STOCHASTIC PROCESSES AND APPLICATIONS

ANALYSIS OF SOME STOCHASTIC MODELS IN INVENTORIES AND QUEUES

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DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

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CERTIFICATE

This is to certify that the work reported in this thesis entitled " ANALYSIS OF SOME STOCHASTIC MODELS IN INVENTORIES AND QUEUES" that is being submitted by Shri. M. Manoharan, for the award of the Degree of Doctor of Philosophy, to Cochin University of Science and Technology, Cochin 682 022, is based on the bonafide research work carried out by him under my supervision and guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Chapter-1

INTRODUCTION

This thesis is devoted to the study of some stochastic models in Inventories and Queues which are physically realizable, though complex. It contains a detailed analysis of the basic stochastic processes underlying these models. Many real-world phenomena require the analysis of system in stochastic rather than deterministic setting. Stochastic models are becoming increasingly important for understanding or making performance evaluation of complex systems in a broad spectrum of fields such as operations research, computer science, economics, telecommunication, engineering etc. Our aim is to have an improved understanding of the behaviour of such models, which may increase their applicability. Some variants of inventory systems with random lead times, non-identically distributed inter-arrival demand times, state dependent demands, perishable commodities, varying ordering levels etc. are considered. Also we study some finite and infinite capacity single server queueing systems with single/bulk services, vacation to the server; transient as well as steady state solutions of the systems are obtained in certain cases. Each chapter in the thesis is provided with self introduction, notations and some important references. This chapter gives a brief introduction to the subject matter and some related topics. It gives a concise survey of some important developments in the area of inventories and queues. Some basic notions in renewal theory and Markov renewal theory are supplemented. Finally an outline of the results obtained is given.

1.1 INVENTORY THEORY - AN OUTLINE

An inventory is an amount of material stored for the purpose of sale or production. Inventory management of physical goods or other products or elements is an integral part of logistic systems common to all sectors of the economy, such as business, industry, agriculture, defence etc. In an economy that is perfectly predictable, inventory may be needed to take advantage of the economic features of a particular technology, or to synchronize human tasks, or to regulate the production process to meet the changing trends in demand. When uncertainity is present, inventories are used as protection against risk of stockout.

The existence of inventory in a system generally implies the existence of an organized complex system involving inflow, accumulation, and outflow of some commodities or goods or items or products. The regulation and control of inventory must proceed within the context of this organized system. Thus inventories, rather than being interpreted as idle resources, should be regarded as a very essential element, the study of which may provide insight to the aggregate operation of the system. The analysis of inventory system defines the degree of interrelationship between inflow, accumulation, and outflow and identifies economic control method for operating such systems.

Analysis of Inventory systems

Inventory systems may be broadly classifed as continuous review systems or periodic review systems. In continuous review systems, the system is monitored continuously over time. In periodic review systems, the system is monitored at discrete, equally spaced instants of time. An analysis of an inventory consists of the following steps: (1) determination of the properties of the system, (2) formulation of the inventory problem, (3) development of a model of the system, and (4) derivation of a solution of the system.

Inventory policies - Decision variables

An inventory policy is a set of decision rules that dictate 'when' and 'how much' to order. Several policies may be used to control an inventory system; of these, the most important policy is the (s,S) policy. Under this particular policy, whenever the position inventory (sum of onhand inventory and outstanding orders) is equal to or less than a value s, a procurement is made to bring its level to S. Under a continuous review system, the (s,S) policy will usually imply the procurement of a fixed quantity Q = S-s of the commodity, while in periodic review systems the procurement quantity will vary. The (s,S) policy incorporates two decision variables s and S. The variable s is called the reorder level, which identifies when to order, while S-s identifies how much to order.

Objective function

In an inventory problem, the objective function may take several forms, and these usually involve the minimization of a cost function or the maximization of a profit function. The planning period or horizon period, which is the length of time over which the system is assumed to operate, may be either finite or infinite. For a finite horizon period, the total cost (profit) experienced over the entire horizon may be the criterion; alternatively the criterion may be average of the total cost (profit) per unit time. On the other hand, if the horizon is infinite, the long run average total cost (profit) experienced over the infinite horizon, is selected as the criterion. In stochastic models expected values of costs are measured. The cost function, in general, consists of the additive contribution of the procurement cost, the holding cost, and the shortage cost.

The inventory models are usually characterized by the demand pattern and the policy for replenishing the stock in the store. The replenishments ordered may arrive after a time lag L, which may be fixed or a random variable. This time lag L is called the 'lead time'. The time interval during which the inventory is empty is termed as a dry period.

The quantitative analysis of the inventory system started with the work of Harris (1915), who formulated and obtained the optimal solution to a simple inventory situation. Wilson (1934) rediscovered the same formula and this was successful in popularizing its use. The formula is an expression for an optimal production lost size given as a square root function of a fixed cost, an investment or holding cost, and the demand. It is often referred to as the 'simple lot-size formula' or the 'economic order quantity (EOQ) formula', or the 'Wilson formula'. A stochastic inventory problem was analysed for the first time by Masse (1946). After that several studies were made in this direction (See Arrow, Harris and Marschak(1951), Dvoretsky, Kiefer and Wolfowitz (1952)). Dvoretsky et al. obtained the conditions under which optimum inventory levels can be found. The development of the theory upto 1952 have been summarized by Whitin (1953).

A valuable review of the problems in the probability theory of storage systems is given by Gani (1957). A systematic account of probabilistic treatment in the study of inventory systems using renewal theoretic arguments is given in Arrow, Karlin and Scarf (1958). Hadley and Whitin (1963) deals with the applications of such models to practical situations. Tijms (1972) gives a detailed analysis of the inventory systems under (s,S) policy. The cost analysis of different inventory systems is given in Naddor (1966). A practical treatment of the (s,S) lost sales model can be found in the recent books by Silver and Peterson (1985) and Tijms (1986).

A detailed review of the work carried out in (s,S) inventory systems upto 1966 can be found in Veinott (1966). We refer to the monograph by Ryshikov (1973) for inventory systems with random lead times. Sivazlian (1974) considers an (s,S) inventory model in which unit demands of items occur with arbitrary interarrival times between demands, but with zero lead time. His results are extended by Srinivasan (1979) to the case in which lead times are i.i.d. random variables having a general distribution. Sahin (1979) considers an (s,S) inventory system in which demand quantities are nonnegative real valued random variables with constant lead time. Again in Sahin (1983) an (s,S) inventory system in which demand quantities (positive integer valued), lead times and interarrival times between consecutive demands are all independent and generally distributed sequences of i.i.d. random variables, is discussed. She obtained binomial moments for the inventory deficit. Thangaraj and Ramanarayanan(1983) consider an inventory system with random lead times and having two ordering levels. Kalpakam and Arivarignan(1985) have studied an (s,S) inventory system having one exhibiting item subject to random failure time and obtained the limiting distribution of position inventory by applying the techniques of semi-regenerative process. Ramanarayanan and Jacob (1987) also analyse an (s,S) inventory system with random lead times and bulk demands. An (s,S) inventory system with rest periods to the server has been analysed by Daniel and Ramanarayanan (1988).

The earliest work on the decay (perishability) problem is due to Ghare and Schrader (1963) who considered the generalization of the standard EOQ model without shortages. Their model was extended to more general types of deterioration by Covert and Philip (1973) and Shah (1977). Nahmias (1982) reviews various models and objective functions in the analysis of such inventory systems. Motivated by the study of blood bank models Kaspi and Perry (1983, 1984) and Perry (1985) have studied inventory systems for perishable commodities in which life time of the items stored are fixed as well as random variables. They utilized the analogy between these systems and a queueing system with impatient customers to study the process of the lost demand, the number of units in the system, etc.

A continuous review (s,S) inventory system in random environment is analysed by Feldman (1978). Richards (1979) analyses an (s,S) inventory system with compound Poisson demand. Algorithms for a continuous review (s,S) inventory system in which the demand is according to a versatile Markovian point process is given by Ramaswami (1981). Approximation for the single-product production-inventory problem with compound Poisson demand and two possible production rates where the product is continuously added to inventory can be seen in De Kok, Tijms and Van der Duyn Schouten (1984). Using Markov decision drift processes, Hordijk and Van der Duyn Schouten (1986) examines the optimality of (s,S) policy in a continuous review inventory model with constant lead time when the demand process is a superposition of a compound Poisson process and a continuous deterministic process.

Stidham (1974) has introduced and studied a wide class of stochastic input-output systems. The system is fed by an exogenous stochastic input process. The quantity in the system builds over time as a result. At a certain (random) time instant all the quantity in the system is instantaneously removed (cleared) and the situation allowed to repeat itself. Such systems are called stochastic clearing systems. Its applications to bulk-service queues and (s,S) inventory systems are given by Stidham (1977,1986). In a generalized stochastic clearing system, the system contents are restored to a level m (> 0), rather than zero, at each clearing instant. With inventory defined as the negative of system contents, the generalized model covers (s,S) inventory systems with continuous or periodic review. In his paper Stidham (1986) discusses the optimality of the clearing parameters.

In the case of random lead times, the concept of vacations to the server during dry period is introduced in inventory system by Daniel and Ramanarayanan (1987, '88).

Several other models with vacations to the server, finite backlog of demands, bulk demands, varying ordering levels etc. can be found in Jacob (1987).

1.2 QUEUEING THEORY - AN OUTLINE

The development of queueing theory started with the publication of Erlang's paper (1909) on the M/D/1 queueing system. For this system, which has constant service times and a Poisson arrival process, Erlang explained the concept of statistical equilibrium. This paper touched the essential points of queueing theory, and for a long time research in queueing theory concentrated on questions, first time discussed by Erlang.

Until 1940, the majority of the contribution to queueing theory was made by people active in the field of telephone traffic problems. After the Second World War, the field of operations research rapidly developed and queueing applications were also found in production planning, inventory control and maintenance problems. In this period, much theoretically oriented research on queueing problems were done.

In the fifties and sixties, the theory reached a very high mathematical level (see Cohen (1969) and Takacs(1962)).

Advanced mathematical techniques like transform methods, Wiener-Hopf decomposition and function theoretic tools were developed and refined. This research resulted in a number of elegant mathematical solutions.

In particular, noting the inadequacy of the equilibrium theory in many queue situations, Pollaczek in 1934 began investigations of the behaviour of the system in a finite interval. Since then, there appeared considerable work in the analytical behavioural study of queueing systems. The trend towards the analytical study of the basic stochastic processes of the system has continued, and queueing theory has proved to be a fertile field for researchers who wanted to do fundamental research on stochastic processes involving mathematical models.

For the time dependent analysis of the system, more sophisticated mathematical procedures are necessary. For instance, for an M/M/1 queue, under statistical equilibrium, the balance of state equations is simple and the limiting distribution of the queue size is obtained by recursive arguments and induction. But for the time dependent solution, the use of transforms is necessary. The time dependent solution was first given by Bailey (1954b) and Ledermann and Reuter (1956). While Bailey used the method of generating functions and the differential equations satisfied by them, Ledermann and Reuter used spectral theory for their solution. Champernowne's(1956) combinatorial method and Conolly's (1958) difference equation techniques are also aimed at the transient solution for the system size in an M/M/1 queue system. Parthasarathy (1987) suggests a simple and direct approach for the same.

To analyse the case of M/G/1 queues, Palm (1943) and Kendall (1953) have used the method of regeneration points and imbedded Markov chain which continue to have a tremendous influence on queueing theory. Kendall's exposition created a new technique for analysing certain queueing models which are not Markovian. His approach made the analysis of the transient behaviour of queueing systems much more accessible. The method of supplimentary variables investigated by Kendall (1951) and Cox (1955) is extensively discussed in the book by Gnedenko and Kovalenko (1968).

The study of bulk queues is considered to be originated with the pioneering work of Bailey (1954a). It may be said to have begun with Erlang's investigations of $M/E_k/1$ queue, for its solution contains implicitly the solution of the model $M^k/M/1$. Bailey studied the stationary behaviour of a single server queue having simple Poisson input, intermittently available server and service in batches of fixed maximum size. The results of this study are given in terms of probability generating functions, the evaluation of which requires determining the zeroes of a polynomial. This study was followed by a series of papers involving the treatment of queueing processes with group arrivals or batch service. Gaver (1959) seems to be the first to take up specifically queues involving group arrivals followed by Jaiswal (1960, 1961, 1962), Bhat(1964) and others. For more details on bulk queues, one may refer to Medhi (1984) and Chaudhry and Templeton (1984). For a detailed treatment of queueing systems and for further references, one may refer any one of the standard books on the subject like Saaty (1961), Takacs (1962), Cohen (1969), Prabhu (1965, 1980), Gnedenko and Kovalenko (1968), Cooper (1972), Gross and Harris (1974), Kleinrock (1975) and Asmussen (1987).

Queueing systems in which the service process is subject to interruptions resulting from unscheduled break-downs of servers, scheduled off periods, arrival of customers with pre-emptive or non-preemptive priorities or the server working on primary and secondary customers arise naturally. The impact of these service interruptions on the performance of a queueing system will depend on the specific interaction between the interruption process and service process.

Queueing models with interruptions and their connection to priority models were first studied by White and Christie (1958), who considered the case with exponential service, on-time and off-time distributions. Their results were extended by Gaver (1962), Keilson (1962), Avi-Itzhak and Naor (1962) and Thiruvengadam (1963) to models with general service time and off-time distributions but exponential on-times. When the on-periods are not exponential, the problem becomes very difficult and one such model is studied by Federgruen and Green (1986). A detailed analysis of single server queueing system with server failures is given in Gnedenko and Kovalenko(1968).

Another variation of the interruption model is the vacation model. In this the queueing system incurs a start-up delay whenever an idle period ends or server takes vacation periods. The vacation model includes server working on primary and secondary customers also. Motivated by the study of cyclic queues, Miller (1964) analysed a system in which the server goes for a vacation (rest period) of random duration whenever it becomes idle. He also considered a system in which server behaves normally but the first customer arriving to an empty system has a special service time. These types of systems and similar ones were also examined by Welsch (1964), Avi-Itzhak, Maxwell and Miller (1965), Cooper (1970), Pakes (1973), Lemoine (1975), Levy and Yechiali (1975), Heyman (1977), Van der Duyn Schouten (1978), Shantikumar (1980, 1982), Scholl and Kleinrock (1983) etc.

All the models having rest periods, set-up time, starter, interruptions etc. can be jointly called as vacation models. While the queue with interruption has preemptive priority for vacation, other types of vacations have least priority among all work with vacation taken when the system is empty. Variations of vacation models are available with single and multiple vacations and exhaustive and non-exhaustive service disciplines.

A queueing system in which the server taking exactly one vacation at the end of each busy period, is called a single vacation system. When the system becomes empty, server starts a vacation and he keeps on taking vacations until, on return from a vacation, atleast one customer is present. This is called a multiple vacation system. We say that a vacation model has the property of exhaustive service in case each time the server becomes available, he works in a continuous manner until the system becomes empty.. Systems with a vacation period beginning after every service completion, or after any vacation period if the queue is empty is known as the single service discipline. According to Bernoulli schedule discipline the server begins a vacation period if the queue is empty. If at a service completion the queue is not empty, then service is resumed with fixed probability p and with probability 1-p a vacation commences. Single service disciplines and exhaustive service disciplines are special cases of the Bernoulli schedule discipline. Another variant of the vacation model is that the server goes for vacation after serving a random number of customers.

Vacation systems with exhaustive service discipline are analysed by several authors. See for example, Levy and Yechiali (1975), Hayman (1977), Courtois (1980), Shantikumar (1980), Scholl and Kleinrock (1983), Lee(1984), Fuhrmann (1984), Doshi (1985), Servi (1986 a), Levy and Kleinrock (1986), Keilson and Servi (1986 b) etc. Systems without exhaustive service discipline are considered by Ali and Neuts (1984), Neuts and Ramalhoto (1984), Fuhrmann and Cooper (1985), Keilson and Servi (1986 b,c) and Servi (1986 a). The case of Bernoulli schedule discipline is introduced by Keilson and Servi (1986 a) and further studied by Servi (1986 b).

The main results in the vacation system is the delay analysis by decomposition. The stochastic decomposition property of M/G/1 queueing system with vacation says that the (stationary) number of customers present in the system at a random point in time is distributed as the sum of two or more independent random variables, one of which is the (stationary) number of customers present in the corresponding standard M/G/1 queue (ie. the server is always available) at a random point in time. For more details on queueing systems with vacations one may refer to Doshi (1986).

All the above models assume the existence of stationary distribution and study the queue length and waiting time distributions. The time dependent behaviour as well as steady state behaviour of M/G/1 and G/M/1 queueing systems are extensively studied by Bhat (1968) in which bulk arrival and bulk service queues are considered and the bahaviour of the waiting time process is obtained. Some aspects of the dynamic behaviour of M/G/1 queues with vacations is studied by Keilson and Servi (1986 c). An attempt to find the transient solution of $M/G^{a,b}/1$ queue with vacation using matrix convolution has been made in Jacob and Madhusoodanan (1987). But they have remarked that the solution in that form is not numerically tractable.

1.3. RENEWAL THEORY

Renewal processes are the simplest, regenerative stochastic processes. Let $\{X_n, n=1,2,\ldots\}$ be a sequence of non-negative independent identically distributed random variables with common distribution function $F(.)$ and assume that $Pr\{X_n=0\} < 1$. Since X_n is non-negative, $E(X_n)$ exists.

Let $S_o = 0$, $S_n = X_1 + X_2 + \ldots + X_n$ for $n \geq 1$, and let

$F_n(x) = Pr\{S_n \leq x\}$ be the distribution function of S_n.

Since X_i's are i.i.d., $F_n(x) = F^{*n}(x)$.

Define the random variable

$$N(t) = \text{Sup}\{n \mid S_n \leq t\}$$

Then the process $\{N(t), t \geq 0\}$ is called a renewal process.

If for some n, $S_n = t$, then the n^{th} renewal is said to occur at time t; S_n gives the time of the n^{th} renewal and is called the n^{th} renewal epoch. The random variable $N(t)$ gives the number of renewals in the interval $(o,t]$.

The function $M(t) = E[N(t)]$ is called the renewal function of the process. It is easy to see that

$$N(t) \geq n \iff S_n \leq t$$

Thus the distribution of $N(t)$ is given by

$$\Pr\{N(t) = n\} = F^{*n}(t) - F^{*n+1}(t)$$

where $F^{*n}(t)$ denotes the n-fold convolution of $F(t)$ with itself $(F^{*o}(t) \equiv 1)$

and the expected number of renewals is given by

$$M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$$

Its derivative

$$m(t) = M'(t) = \sum_{n=1}^{\infty} f^{*n}(t)$$

is the renewal density function, assuming the density function $f(t)$ exists. $m(t)$ is the expected number of renewals per unit time. Let us give another interpretation of renewal density, which is very important in practical applications, in the following way:

$$\begin{aligned} m(t)dt &= M(t+dt) - M(t) \\ &= \sum_{n=1}^{\infty} [F^{*n}(t+dt) - F^{*n}(t)] \\ &= \sum_{n=1}^{\infty} \Pr\{t < S_n \leq t+dt\} \end{aligned}$$

We have Pr {more than one renewal point in $(t,t+dt)$}

$\longrightarrow$ $o(dt)$ as $dt \longrightarrow 0$. Therefore

$$\underset{dt \to 0}{\text{Lt}} \; m(t)dt = \Pr\left\{ S_1 \text{ or } S_2 \text{ or } S_3 \text{ or } \ldots \text{ lies in } (t,t+dt) \right\}$$

ie., $m(t)$ is the probability of a renewal in $(t,t+dt)$.

Now, suppose that the first interoccurrence time X_1 has a distribution $G(\circ)$ which is different from the common distribution $F(\circ)$ of the remaining interoccurrence times X_2, X_3, ...

As before define

$$S_o = 0, \quad S_n = \sum_{i=1}^{n} X_i$$

and

$$N_D(t) = \operatorname{Sup}\left\{ n \mid S_n \leq t \right\}$$

The stochastic process $\left\{ N_D(t), t \geq 0 \right\}$ is called a Delayed or Modified renewal process.

Here we have

$$\Pr\left\{ N_D(t) = n \right\} = G * F^{*(n-1)}(t) - G * F^{*n}(t)$$

so that the modified renewal function is

$$M_D(t) = E\ [N_D(t)] = \sum_{n=o}^{\infty} G*F^{*n}(t)$$

The modified renewal density function is given by

$$m_D(t) = M_D'(t) = \sum_{n=o}^{\infty} g*f^{*n}(t)$$

provided that the density functions $g(x) = G'(x)$ and $f(x) = F'(x)$ exist.

Now, consider a stochastic process $\{X(t), t \geq 0\}$ with state space $\{0,1,2, \ldots\}$, having the property that there exist time points at which the process (probabilistically) restarts itself. That is, suppose that with probability one, there exists a time T_1, such that the continuation of the process beyond T_1 is a probabilistic replica of the whole process starting at 0. Note that this property implies the existence of further times $T_2, T_3, \ldots$, having the same property as T_1. Such a stochastic processes is known as a regenerative process.

From the above it follows that $\{T_1, T_2, \ldots\}$ forms a renewal process; and we shall say that a cycle is completed every time a renewal occurs. For details on renewal theory, one may refer to Cox (1962), Feller (1965), Ross (1970) or Cinlar (1975b) among others.

1.4 MARKOV RENEWAL PROCESSES

A Markov renewal process $\{(X_n,T_n); n \geq 0\}$ has two constituents; $\{X_n : n \geq 0\}$ is a homogeneous Markov chain whilst $(T_{n+1} - T_n)$ is the sojourn time in X_n (throughout, $T_o = 0$). Hence we can think of X_n as the state entered at T_n and left at T_{n+1}. Given $\{X_n: n \geq 0\}$, the $\{T_{n+1} - T_n : n \geq 0\}$ are independent and the distribution of $(T_{n+1} - T_n)$ depends on $\{X_n : n \geq 0\}$ through X_n and X_{n+1} only. We assume that the sojourn times are always strictly positive. When the initial state is i, that is $X_o = i$, the time of returns to state i form an ordinary renewal process; whilst the visits to $j \neq i$ form a delayed renewal process (the delay being the time that elapses until the first visit to j). Thus as Cinlar (1969) puts it 'the theory of Markov renewal processes generalizes those of renewal processes and Markov chains and is a blend of the two'.

The semi-Markov matrix Q has its $(i,j)^{th}$ entry

$$Q(i,j,t) = \Pr\{X_{n+1}=j,\ T_{n+1}-T_n \leq t \mid X_n=i\}$$

so that $\sum_j Q(i,j,t)$ is the distribution function of the sojourn time in i and $P(i,j) = Q(i,j,\infty)$ is the transition matrix of the Markov chain $\{X_n:n \geq 0\}$. The Markov renewal function is $R(i,j,t) = \sum_{n=o}^{\infty} Q^n(i,j,t)$, where

$$Q^{n+1}(i,j,t) = \sum_k \int_o^t Q(i,k,du)\, Q^n(k,j,t-u) \text{ for } n \geq 0$$

and $$Q^o(i,j,t) = I(i,j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

There are processes which are, in general, non-Markovian and yet possess the strong Markov property at certain selected random times. Then, imbedded at such instants, one finds a Markov renewal process.

Let $Y = \{Y(t),\ t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathfrak{X}, P)$ with a topological state space E, and suppose that the function $t \longrightarrow Y(t,\omega)$ is right-continuous and has left-hand limits for almost all $\omega \in \Omega$. A random variable $T:\Omega \longrightarrow [o, \infty]$ is called a stopping time for Y provided that for any $t < \infty$, the occurrence or non-occurrence of the event $\{T \leq t\}$ can be determined once the history $\{Y(u);\ u \leq t\}$ before t is known.

