FLUID MECHANICS

STUDIES OF STABILITY OF FLUID FLOWS USING VARIATIONAL METHODS

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STATEMENT

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any university and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

Geetha S

(GEETHA. S)

CERTIFICATE

This is to certify that the work reported in this thesis entitled " Studies of stability of fluid flows using variational methods " is based on the bonafide work done by Smt. Geetha.S, under my guidance in the Division of Mathematics, School of Mathematical Sciences, Cochin University of Science and Technology, Cochin 682 022, and has not been included in any other thesis, submitted previously for the award of any degree or diploma.

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GEETHA. S

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 THE CONCEPT OF STABILITY

Stability theory has become of dominant importance in the study of dynamical systems. It has many applications in basic fields like meteorology, oceanography, astrophysics and geophysics- to mention few of them. The concept of stability was developed very early in the eighteenth century and was specialized in mechanics to describe some type of equilibrium of a material particle or system. In precise mathematical terms, the equilibrium of a particle, subjected to some forces, is called stable if, after any sufficiently small perturbations of its position and velocity, the particle remains for ever arbitrarily near the equilibrium point with arbitrarily small velocity.

This definition of stability was found useful in many situations, but inadequate in many others so that a host of other important concepts have been introduced in past many years which are more or less related to the first definition and to the common sense meaning of stability.

The mathematical formulation of stability theory proceeds from the nonlinear differential equations which describe the problem of mathematical physics under consideration, under the most general conditions.

The next great advance came in hydrodynamic stability which laid foundations of the stability theory in fluid mechanics. Hydrodynamic stability has been recognized as one of the central problems of fluid mechanics.

In recent years the theoretical developments in the studies of instabilities and turbulence have been as profound as the developments in experimental methods. Classical theory of stability is the linearized theory in which the effect of a small fluctuation away from a solution to the equations is examined as a function of a parameter such as the Reynolds number.

Another major development is the application of new mathematical concepts from the qualitative theory of differential equations, sometimes known as theory of dynamical systems, to the problem of transition to turbulence which have provided new insights in recent years. Traditional method like bifurcation theory have also contributed major new insights.

The exponential instability property of geodesics on manifolds of negative curvature has been studied by many authors beginning with Hardmard,Hopf etc.(see Arnold(1978)). This type of instability leads to the stochasticity of the corresponding geodesic flow. Geodesics are motions of an ideal fluid, therefore the calculation of the curvature of the group of diffeomorphisms gives us some information on the instability of ideal fluid flows.

The study of stability problems is relevant to the study of structure of a physical system. It is particularly important when it is not possible to probe into its interior and obtain information on its structure by a direct method.

1.2 HYDRODYNAMIC STABILITY

The essential problems of hydrodynamic stability were recognized and formulated in the nineteenth century most notably by the pioneers like Helmholtz, Kelvin, Rayleigh and Reynolds. Reynolds (1883) introduced these problems clearly in his own experiments on the instability of flow in a pipe.

Not every solution of the equations of motion, even if it is exact, can actually occur in nature. On the other hand few laminar flows correspond to known solution of the nonlinear equations of motion. The flows that occur in nature must not only obey equations of fluid mechanics, but also be stable. Instability of flows may be caused by various physical aspects of flow namely, disturbance of the equilibrium of the external forces, inertia and viscous stresses etc.. Centrifugal and Coriolis forces are also regarded as external forces in the case of rotation of the whole system in which the fluid moves. Surface tension exerts a stabilizing influence; particularly on disturbances of small length scale by minimizing the area of a surface. A magnetic field can inhibit the motion of an electrically conducting fluid across the magnetic lines of force and thereby stabilize flows. A fluid moves according to the equilibrium between its inertia and internal stresses of pressure in the absence of any external force or of viscosity. A small disturbance may upset this equilibrium. The tendency of fluid to move down pressure gradients may amplify disturbances of certain flows and thereby create instability.

Viscosity has great stabil i zing influence. It dissipates the energy of any disturbance and thereby stabilize a flow. It has also the more complicated effect of diffusing momentum. Due to viscosity, some flows like parallel shear flows become unstable although the same flows of an inviscid fluid are stable. Thermal conductivity or molecular diffusion of heat has also some effects similar to those of viscosity or molecular diffusion of momentum and has usually a stabilizing influence. The boundaries of a flow constrain the development of a disturbance and when they are closer together the flow will be more stable. However, they sometimes give rise to strong shear in boundary layers which leads to instability of the flow.

The problem of hydrodynamic instability originated in the differentiation between stable and unstable patterns of permissible flows. To solve these problems, one must follow the solution of a system of nonlinear partial differential equations. Analysis of dynamic instabilities dates back to the work of Helmholtz and Reynolds. Helmholtz(1890) has analyzed the stability of wave motion along surfaces of discontinuity assuming sharp changes in wind and density along the verticals and showed that the over-all surface is unstable under sufficiently large perturbations. He has also shown that a finite discontinuity in the wind will result in reduced stability. Later Rayleigh (1913) studied the stability of horizontal parallel flows and has shown that the flow stability depends on the shape of the velocity profile. Thus he formulated the result as follows:

> " Parallel flows of an inviscid fluid are stable if the velocity profile has no point of inflection ".

This is known as Rayleigh's theorem. The theorem gives a necessary condition for instability or a sufficient condition for stability for inviscid fluids. Later Tollmien (1936) showed that this condition is also sufficient for velocity distributions of certain types. A physical mechanism for interpreting this result was derived by Lin (1945), using an acceleration formula derived on the basis of von Karman's (1934) mechanism of vorticity redistribution. Rossby (1949) applied these ideas to the motion of polar air masses, fundamental in atmospheric process. A stronger form of Rayleigh's theorem was obtained later by Fjørtoft (1950), who proved that for instability the value of vorticity of the primary flow must have a maximum in the domain of flow. This theorem also gives only a necessary condition for instability.

Some of the instabilities which arise from different causes are Rayleigh - Taylor instability and Kelvin - Helmholtz instability (Chandrasekhar (1961)). The first derives from the character of the equilibrium of an incompressible heavy fluid of variable density (i.e, of heterogeneous fluid). An important special case is that of two fluids of different densities superposed one over the other or accelerated towards each other; the instability of the plane interface between two fluids, when it occurs (particularly in the second context), is called Rayleigh-Taylor instability. The second type of instability arises when the different layers of a stratified heterogeneous fluid are in relative horizontal motion. The special case is when two superposed fluids flow one over the other with a relative horizontal velocity, the instability of the plane interface between two fluids being called Kelvin-Helmholtz instability. The physical mechanism of Kelvin-Helmholtz instability has been described by Batchelor (1967) in terms of the vorticity dynamics. In Rayleigh-Taylor instability, the quantitative observations have been made by Lewis (1950) and others. The method has been applied by Pramod (1989) to study interfacial waves.

1.3 NONLINEAR STABILITY

Nonlinear stability analysis is necessary when one investigate the development of secondary flows and the onset of higher instabilities. Reynolds (1883) has appreciated the importance of nonlinear disturbances of Poiseuille flow in a pipe and Bhor (1909), Noether (1921) and Heisenberg (1951) treated them theoretically for special problems. The main concepts of the theory of nonlinear hydrodynamic stability are due to Landau (1944). Hopf (1948) has developed similar ideas on turbulence as the Reynolds number increases, through the repeated bifurcation of the solution representing the flow.

One of the specific methods in the strongly nonlinear theory of hydrodynamic stability is the energy method, which originated in the early work of Reynolds (1895) and Orr (1907). In the global theory of stability the energy methods have an important place. This method leads to a variational problem and a definite criterion for the stability of basic flow. In fact, any method based on a variational problem can be considered as energy method in a generalized sense. This aspect of subject has been extensively studied by Serrin (1959) and a fuller account of this method till that date has been given by Joseph (1976). The significance of this method is that it provides rigorous criteria for stability with respect to arbitrary disturbances whereas the linear theory provides criteria for instability.

At the end of the last century the celebrated Russian Mathematician Liapunov (1892) elaborated a general method for investigating stability of the solutions of a system of differential equations:

$$\frac{dx_i}{dt} = f_i(t,x_1,\ldots,x_n) \ , \ i = 1,2,\ldots,n.$$

This method is known as Liapunov's direct method (second method), since it yields stability information directly; that is, without solving the differential equations. Liapunov formulated the concept of stability, asymptotic stability and instability in a precise form. As contrasted with mechanical stability, Liapunov's stability has the following features: (1) it pertains no more to a material particle (or the equations thereof), but to a general differential equation , (2) it applies to a solution , that is, not only to an equilibrium or critical point. The major advantage of Liapunov's method is that stability in the large can be obtained without any prior knowledge of solutions.

Recently, this method has become an excellent tool which is widely used not only in the study of differential equations but also in the theory of control systems, systems with time lag, power system analysis, time varying nonlinear feed back systems and so on. Its chief characteristic is the construction of a scalar function, called Liapunov function. In many practical situations, besides nonlinear stability of a solution, it is useful to require that all neighboring solutions tend to the basic solution as time t tends to ∞ . This leads to the notion of asymptotic stability.

The second method of Liapunov for establishing stability is a natural extension of the energy method. The sufficient condition for stability can be deduced by seeking a constant of motion with a local maximum or minimum at the equilibrium. In many examples this constant of motion is energy. Zubov (1957) and Movchan (1959) have generalized the method in order to apply to continuous systems, though it has been used for over sixty years to determine stability of system of ordinary differential equations before them. Pritchard (1968) has derived some criteria for the nonlinear stability of Benard convection and couette flow between rotating cylinders.

The principal draw back of Liapunov's direct method is that no general procedure is known to construct auxiliary functions suiting specific theorems. That is why, in stability problems, one should a priori neglect no available information concerning the solutions. In particular, the first integrals will often be helpful, either to facilitate the search for auxiliary functions or to eliminate part of the variables and thus decrease the number of equations to examine.

In hydrodynamic stability we often consider the stability of steady basic flows and so are interested particularly in autonomous systems. The Euler's equations of motion for a rigid body have as their analog in hydrodynamics the Euler's equations of motion of an ideal fluid. Euler's theorem on the stability of rotation around the large and small axes of the inertia ellipsoid corresponds in hydrodynamics to a slight generalization of Rayleigh's theorem on the stability of flows without inflection points of the velocity profile.

1.4 VARIATIONAL PRINCIPLES AND CONSERVATION LAWS

A variational principle can be used as the basis for the description of a dynamical system. This approach views the motion as a whole and involves a method of searching the path in configuration space which yields a stationary value for a certain integral. Lin (1963) has shown that the governing equations of hydrodynamics can be derived from a variational principle by introducing the requirements that the end points of the particle trajectories or their boundary values are not to be varied.

The concept of conservation laws plays a key role in the analysis of basic properties of the solutions of systems of differential equations. The general principle relating symmetry groups and conservation laws was first determined by Noether (1918). It provides a one to one correspondence between variational symmetries and conservation laws for non-degenerate systems.

