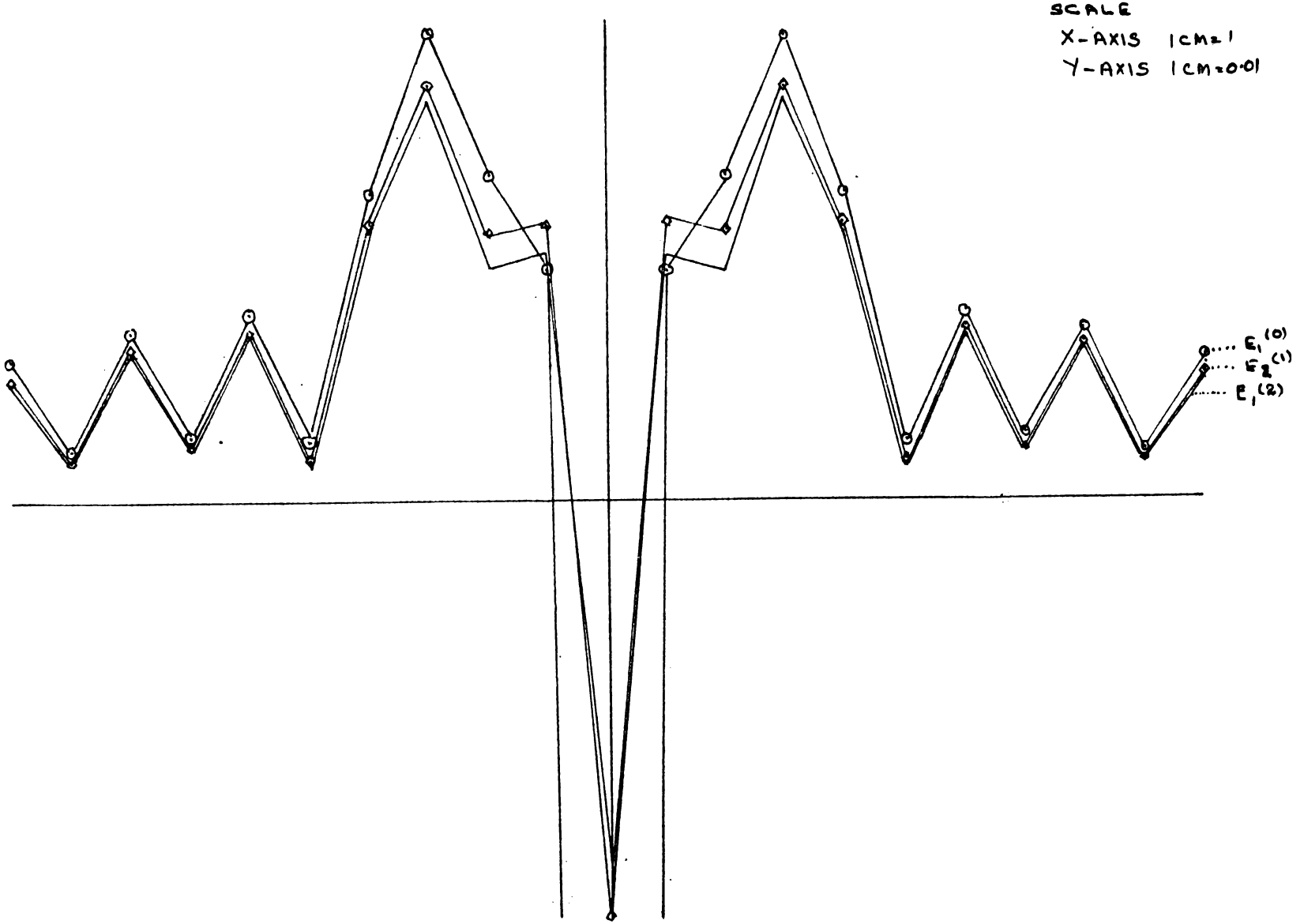


FIG 4.26

ERRORS OF $T_1^{0(i)}(\cos\lambda)$: TABLE 4.72

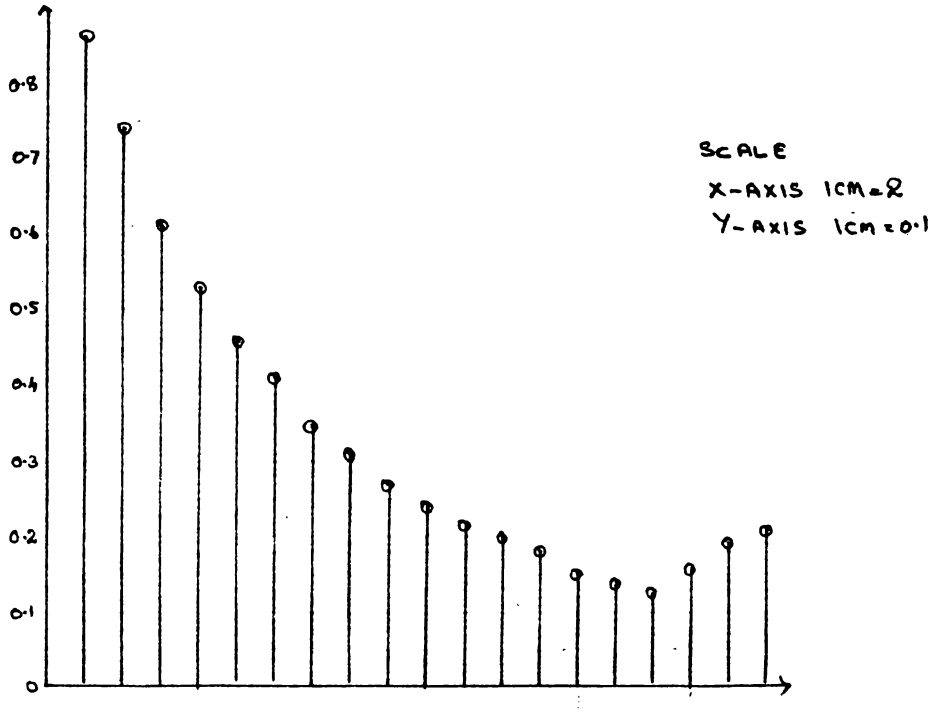


FIG 4·27(a)
ACF OF SERIES D

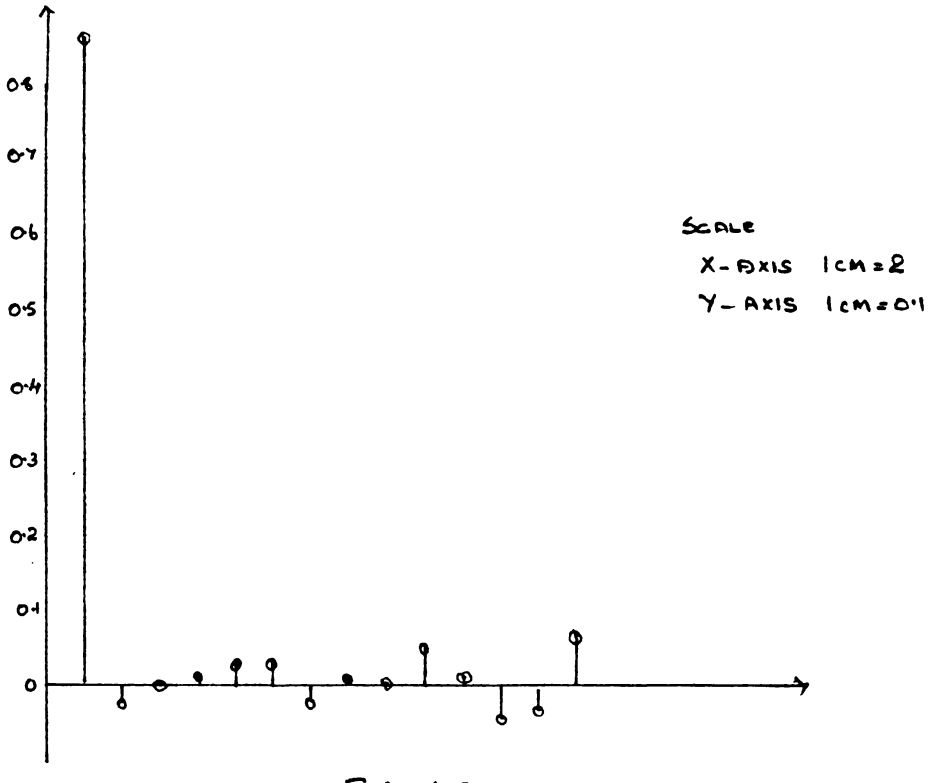


FIG 4·27(b)
PACF OF SERIES D

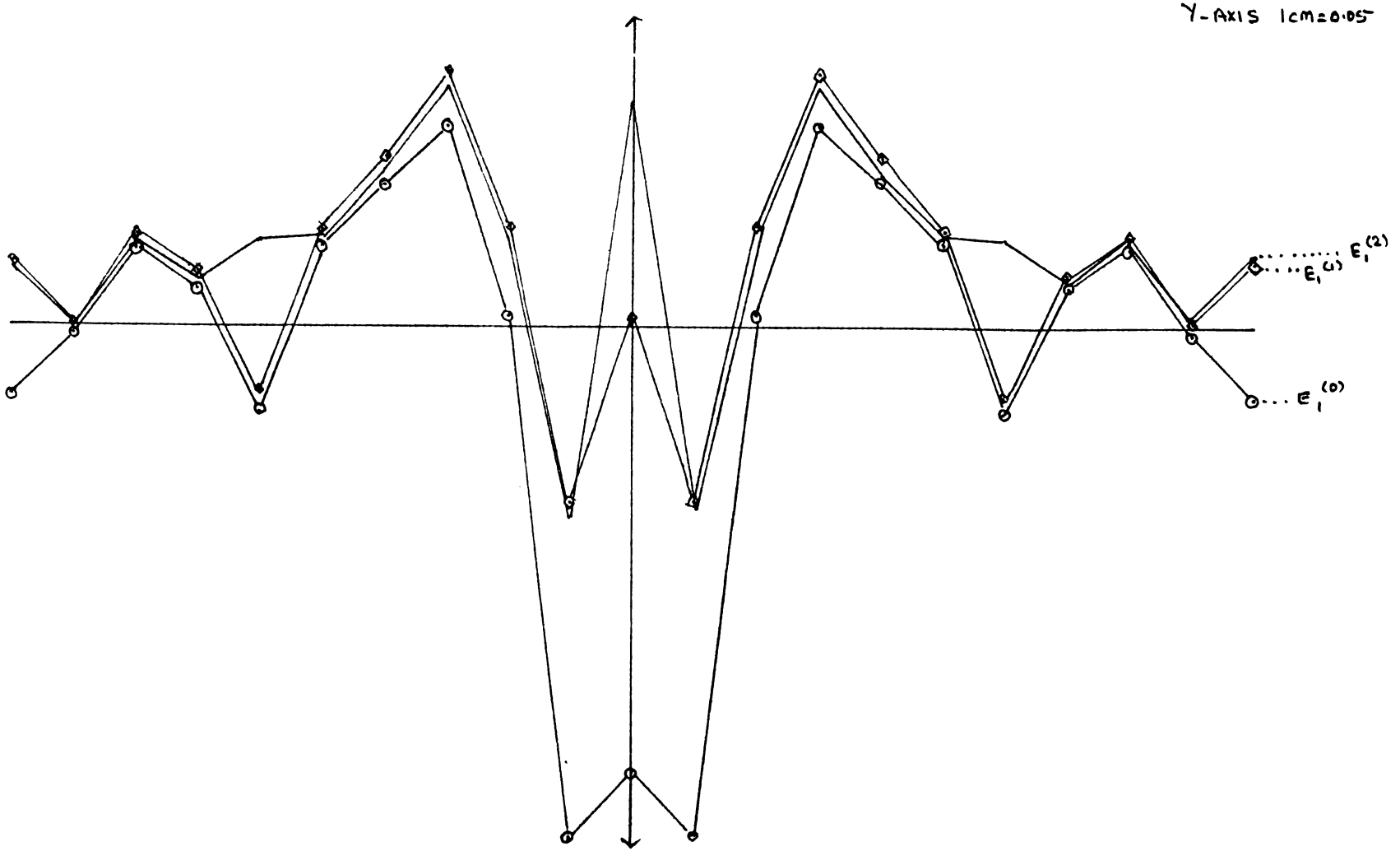


FIG 4.28

ERRORS OF $T_1^{0(i)}(\cos\lambda)$: TABLE 4.77

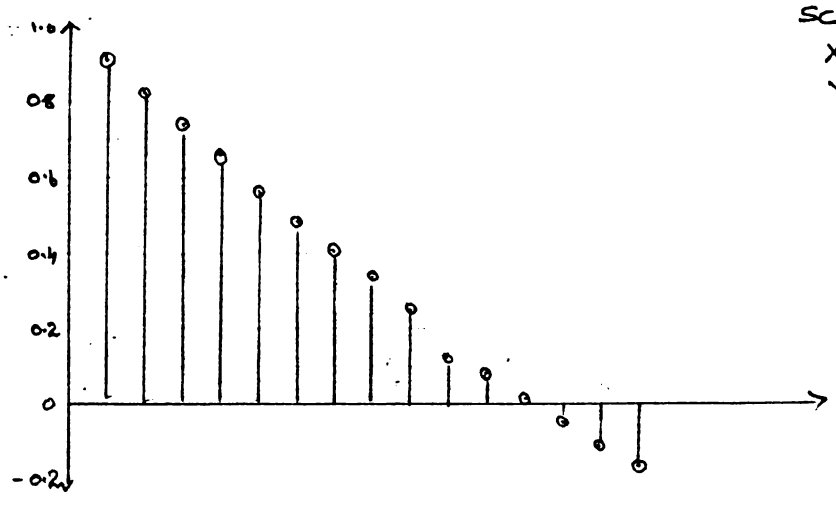


FIG 4.29
ACF OF THE POPULATION DATA

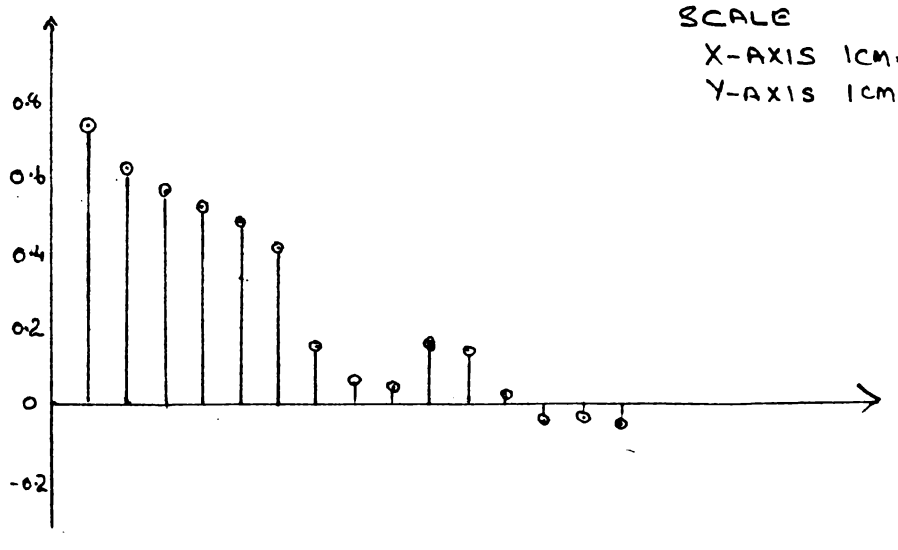


FIG 4.30
ACF OF THE FIRST DIFFERENCED POPULATION DATA