The process $\{Y(t), t \geq 0\}$ is said to be semi-regenerative if there exists a Markov renewal process $(X,T) = \{(X_n, T_n), n \geq 0\}$ with finite state space such that

(a) for each $n \geq 0$, T_n is a stopping time for Y;

(b) for each $n \geq 0$, X_n is determined by $\{Y(u); u \leq T_n\}$

(c) for each $n \geq 0$, $m \geq 1$, $0 \leq t_1 < t_2 < \ldots < t_m$, and function f defined on E^m and positive,

$$E_i[f(Y(T_n+t_1), \ldots, Y(T_n+t_m))/ Y(u); u \leq T_n]$$

$$= E_j \lfloor f(Y(t_1), \ldots, Y(t_m)) \rfloor \text{ on } \{ X_n = j \}$$

where E_i and E_j refer to expectations given the initial state for the Markov chain X.

The theory of Markov renewal processes provides a useful framework for the analysis of many complex stochastic systems. For a summary of basic results and applications of Markov renewal theory one may refer to two excellent survey papers by Cinlar (1969, 1975a).

1.5 AN OVERVIEW OF THE MAIN CONTRIBUTIONS OF THIS THESIS

The main concern of this thesis is the study of some complex stochastic models in Inventories and Queues. By studying the underlying stochastic processes of the models considered, transient state probabilities of the systems are obtained. Steady state results are attempted wherever possible. The associated optimization problems are also discussed for some models.

Renewal theory and Markov renewal theory provide elegant and powerful tools for analysing the underlying stochastic processes of the models considered in this thesis. By identifying the process as a regenerative or semi-regenerative one the transient as well as the steady state solutions are obtained.

Chapter 2 deals with a continuous review (s,S) inventory system with independent non-identically distributed interarrival demand times and random lead times. Explicit expressions are obtained for the distribution of on-hand inventory. An optimization problem associated with this model and also the one associated with the model with zero lead time are discussed. Some numerical examples are considered and the optimal decision variables are obtained.

In chapter 3 we consider two models of (s,S) inventory policy in which the quantity demanded by an arriving customer depends on the availability such that it does not exceed what is available on hand. The interarrival times between demands constitute a family of i.i.d random variables. Model-I assumes zero lead time. Using renewal theoretic arguments, the system state probability distribution at arbitrary time and also the limiting probability distribution are obtained. Optimal decision rule is also indicated.

In Model-II we study the situation with random lead time and in this case the inventory level probability distribution at arbitrary time is derived by applying the techniques of semi-regenerative process. The computation of limiting distribution is also indicated.

Chapter 4 is devoted to a continuous review (s,S) perishable inventory system having exponential life time distribution for the commodities in stock. The demand epochs form a renewal process and the probability distribution of demand magnitude depends on the time elapsed since the previous demand. Lead time is assumed to be zero. For this model the transient and limiting distributions of inventory level are derived by applying the techniques of semi-regenerative process. Some particular cases are also discussed.

In chapter 5 an (s,S) inventory policy with varying ordering levels and random lead times is studied. The quantity ordered is to bring the level back to S and the ordering level is determined based on the number of demands during the previous lead time subject to a maximum level c. Time-dependent system size probabilities are obtained. The correlation between the number of demands during a lead time and the next inventory dry period is obtained. Some illustrations are also given.

The last four chapters are concerned with queueing models. A queueing process of the type $E^k/G^{a,b}/1$ with server vacation is considered in chapter 6. The system is assumed to be of finite capacity. On completing the service of a batch if the server finds less than 'a' units (customers) waiting, he goes on vacation of random duration having a general distribution. If on return from vacation the number of units waiting is again less than 'a', the server extends his vacation for a random length of time independent of and having the same distribution as the previous one. This goes on until on return from vacation there are atleast 'a' units in the system (multiple vacation). The transient system state probability distribution at arbitrary time point is obtained by identifying the regeneration points and using matrix convolutions. Virtual waiting time distribution is also obtained.

Chapter 7 deals with a service system with single and batch services. Customers arriving according to a homogeneous Poisson process enter the service station via a waiting room. At each time when all the customers in the service station are served out, the server scans the waiting room and if he finds less than or equal to a fixed number 'c' of customers he takes them to the service station and serves them one at a time according to FCFS (First Come First Served) rule. If he finds more than 'c' customers the server serves them all together. Single/batch service times have general distributions. Here we consider three models. In the first model the server starts serving as soon as an arrival to an empty system takes place. In the second model when the system becomes empty the server goes on vacation of a random duration. Multiple vacation policy is assumed here. Using Markov renewal theoretic arguments the steady state and transient solution of the system state probabilities and virtual waiting time distributions for the two models are obtained. In Model-III a variant of the standard M/G/1 queue with single and batch services is considered. Here we assume that customers arrive at the service station according to a Poisson process with parameter μ. At the end of each service, if the server finds more than c customers waiting he serves them all together in a batch and if there are less than or equal to c customers, he serves them one at a time according to FCFS rule. Limiting probabilities of the number of customers in the system is obtained explicitly by applying the techniques of semi-regenerative process.

In chapter 8 a single server queueing system with a finite waiting room is considered. The interarrival times of customers and service times have phase type distributions. An arriving customer finding the system full is lost.

Algorithmically tractable matrix formulas are obtained for the computation of stationary queue length distribution.

The last chapter deals with a finite capacity M/G/1 queueing system with server vacation schedules dependent on the number of customers it has served since the completion of the last vacation. Using Markov renewal theory the transient system state probabilities are derived. The virtual waiting time distribution of a customer in the queue is also obtained.

Chapter-2

AN (s,S) INVENTORY SYSTEM WITH NON-IDENTICALLY DISTRIBUTED INTERARRIVAL DEMAND TIMES AND RANDOM LEAD TIMES*

2.1. INTRODUCTION

Inventory systems of (s,S) type had been studied quite extensively in the past. A systematic account of the probabilistic treatment in the study of inventory systems using renewal theoretic arguments has been given by Arrow, Karlin and Scarf (1958). Further details of work carried out in this field can be found in Hadley and Whitin (1963), Veinott (1966), Kaplan (1970), Gross and Harris (1971). Tijms (1972) gave a detailed analysis of (s,S) inventory systems and chapter 3 of his monograph deals with its probabilistic analysis. Sivazlian (1974) has considered an (s,S) inventory model in which unit demands of items occur with arbitrary interarrival times between demands and zero lead time. Srinivasan (1979) examined the same problem with random lead times. Sahin (1979) analysed the model with general interarrival demand distributions and constant lead times. In all the above situations the distribution of on hand inventory

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were computed and associated optimization problems were solved.

In this chapter we consider a continuous review (s,S) inventory model with time between successive unit demands independent but not identically distributed random variables. Specifically X_S, X_{S-1}, ..., X_1,X_0 be the times between successive demands when the inventory levels are S, S-1, ..., 1, 0 respectively. We assume that lead times are independent and identically distributed (i.i.d) random variables and are independent of the arrivals of demands. It is quite natural to expect in a market that time gap between successive demands are non-identically distributed and so this model might be more realistic. Section 2.2 contains a complete description of the model. System state probabilities are derived in Section 2.3. The cost function of the model is formulated in Section 2.4. The steady state behaviour of the system is obtained in Section 2.5 and the last section is concerned with the case when lead times are zero and the associated optimization problem, followed by a numerical example.

The renewal theorem for independent but not identically distributed random variables was given by Smith (1964) which may be used in analysing the model presented here.

2.2. DESCRIPTION OF THE INVENTORY POLICY

Let S be the maximum capacity of a ware house. At time $t = 0$ the inventory level is S. Due to incoming demands the stock level goes on decreasing. The demands are assumed to occur for one unit at a time and the time intervals between the arrivals of two consecutive demands form a family of independent non-identically distributed random variables. As soon as the stock level drops down to s, the reorder level, an order for replenishment is placed for S-s units. We assume that $S > 2s$ to avoid perpetual shortage. The lead time- the time interval measured from the epoch when the stock level drops to s to the epoch when the quantity S-s reaches the ware house- is assumed to be distributed arbitrarily with distribution function $G(.)$ but independent of stock level and demand. Lead times are assumed to be i.i.d. random variables. The market considered here is competitive enough to rule out back-logging of demands and the demands that emanate during the stock out period are deemed to be lost. Thus the stock level can be described by a discrete valued stochastic process $\{ I(t),\ t \geq 0 \}$ with $I(o) = S$.

Let $F_{\alpha}(.)$, $(\alpha = 0,1,2,\ldots,S)$ be the successive distribution function of the time interval X_{α} between the arrivals of two consecutive demands, when there are α ($\alpha = 0,1,2,\ldots,S$) units in the inventory. For the sake of convenience, the underlying distributions are taken as absolutely continuous. The corresponding small letters denote the density functions. All the results can easily be reconstructed, however, for discrete case.

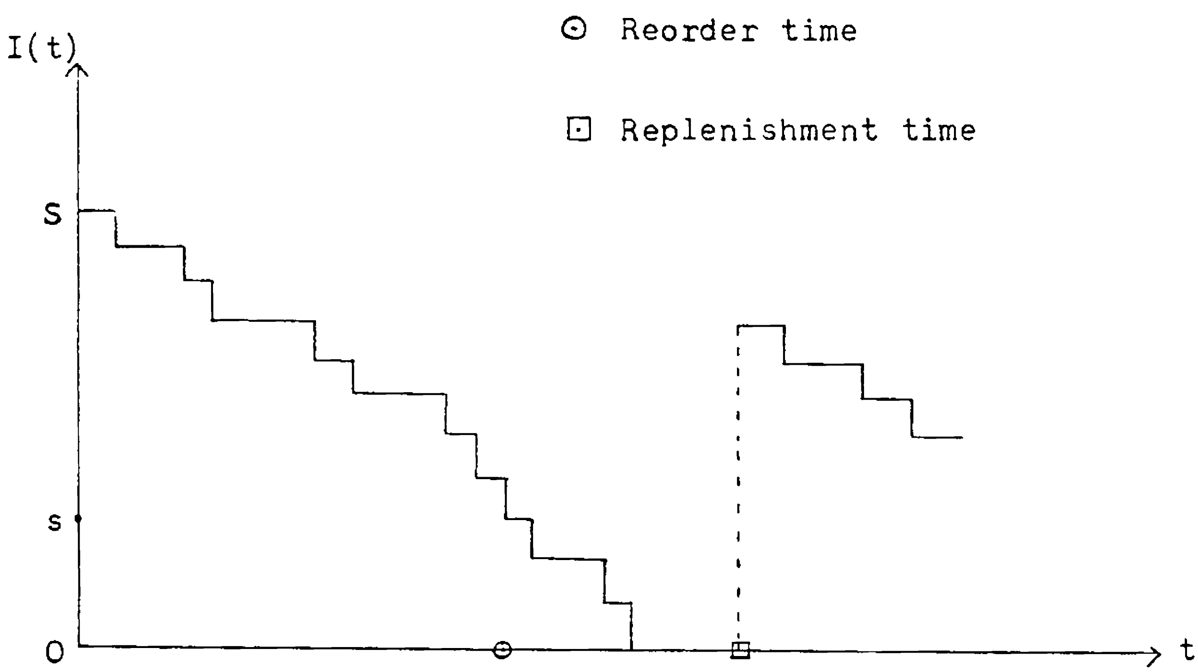

Fig.2.1. A typical plot of the Inventory level against time.

The following notations are used in the sequel:

$I(t)$ - on-hand Inventory level at time t.

$f*g(x)$ - convolution $\int_0^x f(x-y)g(y)dy$ for $f(x)$, $g(x)$ defined on the set of non-negative real numbers.

$f^{*k}(x)$ - k-fold convolution of f(x) with itself, $(f^{*0}(x) \equiv 1)$.

$\bar{F}(.)$ - 1-F(.), the survival function

$\hat{a}(\alpha)$ - Laplace transform $\int_0^\infty e^{-\alpha x} a(x)dx$.

2.3 MAIN RESULTS

Let I(t) denotes the on hand Inventory level at arbitrary time t. The principal quantity of interest is the probability mass function of the inventory level at any arbitrary time t on the time axis. ie., $\Pr\{I(t)=n\}$, $n = 0,1,2,\ldots,S$.

Suppose now that we consider the sequence of times at which the inventory level reaches s (the reorder level) from above. Let Y_1 denote the time elapsed from origin until the first event occured (reaching level s). Y_2 the time-elapsed between the first and second event and so on. The sequence of random variables $\{Y_k\}$, k=1,2,... forms a modified renewal process [See Cox (1962)]. In each of the following expressions we will make use of the renewal density m(u) of the time points at which the inventory level reach s. An explicit expression for m(u) is also given.

(i) $$\Pr\{I(t)=0\} = \int_0^t m(u)\bar{G}(t-u) \int_u^t (f_s * \dots * f_1)(v-u) \left(\sum_{m=0}^{\infty} f_o^{*m}(t-v)\right) dv\, du$$

(ii) For $n=1,2,\dots,s-1$,

$$\Pr\{I(t)=n\} = \int_0^t m(u)\bar{G}(t-u) \int_u^t (f_s * f_{s-1} * \dots * f_{n+1})(v-u)\, \bar{F}_n(t-v)\, dv\, du$$

(iii) $$\Pr\{I(t)=s\} = \int_0^t m(u)\bar{G}(t-u)\bar{F}_s(t-u)\, du$$

(iv) For $n=s+1, s+2, \dots, S-s-1$

$$\Pr\{I(t)=n\} = (F_S * F_{S-1} * \dots * F_{n+1} - F_S * F_{S-1} * \dots * F_n)(t)$$

$$+ \int_0^t m(u) \int_u^t \bar{F}_s(v-u) g(v-u) \int_v^t (f_S * f_{S-1} * \dots * f_{n+1})(w-v)\, \bar{F}_n(t-w)\, dw\, dv\, du$$

$$+ \sum_{k=1}^{s-1} \int_0^t m(u) \int_u^t (f_s * f_{s-1} * \dots * f_{s-k+1})(v-u) \int_v^t \bar{F}_{s-k}(w-v) g(w-u) \int_w^t (f_{S-k} * \dots * f_{n+1})(x-w)\, \bar{F}_n(t-x)\, dx\, dw\, dv\, du$$

$$+ \int_o^t m(u) \int_u^t (f_s * \dots * f_1)(v-u) \int_v^t \Big(\sum_{m=o}^{\infty} f_o^{*m}(w-v) \Big)$$

$$\int_w^t \overline{F}_o(x-w) g(x-u) \int_x^t (f_{S-s} * f_{S-s-1} * \dots * f_{n+1})(y-x)$$

$$\overline{F}_n(t-y) dy\ dx\ dw\ dv\ du.$$

(v) $\Pr\{I(t)=S-s\} = (F_S * F_{S-1} * \dots * F_{S-s+1} - F_S * F_{S-1} * \dots * F_{S-s})(t)$

$$+ \int_o^t m(u) \int_u^t \overline{F}_s(v-u) g(v-u) \int_v^t (f_S * f_{S-1} * \dots * f_{S-s+1})(w-v)$$

$$\overline{F}_{S-s}(t-w) dw\ dv\ du$$

$$+ \sum_{k=1}^{s-1} \int_o^t m(u) \int_u^t (f_s * f_{s-1} * \dots * f_{s-k+1})(v-u) \int_v^t \overline{F}_{s-k}(w-v) g(w-u)$$

$$\int_w^t (f_{S-k} * \dots * f_{S-s+1})(x-w) \overline{F}_{S-s}(t-x) dx\ dw\ dv\ du$$

$$+ \int_o^t m(u) \int_u^t (f_s * f_{s-1} * \dots * f_1)(v-u) \int_v^t \Big(\sum_{m=o}^{\infty} f_o^{*m}(w-v) \Big)$$

$$\int_w^t \overline{F}_o(x-w)\ g(x-u)\ \overline{F}_{S-s}(t-x) dx\ dw\ dv.du.$$

(vi) For $n = S-s+1, S-s+2, \ldots, S-1$

$$\Pr\{I(t)=n\} = (F_S * F_{S-1} * \cdots * F_{n+1} - F_S * F_{S-1} * \cdots * F_n)(t)$$

$$+ \int_0^t m(u) \int_u^t \bar{F}_s(v-u) g(v-u) \int_v^t (f_S * f_{S-1} * \cdots * f_{n+1})(w-v) \bar{F}_n(t-w)\,dw\,dv\,du$$

$$+ \sum_{k=1}^{S-n-1} \int_0^t m(u) \int_u^t (f_s * f_{s-1} * \cdots * f_{s-k+1})(v-u) \int_v^t \bar{F}_{S-k}(w-v) g(w-u) \int_w^t (f_{S-k} * \cdots * f_{n+1})(x-w) \bar{F}_n(t-x)\,dx\,dw\,dv\,du$$

$$+ \int_0^t m(u) \int_u^t (f_s * f_{s-1} * \cdots * f_{s-S+n+1})(v-u) \int_v^t \bar{F}_{s-S+n}(w-v)\, g(w-u)\, \bar{F}_n(t-w)\,dw\,dv\,du$$

and finally

(vii) $$\Pr\{I(t)=S\} = \bar{F}_S(t) + \int_0^t m(u) \int_u^t \bar{F}_s(v-u) g(v-u) \bar{F}_S(t-v)\,dv\,du.$$

Explanation of (i) - (vii):

Since $I(o)=S$, in order to have $I(t)=0$ the inventory must have crossed the level s from above atleast once. Let u be the last instant at which inventory level drops to s. After u, the replenishment of the stock does not materialise upto t and the inventory level reaches zero level at v ($u<v\leq t$) and there may be infinite number of lost demands in $(v,t]$. Using these facts we can arrive at (i). Expressions (ii) and (iii) follow on similar lines.

To prove (iv) we recognise that $I(t) = n$ can happen with or without crossing the level s. The first event can be classified into three mutually exclusive and exhaustive set of events according as (a) after time u (the last instant at which inventory level drops to s) there is replenishment before any demand occurs and after replenishment the inventory level comes down to n at time t, (b) after time u there are exactly k ($k=1,2,\ldots,s-1$) demands, then replenishment takes place and thereafter inventory level drops down to n at time t, and (c) after time u inventory level comes down to zero level, thereafter replenishment occurs and then inventory level drops down to n at time t. Expressions (v), (vi) and (vii) follow similarly.

Let $\emptyset_o(.)$ denotes the probability density function of Y_1 and $\emptyset(.)$ the common probability density function of the random variables $Y_2, Y_3, \ldots$. Then we have

$$\emptyset_o(u) = (f_S * f_{S-1} * \ldots * f_{s+1})(u) \tag{2.3.1}$$

and

$$\emptyset(u) = \int_o^u \bar{F}_s(v) g(v) \; (f_S * f_{S-1} * \ldots * f_{s+1})(u-v) dv$$

$$+ \sum_{k=1}^{s-1} \int_o^t (f_s * f_{s-1} * \ldots * f_{s-k+1})(v) \int_v^u \bar{F}_{s-k}(w-v) g(w)$$

$$(f_{S-k} * \ldots * f_{s+1})(u-w) dw \; dv$$

$$+ \int_o^u (f_s * f_{s-1} * \ldots * f_1)(v) \int_v^u \left(\sum_{m=o}^{\infty} f_o^{*m}(w-v) \right)$$

$$\int_w^u \bar{F}_o(x-w) (f_{S-s} * \ldots * f_{s+1})(u-x) dx \; dw \; dv$$

$$(2.3.2)$$

Then the renewal density of reorder points is given by

$$m(u) = \left(\emptyset_o * \sum_{n=o}^{\infty} \emptyset^{*n} \right)(u). \tag{2.3.3}$$

2.4. COST FUNCTION OF THE MODEL

Having obtained an explicit expression for Pr $\{I(t)=n\}$ in terms of the probability density functions of the basic random variables in question we can obtain the inventory carrying (holding) cost. If h is the holding cost per unit item per unit time, then the total inventory holding cost during the interval (o,t) is

$$H(t) = h \int_{o}^{t} I(u)du \tag{2.4.1}$$

where the above integral can be interpreted in the Ito sense (see Mc Shane (1974)). Taking expected value on both sides of (2.4.1)

we get

$$E[H(t)] = h \sum_{n=1}^{S} n \int_{o}^{t} Pr\{I(t)=n\} du \tag{2.4.2}$$

Let K be the fixed order cost; c = variable procurement cost per unit and k = shortage cost per unit. The average length of time for which there is shortage is $E(L-\sum_{i=1}^{s} X_i)^+$ where L is the lead time and X^+ indicates max (O,X). The expected number of lost demands is therefore equal to $E(L-\sum_{i=1}^{s} X_i)^+/E(X_o)$. So the expected shortage cost per cycle (representing the length of time between two successive epochs at which the inventory level comes to reorder level) is

$$k \quad \frac{E(L-\sum_{i=1}^{s} X_i)^+}{E(X_o)} \qquad (2.4.3)$$

and K+c(S−s) is the fixed cost for procurement per cycle. If we multiply this by M(t), the renewal function corresponding to the renewal process $\{Y_n\}_{n \geq 1}$, we obtain the expected procurement cost over the interval (0,t). The expected shortage cost over the interval (0,t) is

$$k \quad \frac{E(L-\sum_{i=1}^{s} X_i)^+}{E(X_o)} [M(t)-1]$$

Hence we have the total expected cost during the interval (0,t) as

$$C(s,S,t) = h \sum_{n=1}^{S} n \int_{o}^{t} Pr \{I(u)=n\} du + M(t)[K+c(S-s)]$$

$$+ k \quad \frac{E(L-\sum_{i=1}^{s} X_i)^+}{E(X_o)} [M(t)-1] \qquad (2.4.4)$$

2.5. STEADY STATE BEHAVIOUR OF THE SYSTEM

Using the asymptotic results of renewal theory, we obtain the limiting distribution of the discrete valued stochastic process I(t) as follows. The limiting probability mass function $\pi(n)$ is given by

$$\pi(n) = \frac{\int_0^\infty p_o(n,u)\,du}{\int_0^\infty x\,\phi(x)\,dx}, \quad 0 \le n \le S$$

where

$$p_o(n,t) = \lim_{\Delta \to 0} \Pr\left\{I(t_o+t)=n,\ N(t_o+t)-N(t_o)=0 / I(t_o) = s < I(t_o-\Delta)\right\},$$

$$N(t) = \text{Sup}\left\{n \ge 0 : Y_1+Y_2+ \dots + Y_n \le t\right\}.$$

2.6. THE MODEL WITH ZERO LEAD TIME

As a particular case, if we assume that the lead time is zero in the model considered above, then $\{I(t), t \ge 0\}$ is a discrete valued continuous parameter stochastic process taking values s+1, s+2, ..., S. Here the sequence of random variables $\{Y_k\}$, k=1,2,..., forms a renewal process in which distribution of Y_k is given by

$$\Pr\{Y \le y\} = \int_0^y (f_S * f_{S-1} * \cdots * f_{s+1})(u)du$$

Its density be denoted by $f(\circ)$. The probability that the k^{th} order, $k=1,2,3,\ldots$ will be placed in the interval t and t+dt is

$$\Pr\{t < Y_1 + Y_2 + \ldots + Y_k \le t+dt\} = f^{*k}(t),$$
$$k=1,2,3, \ldots$$

Then the probability mass function of $I(t)$ is:

For $n = s+1, s+2, \ldots, S-1$

$$\Pr\{I(t)=n\} = \Big[\int_0^t (f_S * f_{S-1} * \cdots * f_{n+1})(u)du - \int_0^t (f_S * f_{S-1} * \cdots * f_n)(u)du \Big]$$
$$+ \sum_{k=1}^{\infty} \int_0^t \Big[\int_0^x (f_S * f_{S-1} * \cdots * f_{n+1})(u)\, du - \int_0^x (f_S * f_{S-1} * \cdots * f_n)(u)du \Big] f^{*k}(t-x)dx$$

and $\Pr\{I(t)=S\} = [1-\int_0^t f_S(u)du] + \sum_{k=1}^{\infty} \int_0^t [1-\int_0^x f_S(u)du] f^{*k}(t-x)dx$

Let

$$\hat{p}(n,\alpha) = \int_0^{\infty} e^{-\alpha t} \Pr\{I(t)=n\}\, dt$$

$$\hat{f}_n(\alpha) = \int_0^{\infty} e^{-\alpha t} f_n(t)dt$$

and $$\hat{f}(\alpha) = \int_0^{\infty} e^{-\alpha t} f(t)dt$$

Then

For $n=s+1, s+2, \ldots, S-1$

$$\hat{p}(n,\alpha) = \frac{1}{\alpha} [\hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha) - \hat{f}_S(\alpha) \ldots \hat{f}_n(\alpha)]$$

$$+ \sum_{k=1}^{\infty} \frac{1}{\alpha}[\hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha) - \hat{f}_S(\alpha) \ldots \hat{f}_n(\alpha)][\hat{f}(\alpha)]^k$$

$$= \frac{1}{\alpha} \hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha)[1-\hat{f}_n(\alpha)][1-\hat{f}(\alpha)]^{-1} \tag{2.6.1}$$

and $$\hat{p}(S,\alpha) = \frac{1}{\alpha} [1-\hat{f}_S(\alpha)][1-\hat{f}(\alpha)]^{-1} \tag{2.6.2}$$

STEADY STATE DISTRIBUTION OF THE INVENTORY LEVEL

Let p_n be the probability that exactly n units, $n = s+1, s+2, \ldots, S$ are in the inventory in the steady state.

