SOME NON-LINEAR PROBLEMS IN THEORETICAL PHYSICS

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bona fide work done by Mr.B.V.Baby, under my guidance in the Department of Physics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

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DECLARATION

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April 3, 1985.

B.V.Baby.

" ओ वाङ् मे मनसि प्रतिष्ठिता ; मनो मे वाचि
प्रतिष्ठितमाविरावीर्म एधि । वेदस्य म आणीस्थ:
श्रुतं मे मा प्रहासी: । अनेनाधीतेनाहो रात्रान्सन्दधामि
ऋतं वदिष्यामि । सत्यं वदिष्यामि । तन्मामवतु ।
तद्वक्तारमवतु । अवतु मामवतु वक्तारमवतु वक्तारम् ।।
ओं शान्ति: ! शान्ति: !! शान्ति: !!! "

Om! May my speech be rooted in my mind, may my mind be rooted in my speech. Brahman, reveal Thyself to me. Oh! Mind and speech, enable me to grasp the Truth that the Vedas teach. Let not what I have heard forsake me. Let me continuously live my days and nights in my studies. I think; I speak truth. May that (truth) protect me. May that protect the guru; protect me, protect the guru; protect the guru!!

Shanti! Shanti! Shanti!

DEDICATED TO MY LOVING GRADUATE STUDENTS

PREFACE

The work presented in this thesis has been carried out by the author as a part-time research scholar in the Department of Physics, University of Cochin during 1980-1985.

The thesis deals with certain methods of finding exact solutions of a number of non-linear partial differential equations of importance to theoretical physics. Some of these new solutions are of relevance from the applications point of view in diverse branches such as elementary particle physics, field theory, solid state physics and non-linear optics and give some insight into the stable or unstable behaviour of dynamical systems.

The thesis consists of six chapters. The first chapter is introductory and gives a brief survey of non-linear partial differential equations in the context of theoretical physics with emphasis on a new stable particle solution called soliton and some recently developed mathematical tools for finding exact solutions.

In Chapter 2, the solitary wave solutions of double sinh-Gordon equation are studied by using two systematic methods, Hirota's bilinear operator method and the base equation technique.

Chapter 3 contains a new solution of SU(2) Yang-Mills theory which is developed by solving scalar ϕ^4 field by the bilinear operator method.

In Chapter 4 several members of the Klein-Gordon family of non-linear equations are connected by mappings and this procedure yields some new solutions.

Chapters 5 and 6 deal with the similarity method of solving non-linear partial differential equations. In the fifth chapter this method is applied to Klein-Gordon family of equations so as to produce some new type of rotational invariant solutions and explore their common characteristics from the point of view of the similarity group. In the sixth chapter the similarity group method of analysis is applied to coupled equations of SU(2) Yang-Mills theory so as to derive in a uniform manner some known as well as new time-dependent solutions of the Prasad-Sommerfield limit.

A part of these investigations has appeared in the form of the following publications:

1. Solitary wave solutions in double sinh-Gordon system, J.Phys.A16, 2685 (1983).

2. Fluctuations in SU(2) Yang-Mills theory, Pramana, 22, 111 (1984).

3. Composite mapping method for generations of kinks and solitons in the Klein-Gordon family, Phys.Rev.A29, 2899 (1984).

4. Symmetry classification of solutions of two dimensional non-linear Klein-Gordon equations, J.Math.Phys. (submitted).

5. Classical SU(2) Yang-Mills-Higgs system:Time dependent solutions by similarity method, J.Math.Phys (In press).

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DICTIONARY OF ABBREVIATIONS

ABT - Auto Bäcklund transformation
BT - Bäcklund transformation
DsG - Double sine-Gordon
DshG - Double sinh-Gordon
HRI - Hyperbolic rotation invariant
IST - Inverse scattering transform
KdV - Korteweg-de Vries
KG - Klein-Gordon
MKdV - Modified Korteweg-de Vries
NDE - Non-linear differential equation
NLS - Non-linear Schrödinger
NPDE - Non-linear partial differential equation
ODE - Ordinary differential equation
PDE - Partial differential equation
PP - Painlevé property
P-type - Painlevé type
PS - Prasad-Sommerfield
QCD - Quantum chromodynamics
QED - Quantum electrodynamics
sG - sine-Gordon
SI - Similarity invariant
shG - sinh-Gordon
SSB - Spontaneous symmetry breaking
TI - Translation invariant
vev - Vacuum expectation value
YM - Yang-Mills

SYNOPSIS

An immense variety of problems in theoretical physics are of the non-linear type. Non-linear partial differential equations (NPDE) have almost become the rule rather than an exception in diverse branches of physics such as fluid mechanics, field theory, particle physics, statistical physics and optics, and the construction of exact solutions of these equations constitutes one of the most vigorous activities in theoretical physics today. The thesis entitled 'Some Non-linear Problems in Theoretical Physics' addresses various aspects of this problem at the classical level. For obtaining exact solutions we have used mathematical tools like the bilinear operator method, base equation technique and similarity method with emphasis on its group theoretical aspects.

A new era in theoretical physics was ushered in by the discovery of a non-linear transformation called inverse scattering transform (IST) and the collisional stability of a particular solitary wave solution, called soliton, of a class of NPDEs by Gardner, Greene, Kruskal and Miura [99] and Zabusky and Kruskal [29]. Further rapid development added a number of non-linear field theoretical models such as Korteweg-de Vries (KdV), two-dimensional sine-Gordon (sG), non-linear Schrödinger, Thirring model etc. The solitary wave solutions of an integrable system are often called solitons [193] which are either topological or nontopological, depending on the nonvanishing or vanishing of the topological charge. Equations such as KdV, sG etc. belong to this class that is characterised by the existence of an infinite number of conserved quantities. Bäcklund transformation (BT) constitutes one of the oldest approaches to the solution of NPDEs. The method of prolongation structures has been introduced recently to support the studies using IST and BT.

The bilinear operator method pioneered by Hirota [90-98] is closely associated to the numerical method of Padé approximants. We have applied this method to develop single solitary wave solutions of the Double sinh-Gordon (DshG) equation in (1+1) dimensions. The DshG system is a newly introduced system and bears close resemblance to the Liouville and the Toda models. For massive and massless ϕ^4 equations this method yields some previously known solutions.

Non-abelian gauge theories of the Yang-Mills (YM) type are of great interest in contemporary field theory and particle physics, especially in the context of unified models of fundamental interactions. By using some suitable Ansatz one can reduce the SU(2) YM or YM-Higgs theory to non-linear differential equations (NDE) (the massless ϕ^4 equation, or one-dimensional Liouville equation) or to a set of coupled NPDEs. Euclidean space solutions of the massless ϕ^4 equations lead to the celebrated instanton and merons of SU(2) pure YM theory. Monopole solutions of YM-Higgs system can be obtained from the one dimensional Liouville equation or a pair of coupled NPDEs.

We have used the singular solutions of the massless ϕ^4 model to generate a new family of solutions of SU(2) YM theory. These solutions are interpreted as localized YM fields involving no flux transport. It is conjectured that these objects having infinite action and infinite topological charge, closely resemble the merons and may play a tunnelling role. The idea of employing the solutions of a known differential equation to construct a solution of a given NPDE was developed by Pinney, Reid and Burt [80-89] who called it the base equation technique. Using this approach we developed multisolitary wave solutions of the DshG system in arbitrary dimensions, which collapse to a single solitary wave in (1+1) dimensions.

We have developed a generalisation of the base equation method and called it the composite mapping method, wherein a sequence of maps is applied to several members of the non-linear Klein-Gordon (NKG) family to produce new solutions. In a broad scenario like this where one deals with a whole class of NDEs rather than a specific one, besides yielding new solutions, this procedure can expose certain 'family relationships' between different equations which we later confirmed through the similarity group method. Starting from the classical ϕ^4 equation we have generated, through sequential maps, arbitrary dimensional solutions of Liouville, double sine-Gordon (DsG), DshG, massive and massless ϕ^6 equations of the NKG family by imposing simple constraints at each stage. While all other known solutions of the DsG collapse to a single wave in (1+1) dimensions, our solutions behave differently. Since all the four distinct solutions obtained by us can be simultaneously constructed for given values of a parameter, it will be possible to study their interactions. In this context we have also highlighted the appearance of one set of solutions and the disappearance of another set at a critical point.

The Lie point transformation theory has emerged as a most outstanding attempt to study continuous symmetry, particular solutions and dimensional reduction of NPDEs [116-121]. When a second order NPDE is invariant under these transformations, known as similarity transformations, it is possible to reduce the number of independent variables by one, and find similarity solutions of the equation [187]. In general the similarity transformations form an extended group, the similarity group, which upon a suitable redefinition of the generators, leads to the Poincaré group in the case of Klein-Gordon (KG) equations. This suggests a three-fold classification of solutions of two dimensional KG equations into translation invariant, hyperbolic rotation (boost) invariant and similarity invariant types. Similarity invariance denotes invariance under the full similarity group. Such a description emphasises the behaviour of the solutions rather than that of the equations. Most of the known solitary wave solutions are of the translation invariant type. We have produced rotationally invariant solutions of several NKG equations. The group-theoretical meaning of the base equation technique has also been examined. We have found that the similarity groups of the original equation and the constraint equation are identical in all the cases studied in the literature.

It has been conjectured that the existence of the Painlevé property (PP) (i.e., the absence of movable critical points) is a signal to the original equation's integrability. We have shown that the translation-invariant sector of the sG equation does not possess PP whereas the rotation invariant sector does possess PP. This may restrict the Painlevé property of the sG system in some sense.

The SU(2) Yang-Mills field interacting with a Higgs scalar triplet is known to give monopole solutions through the 't Hooft Ansatz. Prasad and Sommerfield developed spherically symmetric solutions to this system in a special limit in the static case. Afterwards several attempts based on guesswork were made to obtain time-dependent exact solutions. We have carried out a similarity group analysis of the coupled system of equations representing the SU(2) YM-Higgs model equations and shown that the equations reduce to the one-dimensional form under the full similarity group or under one of its subgroups. This approach gives two new exact time-dependent solutions along with some previously known solutions.

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1

INTRODUCTION

1.I. Non-linear phenomena in theoretical physics

It seems nature very often delights in signaturing her mysteries in terms of non-linear systems of equations. The non-linear field equations which form the basis of quantum field theory have long been known to possess a rich array of solutions at the classical level. A considerable number of physical applications of non-linear partial differential equations (NPDEs) have been made since the last century, especially in differential geometry and fluid mechanics.

In 1895 Korteweg and de Vries [1] showed that long waves in water of relatively shallow depth, could be modelled approximately by a non-linear equation, which was later named the Korteweg-de Vries (KdV) equation. Solitary wave solutions are some special solutions of the KdV equation, which were historically first observed and recorded by Scott-Russell [2] in 1844. Stokes and Riemann [3,4] had studied some approximate non-linear waves even before Korteweg and de Vries. In 1876, Bäcklund proposed the sine-Gordon (sG) equation [5] to model a non-linear pseudospherical surface in differential geometry. He developed a method of finding arbitrary number of solutions of two dimensional sG equation which was later found to be highly useful in theoretical physics.

Recently, say from 1960 or so, however, applications of non-linear models have blossomed in various disciplines. A vivid example of non-linear phenomena in optics is the observed transformation of a laser beam in a dispersive medium into a sequence of wave lumps [6,7]. The non-linear Schrödinger (NLS) equation has been used for the modelling of stationary two-dimensional self-focusing of plane waves [8,9,10], one-dimensional self-modulation of monochromatic waves [11-14], self-trapping phenomena of non-linear optics, etc.

Non-linear studies were first introduced into plasma physics in 1961 [15]. Today we find a number of areas in plasma physics such as plasma turbulence, relaxation of beams of energy distribution of particles which are optimal for plasma chemistry, anomalous plasma resistance, self-compression of high-intensity waves, namely, the Langmuir, lower hybrid, drift and other waves, space plasma and so on, wherein the non-linearity scenario projects itself [16-19].

Another physical evidence for the existence of non-linear phenomena has been obtained from studies of magnetic flux propagation. The magnetic flux penetrates a very thin ($\sim 25^{o}$A) barrier layer of niobium oxide which separates two superconducting metals (niobium and lead). The simplest non-linear model equation for this magnetic flux dynamics is the one-dimensional sG equation, which has also been used to explain the self-induced transparency phenomenon of non-linear optics.

The sG equation has been proposed as a simple model for elementary particles [27]. The modern theory of elementary particles is a rich repertory of non-linear equations possessing particle-like solutions. The Toda lattice equation [20,21] is a well known non-linear physical model, very extensively studied recently by several authors. Moreover, physicists are increasingly coming to the belief that non-linear phenomena may play an essential part in such different fields as elementary particle physics, solid state physics, hydrodynamics, astrophysics, non-linear optics and even in biology [22,23,24]. Some special non-linear phenomena are of fundamental importance in the system of concepts of a new science called 'synergetics' [25].

Sooner or later, one has to replace the linear oscillator paradigm of classical physics by some non-linear wave models.

1.II. Non-linear differential equations

By a linear operator L we shall mean one having the properties:

$$L(u + v) = L(u) + L(v) \quad (1.1)$$

$$L(ku) = k\,L(u) \quad (1.2)$$

where u and v are arbitrary functions and k is a scalar. By a non-linear operator we shall mean one that is not linear. When a non-linear operator is equated to zero or to a given function, we have a non-linear equation. The principal objective in the study of a non-linear equation is to confirm whether or not a solution can be obtained, either explicitly or implicitly, in terms of classical functions. Studies of classical NPDEs have served as a source of emerging information in theoretical physics.

NPDEs exhibiting wave phenomena can essentially be classified into hyperbolic and dispersive types [26]. The theory of hyperbolic partial differential equations (PDEs) is fairly well studied, whereas that of non-linear dispersive wave equations had received only scant attention till about two decades ago. Equations such as KdV, modified KdV (MKdV), NLS, sG and Boussinesq, belong to the dispersive type.

Since the last century, NPDEs have arisen in large numbers in the study of fluid dynamics. The progressive wave solutions play an important role in the general solution of initial value problems. The prototype for hyperbolic waves is frequently taken to be the equation

$$u_{tt} - c^2 \nabla^2 u = 0 \qquad (1.3)$$

The definition for hyperbolicity depends only on the structure of the equation, but is independent of the properties of the solutions, whereas the prototype for dispersive waves is based on a specific property of solutions rather than the type of equation. A linear dispersive system is characterized by the existence of a solution of the form,

$$u = c \cos(kx - \omega t) \qquad (1.4)$$

where ω, the frequency, is a definite real function of the wave number k and the function $\omega(k)$ is associated to the specific system. The system is said to be dispersive since, the phase velocity $v_p = \omega/k$, depends on k; so the modes with different k will propagate at various speeds, leading to dispersion. In general, for a dispersive wave:

$$\frac{d^2\omega}{dk^2} \neq 0 \qquad (1.5)$$

The general form of a dispersive solution of a linear system can be written as a Fourier integral

$$u(x,t) = \int_0^{\infty} F(k).\cos(kx - \omega t)dk \tag{1.6}$$

F(k) is obtained from the initial and boundary conditions associated to the problem. The function u(x,t) represents a linear superposition of waves of different wave numbers, each travelling with its own phase velocity ω/k. As time evolves a single wave disperses into a whole oscillatory train with different wavelengths. The group velocity is defined by

$$v_g = \frac{d\omega}{dk} \cdot \tag{1.7}$$

For dispersive waves, v_g and v_p are different and it is v_g which plays the dominant role in the propagation. The energy associated to the wave is also propagated with the group velocity v_g.

A simple non-linear wave equation is

$$u_t + 6u\, u_x = -6t\, u_t u_x , \tag{1.8}$$

which has an implicit solution

$$u = u(x - 6ut). \tag{1.9}$$

When u is large and positive, there will be a large effective speed u along x and the wave develops a rising front leading to a 'shock wave'.

Given an underlying wave equation, a travelling wave $u(\eta)$ is a solution which depends upon x and t only through

$$\eta = x - vt \qquad (1.10)$$

where v is a fixed constant [195].

A spatially localized solution of a travelling wave is called a solitary wave, whose transition from one constant asymptotic state as $x \longrightarrow -\infty$ to another as $x \longrightarrow +\infty$, is asymptotically localized in space. In general there are two types of travelling waves, as shown in figs.(1.1) and (1.2).

It is possible to balance the non-linearity against dispersion to get a solitary wave $u(kx - \omega t)$. A typical model for solitary waves in deep and shallow water is provided by the KdV equation [1],

$$u_t + 6u\, u_x - k\, u_{xxx} = 0. \qquad (1.11)$$

A single solitary wave solution of equation (1.11) is:

$$u(x,t) = 2c^2 \operatorname{sech}^2 (cx - 4c^3 t) \cdot \qquad (1.12)$$

The profile corresponding to (1.12) is as shown in fig.(1.1), bell-shaped, asymptotically vanishing and propagates along the x-axis without any change in shape. About 140 years back, Scott-Russell had observed this type of solitary wave in the English Channel [2].

In 1876 Bäcklund developed a transformation theory and demonstrated the possibility of developing arbitrary number of particular solutions of NPDEs of differential geometry [5]. By this procedure he was able to obtain a particular solution of the sG equation

$$u_{xx} - u_{tt} = \sin u \tag{1.13}$$

in the form

$$u = 4 \arctan[\exp(kx - \omega t)] \tag{1.14}$$

The profile of this solution is indicated in fig.(1.2). Nearly 100 years treked and the sG equation (1.13) has become a standard field theoretic model.

In 1962, Perring and Skyrme [27] studied the collision process of solitary wave solutions of the sG equation through computer analysis. They observed that the solitary waves of sG equation emerged from a collision with their shapes

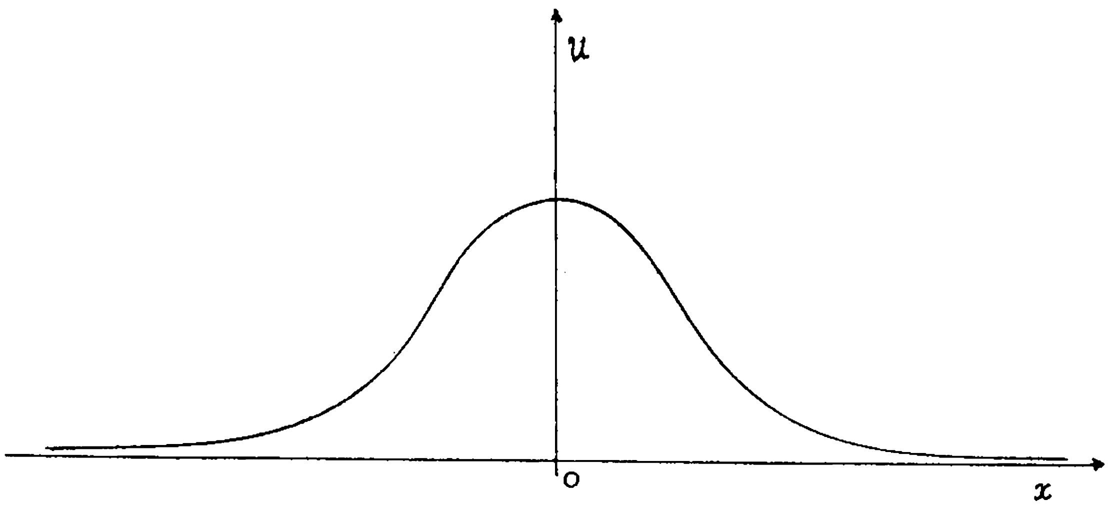

Fig.1.1 An asymptotically vanishing soliton.

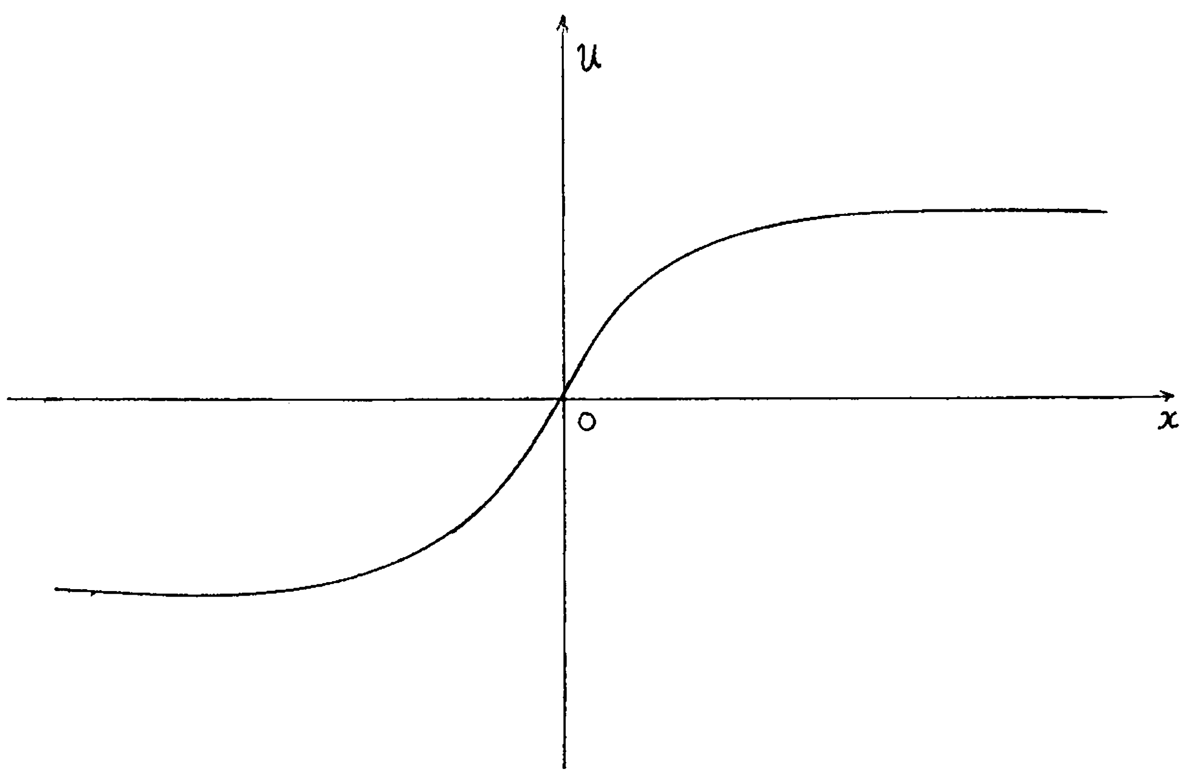

Fig.1.2 A topological soliton or kink.

and velocities unaffected. The wave-wave interactions in the sG system were studied independently by Seeger, Donth and Kochendörfer [28].