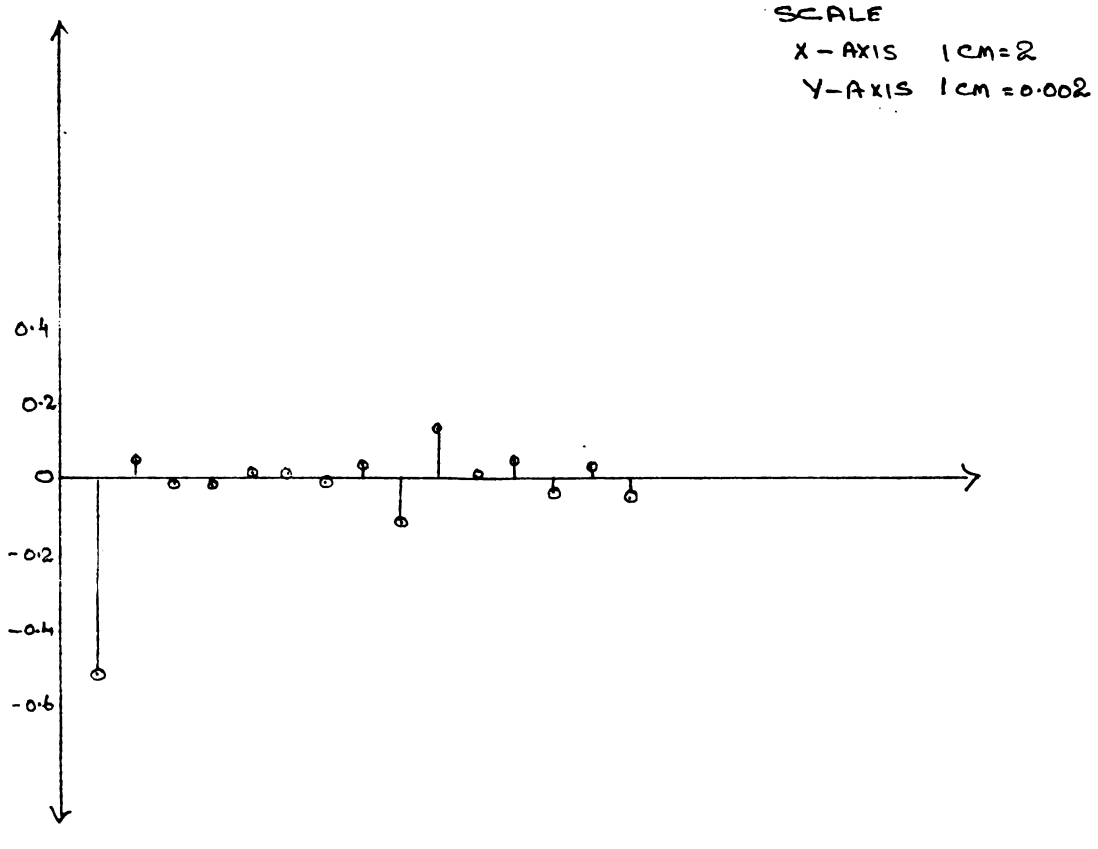


FIG A·31(a)

ACF OF $\nabla^2 P_t$

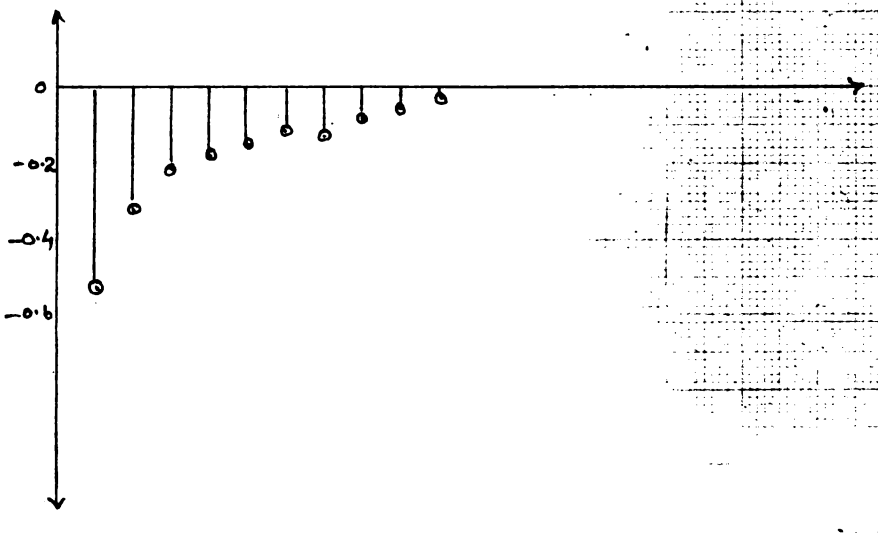


FIG 4.31(b)

PACF OF $\nabla^2 P_t$

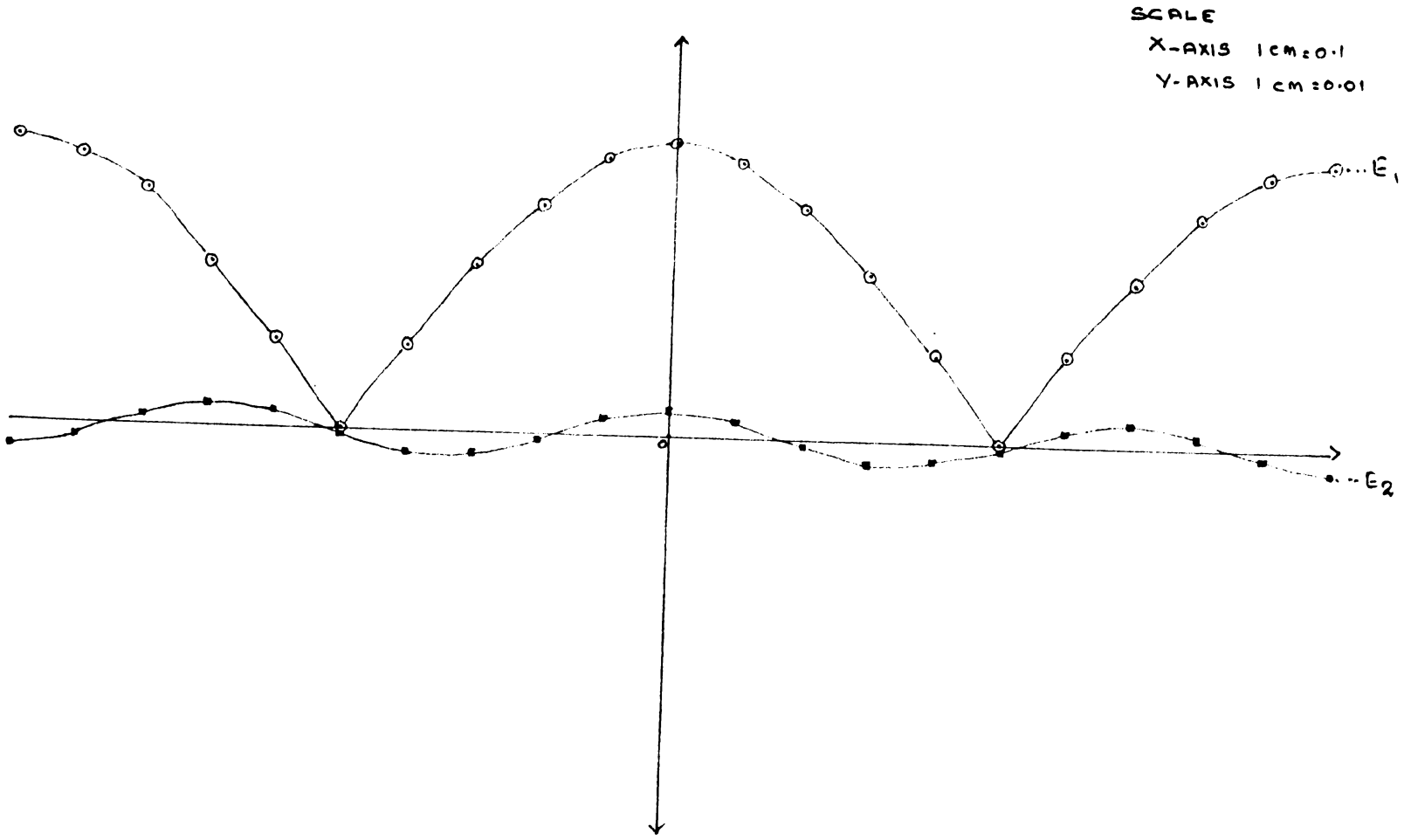


FIG 4·32

ERRORS OF INITIAL R-SPEC : TABLE 4·84

Chapter 5

RESULTS ON MULTIVARIATE TIME SERIES

5.1 Results on the relation between the ACF and the ARMA(p,q) parameters

In chapter 3 the main theorem is obtained by applying the Chebyshev minimax rational approximation theory. If $f(x) \in C[a,b]$ and