Then by a Tauberian theorem, [see Widder (1946)]

$$P_n = \Pr\{I=n\} = \lim_{t \to \infty} \Pr\{I(t)=n\}$$

$$= \lim_{\alpha \to 0} \alpha\, \hat{p}(n,\alpha)$$

For $n=s+1, s+2, \ldots, S-1$,

$$P_n = \lim_{\alpha \to 0} \frac{\hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha)[1-\hat{f}_n(\alpha)]}{[1-\hat{f}(\alpha)]}$$

$$= \lim_{\alpha \to 0} \frac{\hat{f}_S(\alpha) \ldots \hat{f}_{n+1}(\alpha)\ \hat{f}_n{}'(\alpha)}{\hat{f}'(\alpha)} \quad \text{(using L' Hospital's rule)}$$

$$= \frac{E(X_n)}{\sum_{i=s+1}^{S} E(X_i)}$$

$$P_S = \lim_{\alpha \to 0} \frac{1-\hat{f}_S(\alpha)}{1-\hat{f}(\alpha)} = \lim_{\alpha \to 0} \frac{\hat{f}_S{}'(\alpha)}{\hat{f}'(\alpha)}$$

$$= \frac{E(X_S)}{\sum_{i=s+1}^{S} E(X_i)}$$

Thus

$$P_n = \frac{E(X_n)}{\sum_{i=s+1}^{S} E(X_i)}, \quad n=s+1, s+2, \ldots, S. \qquad (2.6.2)$$

Here when we assume X_n's to be independent identically distributed random variables we get the results of Sivazlian (1974).

OBJECTIVE FUNCTION AND OPTIMAL DECISION RULES FOR ZERO LEAD TIME CASE:

If delivery of orders is instantaneous, then no shortage is allowed. Our objective function is the steady state total expected cost per unit time; we have to choose the decision variables s and S so as to minimize the objective function.

The expected time elapsed between two successive orders is

$$E(Y) = \sum_{i=s+1}^{S} E(X_i)$$

Therefore the expected number of orders placed per unit time is

$$\frac{1}{E(Y)} = \frac{1}{\sum_{i=s+1}^{S} E(X_i)} \tag{2.6.3}$$

Expected inventory level at any instant of time is

$$E(I) = \sum_{n=s+1}^{S} nP_n = \sum_{n=s+1}^{S} nE(X_n) / \sum_{i=s+1}^{S} E(X_i) \tag{2.6.4}$$

Total expected cost per unit time is

$$C(s,S) = \frac{K+c(S-s)}{E(Y)} + h\,E(I)$$

where K = fixed order cost, c = variable procurement cost per unit, h = holding cost per unit per unit time.

Substituting for E(Y) and E(I), we have

$$C(s,S) = \Big[\, K+c(S-s)+h \sum_{n=s+1}^{S} n\,E(X_n)\Big] \Big/ \sum_{n=s+1}^{S} E(X_n) \qquad (2.6.5)$$

where s and S are non-negative integers and s < S.

Obviously the above expression is not separable in s and S. The minimization of C(s,S) can be done, knowing the first moments of the interarrival times, with the aid of a computer.

Here s* = 0 is the optimal value. The optimum value of S is obtained by minimizing the function

$$C_1(S) = \Big[\, K+cS+h \sum_{n=1}^{S} n\,E(X_n)\Big] \Big/ \sum_{n=1}^{S} E(X_n) \qquad (2.6.6)$$

over the set of positive integers S. We shall now give an example typical of the above case.

EXAMPLE

Let X_n follow exponential distribution with parameter λ_n. Assume that $S \leq 25$. For specified values of K,c,h and λ_n's the optimal values S* and $C_1(S^*)$ obtained using a computer are given below:

	λ Values	S*	$C_1(S^*)$
K=50, c=1, h=1	$\lambda_n = n$ (n=1,2,...,25)	11	23.842
	$\lambda_n = 1/n$(n=1,2,...,25)	6	7.000
	λ_n's(n=1,2,...,25) are 1.2, 2.1, 1.4, 2.3, 2.5 4.3, 5.0, 2.3, 4.1, 4.0 4.2, 1.5, 2.4, 3.6, 4.6 3.2, 1.7, 4.9, 1.3, 2.0 3.9, 4.2, 1.7, 2.4, 4.9	14	17.625
	λ_n's(n=1,2,...,25) are 1.3, 2.4, 4.2, 5.0, 3.1 4.0, 2.8, 4.2, 3.9, 2.1 1.2, 1.9, 1.7, 2.8, 3.9 4.0, 4.9, 4.8, 3.7, 3.6 2.5, 2.4, 1.3, 1.2, 3.6	14	18.841
	λ_n's reversed in order of IIIrd row	14	19.085
	λ_n's reversed in order of IVth row	16	18.102
K=50, c=2, h=0.5	$\lambda_n = n$, n=1,2,...,25	10	25.606
	$\lambda_n = 1/n$, n=1,2,...,25	8	4.667

Remark:

The model analysed in Section 2.3 can be extended to allow vacations to the server whenever the system becomes empty. In this case also one can write expressions for the inventory level probabilities at arbitrary time points but the optimization part seems to be difficult.

Chapter-3

AN (s,S) INVENTORY SYSTEM WITH STATE DEPENDENT DEMANDS

3.1 INTRODUCTION

Conventional inventory models assume demand and inventory level as independent quantities. In this chapter we consider a continuous review (s,S) inventory model in which quantity demanded by each arriving unit is a positive integer valued random variable that depends on the present inventory level. The time durations between successive demands are i.i.d random variables with finite expectations. It is assumed that quantity demanded will not exceed what is available. In situations like famine etc. the Government directs the shopkeepers to exhibit the quantity of items available with them and its price. Customers rationaly buy items depending on its availability. Sometimes customers may be motivated to procure with the ease of availability. This kind of behaviour may be approximated by a stock dependent demand pattern.

Gupta and Vrat (1986) suggested an EOQ model through cost minimization technique to take care of stock dependent consumption rate. This could not take care of stock-dependent demand rate except where the demand rate is dependent on replenishment size. Mandal and Phaujdar(1989) proposed an EOQ model with instantaneous replenishment, without shortages and demand rate depending upon the current stock level, which is assumed to be linearly increasing with stock status.

Two models are treated in this chapter. In section 3.2 we consider the model with zero lead time. Using renewal theoretic arguments, the system state probability distribution at arbitrary time and also the limiting distribution are obtained. The results are illustrated by a numerical example and a method of finding optimal decision rules is briefly discussed. Section 3.3 is concerned with the model with random lead time. In this case inventory level probability distribution at arbitrary time is derived by applying the techniques of semi-regenerative process. The computation of limiting distribution is also indicated.

We introduce the following notations used in this chapter.

$I(t)$ - Inventory level at time t (≥ 0)

$F(.)$ - Distribution function of time between two successive demands (interarrival time distribution).

$f(.)$ - Density function of $F(.)$.

$F^{*n}(t)$	-	n-fold convolution of F with itself, $n=1,2,\ldots,$ with $F^{*0}(t) \equiv 1$.
$G(.)$	-	Lead time distribution function
$g(.)$	-	Density function of $G(.)$
$k(u)$	-	$\sum_{n=0}^{\infty} f^{*n}(u)$
$P(n,t)$	-	Probability that $I(t)=n$, $n=1,2,\ldots,S$.
$\hat{P}(n,\alpha)$	-	Laplace transform of $P(n,t)$, $n=1,2,\ldots,S$.
q_{ij}	-	Probability that j units are demanded when the inventory level is i
P_{ij}	-	Probability that at a demand epoch there were i units and due to the demand the level is brought to level j.
M	-	$S-s$
E	-	$\{0,1,2,\ldots,s\}$
$\bar{E}$	-	$\{0,1,2,\ldots,S\}$
F	-	$\{s+1,\ s+2,\ \ldots,\ S\}$

$$\phi^n_{i,j} = \begin{cases} \sum\limits_{\substack{i_1, i_2, \ldots, i_{n-1} \\ (i > i_1 > i_2 > \ldots > i_{n-1} > j)}} P_{ii_1} P_{i_1 i_2} \cdots P_{i_{n-1} j}; & s \geq i > j \\ \sum\limits_{\substack{i_1, i_2, \ldots, i_{n-1} \\ (i > i_1 > i_2 > \ldots > i_{n-1} > s)}} P_{ii_1} P_{i_1 i_2} \cdots P_{i_{n-1} j}; & i > s,\ j \leq s \end{cases}$$

3.2 MODEL-I: ZERO LEAD TIME CASE

Here we assume that lead time is zero and no shortage is permitted. As soon as the inventory level falls to s or below an order is placed to bring back the inventory to S. If X_n denotes the inventory level after the n^{th} demand, then $\{X_n\}$ forms a Markov chain with state space $F = \{s+1, s+2, \ldots, S\}$ and its transition probabilities are given by

$$P_{ij} = \Pr\{X_{n+1} = j \mid X_n = i\} = \begin{cases} 0 & \text{for } i \leq j,\ j \neq S \\ q_{i(i-j)} & \text{for } i > j,\ j \neq S \\ \sum\limits_{k=i-s}^{i} q_{ik} & \text{for } i = s+1, s+2, \ldots, S; \\ & \quad\ j = S. \end{cases}$$

First of all, we shall obtain the distribution of a cycle which is defined as the time duration between two successive S to S transitions. We assume that $X_o = I(0) = S$.

Let Z be the length of a cycle. Then

$$h(z) = \Pr\{ z < Z \leq z+dz \}$$

$$= \sum_{n=1}^{M} \Pr\{z < Y_1+Y_2+ \ldots + Y_n \leq z+dz\} \phi^n_{S,S}$$

where $Y_1, Y_2, \ldots$ are i.i.d random variables with distribution F(.) and $\phi^n_{S,S}$ is the probability that starting from S, the inventory level reaches back to S at the n^{th} transition for the first time.

ie. $$\phi^n_{S,S} = \sum_{\substack{i_1, i_2, \ldots, i_{n-1} \in F \\ S > i_1 > i_2 > \ldots > i_{n-1} > s}} P_{Si_1} P_{i_1 i_2} \cdots P_{i_{n-1}S}$$

Thus $$h(z) = \sum_{n=1}^{M} f^{*n}(z)\, \phi^n_{S,S} \qquad (3.2.1)$$

Let $Z_1, Z_2, \ldots$ be the lengths of successive cycles. The distribution of Z_i's are i.i.d with p.d.f. h(.). Then $\{Z_i\}$ forms a renewal process and the corresponding renewal density is given by

$$m(u) = \sum_{r=1}^{\infty} h^{*r}(u) \qquad (3.2.2)$$

Now we can find out the probability distribution of the system size. We have

$$P(S,t) = 1-F(t) + \int_0^t m(u)\,[1-F(t-u)]du \qquad (3.2.3)$$

and for $s+1 \leq i < S$

$$P(i,t) = \sum_{j=1}^{S-i} [F^{*j}(t)-F^{*(j+1)}(t)]\, \phi^j_{S,i}$$

$$+ \int_0^t m(u) \sum_{j=1}^{S-i} [F^{*j}(t-u)-F^{*j+1}(t-u)]\phi^j_{S,i}\, du$$

$$(3.2.4)$$

where $\phi^j_{S,i}$ is the probability of first visit to i in j transitions, starting from S, without visiting the state S in between.

LIMITING PROBABILITY DISTRIBUTION

Taking Laplace transforms of both sides of (3.2.3) we get

$$\hat{P}(S,\alpha) = \frac{1}{\alpha}\,[1-\hat{f}(\alpha)] + \hat{m}(\alpha)\,\frac{1}{\alpha}[1-\hat{f}(\alpha)]$$

But $$\hat{m}(\alpha) = \sum_{r=1}^{\infty} [\hat{h}(\alpha)]^r = \frac{\hat{h}(\alpha)}{1-\hat{h}(\alpha)} \quad (\text{since } \hat{h}(\alpha) < 1)$$

where $\hat{h}(\alpha) = \sum_{n=1}^{M} [\hat{f}(\alpha)]^n \ \phi_{S,S}^n$

Therefore

$$\hat{P}(S,\alpha) = \frac{1}{\alpha}[1-\hat{f}(\alpha)] + \left[1+ \frac{\sum_{n=1}^{M} [\hat{f}(\alpha)]^n \phi_{S,S}^n}{1- \sum_{n=1}^{M} [\hat{f}(\alpha)]^n \phi_{S,S}^n}\right] \tag{3.2.5}$$

Similarly, taking Laplace transforms of both sides of (3.2.4), we get

$$\hat{P}(i,\alpha) = \sum_{j=1}^{S-i} \left[\frac{1}{\alpha}[\hat{f}(\alpha)]^j - \frac{1}{\alpha}[\hat{f}(\alpha)]^{j+1}\right] \phi_{S,i}^j$$

$$+ \sum_{j=1}^{S-i} \hat{m}(\alpha) \left[\frac{1}{\alpha}[\hat{f}(\alpha)]^j - \frac{1}{\alpha}[\hat{f}(\alpha)]^{j+1}\right] \phi_{S,i}^j$$

$$= \sum_{j=1}^{S-i} \frac{1}{\alpha}[\hat{f}(\alpha)]^j \ [1-\hat{f}(\alpha)] \ \phi_{S,i}^j$$

$$\left[1 + \frac{\sum_{n=1}^{M} [\hat{f}(\alpha)]^n \phi_{S,S}^n}{1- \sum_{n=1}^{M} [\hat{f}(\alpha)]^n \phi_{S,S}^n}\right], \tag{3.2.6}$$

$i = s+1, s+2, \ldots, S-1.$

Let P_n be the probability that the inventory level is n (n= s+1, s+2, ..., S) in the steady state. Then,

$$P_n = \lim_{t \longrightarrow \infty} \Pr\{I(t)=n\} = \lim_{\alpha \longrightarrow 0} \alpha \hat{P}(n,\alpha)$$

It is easy to verify from (3.2.5) that

$$P_S = \lim_{\alpha \longrightarrow 0} \frac{[1-\hat{f}(\alpha)] \sum_{n=1}^{M} [\hat{f}(\alpha)]^n \phi^n_{S,S}}{1-\sum_{n=1}^{M} [\hat{f}(\alpha)]^n \phi^n_{S,S}}$$

To obtain the limiting value of this indeterminate expression, we apply L' Hospital's rule once, yielding

$$P_S = \frac{\sum_{n=1}^{M} \phi^n_{S,S}}{\sum_{n=1}^{M} n\, \phi^n_{S,S}} \qquad (3.2.7)$$

Similarly, we get

$$P_i = \frac{\sum_{j=1}^{S-i} \phi^j_{S,i} \sum_{n=1}^{M} \phi^n_{S,S}}{\sum_{n=1}^{M} n\, \phi^n_{S,S}}, \quad i = s+1, s+2, \ldots, S-1 \qquad (3.2.8)$$

Thus, in the steady state, the inventory level is distributed as given in (3.2.7) and (3.2.8) and is independent of the distribution of the interarrival time between demands. One can easily see that this result reduces to Sivazlian (1974) when unit quantities are demanded. Further this reduces to Sahin (1983) and Ramanarayanan and Jacob (1987) with zero lead time when quantity demanded are i.i.d random variables.

Example

Suppose that s=0, S=5. Then $F = \{1,2,3,4,5\}$

Let $Q = (q_{ij})_{i,j \in F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$

Then $P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$

$\emptyset^k_{S,i}$, $i \in F$ and $k = 1,2,\ldots,5$ are obtained as given in the following table.

i \ k	1	2	3	4	5
1	1/5	13/60	3/40	1/120	0
2	1/5	7/60	1/60	0	0
3	1/5	1/20	0	0	0
4	1/5	0	0	0	0
5	1/5	5/12	7/24	1/12	1/120

The steady-state probabilities are calculated using (3.2.7) and (3.2.8) as

n	P_n
1	0.218978134
2	0.145985422
3	0.109489067
4	0.087591253
5	0.437956268
Σ	1.000000144

JOINT DISTRIBUTION OF THE QUANTITY ORDERED AND THE LENGTH OF A CYCLE

Let Q denotes the quantity ordered in a cycle whose length is Z. Then the joint distribution of Q and Z is

$$\eta(n,z) = \Pr\{Q=n,\ z < Z \leq z+dz\}$$

$$= \sum_{k=1}^{M} \Pr\{Q=n,\ z < Z \leq z+dz \mid k \text{ demands}\} \Pr\{k \text{ demands}\}$$

$$= \sum_{k=1}^{M} \sum_{S>i_1>i_2>\ldots>i_{k-1}>s} P_{Si_1} P_{i_1 i_2}, \ldots, P_{i_{k-2} i_{k-1}} q_{i_{k-1}, i_{k-1}-(S-n)} f^{*k}(z)$$

Now the expected value of the quantity ordered per unit time can be calculated as

$$E[Q/Z] = \sum_{n=M}^{S} \int_{0}^{\infty} (n/z)\, \eta(n,z)\, dz \qquad (3.2.9)$$

Also $$E[Z] = \sum_{n=1}^{M} n\, E(Y)\, \phi_{SS}^{n} \qquad (3.2.10)$$

OBJECTIVE FUNCTION AND OPTIMAL DECISION RULE

We assume that the procurement cost consists of a fixed cost K and variable cost c per unit. The holding cost is h per unit per unit time. Our objective function here is the steady state expected total cost per unit time; the decision variables s and S are to be selected so as to minimize the objective function.

Expected inventory at any given time is

$$E(I) = \sum_{i=s+1}^{S} i\, P_i,$$

where P_i's are given by (3.2.7) and (3.2.8).

The total expected cost per unit time is

$$C(s,S) = \frac{K}{E(Z)} + c\, E[Q/Z] + h\, E(I) \qquad (3.2.11)$$

The value of s and S which minimize the above expression are the optimal values (for a given (q_{ij})).

Remark

The model with zero lead time and quantity demanded not restricted to be atmost what is available can be analysed in a similar fashion if we assume that the replenishment is done in such a way as to bring the inventory on hand back to S after meeting the demand that has just taken place.

3.3 MODEL-II: RANDOM LEAD TIME CASE

In this section we consider the model with random lead times. The quantities demanded depend on the inventory level at the demand epoch. Not more than what is available will be demanded (will be sold). Lead times are i.i.d random variables with distribution function $G(.)$ and density $g(.)$. No backlogging is allowed. As soon as the inventory level falls to s or below due to a demand, an order is placed and the quantity ordered for is to bring back the level to S (ie. if inventory on hand is i ($\leq$ s) at the time of ordering, then the quantity ordered is S-i. The demands that arise during a dry period are lost (in fact, by our assumption no item will be demanded by the arrivals during dry period).

Let $Y_o, Y_1, Y_2, \ldots, Y_n \ldots$ be the successive inventory levels at which orders are placed and $0 = T_o < T_1 < T_2 < \ldots < T_n < \ldots$ be the corresponding ordering epochs. Then $\{(Y_n, T_n),\ n = 0,1,2,\ldots\}$ constitutes a Markov renewal process on the state space $E = \{0,1,2,\ldots,s\}$. The semi-Markov kernel of this process is

$$Q(i,j,t) = P[Y_{n+1}=j,\ T_{n+1}-T_n \leq t \ / \ Y_n=i]$$

and is given by

$$Q(i,j,t) = \int_o^t \int_u^t \int_v^t \sum_{n=1}^{i} f^{*n}(u)\ \phi_{i,o}^{n}\ k(v-u)\ g(w)$$

$$\sum_{m=1}^{S-i-j} \frac{[F^{*m}(t-v)-F^{*(m+1)}(t-v)]}{[1-F(w-v)\ ]}\ \phi_{S-i,j}^{m} dw\ dv\ du$$

$$+ \int_o^t \int_u^t \sum_{k=1}^{i} \sum_{n=o}^{i-k} f^{*n}(u)\ \phi_{i,k}^{n}\ g(v)$$

$$\sum_{m=1}^{S-i+k-j} \frac{[F^{*m}(t-v)-F^{*(m+1)}(t-v)]}{[1-F(v-u)]} \phi_{S-i+k,j}^{m} dv\ du,$$

$$i,j \in E \qquad (3.3.1)$$

In the above expression for $Q(i,j,t)$, the first term on the right deals with the case of arrivals (demands) taking place during dry period and second one considers the case of no dry period in between two consecutive replenishments. Let an order placing point be taken as time origin and suppose at such a point the level falls to $i(\leq s)$, another n $(n=1,2,..,i)$ demands bring it to level zero at time u, (if at time 0 the level has not already become zero due to the demand) then follows a dry period with a number of arrivals, this is represented by $k(v-u)$, demand quantity by those arrivals is zero by our assumption- or we may call these unmet demands, the last such taking place at time v. The replenishment takes place at time w(> v). Now the inventory level is S-i. The next demand (the one after time v) takes place after time w and in the interval (w,t) there are exactly m demands, taking away a total of S-i-j units to bring the level to j($\leq$ s) thereby resulting in the next order placing. This explains the first term on the right in (3.3.1). The second term is similarly explained except that in this case there is no dry period.

The Markov renewal functions of the process is given by

$$R(i,j,t) = \sum_{n=o}^{\infty} Q^n(i,j,t), \quad i,j \in E \tag{3.3.2}$$

where

$$Q^n(i,j,t) = \Pr [\, Y_n=j,\ T_n \leq t \,/\, Y_o=i \,]$$

$$= \sum_{k \in E} \int_o^t Q(i,k,du) Q^{n-1}(k,j,t-u)$$

$$\text{for } n \geq 1,\ t \geq 0$$

and $$Q^o(i,j,t) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{for all } t \geq 0$$

TIME DEPENDENT PROBABILITY DISTRIBUTION OF INVENTORY LEVEL.