The Lagrangian variational formulation is comparatively easy and almost straight forward. In fluid dynamics, the Euler description is usually preferred because it reduces the complexity of the governing equations which is not possible in

Lagrangian description. The difficulties with variational principles have become most apparent in this description. Attempts for variational formulations of hydrodynamics have been begun with Bateman (1929), Lichtenstein (1929) and Lamb (1932). Eckart (1938) and Taub(1949) extended this variational principles to adiabatic compressible flows. Both Lagrangian and Eulerian variational formulation for ideal fluid flows have been obtained by Herivel (1955). Following the work of Bateman, Lin (1963) and others, Seliger and Witham (1968) have shown how Euler's equations of motion of inviscid flows might be obtained from a variational principle with a simple Lagrangian function - the pressure. Drobot and Rybarski(1959) have formulated a variational principle for barotropic flows by introducing hydromechanical variations of the fields. Based on this, Mathew and Vedan (1989) and Joseph (1993) have developed a variational principle for non-barotropic flows . This method has the advantage that it avoids such conditions like Lin constraints and provides a systematic approach using Lie group theory leading to conservation laws.

Kelvin (1887) has shown that the Kinetic energy of an incompressible inviscid fluid has a stationary value when the flow is steady, and that the flow is stable if the stationary value is either a maximum or a minimum.

The existence of suitable variational principles for different classes of flows often forms a natural basis for the study of stability of flows . The method leads to a variational problem and a definite criterion for the stability of basic flow. Any such method can be considered as an energy method in a generalized sense.

Arnold (1965a,65b,1966,69) has used such a method to study the stability of stationary flows of an ideal incompressible fluid. Arnold has showed that it is possible to construct variational principles for stationary flows using a special combination of two integrals of motion - integral of energy conservation and integral of vorticity conservation. This functional being a first integral also, has all the properties of a Liapunov function. Thus Arnold substantiated the Rayleigh criterion for stability in an exact nonlinear sense. This method was first developed for two dimensional flows but were later generalized to the case of three dimensioanl flows. The method is called augmented energy method . The success of Arnold's method is based on the possibility of constructing a functional in instantaneous states of hydrodynamic fields which is conserved by virtue of the equations of motion and has a given flow as its extremum. If this extremum is a true maximum or true minimum the flow is stable. In the three dimensional case the problem becomes more complicated and Arnold's method involves consideration of very unwieldy implicit expression for surfaces in functional spaces which are no longer the level surfaces of certain functionals. In this case Dikii (1965b) has used the conservation of potential vorticity in place of vorticity for adiabatic flow of non-homogenous incompressible fluid. Arnold's this method is somewhat similar to that of Fjørtoft(1950), although Fjøortoft used the linearized equations rather than a variational principle. Arnold has used this method also to show that the swirling flow in an irregular annular domain is stable to two-dimensional perturbations if the velocity-profile is concave.

The stability of inviscid fluid flows under finite perturbations evolving according to the nonlinear dynamics of the system has been discussed extensively over the past several years using the augmented energy method, Rayleigh had derived the stability condition for infinitely small perturbations of plane-paralleled flows which sufficed that the velocity profile had no points of inflection. Rayleigh's condition is sufficient also for stability with respect to finite perturbations. But, in addition to that, Arnold has proved the stability of certain flows which have one point of inflection. He has considered only perturbations which do not change the value of velocity circulation along each boundary.

Arnold (1965b) himself has examined further the properties of a flow whose kinetic energy and vorticity are conserved. Drazin and Howard (1966) have obtained results reminiscent of Arnold's by use of the equations of energy and vorticity, rather than by use of a variational principle. Dikii (1965 a,b), and Dikii and Kurganskii (1971) have applied Arnold's method to flows relative to a rotating frame in order to find various criteria of stability.

1.5 SOME DEFINITIONS OF STABILITY

Following Drazin and Reid (1981), We introduce some of the various definitions of stability which are the most widely used and studied.

DEFINITION 1.1

To analyse the stability of any laminar flow we have to consider the fields like velocity $U(x,t)$, pressure $P(x,t)$ and temperature $\theta(x,t)$ which define the basic flow .

If this basic flow is disturbed sightly, the disturbance may either die away, persist as a disturbance of similar magnitude or grow so much that the basic flow becomes a different laminar flow or a turbulent flow. We call such disturbances (asymptotically) stable,neutrally stable or unstable respectively.

DEFINITION 1.2

A basic flow is stable (in the sense of Liapunov) if, for any $\epsilon > 0$, there exists some positive number ζ (depending upon ϵ) such that if

$$\|\bar{u}(x,0) - U(x,0)\|,\ \|\bar{p}(x,0) - P(x,0)\|\ ,\ \text{etc.} < \zeta\ ,\quad \text{then}$$

$$\| \bar{u}(x,t) - U(x,t)\|, \ \| \bar{p}(x,t) - P(x,t)\|, \text{ etc.} < \epsilon, \text{ for all } t \geq 0,$$

where $\bar{u}$ is the velocity field and $\bar{p}$ is the pressure field which satisfy the equations of motion and the boundary conditions. The basic flow is asymptotically stable(in the sense of Liapunov) if,

$$\| \bar{u}(x,t) - U(x,t)\| \text{ , etc } \longrightarrow 0 \text{ as } t \longrightarrow +\infty .$$

These definitions may be not satisfactory when the norm of the basic flow itself decreases or increases substantially in time.

DEFINITION 1.3 (Chandrasekhar,(1961))

Consider a hydrodynamic system in a stationary state, which is defined by a set of parameters $X_1, X_2, \ldots, X_j$. Suppose the system is disturbed. If the disturbance gradually die down, then we say that the system is stable with respect to the particular disturbance and if the disturbance grow in amplitude in such a way that the system progressively departs from the initial state and never reverts to it, then we say that the system is unstable. The locus which separates the two classes of states defines the states of marginal stability of the system (neutral stability).

A system can be considered stable if it is stable with respect to every possible disturbance to which it can be subjected and a system must be considered as unstable even if there is only one special mode of disturbance with respect to which it is unstable.

Following Holm et al. (1985), we identify four interrelated concepts of stability of a dynamical system which are adapted to fluid dynamics.

DEFINITION 1.4

Neutral or Spectral Stability

For a dynamical system

$$\dot{\bar{u}} = \frac{d\bar{u}}{dt} = \bar{X}(\bar{u}) ,$$

an equilibrium point $\bar{u}_e$ satisfying $\bar{X}(\bar{u}_e) = 0$ is called spectrally stable, provided the spectrum of the linearized operator $D\bar{X}(\bar{u}_e)$ has no strictly positive real part. A special case is neutral stability, for which the spectrum is purely imaginary. For Hamiltonian systems spectral stability and neutral stability coincide.

DEFINITION 1.5

Linearized Stability

The equilibrium solution $\bar{u}_e$ is called linearized stable or linearly stable relative to a norm $\|\delta\bar{u}\|$ on infinitesimal variations $\delta\bar{u}$ provided for every $\epsilon>0$, there is a $\zeta>0$ such that if

$\|\delta\bar{u}\| < \zeta$ at $t = 0$, then $\|\delta\bar{u}\| < \epsilon$ for $t>0$, where $\delta\bar{u}$ evolves according to $(\delta\bar{u}) = D\bar{X}(\bar{u}_e)\delta\bar{u}$.

Linearized stability implies spectral stability. The converse is not generally true (For counter example, see Holm et al. (1985)).

DEFINITION 1.6

Formal Stability

The equilibrium solution $\bar{u}_e$ of a system $\dot{\bar{u}} = \bar{X}(\bar{u})$ is formally stable if a conserved quantity is found whose first variation vanishes at the solution and whose second variation at the solution is positive(or negative) definite.

Formal stability implies linearized stability. The converse is not generally true (For counter example, see Holm et al.(1985)).

DEFINITION 1.7

Nonlinear Stability

An equilibrium point $\bar{u}_e$ of a dynamical system is said to be nonlinearly stable if for every neighbourhood U of $\bar{u}_e$ there is a neighbourhood V of $\bar{u}_e$ such that trajectories $\bar{u}(t)$ initially in V never leave U. In terms of a norm $\|\cdot\|$, nonlinear stability means that for every $\epsilon > 0$, there is a $\zeta > 0$ such that if

$\| \bar{u}(0)-\bar{u}_e \| < \zeta$, then $\| \bar{u}(t)-\bar{u}_e \| < \epsilon$ for $t > 0$.

Formal stability need not imply nonlinear stability. Neither formal nor lineraized stability is necessary for nonlinear stability. For a Hamiltonian system, spectral analysis can provide sufficient condition for instability, but it can only give necessary condition for stability. In finite dimensions, formal stability implies stability (a classical result of Lagrange). In infinite dimensional case formal stability need not imply stability. Nonlinear stability requires both formal stability and some convexity estimates to be satisfied. For dissipative systems it has been shown that lineraized stability implies stability.

Formal stability of fluid and plasma has been considered by Fjϕrtoft(1946) , Eliassen and Kleinschmidt (1957), Bernstein et al.(1958), Kruskal and Oberman (1958), Fowler(1963), Gardner (1963), Rosenbluth (1964), Dikii (1965a), Herlitz (1967) and Davidson and Tsai (1973). More recently, formal stability has been established by Blumen (1968), Zakharov and Kuznetsov (1974), Sedenko and Iudovich (1978), Benzi et al.(1982) and Grinfeld (1984), who employed some aspects of Arnold's method (but not the convexity analysis).

Nonlinear stability for conservative fluid and plasma systems has been studied by the Liapunov method by Arnold(1969a), Benjamin (1972), Bona (1975), Mckean (1977), Laedke and Spatschek

(1980), Holm et al.(1983), Holm (1984), Holm et al.(1985), Bennet et al.(1983),Wan (1984) and Hazeltine et al.(1984).

1.6 SCOPE OF THE THESIS

The present thesis is a study of hydrodynamic stability by Arnold's method using variational principles of Drobot and Rybarski (1959) and Mathew and Vedan (1989).

In chapter 2 we present the Hamiltonian formulation of both barotropic and non-barotropic flows. The Lagrangian formulations for barotropic and non-barotropic flows have been developed by Drobot and Rybarski and Mathew and Vedan respectively. We find that by applying Donkin's theorem it is possible to express the evolution equations for hydrodynamics as a finite dimensional Hamiltonian system. Though it is known that Kelvin's circulation theorem follows from the invariance of Poincare-Cartan integral for the Hamiltonian system it is to be noted that the well-known application of Hamiltonian mechanics is treating the evolution equations as an infinite dimensional Hamiltonian system.

Joseph and Vedan have obtained helicity conservation by applying Noether's theorem from the variational principle of Drobot and Rybarski. In chapter 3, we show that their result is valid only for incompressible flow and the result is obtained for a general barotropic flow.