$T_k^l(\cos\lambda) = \dfrac{a_0+a_1(\cos(\lambda) +\ldots+a_l \cos(l\lambda)}{1+b_1\cos(\lambda)+\ldots +b_k \cos(k\lambda)}$ is the rational approximation of $f(\cos\lambda)$, then the error function is given by

$$E(\lambda) = f(\cos\lambda) - T_k^l(\cos\lambda) \qquad (5.1)$$

The rational approximation theory is based on the fact that the approximating function $T_k^l(\cos\lambda)$ of $f(\cos\lambda)$ will be the one which minimises $E(\lambda)$. If $E(\lambda) = 0,$,then

$$f(\cos\lambda) = T_k^l(\cos\lambda). \qquad (5.2)$$

It is shown in Ralston [31] that if $T_k^l(\cos\lambda)$ is that rational approximation, which gives $E(\lambda) = 0,$, then the coefficients a_i, i=0,1,...,l and b_i, i=1,...,k

satisfy the following equations

$$a_o = \frac{1}{2}\left[\rho_o b_o + \rho_1 b_1 + \ldots + \rho_k b_k\right] \tag{5.3}$$

where

$$f(\cos\lambda) = \sum_{i=0}^{\infty} \rho_i T_i(\lambda)$$

$$= \sum_{i=0}^{\infty} \rho_i \cos(i\lambda) \tag{5.4}$$

$$a_r = \frac{1}{2} \sum_{i=1}^{k} b_i (\rho_{|r-i|} + \rho_{r+i}) \quad , \; i=1,\ldots,l \tag{5.5}$$

and $$0 = \frac{1}{2} \sum_{i=1}^{k} b_i (\rho_{|r-i|} + \rho_{r+i}) \quad , \; i=l+1,\ldots,l+k \tag{5.6}$$

The equations defined in (5.3), (5.5) and (5.6) will help to prove the following results, connecting the ARMA parameters and their autocorrelations.

Result 5.1

The autocorrelations of an MA(1) model given by

$$\tilde{x}_t = a_t - \Theta_1 a_{t-1} \tag{5.7}$$

can be expressed in terms of the MA(1) parameter Θ_1, as

$$\rho_1 = \frac{-\Theta_1}{1+\Theta_1^2} \tag{5.8}$$

and $\rho_k = 0, \; k \geq 2$

Proof of 5.1

Let the MA(1) model be $\tilde{x}_t = a_t - \Theta_1 a_{t-1}$ and let $s(\lambda)$ be the spectral density function. The rational form of the spectral density function is given by

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2}\left[1+\Theta_1^2 - 2\Theta_1\cos(\lambda)\right] \tag{5.9}$$

The estimate of the rational form of the spectral density function will be

$$2T_o^1(\cos\lambda) = 2\left[a_o + a_1\cos(\lambda)\right] \tag{5.10}$$

Equations (5.9) and (5.10) give

$$2a_o = \frac{\sigma_a^2}{\sigma_x^2}\left[1 + \Theta_1^2\right] \tag{5.11}$$

and $$\frac{a_1}{2a_o} = \frac{-\Theta_1}{1+\Theta_1^2} \tag{5.12}$$

Now substituting a_o and a_1 obtained from (5.3) and (5.5) respectively in (5.11) and (5.12) the following equations are obtained.

$$1 = \frac{\sigma_a^2}{\sigma_x^2}\left[1+\Theta_1^2\right] \tag{5.13}$$

and

$$\rho_1 = \frac{-\theta_1}{1+\theta_1^2} \tag{5.14}$$

Further for $k > 1$ we get

$$0 = \frac{1}{2} \times 2\rho_k$$

$$\text{i.e., } \rho_k = 0, \quad k > 1 \tag{5.15}$$

Equations (5.14) and (5.15) determine

$$\rho_k = \begin{cases} \dfrac{-\theta_1}{1+\theta_1^2} & k=1 \\ 0 & k > 1 \end{cases} \tag{5.16}$$

Hence the result.

Result 5.2

The autocorrelation function of an MA(q) model given by

$$\tilde{x}_t = a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q} \tag{5.17}$$

can be expressed in terms of the moving average parameters, $\theta_1, \theta_2, \ldots, \theta_q$, as

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \sum_{j=1}^{q-k} \theta_j \theta_{j+k}}{1 + \sum_{i=1}^{q} \theta_i^2} \\ 0 \quad , \quad k > q \end{cases} \tag{5.18}$$

Proof of Result 2

The rational form of the spectral density function of an MA(q) model is given by

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2}\left[1 + \sum_{i=1}^{q}\theta_i^2 + 2\sum_{i=1}^{q}(-\theta_i + \sum_{j=1}^{q-i}\theta_j\theta_{j+i})\cos(i\lambda)\right] \tag{5.19}$$

and the estimate of the rational form of the spectral density function is given by

$$2T_o^2(\cos\lambda) = 2\left[a_o + a_1\cos\lambda + \ldots + a_q\cos(q\lambda)\right] \tag{5.20}$$

On equating coefficients in (5.19) and (5.20) the following equations are obtained.

$$2a_o = \frac{\sigma_a^2}{\sigma_x^2}\left[1 + \sum_{i=1}^{q}\theta_i^2\right]$$

$$\frac{a_i}{2a_o} = \frac{-\theta_i + \sum_{j=1}^{q-i}\theta_j\theta_{j+i}}{1 + \sum_{i=1}^{q}\theta_i^2} \tag{5.21}$$

$$i = 1, \ldots, q$$

Then substituting a_i, i=1,...,q and a_o given in (5.5) and (5.3) respectively, the following equations are obtained.

$$\rho_i = \frac{-\Theta_i + \sum_{j=1}^{q-i} \Theta_j \Theta_{j+i}}{1 + \sum_{i=1}^{q} \Theta_i^2}, \quad i=1,\ldots q \tag{5.22}$$

and $\rho_i = 0, \quad i > q$ (5.23)

Hence the proof of the result.

Result 5.3

The autocorrelations of a series represented by an AR(1) model given by

$$\tilde{x}_t = \Phi_1 \tilde{x}_{t-1} + a_t \tag{5.24}$$

can be expressed in terms of the AR(1) parameter as

$$\rho_1 = \Phi_1$$

and $$\rho_k = \rho_1 \Phi_1^{k-1} \tag{5.25}$$

or $$\rho_k = \Phi_1^k, \quad k \geq 1 \tag{5.26}$$

Proof of Result 5.3

The rational form of the spectral density function of an AR(1) model is given by

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2\left[1+\phi_1^2 - 2\phi_1 \cos\lambda\right]} \tag{5.27}$$

The estimate of the rational form will be of the form

$$2T_1^2(\cos\lambda) = \frac{2a_o}{1+b_1\cos(\lambda)} \tag{5.28}$$

where $a_o = \frac{1}{2}\left[1+b_1\rho_1\right]$ and $b_1 = \frac{-2\rho_1}{1+\rho_2}$. Comparing the corresponding coefficients of $\cos(i\lambda)$, in (5.27) and (5.28) the following equations are obtained.

$$\begin{aligned} 2a_o &= \frac{\sigma_a^2}{\sigma_x^2\left[1+\phi_1^2\right]} \\ b_1 &= \frac{-2\phi_1}{1+\phi_1^2} \end{aligned} \tag{5.29}$$

Substituting a_o and b_1 obtained from (5.3) and (5.6) respectively in (5.29), we get

$$1 + \frac{(-2\rho_1)}{1+\rho_2}\rho_1 = \frac{\sigma_a^2}{\sigma_x^2}\ \frac{1}{1+\phi_1^2} \tag{5.30}$$

and $$\frac{-2\rho_1}{1+\rho_2} = \frac{-2\phi_1}{1+\phi_1^2} \tag{5.31}$$

Simplifying (5.30) and (5.31) it is obtained that

$$\rho_1 = \Phi_1$$

and $\rho_2 = \rho_1 \Phi_1$

i.e., $\rho_k = \Phi_1 \rho_{k-1}$ for $k = 2$

Similarly, we get $\rho_k = \Phi_1 \rho_{k-1}$, $k > 2$.

Hence the proof of the result 3.

Result 5.4

The autocorrelations of an AR(p) model can be expressed in terms of the AR(p) parameters as

$$\rho_k = \Phi_1 \rho_{k-1} + \Phi_2 \rho_{k-2} + \ldots + \Phi_p \rho_{k-p} \qquad (5.32)$$

Proof of Result 5.4

Proof of this result when k = 1 is given in the proof of result (5.3). The method is same, as in the case of k = 1, for k > 1 also. The rational form of the spectral density function is given by

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_z^2} \frac{1}{1 + \sum_{i=1}^{p} \Phi_i^2 + 2 \sum_{i=1}^{p} \left(-\Phi_i + \sum_{j=1}^{q-i} \Phi_j \Phi_{j+i} \right) \cos(i\lambda)} \qquad (5.33)$$

and the estimate of $s(\lambda)$ is given by

$$\hat{s}(\lambda) = 2T_1^o(\cos\lambda)$$

$$= \frac{2a_o}{1+b_1\cos\lambda+\ldots+b_p\cos(p\lambda)} \tag{5.34}$$

Equating the corresponding coefficients in (5.33) and (5.34) we get

$$2a_o = \frac{\sigma_a^2}{\sigma_x^2\left[1+\sum_{i=1}^{p}\phi_i^2\right]} \tag{5.35}$$

$$\frac{b_i}{2} = \frac{-\phi_i+\sum_{j=1}^{p-i}\phi_j\phi_{j+i}}{1+\sum_{i=1}^{p}\phi_i^2} \tag{5.36}$$

Substituting a_o and b_i, i=1,...p obtained from (5.3) and (5.6) in (5.35) and (5.36) and on simplification we get

$$\rho_k = \phi_1\rho_{k-1}+\ldots+\phi_p\rho_{k-p}, \quad k>0$$

Result 5.5

The autocorrelations of a series represented by an ARMA(1,1) model given by

$$\tilde{x}_t = \phi_1\tilde{x}_{t-1} + a_t - \theta_1 a_{t-1} \tag{5.37}$$

can be expressed in terms of the ARMA(1,1) parameters as

$$\rho_1 = \frac{(1-\Phi_1\Theta_1)(\Phi_1 - \Theta_1)}{1+\Theta_1^2 - 2\Phi_1\Theta_1}$$

and $\rho_k = \Phi_1 \rho_{k-1} \quad k \geq 2$

Proof of Result 5.5

The rational form of the spectral density function of an ARMA(1,1) model is given by

$$s(\lambda) = \frac{\sigma_a^2}{\sigma_x^2} \frac{(1+\Theta_1^2) - 2\Theta_1 \cos\lambda}{1+\Phi_1^2 - 2\Phi_1 \cos\lambda} \tag{5.38}$$

and the estimate of the rational form of $s(\lambda)$ is given by

$$\hat{s}(\lambda) = 2 \frac{a_o + a_1 \cos\lambda}{1 + b_1 \cos\lambda} \tag{5.39}$$

Equations (5.38) and (5.39) give

$$2a_o = \frac{\sigma_a^2}{\sigma_x^2} \frac{1+\Theta_1^2}{1+\Phi_1^2}$$

$$\frac{a_1}{a_o} = \frac{-2\Theta_1}{1+\Theta_1^2} \tag{5.40}$$

and $b_1 = \frac{-2\Phi_1}{1+\Phi_1^2}$

Equations (5.3), (5.5) and (5.6) give the following

$$a_o = \frac{1}{2}\left[1 + b_1 \rho_1\right]$$

$$a_1 = \frac{1}{2}\left[2\rho_1 + b_1(1 + \rho_2)\right] \tag{5.41}$$

and $$b_1 = \frac{-2\rho_2}{\rho_1 + \rho_3}$$

For an ARMA(1,1) process σ_x^2 is given by

$$\sigma_x^2 = \frac{1 + \theta_1^2 - 2\theta_1\phi_1}{1 - \phi_1^2}\sigma_a^2 \tag{5.42}$$

Using (5.40), (5.41) and (5.42) we get

$$\frac{1}{2}\left[1 - \frac{2\rho_2\rho_1}{\rho_1 + \rho_3}\right] = \frac{(1 - \phi_1^2)(1 + \theta_1^2)}{(1 + \theta_1^2 - 2\theta_1\phi_1)(1 + \phi_1^2)} \tag{5.43}$$

$$\frac{\frac{1}{2}\left[2\rho_1 - \frac{2\rho_2(1 + \rho_2)}{\rho_1 + \rho_3}\right]}{\frac{1}{2}\left[1 - \frac{2\rho_1\rho_2}{\rho_1 + \rho_3}\right]} = \frac{-2\theta_1}{1 + \theta_1^2} \tag{5.44}$$

and

$$\frac{-2\rho_2}{\rho_1 + \rho_3} = \frac{-2\phi_1}{1 + \phi_1^2} \tag{5.45}$$

Simplifying (5.43), (5.44) and (5.45) we get

$$\rho_1 = \frac{(1-\phi_1\theta_1)(\phi_1-\theta_1)}{1+\theta_1^2 - 2\phi_1\theta_1} \tag{5.46}$$

and $\rho_2 = \phi_1\rho_1$

Then using (5.45) and (5.46) we get

$$\rho_3 = \phi_1\rho_2$$

which implies

$$\rho_k = \phi_1\rho_{k-1} \quad , k \geq 2$$

Hence the proof of the result.