In 1965, Zabusky and Kruskal studied the collision process of solitary waves of KdV equation in plasma [29]. The result was the same: the solitary waves emerged from the collision without any change of shapes and velocities. They called such collisionally stable solitary waves 'solitons' and thus inaugurated a new branch of physics, the non-linear dynamics.

A soliton is essentially a non-linear solitary wave where the dispersion of the group velocity is exactly compensated for by the non-linear self-compression of the wave packet, and as a result, the soliton propagates without spreading and conserves its shape and velocity asymptotically, upon collision with other solitons. Hence for a given solitary wave solution, u(x,t) composed only of solitary waves for large negative time,

$$u(x,t) \sim \sum_{j=1}^{N} u(\eta_j) \text{ as } t \longrightarrow -\infty \qquad (1.15)$$

where $\eta_j = k_j x - \omega_j t$, the name soliton applies if the solitary waves indexed by j emerge from collision with similar waves, with no more than a phase shift, that is,

$$u(x,t) \sim \sum_{j=1}^{N} u(\bar{\eta}_j) \quad \text{as } t \longrightarrow +\infty \tag{1.16}$$

where $\bar{\eta}_j = k_j x - \omega_j t + \delta_j$.

The stability of the soliton, in spite of its large mass, can also be understood as a consequence of a topological charge conservation law. The charge is associated with the trivially conserved current,

$$j^{\mu} = \epsilon^{\mu\nu} \partial_{\nu} u(x,t) \tag{1.17}$$

where

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$$

$$\epsilon^{01} = 1.$$

The topological charge Q for a solution u(x,t):

$$Q = \int j^{0} \, dx$$

$$= [u(+\infty) - u(-\infty)]. \tag{1.18}$$

The conservation of the non-zero charge Q takes care of the stability of the soliton[30].

On the basis of the topological charge Q, solitons are usually classified into two types [30]: topological solitons or kinks ($Q \neq 0$), and non-topological solitons ($Q = 0$). In the literature the term 'soliton' is usually reserved for non-topological solitons of integrable systems.

Some typical examples of kink solutions are those of the sG equation in 1+1 dimensions and the magnetic monopole solution of 't Hooft and Polyakov [31,32] in 3+1 dimensions. The bell-shaped solution (1.12) of KdV equation is a non-topological soliton. The topological charge is the result from non-trivial mapping between the internal field space and the manifold of real space (x,y,z), hence the field configurations of these solitons belong to non-trivial homotopy classes.

A solitary wave solution of any importance must be stable against small linear perturbation, and this property can be checked by a classical linear stability analysis. We shall illustrate this procedure for a Klein-Gordon (KG) type non-linear field equation in 1+1 dimensions:

$$u_{xx} - u_{tt} = V'(u), \tag{1.19}$$

where the field potential V(u) is non-negative to ensure positive definiteness of energy. The prime denotes differentiation with respect to the field u(x,t). For a static configuration u(x,0):

$$u_{xx}(x,0) = V'(u(x,0)) \cdot \qquad (1.20)$$

A time-dependent solution of (1.19) can be written as the sum of a static part and a time-dependent part labelled by a parameter λ_n,

$$u(x,t) = u(x) + \psi_n(x) \exp(i\lambda_n t) \cdot \qquad (1.21)$$

Substitution of this in (1.19) and linearisation around the small perturbation $\psi_n(x)$ gives a Schrödinger equation for ψ_n:

$$-\frac{d^2\psi_n(x)}{dx^2} + V''(u(x,0))\,\psi_n(x) = \lambda_n^2\,\psi_n(x) \qquad (1.22)$$

For classical linear stability, with resonable boundary conditions on $\psi_n(x)$ as $x \to \pm\infty$ the eigenvalues λ_n^2 must be non-negative, so that small perturbation about the static solution u(x,0) do not grow exponentially in time [195].

The Schrödinger equation (1.22) always has a zero frequency solution, called the translation mode, irrespective of the potential [33]. So for linear stability, it is sufficient to demonstrate the existence of a translation mode as the lowest eigenvalue.

The classical stability is believed to ensure the stability of the corresponding quantum state. This procedure of linear stability analysis cannot, however, be extended to higher dimensions [34]. As the dimensions increase, the translation mode eigenvalue becomes degenerate.

A mathematical function of an arbitrary number of soliton solutions or an N-soliton solution was first suggested by Hirota [35]. One particular interesting case occurs when two solitons have the same envelope velocity [36]. This pulse has zero area and is called by different names in the literature: the '0-π' pulse or the 'doublet' or the 'breather'. It can be viewed as a two-soliton bound state or as a localized wave-form with an internal degree of freedom. These breathers translate at a constant velocity without decay and emerge from collision, with atmost a phase shift, as solitons [29]. The energy of the breather is slightly less than that of a two-soliton state; in addition, it pulsates due to the internal degree of freedom [37]. The breather solution of the sG equation (1.13)[37] is expressed in derivative form by

$$\frac{\partial u}{\partial \tau} = E(x,t) = 8\eta \operatorname{sech}\theta \left[\frac{\sin\phi + (\eta/\zeta)\tanh\theta \cos\phi}{1+(\eta/\zeta)^2 \operatorname{sech}^2\theta \cos^2\phi}\right] \tag{1.23}$$

where $\tau = (t - x/c)$, $\phi = \phi_0 + \omega_1 x - 2\zeta t$ and $\theta = \theta_0 + \omega_2 x - 2\eta t$ and c, η, ζ, ω_1, ω_2 are constants. The wave profile of this solution is depicted in fig.(1.3). The breather solution is called a '0-π' pulse in non-linear optics since it takes the field u from zero to zero as x goes from $-\infty$ to $+\infty$ and it has zero area or '0π' area. On the other hand the kink is a '2-π' pulse, it has 'pulse area' 2π when it is associated with the 2π-kink (for example, the sG solution (1.14)).

In more than 1+1 dimensions, one must necessarily deal with models involving spin which lead to complicated systems. Then the translation mode in n-dimensions is n-fold degenerate, whereas for the ordinary Schrödinger equation, the lowest eigenstate is non-degenerate. Hence for space dimensions $n > 1$, the classical scalar system is generally not stable [38,39,40].

In dimensions more than 1+1, fields with spin degrees of freedom have to be imposed to support stable

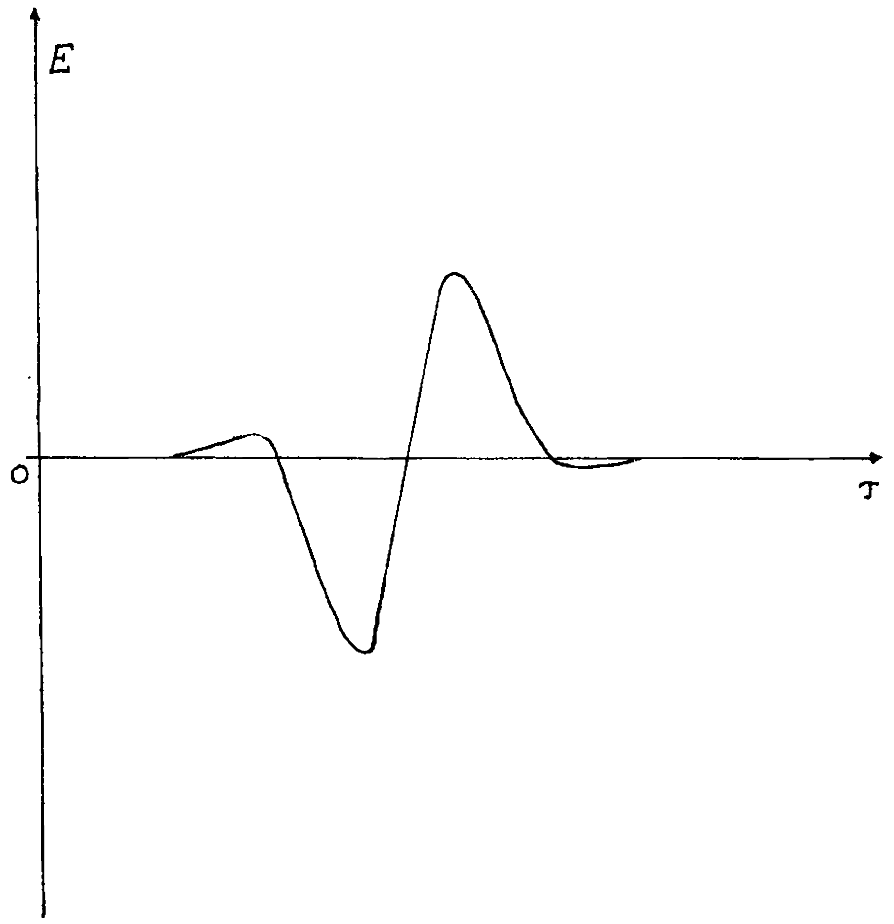

Fig.1.3 Breather solution of sG equation (derivative profile).

static classical configurations. It leads to gauge models that are known to possess soliton solutions in three dimensions [41].

1.III. Four dimensional non-linear theories

As mentioned in 1.I, non-linear field systems are of special interest in the study of elementary particles and quantum field theory as they possess a particle-spectrum which survives quantization. Most interesting of these objects, which are reminiscent of hadrons, arise in theories with spontaneous symmetry breaking (SSB).

The basis of a symmetry principle in physics is that some properties remain invariant under certain transformations. ~~like translation, rotation etc.~~ Gauge theories [42-46] are characterised by their invariance under a group (the gauge group) of symmetry transformations. Based on the type of gauge group that defines the symmetry transformation, gauge theories are classified into abelian and non-abelian types. The simplest gauge group is U(1) and the corresponding gauge theories are called abelian gauge theories. If higher symmetry groups such as SU(2), SU(3) etc. are involved then the theory becomes non-abelian.

Consider the Lagrangian of a free fermion field $\psi(x)$:

$$L_o = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \tag{1.24}$$

which is invariant under the phase transformation,

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x), \tag{1.25}$$

where α is x-independent. This implies that the derivative of the field transforms like the field itself:

$$\partial_\mu \psi(x) \longrightarrow e^{i\alpha}\, \partial_\mu \psi(x) \;. \tag{1.26}$$

The group of transformations (1.25) and (1.26) is the abelian group U(1).

Yang and Mills [47] generalized the principle of gauge invariance to the case where the invariance is associated with a non-abelian internal symmetry group SU(2). Let $\phi^i(x)$, i=1,2,...n be a set of fields. The Lagrangian describing the dynamics of the system will be invariant under a compact Lie group G of transformations of the fields $\phi^i(x)$, given by

$$\phi^i(x) \longrightarrow \phi^i(x) - i\alpha^a T^a_{ij}\phi^j(x) \tag{1.27}$$

where the T^a are the generators of G. The type of gauge invariance discussed above, is known as global or rigid gauge invariance.

When α is made x-dependent,

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x) \ , \tag{1.28}$$

the derivative of the field no longer transforms like the field. This is true for abelian as well as non-abelian groups. But the local gauge invariance can be maintained by defining a covariant derivative D_μ:

$$D_\mu \psi(x) \longrightarrow e^{i\alpha(x)} D_\mu \psi(x) \ . \tag{1.29}$$

To construct the operator D_μ, we define a new vector field $A_\mu(x)$ called a gauge field, which transforms as:

$$A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\,\partial\mu\,\alpha(x) \tag{1.30}$$

where e is the coupling constant. Defining D_μ as

$$D_\mu = \partial_\mu - ieA_\mu \ , \tag{1.31}$$

we can write the locally gauge invariant Lagrangian as

$$\begin{aligned} L_o \longrightarrow L_1 &= \bar{\psi}(x)\,(i\not{D} - m)\psi(x) \\ &= \bar{\psi}(x)(i\not{\partial} - m)\psi(x) + e\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x) \ . \end{aligned} \tag{1.32}$$

If $A_\mu(x)$ is the photon field, then an additional term for the kinetic energy is to be added:

$$L_1 \longrightarrow L_2 = L_1 - 1/4\, F_{\mu\nu} F^{\mu\nu} , \tag{1.33}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \cdot \tag{1.34}$$

The new Lagrangian (1.33) is the Lagrangian for quantum electrodynamics (QED). It can be seen that the local gauge invariance will be destroyed if L_2 contains a term proportional to $A_\mu A^\mu$, which is the mass-term of the field. This implies that local gauge invariance demands the photon to be massless.

The same analysis can be extended to non-abelian gauge groups in the following manner. The matter fields transform according to

$$\phi^i(x) \longrightarrow \phi^i(x) - i\alpha^a(x)\, T^a_{ij}\phi^j(x) \cdot \tag{1.35}$$

The gauge fields $A^a_\mu(x)$ transform as

$$A^a_\mu(x) \longrightarrow A^a_\mu(x) - \frac{1}{e}\, \partial_\mu\alpha^a(x) + f_{abc}\alpha^b A^c_\mu \cdot \tag{1.36}$$

As before, e is the coupling constant and f_{abc} are the structure constants of the compact Lie group G,

$$[T_a, T_b] \equiv i\, f_{abc}\, T_c \ . \qquad (1.37)$$

A gauge-covariant derivative is defined by

$$D_\mu \longrightarrow \partial_\mu - ie\, T^a A^a_\mu(x). \qquad (1.38)$$

The Lagrangian which is locally gauge-invariant can now be written:

$$L_2 = L_1 - 1/4\, F^a_{\mu\nu} F^{\mu\nu}_a \ , \qquad (1.39)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\, f_{abc}\, A^b_\mu A^c_\nu. \qquad (1.40)$$

Once again the gauge field is massless preserving gauge invariance. So the local gauge invariance demands a set of massless vector fields, and the number of massless gauge vector bosons has to be the same as the number of generators of the gauge group. When the Lagrangian $L(\phi, \partial_\mu \phi)$ of the system is invariant under some symmetry group of transformations, by Noether's theorem [48], there exists a well-defined set of conserved current densities $j_\mu^a(x)$ such that

$$\partial_\mu j_\mu^a(x) = 0 \cdot \tag{1.41}$$

The charge associated with the current density $j_\mu^a(x)$ is given by

$$Q^a = \int d^3x \; j_o^a(x) \tag{1.42}$$

which is the generator of a symmetry transformation of the field. The vacuum state $|0\rangle$ in the quantized theory may or may not be invariant under this transformation,

$$Q^a|0\rangle = 0 \tag{1.43}$$

or

$$Q^a|0\rangle \neq 0 \quad . \tag{1.44}$$

When (1.44) holds, the vacuum is not invariant under the symmetry group of transformations, eventhough the Lagrangian is invariant and this case is known as spontaneous symmetry breaking (SSB).

One of the simplest models exhibiting SSB is the one with a single scalar field, defined by the classical ϕ^4

Lagrangian, in 1+1 dimensions [49,50],

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - (\frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4) \ . \qquad (1.45)$$

The Lagrangian has got reflection symmetry $\phi \leftrightarrow -\phi$. The potential function associated to the above Lagrangian is:

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 . \qquad (1.46)$$

The profile of the potential $V(\phi)$ for $m^2 < 0$ is sketched in fig.(1.4). In this case, V has two absolute minima at $\phi = \pm m/\sqrt{-\lambda}$, with a local maximum at $\phi = 0$. This indicates that the symmetry is spontaneously broken by the vacuum state. The vacuum expectation value (vev) of the quantum field ϕ is

$$\langle 0|\phi|0\rangle = \sqrt{m^2/-\lambda} \qquad (1.47)$$
$$= \sigma$$

whereas the meson mass of the ϕ^4 theory is $\sqrt{-2m^2}$.

A more general example is that of an n-component real scalar field ϕ:

$$L = \frac{1}{2}(\partial_\mu \phi^i)^2 - \frac{m^2}{2} \phi^i \phi^i - \frac{\lambda}{4!} (\phi^i \phi^i)^2 \ . \qquad (1.48)$$

This Lagrangian is invariant under the orthogonal group O(n) in n-dimensions. If $m^2 < 0$, the potential has a ring of minima at $\sigma = \sqrt{(-m^2/\lambda)}$. Let the n^{th} component of ϕ be the one which develops a non-vanishing vev namely,

$$\bar{\sigma} = \text{vev} = \langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sigma \end{pmatrix}. \qquad (1.49)$$

The new feature is that there is still a non-trivial group, which leaves the vacuum invariant. This subgroup is O(n-1) with $\frac{1}{2}(n-1)(n-2)$ generators. As stated earlier it can be seen that the Lagrangian (1.48) contains a massive field with bare mass $-2m^2 > 0$ and (n-1) massless fields. Thus to each broken generator of the original group, there corresponds a massless boson, known as Goldstone boson after Goldstone, who conjectured [51] that where there is a spontaneous breaking of a continuous symmetry in a quantum field theory, there must exist massless spin zero particles. If the Lagrangian is invariant under a group G, but the vacuum has a lower symmetry, i.e., it is invariant under a subgroup G_o, then the number of massless Goldstone bosons is given by

$$n = \dim G - \dim G_o. \qquad (1.50)$$

Thus the Lagrangian (1.48), when SSB is taken into account, associates (n-1) Goldstone bosons. Physical illustration of Goldstone bosons is given by excitations of zero frequency modes in solid state physics. Phonons in crystals and liquid helium [52] and magnons in ferromagnets [53] provide examples of excitations with zero frequency.

In conventional gauge theory, all the gauge fields are massless, whereas if gauge theories are to be applied to weak interaction, the gauge fields should be rendered massive. Nambu [54] and Anderson [55], who were stimulated by the idea of SSB in the theory of superconductivity [55], first suggested a possible solution. Towards mid-sixties, it was realised that [56] the Goldstone conjecture, the existence of massless particles associated to SSB, need not be true and the conjecture is valid only for global symmetries, whereas for local gauge theories, through Higgs mechanism, the Goldstone conjecture fails.

In the presence of gauge fields, the Goldstone theorem is no longer applicable because of the fact that the gauge fields can absorb the Goldstone bosons and as a result, the gauge fields become massive. Thus a single mechanism can account for the disappearance of the Goldstone bosons and the emergence of massive gauge fields.

Non-abelian gauge theories, the prototype of which was introduced by Yang and Mills [57], have played a very crucial role in two currently popular models of fundamental physical processes: the 'electroweak' quantum flavour-dynamics, ~~where the gauge fields are identified with the massless photon~~ [58,59], and the 'strong' quantum chromodynamics [60,61,62]. Consequently, SU(3) Yang-Mills (YM) fields coupled to quarks (quantum chromodynamics, QCD) appears to provide the only realistic framework that can accommodate the MIT/SLAC experiments on high energy lepton-nucleon scattering. Even then YM field equations have not been solved in a general setting, let alone in the context of classical field theory.

Although there are several compact Lie groups which find a place in physical applications, we shall be concerned only with SU(2). The basic dynamical variables of SU(2) YM theory are the vector potentials A_a^μ carrying space-time index μ, and internal symmetry index a which ranges over 1,2 and 3. In component notation,

$$A^\mu = \frac{\sigma^a}{2i} A_a^\mu , \qquad (1.51)$$

where the σ^a are the Pauli matrices. The YM fields $F_{\mu\nu}$ are related to the potentials A_μ^a by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\,\epsilon^{abc} A_\mu^b A_\nu^c, \qquad (1.52)$$

where e is the coupling constant. The equation of motion is

$$D_\mu F^{\mu\nu} = 0. \qquad (1.53)$$

The SU(2) gauge theory with a Higgs triplet is defined by the Lagrangian

$$L = -\frac{1}{4} F^{\mu\nu a} F^a_{\mu\nu} - \frac{1}{2} \Pi^{\mu a} \Pi^a_\mu + \frac{1}{2}\mu^2 \phi^a \phi^a - \frac{1}{4} \lambda (\phi^a \phi^a)^2 \qquad (1.54)$$

where

$$\Pi^a_\mu = \partial_\mu \phi^a + e\, \epsilon^{abc} A^b_\mu \phi^c . \qquad (1.55)$$

The Lagrangian L is invariant under local SU(2) transformations, with ϕ_a and A^a_μ both transforming like the adjoint representation.

In this classical theory there is a spontaneous violation of local SU(2) gauge invariance. This is due to the Higgs potential $V(\phi)$,

$$V(\phi) = -\frac{1}{2} \mu^2 \phi^a \phi^a + \frac{1}{4} \lambda (\phi^a \phi^a)^2 . \qquad (1.56)$$

The Higgs field must be nonvanishing at spatial infinity in order that the potential energy be zero there[41]. Thus any physical solution must satisfy

$$\phi_a \longrightarrow \frac{\mu}{\sqrt{\lambda}} n_a(\hat{r}), \quad n_a \cdot n_a = 1, r \longrightarrow \infty \tag{1.57}$$

This defines SSB in the corresponding quantum theory, where one gives the Higgs field a nonzero vacuum expectation value $\langle \phi_a \rangle \neq 0$. If $\phi_a \neq 0$ at infinity, then it necessarily selects a direction n_a in group space. This 'breaks' local SU(2) gauge invariance in the sense that any solution that satisfies (1.57) cannot be invariant under a U(1) subgroup of the SU(2) gauge group. The vector $n_a(\hat{r})$ then determines this subgroup.