In this chapter we also use Arnold's method to study stability of barotropic flows. The infinitesimal generator of transformation group that leaves helicity invariant is used to define the structure that remains invariant under the flow.

In chapter 4, we discuss stability of non-barotropic flows. Though laws of conservation of circulation and helicity have been generalized to non-barotropic flows, these are not applicable as in the case of barotropic flows. Instead, we use the conservation of potential vorticity to define the invariant structure of the flow. The transformation group which corresponds to conservation of potential vorticity is identified and stability criterion is formulated.

The thesis is concluded with a general discussion of the results obtained.

* * *

CHAPTER 2

LAGRANGIAN AND HAMILTONIAN FOR FLUID FLOWS

2.1 INTRODUCTION

In this chapter we present the Hamiltonian formulation of both barotropic and non-barotropic flows. The Lagrangian formulations for the barotropic and non-barotropic flows have been developed by Drobot and Rybarski (1959) and Mathew and Vedan (1989) respectively. The advantage of their method is that it avoids such conditions like Lin constraints and provides a systematic approach using Lie group theory leading to conservation laws.

We develop the Hamiltonian formulation of both barotropic and non-barotropic flows by applying Donkin's theorem and prove that it is possible to express the evolution equations for hydrodynamics as a finite dimensional Hamiltonian system using a non canonical Poisson bracket.

2.2 VARIATIONAL PRINCIPLE FOR BAROTROPIC FLOWS

Following Drobot and Rybarski (1959) and Mathew and Vedan (1989) we consider the Euclidean four dimensional space X_4. A point x in X_4 has coordinates x^{α}, $\alpha = 0,1,2,3$ where x^{o} is the time t and x^{i}, $i = 1,2,3$ are space-like coordinates. F is a function space of 4-dimensional vector valued functions $\bar{p}(x)$ with components $p^{\alpha}(x)$, $\alpha = 0,1,2,3$. Then $X_4 \times F$, the tangent bundle is a manifold. The particular choice $p^{0} = \rho$; $p^{i} = \rho u^{i}$, where ρ is the density and u^{i} (and u_{i}) the velocity components, defines a vector field on X_4 which is section of the fibre bundle.

In Lagrangian approach the fluid flow is the flow generated by the vector field p^{α}. But in an Eulerian approach we are not interested in the motion of the individual fluid particles. Here the governing equations form a system of partial differential equations with independent variables x^{α} and dependent variables p^{α}. The system of equations define a subvariety of the first order jet space $X_4 \times F^{(1)}$ (Olver,(1986), p.98). Though this jet space involves the prolongation of the vector field p^{α}, in our case the computation need not involve the prolongation because of the particular choice of the Lagrangian.

DEFINITION 2.1

Let S be a three dimensional submanifold of X_4 and dS_{α} an oriented element of S. Then $p^{\alpha} dS_{\alpha}$ is a differential 3-form on S. The integral

$$\int_{S} p^{\alpha}\, dS_{\alpha} \quad , \tag{2.1}$$

is called the flux of matter flow across S. We consider a volume τ in X_4. Then the action W is defined as

$$W = \int_T d\tau \; L\,(\bar{x},\bar{p}(\bar{x}))\ , \tag{2.2}$$

where Lagrangian density L is a function of x^α and p^α only.

We consider a one-parameter group of transformations of $X_4 \times F$ into itself with the infinitesimal generator

$$\bar{V} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial p^\alpha}\ , \tag{2.3}$$

where $\xi^\alpha = \xi^\alpha(\bar{x},\bar{p})$ and $\eta^\alpha = \eta^\alpha(\bar{x},\bar{p})$, $\alpha = 0,1,2,3.$
The flow generated by $\bar{V}$ is subjected to the conservation laws of momentum and mass. This leads to the definition of hydromechanical transformation (Joseph).

DEFINITION 2.2

For arbitrary ξ^α, the one-parameter family of transformations generated by

$$\bar{V} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial p^\alpha}$$

where $$\eta^\alpha = \partial_\beta\left(p^\beta \xi^\alpha - p^\alpha \xi^\beta\right) \tag{2.4}$$

is called a hydromechanical transformation. Suffix denotes total derivative with respect to corresponding independent variable. Since η^{α} is defined in terms of ξ^{α}, the independent variation is given by ξ^{α} only and it is known as horizontal variation, since they are in the direction of x^{i} only (Moreau 1981). For barotropic flow of an inviscid fluid the Lagrangian density is chosen as

$$L = \frac{1}{2p^{0}} \left[(p^{1})^{2} + (p^{2})^{2} + (p^{3})^{2}\right] - \epsilon(p^{0}) - p^{0}U , \qquad (2.5)$$

where ϵ is the internal energy and U is the potential of external forces. We consider the action of a transformation $\bar{V}$ (2.3) on W. Then the total variation is defined as

$$\Delta W = \int_{\tau} d\tau \left\{ \frac{\partial L}{\partial p^{\alpha}} \eta^{\alpha} + \partial_{\alpha}(L \xi^{\alpha}) \right\} . \qquad (2.6)$$

Now we state the variational principle:

For all ξ^{α} vanishing on the boundary, the total variation

$$\Delta W = \int_{\tau} d\tau \left\{ \frac{\partial L}{\partial p^{\alpha}} \partial_{\beta}\left(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta}\right) + \partial_{\alpha}\left(L \xi^{\alpha}\right) \right\} = 0 \qquad (2.7)$$

ie.,
$$\int_{\tau} d\tau \left\{ \partial_{\beta}\left(T^{\beta}_{\alpha} \xi^{\alpha}\right) - \psi_{\alpha} \xi^{\alpha} \right\} = 0 ,$$

where

$$T_\alpha^\beta = p^\beta \frac{\partial L}{\partial p^\alpha} + \delta_\alpha^\beta \left(L - p^\gamma \frac{\partial L}{\partial p^\gamma} \right) , \tag{2.8}$$

and

$$\psi_\alpha = p^\beta \left\{ \partial_\beta \left(\frac{\partial L}{\partial p^\alpha} \right) - \partial_\alpha \left(\frac{\partial L}{\partial p^\beta} \right) \right\} , \tag{2.9}$$

The above variational principle gives the hydromechanical Euler-Lagrange equations :

$$\psi_\alpha = 0 , \tag{2.10}$$

Since $\quad p^\alpha \psi_\alpha = 0 , \qquad (2.11)$

the four equations of motion (2.10) are linearly dependent. For $\alpha = 0$, the equations (2.10) give the Bernoulli's equation and for $\alpha = 1,2,3$, the Euler's equations of motion.

2.3 VARIATIONAL PRINCIPLE FOR NON-BAROTROPIC FLOWS

The hydromechanical variational principle of Drobot and Rybarski (1959) has been extended to the case of non-barotropic flows by Mathew and Vedan (1989). In that case we have one more four dimensional vector field $\bar{s}(x)$ with components $s^\alpha(x)$; α = 0,1,2,3, where $s^0 = \rho s$ and s^i are $\rho u^i s$, i = 1,2,3; s being the specific entropy.

Now consider the function space F of vector valued functions $p^\alpha(x)$, $s^\alpha(x)$ and the one-parameter group of transformations of $X_4 \times F$ into itself with the infinitesimal generator

$$\bar{V} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial p^\alpha} + \theta^\alpha \frac{\partial}{\partial s^\alpha} \tag{2.12}$$

where $\xi^\alpha = \xi^\alpha(\bar{x},\bar{p},\bar{s})$, $\eta^\alpha = \eta^\alpha(\bar{x},\bar{p},\bar{s})$, $\theta^\alpha = \theta^\alpha(\bar{x},\bar{p},\bar{s})$.
Here the flow generated by $\bar{V}$ is subjected not only to the conservation laws of mass and momentum but to the conservation of entropy. This leads to the definition of generalized hydromechanical transformation (Joseph).

DEFINITION 2.3

For arbitrary ξ^α, the one-parameter family of transformations generated by

$$\bar{V} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial p^\alpha} + \theta^\alpha \frac{\partial}{\partial s^\alpha} ,$$

where $\eta^\alpha = \partial_\beta\left(p^\beta\xi^\alpha - p^\alpha\xi^\beta\right)$,

and $\theta^\alpha = \partial_\beta\left(s^\beta\xi^\alpha - s^\alpha\xi^\beta\right)$, (2.13)

is called a generalized hydromechanical transformation.

DEFINITION 2.4

The flux of matter-flow across an oriented surface is defined as in the case of barotropic flows. In addition entropy flux of the flow across a surface is defined by

$$\int_S s^\alpha \, dS_\alpha \; . \tag{2.14}$$

For non-barotropic flow of an inviscid fluid the Lagrangian is chosen as

$$L = \frac{1}{2p^0}\left[(p^1)^2 + (p^2)^2 + (p^3)^2\right] - \epsilon(p^0, s^0) - p^0 U \; , \tag{2.15}$$

where the internal energy ϵ is a function of p^0 and s^0 .

The action W is defined as

$$W = \int_\tau d\tau \; L \; (\bar{x}, \bar{p}(\bar{x}), \bar{s}(\bar{x})) \; . \tag{2.16}$$

We consider the action of the transformation $\bar{V}$ (equation(2.12)) on W.

Then the total variation is

$$\Delta W = \int_\tau d\tau \left\{ \frac{\partial L}{\partial p^\alpha} \eta^\alpha + \frac{\partial L}{\partial s^\alpha} \theta^\alpha + \partial_\alpha (L \, \xi^\alpha) \right\} \; . \tag{2.17}$$

We state the variational principle as follows:

For all ξ^α vanishing on the boundary the total variation

$$\Delta W = \int_\tau d\tau \left\{ \frac{\partial L}{\partial p^\alpha} \partial_\beta\left(p^\beta\xi^\alpha - p^\alpha\xi^\beta\right) + \frac{\partial L}{\partial s^\alpha} \partial_\beta\left(s^\beta\xi^\alpha - s^\alpha\xi^\beta\right) + \partial_\alpha\left(L\,\xi^\alpha\right)\right\} = 0 \quad . \tag{2.18}$$

ie.,
$$\int_\tau d\tau \left\{ \partial_\beta\left(T^\beta_\alpha\,\xi^\alpha\right) - \psi_\alpha\,\xi^\alpha \right\} = 0 \quad ,$$

where

$$T^\beta_\alpha = p^\beta \frac{\partial L}{\partial p^\alpha} + s^\beta \frac{\partial L}{\partial s^\alpha} + \delta^\beta_\alpha \left(L - p^\gamma \frac{\partial L}{\partial p^\gamma} - s^\gamma \frac{\partial L}{\partial s^\gamma}\right) \quad , \tag{2.19}$$

and

$$\psi_\alpha = p^\beta \left\{ \partial_\beta\left(\frac{\partial L}{\partial p^\alpha} \right) - \partial_\alpha\left(\frac{\partial L}{\partial p^\beta} \right)\right\} + s^\beta \left\{ \partial_\beta\left(\frac{\partial L}{\partial s^\alpha} \right) - \partial_\alpha\left(\frac{\partial L}{\partial s^\beta} \right)\right\} \tag{2.20}$$

The above variational principle gives the generalized hydromechanical Euler-Lagrange equations :

$$\psi_\alpha = 0. \tag{2.21}$$

Since $p^\alpha\psi_\alpha = 0,$ (2.22)

the four equations of motion (2.21) are linearly dependent. When $\alpha = 0$, we get the generalized form of Bernoulli's equation. For $\alpha = 1,2,3$ the equations (2.21) lead to the Euler equations of motion.