5.2 Bivariate time series modelling

So far we were considering the model identification of a univariate time series. In this section we consider the analysis of a bivariate, stationary time series. A univariate time series is considered as a realization of a stochastic process. In the bivariate case, there will be two time series, usually one is considered as the input series and the other as the corresponding output series. Then the pair of time series is regarded as a realization of a hypothetical population of pairs of discrete time series, called a bivariate stochastic process.

Definition 5.1 Cross covariance and cross correlation functions

A bivariate stochastic process need not be stationary, we assume that the appropriately differenced process (x_t, y_t), where $x_t = \nabla^d X_t$ and $y_t = \nabla^d Y_t$, X_t and Y_t being the given input andoutput series respectively, are stationary. The stationarity assumption implies that the process x_t and y_t have constant mean, μ_x and μ_y and constant variance σ_x^2 and σ_y^2 respectively. The autocovariance coefficients of each constituent series at lag k are defined by the usual formula

$$\gamma_{xx}(k) = E[(x_t-\mu_x)(x_{t+k}-\mu_x)]$$

$$\text{and } \gamma_{yy}(k) = E[(y_t-\mu_y)(y_{t+k}-\mu_y)] \qquad (5.47)$$

where we now used the extended notation $\gamma_{xx}(k)$ and $\gamma_{yy}(k)$ for the autocovariances of the x and y series. The cross covariance coefficients between y_t and x_t at lag k are defined as

$$\gamma_{yx}(k) = E[(y_t-\mu_y)(x_{t+k}-\mu_x)], \quad k=0,\pm1,\pm2,.. \qquad (5.48)$$

and the cross covariance coefficients between x_t and y_t at lag $\pm k$ are

$$\gamma_{xy}(k) = E[(x_t-\mu_x)(y_{t+k}-\mu_y)], \quad k=0,\pm1,\pm2,... \qquad (5.49)$$

In general $\gamma_{yx}(k)$ is not equal to $\gamma_{xy}(k)$. The set $\{\gamma_{xy}(k),\ k=0,\pm1,\ldots\}$ is called the cross covariance function of the bivariate process. The cross correlation coefficient at lag k is defined as

$$\rho_{xy}(k) = \frac{\gamma_{xy}(k)}{\sigma_x \sigma_y}, \quad k=0,\pm1,\pm2,\ldots \tag{5.50}$$

Since $\rho_{xy}(k)$ is not in general equal to $\rho_{yx}(k)$, the cross correlation function, in contrast to the ACF, is not symmetric about k = 0. The cross correlation function is zero over some range $-\infty$ to i or i to ∞. This property of the cross correlation is used to determine the lag of the delayed period of the input series.

Definition 5.2 Estimates of the cross covariances and cross correlations

After differencing the original input and output series d times to obtain stationarity there are N-d pairs of values (y_1,x_1), $(y_2, x_2)\ldots,(y_n,x_n)$, available for analysis. Then the estimate $c_{xy}(k)$ of the cross covariance coefficient at lag k is provided by

$$c_{xy}(k) = \begin{cases} \frac{1}{n} \sum_{t=1}^{n-k} (x_t-\bar{x})(y_{t+k}-\bar{y}), & k=0,1,2,\ldots \\ \frac{1}{n} \sum_{t=1}^{n+k} (y_t-\bar{y})(x_{t-k}-\bar{x}), & k=0,-1,-2,\ldots \end{cases} \tag{5.51}$$

where $\bar{y}$ and $\bar{x}$ are the means of the y series and x series respectively. Similarly the estimate $r_{xy}(k)$ of the cross correlation coefficient $\rho_{xy}(k)$ at lag k is defined by

$$r_{xy}(k) = \frac{c_{xy}(k)}{s_x s_y}, \quad k=0,\pm1,\pm2,... \tag{5.52}$$

s_x and s_y being estimates of σ_x and σ_y respectively.

Definition 5.3 Transfer function models

If x_t and y_t are the stationary series, then the output series y_t in terms of the input variable x_t is given by

$$y_t = v_o x_t + v_1 x_{t-1} + v_2 x_{t-2} + ... \tag{5.53}$$

This model is defined as the transfer function model. Then

$$v(B) = v_o + v_1 B + ... \tag{5.54}$$

$$\text{i.e., } v(B) = \sum_{i=0}^{\infty} v_i B^i$$

is called the linear filter or transfer function. The system is said to stable if the infinite series $v(B)$ converges for $|B| \leq 1$. In this thesis we are considering only stable systems.

It would be often unsatisfactory to parameterize the system in terms of v's. So the transfer functions are parsimoniously represented by

$$v(B) = \frac{\omega(B)}{\delta(B)} \tag{5.55}$$

Then the model becomes

$$\delta(B)y_t = \omega(B)x_t. \tag{5.56}$$

The delayed transfer function model is given by

$$\delta(B)y_t = \omega(B)x_{t-b} \tag{5.57}$$

The condition for stability of $v(B) = \frac{\omega(B)}{\delta(B)}$ is that the roots of the characteristic equation $\delta(B) = 0$ must lie outside the unit circle.

5.3 Estimation of the transfer function

Given the stationary series y_t and x_t we want to determine the transfer function model

$$y_t = A(B)x_{t-b} \tag{5.58}$$

where $A(B) = \frac{\omega(B)}{\delta(B)}$, where $\omega(B)$ and $\delta(B)$ are polynomials in B, the backward shift operator. Now the problem is to obtain A(B).

Definition 5.4 Spectral density function of a transfer function model

The spectral density function of a transfer function model given by (5.58) will be of the form

$$f_{yy}(\lambda) = k|A(e^{i\lambda})|^2 f_{xx}(\lambda), \tag{5.59}$$

where $f_{yy}(\lambda)$ is the spectral density function of the univariate time series y_t and $f_{xx}(\lambda)$ is the spectral density function of the univariate time series x_t. From equation (5.59) we get

$$|A(e^{i\lambda})|^2 = k' \frac{f_{yy}(\lambda)}{f_{xx}(\lambda)} \tag{5.60}$$

The relation obtained in (5.60) is used to estimate the transfer function A(B). The steps are given by (i) obtain the estimates of the rational forms of the spectral densities $f_{yy}(\lambda)$ and $f_{xx}(\lambda)$. Suppose

$$f_{yy}(\lambda) = 2 \frac{a_o + a_1 \cos\lambda + \ldots + a_{l_1} \cos(l_1\lambda)}{1 + b_1 \cos\lambda + \ldots + b_{k_1} \cos(k_1\lambda)} \tag{5.61}$$

and

$$f_{xx}(\lambda) = 2\,\frac{a_o' + a_1'\cos\lambda + \ldots + a_{l_2}'\cos(l_2\lambda)}{1 + b_1'\cos\lambda + \ldots + b_{k_2}'\cos(k_2\lambda)} \tag{5.62}$$

Then

$$\frac{f_{yy}(\lambda)}{f_{xx}(\lambda)} = \frac{\left[a_o + a_1\cos\lambda + \ldots + a_{l_1}\cos(l_1\lambda)\right]\left[1 + b_1'\cos\lambda + \ldots + b_{k_2}'\cos(k_2\lambda)\right]}{\left[1 + b_1\cos\lambda + \ldots + b_{k_1}\cos(k_1\lambda)\right]\left[a_o' + a_1'\cos\lambda + \ldots + a_{l_2}'\cos(l_2\lambda)\right]}$$

On simplification we get

$$\frac{k' f_{yy}(\lambda)}{f_{xx}(\lambda)} = \frac{A_o + A_1\cos\lambda + \ldots + A_L\cos(L\lambda)}{B_o + B_1\cos\lambda + \ldots + B_K\cos(K\lambda)} \tag{5.63}$$

where $L = l_1 + k_2$ and $K = l_2 + k_1$. Considering the transfer function A(B), if the degree of numerator is r and that of the denominator is s then A(B) is given by

$$A(B) = \frac{\omega_o + \omega_1 B + \ldots + \omega_r B^r}{1 + \delta_1 B + \ldots + \delta_s B^s} \tag{5.64}$$

Then

$$\left|A(e^{i\lambda})\right|^2 = \frac{\sum_{i=0}^{r}\omega_i^2 + 2\sum_{i=0}^{r-1}\omega_i\omega_{i+1}\cos\lambda + \ldots + 2\omega_o\omega_r\cos(r\lambda)}{1 + \sum_{i=1}^{s}\delta_i^2 + 2\left[\sum_{i=1}^{s-1}\delta_i\delta_{i+1} + \delta_i\right]\cos\lambda + \ldots + 2\delta_s\cos(s\lambda)} \tag{5.65}$$

Using (5.63) and (5.65) we obtain the following equations:

$$r = l_1 + k_2 = L$$

$$s = l_2 + k_1 = K \tag{5.66}$$

$$A_o = \sum_{i=0}^{r} \omega_i^2$$

$$A_j = 2 \sum_{i=0}^{r-j} \omega_i \omega_{i+j}, \quad j=1,2,\ldots,r \tag{5.67}$$

$$B_o = 1 + \sum_{i=1}^{s} \delta_i^2$$

$$\text{and} \quad B_j = 2\left[\sum_{i=1}^{s-j} \delta_i \delta_{i+j} + \delta_j\right], \quad j=1,2,..s \tag{5.68}$$

Applying the method of iteration given in chapter 3 to the systems of equations (5.67) and (5.68) the values of ω_i, i=0,...,r and δ_i, i=1,...s can be determined.

Here the application of the rational approximation of the spectral density function simultaneously determine the order of the rational form of the transfer function as well as its coefficients. The estimated transfer function

model is

$$\hat{y}_t = \hat{A}(B)x_{t-b} \tag{5.69}$$

where

$$\hat{A}(B) = \frac{\hat{\omega}_o + \hat{\omega}_1 B + \ldots + \hat{\omega}_r B^r}{1 + \hat{\delta}_1 B + \ldots + \hat{\delta}_s B^s} \tag{5.70}$$

5.4 Estimation of transfer function noise model

In practice the output y_t could not be expected to follow exactly the pattern determined by the transfer function model, even if that model were entirely adequate. Disturbances of various kinds other than X normally corrupt the system. A disturbance might originate at any point in the system, but it is often convenient to consider it in terms of the net effect on the output Y. If we assume that the disturbance or noise N_t is independent of the level of X and additive with respect to the influence of X, then we can write

$$Y_t = A(B)X_{t-b} + N_t \tag{5.71}$$

If the noise model can be represented by an ARIMA(p,d,q) process, then

$$\Phi(B)N_t = \Theta(B)a_t, \tag{5.72}$$

where a_t is the white noise. Then the transfer function noise model can be written as

$$Y_t = A(B)X_{t-b} + \Phi^{-1}(B)\,\Theta(B)a_t \tag{5.73}$$

The transformed stationary noise process will be

$$n_t = \nabla^d N_t \tag{5.74}$$

then the estimated values of n_t are obtained as

$$n_t = y_t - A(B)x_t \tag{5.75}$$

Then using R-spec technique, the ARMA(p,q) model representing n_t can be estimated. Hence we obtain

$$n_t = \frac{\hat{\Theta}(B)}{\hat{\Phi}(B)}\, a_t \tag{5.76}$$

Using (5.69) and (5.76) the estimated transfer function model is

$$\hat{y}_t = A(B)x_{t-b} + \hat{\Phi}^{-1}(B)\,\hat{\Theta}(B)a_t \tag{5.77}$$

The identification procedure of the transfer function noise model can be summarised as follows.