In the limit $\mu^2 \longrightarrow 0$, $\lambda \longrightarrow 0$ with $\frac{\mu^2}{\lambda} < \infty$, the Higgs potential $V(\phi)$ in (1.56) vanishes. In this limit the local SU(2) gauge symmetry of the classical solution may or may not be 'restored'. It is restored if the limiting value of μ^2/λ is zero, but not otherwise [41].

A YM theory with a local gauge symmetry breaking potential is characterised by the Lagrangian

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{4} \lambda (A^2 + \mu^2/\lambda)^2. \tag{1.58}$$

To minimize the potential energy at infinity we need [41]

$$A^2 \longrightarrow -\mu^2/\lambda \quad , \text{ as } \quad r \longrightarrow \infty . \tag{1.59}$$

In 1975, Julia and Zee [63] observed that the gauge potential component A_o^a enters the equations of motion very much as a Higgs field does. Moreover, in the limit $\mu^2 = 0$, $\lambda = 0$ and $\mu^2/\lambda < \infty$, one can reinterpret A_o^a as an imaginary Higgs field $i\phi_a$ or conversely ϕ_a as an imaginary gauge potential iA_o^a. This is only true for static fields.

By the Julia-Zee correspondence [63] we can interpret any static solution of the SU(2) gauge theory defined by (1.58) as a solution of the larger theory defined by (1.54).

The first non-abelian YM solution was found by Wu and Yang [64] which is pointlike, where the gauge potential behaves like 1/r everywhere. The Wu-Yang solution describes a pointlike non-abelian magnetic monopole attached to a string.

The equations of motion obtained from the Lagrangian (1.54) are

$$\partial_\mu \Pi^{\mu a} + e\, \epsilon^{abc} A_\mu^b \Pi^{\mu c} - \frac{\partial V(\phi)}{\partial \phi} = 0 \qquad (1.60)$$

and

$$\partial_\mu F^{\mu\nu a} - e\, \epsilon^{abc} F^{\mu\nu b} A_\mu^c + e\, \epsilon^{abc} \Pi^{\nu b} \phi^c = 0\,, \qquad (1.61)$$

where $V(\phi)$ is (1.56). By an ingeneous Ansatz one can reduce these complicated equations to a simple form. In 1968 Wu and

Yang [64] developed an Ansatz, which was subsequently modified by others in which the fields were assumed to be spherically symmetric.

The Wu-Yang-'t Hooft-Julia-Zee Ansatz seeks a solution of (1.60) and (1.61) in the form [65]:

$$\phi^a = \hat{r}^a \, H(r,t)/er \tag{1.62}$$

$$A_o^a = \hat{r}^a \, J(r,t)/er \tag{1.63}$$

$$A_i^a = \epsilon_{aij} \, \hat{r}_j [1 - K(r,t)] \, / \, er \tag{1.64}$$

where $\hat{r} = r^a/r$. Inserting (1.62-1.64) into (1.60) and (1.61) we find

$$r^2 \left[\frac{\partial^2 H}{\partial r^2} - \frac{\partial^2 H}{\partial t^2} \right] = 2HK^2 + \frac{\lambda}{e^2} \left[H^3 - C^2 r^2 H^2 \right] \tag{1.65}$$

where

$$C = \mu e/\sqrt{\lambda} \tag{1.66}$$

For $\nu = 0$ from (1.61)

$$r^2 \frac{\partial^2 J}{\partial r^2} = 2JK^2. \tag{1.67}$$

For $\nu = 1,2,3$ from (1.61)

$$r^2 \left[\frac{\partial^2 K}{\partial r^2} - \frac{\partial^2 K}{\partial t^2} \right] = K(K^2 - 1) + K(H^2 - J^2), \tag{1.68}$$

$$r \frac{\partial^2 J}{\partial t . \partial r} = \frac{\partial J}{\partial t} \tag{1.69}$$

$$\frac{\partial J}{\partial t} K + 2 \frac{\partial K}{\partial t} J = 0 \cdot \tag{1.70}$$

The last pair of equations yield a solution

$$J(r,t) = r f(t) + g(r) \tag{1.71}$$

where f and g are arbitrary functions of t and r respectively.

$$J(r,t) \longrightarrow \text{const.} \quad \text{as} \quad r \longrightarrow \infty \tag{1.72}$$

so that

$$f(t) = 0 , \tag{1.73}$$

which implies

$$J(r,t) = g(r). \tag{1.74}$$

So we have

$$\frac{\partial K}{\partial t} J = 0 \tag{1.75}$$

implying

$$\frac{\partial K}{\partial t} = 0 \tag{1.76}$$

or

$$J = 0 \cdot \tag{1.77}$$

For time-dependent solutions the natural choice is (1.77). In this case (1.65) and (1.68) respectively become

$$r^2\left[\frac{\partial^2 K}{\partial r^2} - \frac{\partial^2 K}{\partial t^2} \right] = K(K^2 - 1) + KH^2 \qquad (1.78)$$

$$r^2\left[\frac{\partial^2 H}{\partial r^2} - \frac{\partial^2 H}{\partial t^2} \right] = 2HK^2 + \frac{\lambda}{e^2}(H^3 - C^2 r^2 H^2). \qquad (1.79)$$

Exact finite energy non-trivial solutions to these coupled radial equations are not known for $\lambda \neq 0$. Yet, the behaviour of solutions [See fig.(1.5)] at the origin and the infinity can be demonstrated [41]:

$$r \longrightarrow \infty \quad K(r,0) \longrightarrow 0 + \text{const.} \exp(-\beta r), \ \beta = (e\mu/\sqrt{\lambda})^{1/2}$$

$$r \longrightarrow \infty \quad H(r,0) \longrightarrow Br + \text{const.} \exp(-\sqrt{2}\mu r), \ B = (\mu e/\sqrt{\lambda})$$

$$r \longrightarrow 0 \quad K(r,0) \longrightarrow 1 + \text{const. } e\, r^2$$

$$r \longrightarrow 0 \quad H(r,0) \longrightarrow \text{const. } e r^2 \qquad (1.80)$$

The particular version of these equations (1.78) and (1.79) that Prasad and Sommerfield [66] considered corresponds to the case $\lambda = 0$ with fixed C (this case being referred to as the Prasad-Sommerfield (PS) limit), and

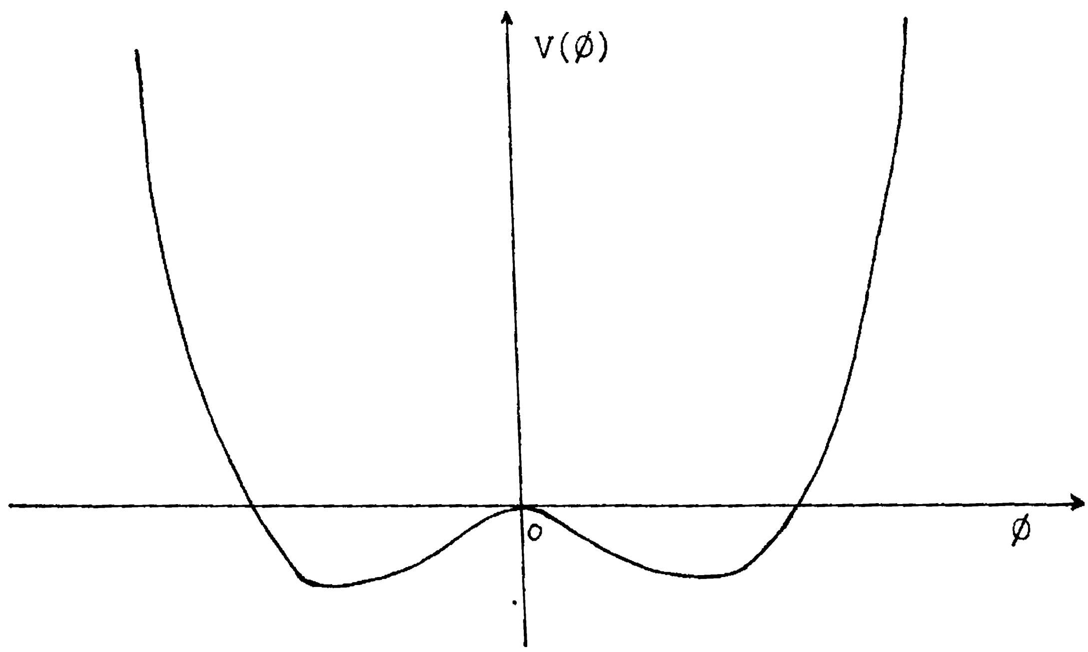

Fig.1.4 Behaviour of $\emptyset^4$ potential for $m^2 < 0$.

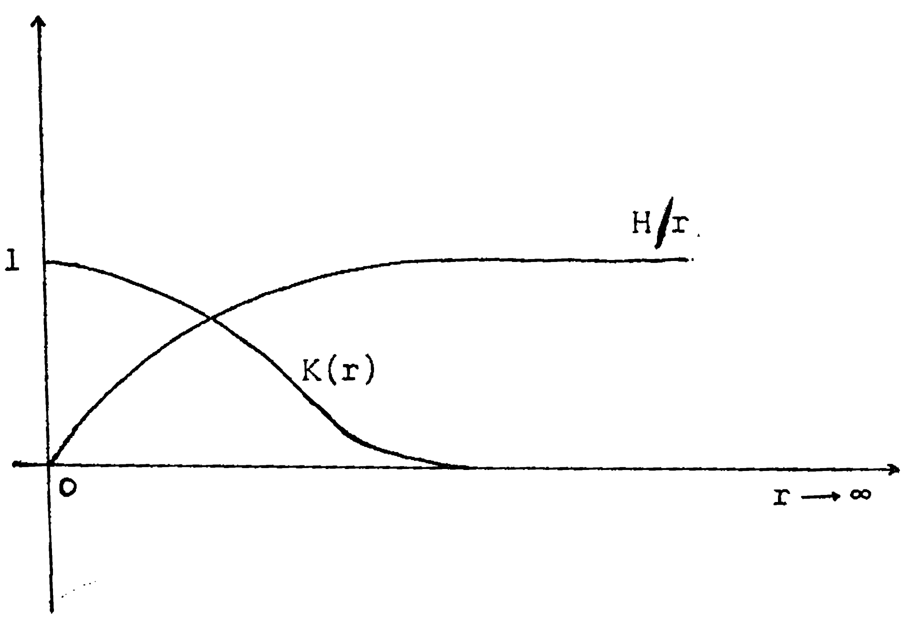

Fig.1.5 Asymptotic behaviour of SU(2) monopole solutions.

$$\frac{\partial H}{\partial t} = 0 \; , \quad \frac{\partial K}{\partial t} = 0 \tag{1.81}$$

giving

$$r^2 \frac{\partial^2 K}{\partial r^2} = K(K^2 - 1) + KH^2 \tag{1.82}$$

$$r^2 \frac{\partial^2 H}{\partial r^2} = 2HK^2. \tag{1.83}$$

They reported [66] finite energy static point monopole solutions of the form

$$K(r) = Cr/\sinh Cr \tag{1.84}$$

$$J(r) = 0 \tag{1.85}$$

$$H(r) = Cr \coth Cr - 1. \tag{1.86}$$

Apart from the spherical symmetry Ansatz other Ansatze are also useful in yielding monopole solutions. Recently, using Bogomolny's cylindrical symmetry Ansatz [67], Forgács et al [68,69] reduced the equations of motion (1.60) and (1.61) to the Ernst equation [70] and obtained N-monopole solutions through an auto-Bäcklund transformation.

By means of a specific Ansatz for the YM potential A_μ^a, one can reduce the equation of motion of pure YM theory to scalar ϕ^4 equation of motion. In Minkowski space it is of the form

$$e\, A_o^a = \pm\, i\, \partial_a \phi/\phi\ , \tag{1.87}$$

$$e\, A_i^a = \epsilon_{ian}\, \partial_n \phi/\phi \pm i\, \delta_{ai}\, \partial_o \phi/\phi \tag{1.88}$$

while in Euclidean space the Ansatz is

$$e\, A_o^a = \mp\, \partial_a \phi/\phi$$

$$e\, A_i^a = \epsilon_{ian}\, \partial_n \phi/\phi \pm \delta_{ai}\, \partial_o \phi/\phi. \tag{1.89}$$

In both cases the equation of motion of pure SU(2) YM theory becomes

$$\frac{1}{\phi}\, \partial_\mu \Box \phi = \frac{3}{\phi^2}\, \partial_\mu \phi\, \Box \phi\ . \tag{1.90}$$

An integration of this equation gives the massless ϕ^4 equation,

$$\Box \phi + \lambda\, \phi^3 = 0\ , \tag{1.91}$$

where λ is an integration constant.

An instanton is an extraordinary solution of the Euclidean SU(2) YM theory found by Belavin, Polyakov, Schwartz and Tyupkin [194] in 1975, and is some kind of a localized vacuum fluctuation with zero energy and characterized by unit topological charge. Moreover, it is of finite action and, a localized, selfdual and non-singular solution.

The selfduality condition in Euclidean space is

$$e\,B_n^a = \pm\, e\,E_n^a \,, \tag{1.92}$$

The signs correspond to selfduality and anti-selfduality respectively. In the general case of non-selfduality, (1.92) does not hold.

Here

$$E_n^a \equiv F_{on}^a \,, \tag{1.93}$$

and

$$B_n^a \equiv -\frac{1}{2}\,\epsilon_{nij}\,F_{ij}^a \tag{1.94}$$

are the SU(2) 'electric' and 'magnetic' YM fields, respectively. The instanton solution is developed from the scalar field ϕ as

$$\phi = C/(x^2 + v^2) \,, \tag{1.95}$$

where v is a constant and $C = \sqrt{8v^2/\lambda}$.

The exact solution of N instantons with arbitrary sizes, centered at points $x = a_n$ in Euclidean space has been constructed [71] from the scalar field ϕ

$$\phi = 1 + \sum_{n=1}^{N} b_n/[x - a_n]^2 . \tag{1.96}$$

This configuration represents either N instantons or N anti-instantons, but not a mixture of both. It is generally believed that no exact solution exists which describes an instanton and an anti-instanton.

Another interesting Euclidean solution is the 'meron', a non-selfdual, singular solution with infinite action and characterized by one half unit of topological charge [72,73,74] and it is a pointlike object. It is believed that merons correspond to tunnelling between two different vacuua in real time. These vacuua have topological charges $n = 0$ and $n = \frac{1}{2}$, respectively: Callen et al [75,76] have suggested that an instanton consists of two merons, and that instanton dissociation into meron pairs signals a phase transition of the YM theory into the confining phase.

A one-meron solution of the YM theory was first given by de Alfaro et al [72] and was developed from the ϕ^4 Ansatz-reduced equation (1.91), where ϕ is

$$\phi = 1/\sqrt{(\lambda x^2)}. \tag{1.97}$$

They also constructed the two meron solution:

$$\phi = \left[\frac{(a-b)^2}{\lambda(x-a)^2(x-b)^2}\right] . \tag{1.98}$$

Clearly solution (1.97) is only a special case of (1.98) when $a \longrightarrow 0$, $b \longrightarrow \infty$, $\lambda \longrightarrow \infty$; $b^2/\lambda < \infty$.

1.IV Methods of exact solution

The first objective in the study of a non-linear differential equation (NDE) is to ascertain whether or not a solution can be obtained either explicitly or implicitly in terms of classical functions. The simplest procedure followed in such an investigation consists in finding a transformation which will reduce the equation to some type that is known to have a solution of the desired kind. Sometimes the solutions of an NDE can be expressed in terms of solutions of a related linear equation or simpler NDE.

The differential equation whose solutions are used to solve another differential equations is often called the 'base equation'. This method was first introduced by Pinney [77] for finding solutions of ordinary NDEs. Kamke [78] studied several NDEs in terms of related base equations.

Ames [79] gives a good summary of references of various studies in this field. Reid and Burt [80-89] have made extensive application of the base qequation technique in a variety of problems in theoretical physics.

In 1950, Pinney used the base equation approach to the NDE,

$$\frac{d^2y}{dx^2} + P(x)\, y + Cy^{-3} = 0 \cdot \qquad (1.99)$$

Its solution

$$y = (au^2 + bv^2)^{1/2} \qquad (1.100)$$

was obtained using the differential equation

$$ab\, \omega^2 = -c, \qquad (1.101)$$

where

$$\omega = u\frac{dv}{dx} - v\frac{du}{dx} \neq 0 \cdot \qquad (1.102)$$

This idea was later generalized [80,81] to develop solutions of the equation

$$\frac{d^2y}{dx^2} + q(x)\, y = r(x)\, y^{1-2n} \qquad (1.103)$$

where ($n \neq 0, -\frac{1}{2}$). The method was subsequently extended to NPDEs [87]. For example, a particular solution of the non-linear KG equation,

$$\partial_\mu \partial^\mu \phi + m^2 \phi + \lambda \phi^3 = 0 \qquad (1.104)$$

can be obtained in terms of two base equations:

$$\partial_\mu \partial^\mu u + m^2 u = 0 \qquad (1.105)$$

and

$$\partial_\mu u \, \partial^\mu u + m^2 u^2 = 0 , \qquad (1.106)$$

where, $\partial^\mu = g^{\mu\nu} \partial_\mu$, $g_{\mu\nu} = (1,-1,-1,-1)$; then

$$\phi = \frac{u}{1 - \frac{\lambda u^2}{8m^2}} \qquad (1.107)$$

Recently, this method has been used to solve double sine-Gordon [89] equations. For the DsG equation,

$$\partial_\mu \partial^\mu \phi + \frac{m^2}{a} \sin(a\phi) + \frac{b}{2a} \sin 2\phi = 0 \qquad (1.108)$$

the basic equations adopted are:

$$\partial_\mu \partial^\mu \psi + (m^2 + b)\psi - 2(m^2 + 2b)\psi^3 + c\psi^5 = 0 \qquad (1.109)$$

and

$$(\partial_\mu\psi)(\partial^\mu\psi) - (m^2 + 3b + A)\psi^2(1-\psi^2) - (2b-B)\psi^4(1-\psi^2) = 0 \qquad (1.110)$$

where,

$$A = -2(m^2 + 2b) \qquad (1.111)$$

$$B = 3b \qquad (1.112)$$

$$\phi = \frac{2}{a} \arcsin\psi \,. \qquad (1.113)$$

One advantage of this method is that one can develop arbitrary dimensional solutions of KG equations. N-solitary wave solutions can also be constructed via the linear superposition principle, when the base equations are linear[89]. A disadvantage of this procedure is that the method has not proved flexible enough to deal with equations outside the KG family.

Hirota developed a direct method [35,90-98] of finding exact solutions of a number of non-linear evolution equations. His method consists in replacing the dependent variable by a ratio of two functions. This approach is very much similar to that of Padé approximants [99]. Hirota defines a new bilinear operator D_z as follows [97]:

$$D_t^n\, g(x,t)\cdot f(x,t) \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n g(x,t)\; f(x,t')\Big|_{t=t'} \qquad (1.114)$$

and

$$D_x^n\, g(x,t)\cdot f(x,t) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n g(x,t)\; f(x',t)\Big|_{x=x'} . \qquad (1.115)$$

Ordinary differential operators and bilinear operators are related:

$$\frac{\partial^n g(x)}{\partial x^n} = D_x^n g\cdot 1 \qquad (1.116)$$

$$\frac{\partial}{\partial x}(g/f) = \frac{D_x\; g\cdot f}{f^2} \qquad (1.117)$$

$$\frac{\partial^2}{\partial x^2}(g/f) = \frac{D_x^2\; g\cdot f}{f^2} - \frac{g}{f}\;\frac{D_x^2 f\cdot f}{f^2} \qquad (1.118)$$

$$\frac{\partial^3}{\partial x^3}(g/f) = \frac{D_x^3\; g\cdot f}{f^2} - 3\,\frac{D_x g\cdot f}{f^2}\;\frac{D_x^2\; f\cdot f}{f^2}$$

$$D_x^n\; (g\cdot f) = D_x^{n-1}(g_x\cdot f - f_x\cdot g) \qquad (1.120)$$

where $g_x = \frac{\partial g}{\partial x}$ and $f_x = \frac{\partial f}{\partial x}$. On replacing the given differential operators and functions by the new bilinear operators and g/f, respectively, we get a coupled bilinear operator equation. This equation is split into two, so that one of the equations is in general of the same structure as the linear part of the original non-linear equation. Functions g and f are then expanded as power series in a parameter $\epsilon \ll 1$ and the coefficients of different powers of ϵ are determined as in perturbation theory. This method can be illustrated for sG equation in 1+1 dimensions:

$$u_{xx} - u_{tt} = \sin u \cdot \tag{1.121}$$

This gives a pair of coupled bilinear equations

$$(D_x^2 - D_t^2) g \cdot f = g \cdot f \tag{1.122}$$

$$(D_x^2 - D_t^2)(f \cdot f - g \cdot g) = 0 \tag{1.123}$$

by the dependent variable transformation,

$$u = 4 \arctan (g/f) \cdot \tag{1.124}$$

Clearly equation (1.122) is very much similar to the linear part of the equation (1.121). Let us consider power series expansions for f and g:

$$f = f_o + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \ldots \tag{1.125}$$

$$g = g_o + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 + \ldots . \tag{1.126}$$

On substituting (1.125) and (1.126) in (1.122) and (1.123) and collecting the coefficients of like powers of ϵ , as in perturbation theory, we get [97]

$$g_o = f_o = 0 \tag{1.127}$$

$$g_1 = \exp(\theta), \tag{1.128}$$

$$f_1 = 1 \tag{1.129}$$

$$g_n = 0, \text{ for all } n \geq 2 \tag{1.130}$$

$$f_n = 0, \text{ for all } n \geq 2 \tag{1.131}$$

where, $\theta = kx - \omega t + \delta$, and

$$k^2 - \omega^2 = 1 . \tag{1.132}$$

This yields the exact solution of sG equation (1.121) as

$$u = 4 \text{ arc } \tan(e^{\theta}) \tag{1.133}$$

which represents a kink-antikink bound pair solution.