Assume that at time zero an order has just been placed with inventory level at i ($\leq$ s) and the demand process starts. Define

$$P(i,j,t) = \Pr\left[I(t)=j \,/\, I(0+)= Y_0 = i\right], \quad i \in E,\ j \in \bar{\bar{E}}.$$

Since the stochastic process $\{I(t),\ t \geq 0\}$ is a semi-regenerative process with the embedded Markov renewal process $(Y,T) = \{(Y_n,T_n),\ n=0,1,2,\ldots\}$, the function $P(i,j,t)$ satisfies the following Markov renewal equation. [see Cinlar (1975)].

$$P(i,j,t) = K(i,j,t) + \int_0^t \sum_{r \in E} Q(i,r,du)\, P(r,j,t-u), \qquad (3.3.3)$$

$$i \in E \text{ and } j \in \bar{\bar{E}}$$

where

$$K(i,j,t) = \Pr[I(t)=j,\ T_1>t/I(0+)=i],\ i \in E,\ j \in \bar{\bar{E}}$$

For any $t \geq 0$, the function $K(i,j,t)$ is given by

i) for $i \in E$ and $i < j \leq s$

$$K(i,j,t) = 0$$

ii) for $i \in E$, $j < i$

$$K(i,j,t) = \overline{G}(t) \sum_{m=1}^{i-j} [F^{*m}(t) - F^{*(m+1)}(t)] \emptyset^{m}_{i,j}$$

iii) for $i \in E$, $s < j \leq S-i$

$$K(i,j,t) = \sum_{k=1}^{i} \int_{o}^{t} \int_{u}^{t} \sum_{n=1}^{i-k} f^{*n}(u)\ \emptyset^{n}_{i,k}\ g(v)$$

$$\sum_{m=1}^{S-i+k-j} \frac{[F^{*m}(t-u) - F^{*(m+1)}(t-u)]}{[1-F(v-w)]}\ dv\ du$$

$$+ \int_{o}^{t} \int_{u}^{t} \int_{v}^{t} \sum_{n=1}^{i} f^{*n}(u)\ \emptyset^{n}_{i,o}\ k(v-u)\ g(w)$$

$$\sum_{m=1}^{S-i-j} \frac{[F^{*m}(t-v) - F^{*(m+1)}(t-v)]}{[1-F(w-v)]}\ dw\ dv\ du$$

and finally

iv) for $i \in E$, $S-i < j \leq S$

$$K(i,j,t) = \int_{o}^{t} \int_{u}^{t} \sum_{k=j-S-i}^{i} \sum_{n=1}^{i-k} f^{*n}(u)\ \emptyset^{n}_{i,k}\ g(v)$$

$$\sum_{m=1}^{S-i+k-j} \frac{[F^{*m}(t-u) - F^{*(m+1)}(t-u)]}{[1-F(v-u)]}\ dv\ du$$

Let $\hat{P}_\alpha$, $\hat{K}_\alpha$ and $\hat{Q}_\alpha$ denote matrices whose $(i,j)^{th}$ elements are $\hat{P}(i,j,\alpha)$, $\hat{K}(i,j,\alpha)$ and $\hat{Q}(i,j,\alpha)$ respectively, where

$$\hat{Q}(i,j,\alpha) = \int_0^\infty \exp(-\alpha t)\, Q(i,j,dt)$$

Then the Laplace transform of the set of Markov renewal equations can be expressed as

$$\hat{P}_\alpha = \hat{K}_\alpha + \hat{Q}_\alpha \hat{P}_\alpha$$

which in turn yields

$$\hat{P}_\alpha = (I-\hat{Q}_\alpha)^{-1} \hat{K}_\alpha = \hat{R}_\alpha \hat{K}_\alpha \tag{3.3.4}$$

where $\hat{R}_\alpha$ is the matrix of Laplace transform of Markov renewal functions of the Markov renewal process (Y,T) which exists for $\alpha > 0$ [see Cinlar (1975)].

LIMITING DISTRIBUTION OF THE INVENTORY LEVEL

In order to obtain the limiting distribution of the stock level, consider the Markov chain $Y = \{Y_n,\ n \geq 0\}$ associated with the Markov renewal process (Y,T). The transition probability matrix $Q' = (Q'(i,j))$ of order s is given by

$$Q'(i,j) = \lim_{t\to\infty} Q(i,j,t)$$

If the chain Y is irreducible, it possesses a unique stationary distribution $\bar{\pi} = (\pi_o, \pi_1, \ldots, \pi_s)$ which satisfies $\bar{\pi}\, Q' = \bar{\pi}$ and $\Sigma\, \pi_j = 1$.

Let $\bar{P} = (P_o, P_1, P_2, \ldots, P_S)$ denotes the steady state probability vector of the inventory level where $P_j = \lim_{t\to\infty} P(i,j,t)$ is the limiting distribution of the inventory level. Now making use of the result given in Cinlar (1975) and assuming that (Y,T) is irreducible recurrent aperiodic we have

$$P_j = \sum_{k\in E} \pi_k \int_o^\infty K(k,j,t)dt \Big/ \sum_{k\in E} \pi_k m_k \qquad (3.3.5)$$

where m_k is the mean sojourn time in state k, given by

$$m_k = \int_o^\infty [\, 1 - \sum_j Q(k,j,t)]\, dt.$$

Chapter-4

MARKOV RENEWAL THEORETIC ANALYSIS OF A PERISHABLE INVENTORY SYSTEM

4.1 INTRODUCTION

Inventory systems of (s,S) type had been studied quite extensively in the past. The details of the work carried out in this field can be found in Arrow, Karlin and Scarf (1958), Hadley and Whittin (1963), Veinott(1966), Sivazlian (1974), Srinivasan (1979), Sahin (1979), Beckmann and Srinivasan (1987) etc.

The earliest work on the decay (perishability) problem is due to Ghare and Schrader(1963) who considered the generalization of the standard EOQ model without shortages. Their model was extended to more general types of deterioration by Covert and Philip (1973) and Shah (1977). In his paper on perishable inventory systems, Nahmias (1982) reviews various models and objective functions in the analysis of such systems. Motivated by the study of blood bank models Kaspi and Perry (1983, 1984) and Perry (1985) have studied inventory systems for perishable commodities in which life time of the items stored are fixed as well as random variables. They utilized the analogy between these systems and a queueing system with impatient customers, to study the process of lost demands, the number of units in the system etc. Kalpakam and Arivarignan (1985a) have studied an (s,S) inventory system having one exhibiting item subject to random failure and obtained the limiting distribution of position inventory by the technique of semi-regenerative process, when quantity demanded is one unit. Again (s,S) inventory system with one exhibiting item subject to exponentially distributed failure time and quantity demanded by an arriving unit depending on the time elapsed since the last arrival is investigated in Kalpakam and Arivarignan (1985b). In both cases they assume zero lead time. The exhibited item on failure is immediately replaced by another item from the stock.

This chapter is devoted to a continuous review (s,S) inventory system in which depletion of stock takes place due to random demand as well as random failures of items, under the assumption that the demand epochs form a renewal process and the demand magnitude have an arbitrary distribution which depend on the interarrival time of demand epochs. Each unit in the stock has a random life time. The replenishment of stock is assumed to be instantaneous.

Here the inventory levels change due to demand or due to depletion of items. Clearly the instants at which units are removed from the stock do not form a renewal process. Hence the usual analysis of the stochastic process of inventory level through renewal theoretic methods fails. We analyse the model by identifying an embedded Markov renewal structure in the stochastic process of the stock level which is seen to be semi-regenerative.

This chapter is organized as follows. Section 4.2 is devoted to problem formulation and analysis of the model. The transient solution of the model is derived explicitly in this section. In section 4.3 we derive the steady state solution making use of the results obtained in Section 4.2. In Section 4.4 we mention some special cases of the model described earlier.

4.2. PROBLEM FORMULATION AND ANALYSIS

The maximum capacity of the warehouse is fixed as S units. Each item in the inventory has negative exponential life time with parameter μ. The failed items are disposed off. The demands are assumed to occur in such a fashion that the time interval between two consecutive demand points constitute a family of i.i.d random variables with common distribution function $F(.)$, an arbitrary conditional probability mass function for demand magnitudes depending only on the time elapsed from the previous demand point.

Further whenever the st ck level drops to a level less than or equal to $s(< S)$, an order for replenishment is placed to bring the st ck level back to S instantaneously.

The following notations are used in this chapter:

$f(t)*g(t)$ - Convolution of functions $f(t)$ and $g(t)$

$f^{*n}(t)$ - n-fold convolution of $f(t)$ $(f^{*0}(t) \equiv 1)$

$\hat{f}(\alpha)$ - Laplace transform of $f(t)$.

$\delta(n)$ - $\begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$

N - $\{1,2,3, \ldots\}$

N^0 - $\{0,1,2, \ldots\}$

M - $S-s$

E - $\{1,2,\ldots,M\}$

$\bar{F}(t)$ - $1-F(t)$

$b_k(t)$ - The conditional probability that the quantity demanded is k, given that the time interval since the last demand is t ($k=1,2,\ldots$)

$\bar{b}_k(t)$ - $\sum_{r=k}^{\infty} b_r(t)$ ($k=1,2,\ldots$)

$\phi(t)$ - $1-\exp(-\mu t)$

$_m\psi_n(t)$ - $\binom{m}{n} [\, \phi(t) \,]^n [\, 1-\phi(t) \,]^{m-n}$,

$n = 0,1,2,\ldots, m$
$m = s+1, s+2, \ldots, S.$

(ie. probability that exactly n out of m stocked items fail in $[0,t]$).

$_m\eta_n(t)$ - $\sum_{k=n}^{m} {_m\psi_k(t)}$

$_m\eta_n'(t)$ - $\frac{d}{dt} (\, _m\eta_n(t) \,)$

$I(t)$ - Inventory level (stock level) at time t.

Let $0 = T_0 < T_1 < T_2 < \ldots$ be the times at which demand occurs. $I(t)$ assumes values in the set $\{s+1, s+2, \ldots, s+M\}$. Let us define

$$I_n = I(T_n+),\ n \in N^0$$

Let $\alpha_1 < \alpha_2 < \ldots$ be the times at which stock is replenished. Define

$$g_n(i,t) = \lim_{\delta \to 0} \Pr\{t < \alpha_n \le t+\delta / I(0+)=s+i, T_1>t\} / \delta \tag{4.2.1}$$

ie, $g_n(i,t)$ denotes the conditional probability that the n^{th} replenishment takes place in $(t,t+\delta)$ given that the stock level is initially s+i and $T_1 > t$.

We prove the following theorem:

<u>Theorem (4.2.1)</u>

The stochastic process $(I,T) = \{(I_n,T_n),\ n \in N^o\}$ is a Markov Renewal Process (MRP) with state space $\{s+1,\ s+2,\ \ldots,\ s+M\}$ and the semi-Markov kernel $\{Q(i,j,t),\ i,j \in E,\ t \geq 0\}$

where

$$Q(i,j,t) = \Pr\{I_{n+1}=s+j,\ T_{n+1}-T_n \leq t \,/\, I_n=s+i\}$$

is given by

$$Q(i,j,t)=\begin{cases} \delta(i-j-1)\displaystyle\sum_{r=1}^{i-j}\int_0^t b_r(u)\ {}_{s+i}\psi_{(i-j-r)}(u)\ dF(u) \\ \quad + \displaystyle\sum_{n=1}^{\infty}\sum_{r=1}^{M-j}\int_0^t b_r(u)\ [g_n(i,u) * {}_{(s+M)}\psi_{(M-j-r)}(u)]\,dF(u) \\ \qquad\qquad \text{for } j \neq M \\ \displaystyle\sum_{r=1}^{i}\int_0^t \bar{b}_r(u)\,{}_{s+i}\psi_{(i-r)}(u)\ dF(u) \\ \quad + \displaystyle\sum_{n=1}^{\infty}\sum_{r=1}^{M}\int_0^t \bar{b}_r(u)\ [g_n(i,u) * {}_{s+M}\psi_{(M-r)}(u)]\ dF(u) \\ \qquad\qquad \text{for } j=M \end{cases} \tag{4.2.2}$$

Proof:

As the demand epochs form a renewal process $T_{n+1}-T_n$ is independent of the intervals T_r-T_{r-1}, $r=1,2,\ldots n$. By assumption, the probability mass function of demand magnitudes at T_{n+1} depends only on the interval length $T_{n+1}-T_n$ and does not depend on either any other intervals T_r-T_{r-1}, $r=1,2,\ldots,n$ or the demand magnitudes. Moreover because of the lack of memory property of exponential distribution, the failures of items in the stock is an interval $(T_n,T_{n+1}]$ is independent of the process $\{I(t), t\leq T_n\}$. Hence,

$$\Pr\{I_{n+1}= s+j,\ T_{n+1}-T_n \leq t \mid I_o,I_1,\ldots,I_n, T_o,T_1,\ldots,T_n\}$$

$$= \Pr\{I_{n+1} = s+j,\ T_{n+1}-T_n \leq t/I_n\} \qquad (4.2.3)$$

which proves the first part of the theorem.

We have $\alpha_n = \alpha_1 + (\alpha_2-\alpha_1) + \ldots + (\alpha_n-\alpha_{n-1})$ with α_1 having the density ${}_{(s+i)}\eta_i'(t)$ and $\alpha_r-\alpha_{r-1}(r=2,3,\ldots)$ having density ${}_S\eta_M'(t)$. Since $\alpha_1,\alpha_2-\alpha_1$, ... are independent random variables we get

$$g_1(i,t) = {}_{(s+i)}\eta_i'(t) \qquad (4.2.4)$$

$$g_n(i,t) = {}_{(s+i)}\eta_i'(t) * {}_S\eta_M'^{*(n-1)}(t), \quad n=2,3,\ldots \qquad (4.2.5)$$

If we denote the number of replenishments in $(0,t)$ by $N(t)$ and defining

$$Q_n(i,j,t) = \Pr\left\{ I_1 = s+j,\ T_1 \leq t,\ N(T_1) = n \,/\, I(0+) = s+i \right\} \qquad (4.2.6)$$
$$n=0,1,2,\ldots$$

Then

$$Q(i,j,t) = \sum_{n=0}^{\infty} Q_n(i,j,t) \qquad (4.2.7)$$

In order to derive the expression for $Q_n(i,j,t)$ $(n=0,1,2,\ldots)$ assume that the next demand after the initial one occurs in $(u,u+du)$ where $u < t$. Making use of the following arguments and the independence of the demand occurrences and failures of items, we have for

(i) $n = 0,\ j \neq M$.

In this case no replenishment takes place in $(0,u]$. Assume that the demand that occurred at time u is for r items ($r = 1,2,\ldots,i-j$; provided $i > j$). In order that the stock level is $s+j$ at time u, $i-j-r$ items should have perished in (o,u) out of the initial $s+i$ ones and

$$Q_o(i,j,t) = \delta(i-j-1) \sum_{r=1}^{i-j} \int_o^t b_r(u)\, {}_{s+i}\psi_{(i-j-r)}(u)\,dF(u)$$

(ii) $n \neq 0$, $j = M$

n^{th} replenishment occurs at some time $v(< u)$ and the stock level at time soon after v is $S(= s+M)$. Assume that the demand at time u is for r items ($r=1,2,\ldots,M-j$). In order that stock level is $s+j$ at time u, $M-j-r$ items out of S should have perished in the interval $(v,u]$

$$Q_n(i,j,t) = \sum_{r=1}^{M-j} \int_o^t b_r(u)[g_n(i,u) * {}_S\psi_{(M-j-r)}(u)]\,dF(u)$$

(iii) $n = 0$, $j = M$

No replenishment in $(0,u]$. In order that the level is S at time u, an order should have occurred at time u and $i-r$ items ($r = 1,2,\ldots,i$) perished out of $s+i$ in $(0,u)$. Then the demand at time u should be for atleast r units.

Therefore,

$$Q_o(i,M,t) = \sum_{r=1}^{i} \int_o^t \bar{b}_r(u)\ {}_{(s+i)}\psi_{(i-r)}(u)\ dF(u)$$

and finally for

(iv) $n \neq 0$, $j = M$.

This deals with the case where the n^{th} $(n=1,2,\ldots)$ replenishment takes place at some time $v(<u)$. As in the previous case there should be an order placed at time u triggered by a demand so that the stock level becomes S instantaneously. Assume that M-r $(r = 1,2,\ldots,M)$ items are perished out of S in (v,u). Then the demand at time u should be for more than r items, and

$$Q_n(i,j,t) = \sum_{r=1}^{M} \int_o^t \bar{b}_r(u)\ [g_n(i,u)*{}_S\psi_{M-r}(u)]dF(u)$$

Now substituting these results in (4.2.7), we get (4.2.2)

[Q.E.D.]

In order to obtain the transient solution of the system let us define

$$p(i,j,t) = \Pr\left\{I(t)=s+j\ /\ I(0+) = s+i\right\},\ i,j \in E.$$

Since the stochastic process $\left\{I(t),t\geq 0\right\}$ is a semi-regenerative process with the embedded MRP (I,T), the function p(i,j,t) satisfies the following Markov renewal equation *(MRE)*

$$p(i,j,t) = k(i,j,t) + \int_{o}^{t} \sum_{r=1}^{M} Q(i,r,du)\, p(r,j,t-u);\ i,j \in E \qquad (4.2.8)$$

where

$$k(i,j,t) = \Pr\{I(t)=s+j,\ T_1>t \;/\; I(0+) = s+i\},\ i,j \in E.$$

<u>Theorem (4.2.2)</u>

The function k(i,j,t) is given by

$$k(i,j,t) = \begin{cases} \bar{F}(t)[\ \delta(i-j)\,{}_{(s+i)}\psi_{(i-j)}(t) + \sum_{n=1}^{\infty} g_n(i,t) * {}_{(s+M)}\psi_{(M-j)}(t)],\ j \neq M \\ \bar{F}(t)[{}_{s+i}\psi_{i}(t) + \sum_{n=1}^{\infty} g_n(i,t) * {}_{S}\psi_{M}(t)],\ j=M \end{cases} \qquad (4.2.9)$$

<u>Proof</u>:

The events along with the condition that the next demand should occur after t units of time, can be classified into the following mutually exclusive and exhaustive cases. The stock level does not drop to s or drops to s atleast once due to the perishability of items. Since the failures of items in no way depend on the demand occurrence, this yields (4.2.9).

[Q.E.D.]

Let $\hat{P}_\alpha$, $\hat{K}_\alpha$ and $\hat{Q}_\alpha$ denote M^{th} order square matrices whose $(i,j)^{th}$ elements are $\hat{p}(i,j,\alpha)$, $\hat{k}(i,j,\alpha)$ and $\hat{Q}(i,j,\alpha)$ respectively, where

$$\hat{Q}(i,j,\alpha) = \int_0^\infty e^{-\alpha t}\, Q(i,j,dt)$$

Then the Laplace transform of the set of MREs can be written as

$$\hat{P}_\alpha = \hat{K}_\alpha + \hat{Q}_\alpha \hat{P}_\alpha \tag{4.2.10}$$

which yields

$$\hat{P}_\alpha = (I-\hat{Q}_\alpha)^{-1} \hat{K}_\alpha = \hat{R}_\alpha \hat{K}_\alpha \tag{4.2.11}$$

where $\hat{R}_\alpha = (I-\hat{Q}_\alpha)^{-1}$ is the matrix of Laplace transforms of Markov renewal functions of the MRP (I,T), which exists for $\alpha > 0$ (Cinlar (1975)).

4.3. STEADY STATE ANALYSIS

In order to obtain the limiting distribution of the inventory level, consider the underlying Markov chain $I = \{I_n,\ n \in N^0\}$ associated with the MRP(I,T).

The transition probability matrix $Q = [Q(i,j)]$ of order M is given by

$$Q(i,j) = \lim_{t\to\infty} Q(i,j,t)$$

$$= \begin{cases} \delta(i-j-1) \sum_{r=1}^{i-j} \int_0^\infty b_r(u)\,{}_{(s+i)}\psi_{(i-j-r)}(u)\,dF(u) \\ \quad + \sum_{n=1}^{\infty} \sum_{r=1}^{M-j} \int_0^\infty b_r(u)[g_n(i,u) *_S \psi_{(M-j-r)}(u)]dF(u); j \neq M \\ \sum_{r=1}^{i} \int_0^\infty \bar{b}_r(u)\,{}_{(s+i)}\psi_{(i-r)}(u)\,dF(u) \\ \quad + \sum_{n=1}^{\infty} \sum_{r=1}^{M} \int_0^\infty \bar{b}_r(u)[g_n(i,u) *_S \psi_{(M-r)}(u)]dF(u); j=M \end{cases} \tag{4.3.1}$$

A necessary and sufficient condition for the chain I to be irreducible is that $b_1(t) \neq 0$ for some interval in $[0, \infty)$. It can be seen by the following arguments.

If $b_1(t) \neq 0$ for some interval in $[0, \infty)$, then from (4.3.1) we have

$$Q(i,j) > 0, \ j \neq M$$

and $\quad Q(i,M) > 0 \qquad$ for $i \in E$.

Thus every state is accessible from all other states and the Markov chain I is irreducible.

Conversely, suppose that $b_1(t) \equiv 0$, then all the entries in the $(M-1)^{th}$ column of Q becomes zero and consequently the state M-1 is inaccessible from any other state, which contradicts the fact that the chain I is irreducible.

Since the chain is irreducible, it possesses a unique stationary distribution $\bar{\pi} = (\pi_1, \pi_2, \ldots, \pi_M)$ which satisfies $\bar{\pi}Q = \bar{\pi}$ and $\Sigma\, \pi_j = 1$.

Let $\bar{p} = (p_1, p_2, \ldots, p_M)$ denotes the steady state probability vector of the inventory level where $p_j = \lim_{t \to \infty} p(i,j,t)$ is the limiting distribution of the inventory level. Now we have the steady state result.

Theorem(4.3.1)

If $b_1(t) \neq 0$ for some interval in $[0, \infty)$ and $F(t)$ is absolutely continuous with finite expectation, then

$$\bar{p} = \bar{\pi} \int_0^\infty K(t)\, dt \,/\, m \qquad (4.3.2)$$

where $m = \int_0^\infty x\, dF(x)$ is the mean interarrival time between demands.

Proof:

As a consequence of the first condition the finite Markov chain I is irreducible and hence recurrent. Hence the embedded MRP (I,T) in the semi-regenerative process $\{I(t),\ t \geq 0\}$ becomes irreducible and recurrent.

By the second condition, the MRP(I,T) is aperiodic, as the derivative of the Semi-Markov kernel exists. Hence making use of the result in Cinlar (1975) we have from equation (4.2.8)

$$p_n = \sum_{j \in E} \pi_j \int_0^\infty k(j,n,t)dt \Big/ \sum_{j \in E} \pi_j m_j$$

where m_j is the mean sojourn time in state j. This can be written as

$$\bar{p} = \bar{\pi} \int_0^\infty K(t)dt \Big/ \bar{\pi}\,\bar{m}$$

where $K(t) = (k(i,j,t))_{i,j \in E}$

and $\bar{m} = (m_1, m_2, \ldots, m_M)^T$

Here $m_j = m = \int_0^\infty x dF(x)$

$$\therefore \quad \bar{p} = \bar{\pi} \int_0^\infty K(t)dt \Big/ m$$

[Q.E.D]

4.4 SPECIAL CASES

As an illustration we consider the following particular case: When the items are not perishable ie, $\mu = 0$ and $F(x)$ is arbitrary.