1. Estimate $\gamma_{yy}(k)$, $\gamma_{xx}(k)$ and $\gamma_{yx}(k)$, $k=0,\pm1,\ldots$

(i) Use the estimated values $\gamma_{yy}(k)$, $\gamma_{xx}(k)$ and $\gamma_{yx}(k)$ to test whether the given series are stationary. If the series is non-stationary obtain the transformed stationary series by differencing d times d = 0 or 1 or 2

(ii) From the estimated values of $\rho_{yx}(k)$, i.e., $\gamma_{yx}(k)$ $k = 0 \pm 1, \pm 2, \ldots$, find the cut off point, which will give the number of delayed lags. Suppose the delayed lag is b, b = 0 or ± 1 or $\pm 2, \ldots$.

(iii) Using $\gamma_{yy}(k)$, $k = 0, \pm 1, \pm 2, \ldots$ and $\gamma_{xx}(k)$, $k=0, \pm 1, \pm 2, \ldots$ estimate the rational form of the spectral density functions $f_{yy}(\lambda)$ of y_t and $f_{xx}(\lambda)$ of x_t respectively, using the method developed in chapter 3.

2. Determine the coefficients of A(B), using the relation

$$|A(e^{i\lambda})|^2 = k \frac{f_{yy}(\lambda)}{f_{xx}(\lambda)} .$$

3. The estimated transfer function model is

$$\hat{y}_t = \hat{A}(B)x_{t-b}$$

4. Find the deviations

$$\hat{n}_t = y_t - \hat{A}(B)x_t \qquad (5.78)$$

5. Estimate the ARMA model for n_t using the R-spec procedure, taking $\hat{n}_t$ as the observed series. Let the ARMA model be

$$\hat{n}_t = \frac{\hat{\Theta}(B)}{\hat{\Phi}(B)} a_t.$$

6. The transfer function noise model estimated is given by

$$\hat{y}_t = \hat{A}(B)x_t + \hat{\Phi}^{-1}(B)\,\Theta(B)a_t$$

$$\text{i.e., } \hat{y}_t = \frac{\hat{\omega}_o + \hat{\omega}_1 B + \ldots + \hat{\omega}_r B^r}{1 + \hat{\delta}_1 B + \ldots + \hat{\delta}_s B^s} x_{t-b} + \frac{\hat{\Theta}(B)}{\hat{\Phi}(B)} a_t$$

$$\text{i.e., } \hat{y}_t = \frac{\hat{\omega}(B)}{\hat{\delta}(B)} x_{t-b} + \frac{\hat{\Theta}(B)}{\hat{\Phi}(B)} a_t$$

which can be written as

$$\hat{\Phi}(B)\hat{\delta}(B)\hat{y}_t = \hat{\Phi}(B)\,\hat{\omega}(B)x_{t-b} + \hat{\delta}(B)\,\hat{\Theta}(B)a_t \qquad (5.79)$$

$$\text{i.e., } \delta^*(B)\hat{y}_t = \phi^*(B)x_{t-b} + \Theta^*(B)a_t \qquad (5.80)$$

where $\delta^*(B) = \hat{\Phi}(B)\hat{\delta}(B)$

$\phi^*(B) = \hat{\Phi}(B)\hat{\omega}(B)$

and $\Theta^*(B) = \hat{\delta}(B)\hat{\Theta}(B)$.

Result 5.6

If x_t and y_t are two jointly stationary time series and $f_{yx}(\lambda)$ is the cross spectral density function, then $|f_{yx}(\lambda)|^2$ can be approximated by a ratio of polynomials in $\cos\lambda$.

Proof of Result 5.6

The proof follows from the following facts.

(i) Since x_t and y_t are jointly stationary $\sum_{k=-\infty}^{\infty} \gamma_{yx}(k)$ is convergent.

(ii) $|f_{yx}(\lambda)|^2$ is an infinite polynomial of the form $c_o+c_1\cos\lambda + c_2\cos(2\lambda)+\ldots$

(iii) Since $|f_{yx}(\lambda)|^2$ is continuous in $[-\pi,\pi]$ and bounded because of (i), $|f_{yx}(\lambda)|^2$ can be approximated by a ratio of polynomials in $\cos\lambda$.

5.5 Analysis of Multivariate time series

The application of result 5.6 is very useful in multivariate time series analysis. In this section we consider the modelling of multivariate time series of the

form

$$y_t = A_1(B)x_{1t-b_1} + A_2(B)x_{2t-b_2} + \ldots + A_n(B)x_{nt-b_n}, \qquad (5.81)$$

where $x_{1t}, x_{2t}, \ldots, x_{nt}$ are pairwise uncorrelated and $A_1(B), \ldots, A_n(B)$ are ratios of polynomials in B.

i.e., $E[x_{it}x_{jt}] = 0, \; i \neq j$

Then spectral density of (5.81) and the cross spectral densities will be given by

$$f_{yy}(\lambda) = |A_1(e^{i\lambda})|^2 f_{x_1x_1}(\lambda) + |A_2(e^{i\lambda})|^2 f_{x_2x_2}(\lambda) + \ldots$$

$$+ |A_n(e^{i\lambda})|^2 f_{x_nx_n}(\lambda)$$

and

$$f_{yx_1}(\lambda) = k_1 A_1(e^{i\lambda}) f_{x_1x_1}(\lambda)$$

$$f_{yx_2}(\lambda) = k_2 A_2(e^{i\lambda}) f_{x_2x_2}(\lambda)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{yx_n}(\lambda) = k_n A_n(e^{i\lambda}) f_{x_nx_n}(\lambda)$$

Or in the matrix form we can write

$$\begin{bmatrix} k f_{x_1x_1}(\lambda) & 0 \ldots\ldots\ldots 0 \\ 0 & k_2 f_{x_2x_2}(\lambda) \ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 0 & 0 \ldots\ldots k_n f_{x_nx_n}(\lambda) \end{bmatrix} \begin{bmatrix} A_1(e^{i\lambda}) \\ A_2(e^{i\lambda}) \\ \ldots\ldots \\ \ldots\ldots \\ A_n(e^{i\lambda}) \end{bmatrix} = \begin{bmatrix} f_{yx_1}(\lambda) \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ f_{yx_n}(\lambda) \end{bmatrix} \tag{5.82}$$

Solving (5.82) we get

$$\begin{aligned} A_1(e^{i\lambda}) &= k_1 \frac{f_{yx_1}(\lambda)}{f_{x_1x_1}(\lambda)} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ A_n(e^{i\lambda}) &= k_n \frac{f_{yx_n}(\lambda)}{f_{x_nx_n}(\lambda)} \end{aligned} \tag{5.83}$$

Now to obtain the coefficients of $A_i(e^{i\lambda})$, i=1,2,...n, estimate the rational form of $|f_{yx_i}(\lambda)|^2$, i=1,2,..n, and $f_{x_ix_i}(\lambda)$ using the R-spec procedure, then equate the coefficients in the equation

$$|A_i(e^{i\lambda})|^2 = k_i \frac{|f_{yx_i}(\lambda)|^2}{|f_{x_ix_i}(\lambda)|^2} \tag{5.84}$$

Then using the method of iteration to the non-linear equations obtain the estimates of the coefficients of $A_i(B)$, i=1,2,...,N. If $\hat{A}_1(B)$, $\hat{A}_2(B)$,..., $\hat{A}_n(B)$ are the estimated transfer functions then the transfer function model is given by

$$\hat{y}_t = \hat{A}_1(B)x_{1t-b_1}+\hat{A}_2(B)x_{2t-b_2}+\ldots+\hat{A}_n(B)x_{nt-b_n} \qquad (5.85)$$

Then the noise series is obtained using

$$\hat{n}_t = y_t - \sum_{i=1}^{n} \hat{A}_i(B)x_{it-b_i} \qquad (5.86)$$

and estimate the ARMA model representing the noise series n_t using the R-spec procedure. The model will be of the form

$$\hat{\Phi}(B)n_t = \hat{\Theta}(B)a_t$$

$$\text{i.e., } n_t = \hat{\Phi}^{-1}(B)\Theta(B)a_t \qquad (5.87)$$

Using (5.85) and (5.87) we obtain the multivariate transfer function noise model as

$$\hat{y}_t = \hat{A}_1(B)x_{1t-b_1}+\ldots+\hat{A}_n(B)x_{nt-b_n}+ \hat{\Phi}^{-1}(B)\hat{\Theta}(B)a_t \qquad (5.88)$$

and the variance of the white noise σ_a^2, which we obtain along with the noise model.

Note 5.1

The transfer function modelling of multivariate time series can be extended to the general case, in the same manner, provided the series are pairwise jointly stationary.

Chapter 6

COMPARISON OF R-SPEC TECHNIQUE

6.1 Introduction

This chapter is completely devoted to error analysis of R-spec technique in comparison with other techniques. For this purpose time series of chemical process concentrations given as series A in Box and Jenkins [8] is presented here. The analysis of the series is done by Box and Jenkins, McIntire, Akaike and others. We selected this series so that we can give a comparison of the R-spec procedure with other prominent techniques.

6.2 Analysis of series A using R-spec technique

The actual data of series A, taken from Box and Jenkins [8] is given in the appendix. A plot of the actual data of series A is given in fig.6.1. ACF of series A is given in table.6.1 and PACF is given in table 6.2 From the correlogram (fig.6.2) and the graph of the PACF (fig.6.3) the upper bounds P and Q are selected as P = 2 and Q = 0.

The initial rational forms are

$$T_1^o(\cos\lambda) = \frac{0.2834}{1-0.76\cos(\lambda)} \tag{6.1}$$

$$T_2^o(\cos\lambda) = \frac{0.25885}{1-0.518167\cos(\lambda)-0.373968\cos(2\lambda)} \tag{6.2}$$

where $\lambda \in [-\pi,\pi]$. The differences of $T_1^o(\cos\lambda)$ and $T_2^o(\cos\lambda)$ from $f(\cos\lambda)$ are given in table 6.3. Table 6.3 shows that the maximum absolute error corresponding to $T_2^o(\cos\lambda)$ is minimum when compared to that of $T_1^o(\cos\lambda)$. So $T_2^o(\cos\lambda)$ is selected as the initial rational approximation of $f(\cos\lambda)$. Table 6.4 gives the differences of $T_2^{o(i)}(\cos\lambda)$, i being the number of iteration, from $f(\cos\lambda)$. The convergence of $|E_2^{(i)}|$, $a_o^{(i)}$, $b_1^{(i)}$ and $b_2^{(i)}$ are shown in table 6.5, which gives $T_2^{o*}(\cos\lambda) = T_2^{o(1)}(\cos\lambda)$. Hence $T_2^{o*}(\cos\lambda)$ is given as

$$T_2^{o*}(\cos\lambda) = \frac{0.2588}{1-0.51839\cos(\lambda)-0.3738\cos(2\lambda)} \tag{6.3}$$

The estimated rational form of the spectral density function

is

$$\hat{s}(\lambda) = 2T_2^{o*}(\cos\lambda) \tag{6.4}$$

From equation (6.3) the order of the ARMA model representing x_t is obtained as p = 2 and q = 0. Further from (6.3) the following equations are obtained:

$$\phi_1 = 0.2592(1 + \phi_1^2 + \phi_2^2) + \phi_1\phi_2 \tag{6.5}$$

$$\phi_2 = 0.1869(1 + \phi_1^2 + \phi_2^2) \tag{.6.6}$$

and

$$\sigma_a^2 = 2x0.2588(1 + \phi_1^2 + \phi_2^2)\sigma_x^2 \tag{6.7}$$

The method of iteration is applied to equations (6.5) and (6.6) to obtain the estimates of ϕ_1 and ϕ_2. The iteration values are given in table 6.6. The estimated values are,

$$\hat{\phi}_1 = 0.41 \tag{6.8}$$

and

$$\hat{\phi}_2 = 0.23 \tag{6.9}$$

Then the residual variance σ_a^2 is estimated using (6.7) and the value obtained is

$$\hat{\sigma}_a^2 = 0.094 \tag{6.10}$$

Hence the model representing series A is obtained, using the R-spec technique as

$$\tilde{x}_t = 0.41\tilde{x}_{t-1} + 0.23\tilde{x}_{t-2} + a_t \qquad (6.11)$$

6.3 Models identified using other methods

Box and Jenkins [8] estimated two models for series A. The models suggested by them are

$$x_t = 1.45+0.92x_{t-1}+a_t-0.58a_{t-1} \qquad (6.12)$$

and $$x_t = x_{t-1}+a_t-0.70a_{t-1} \qquad (6.13)$$

The residual variances of (6.12) and (6.13) are 0.097 and 0.101 respectively.