Hirota's method possesses the advantage of being applicable to equations in any number of dimensions. Equations other than those of the KG family also can be solved by this approach [97]. This method has been used to study three-wave interaction phenomena [97]. A method of developing N-soliton solutions, was also introduced by Hirota for exponential type solutions, by replacing the e^{θ} term by $\sum_{j=1}^{N} e^{\theta_j}$ where $\theta_j = k_j x - \omega_j t + \delta_j$.

A new era in theoretical physics was ushered in by the discovery of 'Inverse scattering transform' (IST) by Gardner, Greene, Kruskal and Miura [99] whereby the initial value problem for the KdV equation could be solved. This method was later formulated in terms of Lax operators [100]. A brief outline of this procedure is presented here.

Let us assume that we are able to find two linear operators L and B which depend on the solution u of a NPDE satisfying the operator relation

$$iL_t = BL - LB. \qquad (1.134)$$

Then the associated eigenvalue problem for the linear operator L is

$$L\psi = E\psi. \qquad (1.135)$$

The eigenvalues E become independent of time when the operator B is selfadjoint. Then the eigenfunction ψ may be shown to evolve in time according to

$$i\psi_t = B\psi. \tag{1.136}$$

There are possibilities to associate a scattering problem with the linear operator L; then for given data $\phi(x,0)$, one can find the $\phi(x,t)$, through a standard procedure. One of the attendant difficulties of this method is the lack of a systematic procedure to identify the linear operators L and B exactly. There are cases where these operators turn out to be trivial. The construction of $\phi(x,t)$ from the scattering data of the linear operator L is also not an easy job, as it leads one to grapple with the Gelfand-Levitan integral equation [101].

There is a close connection between IST and Fourier transformation. The IST provides the exact solution to certain non-linear evolution equation, just as the Fourier transform does for certain linear evolution equations [102,103].

The Bäcklund transformation (BT) had its origin in some studies of Bäcklund [104,105] relating to the simultaneous equations of the first order, arising in differential geometry [106]. The BT provides a method of constructing various classes of equivalent equations, thereby leading to the integrals of the original equation, and this is closely associated to 'contact transformations' [107] of differential geometry.

Let $(x,y,z,\ p=z_x,\ q=z_y)$ be a surface element and (x',y',z',p',q') be an element of any other surface. To connect the two surface elements completely, it is necessary to have five distinct equations. Each set satisfies the total differentials

$$dz = p\,dx + q\,dy \tag{1.137}$$

$$dz' = p'dx' + q'dy' \tag{1.137}$$

Equations (1.137) reduce the number of independent equations to four, namely,

$$F_n(x,y,z,p,q;\ x',y',z',p',q') = 0,\ n=1,2,3,4. \tag{1.138}$$

There are certain cases when the variable z or z' is an integral of Monge-Ampère form [107],

$$R\,r + S\,s + T\,t + U\,(r\,t - s^2) = V \tag{1.139}$$

where $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$. Then the transformation is called a BT. This form of the Monge-Ampère equation is not the most general one, so that not all such equations can be expected to have BTs [107]. Clairin [108,109] developed the BT for the sG equation in two dimensions in the form

$$\frac{1}{2}(q + q') = \frac{1}{\alpha}\sin\left(\frac{z - z'}{2}\right) \tag{1.140}$$

$$\frac{1}{2}(p - p') = \alpha\sin\left(\frac{z + z'}{2}\right) \tag{1.141}$$

where z and z' are two arbitrary solutions of sG equation and α is an integration constant called the BT parameter. This type of BT connects two distinct solutions of the same equation and is called an auto-Bäcklund transformation (ABT). It is known from the theory of surfaces [106] that there exists a relationship among four distinct solutions of sG equations which does not involve quadratures:

$$\tan\left(\frac{z_3 - z_o}{4}\right) = \left(\frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2}\right)\tan\left(\frac{z_1 - z_2}{4}\right) \tag{1.142}$$

where z_o, z_1, z_2, z_3 are particular solutions of the sG equation. This procedure has been exploited [110,111] to develop N-soliton solutions of sG equation. Diagrammatically this procedure can be represented as in fig.(1.6). This diagram can be extended further to develop N-soliton solution without quadratures [110,111].

NPDEs of the type

$$Z_{xt} = f(Z) \tag{1.143}$$

can have a BT as well as an ABT.

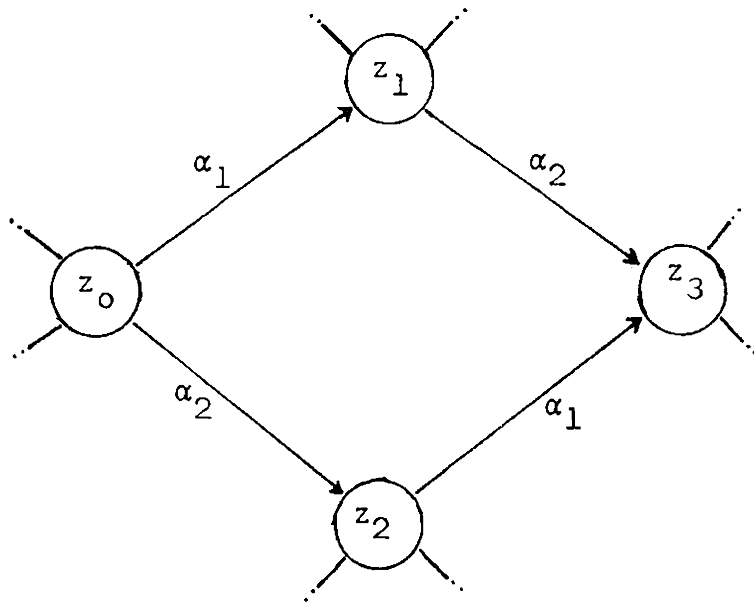

Fig.1.6 Auto-Bäcklund transformations for four arbitrary solutions of sG equation.

For example,

$$Z_{xt} = \exp(mZ) \tag{1.144}$$

which is the two dimensional Liouville equation it can be connected to the wave equation,

$$Z'_{xt} = 0 . \tag{1.145}$$

Thus a solution of Liouville equation can be transformed to the solution of wave equation and vice versa [112].

Apart from various techniques studied in the literature Weiss et al [113,114] have recently introduced a comparatively simple procedure for finding ABTs.

A powerful method of solving ordinary as well as PDEs was developed in the 19th century by Sophus Lie [115] by using continuous transformation or topological groups. For simplicity, let us consider a second order PDE with one dependent variable u and two independent variables x and t

$$H(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt},....) = 0 \tag{1.146}$$

Applying one parameter infinitesimal transformations,

$$x^* = x + \epsilon\, X(x,t,u) + O(\epsilon^2)$$

$$t^* = t + \epsilon\, T(x,t,u) + O(\epsilon^2)$$

$$u^* = u + \epsilon\, U(x,t,u) + O(\epsilon^2)\,. \qquad (1.147)$$

When the equation (1.146) is transformed to

$$H(x^*,t^*,u^*,u^*_{x^*},u^*_{t^*},u^*_{x^*x^*}\ldots) = 0 \qquad (1.148)$$

then we say, equation (1.146) is invarint under the infinitesimal transformation (1.147). X, T and U are the 'infinitesimals' of the transformations associated to the variables x, t and u, respectively.

Let us consider the solution surface $\Omega(x,t,u)$ in (x,t,u) space as in fig.(1.7) where $u^* = u(x^*,t^*)$ on the surface. By the infinitesimal transformation (1.147) we shall have

$$u(x + \epsilon X + O(\epsilon^2);\ t + \epsilon T + O(\epsilon^2))$$
$$= u + \epsilon\, U(x,t,u) + O(\epsilon^2)\,. \qquad (1.149)$$

On expanding and equating the $O(\epsilon)$ terms,

$$X\frac{\partial u}{\partial x} + T\frac{\partial u}{\partial t} = U\,. \qquad (1.150)$$

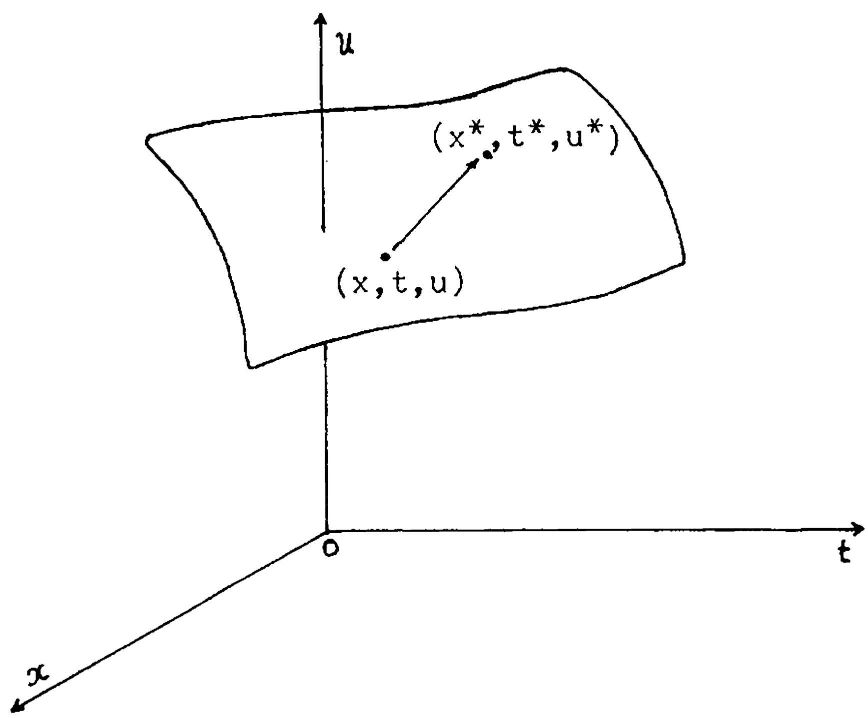

Fig.1.7 The invariant solution surface.

This is called the 'invariant surface condition'. By Lagrange's condition, the solutions of equation (1.150) can be obtained from the characteristic equations,

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} \, . \tag{1.151}$$

This implies

$$\frac{dx}{dt} = \frac{X}{T} = f(x,t,u) \tag{1.152}$$

and

$$\frac{du}{dt} = \frac{U}{T} = g(x,t,u) \, . \tag{1.153}$$

When (1.152) is independent of u we get

$$x = x(t, C_1, C_2) \tag{1.154}$$

$$u = u(t, C_1, C_2) \tag{1.155}$$

where C_1 and C_2 are arbitrary constants. The arbitrary constant of integration C_1 now plays the role of a new independent variable say χ, and C_2 is the dependent variable or 'similarity solution'. Thus we have

$$u(x,t) = F(\chi). \tag{1.156}$$

On inserting (1.156) in (1.146) we can find a equivalent ordinary differential equation (ODE) in terms of a similarity solution $F(\chi)$ and similarity variable χ:

$$K(\chi, F, F', F'', \ldots) = 0 \qquad (1.157)$$

where a prime indicates the differentiation with respect to the similarity variable χ.

The importance of similarity solutions and the similarity approach has been discussed by many authors [116-121]. Two general results seem to emerge from the similarity approach of infinitesimal point transformations:

i. If an ODE is invariant under the transformations, its order can be reduced by one.

ii. If a PDE is invariant under the transformations, it is possible to find similarity solutions of the equation, and the number of variables can be reduced by one.

The similarity method can be extended to arbitrary numbers of independent variables x_j and dependent variables u^i, $j = 1,2,\ldots n$, $i = 1,2,\ldots m$. Let us define infinitesimal transformations of an arbitrary number of variables:

$$x_j^* = x_j + \epsilon X_j(x_1,x_2,..x_n;\ u^1,u^2,...u^m) + O(\epsilon^2)$$

$$u^{i*} = u^i + \epsilon U^i(x_1,x_2,..x_n;\ u^1,u^2,..u^m) + O(\epsilon^2)$$

(1.158)

where X_j are the infinitesimals of the independent variables x_j, and U^i those of the dependent variables u^i. The invariant surface condition associated with the 'infinitesimal operator'

$$\Omega \equiv X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + ...+ X_n \frac{\partial}{\partial x_n}$$

$$+ U^1 \frac{\partial}{\partial u^1} + ...+ U^m \frac{\partial}{\partial u^m} \qquad (1.159)$$

is defined by,

$$\Omega u \Big|_{H=0} \equiv 0 \qquad (1.160)$$

where H = 0 represents the system of PDEs.

Finite transformations corresponding to the infinitesimal set (1.158) are given by

$$u^{i*} = \exp(\epsilon\Omega)\, u^i$$

$$= u^i + \sum_1^\infty \frac{\epsilon^n}{n!} \Omega^n u^i \qquad (1.161)$$

and

$$x_j^* = x_j + \sum_1^\infty \frac{\epsilon^n}{n!} \Omega^n x_j \ . \qquad (1.162)$$

Like the independent variables x_j and dependent variables u^i, the partial derivatives also transform under an infinitesimal transformation. The infinitesimals associated with the partial derivatives $u^i_{x_j}, u^i_{x_j x_k}, \ldots . j,k=1,2,\ldots,n.$ are denoted by $[U^i_{x_j}]$, $[U^i_{x_j x_k}]$, ...and are often called the first extensions, second extensions etc. The partial derivatives transform according to

$$u^{i*}_{x^*_j} = u_{x_j} + \epsilon\,[U^i_{x_j}] + O(\epsilon^2) \tag{1.163}$$

where the extension

$$[U^i_{x_j}] = U^i_{x_j} + U^i_{u^i}\, u^\nu_{x_j} - X_{k\,x_j}\, u^i_{x_k} - X_{k\,u^\nu}\, u^\nu_{x_j}\, u^i_{x_k}\,.$$

$(i = 1,2,3,..m;\ j,\ k = 1,2,..n).$ (1.164)

To simplify the foregoing formalism we shall introduce the total derivative operators

$$\frac{D}{Dx_j} = \frac{\partial}{\partial x_j} + \frac{\partial u^\mu}{\partial x_j}\,\frac{\partial}{\partial u^\mu}$$

and

$$\frac{D}{Dx^*_j} = \frac{\partial}{\partial x^*_j} + \frac{\partial u^{\mu*}}{\partial x^*_j}\,\frac{\partial}{\partial u^{\mu*}}\,. \tag{1.165}$$

The transformation rule (1.163) now becomes

$$u^{i^*}_{x^*_j} = u^i_{x_j} + \epsilon \left[\frac{DU^i}{Dx_j} - \frac{DX_\nu}{Dx_j} u^i_{x_\nu} \right] + O(\epsilon^2) \qquad (1.166)$$

yielding

$$[U^i_{x_j}] = \frac{DU^i}{Dx_j} - \frac{DX_\nu}{Dx_j} u^i_{x_\nu}. \qquad (1.167)$$

Similarly the second derivatives transform as

$$u^{i^*}_{x_j x_k} = u^i_{x_j\ x_j} + \epsilon \left[\frac{DU^i_{x_j}}{Dx_k} - \frac{DX_\nu}{Dx_k} u^i_{x_j x_\nu} \right] + O(\epsilon^2). \qquad (1.168)$$

The second extensions are

$$[U^i_{x_j\ x_k}] = \frac{DU^i_{x_j}}{Dx_k} - \frac{DX_\nu}{Dx_k} u^i_{x_j x_\nu}. \qquad (1.169)$$

In the case of a second order equation, for instance,

$$H(x_j,\ u^i,\ u^i_{x_j},\ u^i_{x_j x_k}) = 0\ ,\ i=1,2,..m;j,k=1,2...n. \qquad (1.170)$$

the invariant surface condition is

$$X_j \frac{\partial H}{\partial x_j} + U^i \frac{\partial H}{\partial u_i} + [U^i_{x_j}] \frac{\partial H}{\partial u^i_{x_j}} + [U^i_{x_j x_k}] \frac{\partial H}{\partial u^i_{x_j x_k}} = 0 \qquad (1.171)$$

On solving (1.171), the infinitesimals X_j and U^i etc., can be found.

1.V Integrability and Painlevé property

Recently much attention has been focused on the classification of non-linear dynamical systems into integrable and non-integrable types. The KdV equation is a simple integrable system [122], whereas DsG model is a non-integrable system [123]. For an integrable system; the equation of motion can be explicitly solved and so these systems are highly regular and well predictable. Classically integrability is defined as the existence of an action-angle representation [124].

For integrable dynamical system the following properties have been noted in the literature [125].

i. The associated initial value problem can be exactly solved by IST.

ii. There exists an infinite number of conservation laws.

iii. The system has an ABT.

iv. The associated 'Lie point vector fields' have 'Lie-Bäcklund vector fields.'

v. They define a pseudospherical surface, that is a surface of constant negative Gaussian curvature.

vi. There exists a covariant exterior derivative of Lie algebra-valued differential forms representation.

The existence of some of the above properties need not imply integrability. For example, Liouville's equation, has an ABT and possesses an IST, but it does not constitute an integrable system. Thus the above conditions are not sufficient to establish integrability. The integrability of a system is closely associated to a property called 'Painlevé property' (PP) of an ODE which can be derived from the PDE modelling the given system [126,127].

Consider a second order ODE in the complex plane, with variable coefficients:

$$\frac{d^2w}{dz^2} + p(z)\,\frac{dw}{dz} + q(z)\,w = 0 \tag{1.172}$$

Let the general solution be

$$w(z;\, A,B) = A\, w_1(z) + B\, w_2(z). \tag{1.173}$$

If the singularities of w(z) do not depend on A or B, then they are said to be fixed. A singularity is said to be movable when its location depends on the constants of integration. Linear differential equations have only fixed singularities, whereas NDEs can have both fixed and movable singularities.

The absence of movable branch points or essential singularities of an ODE is described as the Painlevé property (PP) and the equations that possess PP are said to be of Painlevé type or P-type.

The first order equations with PP are generalized Riccati equations:

$$\frac{dw}{dz} = P_o(z) + P_1(z)w + P_2(z)w^2. \tag{1.174}$$

A second order equation

$$\frac{d^2w}{dz^2} = F\left(\frac{dw}{dz}, w, z \right) \tag{1.175}$$

is P-type if it can be reduced to any one of the six Painlevé transcendents (PI-PVI) listed below [128]:

PI, $$\frac{d^2w}{dz^2} = 6w^2 + z$$

PII, $$\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha$$

PIII, $$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}$$

PIV, $$\frac{d^2w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \beta\frac{1}{w}$$

PV, $$\frac{d^2w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right)$$

PVI, $$\frac{d^2w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz}$$

$$+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right] \tag{1.176}$$

where α, β, γ and δ are constants.

Ablowitz et al [126,127] have conjectured that every non-linear ODE obtained by an exact similarity reduction of a NPDE solvable by some inverse scattering transform has PP. It is believed that if an ODE possesses PP, then this system is integrable.

In classical mechanics, the Toda lattice [20] is a well known example of an integrable system. In field theory the sG equation is an integrable system in 1+1 dimensions. In quantum field theory a best known example of an integrable system is the quantum non-linear Schrödinger equation [129].

Recently a number of studies on non-linear operators and integrability of evolution equations have also appeared. Let u be a smooth function on the real line vanishing rapidly at infinity, and K(u) be a smooth vector field on the space S

[130-136]
of these functions. For integrability, the evolution equation.

$$u_t - K(u) = 0 \quad (1.177)$$

has dependence on a certain integro-differential operator T(u). This operator T(u) has been given various names in the literature: squared eigenfunction operators [130], recursion operators [131-134], strong symmetries [135], hereditary symmetries [135], Kähler operators [136] or regular operators [137]. Several investigators, especially Magri [136], Gelfand and Dorfmom [137] Fokas and Fuchssteiner [138-139] and Aiyer [132-134] have extensively studied the structure of these operators and their connection to the Hamiltonian formulation.

2

SOLITARY WAVES IN DOUBLE SINH-GORDON SYSTEM

2.I The double sinh-Gordon equation

Skyrme [140] proposed a non-linear field theory which, for the scalar case and in 1+1 dimensions, reduces to a non-linear extension of the Lagrangian density corresponding to the linear KG equation. The equation considered by him has subsequently become known as the sG equation which is characterised by a sine function in the equation of motion. In 1962, Perring and Skyrme [27] found by a computer analysis that the solitary wave solutions of the sG equation are collisionally stable and thereby paved the way for the introduction of the soliton concept. Later this was recognised as an important model in solid state physics [142] and high energy particle physics [49,141]. An equation of motion with two sine functions was subsequently introduced [142-147] and named the double sine-Gordon (DsG) equation which has led to several applications in non-linear optics [143,145] such as the study of the B-phase of liquid helium [145,146] and the treatment of quasi-one-dimensional charge-density wave condensates of organic linear conductors like TTF-TCNQ [147]. In 1+1 dimensions the sG field system undergoes a second order phase transition [148]. The hyperbolic version of the sG family of equations has been discussed recently. Unlike the sG equation, the sinh-Gordon (shG) equation has no soliton solutions [149] although like the sG equation, this has got an ABT and an infinite number of conservation laws.