Then

$$Q = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \beta_1 & 0 & \cdots & \cdots & \cdots & 0 & \bar{\beta}_2 \\ \beta_2 & \beta_1 & \cdots & \cdots & \cdots & 0 & \bar{\beta}_3 \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \beta_{M-1} & \beta_{M-2} & \cdots & \cdots & \cdots & \beta_1 & \bar{\beta}_M \end{bmatrix} \qquad (4.4.1)$$

where

$$\beta_i = \int_0^\infty b_i(t)\, dF(t)$$

and $$\bar{\beta}_i = \sum_{k=i}^{\infty} \int_0^\infty b_k(t)\, dF(t) = \sum_{k=i}^{\infty} \beta_k$$

The stationary distribution $\bar{\pi}$ can be obtained following the usual method of solving

$$\bar{v}\, Q = \bar{v}$$

and normalizing

$$\bar{v} = (v_1, v_2, \ldots, v_M)$$

which then gives

$$\pi_j = r_{M-j} / (1+R_{M-1}),\ j=1,2,\ldots,M-1$$

and $\pi_M = 1 / (1+R_{M-1})$ (4.4.2)

where r_j is the discrete analogue of the renewal density of the sequence $\{\beta_j,\ j \geq 1\}$ which is equal to $\sum_{k=1}^{\infty} \beta_j^{*k}$ and $R_j = \sum_{k=1}^{j} r_j$ is the renewal function corresponding to the sequence $\{\beta_n,\ n \geq 1\}$ (see for eg. Feller (1968)).

The steady state distribution of the stock level is given by

$$\bar{p} = \int_0^{\infty} \bar{F}(t)\, \bar{\pi}\, I\, dt / m$$

$$= \bar{\pi}, \quad \text{since } \int_0^{\infty} \bar{F}(t)dt = m$$

Hence p_j, $j = 1,2,\ldots,M$ are given by (4.4.2).

<u>Corollary (1)</u>

If the demand process is compound renewal, ie., $b_r(t)=b_r$, then $\beta_i = b_i$. The limiting distribution

p_n, $n = 1,2,\ldots,M$ is given by (4.4.2) where r_j and R_j correspond to the sequence $\{b_n, n \geq 1\}$ which agrees with the result of Sahin (1979) in the discrete case.

Corollary (2)

If the items demanded are unit quantities we have $b_1 = 1$ and $b_r = 0$ for $r > 1$. Then

$$r_j = 1 \quad \text{and } R_j = j, \; j = 1,2, \ldots$$

Then (4.4.2) reduces to $\pi_j = 1/M$, $j = 1,2,\ldots,M$ which agrees with the result of Sivazliam (1974).

Remark

Several extensions of the model considered in this chapter are possible. For example, one may consider a model with a positive lead time. For the analysis of such models one needs more sophisticated techniques. The assumption that life time distribution is exponential is crucial in this chapter. As one may expect, the results in the general life time distribution will not be as neat and easy to compute as they are in this rather simple model.

Chapter-5

AN INVENTORY SYSTEM WITH UNIT DEMAND AND VARYING ORDERING LEVELS

5.1 INTRODUCTION

In the previous chapter we discussed state dependent demand process. In this chapter we shall introduce another type of dependence in the basic process. The interarrival times of demands are i.i.d random variables with distribution function $G(.)$ that is absolutely continuous with density $g(.)$. Each arrival demands exactly one unit. Initially the inventory level is s resulting in an order placing. Order is placed for M units and let $S = M+s$. Lead times are i.i.d random variables which are independent of the quantity ordered for and inventory level, having absolutely continuous distribution function $F(.)$ and its density be $f(.)$. We fix a $c(> s)$. The ordering levels other than the initial one are determined as follows. Suppose the number of demands during a lead time is J then the next ordering level is $I = \min(J,c)$. Thus the ordering level can be $0.1,2,\ldots,c$. The order quantity will be such as to bring back the inventory level to S at the ordering epoch. Thus S-I units are ordered if I is the ordering level. No backlog is permitted. We discuss the time dependent probability distribution of the inventory level in Section 5.2. Correlation between the number of demands during a lead time and the length of the next inventory dry period is obtained in Section 5.3. Some illustrations are also given. This model has been discussed earlier by Ramanarayanan and Jacob (1986). However their method has a drawback that computation is hard and further passage to the limit is rather difficult.

In the sequel we use the following notations.

$G(.)$, $g(.)$ - Cumulative distribution function (c.d.f) and probability density function (p.d.f), respectively, of the interarrival time between demands.

$F(.)$, $f(.)$ - c.d.f and p.d.f, respectively, of the lead times.

denotes convolution

$f^{*n}(x)$ = n-fold convolution of $f(x)$ with itself ($f^{*0}(x) \equiv 1$).

E = $\{0,1,2,\ldots,s, \ldots c\}$

R^+ = Set of non-negative real numbers

N = Set of natural numbers

$N^o \quad = \quad \{0\} \cup N$

$\bar{G}(.) \quad = \quad 1-G(.)$

5.2 ANALYSIS OF THE MODEL

Let $T_o(=0)$, $T_1, T_2, \ldots, T_n, \ldots$ be the epochs at which the initial, first, ... n^{th} orders are placed for replenishment and $X_o(=s)$, $X_1, X_2, \ldots, X_n, \ldots$ be the corresponding ordering levels. Assume that Y_o, Y_1, $Y_2, \ldots, Y_n, \ldots$ be respectively the number of demands during the lead times these start at $T_o, T_1, T_2, \ldots, T_n, \ldots$. Then $\{(X_n, Y_n),\ n=0,1,2,\ldots\}$ constitutes a Markov chain on the set $E \times N^o$.

Now define $Z_n = (X_n, Y_n)$. The process $\{(Z_n, T_n),\ n \in N^o\}$ constitutes a Markov renewal process with the underlying semi-Markov process $\{Z_t,\ t \in R^+\}$ where

$$Z_t = (X_n, Y_n) \text{ for } T_n \leq t < T_{n+1}$$

The semi-Markov kernel is given by

$$Q((i,I),(j,J),t) = P\left\{(X_{n+1}, Y_{n+1}) = (j,J),\ T_{n+1}-T_n \leq t / (X_n, Y_n) = (i,I)\right\}$$

$$i,j \in E,\ I,J \in N^o$$

Now

$$Q((i,I),(j,J),t) = Q_1((i,I),(j,J),t) + Q_2((i,I),(j,J),t) \qquad (5.2.1)$$

where $Q_1(\cdot,\cdot,\cdot,t)$ and $Q_2(\cdot,\cdot,\cdot,t)$ correspond to, respectively, transition from (i,I) to (j,J) in time t without and with a dry period in between. Note that if $I \leq c$ then $j = I$ and for $I > c$, $j = c$. Further, if $i > I$, then

$$Q((i,I), (j,J),t) = Q_1((i,I),(j,J),t)$$

and if $i \leq I$, then

$$Q((i,I),(j,J),t) = Q_2((i,I),(j,J),t)$$

where $j = \min\{c,I\}$.

Now $Q_1((i,I),(j,J),t)$ is given by

$$Q_1((i,I),(I,J),t) = \int_o^t \int_u^t \int_v^t \int_w^\infty g^{*I}(u)f(v)\,\frac{g^{*(S-2I)}(w-u)}{1-G(v-u)}\, g^{*J}(x-w)\,[1-F(x-w)]dx\,dw\,dv\,du$$

which is valid for $i > I$.

For $c > I \geq i$, we have

$$Q_2((i,I),(I,J),t) = \int_o^t \int_u^t \int_v^t \int_w^\infty g^{*I}(u)\ f(v)\ \frac{g^{*(S-i-I)}(w-u)}{1-G(v-u)}$$

$$g^{*J}(x-w)\ [1-F(x-w)]dx\ dw\ dv\ du$$

Finally for $I \geq c$, we have

$$Q_2((i,I),\ (c,J),t) = \int_o^t \int_u^t \int_v^t \int_w^\infty g^{*I}(u)\ f(v)\ \frac{g^{*(S-i-c)}(w-u)}{1-G(v-u)}$$

$$g^{*J}(x-w)\ [1-F(x-w)]dx\ dw\ dv\ du.$$

Now we are in a position to find the system size probability distribution at arbitrary time point t. For this purpose consider the Markov renewal function $R(\ (s,I),\ (j,J),t)$ of the process under consideration.

This is given by

$$R((s,I),(j,J),t) = \sum_{m=o}^{\infty} Q^{*m}((s,I),(j,J),t) \qquad (5.2.2)$$

Note that $R((s,I),\ (j,J),t)$ represents the number of visits to (j,J) from (s,I) in $(o,t]$.

Let its density be represented by $r((s,I),(j,J),t)$.

Let $P_{(n,j)}^{(s,I)}(t) = Pr\{$Inventory level at time t is n and the last reorder level is j given that initially the system was in state $(s,I)\}$.

Then for $c < n \leq S$,

$$P_{(n,j)}^{(s,I)}(t) = \int_o^t \int_u^t \int_w^t \sum_{k=o}^{j-1} r((s,I),(j,k),u) g^{*k}(w-u)\ f(v-u)$$

$$\left[\frac{G^{*(S-k-n)}(t-w) - G^{*(S-k-n+1)}(t-w)}{1 - G(v-w)} \right] dv\ dw\ du$$

$$+ \int_o^t \int_u^t \int_w^t \sum_{k>j} r((s,I),(j,J),u) g^{*k}(w-u) f(v-u)$$

$$\left[\frac{G^{*(S-j-n)}(t-w) - G^{*(S-j-n+1)}(t-w)}{1 - G(v-w)} \right] dv\ dw\ du$$

For $j < n \leq c$,

$$P_{(n,j)}^{(s,I)}(t) = \int_o^t \int_u^t \int_w^t \sum_{k<n} r((s,I),(j,k),u)\ g^{*k}(w-u) f(v-u)$$

$$\left[\frac{G^{*(S-k-n)}(t-w) - G^{*(S-k-n+1)}(t-w)}{1 - G(v-w)} \right] dv\ dw\ du$$

For $1 \leq n \leq j$

$$P^{(s,I)}_{(n,j)}(t) = \int_o^t \int_u^t \int_w^t \sum_{k<j-n} r(s,I),(j,k),u)g^{*k}(w-u)f(v-u)$$

$$\left[\frac{G^{*(S-k-n)}(t-w)-G^{*(S-k-n+1)}(t-w)}{1 - G(v-w)} \right] dv\ dw\ du$$

$$+ \int_o^t \int_u^t \int_t^\infty \sum_{k\geq j-n} r((s,I),(j,k),u)g^{*(j-n)}(w-u)$$

$$\left[\frac{G^{*k-(j-n)}(v-w)-G^{*k-(j-n)+1}(v-w)}{1 - G(t-w)} \right] f(v-u)dv\ dw\ du$$

Finally for $n = 0$,

$$P^{(s,I)}_{(o,j)}(t) = \int_o^t \int_u^t \int_t^\infty \sum_{k\geq j} r(s,I),(j,k),u)g^{*j}(w-u)$$

$$\left[\frac{G^{*k-j}(v-w)-G^{*k-j+1}(v-w)}{1 - G(t-w)} \right] f(v-w)dv\ dw\ du$$

5.3 CORRELATION BETWEEN THE NUMBER OF DEMANDS DURING A LEAD TIME AND THE INVENTORY DRY PERIOD

Let J be the random variable representing an ordering level and Z be the length of the subsequent dry period. Then J=j, for j=0,1,2,...,c-1 if the number of demands during the previous lead time was j and J=c if the number of demands during the previous lead time was larger than or equal to c.

Now for $0 \leq j \leq c-1$,

$$p_j = \Pr\{J=j\} = \int_0^\infty f(y)\ [G^{*j}(y)-G^{*(j+1)}(y)]dy$$

and $$p_c = \Pr\{J=c\} = \int_0^\infty f(y)\ G^{*c}(y)dy$$

For $0 \leq j \leq c$,

$$\Pr\{J = j,\ Z = 0\} = p_j \int_0^\infty f(y)\ \overline{G^{*j}}(y)dy$$

and

$$\Pr\{J = j,\ z < Z \leq z+dz\} = p_j(\int_z^\infty f(y)g^{*j}(y-z)dy)dz$$

for $z > 0$ and $0 \leq j \leq c$.

Then

$$\begin{aligned}E(e^{-\alpha Z}r^J) &= \sum_{j=o}^{c} p_j r^j \int_0^\infty f(y)\overline{G^{*j}}(y)dy \\ &\quad + \sum_{j=o}^{c} p_j r^j \int_0^\infty e^{-\alpha z}(\int_z^\infty f(y)g^{*j}(y-z)dy)dz \\ &= \sum_{j=o}^{c} p_j r^j - \alpha \sum_{j=o}^{c} p_j r^j \int_0^\infty e^{-\alpha y}f(y) \\ &\qquad (\int_0^y e^{\alpha x}\ G^{*j}(x)dx)dy. \qquad (5.2.3)\end{aligned}$$

From this we have

$$E(Z) = -\frac{\partial}{\partial\alpha} E\,(e^{-\alpha Z}\, r^{J})\Big|_{\alpha=o,\ r=1}$$

$$= \sum_{j=o}^{c} p_j \int_o^{\infty} f(y)\, \left(\int_o^{y} G^{*j}(x)dx\right)dy$$

$$E(Z^2) = \frac{\partial^2}{\partial\alpha^2} E\,(e^{-\alpha Z}\, r^{J}\,)\Big|_{\alpha=o,\ r=1}$$

$$= 2 \sum_{j=o}^{c} p_j \int_o^{\infty} y\ f(y) \int_o^{x} G^{*j}(u)du\ dy$$

Similarly,

$$E(J) = \frac{\partial}{\partial r} E\,(e^{-\alpha Z}\, r^{J})\ \Big|_{\alpha=o,\ r=1}$$

$$= \sum_{j=o}^{c} j\ p_j$$

and $$E(J^2) = \sum_{j=o}^{c} j(j-1)p_j.$$

Finally,

$$E(ZJ) = \sum_{j=o}^{c} j\ p_j \int_o^{\infty} f(y)\ \left(\int_o^{y} G^{*j}(x)dx\right)dy$$

Substituting these values in $Cov(Z,J) = E(ZJ)-E(Z)\ E(J)$ and using the fact that correlation coefficient between Z and J is $Cov(Z,J) \,/\, \sqrt{Var(Z)}\ \sqrt{Var(J)}$ we get the required expression.

AN ILLUSTRATION

Assume that $G(x) = 1-e^{-x}$

$F(x) = 1-e^{-2x}$, and

$c = 3.$

Then one easily computes

$p_o = 2/3, \quad p_1 = 2/9, \quad p_2 = 2/27, \quad p_3 = 1/27$

$E(Z) = 0.489026063, \quad E(Z^2) = 0.852537722$

$E(J) = 13/27 \quad , \quad E(J^2) = 10/27$

$\text{Cov}\ (Z,j) = -\ 0.18813189$

$\text{Var}\ (J) = 0.138545953$

$\text{Var}\ (Z) = 0.613391231$

and Correlation coefficient $= -\ 0.64535208.$

The negative correlation indicates that when the ordering level increases the length of the subsequent dry period decreases.

Chapter-6

TRANSIENT SOLUTION OF $E^k/G^{a,b}/1$ QUEUE WITH VACATION *

6.1 INTRODUCTION

This chapter is devoted to the study of the following single server queueing model with finite capacity.

We assume that the interarrival times of customers (units) at a counter are distributed as an Erlang of order k, the density being

$$e(x) = \begin{cases} \dfrac{\mu^k x^{k-1} e^{-\mu x}}{(k-1)!} & , x > 0, \mu > 0, k \geq 2 \\ 0 & , \text{otherwise} \end{cases}$$

We may consider each arrival as consisting of k stages 0,1,2,...,k-1 in each of which the customers spend an exponentially distributed time (μ) before proceeding to the next stage. The physical arrival of a customer to the system corresponds to his reaching the k^{th} stage.

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* This appeared in Proc. ICMMST-88, Vol.2, World Scientific (Singapore).

The capacity of the waiting room (W.R) is assumed to be equal to b (finite). The customers who arrive when W.R is full are deemed to be lost. Customers are served in batches of size lying between a and b ($a < b$) (both included). We assume that the successive service times are independent random variables, but their distributions depend on the batch sizes. We denote the distribution function of service time for a batch of j ($a \leq j \leq b$) customers by $G_j(.)$ and the corresponding density function by $g_j(.)$.

When a service terminates with less than a customers waiting in the system the server leaves for vacation. On return from vacation if the server finds again less than a units he extends his vacation. This process continues until on return he finds at least a units waiting. The vacation times are i.i.d random variables with probability distribution function $H(.)$ and density $h(.)$.

It is of interest to give some actual situations which may be described by this model. Many types of transportation processes involving buses, trains, aeroplanes, ships etc. all have a feature of bulk service in common. The incorporation of vacation to the server in these situations might be more realistic. If an item entering the system is considered to have a set of k phases of arrivals, all of them have identical exponential distributions, the afore mentioned situations could be best approximated by the present model.

Queueing systems with general bulk service rule with range (a,b) has been studied in great detail by Neuts (1967), Kambo and Chaudhry (1982), Medhi (1984) and others. The system $E^k/M^{a,b}/1$ has been discussed in Easton et al.(1982) and Holman et al.(1981). An extensive survey of vacation models is given in Doshi (1986). An attempt to find the time dependent solution of $M/G^{a,b}/1$ queue can be seen in Jacob, Krishnamoorthy and Madhusoodanan (1988). Jacob and Madhusoodanan (1987) extend the above to a vacation model, having arbitrary distribution. The virtual waiting time is also discussed in these papers.

In this chapter we analyse the queueing model described at the beginning of this section. Section 6.3 gives the transient system state probability distribution and in section 6.4 we derive the virtual waiting time distribution in the queue at time t given the state of system at that instant. The renewal theoretic argument is used throughout the analysis.

6.2 NOTATIONS AND PRELIMINARIES

Let * denote the convolution operator. $f^{*n}(.)$ stands for the n-fold convolution of f(.) with itself ($f^{*0}(.) \equiv 1$).

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Let $A(x) = [\ a_{ij}(x)\ ]$ is an m x p matrix, and

$B(x) = [\ b_{ij}(x)\ ]$ is a p x n matrix, then

$(A*B)(x) = [\ c_{ij}(x)\ ]$ is an m x n matrix whose

elements are given by

$$c_{ij}(x) = \sum_{k=1}^{p} (a_{ik} * b_{kj})\ (x) \tag{6.2.1}$$

Let the state of the input process (arrival) be represented by (ℓ ,n), where ℓ gives the stage number of arriving customer since the last physical arrival and n is the number of units present in the system. Thus $0 \leq \ell < k$ and the transition $(k-1,n) \longrightarrow (0,n+1)$ characterizes the input scheme. If we write $i = kn+\ell$, the total number of stages in the system, then (ℓ ,n) is defined by

$$n = [\, i/k \,]$$

and $$\ell = i - [i/k]k$$

where $[x]$ denotes the greatest integer less than or equal to x.

Write

$$\mu_n(x) = e^{-\mu x}(\mu x)^n \,/\, n!, \quad n=0,1,2,\ldots,bk-1$$

$$\mu_{bk}(x) = \sum_{n=bk}^{\infty} e^{-\mu x}(\mu x)^n \,/\, n! \qquad (6.2.2)$$

Define for any real $x > 0$.

$f_{ij}(x)dx$ = Probability that, starting at time zero, the service of a batch of size $[i/k]$ terminates in $(x, x+dx)$ and $j-(i-[i/k]k)$ stages arrive during $(o,x]$;
$ak \leq i \leq bk$, $j = i-[i/k]\,k, \ldots, bk$

Then we have

$$f_{ij}(x) = g_n(x)\,\mu_{j-\ell}(x); \quad ak \leq i \leq bk, \quad \ell \leq j \leq bk \qquad (6.2.3)$$

where $n = [i/k]$ and $\ell = i-[i/k]\,k$

and $f_{ij}(x) = 0$, otherwise.

Let $\mathbb{F}(x) = [f_{ij}(x)]$, $i,j = ak, ak+1, \ldots, (a+1)k, \ldots, bk$;

be a square matrix of order $[(b-a)k+1]$ and $\mathbb{H}$ be a matrix of order $[(b-a)k+1] \times ak$ given by

$$\mathbb{H}(x) = [f_{ij}(x)], \quad \begin{array}{l} i = ak, ak+1, \ldots, (a+1)k, \ldots, bk \\ j = 0, 1, \ldots, k, k+1, \ldots, ak-1 \end{array}$$

Also write

$$\underline{f}_i = (f_{i,ak}(x), f_{i,ak+1}(x), \ldots, f_{i,bk}(x)); \quad ak \leq i \leq bk$$

Let $\mathbb{F}^{*o}(x)$ be the identity matrix of order $(b-a)k+1$ and for $n \geq 1$, $\mathbb{F}^{*n}(x)$ be the n-fold convolution of $\mathbb{F}(x)$ with itself.

Denote by $Q_j^i(x)$ the j^{th} co-ordinate of the row vector $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ $(ak \leq j \leq bk)$.

Then $Q_j^i(x)dx$ is the probability that starting with i stages initially (at time 0), several batches of customers were served out continuously and the last batch service terminates in $(x, x+dx)$ with j stages remaining in the W.R.

Now consider the ak-component row vector

$$\underline{K}^i(x) = (\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{H})(x)$$

and write

$k^i(x) = \underline{K}^i(x)\, \underline{1}$, where $\underline{1}$ is an ak-component column vector of ones.

The time interval during which there were uninterupted service is termed as a busy period.

If $b_i(x)$ is the probability density function of a busy period initiated by i customers, then

$$b_i(x) = \sum_{j=o}^{ak-1} f_{ij}(x) + k^i(x) \tag{6.2.4}$$

Assume that at time 0, the process starts with the commencement of service of a batch of size m ($a \leq m \leq b$) customers. The termination of a busy period leads to a vacation period for the server. By our assumption the process starts with a busy period. Let $T_1, T_2, T_3, \ldots$ be the successive epochs at which busy periods terminate (or vacation period begins). That is, these are the time points at which the total number of stages waiting at the completion of a service is an element of the set $\{0, 1, \ldots, k, k+1, \ldots, ak-1\}$. Then the sequence $\{T_n\}$ forms a delayed renewal process.

The probability density function of T_1 is $b_m(x)$. Let Z be the time between any two consecutive renewal points. Then the probability density function of Z is given by

$$\phi(z) = \Pr\{z \le Z \le z+dz\}$$

$$= \sum_{j=o}^{ak-1} \int_{o}^{z} \sum_{m=o}^{\infty} h^{*m}(u) \sum_{p=o}^{ak-j-1} \mu_p(u) \int_{u}^{z} h(v-u) \sum_{q=ak-j-p}^{bk-j-p} \mu_q(v-u) b_{j+p+q}(z-v)\,dv\,du \qquad (6.2.5)$$

Then the renewal density function of the delayed renewal process is given by

$$m(u) = (b_m * \sum_{n=o}^{\infty} \phi^{*n})(u) \qquad (6.2.6)$$

6.3 TRANSIENT STATE PROBABILITIES OF THE SYSTEM

The state space of the continuous time stochastic process underlying the model can be described as

$$S = \left\{ (p,q) : a \leq p \leq b,\ o \leq q \leq bk \right\} \cup \left\{ (o,q) : o \leq q \leq bk \right\}$$

where (p,q) denotes the state that the service of a batch of size p is in progress and there are q stages in the W.R., state (o,q) denotes that the server is on vacation and there are q stages in the W.R.

Let $P_{(p,q)}(t) = \Pr\left\{\text{state of the system at time } t \text{ is } (p,q)\right\}$.