In conventional calculus of variations we consider the vertical variations , that is, variations of the dependent variables. Moreau (1982) has introduced the concept of horizontal variations, that is, the variations of the independent variables which was later named as transport method. In both these cases the variations are infinitesimal transformations acting either on the spacë of dependent variables or on the space of independent variables.

The method of Drobot & Rybarski (1959) and Mathew and Vedan (1989) amount to considering transformation groups acting on the space of dependent and independent variables in an Eulerian frame work. When the transformation is restricted to hydromechanical ones, the variations of the dependent variables are expressed only in terms of the variations of the independent variables. Thus this method has closest analog in literature to horizontal variations or transport method of Moreau (1982). A closely related variational principle has been discussed by Zaslavski and Perfilev (1969).

2.4 HAMILTONIAN FORMULATION FOR BAROTROPIC FLOWS

In the theory of classical mechanics Hamiltonian formulation is restricted by their excessive reliance on canonical coordinates. The advances in the study of dynamical systems have led to the concept of the Hamiltonian system of differential equations and has formed the basis of much of the more advanced work in classical mechanics including motion of a rigid body , celestial mechanics and quantization theory. A coordinate free approach to Hamiltonian system has led to the development of the theory with Poisson bracket as the fundamental object of study. This approach to Hamiltonian system admits Hamitonian structures of varying rank which are important in the study of stability. The special case of Lie Poisson bracket on the dual to a Lie algebra plays a key role in representation theory and geometric quantization and provides a theoretical basis for the general theory of reduction of Hamiltonian systems.

In the case of finite dimensional system Darboux's theorem assures that it is always possible to introduce canonical coordinates. But the theorem is no longer valid for an infinite dimensional system to which system of evolution equations of continuum mechanics belongs. In this case the concept of Poisson manifold for finite dimensional system has led to a natural generalization to infinite dimensional system.

In classical mechanics the transformation from a Lagrangian system to the corresponding Hamiltonian system is accomplished by Legendre transformation. It is to be noted that there exist a Hamiltonian formulation of hydromechanics as a finite dimensional system (Arnold (1988) and Gantmacher (1975)). Following Gantmacher, we show that starting from the above Lagrangians for barotropic and non-barotropic flows the evolution equations for hydrodynamics form a Hamiltonian system.

Now we state Donkin's theorem:

<u>DONKIN'S THEOREM</u> (Gantmacher (1975),p.74)

Given a certain function $X\ (x_1,\dots,x_n)$, the Hessian of which is different from zero:

$$\det \left[\frac{\partial^2 X}{\partial x_i \partial x_k} \right]^n_{i,k=1} \neq 0 \tag{2.23}$$

let there exists a transformation

$$x_i \longrightarrow y_i$$

of the variables generated by the function $X\ (x_1,\dots,x_n)$:

$$y_i = \frac{\partial X}{\partial x_i} \ , \qquad (i = 1,\dots\dots,n) \ . \tag{2.24}$$

Then there exists a transformation, the inverse of transformation (2.24), generated by a function $Y\ (y_1,\dots,y_n)$

$$x_i = \frac{\partial Y}{\partial y_i} \ , \qquad (i = 1,\dots,n) \ . \tag{2.25}$$

Here the generating function Y of the inverse transformation is related to the generating function X of the direct transformation

$$Y = \sum_{i=1}^{n} x_i \, y_i - X \quad . \tag{2.26}$$

If the function X contains the parameters $\alpha_1, \dots, \alpha_m$, then Y also contains these parameters and

$$\frac{\partial Y}{\partial \alpha_j} = -\frac{\partial X}{\partial \alpha_j} \, , \quad (j = 1, \dots, m) \tag{2.27}$$

We utilize Donkin's theorem to make the transition from the Lagrangian variables to the Hamiltonian variables by replacing in the theorem

the function X by L ,

the variables x_i by p^i ,

y_i by $\bar{p}_i$,

and the parameters $\alpha_1, \dots, \alpha_m$ by x^α and ρ .

Now we define $\bar{p}_i$ corresponding to the field p^i:

$$\bar{p}_i = \frac{\partial L}{\partial p^i} \quad \text{and} \quad H = \sum_{i=1}^{n} p^i \, \bar{p}_i - L \; ,$$

where L is given by (2.5)

By actual computation we have $\bar{p}_i = u^i$ and the Hamiltonian density

$$H = \frac{1}{2} \rho \left[(u_1)^2 + (u_2)^2 + (u_3)^2 \right] + \epsilon(\rho) + \rho U \; , \tag{2.28}$$

which is same as the total energy of unit volume. Clearly if we substitute $p^0 = \rho$ and $p^i = \rho u^i$ the hydromechanical Euler-Lagrange equations reduce to the equations of motion. They are in canonical form. Kelvin's circulation theorem has been obtained as a consequence of the invariance of Poincare-Kartan integral associated with this Hamiltonian.

It has been shown by Holm et al.(1985) that the equations for barotropic flow are Hamiltonian with Poisson bracket

$$\{F,G\} = \int_V dV \Big\{ \bar{M} \cdot \Big[\Big(\frac{\delta G}{\delta M} \cdot \nabla\Big) \frac{\delta F}{\delta M} - \Big(\frac{\delta F}{\delta M} \cdot \nabla\Big) \frac{\delta G}{\delta M} \Big]$$

$$+ \rho \Big[\frac{\delta G}{\delta M} \cdot \Big(\nabla \frac{\delta F}{\delta \rho}\Big) - \frac{\delta F}{\delta M} \cdot \Big(\nabla \frac{\delta G}{\delta \rho}\Big)\Big]\Big\} , \qquad (2.29)$$

where F and G are functionals and $\bar{M} = (p^1, p^2, p^3)$ and $\frac{\delta}{\delta M}$ & $\frac{\delta}{\delta \rho}$ variational derivatives. Thus the evolution equations are given by

$$\frac{\partial F}{\partial t} = \{F,H\} , \qquad (2.30)$$

in which H is given by the equation (2.28).

This bracket is earlier found in Morrison and Green (1980), Holm, Marsden, Ratiu and Weinstein (1985) and Marsden (1982). This bracket is the Lie-Poisson bracket for a semi-direct product.

2.5 HAMILTONIAN FORMULATION FOR NON-BAROTROPIC FLOWS

In the case of non-barotropic flows we apply Donkin's theorem with the following transformations:

Replace the function X by L ,

the variables x_i by p^i ,

y_i by $\bar{p}_i$,

and the parameters $(\alpha_1, \ldots, \alpha_m)$ by x^α, ρ and s^0.

Here only s^0 is entering into our variational principle

We define $H = \sum_{i=1}^{n} p^i \bar{p}_i - L$,

where L is given by (2.15).

Thus we get

$$H = \frac{1}{2} \rho \left[(u_1)^2 + (u_2)^2 + (u_3)^2\right] + \epsilon(\rho, \rho s) + \rho U , \qquad (2.31)$$

as the Hamiltonian for the non-barotropic flows. As in the case of barotropic flows, the Hamiltonian is the total energy of unit volume.

Following Morrison and Green (1980) and Holm et al (1985) it can be found that the equations of non-barotropic flow can be treated as an infinite dimensional Hamiltonian system with Poisson bracket

$$\{F,G\} = \int_V dV \left\{ \bar{M}\cdot\left[\left(\frac{\delta G}{\delta M}\cdot\nabla\right)\frac{\delta F}{\delta M} - \left(\frac{\delta F}{\delta M}\cdot\nabla\right)\frac{\delta G}{\delta M}\right]\right.$$

$$+ \rho\left[\frac{\delta G}{\delta M}\cdot\left(\nabla\frac{\delta F}{\delta\rho}\right) - \frac{\delta F}{\delta M}\cdot\left(\nabla\frac{\delta G}{\delta\rho}\right)\right]$$

$$\left. + \sigma\left[\frac{\delta G}{\delta M}\cdot\left(\nabla\frac{\delta F}{\delta\sigma}\right) - \frac{\delta F}{\delta M}\cdot\left(\nabla\frac{\delta G}{\delta\sigma}\right)\right]\right\}, \qquad (2.32)$$

where $\bar{M} = (p^1, p^2, p^3)$ and $\sigma = s^0 = \rho s$.

2.6 DISCUSSION

In the theory of fields it is well-known that the Lagrangian formulation is preserved in a natural way when we go from the discrete to the continuous case. It is general to use superscript 1,2,3 to denote the spacial coordinate and by setting $t = x^0$, the four vector $\bar{x} = x^0, x^1, x^2, x^3$ denotes a point or event in the four dimensional space time. ie, four-space. This treatment leads to Euler-Lagrange equations which involve partial derivatives of the gradient of the fields.

Abarbanel and Holm (1987) have studied non-linear stability for inviscid, incompressible and barotropic flow. They use both Lagrangian and Eulerian treatment. It is well-known that a variational principle for Eulerian fluid flow cannot be given fully in terms of the field variables velocity and density. Following Lin, they are considering Lagrangian Markers in their Eulerian treatment. Further analysis also involves the Lagrangian Markers . The enlargement of fluid phase space by adding the Lagrangian labels is a return to a full set of phase space coordinates from the reduced space of coordinates $\bar{u}$ and ρ.

Passing from the Lagrangian to Hamiltonian formulation for a system of particles one set of canonical equations is the Lagrangian equations of motion expressed in terms of conjugate variables and Hamiltonian and the remaining are following from the definition of conjugate momenta. This is precisely in the case of Hamiltonian system we have obtained also.

* * *

CHAPTER 3

HELICITY CONSERVATION AND STABILITY OF BAROTROPIC FLOWS

3.1 INTRODUCTION

This chapter focuses on the stability studies of barotropic flows based on Arnold's (1965a,b) method using the variational principles of Drobot and Rybarski (1959). The infinitesimal generator of transformation group that leaves helicity invariant is used to define the structure that remains invariant under the flow. We show that the helicity conservation obtained by Joseph (1993) by applying Noether's theorem from the variational principle of Drobot and Rybarski (1959) is valid only for incompressible flow and the result is obtained for a general barotropic flow.