A new member called the double sinh-Gordon (DshG) model has recently been added to the KG family of equations by Behera and Khare [150]. They found a kink solution for this model and demonstrated the possibility of calculating the exact free energy associated with the second order phase transition that the system undergoes. Minami [151] has recently studied this model and established its relation to the Toda lattice model [20].

Morse [152] introduced an anharmonic potential of the exponential type:

$$V(\phi) = a/b \cdot \exp(-b\phi) - 2a/b \exp(-b/2 \cdot \phi) \tag{2.1}$$

which was later called the Morse potential. A more general form of an anharmonic exponential type potential is

$$V(\phi) = a_1/b \exp(-b\phi) + 2a_2/b \exp(-b\phi/2) + a_3/b \exp(b\phi) + 2a_4/b \exp(b\phi/2) \cdot \tag{2.2}$$

When

$$a_1 = a_2 = a_4 = 0 , \qquad (2.3)$$

the potential yields the classical Liouville equation [142],

$$\phi_{xx} - \phi_{tt} = a_3 \exp(b\phi) \cdot \qquad (2.4)$$

This is a well studied field-theoretic model and widely used in fluid mechanics and differential geometry [41,154-157].

For

$$a_1 = a_3 = \alpha$$

$$a_2 = a_4 = 0$$

$$b = 1 , \qquad (2.5)$$

we obtain the equation of motion

$$\phi_{xx} - \phi_{tt} = \alpha \sinh \phi , \qquad (2.6)$$

which is generally known as the sinh-Gordon equation.
For the choice

$$b_4 = 4$$

$$a_1 = a_3 = \eta^2/2$$

$$a_2 = a_4 = -2\eta , \qquad (2.7)$$

(2.2) gives

$$V(\emptyset) = \eta^2/8 \cosh 4\emptyset - \eta \cosh 2\emptyset , \tag{2.8}$$

where η is a real parameter. To ensure the vanishing of the potential as $\emptyset \longrightarrow 0$, we may modify this trivially into the form

$$V(\emptyset) = \eta^2/8 \cosh 4\emptyset - \eta \cosh 2\emptyset - \tfrac{1}{8}\eta^2 + \eta. \tag{2.9}$$

This represents the potential corresponding to the DshG equation and has minima at

$$\emptyset = 0 \qquad \text{for } \eta > 2 , \tag{2.10}$$

and

$$\cosh 2\emptyset = 2/\eta \quad \text{for } \eta < 2. \tag{2.11}$$

For the second condition (2.11) there are two degenerate minima. The values of the potential at the minima are

$$V_{min}(\emptyset = 0) = 0 \tag{2.12}$$

and

$$V_{min}(\cosh 2\emptyset = 2/\eta) = -\tfrac{1}{4}(\eta^2 - 4\eta + 4). \tag{2.13}$$

The equation of motion corresponding to the potential (eq.(2.9)) is

The first known solution for this model is a kink-like solution [150]:

$$\phi(x,t) = \text{arc tanh} \left\{ \frac{(1-\eta/2)}{\sqrt{(1-\eta^2/2)}} \tanh \left[\frac{\sqrt{2}\,(x-ut)}{\sqrt{(1-\eta^2/4)}\sqrt{[m(c^2-u^2)]}} \right] \right\}, \tag{2.15}$$

which is defined for the values $|\eta| < 2$. As $x \longrightarrow \infty$, this behaves according to

$$\tanh\,[\phi(\pm\infty)] = \mp \sqrt{[(2-\eta)/(2+\eta)]}\,, \tag{2.16}$$

which are the values of the field ϕ corresponding to the two degenerate minima characterising the kink solution (2.15).

In this chapter we first show that the DshG field system possesses other types of solution besides the large amplitude kink-like solutions. For this purpose we use the bilinear operator method as well as the base equation technique. We also examine the asymptotic behaviour of multisolitary wave solutions and carry out a linear stability analysis of the single solitary wave solution. This system is shown to possess stable solitary wave solutions in 1+1 dimensions. Characterised as they are by a vanishing topological charge, these new solutions can be considered non-topological objects [30].

2.II. Solitary waves by the bilinear operator method

Define a transformation

$$\emptyset = \text{arc tanh } (g/f) \tag{2.17}$$

so that the equation of motion (2.14) yields the bilinear differential equation,

$$(f^2+g^2)(D_x^2-D_t^2)f\cdot g - f\cdot g(D_x^2-D_t^2)(f\cdot f + g\cdot g)$$

$$= 2\eta^2 f\cdot g(f^2+g^2) - 4\eta(f^2-g^2)f\cdot g \tag{2.18}$$

where D_z^2 is the bilinear differential operator. On splitting (2.18) so that one is linear and the other may be nonlinear in f and g, we find:

$$(D_x^2-D_t^2)f\cdot g = 2\eta(\eta-2)f\cdot g, \tag{2.19}$$

$$(D_x^2-D_t^2)(f\cdot f + g\cdot g) = -8\eta\ g\cdot g. \tag{2.20}$$

We introduce power series expansions for f and g in a parameter ϵ which is very close to unity:

$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \ldots. \tag{2.21}$$

$$g = \epsilon g_1 + \epsilon^2 g_2 + \ldots. \tag{2.22}$$

On equating the coefficients of same powers of ϵ we obtain a set of differential equations:

$$(D_x^2-D_t^2)g_1 = 2\eta(\eta-2)g_1 \tag{2.23}$$

$$(D_x^2-D_t^2)(2f_2 + g_1g_1) = -8\eta g_1^2 \tag{2.24}$$

$$(D_x^2 - D_t^2)\, g_2 = 2\eta\,(\eta-2)\, g_2$$

$$(D_x^2 - D_t^2)\,(g_3+f_2g_1) = 2\eta\,(\eta-2)\,(g_3 + f_2g_1)$$

$$(D_x^2 - D_t^2)\,(g_4+f_2g_2) = 2\eta\,(\eta-2)\,(g_4 + f_2g_2)$$

$$\vdots \qquad \vdots \qquad \vdots$$

Equation (2.23) implies,

$$\frac{\partial^2 g_1}{\partial x^2} - \frac{\partial^2 g_1}{\partial t^2} = 2\eta(\eta-2)g_1 \ . \tag{2.25}$$

A simple solution for this equation (2.25) is

$$g_1 = \exp(\theta) \tag{2.26}$$

where, $\theta = kx - \omega t + \delta$ and

$$k^2 - \omega^2 = 2\eta(\eta-2). \tag{2.27}$$

Equation (2.24) yields,

$$2\left[\frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial t^2}\right] + 2\left[g_1\frac{\partial^2 g_1}{\partial x^2} - g_1\frac{\partial^2 g_1}{\partial t^2} - \left(\frac{\partial g_1}{\partial x}\right)^2 + \left(\frac{\partial g_1}{\partial t}\right)^2\right] = 8\eta g_1^2 \ . \tag{2.28}$$

Inserting (2.26) in (2.28),

$$f_2 = -\exp(2\theta)/(2\eta-4) \cdot \qquad (2.29)$$

We ~~set~~ can consistently set [97]

$$g_n = 0 \quad \text{for all } n \geqslant 2$$

$$f_n = 0 \quad \text{for all } n \geqslant 4$$

$$\epsilon = 1 \qquad (2.30)$$

Combining (2.17), (2.21), (2.22), (2.26), (2.29) and (2.30) an exact solitary wave solution of DshG equation (2.14) is obtained:

$$\phi(x,t) = \text{arc tanh}\left(\frac{\pm \exp(\theta)}{1-\exp(2\theta)/(2\eta-4)}\right). \qquad (2.31)$$

This solution for $\eta < 2$ behaves qualitatively as sketched in fig.(1.1).

~~In contrast with the kink solution (eq.(2.16)),~~ This new solution is defined for $\eta < 2$ and can readily be extended to arbitrary dimensions. We might expect to obtain the multi-solitary wave solution by setting,

$$g = \sum_{j=1}^{N} \exp(\theta_j). \qquad (2.32)$$

However, the corresponding power series of the form (2.23) and (2.24) do not terminate, exposing the failure of the bilinear operator method to provide any such solutions.

2.III. Multisolitary wave solutions

The base equation technique is found useful for the construction of multisolitary or N-solitary wave solutions of the DshG equation in arbitrary dimensions.

Let us accordingly rewrite equation (2.14) as

$$\partial_\mu \partial^\mu \phi = \eta^2/2 \sinh 4\phi - 2\eta \sinh 2\phi \ , \tag{2.33}$$

where $\mu = 0,1,2,\ldots(n-1)$ and the n dimensional D'Alembertian,

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} \ . \tag{2.34}$$

The transformation

$$\phi = \text{arc} \sinh \psi \tag{2.35}$$

converts (2.33) into the form

$$[1+\psi^2]^{-1/2} \partial_\mu \partial^\mu \psi - [1+\psi^2]^{-3/2} \psi \ \partial_\mu \psi \partial^\mu \psi - 2\eta^2 \psi [1+\psi^2]^{1/2} [1+2\psi^2]$$

$$+ 4\eta \psi [1+\psi^2]^{1/2} = 0 \ , \tag{2.36}$$

where

$$\partial_\mu\psi\,\partial^\mu\psi = \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial t} - \sum_{i=1}^{n-1}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_i}. \tag{2.37}$$

We may take the equation

$$\partial_\mu\psi\,\partial^\mu\psi = [1+\psi^2](4\eta-6\eta^2-D)\psi^2 - [1+\psi^2](4\eta^2+B)\psi^4 \tag{2.38}$$

as the base equation. The function ψ can then be expressed as [89],

$$\psi = u\,A^{-1/2}, \tag{2.39}$$

where

$$A = (1 - Bu^2/8m) - Cu^4/12D^2 , \tag{2.40}$$

$$B = (8\eta - 8\eta^2)$$

$$C = -6\eta^2$$

$$M = (\eta - \eta^2/2)$$

$$D^2 = (4\eta - 2\eta^2), \tag{2.41}$$

and u satisfies the equations

$$\partial_\mu\,\partial^\mu u + D^2 u = 0 \tag{2.42}$$

$$\partial_\mu u\,\partial^\mu u + D^2u^2 = 0. \tag{2.43}$$

Henceforth, the last two equations (2.42) and (2.43) can be employed as base equations for solving (2.38). These equations admit a simple exponential type solution:

$$u = a \exp(\alpha k x), \tag{2.44}$$

where a is an arbitrary parameter and

$$k = (k_o, k_1, \ldots k_{n-1}) \tag{2.45}$$

$$x = (t, x_1, x_2, \ldots x_{n-1}) \tag{2.46}$$

$$\alpha = [-(4\eta - 2\eta^2)/k^2]^{1/2}, \tag{2.47}$$

so that $(4\eta-2\eta^2) < 0$ or $\eta \notin (0,2)$.

Equations (2.39)-(2.46) imply an exact solution of the DshG system (2.33):

$$\phi = \text{arc sinh} \left(u/[1- (1-2\eta)u^2/(8-4\eta) + u^4/2(4-2\eta)^2]^{1/2} \right). \tag{2.48}$$

All the solutions of (2.42) and (2.43) are automatically the solutions of the DshG equation. Since (2.42) is a linear equation, the linear superposition

$$u = \sum_{j=1}^{N} a_j \exp(\alpha_j k_j x) \tag{2.49}$$

is also a solution of (2.42) and (2.43). On substituting this form in (2.48), a multisolitary wave solution of the DshG equation emerges with the additional set of conditions:

$$\alpha_j \alpha_i k_j k_i + (4\eta - 2\eta^2) = 0 , \tag{2.50}$$

where

$$k_i = (k_{io}, k_{i1}, k_{i2}, \ldots k_{in-1})$$

$$= (k_{io}, \vec{K}_i) \tag{2.51}$$

and

$$k_i \neq k_j , \vec{K}_i \cdot \vec{K}_j \neq 0 , \tag{2.52}$$

for any i and j.

For a multisolitary wave solution in an n dimensional space-time, the number of independent components of the wave vector $\alpha_j k_j x$ is N(n-1), whereas the number of constraints in equation (2.50) is N(N-1)/2, for $i \neq j$. Hence for the system not to be overdetermined,

$$N \leqslant (2n-1). \tag{2.53}$$

Thus the solitary wave index N is restricted by the dimensionali of the space-time n. Nevertheless, in 1+1 dimensions only one solitary wave can be found as there is only one independent wave vector and any other wave vector is necessarily parallel to it, as can be verified in the following way [89].

For any two vectors k_i and k_j, if θ_{ij} be the angle between them, then [89]

$$\cos\theta_{ij} = \frac{k_{io}\,k_{jo}}{|\vec{K}_i|\,|\vec{K}_j|} \pm \frac{(k_i^2\,k_j^2)^{1/2}}{|\vec{K}_i||\vec{K}_j|}$$

$$= v_i\,v_j \pm \sqrt{[(v_i^2-1)(v_j^2-1)]}, \qquad (2.54)$$

where $v_i^2 = k_{io}^2/\vec{K}_i^2$ and $k_i^2 = k_{io}^2 - \vec{K}_i^2$.

For a time-like k_i, $k_i^2 > 0$ implies

$$k_{io}^2 - \vec{K}_i^2 > 0, \qquad (2.55)$$

giving

$$\sqrt{(k_{io}^2/\vec{K}_i^2)} = |v_i| > 1. \qquad (2.56)$$

Similarly for a space-like k_i, we have

$$|v_i| < 1. \qquad (2.57)$$

To ensure the consistency of the trigonometric function

$$|\cos \theta_{ij}| \leqslant 1. \tag{2.58}$$

Equation (2.54) now gives

$$-1 - v_i v_j \leqslant \pm \sqrt{[(v_i^2 - 1)(v_j^2 - 1)]} \leqslant 1 - v_i v_j. \tag{2.59}$$

For time-like k_i we have $|v_i| > 1$, implying

$$v_i v_j > 1$$

or

$$1 - v_i v_j < 0. \tag{2.60}$$

From (2.59) and (2.60) we find

$$(1 + v_i v_j)^2 \geqslant (v_i^2 - 1)(v_j^2 - 1) \geqslant (1 - v_i v_j)^2, \tag{2.61}$$

which gives

$$(v_i - v_j)^2 \leqslant 0 \text{ , for all i and j.} \tag{2.62}$$

For real values of v_i and v_j (2.62) is true only for the equality, implying

$$v_i = v_j \text{ for all i and j.} \tag{2.63}$$

Equations (2.54) and (2.63) yield,

$$\cos \theta_{ij} = 1 , \tag{2.64}$$

and thus

$$\theta_{ij} = 0. \tag{2.65}$$

Hence in 1+1 dimensions all vectors are necessarily parallel and therefore all are dependent and so the multi-solitary waves reduce to a single solitary wave. For a space-like k_i, this argument can be repeated and the result will not be contradicted.

2.IV. Linear stability of solutions in 1+1 dimensions

To investigate the linear stability of a particular solution of a NDE a small perturbation is applied to the solution, and examine whether or not this small perturbation grows with time. If the perturbation remains small enough, the non-linear equation that it obeys may be approximated by a linear equation. In this section we analyse the linear stability of the single solitary wave solution (2.31).

The static form of the single solitary wave solution (2.31) is

$$\phi_s(x) = \text{arc tanh}\left(\frac{\exp(kx)}{1 - \exp(2kx)/(2\eta-4)}\right). \tag{2.66}$$

Let

$$\phi(x,t) = \phi_s(x) + \phi_p(x,t) , \tag{2.67}$$

where $\phi_p(x,t)$ is a perturbation such that $|\phi_p| << 1$.

On substituting (2.67) in (2.14) we obtain,

$$\phi_{p,xx} - \phi_{p,tt} = \eta^2/2 \sinh[4(\phi_s + \phi_p)] - 2\eta \sinh[2(\phi_s + \phi_p)]$$
$$- [\eta^2/2 \sinh 4\phi_s - 2\eta \sinh 2\phi_s]. \tag{2.68}$$

By the linearity assumption $|\phi_p| << 1$, so

$$\phi_{p,xx} - \phi_{p,tt} = \phi_p[2\eta^2 \cosh 4\phi_s - 4\eta \cosh 2\phi_s]. \tag{2.69}$$

This is not of the same form as (2.14); nevertheless, it is linear and therefore easier to solve. Now consider a separable form of the solution,

$$\phi_p(x,t) = f(x) \exp(\lambda t). \tag{2.70}$$

This leads to the Schrödinger eigenvalue problem,

$$[-\frac{\partial^2}{\partial x^2} + V_0 \;(\phi_s) \;+(\lambda^2+2\eta^2-4\eta)]f(x) = 0 \qquad (2.71)$$

for the potential

$$V_0(\phi_s) \;=\; 2\eta^2 \cosh 4\phi_s - 4\eta \cosh 2\phi_s - (2\eta^2-4\eta) \qquad (2.72)$$
$$= V''(\phi_s) - (2\eta^2 - 4\eta) .$$

$V_0(\phi_s)$ is smooth and bounded and tends to zero as $x \longrightarrow \pm\infty$. Thus, there exists atmost a finite number of bound product solutions for which $|f| \longrightarrow 0$ as $x \longrightarrow \pm\infty$. But corresponding to the eigenvalue $\lambda^2+2\eta^2-4\eta=0$ there exists a non-zero eigenfunction f(x) given by

$$f(x,0) \;=\; \frac{\partial\phi_s(x)}{\partial x} . \qquad (2.73)$$

The nodes of f(x,0) are infinitely separated; so $\lambda = 0$ is the lowest eigenvalue [158]. This demonstrates the linear stability [34] of the solution (2.31).

2.V. Asymptotic behaviour of multisolitary wave solutions

The multisolitary wave solutions in more than 1+1 dimensions are of the form

$$\phi_N(x,t) = \text{arc sinh}\left(\frac{u_N}{\left[1 - \left(\frac{1-2\eta}{8-4\eta}\right)u_N^2 + \frac{u_N^4}{2(4-2\eta)^2}\right]^{1/2}}\right) \tag{2.74}$$

where

$$u_N = \sum_{j=1}^{N} a_j \exp(\alpha_j k_j x_j). \tag{2.75}$$

This can be seen to break up into N simple waves in the asymptotic regions. For as

$$\alpha_j k_j x_j \longrightarrow -\infty$$

$$\phi_N \longrightarrow \text{arc sinh } u_N \tag{2.76}$$

and as $\alpha_j k_j x_j \longrightarrow +\infty$, the dominant term in the braces of (2.74) is

$$u_N^2 \,/\, 2(4-2\eta)^2. \tag{2.77}$$

Consequently,

$$\phi_N \approx \text{arc sinh}(u_N/(4-2\eta)) \text{ as } \alpha_j k_j x_j \longrightarrow +\infty. \tag{2.78}$$

To calculate the phase shift we consider the j^{th} wave in the asymptotic regions,

$$\phi_j = \text{arc sinh}[a_j \exp(\alpha_j k_j x_j)] \text{ as } x \longrightarrow -\infty \qquad (2.79)$$

and

$$\phi_j \approx \text{arc sinh}[a_j \exp(-\alpha_j k_j x_j)/(4-2\eta)] \text{ as } x \longrightarrow +\infty . \qquad (2.80)$$

Defining the corresponding phases [89] as,

$$\delta_-^j = \log a_j \qquad (2.81)$$

$$\delta_+^j = \log [(4-2\eta)a_j^{-1}] \qquad (2.82)$$

the phase shift for the j^{th} wave is given by

$$\Delta_j = \delta_+^j - \delta_-^j \qquad (2.83)$$

$$\approx \log[(4-2\eta)/a_j^2]. \qquad (2.84)$$

The multisolitary wave solutions behave as if they were simple waves both at $-\infty$ and $+\infty$, and each component wave nearly undergoes a phase shift given by (2.84). However, there is no loss of stability for the multisolitary wave profile as a whole in the asymptotic regions. Similar behaviour has been noted [29] for KdV solitons in one space dimension.

2.VI. Topological charge

The conserved topological charge Q associated with a solitary wave in 1+1 dimensions is defined as

$$Q = \int_{-\infty}^{\infty} J^{o}\, dx \ , \tag{2.85}$$

where

$$J^{\mu} = \epsilon^{\mu\nu}\, \partial_{\nu} \phi \ , \tag{2.86}$$

and

$$\epsilon^{o1} = 1, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \ . \tag{2.87}$$

The Behera-Khare kink (eq.(2.15)) can be shown to possess a topological charge

$$Q = 2 \text{ arc tanh } \sqrt{[(2-\eta)/(2+\eta)]} \ , \ |\eta| < 2. \tag{2.88}$$

However, the solitary wave solutions reported herein are associated with vanishing topological charge and are, therefore, non-topological configurations [30].

Even though the multisolitary wave solutions which are defined in more than one space dimension possess very good stability properties, Derrick's theorem [38] does not permit them to possess finite energy.

3

SU(2) YANG-MILLS FIELDS WITH NO ENERGY TRANSPORT

3.I. Classical SU(2) Yang-Mills gauge theory

Non-abelian gauge theories of the type first introduced by Yang and Mills [57] play a very crucial role in currently popular models of electroweak and strong interactions. Although in principle, any compact Lie group can be used to construct a non-abelian gauge theory, the simplest SU(2) model [41] suffices to bring out several characteristic features of such theories.