Considering the mutually exclusive and exhaustive cases and making use of renewal theoretic argument, we derive the following expressions for transient state probabilities.

i) For $a \leq p \leq b$ and $o \leq q \leq bk$

$$P_{(p,q)} = \int_o^t \sum_{j=o}^{k-1} Q_{pk+j}^{mk}(u)\ [1-G_p(t-u)]\ \mu_{q-j}(t-u)du$$

$$+ \int_o^t m(u) \sum_{j=o}^{ak-1} \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{r=o}^{ak-j-1} \mu_r(v-u) \int_v^t h(w-v)$$

$$\sum_{s=ak-j-r}^{bk-j-r} \mu_s(w-v) \int_w^t \sum_{\ell=o}^{k-1} Q_{pk+\ell}^{j+r+s}(x-w)[1-G_p(t-x)]\ \mu_{q-\ell}(t-x)dx\ dw\ dv\ du$$

$$+ \delta_{pm}[1-G_m(t)]\ \mu_q(t).$$

(ii) For $o \leq q \leq bk$

$$P_{(o,q)}(t) = \int_o^t m(u) \sum_{j=o}^{ak-1} \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{r=o}^{ak-j-1} \mu_r(v-u)$$

$$[1-H(t-v)]\, \mu_{q-j-r}(t-v)dv\, du.$$

6.4 VIRTUAL WAITING TIME IN THE QUEUE

Let the random variable $W_q(t)$ denote the virtual waiting time at time t (see for example Takac's (1962)). $W_q(t)$ is the interval of time that a unit would have to wait in the queue before starting its service, if it were to arrive at time t. Here we find the probability distribution of $W_q(t)$ conditioned on the state of the system at time t.

Let $\psi_t(z/(p,q)) = \Pr\{W_q(t) < z \mid \text{state of the system at time } t \text{ is } (p,q)\},\ z > o$

Then,

(i) $a \leq p \leq b,\ ak-1 \leq q \leq bk-1$

$$\psi_t(z/(p,q)) = \int_o^t \sum_{j=o}^{k-1} Q^{mk}_{pk+j}(u)\mu_{q-j}(t-u)[G_p(t+z-u)-G_p(t-u)]du$$

$$+ \int_{o}^{t} m(u) \sum_{j=o}^{ak-1} \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{r=o}^{ak-j-1} \mu_r(v-u) \int_{v}^{t} h(w-v)$$

$$\sum_{s=ak-j-r}^{bk-j-r} \mu_s(w-v) \int_{w}^{t} \sum_{\ell=o}^{k-1} Q_{pk+\ell}^{j+r+s} (x-w) \ \mu_{q-\ell} \ (t-x)$$

$$[\ G_p(t+z-x)-G_p(t-x)] \ dx \ dw \ dv \ du$$

The virtual waiting time distribution conditioned on other different states (p,q) of the system can be easily obtained on similar lines.

Chapter-7

A SERVICE SYSTEM WITH SINGLE AND BATCH SERVICES

7.1 INTRODUCTION

In this chapter we consider the following single and batch service queueing system. For convenience in describing the model we assume that the system consists of a waiting room (W.R) and a service station (S.S) both of unlimited capacity. The service station is manned by a single server. Customers (units) arrive according to a Poisson process with rate μ. Upon arrival a customer enters the W.R. This is the first stage of the queue. The second stage of the queue resides in the S.S. As soon as all units in the S.S are served out the server scans the W.R. If he finds less than or equal to c (fixed number) customers, he will serve them at the S.S, one at a time, according to FCFS rule, with service time of each customer i.i.d random variables with distribution function $G_1(.)$ and density function $g_1(.)$. If the server finds more than c customers in the W.R., he will serve them in batches with batch service times i.i.d random variables with a distribution function $G_2(.)$ and density $g_2(.)$, independent of the batch sizes. The time required to transfer customers from the W.R to the S.S is assumed to be negligible.

If, at the time when the services of all units in the S.S are completed and the W.R is empty, the server goes on vacation of random duration, independent of the number of units served. The vacation times are i.i.d random variables with distribution function H(.) and density function h(.). On return from vacation, if the server finds the W.R again empty he goes for another vacation independent of and identically distributed as the previous one; else he starts service.

There are many real-life queueing situations in which service is rendered with a control limit policy (see for eg. Crabil et al.(1977) and Ignall and Kolesar (1974)). For example, it may be possible to process jobs manually or by machine. When the number of jobs to be processed is not more than a fixed number it will be profitable to do them manually. When the number of jobs exceeds a certain quantity, processing by machine turns out to be cheaper.

In this chapter we consider three models. In Model-I we analyse the situation where the server is always present at the service station, serving or ready to serve. Model-II deals with the case where the server goes on vacation when the system becomes empty. These are to be analysed separately since the regeneration points of the processes corresponding to the models differ. We derive the transient system state probabilities and virtual waiting time distribution of a customer in the queue corresponding to Model-I and Model-II in Section 7-3 and Section 7.4, respectively. The renewal theoretic argument is used throughout the analysis. In Section 7.5 we analyse the third model which is a variant of the standard M/G/1 queue with single and batch services. The transient as well as the steady state distribution of the number of customers in the system are obtained for this model. The 'c' considered here has significance in problems concerned with control of queues.

Neuts and Ramalhoto (1984) discuss a process with Poisson arrival and arbitrary service time distribution. After each service the server locates the next customer to be served and the location times has exponential distribution. This location time can be regarded as the rest time discussed in Model-II of this chapter. Ali and Neuts (1984) discuss a queueing problem with a waiting room and a service station. To ensure uninterrupted service they assume the addition of 'overhead customers' to the service station and obtained stationary distribution of queue length and waiting time distribution.

7.2 NOTATIONS AND PRELIMINARIES

Let * denote convolution. Then $f^{*n}(\cdot)$ stands for the n-fold convolution of $f(\cdot)$ with itself ($f^{*o}(\cdot) \equiv 1$).

Denote $S(x) = \sum_{m=o}^{\infty} h^{*m}(x)\, q_o(x)$

Let $A(x) = [\, a_{ij}(x) \,]$ be a matrix of order m x p

and $B(x) = [\, b_{ij}(x) \,]$ be a matrix of order p x n.

Then the convolution product of A and B is defined as

$A * B(x) \doteq [\, \sum_{k=1}^{p} a_{ik} * b_{kj}(x) \,]$ which is a matrix of order m x n.

Let $q_j(x)$ = Probability that there are j arrivals in (o,x]

$$= \frac{e^{-\mu x}(\mu x)^j}{j!}, \quad j = 0,1,2,\ldots,c \qquad (7.2.1)$$

$\underline{c}$ stands for the set $\{c+1,\ c+2,\ \ldots\}$ and $i = \underline{c}$ means that $i \geq c+1$

$$q_{\underline{c}}(x) = \sum_{j=c+1}^{\infty} \frac{e^{-\mu x}(\mu x)^j}{j!} \qquad (7.2.2)$$

Define for $i = 1,2,\ldots,c,\ \underline{c}$; $j = 0,1,2,\ldots,\ c,\ \underline{c}$

$f_{ij}(x)dx$ = Probability that starting at time zero, the service of the i^{th} customer terminates in $(x,x+dx)$ and there are j arrivals in $(o,x]$.

For $i = 1,2,\ldots,\ c,\ \underline{c}$ let

$$\underline{f}_i(x) = (f_{i1}(x),\ f_{i2}(x),\ \ldots,\ f_{ic}(x),\ f_{i\underline{c}}(x))$$

Also let

$$\underline{f}_o(x) = (f_{1o}(x),\ f_{2o}(x),\ \ldots,\ f_{co}(x),\ f_{\underline{c}o}(x))^T$$

Define the matrix

$$\mathbb{F}(x) = [f_{ij}(x)]_{i,j} = 1,2,\ldots,\ c,\ \underline{c}.$$

Let $\mathbb{F}^{*o}(x)$ be the identity matrix of order c+1 and for $n \geq 1$,

$$\mathbb{F}^{*n}(x) = \text{the n-fold convolution of } \mathbb{F}(x) \text{ with itself.}$$

The basic model is a semi-Markov process which refers to the process of transitions at the times when the customers move from the W.R to the S.S. The state of this process is the number of customers which entered the S.S at the last transition. In order to calculate state probabilities one uses standard techniques.

Let us define a different state (i,j), where i and j are the number of customers in service and waiting room, respectively. We call these states "micro-states". The probabilities of these micro-states can be calculated by conditioning on the last transition time and state.

Now for $i = 1,2,\ldots, c, \underline{c}$ and $\ell = 1,2, \ldots, c, \underline{c}$, let $Q^i_\ell(x)$ be the ℓ^{th} coordinate of the vector $(\underline{f}_i * \sum_{n=0}^{\infty} \mathbb{F}^{*n})(x)$.

This is the probability that the semi-Markov process is at state ℓ at time x, given that at time 0 it was in state i, and that it did not visit state 0 in the interval (o,x].

Also let $\quad k^i_o(x) = (\underline{f}_i * \mathbb{F}^{*n} * \underline{f}_o)(x).$

If $b_i(x)$, ($i=1,2,\ldots, c, \underline{c}$) is the probability density function of a busy period initiated by i customers, then we have,

$$b_i(x) = f_{io}(x) + k^i_o(x) \tag{7.2.3}$$

7.3. MODEL-I: SERVER WITHOUT VACATION

In this model we assume that the server is always available for service. We have

$$f_{ij}(x) = \begin{cases} g_1^{*i}(x) q_j(x), & i=1,2,\dots,c;\ j=0,1,2,\dots,\ c,\underline{c} \\ g_2(x) \quad q_j(x), & i=\underline{c};\ j=0,1,2,\dots,\ c,\underline{c} \end{cases} \tag{7.3.1}$$

The state space of the process is

$$\{(i,j) : i=1,2,\dots,\ c,\underline{c};\ j=0,1,2,\dots,\ c,\underline{c}\} \cup \{0\}$$

where (i,j) is the state that there are i units in the S.S (including the one being served) and j units in the W.R. State 0 means no unit in the W.R. (hence the server is idle). Clearly the time at which an idle period is terminated (commencement of busy period) is a regeneration point of the stochastic process under consideration.

Assume that initially there were 'a' units in the W.R. Then the probability density function of the initial busy cycle (busy period + idle period) is

$$h_1(x) = (b_a * e_\mu)\ (x)$$

where $e_\mu(.)$ denotes the exponential density function with parameter μ.

Probability density function of a busy cycle other than the initial one is

$$h_2(x) = (b_1 * e_\mu)(x)$$

Therefore, the renewal density function of the delayed renewal process is

$$m(u) = (h_1 * \sum_{n=o}^{\infty} h_2^{*n})(u) \tag{7.3.2}$$

TRANSIENT PROBABILITIES OF THE SYSTEM STATE

Let $P_{ij}(t)$ be the probability that the system is in state (i,j) at time t. Considering the mutually exclusive and exhaustive cases and making use of renewal theoretic arguments the following expressions are obtained.

(i) For $i=1,2,\ldots,a$ and $j=0,1,2,\ldots, c,\underline{c}$

$$\begin{aligned}
P_{ij}(t) = {} & [\, G_1^{*(a-i)}(t) - G_1^{*(a-i+1)}(t)]\; q_j(t) \\
& + \int_o^t \sum_{\ell=i}^{c} Q_\ell^a(u)\; [G_1^{*(\ell-i)}(t-u) - G_1^{*(\ell-i+1)}(t-u)] \\
& \qquad q_j(t-u)\,du \\
& + \int_o^t m(u) \int_u^t \sum_{\ell=i}^{c} Q_\ell^1(v-u)[G_1^{*(\ell-i)}(t-v) - G_1^{*(\ell-i+1)}(t-v)] \\
& \qquad q_j(t-v)\,dv\; du.
\end{aligned}$$

(ii) For $i=a+1,\ a+2,\ \ldots,\ c$ and $j=0,1,2,\ldots,c,\underline{c}$

$$P_{ij}(t) = \int_0^t \sum_{\ell=i}^{c} Q_\ell^a(u)\left[G_1^{*(\ell-i)}(t-u)-G_1^{*(\ell-i+1)}(t-u)\right]q_j(t-u)\,du$$

$$+ \int_0^t m(u) \int_u^t \sum_{\ell=i}^{c} Q_\ell^1(v-u)\left[G_1^{*(\ell-i)}(t-v)-G_1^{*(\ell-i+1)}(t-v)\right] q_j(t-v)\,dv\,du.$$

(iii) For $i=\underline{c}$ and $j=0,1,2,\ldots,c,\underline{c}$

$$P_{ij}(t) = \int_0^t Q_i^a(u)\,[1-G_2(t-u)]q_j(t-u)\,du$$

$$+ \int_0^t m(u) \int_u^t Q_i^1(v-u)\,[1-G_2(t-v)]q_j(t-v)\,dv\,du$$

Finally,

(iv)

$$P_o(t) = G_1^{*a}(t)\,q_o(t) + \int_0^t b_a(u)\,q_o(t-u)\,du$$

$$+ \int_0^t Q_{\underline{c}}^a(u)\,G_2(t-u)\,q_o(t-u)\,du$$

$$+ \int_0^t m(u) \int_u^t b_1(v-u)q_o(t-v)\,dv\,du$$

$$+ \int_0^t m(u) \int_u^t Q_{\underline{c}}^1(v-u)\,G_2(t-v)\,q_o(t-v)\,dv\,du$$

VIRTUAL WAITING TIME IN THE QUEUE

Let $W_q(t)$ denote the virtual waiting time at time t. It is the amount of time that a unit would wait (in queue) before starting its service, if it were to arrive at time t. We shall obtain the distribution of $W_q(t)$ conditioned on the state of the system at time t.

Let $\quad \phi_t(x/(i,j)) = \Pr\{W_q(t) \leq x \mid \text{state of the system at time } t \text{ is } (i,j)\}$

$$i=1,2,\ldots,\ c,\underline{c};\ j=0,1,2,\ldots,\ c,\underline{c}.$$

and

$$\phi_t(x/o) = \Pr\{W_q(t) \leq x \mid \text{state of the system at time } t \text{ is } 0\}$$

Then

(i) For $i = 1,2,\ldots,c$; $j = 0,1,2,\ldots,\ c-1$.

$$\phi_t(x/(i,j)) = \int_o^t \sum_{\ell=i}^{c} Q_\ell^a(u) \int_t^{t+x} g_1^{*\ell}(v-u) \sum_{k=o}^{c-j-1} q_k(v-t)\, g_1^{*j}(t+x-v)\,dv\,du$$

$$+ \int_o^t m(u) \int_u^t \sum_{\ell=i}^{c} Q_\ell^1(v-u) \int_t^{t+x} g_1^{*\ell}(w-v) \sum_{k=o}^{c-j-1} q_k(w-t)\, g_1^{*j}(t+x-w)\,dw\,dv\,du$$

$$+ \int_{o}^{t} \sum_{\ell=i}^{c} Q_{\ell}^{a}(u) \int_{t}^{t+x} g_1^{*\ell}(v-u) \sum_{k=c-j}^{\infty} q_k(v-t)dv\ du$$

$$+ \int_{o}^{t} m(u) \int_{u}^{t} \sum_{\ell=i}^{c} Q_{\ell}^{1}(v-u) \int_{t}^{t+x} g_1^{*\ell}(w-v) \sum_{k=c-j}^{\infty} q_k(w-t)$$

$$dw\ dv\ du$$

(ii) For $i=1,2,...,c$ and $j = c,\underline{c}$

$$\emptyset_t(x/(i,j)) = \int_{o}^{t} \sum_{\ell=i}^{c} Q_{\ell}^{a}(u)\ g_1^{*\ell}(t+x-u)du$$

$$+ \int_{o}^{t} m(u) \int_{u}^{t} \sum_{\ell=i}^{c} Q_{\ell}^{1}(v-u) g_1^{*\ell}(t+x-v)dv\ du$$

(iii) For $i = \underline{c}$ and $j = 0,1,2,..., c-1$.

$$\emptyset_t(x/(i,j)) = \int_{o}^{t} Q_{\underline{c}}^{a}(u) \int_{t}^{t+x} g_2(v-u) \sum_{k=o}^{c-j-1} q_k(v-t) g_1^{*j}(t+x-v)dv\ du$$

$$+ \int_{o}^{t} m(u) \int_{u}^{t} Q_{\underline{c}}^{a}(v-u) \int_{t}^{t+x} g_2(w-v) \sum_{k=o}^{c-j-1} q_k(w-t)$$

$$g_1^{*j}(t+x-w)dw\ dv\ du$$

$$+ \int_{o}^{t} Q_{\underline{c}}^{a}(u) \int_{t}^{t+x} g_2(v-u) \sum_{j=c-j}^{\infty} q_k(v-t)\ dv\ du$$

$$+ \int_{o}^{t} m(u) \int_{u}^{t} Q_{\underline{c}}^{1}(v-u) \int_{t}^{t+x} g_2(w-v) \sum_{k=c-j}^{\infty} q_k(w-t)dw\ dv\ du$$

(iv) For $i = \underline{c}$ and $j = c, \underline{c}$

$$\phi_t(x/(i,j)) = \int_0^t Q^a_{\underline{c}}(u)\ g_2(t+x-u)du + \int_0^t m(u) \int_u^t Q^a_{\underline{c}}(v)\ g_2(t+x-v)dv\ du$$

Finally,

(v) $\phi_t(x/o) \equiv 1$

7.4 MODEL-II: WITH MULTIPLE VACATION POLICY

Here we assume that the server goes on vacation as soon as the system becomes empty. The vacation times are i.i.d random variables with distribution function $H(.)$ and density function $h(.)$. On return from vacation if the system is again found to be empty, the server extends his vacation by a time having the same distribution and independent of the previous vacation duration. This process continues until on return from vacation there is at least one unit in the waiting room.

As in the first model, here again, we have

$$f_{ij}(x) = \begin{cases} g_1^{*i}(x)\ q_j(x) & \text{for } i=1,2,\ldots,c;\ j=0,1,2,\ldots,c,\underline{c} \\ g_2(x)\ q_j(x) & \text{for } i=\underline{c};\ j=0,1,2,\ldots,\ c,\underline{c} \end{cases} \qquad (7.4.1)$$

The state-space of the process is $\{(i,j) \mid i,j=0,1,2,\ldots c,\underline{c}\}$ where by state (i,j) we mean that there are i units in the S.S (including the one being served) and j units in the W.R. State (o,j) $j=0,1,2,\ldots,\underline{c}$, indicates that the server is on vacation and there are j units in the W.R.

If we denote by $T_1, T_2, \ldots$ the successive time points at which the server goes for rest after a busy period, then the sequence $\{T_n\}$ forms a delayed renewal process.

The probability density function of T_1 is $b_a(x)$. Let Z be the time between two such renewal points. Then the probability density function of Z is given by

$$\eta(z) = \Pr\{z < Z \leq z + dz\}$$

$$= \int_o^z S(u) \int_u^z h(v-u) \sum_{i=1}^{c} q_i(v-u) b_i(z-v) dv\, du$$

Then the renewal density function of the delayed renewal process is given by

$$m(u) = (b_a * \sum_{n=o}^{\infty} \eta^{*n})(u) \tag{7.4.2}$$

TRANSIENT PROBABILITIES OF THE SYSTEM STATE

Let $P_{ij}(t) = \Pr\{\text{system is in state } (i,j) \text{ at time } t\}$

$i,j = 0,1,2,\ldots, c,\underline{c}$.

Considering mutually exclusive and exhaustive cases we get the expressions for $P_{ij}(t)$ as follows:

(i) For $i=1,2,\dots,a;\quad j=0,1,2,\dots, c,\underline{c}$

$$P_{ij}(t) = [G_1^{*(a-i)}(t) - G_1^{*(a-i+1)}(t)]\,q_j(t)$$

$$+ \int_0^t \sum_{\ell=i}^{a} Q_\ell^{a}(u)\,[G_1^{*(\ell-i)}(t-u)-G_1^{*(\ell-i+1)}(t-u)]\,q_j(t-u)\,du$$

$$+ \int_0^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=i}^{c} q_\ell(w-v)$$

$$[G_1^{*(\ell-i)}(t-w)-G_1^{*(\ell-i+1)}(t-w)]\,q_j(t-w)\,dw\,dv\,du$$

$$+ \int_0^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{c} q_\ell(w-v)$$

$$\int_w^t \sum_{k=i}^{c} Q_k^{\ell}(y-w)\,[G_1^{*(k-i)}(t-y)-G_1^{*(k-i+1)}(t-y)]$$

$$q_j(t-y)\,dy\,dw\,dv\,du.$$

(ii) For $i = a+1, a+2, \dots, c$ and $j=0,1,2,\dots,c,\underline{c}$

$$P_{ij}(t) = \int_0^t \sum_{\ell=i}^{c} Q_\ell^{a}(u)\,[G_1^{*(\ell-i)}(t-u)-G_1^{*(\ell-i+1)}(t-u)]\,q_j(t-u)\,du$$

$$+ \int_o^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=i}^{c} q_\ell(w-v)$$

$$[G_1^{*(\ell-i)}(t-w)-G_1^{*(\ell-i+1)}(t-w)]q_j(t-w)dw\ dv\ du$$

$$+ \int_o^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{c} q_\ell(w-v)$$

$$\int_w^t \sum_{k=i}^{c} Q_k^\ell(y-w)\ [G_1^{*(k-i)}(t-y)-G_1^{*(k-i+1)}]q_j(t-y)$$

$$dy\ dw\ dv\ du$$

(iii) For $j=0,1,2,\ldots, c, \underline{c}$

$$P_{\underline{c}j} = \int_o^t Q_{\underline{c}}^a(u)\ [1-G_2(t-u)]\ q_j(t-u)\ du$$

$$+ \int_o^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{c} q_\ell(w-v)$$

$$\int_w^t Q_{\underline{c}}^\ell(y-w)\ [1-G_2(t-y)]q_j(t-y)dy\ dw\ dv\ du$$

and finally

(iv) For $j=0,1,2,\ldots, c,\underline{c}$

$$P_{oj}(t) = \int_o^t m(u) \int_u^t S(v-u)[1-H(t-v)]q_j(t-v)\ dv\ du\ .$$

VIRTUAL WAITING TIME IN THE QUEUE

The expressions for the distribution of virtual waiting time $W_q(t)$ conditioned on the state of the process at time t are obtained below.