The R and S procedure due to McIntire [28] also suggests two models for series A. The models are given by

$$x_t = 0.42x_{t-1}+0.17x_{t-2}+0.26x_{t-7}+a -0.26a_{t-1} \qquad (6.14)$$

and

$$x_t = 0.32x_{t-1}+0.18x_{t-2}+0.50x_{t-7}+a_t \qquad (6.15)$$

with residual variances 0.098 and 0.097 respectively.

Box and Jenkins as well as McIntire suggests two different models for series A, whereas the R-spec technique determines a unique model.

Table 6.7 gives models identified for series A using different identification techniques.

Table 6.1

The estimated ACF of Series A

Lags	1	2	3	4	5
1-5	0.57	0.50	0.40	0.36	0.33
6-10	0.35	0.39	0.32	0.30	0.26
11-15	0.19	0.16	0.20	0.24	0.14

Table 6.2

The estimated PACF of Series A

Lags	1	2	3	4	5
1-5	0.57	0.25	0.08	0.09	0.07
6-10	0.15	0.19	-0.03	0.01	-0.01
11-15	-0.09	-0.04	0.04	0.08	-0.10

Table 6.3*

Errors in various rational approximations to $f(\cos\lambda)$

k	E_1	E_2
-1.0	0.05098	-0.01423
-0.9	0.07741	-0.02443
-0.8	0.02450	0.00144
-0.7	0.00706	0.02068
-0.6	-0.00406	0.04847
-0.5	-0.0434	0.05160
-0.4	-0.12657	0.01724
-0.3	-0.04579	0.14724
-0.2	-0.22976	-0.05036
-0.1	0.36240	0.08386
0	0.61970	0.02825
0.1	0.36240	0.08386
0.2	-0.22976	-0.05036
0.3	-0.04579	0.14724
0.4	-0.12657	0.01724
0.5	-0.0434	0.05160
0.6	-0.00406	0.04847
0.7	0.00706	0.02068
0.8	0.02450	0.00144
0.9	0.07741	-0.02443
1.0	0.05098	-0.01423

$$\lambda = k\pi$$

$$E_1 = f(\cos\lambda) - \frac{0.2834}{1-0.76\cos(\lambda)}$$

$$E_2 = f(\cos\lambda) - \frac{0.25885}{1-0.518167\cos(\lambda)-0.373968\cos(2\lambda)}$$

* The errors E_1 and E_2 were calculated for 201 values of $\lambda \in [-\pi,\pi]$ and twentyone values are included in table.

Table 6.4

Errors of $T_2^{o(i)}(\cos\lambda)$ from $f(\cos\lambda)$, i being the number of iteration

k	$E_2^{(o)}$	$E_2^{(1)}$
-1.0	-0.01423	-0.01410
-0.9	-0.02443	-0.02454
-0.8	0.00144	0.00151
-0.7	0.02068	0.02073
-0.6	0.04847	0.04849
-0.5	0.05160	0.05161
-0.4	0.01724	0.01724
-0.3	0.14724	0.14697
-0.2	-0.05036	-0.05037
-0.1	0.08386	0.08361
0	0.02825	0.02748
0.1	0.08386	0.08361
0.2	-0.05036	-0.05037
0.3	0.14724	0.14697
0.4	0.01724	0.01724
0.5	0.05160	0.05161
0.6	0.04847	0.04849
0.7	0.02068	0.02073
0.8	0.00144	0.00151
0.9	-0.02443	-0.02454
1.0	-0.01423	-0.01410

$\lambda = k\pi$

$$E_2^{(o)} = f(\cos\lambda) - \frac{0.25885}{1-0.51818\cos(\lambda)-0.3740\cos(2\lambda)}$$

$$E_2^{(1)} = f(\cos\lambda) - \frac{0.2588}{1-0.51839\cos(\lambda)-0.3738\cos(2\lambda)}$$

Table 6.5

Iteration table for the rational approximation $T_2^{o*}(\cos\lambda)$

Iteration Number	$\lvert E_2^{(i)} \rvert$	$a_o^{(i)}$	$b_1^{(i)}$	$b_2^{(i)}$
1	0.14724	0.25885	-0.51818	-0.3740
2	0.14697	0.2588	-0.51839	-0.3738

Table 6.6

Iteration table for Φ_1 and Φ_2

Iteration Number	Φ_1	Φ_2
0	0.2592	0.1869
1	0.334113	0.2059855
2	0.3679551	0.2156941
3	0.3857181	0.2208999
4	0.3956165	0.2238268
5	0.4013031	0.2255155
6	0.4046249	0.225043
7	0.4065838	0.2270882
8	0.4077455	0.2274347
9	0.4084366	0.227641
10	0.4088486	0.2277639
11	0.4090944	0.2277373
12	0.4092412	0.227812
13	0.4092924	0.2279015
14	0.4093622	0.2279169
15	0.409401	0.2279289
16	0.4094244	0.2279359
17	0.4094384	0.22794
18	0.409447	0.2279425
19	0.4094519	0.2279441
20	0.4094549	0.2279449

Table 6.7

Models identified for Series A using different techniques

Selection technique	Order	Model	Residual*
AIC	(1,0,1)	$x_t = 0.87x_{t-1} + a_t - 0.48a_{t-1}$	0.098
ANDER	(2,0,0)	$x_t = 0.42x_{t-1} + 0.25x_{t-2} + a_t$	0.099
Box-Jenkins	(0,1,1)	$x_t = x_{t-1} + a_t - 0.70a_{t-1}$	0.101
	(1,0,1)	$x_t = 0.92x_{t-1} + a_t - 0.58a_{t-1}$	0.098
FPE	(7,0,0)	$x_t = 0.37x_{t-1} + 0.20x_{t-2} + 0.02x_{t-3} + 0.01x_{t-4} - 0.01x_{t-5} + 0.06x_{t-6} + 0.16x_{t-7} + a_t$	0.091
R and S	(7,0,)	$x_t = 0.42x_{t-1} + 0.17x_{t-2} + 0.26x_{t-7} + a_t - 0.26a_{t-1}$	0.098
	(7,0,0)	$x_t = 0.32x_{t-1} + 0.18x_{t-2} + 0.50x_{t-7} + a_t$	0.097
R-spec	(2,0,0)	$x_t = 0.41x_{t-1} + 0.23x_{t-2} + a_t$	0.094
White Noise	(0,0,1)	$x_t = a_t$	0.159
Yule-Walker	(7,1)	$x_t = 0.17x_{t-1} + 0.27x_{t-2} + 0.06x_{t-3} + 0.02x_{t-4} - 0.01x_{t-5} + 0.07x_{t-6} + 0.18x\ x_{t-7} + a_t + 0.20a_{t-1}$	0.87

* Residuals are calculated as the mean of squares of one-step-ahead forecast errors.

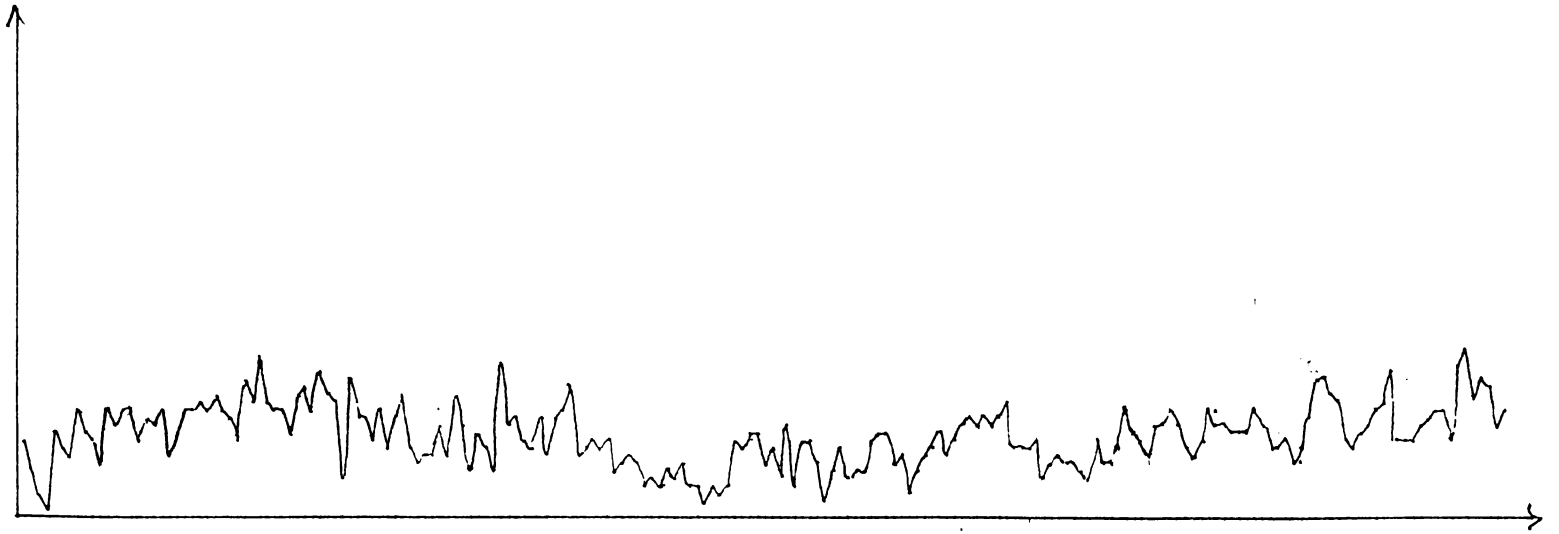


FIG. 6.1

PLOT OF SERIES A:

CHEMICAL PROCESS CONCENTRATION READINGS
EVERY TWO HOURS

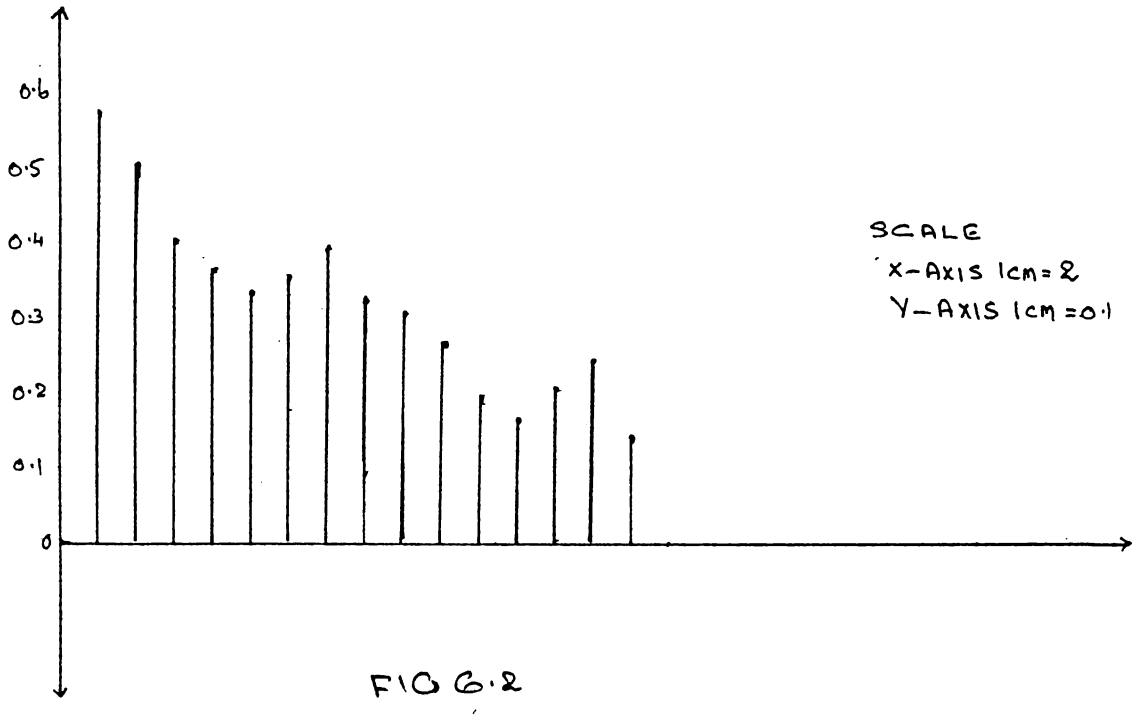


FIG 6.2

ACF OF SERIES A

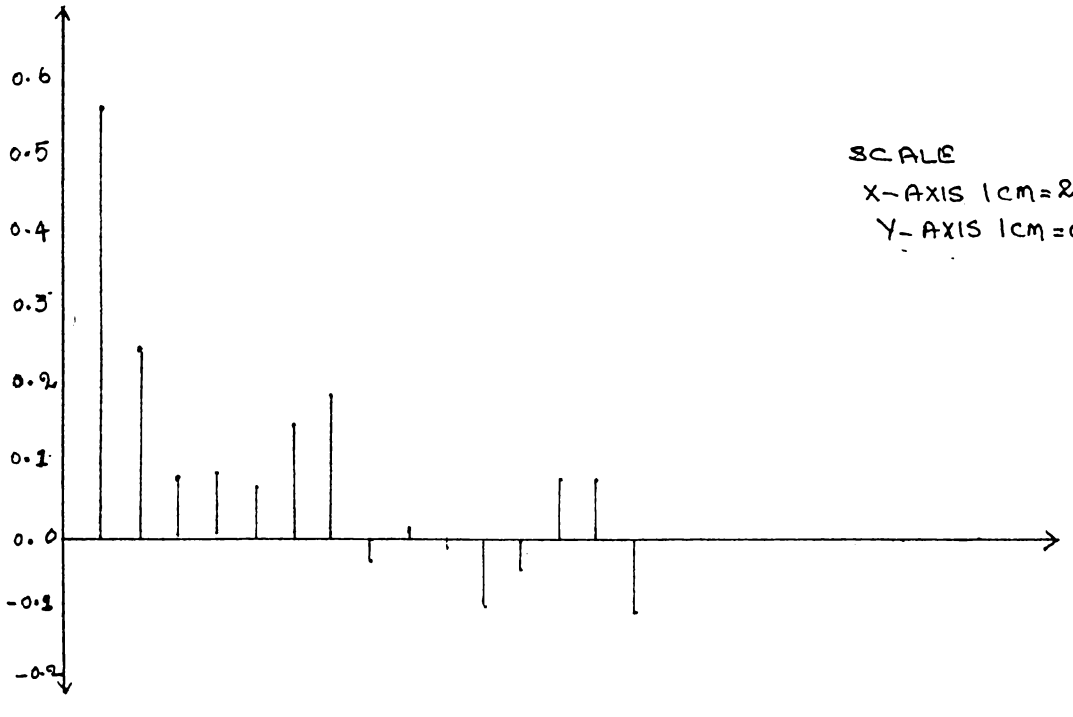


FIG 6.3

PACF OF SERIES A

Chapter 7

CONCLUSION

A new technique for ARMA(p,q) model identification of stationary time series, using the rational approximation of the spectral density function is presented in this thesis. Analysis of a number of stationary time series are carried out. Table 6.7 shows that the R-spec procedure is equally good as any other prominent techniques used for the analysis. It has got the additional advantage that this procedure determines a unique model for a given series. Moreover, the order of the model is determined more scientifically since it is obtained by computation.

In chapter 5 the method is extended to multivariate series provided the component series are pairwise jointly stationary. This result has wide applications in the analysis of industrial data. The estimation solutions of multivariate time-series analysis appears to be an interesting application.

Some more work can be done in the multivariate time series analysis like the error analysis, the case of non-stationary components etc.

APPENDIX

A.1 PROOF OF LEMMA 3.1: (See Blum [7] page 313)

Consider $R_m^n[a,b]$ to be a subset of the metric space $C[a,b]$, with the uniform metric. Let d be the distance from f to $R_m^n[a,b]$ such that $\|f-R_i\|_\infty \longrightarrow d$. Hence $\|R_i\|_\infty \leq \|f-R_i\|_\infty + \|f\|_\infty \leq 1+d+\|f\|_\infty = k$ for all i sufficiently large. Now, $R_i = \frac{P_i}{Q_i}$, where deg $P_i \leq n$ and deg $Q_i \leq m$. Furthermore by dividing numerator and denominator by $\|Q_i\|_\infty$, which is non-zero by virtue of $Q_i(x) > 0$, we may take $\|Q_i\|_\infty = 1$. Therefore, the coefficient vectors $b_i = (b_{oi},\ldots,b_{mi})$ of Q_i lie in some closed bounded, hence compact set in (m+1)-dimensional space. Also

$$|P_i(x)| = |Q_i(x)||R_i(x)| \leq \|Q_i\|_\infty \times \|R_i\|_\infty \leq k.$$

Therefore $\{\|P_i\|_\infty\}$ is a bounded set and the same holds for the coefficient vectors $a_i = (a_{oi},\ldots,a_{ni})$ of the P_i. By sequential compactness, there exist subsequences of (b_i) and (a_i) which converge. Taking a common subsequence and reindexing, we may write $P_i \rightarrow P$ and $Q_i \rightarrow Q$. Since all $\|Q_i\|_\infty = 1$ it follows that $\|Q\|_\infty = 1$. If $Q(x) \neq 0$, then $R(x) = \frac{P(x)}{Q(x)}$ is the limit of $R_i(x)$. Suppose $Q(x_j) = 0$

for some x_j in [a,b]. (There are atmost m such points because $\|Q\|_\infty = 1$). Since $R_i(x) \rightarrow R(x)$ for any x such that $Q(x) \neq 0$, it follows that for such x, $|P(x)| \leq k|Q(x)|$. Now by the continuity ofP and Q at x_j, this inequality must remain valid at x_j. This implies $P(x_j) = 0$. Thus the linear factors $(x-x_j)$ may be cancelled from both P and Q, yielding a rational function $\frac{P^*}{Q^*}$ which is equal to R and such that $Q^*(x) > 0$ for all x in [a,b]. Therefore $\frac{P^*}{Q^*}$ is in R_m^n [a,b] and $\left\| f - \frac{P^*}{Q^*} \right\|_\infty = \lim \|f - R_i\|_\infty = d$, i.e., $\frac{P^*}{Q^*}$ is the best approximation which proves the lemma.

A.2 PROOF OF LEMMA 3.2

Sufficiency is established first. Suppose R^* is not the best approximation. Then there exists $\frac{P}{Q} = R \in R_m^n$ [a,b] such that

$\|f-R\|_\infty \leq \|f-R^*\|_\infty$. Let $E = \{ x \mid |r(x)| = \|f-R^*\|_\infty \}$

For $x \in E$, $(\operatorname{sgn} r^*(x) = f-R^*$ and

$$\operatorname{sgn} r^*(x)\,(f(x)-R(x)) \leq \|f-R\|_\infty < \operatorname{sgn} r^*(x)(f(x)-R^*(x))$$

Hence $\operatorname{sgn} r^*(x)\,(R(x)-R^*(x)) > 0$ on E, so that $r^*(x)$ and $Q(x)(R(x)-R^*(x))$ have the same sign on E. $(Q(x) > 0)$. But

we have seen that

$QR-QR^* = P-R^*Q \in V_P+R^*V_Q$ has atmost

$\max\{n+\deg Q^*,\ m+\deg P^*\}$ changes of sign in $[a,b]$. Therefore, $r^*(x)$ changes sign on E atmost that many times i.e., has less than $1+\max\{n+\deg Q^*,\ m+\deg P^*\}$ successive changes in sign on E. This completes the proof of sufficiency.

To prove the alternating sign condition is necessary, let $R^* = \frac{P^*}{Q^*} \in R_m^n[a,b]$ be such that the residual $r^* = f-R^*$ does not satisfy the condition. We shall show that this implies the existence of a better rational approximation in $R_m^n[a,b]$. In fact, such an improved approximation will be found in the family

$$R_\alpha = \frac{P^* + \alpha P}{Q^* - \alpha Q}, \text{ where } \alpha \text{ is a positive real parameter}$$

and $\deg P \le n$, $\deg Q \le m$. To determine the value of α, we compute the residual,

$$r_\alpha = f-R_\alpha = f-R^* +R^*-R_\alpha = r^*+(R^*-R_\alpha)$$

$$\text{Now } R^*-R_\alpha = \frac{-\alpha(P+R^*Q)}{Q^* - \alpha Q} = \frac{-\alpha g}{Q^* - \alpha Q} \qquad (1)$$

where the function $g = P+R^*Q$ is an arbitrary element of $V_{R^*} = V_P+R^*V_Q$. For any Q we can choose α sufficiently small so that $Q^*(x) - \alpha Q(x) > 0$ on $[a,b]$. For such α the

sign of $R^*(x)-R_\alpha(x)$ is opposite to the sign of $g(x)$. Therefore, if we could find P and Q such that g has the same sign as r^*, then $|r_\alpha| < |r^*|$. Actually something less suffices.

Since r^* does not satisfy the necessary condition, the maximum number, k, of points in E on which r^* assumes its extreme values with alternating sign must satisfy the inequality,

$$1 \leq k < 2 + \max\{n+\deg Q^*, m+\deg P^*\}$$

$$= 1 + d \qquad (2)$$

Hence, there exists $x_1 > a$ such that $r^*(x)$ has the same sign for all $x \in E \cap [a,x_1)$ (otherwise, $r^*(x)$ alternates infinitely many times). If $r^*(x)$ has the same sign for all $x \in E$, take $x_1 = b$. Otherwise, let

$$x^* = \text{Sup}\{x_1 : r^*(x) \text{ has constant sign in } E \cap [a,x_1)\}$$

Let $y_1 = \text{Sup}\{x < x^* : r^*(x) = 0\}$

Then $r^*(y_1) = 0$ and $r^*(x)$ has constant sign for all $x \in E \cap [a,y_1)$. Continuing in the same way with the interval $[y_1,b]$, we find y_2 such that $r^*(y_2) = 0$ and $r^*(x)$ has constant

sign in $[y_1, y_2] \cap E$, this being opposite to the sign in $[a, y_1)$. Since there are k points in E on which $r^*(x)$ alternates in sign, the above process yields a partition $a < y_1 < \ldots < y_{k-1} < b$ such that $r^*(y_i) = 0$ $(1 \leq i \leq k-1)$ and $r^*(x)$ changes sign only on those points of E in successive subintervals. We can now show that there exists $g \in V_{R^*}$ which agrees in sign with r^* on the set E. From (1), the dimension, d, of V_{R^*} is greater than k-1. Hence there is a basis $\{g_1, \ldots g_d\}$. Furthermore, we have seen that any function in V_{R^*} has atmost d-1 zeros. But $\{g_1, \ldots g_d\}$ satisfy the Haar condition, which implies any linear combination $g = \sum_{i=1}^{d} a_i g_i$ is determined by its values at d distinct points. Choose points $y_1, \ldots, y_{k-1}$ to be zeros of g. Choose any $y_k, \ldots, y_{d-1}$ to be zeros of g outside of $[a,b]$, but within an interval in which $Q^*(x) > 0$. Finally, choose $y_d \in E \cap [a,b]$ and set

$$g(y_d) = \operatorname{sgn} r^*(y_d).$$

The function g so defined has sign changes at and only at the points y_i; $1 \leq i \leq d-1$. Since it agrees in sign with r^* at y_d, it must agree on all of E. We take this g in (1). It remains to choose α. Let $\mu = \inf \{|g(x)| : x \in E\}$. Since $g(x) \neq 0$ for $x \in E$, it follows (E being closed and hence compact) that $\mu > 0$.