The Lagrangian for a pure YM theory is

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{3.1}$$

The fields $F_{\mu\nu}$ are related to the potentials A_μ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e\,\epsilon_{abc} A_\mu A_\nu , \tag{3.2}$$

where e is the gauge coupling constant and the Levi-Civita symbol ϵ_{abc} represents the structure constants of the SU(2) group. The equation of motion is

$$D_\mu F^{\mu\nu} = 0, \tag{3.3}$$

with D_μ defining the covariant derivative,

$$D_\mu = \partial_\mu - e A_\mu . \tag{3.4}$$

A Minkowski-space SU(2) gauge field configuration is said to be self-dual if

$$\tilde{F}^a_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_a^{\alpha\beta} = \pm i F^a_{\mu\nu} , \tag{3.5}$$

or equivalently

$$i E^a_n = \pm B^a_n , \tag{3.6}$$

where

$$E^a_n \equiv F^a_{on} , \quad B^a_n = - \frac{1}{2} \epsilon_{nij} F^a_{ij} \tag{3.7}$$

are the SU(2) 'electric' and 'magnetic' YM fields. The Lagrangian (3.1) can be expressed as

$$L = \frac{1}{2} [E^a_n E^a_n - B^a_n B^a_n]. \tag{3.8}$$

This is minimized (or maximized) by a self-dual solution.

Quite a large variety of exact solutions of pure YM theory are known, the first explicit static solution of which was found by Ikeda and Miyachi [159]. This solution can be written in the form

$$A_o^a = \delta_{a3}\left(\frac{C}{r+D}\right) , \quad A_i^a = 0. \tag{3.9}$$

with C and D constants. For D = 0 this essentially represents a static, point-like Coulomb potential.

Through a specific Ansatz for the potential, there exists a useful connection between the SU(2) YM theory and the scalar ϕ^4 equation of motion. It was first discovered by 't Hooft [160] in connection with the instanton problem, which was further simplified by Corrigan and Fairlie [161] and Wilczek [162]. This Ansatz, quoted in Chapter 1, (see equations (1.87)-(1.89)) reduces the equation of motion (3.3) of the SU(2) YM theory to a single equation of motion for the ϕ^4 scalar theory

$$\Box\phi + \lambda\phi^3 = 0. \tag{3.10}$$

The YM field strengths in Minkowski space take the following form:

$$\begin{aligned} e\,E_n^a &\equiv e\,F_{on}^a \\ &= \epsilon_{nam}\left[\frac{1}{\phi}\,\partial_o\partial_m\phi - \frac{2}{\phi^2}\,\partial_o\phi\;\partial_m\phi\right] \\ &\pm i\,\delta_{na}\left[\frac{1}{\phi}\,\partial_o^2\phi - \frac{1}{\phi^2}(\partial_o\phi\;\partial_o\phi + \partial_m\phi\;\partial_m\phi)\right] \\ &\pm i\left[\frac{1}{\phi}\,\partial_n\partial_a\phi - \frac{2}{\phi^2}\,\partial_n\phi\;\partial_a\phi\right], \end{aligned} \tag{3.11}$$

$$e B_n^a = -\frac{1}{2} e \epsilon_{nij} F_{ij}^a$$

$$= \pm ie E_n^a + \delta_{an} \frac{1}{\phi} \Box \phi. \qquad (3.12)$$

The Lagrangian density is given by

$$2e^2 L = \Box \partial_\alpha (\partial^\alpha \phi / \phi) - 3 \lambda^2 \phi^4. \qquad (3.13)$$

The pseudoscalar density D(x) and L are related:

$$e^2 (L \pm D) = \frac{1}{2} e^2 (B_n^a - i E_n^a)^2$$

$$= \frac{3}{2} (\Box \phi / \phi)^2, \qquad (3.14)$$

where the $\pm$ signs correspond to selfduality and anti-selfduality, respectively, of the gauge fields. The selfduality condition in Minkowski space

$$B_n^a = \pm i E_n^a, \qquad (3.15)$$

(the + sign corresponds to selfduality and − sign corresponds to antiselfduality)

yields

$$D = \pm L \qquad (3.16)$$

or

$$\Box \phi = 0, \qquad (3.17)$$

or $\phi = 0$

(Non-selfduality implies violation of (3.15)) [41].

The energy momentum tensor $\theta_{\mu\nu}(x)$ is

$$\theta_{\mu\nu} = - F^a_{\mu\lambda} F^{a\lambda}_{\nu} + \frac{1}{4} g_{\mu\nu} F^a_{\alpha\beta} F^{\alpha\beta}_a . \quad (3.18)$$

In component form

$$\theta_{oo} = \frac{1}{2}(E^a_n E^a_n + B^a_n B^a_n) \quad (3.19)$$

$$\theta_{oj} = - \epsilon_{jmn} E^a_m B^a_n \quad (3.20)$$

$$\theta_{ij} = - E^a_i E^a_j - B^a_i B^a_j + \delta_{ij} \frac{1}{2} (E^a_n E^a_n + B^a_n B^a_n) . \quad (3.21)$$

In terms of the scalar field $\emptyset$, (3.18) becomes

$$e^2 \theta_{\mu\nu} = \frac{\Box \emptyset}{\emptyset} [\frac{4}{\emptyset^4} \partial_\mu \emptyset \partial_\nu \emptyset - \frac{2}{\emptyset} \partial_\mu \partial_\nu \emptyset$$

$$+ g_{\mu\nu} (\frac{1}{2\emptyset} \Box \emptyset - \frac{1}{\emptyset^2} \partial^\alpha \emptyset \partial_\alpha \emptyset)] . \quad (3.22)$$

For selfduality the condition (3.17) implies

$$\theta_{\mu\nu} = 0. \quad (3.23)$$

This may be realized in two distinct ways. Thus, when either $\Box \phi = 0$ or when the expression in square bracket in (3.22) vanishes, then the solution becomes selfdual.

The total energy of the YM field is then

$$e^2 E = \int d^3x \, e^2 \theta_{oo}$$

$$= - 6\lambda \int d^3x [\frac{1}{2}(\partial_o \emptyset)^2 + \frac{1}{2}(\nabla \emptyset)^2 + \frac{1}{4} \lambda \emptyset^4] . \quad (3.24)$$

In Euclidean space the YM field strengths are given by

$$e\,E^a_n = \epsilon_{nam}\,[\tfrac{1}{\phi}\,\partial_o\partial_m\phi - \tfrac{2}{\phi^2}\,\partial_o\phi\,\partial_n\phi]$$

$$\pm\,\delta_{na}\,[\tfrac{1}{\phi}\,\partial_o^2\phi - \tfrac{1}{\phi^2}\,(\partial_o\phi\,\partial_o\phi - \partial_m\phi\,\partial_m\phi)]$$

$$\pm\,[\tfrac{1}{\phi}\,\partial_a\partial_n\phi - \tfrac{2}{\phi^2}\,\partial_a\phi\,\partial_n\phi]\,, \tag{3.25}$$

$$e\,B^a_n = \pm\,E^a_n - \delta_{na}\,\tfrac{1}{\phi}\,\Box\,\phi\,. \tag{3.26}$$

The selfduality condition is

$$E^a_n = \pm\,B^a_n, \tag{3.27}$$

Non-selfduality implies violation of (3.27).
The Lagrangian density L is given by

$$-\,L = \pm\,D - \frac{3}{2e^2}\,(\Box\phi/\phi)^2, \tag{3.28}$$

where the two signs correspond to selfduality and anti-selfduality cases, respectively.

The pseudoscalar density

$$D = \pm\,\frac{1}{2e^2}\,\partial_\mu\,[\Box(\partial_\mu\phi/\phi)]$$

$$= -\,E^a_n\,B^a_n\,. \tag{3.29}$$

The expression for the energy momentum tensor $\theta_{\mu\nu}$ is the same as in Minkowski space.

The recent flurry of activity in exploring the classical sector of YM theory has unravelled a rich variety of nontrivial classical solutions having striking implications at the quantum level [41]. Both the selfdual and anti-selfdual classes of solutions are equally important.

It is possible to classify solutions of YM theory into propagating and non-propagating types, based on their velocity of transport [163,164]. ~~The non-propagating (zero velocity) solutions may be considered~~ . A self-dual solution then turns out to be a YM field with zero energy transport, because the momentum density vanishes. This point seems to have been overlooked by Kovacs and Lo [165] while reporting a set of selfdual and anti-selfdual solutions in SU(2) YM theory.

In the present work we present a class of anti-selfdual and non-selfdual solutions which can be interpreted as non-linear YM field with no energy transport. They are singular, infinite action configurations whose actual role at the quantum level, is not clear. Their contribution to the exact functional integral could be different from zero, and in the infrared region may even be more important than that of any finite action configuration [166].

3.II. Application of the bilinear method

Non-singular solutions of (3.10) in terms of elliptic functions have recently been used to construct progressive wave solutions of YM theory [164,167]. Employing Hirota's bilinear operator method, a class of singular solutions can be obtained. Define a transformation

$$\phi = g/f \tag{3.30}$$

which converts equation (3.10) into the bilinear form:

$$f\, D_\mu D^\mu\, g\cdot f - g\, D_\mu D^\mu\, f\cdot f + \lambda g^3 = 0, \tag{3.31}$$

where

$$D_\mu D^\mu = D_t^2 - D_x^2 - D_y^2 - D_z^2 , \tag{3.32}$$

D_ω^2 being a Hirota-type bilinear operator.

On splitting, (3.31) yields,

$$D_\mu D^\mu\, g\cdot f = 0 , \tag{3.33}$$

$$D_\mu D^\mu\, f\cdot f = \lambda g^2. \tag{3.34}$$

We introduce power series expansions of g and f in a parameter ϵ

which is assumed to be close to unity:

$$g = g_o + \epsilon g_1 + \epsilon^2 g_2 + \ldots \tag{3.35}$$

$$f = f_o + \epsilon f_1 + \epsilon^2 f_2 + \ldots \tag{3.36}$$

where the g_n and f_n are functions to be suitably determined.

Introducing these expansions into (3.33) and (3.34), the following equations emerge:

$$\begin{aligned}
&f_o \Box f_o - [(\partial_o f_o)^2 - (\partial_i f_o)^2] = 0 \\
&g_1 \Box f_o + f_o \Box g_1 + 2(\partial_i f_o)(\partial_i g_1) - 2(\partial_o f_o)(\partial_o g_1) = 0 \\
&f_o \Box f_1 + f_1 \Box f_o + 2(\partial_i f_o)(\partial_i f_1) - 2(\partial_o f_o)(\partial_o f_1) = 0 \\
&g_1 \Box g_1 - (\partial_o g_1)(\partial_o g_1) + (\partial_i g_1)(\partial_i g_1) = 0 \\
&2f_1 \Box f_1 - 2(\partial_o f_1)(\partial_o f_1) + 2(\partial_i f_1)(\partial_o f_1) - \lambda g_1^2 = 0 \, , \\
&\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots \\
&\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots
\end{aligned} \tag{3.37}$$

where $\Box = \partial_t^2 - \nabla^2$. A consistent solution for the system of equations (3.37) is then obtained as follows:

$$g_1 = \pm B \tag{3.38}$$

$$f_o = C \tag{3.39}$$

$$f_1 = \pm kx , \tag{3.40}$$

provided,

$$g_o = 0 = g_n = 0 \quad \text{for all } n \geqslant 2 \tag{3.41}$$

$$f_n = 0 \quad \text{for all } n \geqslant 2 , \tag{3.42}$$

where B and C are arbitrary non-negative real constants, and

$$k \equiv (k_o, \vec{K}_i) \quad \text{and} \quad x \equiv (x_o, \vec{X}_i). \tag{3.43}$$

With ϵ being set equal to unity, we arrive at the solution

$$\phi = \pm B/(C \pm kx). \tag{3.44}$$

The associated dispersion relation is

$$k^2 = -\lambda B^2/2. \tag{3.45}$$

The solution conjectured earlier [165] corresponds to the case, $\pm B = 1$, with $\lambda = -2$.

3.III Yang-Mills fields with no energy transport

The solution of the auxiliary ϕ^4 equation we have obtained (see eq.(3.44)) is singular on the hyperplane $B \pm kx = 0$ and yields YM fields exhibiting analogous behaviour. The YM potentials corresponding to the solution (3.44) in Minkowski space are

$$e A_o^a = \pm i \frac{k_a}{B} \phi \qquad (3.46)$$

$$e A_i^a = \epsilon_{ian} k_a \phi/B \pm i \delta_{ai} k_o \phi/B. \qquad (3.47)$$

The corresponding electric and magnetic fields are then

$$e E_n^a \equiv e F_{on}^a$$

$$= \pm i \, \delta_{na} (-\lambda/2) \phi^2 , \qquad (3.48)$$

$$e B_n^a \equiv -\frac{1}{2} e \, \epsilon_{nij} F_{ij}^a$$

$$= \pm \delta_{na} \lambda \phi^2/2 - \delta_{na} \lambda \phi^2. \qquad (3.49)$$

It can be easily shown that the scalar field solution (3.44) is also valid in Euclidean space. Using the Euclidean version of the potentials (1.89) we have

$$e\, A^a{}_o \;=\; \pm\, k_a \emptyset/B \qquad (3.50)$$

$$e\, A^a_i \;=\; -\, \epsilon_{ian}\, k_n \emptyset/B \;\pm\; \delta_{ai}\, k_o \emptyset/B\,. \qquad (3.51)$$

The corresponding electric and magnetic fields are

$$e\, E^a_n \;=\; \pm\, \delta_{na}(-\lambda/2)\emptyset^2\,, \qquad (3.52)$$

$$e\, B^a_n \;=\; \pm\, \delta_{na}(-\lambda/2)\emptyset^2 + \delta_{na}\,\lambda\,\emptyset^2. \qquad (3.53)$$

The square of the electric field in Minkowski space is

$$E^a_n\, E^a_n \;=\; -(3\,\lambda^2/4e^2)\emptyset^4\,, \qquad (3.54)$$

while in Euclidean space it only differs in sign:

$$E^a_n\, E^a_n = \quad (3\,\lambda^2/4e^2)\emptyset^4. \qquad (3.55)$$

For the selfdual fields in Minkowski space we consider the plus sign of the first term in (3.49), giving

$$e\, B^a_n \;=\; \delta_{na}(-1/2)\,\lambda\,\emptyset^2$$

$$\;=\; -ie\, E^a_n\,, \qquad (3.56)$$

while in Euclidean space,

$$e\,B_n^a = -e\,E_n^a\,, \tag{3.57}$$

which is incidentally the anti-selfdual condition in Euclidean space.

The pseudoscalar density D(x) corresponding to this in Minkowski space is given by

$$D(x) = -\,i\,E_n^a\,B_n^a$$

$$= (3\,\lambda^2/4e^2)\phi^4\,, \tag{3.58}$$

and in Euclidean space

$$D(x) = -\,E_n^a\,B_n^a$$

$$= (3\,\lambda^2/4e^2)\phi^4. \tag{3.59}$$

If we consider the negative sign in (3.49) we obtain

$$e\,B_n^a = 3\,\delta_{na}(-1/2)\lambda\,\phi^2$$

$$= 3\,ie\,E_n^a. \tag{3.60}$$

This is the non-selfdual condition in Minkowski space. Similarly taking the negative sign in (3.51), the non-selfdual condition

$$e\, B_n^a \;=\; 3e\, E_n^a \tag{3.61}$$

in Euclidean space is satisfied.

These results imply that the fields can be non-selfdual in Minkowski as well as Euclidean space. The pseudoscalar density corresponding to the non-selfdual field in both Euclidean and Minkowski spaces are of the same form:

$$D(x) \;=\; -(9\,\lambda^2/4e^2)\phi^4. \tag{3.62}$$

Obviously, the energy momentum tensor, $\theta_{\mu\nu}$, vanishes for the anti-selfdual case, whereas when the fields are non-selfdual we have the components of energy momentum tensor:

$$\theta_{oo} \;=\; -(3\,\lambda^2/e^2)\phi^4 \tag{3.63}$$

$$\theta_{ij} \;=\; -\,\delta_{ij}(5\,\lambda^2/e^2)\phi^4 \tag{3.64}$$

$$\theta_{oj} \;=\; 0. \tag{3.65}$$

It may be noted that the selfdual or anti-selfdual solutions of Kovacs and Lo [165] correspond to the choice of the plus sign or minus sign in the denominator of the solution (3.44), whereas, in the present work the corresponding property, namely anti-selfduality or non-selfduality, does not depend on the choice of the sign of kx in (3.44).

Following Brillouin [163], a flux velocity v_j can be defined [153]:

$$v_j = \theta_{oj}/\theta_{oo}. \tag{3.67}$$

The flux velocity v_j is identically zero in the present case. This implies that the YM solutions herein obtained are non-propagating waves, and are best described as localized fields. Nevertheless, a non-vanishing phase velocity can be associated with these waves, as seen from the dispersion relation (3.45) which remains valid in Euclidean space as well. For $\lambda < 0$, the phase velocity is greater than c and for $\lambda > 0$ it is less than c, where c is the velocity of light.

Since the pseudoscalar density D(x) is singular the topological charge becomes infinite [41]. An interesting property of the YM fields is their singular behaviour on the hyperplane $B \pm kx = 0$. That this is an essential feature of the solution is clear from the observation that any gauge invariant quantity, such as θ_{oo}, exhibits this singularity. The hyperplane singularity of the YM fields, like the Dirac string [168], is not an artifact of the gauge. Nevertheless, there is no magnetic monopole attached to the YM fields.

Though the Minkowski form of the YM fields is complex, the square of the electric field is real. Being of infinite action and infinite topological charge it would be tempting to compare these localized fields with merons which arise in Euclidean space and are believed to correspond to tunnelling between different vacuua in real time, characterized by topological charges n = 0 and n = 1/2, respectively. The YM fields with their infinite topological charge may be assigned a tunnelling role in Minkowski space, between the vacuua n = 0 and n = ∞.

4

COMPOSITE MAPPING METHOD FOR GENERATION OF SOLUTIONS IN THE KLEIN-GORDON FAMILY

4.I. Relationships between KG type equations

Equations of the KG family featuring scalar fields possess Lorentz invariance as a common property. This suggests the possibility of mapping a particular solution of one such equation into that of another, of the same family. For selected pairs of KG type equations this procedure has been successfully implemented before [88,169], constituting applications of the base equation technique [77-89]. The recent identification of a large variety of NDEs in the KG family possessing kink or soliton solutions with interesting properties and physical applications calls for a more detailed exploration of the base equation method. Lorentz invariance being a common trait of such equations, there can exist relationships between different members of the KG family, realizable in the form of maps between particular solutions of classes of KG type equations. Naturally, for a specified class of such equations, this would imply a multiplicity of maps or a composition of maps which generates particular solutions.

In the present chapter extensive use of the technique of composite maps is made to produce kinks and solitons in the KG family. Specifically, this approach is applied to equations such as sG, DsG, ϕ^4, Liouville and ϕ^6. The non-linear correspondence between particular solutions of some of the members of the KG family through a transformation in terms of the arc sine [88] or arc tangent [169] function has been studied in the literature. We combine both of these transformations and develop a non-linear composite map that takes solutions of one KG equation to those of two other equations of the same family. A solution of the original KG equation is found by solving the transformed equations simultaneously. The results obtained expose several 'family relationships' existing within the KG family.

4.II. Solution of ϕ^4 equation by bilinear operator method

Since some solutions of the ϕ^4 equation play an important role in our programme of composite maps, we obtain them here by the bilinear operator method [90-98]. We write the scalar field ϕ satisfying the massive ϕ^4 equation,

$$\partial_\mu \partial^\mu \phi + \alpha\phi + \beta\phi^3 = 0 , \quad \mu=0,1,2,\ldots n. \tag{4.1}$$

as the quotient of two functions

$$\phi = g/f. \tag{4.2}$$

This leads to the bilinear differential equation

$$f\, D_\mu D^\mu g\cdot f + \alpha\, g\, f^2 - g\, D_\mu D^\mu f\cdot f + \beta g^3 = 0, \tag{4.3}$$

where

$$D_\mu D^\mu = D_t^2 - \sum_{i=1}^{n} D_{x_i}^2, \tag{4.4}$$

and D_ω^2 is a Hirota-type bilinear operator.

On splitting (4.3), we have

$$D_\mu D^\mu \; g \cdot f + \alpha \, g \, f = 0, \tag{4.5}$$

$$D_\mu D^\mu \; f \cdot f + \beta \, g \, g = 0. \tag{4.6}$$

Expanding g and f as power series,

$$g = g_o + \epsilon \, g_1 + \epsilon^2 g_2 + \ldots. \tag{4.7}$$

$$f = f_o + \epsilon \, f_1 + \epsilon^2 f_2 + \ldots. \tag{4.8}$$

yields a set of equations for the g_i and f_i, with the conditions

$$g_j = 0 \; ; \quad f_j = 0, \quad \text{for all } j \geqslant 2. \tag{4.9}$$

Choosing

$$g_o = f_o = 0, \tag{4.10}$$

the solution

$$g_1 = \pm \sqrt{(2\alpha/\beta)} \tag{4.11}$$

$$f_1 = 2\exp(\theta) / (\exp(2\theta) - 1), \tag{4.12}$$

follows, where $\theta = k_o t - \sum_{j=1}^{n} k_j x_j + \delta$. Assuming a dispersion relation

$$\sum_{j=1}^{n} k_j^2 - k_o^2 = \alpha , \tag{4.13}$$

we find the solution

$$\phi = \pm \sqrt{(2\alpha/\beta)}\ 2\exp(\theta) / [\exp(2\theta) - 1]. \tag{4.14}$$

This is a new solution which turns out to be singular at $\theta = 0$. However, as discussed later, this singularity gets eliminated under suitable transformation to other non-linear equations like the DsG and ϕ^6.