(i) For i=1,2,...,c and j=0,1,2,..., c-1.

$$\phi_t(x/(i,j)) = \int_0^t \sum_{\ell=i}^{c} Q_\ell^a(u) \int_t^{t+x} g_1^{*i}(v-u) \sum_{k=0}^{c-j-1} q_k(v-t)$$

$$g_1^{*j}(t+x-v)\,dv\,du$$

$$+ \int_0^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{\underline{c}} q_\ell(w-v)$$

$$\int_w^t \sum_{k=i}^{c} Q_k^\ell(y-w) \int_t^{t+x} g_1^{*k}(z-y) \sum_{k=0}^{c-j-1} q_k(z-t)$$

$$g_1^{*j}(t+x-z)\,dz\,dy\,dw\,dv\,du$$

(ii) For i=1,2,...,c and j=c, $\underline{c}$.

$$\phi_t(x/(i,j)) = \int_0^t \sum_{\ell=i}^{c} Q_\ell^a(u) g_1^{*\ell}(t+x-u)\,du$$

$$+ \int_0^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{\underline{c}} q_\ell(w-v)$$

$$\int_w^t \sum_{k=i}^{c} Q_k^\ell(y-w)\; g_1^{*k}(t+x-y)\,dy\,dw\,dv\,du.$$

(iii) For $i=\underline{c}$ and $j=0,1,2,...,c-1$.

$$\phi_t(x/(i,j)) = \int_0^t Q_{\underline{c}}^a(u) \int_t^{t+x} g_2(v-u) \sum_{k=o}^{c-j-1} q_k(v-t) g_1^{*j}(t+x-v)dv\ du$$

$$+\int_0^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{c} q_\ell(w-v) \int_w^t Q_{\underline{c}}^\ell(y-w)$$

$$\int_t^{t+x} g_2(z-y) \sum_{k=o}^{c-j-1} q_k(z-t)\ g_1^{*j}(t+x-z)dz\ dy\ dw\ dv\ du$$

$$+ \int_0^t Q_{\underline{c}}^a(u) \int_t^{t+x} g_2(v-u) \sum_{k=c-j}^{\infty} q_k(v-t)dv\ du$$

$$+ \int_0^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{c} q_\ell(w-v) \int_w^t Q_{\underline{c}}^\ell(y-w)$$

$$\int_t^{t+x} g_2(z-y) \sum_{k=c-j}^{\infty} q_k(z-t)dz\ dy\ dw\ dv\ du$$

(iv) For $i=\underline{c}$ and $j = c, \underline{c}$

$$\phi_t(x/(i,j)) = \int_0^t Q_{\underline{c}}^a(u)\ g_2(t+x-u)du$$

$$+\int_0^t m(u) \int_u^t S(v-u) \int_v^t h(w-v) \sum_{\ell=1}^{\underline{c}} q_\ell(w-v)$$

$$\int_w^t Q_{\underline{c}}^\ell(y-w)\ g_2(t+x-y)dy\ dw\ dv\ du.$$

(v) For $i=0$ and $j=0,1,2,\ldots,c-1$

$$\phi_t(x/(i,j)) = \int_0^t m(u) \int_u^t S(v-u) \int_t^{t+x} h(w-v) \sum_{k=0}^{c-j-1} q_k(w-t)$$

$$g_1^{*j}(t+x-w)\,dw\,dv\,du$$

$$+ \int_0^t m(u) \int_u^t S(v-u) \int_t^{t+x} h(w-v) \sum_{k=c-j}^{\infty} q_k(w-t)\,dw\,dv\,du$$

and finally,

(vi) For $i=0$ and $j=c$, $\underline{c}$

$$\phi_t(x/(i,j)) = \int_0^t m(u) \int_u^t S(v-u)\ h(t+x-v)\,dv\,du.$$

7.5 MODEL III: A VARIANT OF STANDARD M/G/1 QUEUE WITH SINGLE AND BATCH SERVICES

In Model-I we have discussed an M/G/1 queueing system with two stages of services, one at the waiting room and the other at the service station. The type of service to be rendered is decided at the waiting room-single service or bulk service-according as there are less than or equal to 'c' customers or there are more than c customers. In this model we assume that customers arrive at the service station according to a

Poisson process with parameter μ. At the end of each service, if the server finds more than c customers waiting he serves them all together in a batch according to a general service time distribution $G_2(.)$ which is independent of the batch size and if there are less than or equal to c customers, he serves them one at a time according to FCFS rule, with service time of each having a general distribution $G_1(.)$, independent of the system size.

We are interested in the transient as well as steady state distribution of the number of customers in the system at time t.

Let $0 = T_0 < T_1 < T_2 \ldots$ be the successive epochs of completion of services (single or batch) and X(t) denotes the number of customers present in the system at time t. Then X(t) assumes values in the set

$$E = \{0,1,2, \ldots\}$$

Let $X(T_n+) = X_n$, $n \geq 0$.

By our assumption, the discrete parameter stochastic process $(X,T) = \{(X_n,T_n), n \geq 0\}$ is a Markov renewal process with statespace E and the corresponding Semi-Markov kernel

$\{Q(i,j,t),\ i,j \in E,\ t \geq 0\}$ where

$$Q(i,j,t) = \Pr\{X_{n+1}=j,\ T_{n+1}-T_n \leq t / X_n=i\}$$

is given by

$$Q(i,j,t)=\begin{cases} \int_0^t \mu e^{-\mu s} \int_0^{t-s} \frac{e^{-\mu u}(\mu u)^j}{j!} dG_1(u)ds; & i=0,\ j\geq 0 \\ \int_0^t \frac{e^{-\mu u}(\mu u)^{j-i+1}}{(j-i)!} dG_1(u) & ; 1\leq i\leq c,\ j\geq i-1 \\ \int_0^t \frac{e^{-\mu u}(\mu u)^j}{j!} dG_2(u) & ; i>c,\ j\geq 0 \end{cases} \tag{7.5.1}$$

Now $\{X(t), t \geq 0\}$ is a semi-regenerative process with state space E with embedded Markov renewal process (X,T) described above. Let $\{R(i,j,t),\ i,j \in E,\ t \geq 0\}$ denote the Markov renewal kernel corresponding to the semi-Markov kernel in (7.5.1).

For each $i,k \in E$, $t \geq 0$, define

$$P(i,k,t) = \Pr\{X(t) = k \mid X(o) = i\}$$

Then it is easy to see that

$$P(i,k,t) = \sum_{j=o}^{k} \int_o^t R(i,j,ds)\, K(j,k,t-s) \tag{7.5.2}$$

where

$$K(j,k,t) = \Pr\left\{X(t)=k,\ T_1>t \mid X(o)=j\right\}$$

and it is given by

$$K(j,k,t)=\begin{cases} e^{-\mu t} & ,\ j=0,\ k=0; \\ \int_0^t \mu e^{-\mu(t-s)}[1-G_1(s)]\frac{e^{-\mu s}(\mu s)^{k-1}}{(k-1)!}\,ds, & j=0,\ k>0; \\ [1-G_1(t)]\ \frac{e^{-\mu t}(\mu t)^{(k-j)}}{(k-j)!} & ,\ 1\leq j\leq c,\ k\geq j; \\ [1-G_2(t)]\ \frac{e^{-\mu t}(\mu t)^{k-j}}{(k-j)!} & ,\ j>c,\ k\geq j\ ; \\ 0 & ,\ \text{otherwise} \end{cases} \quad (7.5.3)$$

STEADY STATE ANALYSIS

In order to obtain the limiting distribution of the number of customers in the system, consider the underlying Markov chain $X = \{X_n,\ n\geq 0\}$ associated with the Markov renewal process (X,T). The transition probability matrix $P = [P(i,j)]$ is given by

$$P(i,j) = Q(i,j,\infty) = \begin{cases} q_j^1 & i=0,\ j\geq 0\ ; \\ q_{j-i+1}^1 & 1\leq i\leq c,\ j\geq i-1; \\ q_j^2 & i>c,\ j\geq 0 \end{cases} \quad (7.5.4)$$

where

$$q_n^i = \int_0^\infty \frac{e^{-\mu u}(\mu u)^n}{n!}\,dG_i(u),\ i=1,2,\ n\geq 0.$$

ie, the matrix P has the form

$$P=\begin{array}{cc} & \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & c & c+1 & c+2 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ c \\ c+1 \\ c+2 \\ \vdots \end{array} & \begin{bmatrix} q_o^1 & q_1^1 & q_2^1 & \cdots & q_c^1 & q_{c+1}^1 & q_{c+2}^1 & \cdots \\ q_o^1 & q_1^1 & q_2^1 & \cdots & q_c^1 & q_{c+1}^1 & q_{c+2}^1 & \cdots \\ 0 & q_o^1 & q_1^1 & \cdots & q_{c-1}^1 & q_c^1 & q_{c+1}^1 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & q_1^1 & q_2^1 & q_3^1 & \cdots \\ q_o^2 & q_1^2 & q_2^2 & \cdots & q_c^2 & q_{c+1}^2 & q_{c+2}^2 & \cdots \\ q_o^2 & q_1^2 & q_2^2 & \cdots & q_c^2 & q_{c+1}^2 & q_{c+2}^2 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{bmatrix} \end{array}$$

The stationary probability vector $\pi = (\pi_o,\pi_1,\ldots,\pi_c,\pi_{c+1},\ldots)$ is the unique solution of $\pi = \pi P$, $\Sigma\ \pi_i=1$.

ie. $$\pi_j = \pi_o\ q_j^1 + \sum_{r=1}^{j+1} \pi_r\ q_{j-r+1}^1 + q_j^2 \sum_{r=c+1}^{\infty} \pi_r,$$

$$0 \leq j \leq c-1; \qquad (7.5.5)$$

and

$$\pi_j = \pi_o q_j^1 + \sum_{r=1}^{c} \pi_r \, q_{j-r+1}^1 + q_j^2 \sum_{r=c+1}^{\infty} \pi_r, \; j \geq c.$$

The above system of linear equations, after some rearrangements and using the condition $\Sigma \, \pi_i = 1$, reduces to

$$
\begin{array}{ll}
(1+q_o^2-q_o^1)\pi_o + (q_o^2-q_o^1)\pi_1 + q_o^2\pi_2 + q_o^2\pi_3 + \cdots + q_o^2\pi_c & =q_o^2 \\
(q_1^2-q_1^1)\pi_o + (1+q_1^2-q_1^1)\pi_1 + (q_1^2-q_o^1)\pi_2 + q_1^2\pi_3 + \cdots + q_1^2\pi_c & =q_1^2 \\
(q_2^2-q_2^1)\pi_o + (q_2^2-q_2^1)\pi_1 + (1+q_2^2-q_1^1)\pi_2 + (q_2^2-q_o^1)\pi_3 + \cdots + q_2^2\pi_c & =q_2^2 \\
\vdots & \vdots \\
(q_c^2-q_c^1)\pi_o + (q_c^2-q_c^1)\pi_1 + (q_c^2-q_{c-1}^1)\pi_2 + \cdots + (1+q_c^2-q_1^1)\pi_c & =q_c^2 \\
(q_{c+1}^2-q_{c+1}^1)\pi_o + (q_{c+1}^2-q_{c+1}^1)\pi_1 + (q_{c+1}^2-q_c^1)\pi_2 + \cdots + (q_{c+1}^2-q_2^1)\pi_c + \pi_{c+1} & =q_{c+1}^2 \\
(q_{c+2}^2-q_{c+2}^1)\pi_o + (q_{c+2}^2-q_{c+2}^1)\pi_1 + (q_{c+2}^2-q_{c+1}^1)\pi_2 + \cdots + (q_{c+2}^2-q_3^1)\pi_c + \pi_{c+2} & =q_{c+2}^2 \\
\vdots & \vdots
\end{array}
$$

Solving the first c+1 equations we can find $\pi_o, \pi_1, \ldots, \pi_c$, which then determine $\pi_{c+1}, \pi_{c+2}, \ldots$ from the remaining equations.

LIMITING DISTRIBUTION OF NUMBER OF CUSTOMERS IN THE SYSTEM

Let $\{p_k\}$ denote the limiting distribution of the number of customers in the system

ie., $p_k = \lim_{t \to \infty} P(i,k,t)$

If the Markov chain X is transient, then from equation (7.5.2) we have

$$p_k = \lim_{t \to \infty} p(i,k,t) = \sum_{j=o}^{k} R(i,j,\infty)\ K(j,k,\infty)$$

$$= 0.$$

Suppose that X is recurrent. From the form of P it is clear that X is irreducible. The sojourn time in state O has an exponentially distributed component. ie, $Q(o,j,t)$ is not a step function and therefore, all states are aperiodic in (X,T).

One can easily see that the function $t \to K(j,k,t)$ is directly Rieman integrable for every $j,k \in E$. Applying the key renewal theorem to each one of the terms in (7.5.2) (see Cinlar (1975b)), we have

$$p_k = \lim_{t \to \infty} P(i,k,t) = \frac{\pi_k\ m_k}{\sum_j \pi_j m_j} \qquad (7.5.6)$$

independent of the initial state i, where m_j is the mean sojourn time in state j of the Markov renewal process (X,T).

Here

$$m_j = \begin{cases} \frac{1}{\mu} + \mu_1 , & j = 0 ; \\ \mu_1 , & 1 \leq j \leq c; \\ \mu_2 , & j > c . \end{cases}$$

where $\mu_i = \int_0^\infty x \, dG_i(x)$

Therefore the limiting probabilities are given by

$$p_k = \begin{cases} \dfrac{\pi_o\left(\frac{1}{\mu} + \mu_1\right)}{\mu_1 \sum_{j=o}^{c} \pi_j + \mu_2 \sum_{j=c+1}^{\infty} \pi_j} & k = 0 \\[2ex] \dfrac{\pi_k \mu_1}{\mu_1 \sum_{j=o}^{c} \pi_j + \mu_2 \sum_{j=c+1}^{\infty} \pi_j} & 1 \leq k \leq c \\[2ex] \dfrac{\pi_k \mu_2}{\mu_1 \sum_{j=o}^{c} \pi_j + \mu_2 \sum_{j=c+1}^{\infty} \pi_j} & k \geq c. \end{cases} \qquad (7.5.7)$$

Chapter-8

A FINITE CAPACITY PH/PH/1 QUEUE

8.1 INTRODUCTION

A GI/G/1 queue in which the interarrival and service time distributions are both of phase type (PH-distribution) is called a PH/PH/1 queue. It is a particular case of both PH/G/1 and GI/PH/1 queue which are treated through various approaches by various researchers in the past (See for eg: Neuts (1981)). There are merits in presenting these approaches separately. Some quantities of interest are easier in one setting than another. Particular features of PH-distributions could be exploited in obtaining algorithmically tractable solutions of the system.

In this chapter a single server queueing system with a waiting room of finite capacity N is studied. There can be atmost N+1 customers present in the system, including the one being served. Arrivals of customers is according to a PH-renewal process with the probability distribution F(.) of the interarrival times has the irreducible representation (α, T) of order m and is given by

$$F(x) = 1 - \alpha \exp(Tx)e, \quad \text{for } x \geq 0 \qquad (8.1.1)$$

The row vector α and the matrix T are of dimension m. The vector e is a column vector with all its components equal to 1. The vector T^{o} is defined by $T^{o} = - Te$.

The successive service times are mutually independent with common probability distribution G(.) of phase type with the irreducible representation (β, S) of order n and is given by

$$G(x) \quad = \quad 1 - \beta \exp(Sx)e \quad \text{for} \quad x \geq 0 \qquad (8.1.2)$$

The row vector β and the matrix S are of dimension n. The vector S^{o} is defined by $S^{o} = - Se$.

Here T and S are generators of Markov processes describing the generation of customers and their service. The PH-distribution has a probabilistic interpretation, by its definition so that they can be considered as arrivals and service processes in some networks where the sojourn time at each node has an exponential distribution. F(x) and G(x) could be interpreted as the distribution of time spent by the customer until its departure from the network. So the results in this chapter may be applied, for example, for modelling computer networks etc. Many researchers have been investigating PH-distributions and developing procedures to fit these distribution to given data sets (See Harris and Sykes (1984), Altiok (1985), Khoshgoftaar and Perros (1985) etc.). Also, various algorithms have been already developed for problems in applied probability, where these distributions are involved (See Neuts (1981), Kao (1988) etc.).

In this chapter we analyse the queueing system PH/PH/1/N+1. An arriving customer finding the system full is assumed to be lost. Section 8.2 introduces the notations used and some preliminaries. In Section 8.3 the system is analysed in detail to obtain the stationary distribution of queue length. Using matrix theory algorithmically tractable solution is obtained. The result is illustrated in Section 8.4 by some numerical examples.

8.2 NOTATIONS AND PRELIMINARIES

I denotes the identity matrix

$A \otimes B$ denotes the Kronecker (tensor) product of the matrices A and B:

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are rectangular matrices of dimensions $m_1 \times n_1$ and $m_2 \times n_2$, their Kronecker product $A \otimes B$ is the matrix of dimensions $m_1m_2 \times n_1n_2$, written in block partitioned form as

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ \vdots & & & \\ \vdots & & & \\ \vdots & & & \\ a_{m_1 1}B & a_{m_1 2}B & \cdots & a_{m_1 n_1}B \end{bmatrix}$$

Some useful properties of the Kronecker product that are repeatedly used in the sequel are the following (Ref. Bellman (1974)).

(i) $A \otimes (B+C) = A \otimes B + A \otimes C$

(ii) $(A+B) \otimes (C+D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$

(iii) $(A \otimes B)(C \otimes D) = AC \otimes BD$

8.3 STATIONARY DISTRIBUTION OF QUEUE LENGTH

In this section we derive the stationary distribution of the system at arbitrary instants of time.

Let $X(t)$ denote the state of the system at time t. Then $\{X(t), t \geq 0\}$ may be considered as a homogeneous

Markov process on the state space

$$S = \{(o,j) \mid j=1,2,\ldots,m\} \cup \{(i,j,k) \mid i=1,2,\ldots,N+1;\ j=1,2,\ldots,m;\ k=1,2,\ldots,n\}$$

where state (o,j) represents that the system is empty and the arriving customer is at phase j, while the state (i,j,k) represents and that there are i customers in the system and the arriving and the being served customers are at phases j and k respectively. The states are labeled in the lexicographic order.

The infinitesimal generator of the process is (in block partitioned form):

$$\begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & N & N+1 \end{array} \\
\begin{array}{c} 0\\ 1\\ 2\\ 3\\ \vdots\\ N\\ N+1 \end{array} &
\begin{bmatrix}
T & T^o\alpha \otimes \beta & 0 & 0 & \cdots & 0 & 0 \\
I \otimes S^o & T \otimes I + I \otimes S & T^o\alpha \otimes I & 0 & \cdots & 0 & 0 \\
0 & I \otimes S^o\beta & T \otimes I + I \otimes S & T^o\alpha \otimes I & \cdots & 0 & 0 \\
0 & 0 & T \otimes I + I \otimes S & T^o\alpha \otimes I & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & T \otimes I + I \otimes S & T^o\alpha \otimes I \\
0 & 0 & 0 & \cdots & \cdots & I \otimes S^o\beta & (T+T^o\alpha) \otimes I + I \otimes S
\end{bmatrix}
\end{array}$$

Let π_s denote the stationary probability of the state $s \in S$.

Define

$$\pi_o = (\pi_{o1}, \pi_{o2}, \ldots, \pi_{om})$$

$$\pi_i = (\pi_{i,1,1}, \pi_{i,1,2}, \ldots, \pi_{i,1,m}, \pi_{i,2,1}, \ldots, \pi_{i,m,n}),$$
$$i = 1,2,\ldots, N+1.$$

Under our assumption, the stationary distribution π_s, $s \in S$ is the unique solution of the steady state equilibrium equation [See Neuts (1981)].

$$\pi Q = 0, \quad \pi e = 1$$

where $\pi = (\pi_o, \pi_1, \pi_2, \ldots, \pi_{N+1})$

ie.,

$$\pi_o T + \pi_1 (I \otimes S^o) = 0 \quad (8.3.1)$$

$$\pi_o (T^o \alpha \otimes \beta) + \pi_1 (T \otimes I + I \otimes S) + \pi_2 (I \otimes S^o \beta) = 0 \quad (8.3.2)$$

$$\pi_{i-1} (T^o \alpha \otimes I) + \pi_i (T \otimes I + I \otimes S) + \pi_{i+1} (I \otimes S^o \beta) = 0 \quad (8.3.3)$$
$$i = 2,3,\ldots,N.$$

$$\pi_N (T^o \alpha \otimes I) + \pi_{N+1} ((T + T^o \alpha) \otimes I + I \otimes S) = 0 \quad (8.3.4)$$

$$\sum_{i=o}^{N+1} \pi_i e = 1 \quad (8.3.5)$$

Multiplying on the right both sides of (8.3.2), (8.3.3) and (8.3.4) by the matrix $I \otimes e$ and using the properties of Kronecker product we get

$$\pi_o(T^o\alpha) + \pi_1(T \otimes e - I \otimes S^o) + \pi_2(I \otimes S^o) = 0 \qquad (8.3.6)$$

$$\pi_{i-1}(T^o\alpha \otimes e)+\pi_i(T \otimes e - I \otimes S^o)+\pi_{i+1}(I \otimes S^o)=0 \qquad (8.3.7)$$

$$i=2,3,\ldots,N.$$

$$\pi_N(T^o\alpha \otimes e)+\pi_{N+1}((T + T^o\alpha) \otimes e - I \otimes S^o) = 0 \qquad (8.3.8)$$

Adding equations (8.3.1), (8.3.6), (8.3.7) and (8.3.8) we get

$$\pi_o(T+T^o\alpha)+\pi_1[(T+T^o\alpha) \otimes e] + \pi_2[(T+T^o\alpha) \otimes e]+ \ldots$$
$$+\pi_N[(T+T^o\alpha) \otimes e] + \pi_{N+1}[(T+T^o\alpha) \otimes e]= 0.$$

Since $A \otimes e = (I \otimes e)A$, we can write the above equation as

$$\left[\pi_o + \sum_{i=1}^{N+1} \pi_i(I \otimes e) \right] (T+T^o\alpha) = 0 \qquad (8.3.9)$$

Now multiplying on the right both sides of (8.3.1), (8.3.6), (8.3.7) and (8.3.8) by the m-component vector e, we get

$$-\pi_oT^o + \pi_1(e \otimes S^o) = 0$$

$$\pi_o T^o - \pi_1[(T^o \otimes e) + (e \otimes S^o)] + \pi_2(e \otimes S^o) = 0$$

$$\pi_{i-1}(T^o \otimes e) - \pi_i[(T^o \otimes e)+(e \otimes S^o)] + \pi_{i+1}(e \otimes S^o) = 0$$

$$i = 2,3,\ldots, N.$$

and $\pi_N(T^o \otimes e) - \pi_{N+1}(e \otimes S^o) = 0$

From the above equations we have

$$\pi_1(e \otimes S^o) = \pi_o T^o \qquad (8.3.10)$$

and $\pi_{i+1}(e \otimes S^o) = \pi_i(T^o \otimes e), \; i=1,2,\ldots,N$

Again, multiplying on the right both sides of equations (8.3.2) and (8.3.3) by the matrix $I \otimes (e\beta-I)$, we have

$$\pi_1 \; ((T \otimes I) + (I \otimes S)) \; (I \otimes (e\beta-I)) = 0$$

and $\pi_{i-1}(T^o\alpha \otimes (e\beta-I))+\pi_i(T \otimes I+I \otimes S)(I \otimes (e\beta-I))= 0$

$$i = 2,3,\ldots,N.$$

The first equation reduces to

$$\pi_1 \; [T \otimes (e\beta-I) + I \otimes S(e\beta-I)] = 0$$

ie. $\pi_1 \; [T \otimes (e\beta-I) - I \otimes S \;] = \pi_1(I \otimes S^o\beta)$

$$= \pi_1[(I \otimes S^o)(I \otimes \beta)]$$

$$= -\pi_o(T \otimes \beta) \; (\text{using}(8.3.1))$$

and the second equation becomes,

$$\pi_i[T \otimes (e\beta - I) - I \otimes S] = \pi_i(I \otimes S^o\beta) + \pi_{i-1}(T^o\alpha \otimes I) - \pi_{i-1}(T^o\alpha \otimes e\beta)$$

$$= \pi_i(I \otimes S^o\beta) + \pi_{i-1}(T^o\alpha \otimes I) - \pi_{i-1}(T^o \otimes e)(\alpha \otimes \beta)$$

$$= \pi_i(I \otimes S^o\beta) + \pi_{i-1}(T^o\alpha \otimes I) - \pi_i(e \otimes S^o)(\alpha \otimes \beta)$$

(using (8.3.10))

$$= \pi_i(I \otimes S^o\beta) + \pi_{i-1}(T^o\alpha \otimes I) - \pi_i(e\alpha \otimes S^o\beta)$$

$i=2,3,\ldots,N.$

Putting $S^* = T \otimes (e\beta - I) - I \otimes S$, the above equations become,

$$\pi_1 S^* = -\pi_o(T \otimes \beta) \tag{8.3.11}$$

$$\pi_i S^* = \pi_i(I \otimes S^o\beta) + \pi_{i-1}(T^o\alpha \otimes I) - \pi_i(e\alpha \otimes S^o\beta) \tag{8.3.12}$$

$i=2,3,\ldots,N.$

Similarly after multiplying on the right both sides of (8.3.2) and (8.3.3) by the matrix $(e\alpha - I) \otimes I$, and putting $T^* = (e\alpha - I) \otimes S - T \otimes I$, we get

$$\pi_i T^* = \pi_i(T^o\alpha \otimes I) + \pi_{i+1}((I - e\alpha) \otimes S^o\beta), \tag{8.3.13}$$

$i=1,2,\ldots,N.$

Now combining (8.3.11)-(8.3.13) and (8.3.4), we have

$$\pi_1 S^* = -\pi_o(T \otimes \beta) \tag{8.3.14}$$

$$\pi_i S^* = \pi_{i-1} T^*, \quad i=2,3,\ldots,N \tag{8.3.15}$$

and

$$\pi_{N+1} = [(T+T^o\alpha) \otimes I+I \otimes S] = -\pi_N(T^o\alpha \otimes I) \tag{8.3.16}$$

Following Neuts (1981), we see that the irreducible matrix $Q^* = T+T^o\alpha$ is the generator of a homogeneous Markov process and it is substable (A matrix is stable if all its eigen values have their real parts strictly less than zero; it is substable if their real parts are less than or equal to zero). The eigen value of $Q^* \otimes I + I \otimes S$ are the sum of the eigen values of the substable matrix Q^* and the stable matrix S (see Bellman (1974)). Hence $Q^* \otimes I + I \otimes S$ is stable and invertible.