The complete analysis of three dimensional stability problem is too complicated for mathematical treatment. Even in two dimensional case we have to resort to certain assumptions for the problem to be mathematically amenable. Thus in the case of stability of atmospheric flows two approaches are used. In baroclinic stability problem, the current is assumed to vary in the vertical direction only and latitude variations are neglected [Charney (1947) and Kuo (1952)]. On the other hand in barotropic stability problem, the current varies in the latitude direction only and the vertical variations are neglected. It has been shown by Foot and Lin (1950) and Kuo (1949,1951) that barotropic basic current is stable if the absolute vorticity profile is monotonic.

In the general theory of inviscid fluid dynamics barotropic flows are singled out because of the special conservation laws associated with such flows. Instability studies of these flows have drawn special attention due to simplicity in analysis.

The stability of inviscid barotropic flow has been studied by Lynden-Bell and Katz (1981) based on the invariance of the classical integrals of energy, momentum, angular momentum and the initial position supplemented by all the invariants implied by Kelvin's circulation theorem. It is shown that all states of steady flow, even those that are only steady when observed from rotating axes,are stationary states of an energy functional. The minimization of the energy is clearly sufficient for stability. They have also developed a Lagrangian formulation based on Clebsch's variables so that the conserved circulation appears as the momenta conjugate to ignorable coordinates and then proceeded to a Ruthian. The method is truly Lagrangian. Some examples are also discussed.

Islamov (1982) has analyzed the stability of barotropic flows on the basis of a finite-difference analog of the linearized vorticity equation. The conditional formal stability of two dimensional equivalent barotropic modon has been investigated by Swater's (1986) and he has obtained that the criteria for stability depend on the wave number of the initial disturbance.

Arnold's (1965) method has been extensively used in the study of stability of barotropic and non-barotropic flows.

3.2 ARNOLD'S METHOD (Arnold (1965))

The equations of three dimensional hydrodynamics of an ideal fluid are infinite dimensional analog to the following finite dimensional situation. Consider a dynamical system

$$\dot{x} = f(x), \quad x = (x_1, \cdots, x_n) \quad . \tag{3.1}$$

Assume that this space is decomposed into k-dimensional sheets, each of which is an integrable manifold. A point x of a sheet F is regular, if in the neighborhood of this point there exists (a system of) coordinates $y_1, \cdots, y_n$ such that the sheets are given by

$$y_{k+1} = c_{k+1}, \ldots\ldots\ldots, y_n = c_n \quad . \quad (c_i\text{'s constants}).$$

Arnold's stability arguments for fluid flows are based on well-known results for the stability of Eulerian rotation of a rigid body (top) around its large or small axis of inertia (Landau and and Lifshitz (1976),p.116-117). Let the principal axes of inertia I_1, I_2 and I_3 be such that $I_3 > I_1 > I_2$. The Euler's equations for rigid body rotation have two constants of motion namely, energy E and angular momentum M. These are

$$\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = 2E \,,$$

$$\text{and} \qquad M_1^2 + M_2^2 + M_3^2 = M^2 \,, \tag{3.2}$$

where (M_1,M_2,M_3) is the angular momentum vector. These are the equations of an ellipsoid with semi axes $\sqrt{2EI_1}, \sqrt{2EI_2}, \sqrt{2EI_3}$ and a sphere of radius M in the (M_1,M_2,M_3) space respectively. When the angular momentum vector moves relative to axes of inertia of top, its terminus moves along the line of intersection of these two surfaces. Let the axes of ellipsoid be in the direction of principal axes. It is noted that for M^2 near to $2EI_1$ and $2EI_3$, the paths of the terminus are closed curves along the X_1 and X_3 axes respectively near the poles. These correspond to stability of top motion. For M^2 near $2EI_2$ the paths are ellipses intersecting at the poles of the X_2 axis and so correspond to instability. Conditional maximum and minimum correspond to $M^2 = 2EI_3$ and $M^2 = 2EI_1$ respectively which correspond to uniform rotation about the X_3 and X_1 axes respectively. These give the equilibrium of the Euler's equations for the rigid body rotation.

Coming to the system (3.1), suppose it has a first integral E . Let a point x_0 on F be a local conditional extremum of the constant E, x_0 is a regular point and the quadratic form d^2E is non-singular on F . Then x_0 is an equilibrium of the system. If this extremum is maximum or minimum then the equilibrium is stable for small finite perturbations.

Euler's equations for inviscid flows form a system like (3.1) in the infinite dimensional space of the vector field of possible velocities $\bar{u}$; that is, satisfy equation of continuity and boundary conditions. For steady flows , $\frac{\partial \bar{u}}{\partial t} = 0$ which correspond to equilibrium position of the system. The space of the field $\bar{u}$ is decomposed into sheets based on Kelvin's circulation theorem. Based on the total energy which is a first integral, Arnold formulated the stability criterion. The results are summarized in Arnold (1978).

Some well-known results of Arnold has been generalized by Grinfeld (1984) and proved that stationary three dimensional barotropic flow of an ideal fluid yields an extremum of the total mechanical energy with respect to variations of the hydrodynamic fields that possess the same vorticity and derived some sufficient conditions for the stability of corresponding stationary flows.

Arnold's method for nonlinear stability of ideal incompressible flow in two dimensions has been extended to the barotropic compressible case by Holm et al. (1983) and the results applied to planar shear flows. Abarbanel et al.(1984) have derived the necessary and sufficient conditions for the formal stability of a parallel shear flow in a three dimensional stratified fluid. Holm et al.(1985) established nonlinear stability of fluid and plasma problems. Nonlinear stability of stationary solutions of incompressible inviscid stratified fluid flow in two and three dimensions has been analyzed by Abarbanel et al.(1986). They have treated both the Euler's equations and their Boussinesq approximation. The resulting nonlinear stability criteria involve standard quantities such as the Richardson number, but they differ from the linearized stability criteria.

Abarbanel and Holm (1987) have investigated the nonlinear stability of a homogeneous fluid and of a barotropic fluid in three dimensions. It is shown that three dimensional flows are not formally stable due to a particle vortex stretching mechanism.

In this chapter we follow Arnold's method to study stability of barotropic flows. The invariance criterion for helicity based on the variational principle of Drobot and Rybarski (1959) is used to define equihelicity sheets in the space of velocity vector fields.

3.3 NOETHER'S THEOREMS AND CONSERVATION LAWS

Associated with a variational problem we can consider a variational symmetry which is a local transformation group under which the action integral is invariant. The relation between such variational symmetries and conservation laws associated with the corresponding classical Euler-Lagrange equations is embodied in Noether's theorem.

The system corresponding to our variational principle is under determined as is clear from equations (2.22). Classical Noether's second theorem is concerned with such systems for which there may be trivial conservation laws determined by non-trivial variational symmetry groups. In this sense theorem 2 of Drobot and Rybarski (1959) and theorem 4.9 of Mathew and Vedan (1989) are essentially classical Noether's second theorem adapted to hydromechanical variational principle (In Drobot and Rybarski Noether's first theorem corresponds to transformations depending on scalar parameters and second theorem, transformations depending on scalar functions). The symmetries under consideration are called generalized symmetries.

Let us consider the action integral

$$W = \int_{\tau} d\tau \; L \left(\bar{x}, \bar{p}(\bar{x}) \right) \quad . \tag{3.3}$$

We note that definition (4.10) of Olver ((1986),p.257) define a variational symmetry and theorem (4.12) gives the condition for

$$\overline{V} = \xi^{i}(\bar{x},\bar{u}) \frac{\partial}{\partial x^{i}} + \eta^{\alpha}(\bar{x},\bar{u}) \frac{\partial}{\partial u^{\alpha}} \quad ,$$

to be the infinitesimal generator of an ordinary variational symmetry. Theorem (4.29)(p.278) is the Noether's first theorem connecting ordinary symmetry to conservation laws of Euler-Lagrange equations and definition (4.33)(p.283) is used to relax conditions on variational symmetry so that Noether's theorem follows. This defines a divergence symmetry. In the case of generalized symmetries this is used to define a generalized variational symmetry. Thus Noether's theorems(5.42) and (5.50) (p.328,337) are not based on the invariance of the action integral but on the definition of variational symmetry.

Joseph (1993) has used the criterion for the invariance of the action integral (3.3) to derive the conservation law of helicity for a barotropic flow. He points out that the infinitesimal criterion for the invariance leads to

$$\partial_\beta (T^\beta_\alpha \, \xi^\alpha) - \psi_\alpha \, \xi^\alpha - \xi^\alpha \, \partial_\alpha p^\beta \, \frac{\partial L}{\partial p^\beta} = 0 . \tag{3.4}$$

Thus a linear combination of usual Euler-Lagrange expressions and the hydromechanical Euler expressions is a divergence. This leads us to the following definitions.

Let $\quad \bar{V} = \xi^\alpha \dfrac{\partial}{\partial x^\alpha} + \eta^\alpha \dfrac{\partial}{\partial p^\alpha} ,$

where $\quad \eta^\alpha = \partial_\beta (p^\beta \xi^\alpha - p^\alpha \xi^\beta),$

DEFINITION 3.1

$\bar{V}$ is a hydromechanical variational symmetry if

$$\bar{V}(L) + L \, \mathrm{Div}\, \bar{\xi} + \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha = 0 , \qquad (3.5)$$

or is a divergence.

DEFINITION 3.2

The action **W** (equation (3.3)) is said to be hydromechanically div-invariant if

$$\bar{V}(L) + L \, \mathrm{Div}\, \bar{\xi} + \partial_\alpha p^\alpha \frac{\partial L}{\partial p^\beta} \xi^\alpha ,$$

is a divergence.

DEFINITION 3.3

The action **W** is said to be hydromechanically invariant if

$$\bar{V}(L) + L \, \mathrm{Div}\, \bar{\xi} + \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha = 0 .$$

THEOREM 3.1

If $\eta^\alpha = 0$,then the action **W** is hydromechanically div invariant.

PROOF:

In this case we have

$$\bar{V}(L) + L \operatorname{Div} \bar{\xi} + \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha = \partial_\alpha \left(L \xi^\alpha \right) . \tag{3.6}$$

Hence the theorem.

THEOREM 3.2 (Drobot and Rybarski (1959))

The hydromechanical variation $\eta^\alpha = 0$ if and only if

$$\xi^\alpha = p^\alpha \phi + \frac{q_\beta}{p^\nu q_\nu} e^{\alpha\beta\lambda\mu} \partial_\lambda \phi_\mu \tag{3.7}$$

where ϕ is an arbitrary scalar function, q_β is an arbitrary vector, and ϕ_μ is any vector satisfying the equations

$$p^\lambda \left(\partial_\lambda \phi_\mu - \partial_\mu \phi_\lambda \right) = 0 . \tag{3.8}$$

PROOF:

Refer Drobot and Rybarski (1959, p.405)

THEOREM 3.3

The variations $\xi^0 = 0$ and $\xi^i = \dfrac{\omega^i}{\rho}$, $i = 1,2,3$ where $\bar{\omega} = \nabla \times \bar{u}$ preserves vorticity.