A known multi-dimensional solution reported earlier [169] can also be obtained by this procedure. Setting

$$g_o = 0; \quad f_1 = 1. \tag{4.15}$$

Equation (4.5) yields

$$\partial_\mu \partial^\mu g_1 + \alpha g_1 = 0. \tag{4.16}$$

A simple solution for this equation is

$$g_1 = \exp(\theta). \tag{4.17}$$

When we set

$$g_j = 0 \; ; \; f_j = 0 \quad \text{for all } j \geqslant 2, \tag{4.18}$$

(4.6) gives

$$f_1 = (-\beta/8\alpha)\exp(2\theta).$$

Then an exact solution of ϕ^4 equation is,

$$\phi = \pm \exp(\theta)/[1 - (\beta/8\alpha)\exp(2\theta)]. \tag{4.19}$$

If we change α to $-\alpha$ in (4.1), a non-singular soliton-like solution (i.e., asymptotically vanishing),

$$\phi = \pm \exp(\theta)/[1 + (\beta/8\alpha)\exp(2\theta)], \tag{4.20}$$

is obtained.

Setting $\alpha = 0$ in (4.1) gives a solution of massless ϕ^4 equation:

$$\phi = \pm A/(B \pm \theta) , \tag{4.21}$$

(A and B are arbitrary constants) which coincides with (3.45), obtained by the bilinear method.

4.III. Maps from ϕ^4 to sG and Liouville's equations

As a first example of the composite mapping method, we shall consider maps from ϕ^4 to sG and Liouville equations. The ϕ^4 and sG systems are two model field theories widely employed in different branches of theoretical physics and adequate references to these models have been given in the preceding chapters. The Liouville equation was introduced by Liouville [163] in 1853 and has a variety of applications such as in the Lagrange stream function model for two dimensional steady vortex motion of an incompressible fluid [170], the theory of thermionic emission and the problem of the isothermal gas sphere [171]. It has been noticed that several coupled equations in modern gauge theories reduce to the one or two dimensional Liouville equation [41,172].

The sG equation in arbitrary dimensions may be written in the form

$$\partial_\mu \partial^\mu \psi + \alpha \sin \psi = 0. \tag{4.22}$$

The non-linear map

$$\psi = 2 \text{ arc sin } \phi \tag{4.23}$$

transforms (4.22) into

$$[1-\phi^2]\, \partial_\mu \partial^\mu \phi + \phi\, \partial_\mu \phi\, \partial^\mu \phi + \alpha\phi[1-\phi^2]^2 = 0. \tag{4.24}$$

A suitable splitting of (4.24) gives ϕ^4,

$$\partial_\mu \partial^\mu \phi + \alpha\phi - 2\alpha\phi^3 = 0 , \tag{4.25}$$

and the constraint:

$$\partial_\mu \phi\, \partial^\mu \phi + \alpha\phi^2 - \alpha\phi^4 = 0. \tag{4.26}$$

Choosing ϕ as in (4.14) where β is replaced by -2α gives a complex solution of sG equation:

$$\psi = 2 \text{ arc sin}[\pm 2i \exp(\theta)/(1+ \exp(2\theta))]. \tag{4.27}$$

Choosing ϕ as in (4.19) with the replacement $\beta \longrightarrow -2\alpha$, gives an analytic solution of sG equation:

$$\psi = 2 \text{ arc sin}[\pm \exp(\theta)/(1+ \exp(2\theta)/4)]. \tag{4.28}$$

The Liouville equation in arbitrary dimensions may be written,

$$\partial_\mu \partial^\mu \psi + \beta e^{2\psi} = 0. \tag{4.29}$$

Under the transformation

$$\psi = \log \phi, \tag{4.30}$$

we obtain the massive ϕ^4, and the constraint equations:

$$\partial_\mu \partial^\mu \phi + \alpha\phi + 2\beta\phi^3 = 0 \tag{4.31}$$

$$\partial_\mu \phi \, \partial^\mu \phi + \alpha\phi^2 + \beta\phi^4 = 0, \tag{4.32}$$

where α can be zero, or a real constant. Using (4.14) we obtain an exact solution of the Liouville equation for $\alpha \neq 0$:

$$\psi = \log [2\sqrt{(\alpha/\beta)} \exp(\theta)/(\exp(2\theta)-1)]. \tag{4.33}$$

Inserting (4.20) and replacing α by $-\alpha$ gives another solution of (4.29):

$$\psi = \log [\exp(\theta)/(1+(\beta/4\alpha)\exp(2\theta))], \text{ where } \alpha \neq 0. \tag{4.34}$$

Setting $\alpha = 0$ in (4.31), the resulting equation is analogous to massless ϕ^4. From (4.21) we can find the third independent solution of the Liouville equation:

$$\psi = \log [\pm A/(B \pm \theta)], \tag{4.35}$$

provided $k^2 = -\beta A^2$.

4.IV. The composite map : $\phi^4 \longrightarrow$ DsG $\longrightarrow \phi^6$

It has been shown that [173] the ϕ^6 model has soliton-like solutions in 1+1 dimensions. The DsG equation involving arbitrary parameters α and β,

$$\partial_\mu \partial^\mu \psi + \alpha \sin \psi + \beta \sin(\psi/2) = 0, \tag{4.36}$$

transforms under the map,

$$\psi = 4 \arc \tan \phi \tag{4.37}$$

into:

$$2[1+\phi^2]\partial_\mu \partial^\mu \phi - 4\phi\partial_\mu \phi \, \partial^\mu \phi + (2\alpha+\beta)\phi + (-2\alpha+\beta)\phi^3 = 0. \tag{4.38}$$

On splitting this equation, there emerge the ϕ^4 equation

$$\partial_\mu \partial^\mu \phi + (\alpha + \beta/2)\phi + \beta\phi^3 = 0, \tag{4.39}$$

and a constraint:

$$\partial_\mu \phi \, \partial^\mu \phi + (\alpha + \beta/2)\phi^2 + \beta/2 \, \phi^4 = 0 \qquad (4.40)$$

Inserting the form given in (4.14) with the replacement $\alpha \longrightarrow (\alpha + \beta/2)$, a '0π-pulse' like solution of the DsG equation is obtained:

$$\psi = 4 \text{ arc } \tan[\pm 2\sqrt{(2\alpha+\beta)/\beta}\,]\exp(\theta)/(\exp(2\theta)-1)] \qquad (4.41)$$

Also corresponding to (4.19) we get another solution:

$$\psi = 4 \text{ arc } \tan\left(\frac{\pm \exp(\theta)}{1-[\beta/(8\alpha+4\beta)]\exp(2\theta)}\right). \qquad (4.42)$$

The dispersion relation associated with both these solutions is

$$\sum_{j=1}^{n} k_j^2 - k_o^2 = \alpha + \beta/2. \qquad (4.43)$$

For $\alpha = -\beta/2$, the massless ϕ^4 equation follows from (4.30). The corresponding DsG equation is

$$\partial_\mu \partial^\mu \psi - \beta/2 \sin\psi + \beta \sin(\psi/2) = 0. \qquad (4.44)$$

Using (4.21) we obtain two pairs of '0π-pulse' like solutions of DsG equation (4.44):

$$\psi = 4 \text{ arc } \tan[\pm A/(B \pm \theta)]. \tag{4.45}$$

By the non-linear transformation

$$\psi = 4 \text{ arc } \sin \phi , \tag{4.46}$$

one can pass from the DsG equation (4.36) to the equation

$$2(1-\phi^2)\partial_\mu \partial^\mu \phi + 2\phi\, \partial_\mu \phi\, \partial^\mu \phi + 2\alpha\phi(1-\phi^2)^2 (1-2\phi^2) + \beta\phi(1-\phi^2)^2 = 0. \tag{4.47}$$

This yields the ϕ^6 model defined by

$$\partial_\mu \partial^\mu \phi + (\alpha+\beta/2)\phi - (4\alpha+\beta)\phi^3 + 3\alpha\, \phi^5 = 0 \tag{4.48}$$

plus the constraint:

$$\partial_\mu \phi\, \partial^\mu \phi + (\alpha+\beta/2)\phi - (2\alpha+\beta/2)\phi^4 + \alpha\, \phi^6 = 0. \tag{4.49}$$

The DsG solution (4.41) now yields an exact solution of ϕ^6:

$$\phi = \pm \frac{2\sqrt{(2\alpha+\beta)\exp(\theta)}}{\sqrt{[\beta+(8\alpha+2\beta)\exp(2\theta)+\beta\exp(4\theta)]^{1/2}}}. \tag{4.50}$$

The associated dispersion relation is

$$\sum_{j=1}^{n} k_j^2 - k_o^2 = (2\alpha + \beta)/2. \qquad (4.51)$$

The solution (4.50) can be interpreted as a nontopological soliton-like configuration.

Corresponding to (4.42) we get another solution of the ϕ^6 equation (4.48):

$$\phi = \frac{\pm \exp(\theta)}{\left\{[1-\beta \exp(2\theta)/8(\alpha+\beta/2)]^2 + \exp(2\theta)\right\}^{1/2}}. \qquad (4.52)$$

The associated dispersion relation is the same as (4.51).

4.V. The composite map: $\phi^2 \longrightarrow sG \longrightarrow \phi^4$

Starting from solutions of the linear KG or ϕ^2 equation, a pair of maps can be constructed yielding solutions of sG and ϕ^4 equations. A solution of the ϕ^2 equation:

$$\phi = e^{\theta}, \qquad (4.53)$$

where θ is defined as in (4.12), can be used to obtain a solution of sG equation in the following manner. The non-linear map (4.37) converts sG equation (4.22) into:

$$[1+\phi^2]\partial_\mu\partial^\mu\phi \; - 2\phi\, \partial_\mu\phi\, \partial^\mu\phi \; + \alpha\phi[1-\phi^2] \;=\; 0\cdot \qquad (4.54)$$

On splitting this gives

$$\partial_\mu\partial^\mu\phi \;+ \alpha\phi \;=\; 0 \qquad (4.55)$$

$$\partial_\mu\phi\, \partial^\mu\phi \;+ \alpha\phi^2 \;=\; 0. \qquad (4.56)$$

Inserting (4.53), which is a solution of ϕ^2, the well known solution of the sG, namely:

$$\psi \;=\; 4 \text{ arc tan } [\exp(\theta)] \qquad (4.57)$$

follows. Using (4.23) we obtain a well behaved soliton-like solution of the ϕ^4 equation (4.25):

$$\phi \;=\; 2 \exp(\theta)/[1+\exp\,(2\theta)]. \qquad (4.58)$$

4.VI. Multisolitary wave solutions

A fact of some importance is that N solitary wave solutions or multisolitary wave solutions can easily be constructed for cases where e^θ appears, by the replacement

$$e^\theta \longrightarrow \sum_{j=1}^{N} e^{\theta_j}\,, \qquad (4.59)$$

where $\theta_j = k_{oj}t - \sum_{i=1}^{n} k_{ij}x_i + \delta_j$, $j=1,2....N$, and by imposing the additional constraints

$$(k_{ij} - k_{s\ell})^2 - (k_{oj} - k_{o\ell})^2 = 0, \qquad (4.60)$$

where i, s = 1,2...n, j, ℓ = 1,2,...N. As discussed in Chapter 2, the relation (4.60) restricts the number of solitary wave solutions according to $N \leq (2n-1)$, where n is the space-time dimensionality. However, in 1+1 dimensions there will exist only one solitary wave as there is only one independent wave vector and any other vector is necessarily parallel to it.

4.VII. Discussion

The composite mapping method has been shown to be a powerful tool for exposing family relationships among non-linear differential equations of importance to physics and also for generating kink and soliton-like solutions. Of the several new solutions herein reported, the solution (4.45) for DsG and solutions (4.50) and (4.52) for ϕ^6, are of special interest. The other known solutions for the DsG collapse to a single kink or anti-kink in 1+1 dimensions [89]. Our solution (4.45) is an exception to this behaviour. Since all the four distinct solutions specified by (4.45) can be simultaneously constructed for given values of the parameter β, it should be possible to study their interactions

Another interesting feature is that when the parameters α and β in (4.36) are varied, the solution (4.41) disappears at $\alpha = -\beta/2$, but the solution (4.45) arises precisely at this point in such a manner that corresponding to a kink or antikink given by (4.41), there are now two pairs of '0π-pulse' like solutions given by (4.45) of the DsG equation. Similar phenomena have been studied for the KdV system [174,175], in which the varied parameter is the depth.

The ϕ^6 model, apart from being a classical field theory in its own right, is a model of the first order ferroelectric phase transition discussed in condensed-matter physics [176], whose finite-temperature behaviour has recently been studied [177]. The only other known (time-dependent) soliton-like solution of the ϕ^6 equation is that reported in Ref [89].

When the composite mapping method is applied to the hyperbolic counterparts of the KG models discussed here it is found that resulting solutions are all singular.

5

SYMMETRY CLASSIFICATION OF SOLUTIONS OF NON-LINEAR KLEIN-GORDON EQUATIONS

5.I. The similarity approach

It is now clear that Lie's point transformation theory has been one of the most outstanding attempts to study continuous symmetry, particular solutions, and dimensional reduction of NPDEs [175]. Recently the importance of similarity solutions has been stressed by many authors in different contexts [116-121]. Similarity solutions are believed to correspond to solutions belonging to the continuous spectrum part of the IST solvable equations. In early studies of soliton concepts, it was merely shown that similarity solutions of the KdV equation satisfied a third order non-linear ODE [178,179]. Later several investigations were carried out to identify integrable systems using similarity transformations.

In classical mechanics the so called Painlevé property (PP) serves to distinguish between integrable and non-integrable systems. Such connections have been studied by the similarity reduction of PDEs to ODEs using infinitesimal transformations. For example, the sG equation in 1+1 dimensions can be reduced to PIII [180], whereas the KdV [181] and modified KdV [182] are reduced to PII and the Boussinesq [183] is reduced to PI. This study has also been extended to multidimensional systems [184-187].

In this chapter we study certain group-theoretic properties of solutions of 1+1 dimensional non-linear KG equations. In general the similarity transformations form an extended group, the similarity group, which upon a suitable redefinition of the generators, leads to the Poincaré group in two dimensions. This suggests a three-fold classification of solutions of two dimensional KG equations into translation invariant (TI) hyperbolic rotation (boost) invariant (HRI) and similarity invariant (SI) types. Here the phrase 'similarity invariant' is used in the sense of invariance under the full similarity group. Such a description, which focuses on the behaviour of the solutions rather than that of the equation, is physically important in the sense of establishing links with certain conservation laws, whenever such laws exist.

The similarity reduced equation obtained from a two dimensional KG equation is an ODE which may or may not possess PP. When the reduced equation is of P-type, it would be reasonable to seek the origin of the PP in one of the restricted symmetry classes like translation or hyperbolic rotation.

Most of the known solutions of KG equations are of the TI type (i.e., solitary waves). In five different cases of non-linearity HRI solutions are obtained, most of them new. In the sG case, we find that the PP arises from hyperbolic rotation invariance. The group-theoretical meaning of the Pinney-Reid-Burt base equation method [89] of solving NPDEs is also examined. We point out that the similarity groups of the given equation, the base equation and the constraint are identical.

5.II. Similarity group of Klein-Gordon type equations

If a KG type differential equation in 1+1 dimensions

$$u_{xx} - u_{tt} = F(u) \tag{5.1}$$

remains invariant under the infinitesimal transformations

$$\begin{aligned} x^* &= x + \epsilon X(x,t,u) + O(\epsilon^2) \\ t^* &= t + \epsilon T(x,t,u) + O(\epsilon^2) \\ u^* &= u + \epsilon U(x,t,u) + O(\epsilon^2) , \end{aligned} \tag{5.2}$$

then

$$u^*_{x^*x^*} - u^*_{t^*t^*} = F(u^*) \quad . \tag{5.3}$$

The derivatives u_{xx} and u_{tt} transform according to

$$u^*_{x^*x^*} = u_{xx} + \epsilon\,[U_{xx}] + O(\epsilon^2)$$

$$u^*_{t^*t^*} = u_{tt} + \epsilon\,[U_{tt}] + O(\epsilon^2) \;, \tag{5.4}$$

where the bracketed symbols denote the <u>extensions</u>.

Equating coefficients of first order in ϵ , in (5.3) ~~we~~ find

$$[U_{xx}] - [U_{tt}] = \frac{\partial F}{\partial u}(u). \tag{5.5}$$

The extension $[U_{xx}]$ is calculated to be of the form:

$$[U_{xx}] = U_{xx} + 2(U_{xu} - X_{xx})\,u_x + (U_{uu} - 2X_{xu})u_x^2 - X_{uu}u_x^3$$

$$+ (U_u - 2X_x)u_{xx} - 3X_u\,u_{xx}\,u_x - T_{xx}\,u_t$$

$$- 2T_{xu}\,u_x\,u_t - T_{uu}\,u_x^2\,u_t - 2T_x\,u_{xt}$$

$$- T_u\,u_{xx}\,u_t - 2T_u\,u_{xt}\,u_x.$$

The corresponding expression for $[U_{tt}]$ can be obtained from this by the permutations (x,t) and (X,T).

Substituting for the extensions in (5.5) and equating coefficients of different orders of derivatives of u to zero, the following constraints are obtained.

$$U_{xx} - U_{tt} = 0, \qquad 2\,U_{tu} - T_{tt} + T_{xx} = 0$$

$$X_{xx} - 2\,U_{xu} - X_{tt} = 0, \qquad T_{xu} - X_{tu} = 0$$

$$T_x - X_t = 0, \quad T_t = T_u = 0$$

$$X_x = X_u = 0, \quad U_x = U_t = U_u = 0\,. \tag{5.6}$$

Solving the above constraints consistently the infinitesimals are obtained as

$$X = \alpha t + \beta$$

$$T = \alpha x + \delta$$

$$U = 0, \tag{5.7}$$

where α, β, δ are constant parameters. Thus the infinitesimals do not depend on the form of F(u). In (5.7) the parameter α defines a hyperbolic rotation (i.e., one leaving $(x^2 - t^2)$ invariant), while the parameters β and δ define space and time translations respectively.

Following the standard procedure, we define three generators X_i, $(i = 1,2,3)$:

$$X_1 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}$$

$$X_2 = \frac{\partial}{\partial x}$$

$$X_3 = \frac{\partial}{\partial t} \, . \qquad (5.8)$$

These generators obey the Lie algebra given by

$$[X_1, X_2] = -X_3$$

$$[X_3, X_1] = X_2$$

$$[X_2, X_3] = 0. \qquad (5.9)$$

However, if we replace $t \longrightarrow i\,t$ a redefinition of the generators X_i is required, and they can then be shown to obey the Poincaré group Lie algebra for two dimensions.

~~Hyperbolic rotation generator X_1 does not give any conservation law.~~

If we define an infinitesimal operator Ω associated with the similarity group $\mathcal{G}$ of KG equations by the relation

$$\Omega = X \frac{\partial}{\partial x} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u} , \qquad (5.10)$$

then the invariance of a solution u under $\mathcal{G}$ is expressed by

$$\Omega u = 0. \qquad (5.11)$$

5.III. The ternary classification of solutions

From the Lie algebra associated with the similarity group $\mathcal{G}$ of two dimensional KG equations, we can isolate two trivial abelian subalgebras, one generated by X_1 and the other by X_2 and X_3. The corresponding groups of transformations $\mathcal{G}_r$ and $\mathcal{G}_s$ include hyperbolic rotations and translations, respectively. This motivates a classification of particular solutions of a KG equation into three sets: the TI, HRI and SI types. The SI solutions are Poincaré-invariant, but the remaining two classes also merit sufficient attention, as the following discussion shows.

Firstly, we consider the TI class of solutions. The corresponding infinitesimals are

$$X = \beta$$

$$T = \delta$$

$$U = 0. \qquad (5.12)$$

The similarity variable χ and the similarity solution are obtained from the Lagrange condition

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U}. \qquad (5.13)$$

This gives

$$\chi = \delta x - \beta t. \qquad (5.14)$$

The similarity solution is then expressed by

$$u = f(\chi). \qquad (5.15)$$

On inserting (5.15) into (5.1) the invariant equation results:

$$(\delta^2 - \beta^2)\frac{d^2 f}{d\chi^2} = F(f). \qquad (5.16)$$

All travelling wave solutions (solitary waves) of (5.1) satisfy (5.16) and are invariant under the translation subgroup $\mathcal{G}_s$, so that

$$\Omega_s u_s = 0 , \qquad (5.17)$$

where Ω_s is the infinitesimal operator corresponding to translations,

$$\Omega_s = \beta \frac{\partial}{\partial x} + \delta \frac{\partial}{\partial t} , \qquad (5.18)$$

and u_s is a TI solution.

To generate HRI solutions, we take the infinitesimals in the form

$$X = \alpha t$$

$$T = \alpha x$$

$$U = 0. \qquad (5.19)$$

The Lagrange condition (5.13) yields the similarity variable η and similarity solution $g(\eta)$:

$$\eta = (\alpha/2)(x^2 - t^2) \qquad (5.20)$$

$$u = g(\eta). \qquad (5.21)$$

The corresponding invariant equation is obtained by using (5.21) in (5.1):

$$\eta \frac{d^2 g}{d\eta^2} + \frac{dg}{d\eta} = (1/2\alpha)F(g). \tag{5.22}$$

By a change of variable

$$\eta = e^{\tau} , \tag{5.23}$$

(5.22) reduces to

$$\frac{d^2 g}{d\tau^2} = \frac{1}{2\alpha} e^{\tau} F(g). \tag{5.24}$$

HRI solutions u_r are invariant under the hyperbolic rotation subgroup $\mathcal{G}_r$ in the sense,

$$\Omega_r u_r = 0 , \tag{5.25}$$

where the infinitesimal operator Ω_r of hyperbolic rotations is of the form

$$\Omega_r = \alpha t \frac{\partial}{\partial x} + \alpha x \frac{\partial}{\partial t} . \tag{5.26}$$

To the best of our knowledge, $\mathcal{G}_r$-invariant solutions have been rarely quoted in the literature. A few examples are given in the next section.