Now define the open set

$$F = \{x: (\operatorname{sgn} r^*(x))g(x) > \mu/2 \text{ and}$$

$$|r^*(x)| > \|r^*\|_\infty/2$$

Clearly, $E \subset F$. Let $G = [a,b] - F$. Since G is closed, $|r^*(x)| < \|r^*\|_\infty - \epsilon$ for some $\epsilon > 0$. Hence, for α sufficiently small, and all $x \in G$.

$$|r_\alpha(x)| \leq |r^*(x)| + |R^*(x) - R_\alpha(x)| \leq \|r^*\|_\infty$$

Finally, choose $\alpha > 0$ so small that $r_\alpha(x)$ and $r^*(x)$ have the same sign on F. For $x \in F$, this yields

$$|r_\alpha(x)| = (\operatorname{sgn} r^*(x)r_\alpha(x) = (\operatorname{sgn} r^*(x))r^*(x)$$

$$- (\operatorname{sgn} r^*(x)) \frac{\alpha g(x)}{Q^*(x)-\alpha Q(x)}$$

$$\leq \|r^*\|_\infty - \frac{\mu\alpha}{2|Q^*(x) - \alpha Q(x)|} < \|r^*\|_\infty$$

Hence R_α is a better approximation than R^*. This completes the proof.

A.3 PROOF OF LEMMA 3.3

Let $f \in C[a,b]$ and suppose R^* and $R = \frac{P}{Q}$ are best approximations in $R_m^n[a,b]$. The function $g = Q(R-R^*)$ $= P-R^*Q$ is in $V_P + R^*V_Q$. Furthermore, $g = Q(f-R^*)-(f-R))$ $= Q(r^*-r)$. On set E, r^* assumes its extreme values, we

have $g(x) = Q(x)(\operatorname{sgn} r^*(x) \| f-R\|_\infty - r(x))$. Since $Q(x) > 0$ and $|r(x)| \leq \| f-R\|_\infty$, it follows that $g(x)$ and $r^*(x)$ have the same sign on E whenever $g(x) \neq 0$. By lemma 3.2, $r^*(x)$ must alternate in sign on atleast $2+\max\{n+\deg Q^*, m+\deg P^*\} = d+1$ points of E. This implies that g has atleast d zeros, counting multiplicities. (If x_i, x_{i+1}, x_{i+2} are three successive points on which r^* alternates in sign say $r^*(x_i) > 0$, $r^*(x_{i+1}) < 0$, $r^*(x_{i+2}) > 0$, and $g(x_{i+1}) = 0$, then either $g(x)$ changes sign at x_{i+1}, in which case there are atleast two distinct zeros in x_i, x_{i+2} or x_{i+1} is a zero of multiplicity two). Since the degree of Q^*g is atmost d-1, this implies that g is identically equal to zero. Therefore, $R = R^*$, establishing uniqueness.

A.4 RANDOM NUMBERS USED FOR SIMULATION*

4.69	4.42	1.24	3.86	5.65
4.25	7.19	9.69	1.46	7.41
3.29	9.65	1.04	5.32	0.62
0.50	7.78	3.58	6.76	2.33
3.16	5.37	4.53	4.07	0.10
1.73	6.71	0.97	9.51	3.51
7.70	0.29	3.36	2.25	1.62
9.81	7.97	0.50	4.36	2.08
5.24	8.77	3.93	9.34	0.50
4.91	3.48	5.58	5.37	8.63
7.98	1.10	8.63	2.78	1.94
8.31	5.22	3.32	1.38	3.02
0.74	0.76	0.79	7.69	7.48
0.05	0.40	3.72	4.57	1.33
4.23	0.19	2.43	5.91	7.59
1.38	9.20	7.93	9.00	7.93
9.71	6.17	4.74	4.37	2.46
1.66	7.33	0.50	8.45	3.27
4.55	4.21	2.07	5.66	2.82
2.01	2.67	7.46	6.82	5.78

* Random numbers are selected from Fisher and Yates [13], pp.134-137.

A.5 WHITE NOISE FOR SIMULATION

The random numbers given in A.4 are transformed to N(0,1) deviates, using the transformation $y_t = \frac{x_t - \mu}{\sigma_x}$, which are used as white noises in simulating MA(2), AR(1) and ARMA(1,1) series considered in examples 4.10, 4.11 and 4.12 respectively.

A.6 SERIES C - CHEMICAL PROCESS:TEMPERATURE READINGS EVERY MINUTE*

26.6	22.4	19.7	26.0	24.0	21.3	21.7	24.5	23.4	18.8
27.0	22.2	19.9	25.8	23.9	21.2	21.8	24.5	23.3	
27.1	22.0	20.0	25.6	23.7	21.2	21.9	24.5	23.3	
27.1	21.8	20.1	25.4	23.6	21.1	22.2	24.5	23.3	
27.1	21.4	20.2	25.2	23.5	21.0	22.5	24.4	23.4	
27.1	20.9	20.3	24.9	23.5	20.9	22.8	24.4	23.4	
26.9	20.3	20.6	24.7	23.5	21.0	23.1	24.2	23.3	
26.8	19.7	21.6	24.5	23.5	21.0	23.4	24.2	23.2	
26.7	19.4	21.9	24.4	23.5	21.1	23.8	24.1	23.3	
26.4	19.3	21.7	24.4	23.7	21.2	24.1	24.1	23.3	
26.0	19.2	21.3	24.4	23.8	21.1	24.6	24.0	23.2	
25.8	19.1	21.2	24.4	23.8	20.9	24.9	24.0	23.1	
25.6	19.0	21.4	24.4	23.9	20.8	24.9	24.0	22.9	
25.2	18.9	21.7	24.3	23.9	20.8	25.1	23.9	22.8	
25.0	18.9	22.2	24.4	23.8	20.8	25.0	23.8	22.6	
24.6	19.2	23.0	24.4	23.7	20.8	25.0	23.8	22.4	
24.2	19.3	23.8	24.4	23.6	20.9	25.0	23.7	22.2	
24.0	19.3	24.6	24.4	23.4	20.8	25.0	23.7	21.8	
23.7	19.4	25.1	24.4	23.2	20.8	24.9	23.6	21.3	
23.4	19.5	25.6	24.5	23.0	20.7	24.8	23.7	20.8	
23.1	19.6	25.8	24.5	22.8	20.7	24.7	23.6	20.2	
22.9	19.6	26.1	24.4	22.6	20.8	24.6	23.6	19.7	
22.8	19.6	26.3	24.3	22.4	20.9	24.5	23.6	19.3	
22.7	19.6	26.3	24.2	22.0	21.2	24.5	23.5	19.1	
22.6	19.6	26.2	24.2	21.6	21.4	24.5	23.5	19.0	

* 226 observations (Read downwards).

A.7 SERIES D - CHEMICAL PROCESS VISCOSITY READINGS EVERY HOUR*

8.0	8.9	8.4	9.5	9.0	9.5	9.4	9.6	9.6	9.8	9.6	8.4	8.6
8.0	9.1	8.3	9.5	8.8	9.3	9.4	9.6	9.8	9.8	9.2	8.8	8.3
7.4	9.5	8.3	9.9	8.6	9.5	9.0	9.2	10.2	9.7	9.6	8.4	7.9
8.0	8.5	8.1	9.5	8.6	9.5	9.4	9.2	10.0	9.6	9.2	8.4	8.5
8.0	8.4	8.2	9.7	8.0	9.1	9.4	9.0	10.0	9.4	9.2	9.0	8.7
8.0	8.3	8.3	9.1	8.0	9.3	9.6	9.0	10.0	9.2	9.6	9.0	8.9
8.0	8.2	8.5	9.1	8.0	9.5	9.4	9.0	9.4	9.0	9.6	9.4	9.1
8.8	8.1	8.1	8.9	8.0	9.3	9.2	9.4	9.2	9.4	9.6	10.0	9.1
8.4	8.3	8.1	9.3	8.6	9.1	8.8	9.0	9.6	9.6	9.6	10.0	9.1
8.4	8.4	7.9	9.1	8.0	9.3	8.8	9.0	9.7	9.6	9.6	10.0	9.0
8.0	8.7	8.3	9.1	8.0	9.1	9.2	9.4	9.7	9.6	9.6	10.2	
8.2	8.8	8.1	9.3	8.0	9.5	9.2	9.4	9.8	9.6	10.0	10.0	
8.2	8.8	8.1	9.5	7.6	9.4	9.6	9.6	9.8	9.6	10.0	10.0	
8.2	9.2	8.1	9.3	8.6	9.5	9.6	9.4	9.8	9.6	10.4	9.6	
8.4	9.6	8.4	9.3	9.6	9.6	9.8	9.6	10.0	9.0	10.4	9.6	
8.4	9.0	8.7	9.3	9.6	10.2	9.8	9.6	10.0	9.4	9.8	9.6	
8.4	8.8	9.0	9.9	10.0	9.8	10.0	9.6	8.6	9.4	9.0	8.6	
8.6	8.6	9.3	9.7	9.4	9.6	10.0	10.0	9.0	9.4	9.6	9.0	
8.8	8.6	9.3	9.1	9.3	9.6	9.4	10.0	9.4	9.6	9.8	9.6	
8.6	8.8	9.5	9.3	9.2	9.4	9.8	9.6	9.4	9.4	9.6	9.6	
8.6	8.8	9.3	9.5	9.5	9.4	9.8	9.2	9.4	9.6	8.6	9.0	
8.6	8.6	9.5	9.4	9.5	9.4	8.8	9.2	9.4	9.6	8.0	9.0	
8.6	8.6	9.5	9.0	9.5	9.4	8.8	9.2	9.0	9.8	8.0	8.9	
8.6	8.4	9.5	9.0	9.9	9.6	8.8	9.0	10.0	9.8	8.0	8.8	
8.8	8.3	9.5	8.8	9.9	9.6	9.6	9.0	10.0	9.8	8.0	8.7	

* 310 Readings (Read downwards).

A.8 POPULATION DATA* (INDIA)

Year	Data in millions	Year	Data in millions
1947	33.9	1964	47.2
1948	34.2	1965	48.3
1949	35.4	1966	49.3
1950	35.8	1967	50.4
1951	36.3	1968	51.5
1952	36.8	1969	52.7
1953	37.2	1970	53.9
1954	37.7	1971	55.1
1955	38.2	1972	56.2
1956	38.7	1973	57.4
1957	39.2	1974	58.6
1958	39.8	1975	60.1
1959	42.4	1976	61.3
1960	43.2	1977	62.6
1961	44.2	1978	63.8
1962	45.2	1979	65.1
1963	46.2	1980	66.4

* The data is taken from census figures published by the Central Statistical Organisation.

A.9 SERIES A - CHEMICAL PROCESS CONCENTRATION READINGS EVERY TWO HOURS*

17.0	17.6	17.6	17.0	16.5	17.3	17.2	17.0
16.6	17.4	16.9	16.9	17.2	17.2	17.2	16.9
16.3	17.3	16.7	17.0	16.4	17.3	17.4	17.1
16.1	17.0	16.8	16.6	17.0	17.2	17.2	17.2
17.1	17.8	16.8	16.7	17.0	17.2	16.9	17.4
16.9	17.5	17.2	16.8	16.7	17.5	16.8	17.5
16.8	18.1	16.8	16.7	16.2	16.9	17.0	17.9
17.4	17.5	17.6	16.4	16.6	16.9	17.4	17.0
17.1	17.4	17.2	16.5	16.9	16.9	17.2	17.0
17.0	17.4	16.6	16.4	16.5	17.0	17.2	17.0
16.7	17.1	17.1	16.6	16.6	16.5	17.1	17.2
17.4	17.6	16.9	16.5	16.6	16.7	17.1	17.3
17.2	17.7	16.6	16.7	17.0	16.8	17.1	17.4
17.4	17.4	18.0	16.4	17.1	16.7	17.4	17.4
17.4	17.8	17.2	16.4	17.1	16.7	17.2	17.0
17.0	17.6	17.3	16.2	16.7	16.6	16.9	18.0
17.3	17.5	17.0	16.4	16.8	16.5	16.9	18.2
17.2	16.5	16.9	16.3	16.3	17.0	17.0	17.6
17.4	17.8	17.3	16.4	16.6	16.7	16.7	17.8
16.8	17.3	16.8	17.0	16.8	16.7	16.9	17.7
17.1	17.3	17.3	16.9	16.9	16.9	17.3	17.2
17.4	17.1	17.4	17.1	17.1	17.4	17.8	17.4
17.4	17.4	17.7	17.1	16.8	17.1	17.8	
17.5	16.9	16.8	16.7	17.0	17.0	17.6	
17.4	17.3	16.9	16.9	17.2	16.8	17.5	

* 197 Observations (Read downwards).

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