PROOF:

In theorem 3.2 let $\phi_\mu = \frac{\partial L}{\partial p^\mu}$ and $\phi = 0$. Then

$$p^\lambda \left(\partial_\lambda \phi_\mu - \partial_\mu \phi_\lambda \right) = \phi_\mu = 0 ,$$

for the fluid flow. Let us choose $q_o = -1$ and $q_i = 0$, $i = 1, 2,3$. Then

$$\xi^i = \frac{1}{p^o} e^{ijk} \partial_j \phi_k = \frac{\omega^i}{p^o} . \tag{3.9}$$

Thus when $\xi^i = \frac{\omega^i}{p^c}$, $\eta^i = 0$.

In this case conservation of helicity can be obtained for a barotropic flow, where we can relax the conditions on L and incompressibility, by applying Noether's theorem.

3.4 CONSERVATION OF HELICITY

THEOREM 3.4

In the case of barotropic flows the total helicity

$$\int_V dV \quad \bar{u} \cdot \bar{\omega} , \tag{3.10}$$

is a constant of motion, where V is the three dimensional domain of flow.

PROOF:

Let $\xi^{\alpha} = \frac{\omega^{\alpha}}{\rho}$. Then hydromechanical variations η^{α} vanish. Thus $\bar{V} = \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}}$. By theorem 3.1, we have

$$\bar{V}(L) + L \operatorname{Div} \bar{\xi} + \partial_{\alpha} p^{\beta} \frac{\partial L}{\partial p^{\beta}} \xi^{\alpha} = \partial_{\alpha}\left(L \xi^{\alpha}\right) .$$

Then we have

$$\Delta W = \int_{\tau} d\tau \quad \partial_{\beta}\left(L \xi^{\beta}\right) .$$

ie., $$\int_{\tau} d\tau \left[\partial_{\beta}\left(T^{\beta}_{\alpha} \xi^{\alpha}\right) - \psi_{\alpha} \xi^{\alpha}\right] = \int_{\tau} d\tau \ \partial_{\beta}\left(L \xi^{\beta}\right) ,$$

for arbitrary volume τ .

ie., $$\partial_{\beta}\left(T^{\beta}_{\alpha} \xi^{\alpha}\right) - \psi_{\alpha} \xi^{\alpha} = \partial_{\beta}\left(L \xi^{\beta}\right) ,$$

$$\partial_{\beta}\left\{ \frac{\partial L}{\partial p^{\alpha}} \left(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta}\right)\right\} - \psi_{\alpha} \xi^{\alpha} = 0 ,$$

$$\psi_{\alpha} \xi^{\alpha} = \partial_{\beta}\left\{ \frac{\partial L}{\partial p^{\alpha}} \left(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta}\right)\right\} .$$

During motion, $\psi_{\alpha} = 0$,so that we have

$$\partial_{\beta}\left\{ \frac{\partial L}{\partial p^{\alpha}} \left(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta}\right)\right\} = 0 .$$

The corresponding conserved quantity is

$$\int_V dV \quad p^o \xi^i \frac{\partial L}{\partial p^i} \quad .$$

Substituting the values of ξ^i and $\frac{\partial L}{\partial p^i}$, we get the above integral as

$$\int_V dV \quad \bar{u} \cdot \bar{\omega}$$

which is the helicity integral. Though it is the total helicity which is seem to be conserved, it has been shown by Moffat (1969) that the result holds when V is any volume with surface on which $\bar{\omega} . \bar{n} = 0$.

3.5 <u>EQUIHELICITY FIELDS OF FLOWS</u>

<u>DEFINITION</u> <u>3.4</u>

Two velocity fields $\bar{u}$ and $\bar{u}'$ are equihelicity fields if there exist a smooth, volume preserving mapping g of the domain V into itself such that

$$\int_V dV \quad \bar{u} \cdot (\nabla \times \bar{u}) = \int_{gV} dV \quad \bar{u}' \cdot (\nabla \times \bar{u}') \qquad (3.11)$$

Then the law of conservation of helicity takes the following form :

THEOREM 3.5

Let $\bar{u}(\bar{x},t)$ be the velocity field of a barotropic fluid flow. Let $\bar{x}(t)$ be the trajectory of a fluid particle and g be the flow map

$$g : \bar{x}(0) \longrightarrow \bar{x}(t)$$

Then the fields $\bar{u}(\bar{x},0)$ and $\bar{u}(\bar{x},t)$ are equihelicity fields.

PROOF:

The proof follows from the conservation of helicity.

Drobot and Rybarski (1959) have stated the variational principle from which the equations of motion follows. Conversely we can state the theorem as follows:

THEOREM 3.6

In the case of barotropic flows the action integral

$$W = \int_{T} d\tau \; L \; ,$$

is invariant under all hydromechanical transformations, ξ^{α} vanishing on the boundary.

The absolute invariance of the action integral W under Galilean transformation shows that the system has the total energy as a first integral. That is,

$$\int_V E \, dV \tag{3.12}$$

where

$$E = \frac{1}{2p^0}\left[(p^1)^2 + (p^2)^2 + (p^3)^2\right] + \epsilon(p^0) + p^0 U , \tag{3.13}$$

is a constant. Hence the total energy is a constant of motion. Now we consider the Euler's equations as a system of evolution equations in the infinite dimensional space of the vector fields p^α. Following Arnold we give a structure to the space of p^α as follows:

Two fields belong to the same sheet if there exist a transformation between them which leaves the helicity integral invariant.

By theorem (3.5) this structure is invariant under the flow. The steady state flow is the equilibrium position of the system. It is to be noted that we can obtain steady state flow equations from the variational principle by considering a 3-dimensional volume instead of 4-dimensional space considered by Drobot and Rybarski. Also we have obtained the helicity conservation from the variational principle by considering variation in which $\xi^0 = 0$ and η^α vanishing. Now we consider the energy integral E (equation (3.12)) of steady flow.

THEOREM 3.7

$$\Delta \int_V dV\ E = 0 \quad , \tag{3.14}$$

ξ^i being the variation corresponding to which $\eta^\alpha = 0$ (and vanishing on the boundary of V).

PROOF:

$$\Delta \int_V dV\ E = \int_V dV\ \partial_i (E\ \xi^i)$$

$$= \int_V dV\ \partial_i \left\{ \left(\frac{1}{2p^0} \left[(p^1)^2 + (p^2)^2 + (p^3)^2 \right] + \epsilon(p^0) + p^0 U \right) \xi^i \right\}$$

$$= 0$$

Taking $\xi^i = \dfrac{\omega^i}{\rho}$, we get the following result:

THEOREM 3.8

E has stationary value for steady flow compared to all equihelicity flows.

PROOF:

Proof follows from theorem 3.7 with ξ^i given by (3.9).

Now by Arnold's method stability of barotropic flows can be studied based on the positive or negative definiteness of the second variation of total energy integral.

3.6 STABILITY OF STEADY BAROTROPIC FLOWS

In order to study the stability we find the second variation of the energy integral on the sheet of equihelicity flows. The stability criterion can be obtained if the second variation is of definite sign.

Let us denote the energy integral (3.12) by I. Then

$$\Delta^2 I = \Delta \int_V dV \ \partial_i \left(E \, \xi^i \right)$$

Without giving its derivation we merely set down the second variation as

$$\Delta^2 I = \int_V dV \left\{ \left(\bar{\omega} \cdot \nabla \right) \delta U + \frac{1}{\rho} \left(P' - u^j u_j \right) (\delta\rho)^2 - 2u_j \ \delta u^j \delta\rho \right.$$

$$\left. + \frac{1}{\rho} \delta(\rho u^j) \ \delta(\rho u_j) + \rho u_j \ \delta^2 u^j + P \, \partial_k \left(\frac{1}{\rho^2} \omega^k \ \delta\rho \right) \right\}, \tag{3.15}$$

$\delta = \xi^i \dfrac{\partial}{\partial x^i}$, being the local variation and $P = p^o \dfrac{\partial \epsilon}{\partial p^o} - \epsilon$,the pressure.

The integrand cannot be of definite sign for an arbitrary three dimensional flow due to vortex stretching. Thus the steady flow is potentially unstable.

EXAMPLE 3.1

As an example we consider stability of three dimensional steady barotropic flows with a free surface above a plane bottom with respect to two dimensional disturbances of fixed period. We use z and x for vertical and horizontal coordinates respectively. This is the classic example which Arnold used in his original nonlinear stability analysis. In strictly two dimensional case clearly vorticity is in the direction normal to the (x,z) plane and second variation is identically zero. But here we permit vorticity corresponding to the perturbation in the (x,z) plane also. For the steady undisturbed flow the velocity $\bar{u} = (u(z),0,0)$ and vorticity $\bar{\omega} = (0,u'(z),0)$. We shall choose the inertial reference system in which the free boundary of the stationary motion is at rest. Then the formula for the second variation of the energy is

$$\Lambda^2 I = \int_V dV \left\{ \left(\bar{\omega} \cdot \nabla \right) \delta U + \frac{1}{\rho} \left(P' - u^2\right) (\delta\rho)^2 - 2u \;\; \delta u^1 \delta\rho \right.$$

$$\left. + \frac{1}{\rho} \, \delta(\rho u^j) \, \delta(\rho u_j) + \rho u \, \delta^2 u^1 + P \, \partial_k \left(\frac{1}{\rho^2} \, \omega^k \, \delta\rho \right) \right\}. \tag{3.16}$$

The flow is stable if the integrand is positive definite or negative definite.

Grinfeld (1984) has studied this problem for two dimensional disturbances and Abarbanel et al.(1987) has considered this problem to study the effect of vortex stretching. While Grinfeld uses equivorticity flows to study barotropic flows Abarbanel et al. use conservation of potential vorticity.

Grinfeld's analysis does not involve study of effect of vortex stretching. He has obtained sufficient conditions for stability.

The first four terms in (3.16) are comparable with the terms of equation (83) of Grinfeld. But equation (3.16) shows that in the presence of vortex stretching the conditions given by Grinfeld are not sufficient for stability. The role of vortex stretching in stabilizing or destabilizing flows under two dimensional perturbations is evident from this. In this context it is worth to recall that Abarbanel et al.have noted that three dimensionality of the equilibrium flow (1986) is required for stability norm to exist in the shear flow examples.