Next we prove that T^* and S^* are invertible. Consider $T^* = -(T \otimes I + (I-e\alpha) \otimes S)$. By our assumption T and S are stable matrices. T^* can be rearranged as

$$T^* = -(T \otimes I)\ [I \otimes S^{-1} + T^{-1}(I-e\alpha) \otimes I]\ (I \otimes S) \tag{8.3.17}$$

Firstly we prove that $T^{-1}(I-e\alpha)$ is substable.

Let λ be an eigen value of $T^{-1}(I-e\alpha)$ and v be the corresponding eigen vector. Assume that $Re(\lambda) > 0$.

We have $\quad \lambda v = T^{-1}v - (\alpha v)T^{-1}e$

Since T and T^{-1} are stable, $1/\lambda$ cannot be an eigen value of T and λ cannot be an eigen value of T^{-1}.

Hence $\quad \alpha v \neq 0.$

Further $\quad v = \left(\frac{\alpha v}{\lambda}\right)\left(\frac{1}{\lambda}I-T\right)^{-1} e$

$$\therefore \quad \alpha v = \left(\frac{\alpha v}{\lambda}\right)\alpha\left(\frac{1}{\lambda} I-T\right)^{-1} e$$

$$\Longrightarrow \quad \frac{1}{\lambda}\alpha\left(\frac{1}{\lambda} I-T\right)^{-1} e = 1 \tag{8.3.18}$$

The Laplace-Stieltjes transform of F(x) is given by

$$F^*(s) = 1-s\,\alpha(sI-T)^{-1}e$$

$$\therefore \quad F^*(1/\lambda) = 1-\frac{1}{\lambda}\alpha\left(\frac{1}{\lambda} I-T\right)^{-1}e = 0 \text{ (by (8.3.18))}$$

But the Laplace-Stieltjes transform of a non-negative random variable is strictly positive in the right plane. Hence $T^{-1}(I-e\alpha)$ is substable.

Since S^{-1} is stable and $T^{-1}(I-e\alpha)$ is substable, $I \otimes S^{-1} + T^{-1}(I-e\alpha) \otimes I$ is stable and invertible. Moreover $T \otimes I$ and $I \otimes S$ are invertible. Hence from (8.3.17) we conclude that T^* is invertible.

By the same arguments S^* is invertible.

Now the stationary distribution $\pi_i (i = 1,2,\ldots,N+1)$ is given by

$$\pi_1 = -\pi_o (T \otimes \beta) S^{*-1}$$

$$\pi_i = \pi_{i-1} T^* S^{*-1}, \quad i=2,3,\ldots,N.$$

and

$$\pi_{N+1} = -\pi_N (T^o\alpha \otimes I)[(T+T^o\alpha) \otimes I + I \otimes S]^{-1}.$$

By defining the matrices

$$M_o = -(T \otimes \beta) S^{*-1} \tag{8.3.19}$$

$$M = T^* S^{*-1} \tag{8.3.20}$$

$$M_N = -(T^o\alpha \otimes I)[(T+T^o\alpha) \otimes I + I \otimes S]^{-1} \tag{8.3.21}$$

which are respectively of dimensions $m \times mn$, $mn \times mn$ and $mn \times mn$. It is clear that the mn-component row vector π_i can be obtained by the following formulae.

$$\pi_i = \begin{cases} \pi_o M_o M^{i-1} & , i = 1,2,\ldots,N \\ \pi_o M_o M^{N-1} M_N & , i = N+1 \end{cases} \tag{8.3.22}$$

In order to calculate π_o, we substitute the above values in (8.3.9) to get

$$\pi_o[\ I + (\sum_{i=o}^{N-1} M_oM^i + M_oM^{N-1}M_N)(I \otimes e)]\ (T+T^o\alpha) = 0$$

ie., $\quad \pi_o\ W(T+T^o\alpha) \ = \ 0 \qquad (8.3.23)$

where

$$W = I + (\sum_{i=o}^{N-1} M_oM^i + M_oM^{N-1}M_N)(I \otimes e) \qquad (8.3.24)$$

Since $T+T^o\alpha$ is irreducible, the system

$$u(T+T^o\alpha) = 0, \quad ue = 1 \qquad (8.3.25)$$

has a unique solution. Therefore from (8.3.23) we get

$\pi_oW \ = cu,$ where c is a constant.

On applying the normalizing condition (8.3.5) we get c=1.

Hence in order to obtain π_o we need to solve for the system

$$\pi_oW \ = \ u$$

and the stationary probabilities can be calculated using the formulae (8.3.22).

If we denote $\quad p_i \ = \sum_{j=1}^{m} \ \sum_{k=1}^{n} \pi_{ijk} \ (i=1,2,\ldots,N+1)$

and $\quad p_o \ = \sum_{j=1}^{m} \pi_{oj}$

then $\{p_i,\ i=0,1,2,\ldots,N+1\}$ represents the stationary distribution of the system size

8.4 NUMERICAL EXAMPLES

We illustrate the results of this chapter by giving numerical results for certain queueing systems. The calculations are done on a computer with programs written in Pascal.

Example (1)

$F(.)$ is hyper exponential with

$$\alpha = (0.2,\ 0.3,\ 0.5) \text{ and } T = \begin{bmatrix} -1.0 & 0 & 0 \\ 0 & -2.0 & 0 \\ 0 & 0 & -3.0 \end{bmatrix}$$

$G(.)$ is an Erlang distribution with

$$\beta = (1,0,0,0) \text{ and } S = \begin{bmatrix} -3.0 & 3.0 & 0 & 0 \\ 0 & -3.0 & 3.0 & 0 \\ 0 & 0 & -3.0 & 3.0 \\ 0 & 0 & 0 & -3.0 \end{bmatrix}$$

and assume that $N = 10$.

Example (2)

$F(.)$ is an Erlang distribution of order 3 and with parameter 2.5, $G(.)$ is an Erlang distribution of order 4 and with parameter 3.5. Assume $N = 10$.

$\alpha = (1,0,0)$ $\qquad\qquad \beta = (1,0,0,0)$

$$T = \begin{bmatrix} -2.5 & 2.5 & 0 \\ 0 & -2.5 & 2.5 \\ 0 & 0 & -2.5 \end{bmatrix} \quad \text{and } S = \begin{bmatrix} -3.5 & 3.5 & 0 & 0 \\ 0 & -3.5 & 3.5 & 0 \\ 0 & 0 & -3.5 & 3.5 \\ 0 & 0 & 0 & -3.5 \end{bmatrix}$$

The stationary system size probabilities corresponding to Examples (1) and (2) are given below.

k	p_k	
	Example (1)	Example (2)
0	0.3555503	0.2857161
1	0.2486521	0.4315060
2	0.1626867	0.1895425
3	0.0978271	0.0634414
4	0.0573769	0.0203101
5	0.0334362	0.0064690
6	0.0194558	0.0020593
7	0.0113212	0.0006573
8	0.0066011	0.0002085
9	0.0038559	0.0000661
10	0.0022790	0.0000212
11	0.0009621	0.0000051
$\Sigma\ p_k$	1.0000044	1.0000016

Chapter-9

A FINITE CAPACITY M/G/1 QUEUE WITH VACATION

9.1 INTRODUCTION

In the ordinary M/G/1 queue, a poisson stream of customers with i.i.d. service times of arbitrary distribution enters a single server facility with continuous service. When the system capacity is finite with room for atmost N customers in the system, ie. in queue or in service, the system will be designated by M/G/1/N.

Queueing systems with server vacations arise naturally as models of many diverse fields such as computer, communication and production systems. Under specified conditions, the server after finishing the customer in service, discontinues service for an independent vacation period. When a vacation period ends, either customers are present in the queue or the queue is empty. If customers are present, service is resumed. Otherwise, a new vacation period, having the same distribution as the previous one and independent of it, begins. Customers are served in the order of their arrival. Different models are distinguished by the rules which determine when service stops and a vacation begins. For an M/G/1 vacation system with exhaustive service, i.i.d vacations are performed whenever the queue is empty. In a single service discipline a vacation period begins after every service completion, or after any vacation period if the queue is empty. In a Bernoulli schedule discipline the server begins a vacation if the queue is empty. If at a service completion epoch the queue is not empty, the service is resumed with probability p and with probability 1-p a vacation commences.

An extensive literature on single server vacation systems has been developed in recent years. Notable among them are Courtois (1980), Scholl and Kleinrock (1983), Fuhrman and Cooper (1985), Keilson and Servi (1986a), Levy and Yechiali (1975), Heymann (1977), Ramachandran Nair (1987) etc. For a survey of Queueing vacation models one may refer to Doshi (1986). While various server vacation policies have been considered for the M/G/1 system with infinite capacity, very few papers deal with the finite system. Finite capacity systems have been studied previously in Hoskstad (1977), Miller (1975), Levenberg (1975), Courtois (1980) and Loris-Teghem (1988). Lee (1984) studied an M/G/1 queue with finite waiting space and server vacation where he considered two types of service discipline: viz. 1) exhaustive service discipline, and 2) limited service discipline ie., the server will begin a vacation if either the queue has been emptied or M customers have been served during a visit. In all these studies the steady state behaviour of the system is of major concern.

In this chapter we consider a rather large class of vacation policies which contains, in particular, the policies described above. A precise description of the model is given below. The model is analysed, using Markov renewal theory, in section 9.2. The time dependent queue size distribution is obtained in section 9.3 and virtual waiting time distribution is attempted in section 9.4.

We consider an M/G/1/N queueing system where an arriving customer finding N customers present in the system may not enter the system and is lost. Customers arrive according to a Poisson process of rate μ. The successive service times are i.i.d random variables with distribution function G(.) and they are also independent of interarrival times. The queue discipline is FIFO. The server goes for vacation either when the queue becomes empty or after serving a random number of customers. ie, if the queue is empty after a service completion then the server begins a vacation period for a duration having a general distribution. At the end of a vacation period service begins if at least one customer is present in the queue. Otherwise, one or more additional vacations are repeated until at least one customer is present (multiple vacation). If k units were served continuously since his arrival to the system after the last vacation, the server may go for vacation with probability p_k and resumes his service with probability $1-p_k$. We have $p_1 < p_2 < \ldots < p_M = 1$, where M is the maximum number of customers served in a busy period. The vacation times are i.i.d random variables with a common probability distribution function $H(\circ)$ and they are assumed to be independent of the interarrival times and service times. Without loss of generality we assume that $N < M$.

One can model many service systems using the present model. In certain cases it often happens that the service rate decreases with the increasing number of units served. In that case the service time of customers also increases and thereby results in increased cost to the system. Hence there will be an optimal number of units the server can be allowed to serve continuously so as to make the system most efficient and least expensive.

The following notations are used in the sequel:-

The lower case letters denote the probability density functions (assuming that they exist).

$$\mu_n(x) = \frac{e^{-\mu x}(\mu x)^n}{n!}, \quad n = 0,1,2, \ldots, N-1$$

$$\mu_N(x) = \sum_{n=N}^{\infty} \frac{e^{-\mu x}(\mu x)^n}{n!}$$

$*$ denotes the convolution

$f^{*n}(.)$ = n-fold convolution of $f(.)$ with itself $(f^{*o}(.) \equiv 1)$

N^o = $\{0,1,2,\ldots,\}$

E = $\{0,1,2,\ldots, N\}$

$b(x)$ = $\sum_{m=o}^{\infty} h^{*m}(x)$

$\langle r \rangle$ = $\max(o,r)$.

9.2 ANALYSIS OF THE MODEL

Let us denote by $X(t)$ the number of customers present in the system at time t. Then the process

$$X = \{X(t), \; t \geq 0\}$$

is a semi-regenerative process with state space $E = \{0,1,2,\ldots N\}$ and the following embedded Markov renewal process.

Let $0 = T_o, T_1, T_2, \ldots$ be the instants of successive busy period terminations, and X_n be the number of customers left behind at the termination of the n^{th} busy period. Then $(X,T) = \{(X_n, T_n); \; n \in N^o\}$ is a time homogeneous Markov renewal process. The associated semi-Markov kernel over the set E is

$$Q = \{Q(i,j,t) : i,j \in E, \; t \geq 0\}$$

where $Q(i,j,t) = Pr\{X_{n+1}=j,\ T_{n+1}-T_n \leq t \mid X_n=i\}$ is given by

$$Q(i,j,t) = \begin{cases} \int_o^t \sum_{k=\langle j-i\rangle}^{M-i+j} \mu_k(u)(h*g^{*(i+k-j)})(u)p_{i+k-j}du, \\ \qquad\qquad i \neq 0,\ j \in E; \\ \int_o^t e^{-\mu u}b(u) \sum_{k=o}^{M} \mu_{j+k}(t-u)(h*g^{*(k-j)})(t-u)p_{k-j}\,du, \\ \qquad\qquad i = 0,\ j \in E \end{cases} \tag{9.2.1}$$

Let $\mathcal{Q}(t) = [Q(i,j,t)]_{i,j \in E}$

Define for $n \in N^o$

$$Q^n(i,j,t) = Pr\{X_n=j,\ T_n \leq t \mid X_o=i\},\ i,j \in E,\ t \geq 0.$$

Then

$$Q^o(i,j,t) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{for all } t \geq 0,$$

and we have, for $n \geq 0$, the recurrence relation

$$Q^{n+1}(i,k,t) = \sum_{j \in E} \int_o^t Q(i,j,du)Q^n(j,k,t-u)$$

Then the Markov renewal functions are given by

$$R(i,j,t) = \sum_{n=0}^{\infty} Q^n(i,j,t), \quad i,j \in E, \quad t \geq 0 \qquad (9.2.2)$$

Denote $R(t) = [\, R(i,j,t) \,]_{i,j \in E}$. It is the Markov renewal kernel corresponding to $Q(t)$.

Here we note that, since the state space E is finite we can compute the Markov renewal kernel by the relation

$$\hat{R}_\alpha = (\, I - \hat{Q}_\alpha \,)^{-1} \qquad (9.2.3)$$

where $\hat{Q}_\alpha = [\, \hat{Q}_\alpha(i,j)]_{i,j \in E}$ and $\hat{R}_\alpha = [\hat{R}_\alpha(i,j)]_{i,j \in E}$,

$$\hat{Q}_\alpha(i,j) = \int_0^\infty e^{-\alpha t} Q(i,j,dt),$$

$$\hat{R}_\alpha(i,j) = \int_0^\infty e^{-\alpha t} R(i,j,dt).$$

9.3 THE STATE SPACE AND TRANSIENT SYSTEM SIZE PROBABILITIES

Our basic concern is with the process $X(t)$, the number of customers in the system at time t. Consider the trivariate stochastic process $\underline{W}(t) = (X(t), Y(t), Z(t))$, where

$X(t) =$ number of customers in the system

$$Y(t) = \begin{cases} 0 \text{ if a vacation is in progress at time } t \\ 1 \text{ if a service is in progress at time } t \end{cases}$$

and

$$Z(t) = \begin{cases} \text{number of customers served upto } t \text{ since the commencement of the current busy period (termination of the last vacation period), if a service is in progress at time } t \\ 0, \text{ if a vacation is in progress at time } t. \end{cases}$$

This process may be discussed on its state space

$$S = \{(i,j,k);\ 0 \le i \le N,\ j = 0,1,\ 0 \le k \le M-1\}.$$

We assume that at time $t = 0$, the server just completes a busy period and enters a vacation period so that the initial state of the process is

$$\underline{W}(0) = (X(0), Y(0), Z(0)) = (a,0,0) \text{ for some } a \in E$$

For each $a,i \in E$, $j \in \{0,1\}$, $k \in \{0,1,2,\ldots,M-1\}$, $t \ge 0$ define

$$P(t,a,i,j,k) = Pr\{\underline{W}(t) = (i,j,k) \mid \underline{W}(0) = (a,0,0)\}$$

Then we have

<u>Theorem 9.3.1</u>. For any $a, i \in E$, $j \in \{0,1\}$, $0 \leq k \leq M-1$ and $t \geq 0$,

$$P(t,a,i,j,k) = \sum_{\ell \in E} \int_0^t R(a,\ell,du)\, K(t-u,\ell,i,j,k)$$

where

$$K(t,\ell,i,j,k) = \Pr\left\{ \underline{W}(t)=(i,j,k),\ T_1>t \,|\, \underline{W}(0)=(\ell,0,0)\right\}$$

are as given below.

$$K(t,\ell,i,j,k)=\begin{cases} \int_0^t e^{-\mu u} b(u)\mu_i(t-u)[1-H(t-u)]du \\ \qquad \text{for } \ell=0,\ 0\leq i\leq N,\ j=0,\ k=0; \\ \int_0^t e^{-\mu u} b(u) \int_u^t h(v-u)\mu_{i+k}(t-u)[G^{*k}(t-v)-G^{*(k+1)}(t-v)] \\ \qquad (1-p_k)dv\, du \\ \qquad \text{for } \ell=0,\ 1\leq i\leq N,\ j=1,\ 0\leq k\leq M-1; \\ [1-H(t)]\mu_{i-\ell}(t) \quad \text{for } 1\leq \ell\leq N,\ \ell\leq i\leq N,\ j=0,\ k=0; \\ \mu_{i+k-\ell}(t) \int_0^t h(u)\ [G^{*k}(t-u)-G^{*(k+1)}(t-u)](1-p_k)du \\ \qquad \text{for } 1\leq \ell\leq N,\ 1\leq i\leq N,\ j=1,\ 1\leq k\leq M-1 \\ \mu_{i-\ell}(t) \int_0^t h(u)[1-G(t-u)]du \\ \qquad \text{for } 1\leq \ell\leq N,\ \ell\leq i\leq N,\ j=1,\ k=0 \\ 0 \qquad \text{otherwise} \end{cases}$$

Proof:

$$P(t,a,i,j,k) = \Pr\left\{\underline{W}(t)=(i,j,k),\ T_1>t \mid \underline{W}(0) = (a,0,0)\right\}$$

$$+\Pr\left\{\underline{W}(t)=(i,j,k),\ T_1\leq t \mid \underline{W}(0) = (a,0,0)\right\}$$

$$= K(t,a,i,j,k) + \sum_{\ell\in E} \int_0^t Q(a,\ell,du)\ P(t-u,\ell,i,j,k)$$

which is a Markov renewal equation (see for eg. Cinlar (1975b)) and its solution is given by

$$P(t,a,i,j,k) = \sum_{\ell\in E} \int_0^t R(a,\ell,du)\ K(t-u,\ell,i,j,k)$$

Since there are only finitely many states, this solution is unique.

Now the expressions for $K(t,\ell,i,j,k)$ can be seen to be as given in the statement of the theorem by considering different possible values of ℓ,i,j,k.

[Q.E.D.]

9.4 VIRTUAL WAITING TIME DISTRIBUTION

Let $\eta(t)$ be the virtual waiting time of a customer in the queue. Here we obtain the probability distribution of $\eta(t)$ conditioned on the state of the system at time t.

Denote $\emptyset_t(x/(i,j,k)) = \Pr\{\eta(t)\leq x \mid \underline{W}(t) = (i,j,k),$
$\underline{W}(0) = (a,0,0)\}$

Then we consider the following cases separately.

Case (i): $i=0,\ j=0,\ k=0,$

$$\emptyset_t(x|(i,j,k)) = \int_0^t R(a,0,du)\quad \mu_0(t-u)\int_u^t b(v-u)\,[H(t+x-v)-H(t-v)]dv\ du$$

Case (ii): $1\leq i\leq N,\ j=0,\ k=0$

$$\emptyset_t(x|(i,j,k) = \int_0^t R(a,0,du)\mu_i(t-u)\int_u^t b(w-u)\int_t^{t+x} h(v-w)\sum_{k=0}^{i}(G^{*i}*H^{*k})(t+x-v)dv\ dw\ du$$

$$+\sum_{\ell=1}^{i}\int_0^t R(a,\ell,du)\mu_{i-\ell}(t-u)\int_t^{t+x} h(v-u)\sum_{n=0}^{i}(G^{*i}*H^{*n})(t+x-v)dv\ du$$

and finally,

Case (iii): $1\leq i\leq N,\ j=1,\ 0\leq k\leq M-1$

$$\emptyset_t(x|(i,j,t) = \int_0^t\sum_{\ell=1}^{j+k} R(a,\ell,du)\mu_{i+k-\ell}(t-u)\int_t^{t+x} h(v-u)\sum_{n=1}^{i+k}(G^{*(k+i)}*H^{*n})(t+x-v)dv\ du$$

$$+ \int_{o}^{t} R(a,0,du)\ \mu_{i+k}(t-u) \int_{u}^{t} b(w-u) \int_{t}^{t+x} h(v-w)$$

$$\sum_{n=1}^{i+k} (G^{*(k+i)} * H^{*n})(t+x-v)dv\ du.$$

Remark:

In this model if we put $M=1$ and $p_1=1$ we get single service discipline and if we put $M=\infty$ and $p_i=p$ $(i=1,2,...)$ we have the Bernoulli schedule discipline and when $p=0$ it becomes the exhaustive service discipline.

CONCLUDING REMARKS

In this thesis, (s,S) inventory systems with non-identically distributed interarrival demand times and random lead times, state dependent demands, varying ordering levels and perishable commodities with exponential life times have been studied. The queueing system of the type $E^k/G^{a,b}/1$ with server vacations, service systems with single and batch services, queueing system with phase type arrival and service processes and finite capacity M/G/1 queue when server going for vacation after serving a random number of customers are also analysed.

In inventory theory, one can extend the present study to the case of multi-item, multi-echelon problems. The study of perishable inventory problem when the commodities have a general life time distribution would be a quite interesting problem. The analogy between the queueing systems and inventory systems could be exploited in solving certain models.

Consider an inventory system with more than one ordering levels and more than one server. Assume that some of the servers take vacation when the inventory is less than a prescribed quantity. Here again one can investigate the transient as well as steady state solutions.

The techniques used to derive the time dependent solutions may be of special interest to any stochastic system having regenerative or semi-regenerative structure. The most important problem one can think of is to develop an algorithm to compute the given transient solutions numerically. To the application point of view this is a quite worthwhile work. For developing the algorithm, possibly one can effectively use some fast transform techniques (see, Elliott and Rao (1982)) because here we cannot use the usual procedure of Laplace transforms.

In vacation models, one important result is the stochastic decomposition property of the system size or waiting time. One can think of extending this to the transient case. The distribution of virtual waiting time may be used for the decomposition property of the waiting time, since it can be defined as the unfinished work (see Kleinrock (1975)).

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