The similarity variable ζ, and the similarity solution $h(\zeta)$ corresponding to the infinitesimals of the full group $\mathcal{G}$ are given by

$$\zeta = \frac{1}{2}\alpha(x^2 - t^2) + (x - \beta t) + (-\beta^2/2\alpha) , \qquad (5.27)$$

and

$$u = h(\zeta). \qquad (5.28)$$

However, the resulting invariant equation takes the same form as (5.22) with the replacements $\eta \longrightarrow \zeta$ and $g \longrightarrow h$. As a consequence, it is possible to transform a $\mathcal{G}_r$ invariant solution into a $\mathcal{G}$ invariant one of the form (5.28) by redefining the similarity variable. This procedure is not applicable to $\mathcal{G}_s$-invariant solutions.

5.IV. Examples of HRI solutions

1. Double sine-Gordon equation

An HRI solution of the DsG equation,

$$u_{xx} - u_{tt} = -\alpha \sin u + 2\alpha \sin u/2 , \qquad (5.29)$$

has been obtained by the procedure described in the preceding section. This new solution is given by

$$u = 4 \text{ arc } \tan\left\{\pm 1/\sqrt{[\alpha(x^2 - t^2)]}\right\}. \tag{5.30}$$

Replacing the sines in (5.29) by hyperbolic sines we are led to a new solution of the DshG equation

$$u = 4 \text{ arc } \tanh\left(\pm 1/\sqrt{[\alpha(x^2 - t^2)]}\right). \tag{5.31}$$

2. <u>Massless u^4 equation</u>

An HRI solution of the massless u^4 equation [41],

$$u_{xx} - u_{tt} = 2\alpha u^3 , \tag{5.32}$$

already exists in the literature in the context of the de Alfaro-Fubini-Furlan one meron solution [72],

$$u = \pm 1/\sqrt{[\alpha(x^2 - t^2)]}. \tag{5.33}$$

3. <u>Massless u^6 equation</u>

The massless u^6 equation,

$$u_{xx} - u_{tt} = 2\alpha u^3 - 3\alpha u^5 , \tag{5.34}$$

is found to possess an HRI solution,

$$u = \pm 1\sqrt{[1 + \alpha(x^2 - t^2)]}. \tag{5.35}$$

This solution vanishes asymptotically with a singularity on the surface $[\alpha(x^2 - t^2) + 1] = 0$.

4. Liouville equation

The invariant equation for the Liouville equation [153]

$$u_{xx} - u_{tt} = 4\alpha \exp(2u) \tag{5.36}$$

can be exactly reduced to the one dimensional Liouville form

$$\frac{d^2 s}{d\tau^2} = \exp(2s) \tag{5.37}$$

where $2s = \tau + 2g$. Reducing further to an integral we have

$$\tau = 2\int \frac{dz}{(z^2 - c^2)} \tag{5.38}$$

with $(z^2 - c^2) = \exp(2s)$, and c is an arbitrary constant. The three known solutions of the one dimensional Liouville equation [172] can therefore be transformed into the corresponding three HRI solutions of the two dimensional Liouville equation listed below:

$$u = -\log[\log(\alpha(x^2 - t^2))] - \tfrac{1}{2}\log[\alpha(x^2 - t^2)], \quad (5.39)$$

$$u = \log[c \operatorname{sech}(c \log(\alpha(x^2 - t^2))] - \tfrac{1}{2}\log[\alpha(x^2 - t^2)], \quad c \neq 0 \quad (5.40)$$

$$u = \log[c \operatorname{cosech}(c \log[\alpha(x^2 - t^2)])] - \tfrac{1}{2}\log(\alpha(x^2 - t^2)), \quad c \neq 0 . \quad (5.41)$$

5.V. Painlevé property

Ablowitz et al [126,127] conjectured that, when the ODEs obtained by the similarity reduction from a given PDE is of the P-type, then the PDE would be integrable. However this connection does not seem to hold in general. ~~In the symmetry scenario sketched in the preceding section, it would be instructive to trace the origin of an equation's integrability in the different symmetry classes into which its solutions fall.~~ We show, by means of the example of the sG equation, that the PP resides in one of the restricted svmmetrv classes such as rotation or translation.

It is well-known that the similarity-reduced equation corresponding to the sG equation possesses PP. It will be shown below that the PP is characteristic of HRI solutions, whereas it is not shared by the purely TI solutions.

The sG equation is written in the form

$$u_{xx} - u_{tt} = \sin u \, . \tag{5.42}$$

Corresponding to the sG equation (5.42), the invariant equation (5.22) reads

$$\eta \frac{d^2 g}{d\eta^2} + \frac{dg}{d\eta} = \frac{1}{2\alpha} \sin g. \tag{5.43}$$

A change of variable [180],

$$\omega = \exp(ig), \tag{5.44}$$

yields the P-type (PIII) equation [180]:

$$\frac{d^2\omega}{d\eta^2} = (1/\omega)\left(\frac{d\omega}{d\eta}\right)^2 - (1/\eta)\frac{d\omega}{d\eta} - (1/4\alpha\eta)(\omega^2 - 1). \tag{5.45}$$

On the other hand, the TI similarity equation (5.16) with a replacement $F(f) \longrightarrow \sin f$, becomes the non-P-type:

$$\frac{d^2\omega}{d\eta^2} = (1/\omega)\left(\frac{d\omega}{d\eta}\right)^2 - (\omega^2 - 1)/[\eta(\delta^2 - \beta^2)] \qquad (5.46)$$

under the substitution,

$$\omega = \exp(i f). \qquad (5.47)$$

This shows that the purely TI solutions of sG do not exhibit PP.

In the light of the ~~conjectured connection between PP and integrability~~ results herein obtained, it follows that for the sG, there exists a Painlevé sector as well as a non-Painlevé sector, of solutions, these being identified with the HRI and TI classes, respectively.

5.VI. Similarity group and base equation method

In the base equation method a solution of a KG equation is used to construct a solution of another KG equation. This mapping is governed by a constraint equation which is to be respected by both members of the solution pair. As shown above, since the KG family of equations is characterised by the same similarity group $\mathcal{G}$, it follows that both members of the solution pair must belong to this group. Consequently, the constraint equation which is usually a different equation, must also belong to the same similarity group.

To illustrate the above ideas, let us consider the pair of KG equations,

$$u_{xx} - u_{tt} = F(u), \tag{5.48}$$

and

$$v_{xx} - v_{tt} = G(v). \tag{5.49}$$

If a solution u of (5.48) is to be mapped into a solution v of (5.49) according to

$$v = v(u), \tag{5.50}$$

then the usually imposed constraint is

$$v_x^2 - v_t^2 = 2\int G(v)dv. \tag{5.51}$$

This equation belongs to the same similarity group $\mathcal{G}$ as (5.48) or (5.49). As a specific example, we consider the mapping of a class of particular solutions of the Liouville equation

$$u_{xx} - u_{tt} = \lambda \exp(2u) \tag{5.52}$$

to those of the massless u^4 equation

$$v_{xx} - v_{tt} = \lambda v^3 , \tag{5.53}$$

through a relation

$$u = \frac{1}{2} \log v. \tag{5.54}$$

The constraint that permits such a map is

$$v_x^2 - v_t^2 = \frac{1}{2} \lambda v^4 . \tag{5.55}$$

It can be readily verified that the similarity group of (5.55) is identical with $\mathcal{G}$.

5.VII. Discussion

We have pointed out that, by a suitable redefinition of the generators, the similarity group $\mathcal{G}$ of KG equations can be made isomorphic to the Poincaré group. If that be the case, then $\mathcal{G}$ must coincide with the invariance group of the Hamiltonian,

$$H = \frac{1}{2}(u_x^2 - u_t^2) + \frac{1}{2}\int F(u)du \,. \tag{5.56}$$

This property may be traced to the vanishing of the infinitesimal U, generating infinitesimal transformations of u.

In the preceding sections we have studied a three-fold symmetry categorization of solutions of two dimensional KG equations. The KG family is characterized by a unique similarity group $\mathcal{G}$ with subgroups $\mathcal{G}_r$ and $\mathcal{G}_s$ representing hyperbolic rotations and translations, respectively. Kinks and other solitary waves belong to the TI class, while in the only known integrable KG type system the sG, the solitons, through their conjectured property of satisfying a P-type similarity equation, belong to the HRI class. It should be emphasized that PP is not always characteristic of HRI solutions, though this may be true with KG equations. For a non-KG type equation such as Boussinesque equation [183] the only admissible transformations turn out to be translations and therefore, its PP is associated with translation symmetry.

We have exhibited new HRI solutions in several interesting cases such as DsG, DshG, massless u^4, massless u^6 and Liouville equations. HRI solutions can be made SI by imposing translational invariance on them, but TI solutions do not permit this kind of global extension.

The two dimensional solutions reported in this chapter can be extended straightforwardly to higher dimensions.

6

TIME-DEPENDENT EXACT SOLUTIONS OF CLASSICAL SU(2) YANG-MILLS-HIGGS SYSTEM

6.I. Classical solutions of SU(2) Yang-Mills-Higgs system

According to current views the fundamental physical interactions are described by unified gauge theories. A great deal of progress has currently been made towards understanding of the classical YM field equations [57]. These properties have turned out to be of interest not only in their own right but also in connection with confinement [58-62]. In four dimensional spontaneously broken nonabelian gauge theories, solitons appear as monopoles corresponding to a suitable generalization of the Dirac quantization condition [168]. They predict the existence of heavy monopoles.

A physically significant prototype of a spontaneously broken nonabelian gauge theory is the SU(2) Yang-Mills-Higgs theory. Static monopole solutions were discovered by Prasad and Sommerfield in the limit of vanishing Higgs coupling ($\lambda \rightarrow 0$), this limit being nowadays known as the Prasad-Sommerfield (PS) limit [66]. Various workers have since obtained time-dependent, singular solutions of this system [65,188-192]. No systematic procedure was adopted in arriving at any of these solutions with the result that both the static and time-dependent solutions emerged as products of ingenious guesswork.

Herein it is proposed to carry out a similarity analysis of the time-dependent non-linear coupled differential equations associated with a spontaneously broken SU(2) YM theory [60-66]. This gives the invariance group of the system of equations, and we use this information to generate some of the previously known time-dependent solutions in the PS limit. We also present two new time-dependent solutions possessing surface singularities.

6.II. Similarity group of SU(2) YM-Higgs system

The SU(2) YM field coupled to an SU(2) Higgs field is given by the Lagrangian density:

$$L = -(1/4)F^{\mu\nu a} F^{a}_{\mu\nu} - (1/2)\Pi^{\mu a}\Pi^{a}_{\mu} + (\mu^2/2)\phi^a\phi^a - (\lambda/4)(\phi^a\phi^a)^2 , \qquad (6.1)$$

where the symbols have been defined in chapter 1. At the PS limit, the time-dependent spherically symmetric Ansatz [65] reduces the equations of motion to the form

$$r^2(H_{,rr} - H_{,tt}) = 2\,HK^2 \qquad (6.2)$$

$$r^2(K_{,rr} - K_{,tt}) = K(K^2 - 1) + KH^2 , \qquad (6.3)$$

where

$$H_{,xx} = \frac{\partial^2 H}{\partial x^2} \quad \text{and} \quad K_{,xx} = \frac{\partial^2 K}{\partial x^2} , \; (x = r,t). \qquad (6.4)$$

To study the similarity group $\mathcal{G}$ of this system, we define a generic dependent variable $u^\alpha (\alpha = 1,2)$ such that $u^1 = K$ and $u^2 = H$, and consider a one parameter family of infinitesimal transformations defined by

$$r^* = r + \epsilon\, R(r,t,u^1,u^2) + O(\epsilon^2)$$

$$t^* = t + \epsilon\, T(r,t,u^1,u^2) + O(\epsilon^2)$$

$$u^{\alpha *} = u^\alpha + \epsilon\, U^\alpha(r,t,u^1,u^2) + O(\epsilon^2) . \qquad (6.5)$$

The infinitesimals R, T and U^α must ensure the invariance of (6.2) and (6.3) under the transformations (6.5). The derivatives $\frac{\partial^2 u^\alpha}{\partial r^2}$ and $\frac{\partial^2 u^\alpha}{\partial t^2}$ transform according to

$$u^{\alpha *}_{,r^* r^*} = u^\alpha_{,rr} + \epsilon\, [U^\alpha_{,rr}] + O(\epsilon^2) \qquad (6.6)$$

$$u^{\alpha *}_{,t^* t^*} = u^\alpha_{,tt} + \epsilon\, [U^\alpha_{,tt}] + O(\epsilon^2) , \qquad (6.7)$$

where $u^{\alpha}_{,xx} = \frac{\partial^2 u^{\alpha}}{\partial x^2}$ etc. and $[U^{\alpha}_{,xx}]$ denotes an 'extension' $(x = r,t)$:

$$[U^{\alpha}_{,xx}] = \frac{\partial^2 U^{\alpha}}{\partial x^2} + 2\frac{\partial^2 U^{\alpha}}{\partial x\, \partial u^{\beta}} u^{\beta}_{,x} - \frac{\partial^2 X^i}{\partial x^2} u^{\alpha}_{,i} + \frac{\partial u^{\alpha}}{\partial u^{\beta}} u^{\beta}_{,xx}$$

$$- 2\frac{\partial X^i}{\partial x} u^{\alpha}_{,xi} + \frac{\partial^2 u^{\alpha}}{\partial u^{\beta} \partial u^{\gamma}} u^{\beta}_{,x} u^{\gamma}_{,x} - 2\frac{\partial^2 X^i}{\partial x\, \partial u^{\beta}} u^{\beta}_{,x} u^{\alpha}_{,i}$$

$$- \frac{\partial X^i}{\partial u^{\beta}} (u^{\alpha}_{,i} u^{\beta}_{,xx} + 2 u^{\beta}_{,x} u^{\alpha}_{,ix})$$

$$- \frac{\partial^2 X^i}{\partial u^{\beta} \partial u^{\gamma}} u^{\beta}_{,x} u^{\gamma}_{,x} u^{\alpha}_{,i} \qquad (6.8)$$

with

$$X^1 = R\,, \quad X^2 = T. \qquad (6.9)$$

When equations (6.5) are substituted into the transformed system corresponding to (6.6) and (6.7) and coefficients of first order in ϵ are equated to zero, we find:

$$r^2[U^1_{,rr}] - r^2[U^1_{,tt}] + 2rX^1(K_{,rr} - K_{,tt})$$

$$+ U^1(1 - 3K^2 - H^2) - 2\,U^2 KH = 0, \qquad (6.10)$$

$$r^2[U^2_{,rr}] - r^2[U^2_{,tt}] + 2r\, X^1(H_{,rr} - H_{,tt})$$

$$- 2\, K^2U^2 - 4HKU^2 = 0. \qquad (6.11)$$

Solving (6.10) and (6.11) for the infinitesimals (see Appendix),

$$R = 2\lambda rt + \varkappa r \qquad (6.12)$$

$$T = \lambda(r^2 + t^2) + \varkappa t + \sigma \qquad (6.13)$$

$$U^\alpha = 0\ , \qquad (6.14)$$

where λ, $\varkappa$ and σ are constants.

Equation (6.14) states the invariance of any solution of the SU(2) YM-Higgs system reduced by the Ansatz (1.62-1.64) under the similarity transformation (6.5).

The occurrence of three independent parameters λ, $\varkappa$ and σ above permits us to define the generators G_a, $a = 1,2,3$.

$$G_1 = 2rt \frac{\partial}{\partial r} + (r^2 + t^2) \frac{\partial}{\partial t}\ , \qquad (6.15)$$

$$G_2 = r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t}\ ,$$

$$G_3 = \frac{\partial}{\partial t}\ , \qquad (6.16)$$

which satisfy the Lie algebra,

$$[G_1, G_2] = -G_1 \tag{6.17}$$

$$[G_2, G_3] = -G_3 \tag{6.18}$$

$$[G_3, G_1] = 2G_2. \tag{6.19}$$

We identify this as the Lie algebra associated with the similarity group $\mathcal{G}$ of equations (6.2) and (6.3).

6.III. Time-dependent solutions

In the preceding chapter we developed a method of constructing particular solutions of non-linear KG equations under various subgroups of the similarity group. That procedure may be extended to the SU(2) YM-Higgs system reduced by the Ansatz (1.62-1.64). The idea is to consider different subgroups of the similarity group $\mathcal{G}$, define a similarity variable for each subgroup, set up the corresponding similarity-reduced equations and solve them. Solutions are obtained in cases where the reduced equations are of the PS type. The different cases are discussed below.

A. Full group $\mathcal{G}$: $\lambda \neq 0, \varkappa \neq 0, \sigma = \varkappa^2/4\lambda$

Equations (6.12-6.14) yield the similarity variable

$$\chi = r/(t^2 - r^2 + \frac{\varkappa t}{\lambda} + \frac{\varkappa^2}{4\lambda}). \qquad (6.20)$$

The similarity-reduced system of equations are

$$\chi^2 \frac{d^2K}{d\chi^2} = K(K^2 - 1) + KH^2, \qquad (6.21)$$

$$\chi^2 \frac{d^2H}{d\chi^2} = 2HK^2. \qquad (6.22)$$

This is of the same form as the equation considered by Prasad and Sommerfield for the static case ((1.82) and (1.83)). A solution of (6.21) and (6.22) is ((1.84) and (1.86)):

$$K(\chi) = C\chi/\sinh(C\chi), \qquad (6.23)$$

$$H(\chi) = C\chi \coth(C\chi) - 1, \qquad (6.24)$$

where χ has been defined in (6.20). This solution coincides with that reported in Ref [65].

However, a new solution can be obtained by replacing r by χ in the static solution reported in Ref [172]. Thus we are led to the solution

$$K(\chi) = \chi/(A + \chi) \tag{6.25}$$

$$H(\chi) = A/(A + \chi), \tag{6.26}$$

where A is a nonzero arbitrary constant. Both $K(\chi)$ and $H(\chi)$ are singular on the surface $(A + \chi) = 0$.

B. Subgroup $\mathcal{G}_1$: $\mathcal{X} = \sigma = 0$

Under the subgroup $\mathcal{G}_1 \subset \mathcal{G}$ specified by $\mathcal{X} = \sigma = 0$, the infinitesimals read

$$R = 2\lambda rt \tag{6.27}$$

$$T = \lambda (r^2 + t^2). \tag{6.28}$$

With a similarity variable

$$\eta = r/(t^2 - r^2) \tag{6.29}$$

the reduced system assumes the form of (6.21) and (6.22) on the replacement $\chi \rightarrow \eta$. We note that there exist two families of solutions just as in the case of the full group $\mathcal{G}$ mentioned above. They are

$$K(\eta) = C\,\eta/\sinh(C\eta) \tag{6.30}$$

$$H(\eta) = C\,\eta\,\coth(C\eta) - 1. \tag{6.31}$$

as found in Ref [65], and

$$K(\eta) = \eta/(A + \eta) \tag{6.32}$$

$$H(\eta) = A/(A + \eta)\ ,\ A \neq 0 \tag{6.33}$$

which constitutes a new time-dependent solution. The later is singular on the surface $(A + \eta) = 0$.

C. Subgroup $\mathcal{G}_2$: $\lambda = 0$, $\varkappa \neq 0$, $\sigma \neq 0$

For the subgroup $\mathcal{G}_2 \subset \mathcal{G}$, $\lambda = 0$, and the infinitesimals are

$$R = \varkappa\, r \tag{6.34}$$

$$T = \varkappa\, t + \sigma. \tag{6.35}$$

The corresponding similarity variable is

$$\xi = r/(t - a) \tag{6.36}$$

where a is a nonzero arbitrary constant. The similarity equations read

$$(\xi^2 - \xi^4)\, K'' - 2\xi^3 K' = K(K^2 - 1 + H^2) \tag{6.37}$$

$$(\xi^2 - \xi^4)\, H'' - 2\xi^3 H' = 2\, HK^2 \tag{6.38}$$

where a prime denotes differentiation with respect to ξ. It has not been possible for us to find an exact solution for this system.

6.IV. Discussion

The similarity method of analysis of the non-linear coupled differential equations equivalent to the classical SU(2) YM-Higgs system gives the similarity group $\mathcal{G}$, which evidently depends on the Ansatz employed. There is an explicit time-dependent similarity variable for each subgroup of $\mathcal{G}$. Under the full group $\mathcal{G}$ as well as under one of its subgroups $\mathcal{G}_1$, time-dependent solutions arise as generalizations of the well known static solutions of Refs[66] and [172]. This indicates the possibility of transforming any static solution of (6.2) and (6.3) into a non-trivial time-dependent form. The two new solutions herein obtained as well as those reported earlier in the literature, can be continued to the Euclidean space.

APPENDIX

The equations (6.12)-(6.14) are obtained by solving nearly 100 constraint relations obtained from (6.10) and (6.11) by equating coefficients of different orders of derivatives of K and H. These constraints can be divided into primary and secondary types such that the latter arise as combinations of the former. We list only the primary constraints.

$$r^2 U^1_{rr} - r^2 U^1_{tt} + (1-3K-H^2)U^1 - 2KHU^2 = 0$$

$$2r^2 U^2_{rH} - r^2 R_{rr} - r^2 R_{tt} = 0$$

$$2r^2 U^1_{rK} - r^2 R_{rr} - r^2 R_{tt} = 0$$

$$r^2 U^2_{rr} - r^2 U^2_{tt} - 2K^2 U^2 - 4KHU^1 = 0$$

$$r U^1_K - 2rR_r + 2R = 0, \quad rU^2_H - 2rT_t + 2R = 0$$

$$T_t + rT_{rt} - R_r = 0, \quad rR_r - R = 0, \quad rT_t - R = 0$$

$$T_{rr} - T_{tt} = 0, \quad T_{rt} = T_K = T_H = 0$$

$$U^1 = U^2 = 0, \quad R_K = R_H = 0 \cdot$$

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