* * *

CHAPTER 4

CONSERVATION OF POTENTIAL VORTICITY AND STABILITY OF NON-BAROTROPIC FLOWS

4.1 INTRODUCTION

Unlike barotropic flows the stability of non-barotropic flows is rarely treated in literature. One of the reasons is that till recently non-barotropic flows were not known to have sufficient conserved quantities as in the case of barotropic flows. But it has been shown by Eckart (1960), Bretherton (1970) and Mobbs (1981) that well-known conservation laws associated with vorticity for barotropic flows can be generalized to the case of non-barotropic flows by replacing velocity $\bar{u}$ in some of their quantities by $\bar{u} - \eta\nabla s$ where η is thermacy and s the specific entropy. Further, it is to be noted that Kelvin's circulation theorem for barotropic flows is a special case of a more general one in which the closed curve is lying on the surfaces s = constant (Pedlosky (1979)). It has been shown by Joseph (1993) that the basis of these conservation laws is that the flow considered is isentropic. The only available results of stability of non-barotropic flows are the stability of adiabatic flows by Dikii (1965b) and Holm et al.(1985)

In this chapter we obtain the stability criterion for non-barotropic flows based on the variational principles due to Mathew and Vedan (1989). The infinitesimal generator of transformation group that leaves the potential vorticity invariant is used to define the structure that remains invariant under the flow.

4.2 CONSERVATION OF POTENTIAL VORTICITY

Although we have introduced a new four-vector $s^{\alpha}, (\alpha = 0,1,2,3)$ in chapter 2 for non-barotropic flows, the Lagrangian contains only s^{0} so that only s^{0} enters into our variational principle. Following Mathew and Vedan (1989) we consider $\eta^{\alpha} = 0$, and $\theta^{\alpha} = 0$, $(\alpha = 0,1,2,3)$. We use generalized hydromechanical transformation with $\xi^{0} = 0$. Let $(\xi^{1},\xi^{2},\xi^{3})$ be the components of the three-dimensional vector $\bar{\xi}$ and (u^{1},u^{2},u^{3}) be the components of $\bar{u}$. Let ∇ denote the spacial divergence operator

$$\nabla = [\partial/\partial x^{1},\ \partial/\partial x^{2},\ \partial/\partial x^{3}] \quad .$$

Then we find that in equation (2.17)

$$\eta^{\alpha} = 0 \quad \text{and} \quad \theta^{\alpha} = 0 \ , \ \alpha = (0,1,2,3) \ , \tag{4.1}$$

provided

$$\nabla\cdot\left(\rho\bar{\xi} \right) = 0, \quad \rho\bar{\xi}\cdot\nabla s = 0 \quad \text{and} \quad \frac{\partial}{\partial t}(\rho\bar{\xi}) + \nabla\times\left(\rho\bar{\xi}\times\bar{u} \right) = 0 \, . \tag{4.2}$$

Then

$$\bar{\xi} = \frac{1}{\rho}\ \nabla f \times \nabla s \ , \tag{4.3}$$

is a solution of equations (4.2), where f satisfies the equation

$$\nabla [Df/Dt] \times \nabla s = 0 , \tag{4.4}$$

where D/Dt is the material differentiation operator.

Equation (4.2) and its solution (4.3) have appeared in Katz and Lynden-Bell (private communication) and Friedman and Schutz (1978). Joseph (1993) has pointed out that conservation of potential vorticity follows from the above equations by comparing the derivation of Katz and Lynden-Bell. Here we give the details as follows:

Let $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be any three vector such that $D\bar{\alpha}/Dt = 0$. Then the equation (4.4) is satisfied if $f = f(\bar{\alpha})$. Mathew and Vedan (1991) have proved the following theorem.

THEOREM 4.1

If there exists a divergence symmetry for the action integral

$$W = \int_{\tau} d\tau \ L \ (\bar{x}, \bar{p}(\bar{x}), \bar{s}(\bar{x})) \ , \tag{4.5}$$

depending on r arbitrary functions and their derivatives up to a given order q , there exist exactly r linearly independent identities between the Euler-Lagrange expressions ψ_α and their derivatives, provided the symmetry corresponds to generalized hydromechanical transformations. Theorem 4.1 leads to the equation

$$\partial_\beta (T^\beta_\alpha \xi^\alpha - L \xi^\beta) = \psi_\alpha \xi^\alpha .$$

The corresponding conserved density is

$$T^o_\alpha \xi^\alpha - L \xi^o .$$

From the above choice of ξ^α we have $\xi^o = 0$ and ξ^i given by the equation (4.3), $i = 1,2,3$. Then

$$T^o_\alpha \xi^\alpha - L \xi^o = \bar{u} \cdot \nabla f \times \nabla s .$$

Thus we have

$$\int_V dV \ \bar{u} \cdot \nabla f \times \nabla s ,$$

is a constant.

Using Green's theorem ,

$$\int_T dV \ \bar{u} \cdot \nabla f \times \nabla S = - \int_S dS \ \bar{n} \cdot (\bar{u} \cdot \nabla s) f + \int_V dV \ f(\nabla \times \bar{u}) \ \nabla s \tag{4.6}$$

where V is a three dimensional volume with surface S.

We choose f which is non-zero only within volume V . Then the first term on the right hand side vanishes. Since f is arbitrary we get

$$\frac{\bar{\omega} \cdot \nabla s}{\rho} \quad , \tag{4.7}$$

is a constant, where $\bar{\omega}$ is the vorticity. This is the law of conservation of potential vorticity. Thus we have seen that the infinitesimal generator of the transformation of the domain for which the potential vorticity is constant is $\bar{\xi} = \frac{1}{\rho} \nabla f \times \nabla s$.

Potential vorticity conservation can also be obtained directly from the infinitesimal criterion for hydromechanical invariance (Chapter 3).

The generalized form of the variational principles of Drobot and Rybarski (1959) and Mathew and Vedan (1989) is based on extending the field of dependent variables by considering the entropy flux vector s^α in addition to the momentum flux vector p^α in the four dimensional manifold X_4. These lead to the following definitions in the case of non-barotropic flows .

$$\text{Let} \qquad \bar{V} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial p^\alpha} + \theta^\alpha \frac{\partial}{\partial x^\alpha} \quad , \tag{4.8}$$

$$\text{where} \qquad \eta^\alpha = \partial_\beta (p^\beta \xi^\alpha - p^\alpha \xi^\beta) \ .$$

$$\text{and} \qquad \theta^\alpha = \partial_\beta (s^\beta \xi^\alpha - s^\alpha \xi^\beta) \quad . \tag{4.9}$$

DEFINITION 4.1

$\bar{V}$ is a hydromechanical variational symmetry if

$$\bar{V}(L) + L \operatorname{Div} \bar{\xi} + \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha + \partial_\alpha s^\beta \frac{\partial L}{\partial s^\beta} \xi^\alpha = 0 , \qquad (4.10)$$

or is a divergence.

DEFINITION 4.2

The action integral W (equation (4.5) is said to be hydromechanically div invariant if

$$\bar{V}(L) + L \operatorname{Div} \bar{\xi} + \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha + \partial_\alpha s^\beta \frac{\partial L}{\partial s^\beta} \xi^\alpha ,$$

is a divergence.

DEFINITION 4.3

The action W is said to be hydromechanically invariant if

$$\bar{V}(L) + L \operatorname{Div} \bar{\xi} + \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha + \partial_\alpha s^\beta \frac{\partial L}{\partial s^\beta} \xi^\alpha = 0 .$$

THEOREM 4.2

If η^α and θ^α vanish, then the action W is hydromechanically div invariant.

PROOF:

We have

$$\bar{V}(L) + L \text{ Div } \bar{\xi} + \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha + \partial_\alpha s^\beta \frac{\partial L}{\partial s^\beta} \xi^\alpha = \partial_\alpha \left(L \xi^\alpha \right) \qquad (4.11)$$

Hence the theorem.

THEOREM 4.3

The hydromechanical variations η^α and θ^α vanish, if

$$\bar{\xi} = \frac{1}{\rho} \nabla f \times \nabla s \ ,$$

where f satisfies the equation

$$\nabla \, [Df/Dt] \times \nabla s = 0 \ ,$$

s being the entropy.

PROOF:

Follows from equations (4.1)-(4.4)

THEOREM 4.4

In the case of non-barotropic flows potential vorticity

$$\int_V dV \ \frac{\bar{\omega} \cdot \nabla s}{\rho} \ ,$$

is a constant of motion.

PROOF:

$$\text{Let} \quad \bar{\xi} = \frac{1}{\rho} \nabla f \times \nabla s$$

Then by theorem 4.3 we have $\eta^{\alpha} = 0$ and $\theta^{\alpha} = 0$, $\alpha = (0,1,2,3)$.

Thus $$\bar{V} = \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} .$$

By theorem 4.2 we have

$$\bar{V}(L) + L \,\mathrm{Div}\, \bar{\xi} + \partial_{\alpha} p^{\beta} \frac{\partial L}{\partial p^{\beta}} \xi^{\alpha} + \partial_{\alpha} s^{\beta} \frac{\partial L}{\partial s^{\beta}} \xi^{\alpha} = \partial_{\alpha}\left(L \xi^{\alpha}\right) .$$

Then we have the variation

$$\Delta W = \int_{\tau} d\tau \quad \partial_{\beta}\left(L \xi^{\beta}\right) .$$

ie., $$\int_{\tau} d\tau \left\{ \partial_{\beta}\left(T^{\beta}_{\alpha} \xi^{\alpha}\right) - \psi_{\alpha} \xi^{\alpha} \right\} = \int_{\tau} d\tau \quad \partial_{\beta}\left(L \xi^{\beta}\right) ,$$

where T^{β}_{α} and ψ_{α} are given by equations (2.19) and (2.20).
Since τ is arbitrary ,

$$\partial_{\beta}\left(T^{\beta}_{\alpha} \xi^{\alpha}\right) - \psi_{\alpha} \xi^{\alpha} = \partial_{\beta}\left(L \xi^{\beta}\right) .$$

ie., $$\partial_\beta\left\{ \frac{\partial L}{\partial p^\alpha} \left(p^\beta\xi^\alpha - p^\alpha\xi^\beta\right) + \frac{\partial L}{\partial s^\alpha} \left(s^\beta\xi^\alpha - s^\alpha\xi^\beta\right)\right\} - \psi_\alpha \xi^\alpha = 0 .$$

ie., $$\psi_\alpha \xi^\alpha = \partial_\beta \left\{ \frac{\partial L}{\partial p^\alpha} \left(p^\beta\xi^\alpha - p^\alpha\xi^\beta\right) + \frac{\partial L}{\partial s^\alpha} \left(s^\beta\xi^\alpha - s^\alpha\xi^\beta\right)\right\} .$$

During motion $\psi_\alpha = 0$, so that we have

$$\partial_\beta \left\{ \frac{\partial L}{\partial p^\alpha} \left(p^\beta\xi^\alpha - p^\alpha\xi^\beta\right) + \frac{\partial L}{\partial s^\alpha} \left(s^\beta\xi^\alpha - s^\alpha\xi^\beta\right) \right\} = 0 .$$

Thus the corresponding conserved density is

$$\int_V dV \ p^0\xi^i \frac{\partial L}{\partial p^i} .$$

Substituting $\bar{\xi}$ from equation (4.3) and $\frac{\partial L}{\partial p^i} = \bar{u}$, we get the above integral as

$$\int_V dV \ \bar{u} \cdot \nabla f \times \nabla s .$$

Thus we have $\int_V dV \ \bar{u} \cdot \nabla f \times \nabla s$ is a constant.

Comparing with equation (4.5), the result follows.

It is to be noted that though vorticity conservation and Helmholtz theorem were obtained by Mathew and Vedan directly from Noether's theorem, the derivation of conservation of potential vorticity (1991) was not straight forward. Helicity conservation was obtained by Joseph (1993) from the invariance criterion, but potential vorticity conservation was obtained by relating the equations (4.1,4.2,4.3) to a corresponding result by Katz and Lynden-Bell. Here we complete the proof first by investigating the correct relation with equations of Katz and Lynden-Bell and then show that this can be easily obtained from the invariance criterion itself. The more general case of symmetries corresponding to non-vanishing hydromechanical variations is still an open problem.

4.3 EQUI-POTENTIAL VORTICITY FLOWS

DEFINITION 4.4

Two fields $(\rho,\bar{u},\bar{s})$ and $(\rho',\bar{u}',\bar{s}')$ are equi-potential vorticity fields if there exists a smooth, volume preserving mapping g of the domain V into itself such that

$$\int_V dV \frac{(\nabla \times \bar{u})\cdot\nabla s}{\rho} = \int_{gV} dV \frac{(\nabla \times \bar{u}')\cdot\nabla s'}{\rho'} \tag{4.12}$$

The law of conservation of potential vorticity has the following form:

THEOREM 4.5

Let $\bar{x}(t)$ be the trajectory of a fluid particle and g be the flow map

$$g : \bar{x}(0) \longrightarrow \bar{x}(t)$$

Then the fields $\{\rho(\bar{x},0),\bar{u}(\bar{x},0),\bar{s}(\bar{x},0)\}$ and $\{\rho(\bar{x},t),\bar{u}(\bar{x},t),\bar{s}(\bar{x},t)\}$

are equi-potential vorticity fields.

PROOF:

The proof follows from the conservation of potential vorticity.

Mathew and Vedan (1989) have stated the variational principle from which the equations of motion follows. Conversely we can state the theorem as follows:

THEOREM 4.6

In the case of non-barotropic flows

$$W = \int_{\tau} d\tau \ L$$

is invariant under all hydromechanical transformations, ξ^{α} vanishing on the boundary.

The action integral W is absolutely invariant under the Galilean transformation. This shows that the system has a first integral

$$\int_{V} dV \ E \ , \tag{4.13}$$

where

$$E = \frac{1}{2p^0}\left[(p^1)^2 + (p^2)^2 + (p^3)^2\right] + \epsilon(p^0, s^0) + p^0 U , \qquad (4.14)$$

is the total energy. Hence the total energy is a constant of motion.

As in the case of barotropic flows the equations for non-barotropic flows form a system like (3.1). The steady state corresponds to equilibrium position of the system.

Following Arnold we give a structure to the space of p^α and s^α as follows:

Two fields belong to the same sheet if they are equi-potential vorticity fields. That is, two fields belong to the same sheet if there exists a transformation between them which leaves the potential vorticity invariant.

By theorem 4.5 this structure is invariant under the flow. As in the case of barotropic flows we can obtain steady state flow equations from the variational principle by considering three dimensional volume instead of four dimensional space considered by Mathew and Vedan (1989). Also we have obtained the conservation of potential vorticity. Now we consider the energy integral (4.13) of steady flow.

THEOREM 4.7

$$\Delta \int_V dV\ E = 0 ,$$

ξ^i being the variations corresponding to which $\eta^\alpha = 0$ and $\theta^\alpha = 0$ and vanishing on the boundary of V.

PROOF:

$$\Delta \int_V dV\ E = \int_V dV\ \partial_i (E\ \xi^i)$$

$$= \int_V dV\ \partial_i \left\{ \left(\frac{1}{2p^o} \left[(p^1)^2 + (p^2)^2 + (p^3)^2 \right] + \epsilon(p^o, s^o) + p^o U \right) \xi^i \right\}$$

$$= 0$$

Taking $\bar{\xi} = \frac{1}{\rho} \nabla f \times \nabla s$, where $\bar{\xi} = (\xi^1, \xi^2, \xi^3)$, we get the following result:

THEOREM 4.8:

E has stationary value for steady flows compared to all close equipotential vorticity flows.

PROOF:

Proof follows from theorem 4.7 with $\bar{\xi}$ given by (4.3).

4.4 STABILITY OF NON-BAROTROPIC FLOWS

Using the variational principle (Chapter2) developed by Mathew and Vedan (1989) we have found out the ξ^{α} which corresponds to the invariance of potential vorticity. Among all fields p^{α}, s^{α} and which correspond to a constant potential vorticity, steady flow has an extremum for the total energy. We have to find out the second variation of the energy integral (4.13) to study the stability of non-barotropic flows. If it is of definite sign the flow is stable.

Let $$J = \int_V dV\ E ,$$

where E is given by equation (4.14). Then the second variation of the functional J is

$$\Delta^2 J = \Delta \int_V dV \quad \partial_i (E\ \xi^i) .$$

Without writing the derivation we give the simplified form of the variation as

$$\Delta^2 J = \int_V dV \left\{ (\ \bar{\omega} \cdot \nabla\ ) \delta U + \frac{1}{\rho} \left(P' - u^j u_j \right) (\delta\rho)^2 - 2u_j \quad \delta u^j \delta\rho \right.$$

$$\left. + \frac{1}{\rho}\ \delta(\rho u^j)\ \delta(\rho u_j) + \rho u_j\ \delta^2 u^j + P\ \partial_k \left(\frac{1}{\rho^2}\ \omega^k\ \delta\rho \right) \right\} , \tag{4.15}$$

where pressure $P = p^{o} \frac{\partial \epsilon}{\partial p^{o}} + s^{o} \frac{\partial \epsilon}{\partial s^{o}} - \epsilon$.

The integrand in the second variation has the same form as for the barotropic flow. But it is to be noted that the variations to be considered are different as they correspond to equi-potential vorticity flows.

As pointed out in the beginning the study of stability of non-barotropic flows is still in the initial stages. Though we are not giving any specific examples, as for barotropic flows the role of vortex stretching seems to have a significant role in destabilizing flows.

* * *

CONCLUSION

CONCLUSION

The results obtained in this thesis can now be summarized as follows:

Arnold's method for stability study is based on a suitable variational formulation for fluid flows. Following Drobot and Rybarski, Mathew and Vedan, Joseph and Vedan have studied the variational formulations of barotropic and non-barotropic flows. The use of a Euclidean space X_4 to represent the space-time configuration space of system leads to a systematic method for deriving governing equations for fluid flows form a suitable action integral.

In Lagrangian approach the configuration space is essentially Riemannian and not Euclidean. But the curved Riemannian space flattens out more and more if we restrict ourselves to smaller and smaller region. This is the case when we consider the 4-dimensional manifold X_4.

In the case of Hamiltonian formulation the phase space is Euclidean. The concept of phase flow is based on the motion of a system in the phase space. This motion is, in terms of hydrodynamics, a Lagrangian description while the Liouville's theorem for phase flow is based on Eulerian equation of continuity. On the basis of this analogy it is natural to expect a simpler theory of fields in Lagrangian and Hamiltonian formulation for hydrodynamics compared to other physical systems. It seems that the Lagrangian and Hamiltonian formulations of fluid dynamics obtained above can be justified in this sense and the evolution equations written in terms of material derivative in chapter 2 can be considered finite dimensional.

Poisson bracket formulation of field theory is not carried out in step by step correspondence with that for discrete systems. For example, Poisson bracket in field theory are defined only in terms of a pair of densities. A way for doing this is to define Poisson bracket as an integral, the integrand being variational derivatives. But Arnold uses the Poisson bracket with the gradients of the functions

$$\frac{\partial \bar{r}}{\partial t} = \{\bar{v}\ \bar{r}\},$$

where $\{A\ B\}$ is the Poisson bracket of the vector fields defined by

$$\{A\ B\}_i = \sum (\partial A_i / \partial x_j) B_j - (\partial B_i / \partial x_j) A_j$$

This can be compared to the Hamiltonian system we have obtained in chapter 2.

The equilibrium solution of the equations of non-dissipative continuum mechanics are usually found by minimizing appropriate variational integral. However, when presented with a dynamical problem one encounters systems of evolution equations for which the Lagrangian view point, even if applicable is no longer appropriate or natural to the problem. In this case, the Hamiltonian formulation of systems of evolution equations assumes the natural variational role for the system. The excessive reliance on canonical coordinates guaranteed by the Darboux theorem in finite dimensions, is no longer valid for the evolution equations. The Poisson bracket approach generalizes in this context.

The Poisson brackets of the Hamiltonian system of two dimensional incompressible inviscid flow, two dimensional barotropic flow and three dimensional adiabatic (non-barotropic) flow are given in Holm et al. (1985). Here we note that for barotropic and non-barotropic flows the Poisson brackets are defined in terms of the variable p^{α} of Drobot and Rybarski (1959). These Hamiltonian structures are used by them in the stability studies.

But it is to be noted that the Hamiltonian structure is used only for obtaining integrals of motion in the study of stability. Instead the variational formulation developed by Mathew, Joseph and Vedan can be used to get known conservation laws of motion and the corresponding infinitesimal generators can be used to define flows with given constants of motion. These are used to define concepts like equivorticity used by Arnold.

In the case of two dimensional flows it is shown by Arnold (1965) that a suitable combination of two integrals of motion, being a first integral, has all the properties of a Liapunov function in a suitable metric and may be used to establish stability in an exact nonlinear sense. In the case of three dimensional flow the conservation of vorticity does not permit the construction of a Liapunov function, instead he uses the property that a stationary flow possesses an extremum in kinetic energy with respect to the variations of velocity fields with the same prescribed vorticity. Arnold has proved this for incompressible flow. Later Grinfeld has generalized this to the case of inviscid barotropic flow in a potential field. He uses only the condition on constancy of sign to establish sufficient condition for stability of flow. The formula for the second variation of fields of equivorticity is derived and used in stability analysis.

In chapter 3, we have generalized the result of Joseph in finding the infinitesimal generator for the variational principle from which the conservation of helicity follows. The stability criterion obtained refers only to formal stability but shows that the conditions obtained by Grinfeld in his example may not be sufficient for stability.

In chapter 4 again we have obtained infinitesimal generator for variational symmetry. This leads to conservation of potential vorticity in the case of non-barotropic flows. The stability criterion is obtained.

Our studies point to a new direction for stability studies based on Lagrangian formulation instead of the Hamiltonian formulation used by other authors. The role and applicability of Arnold's method are being widely discussed in the literature. It is interesting to note that after more than two decades Rouchon (1991) has given a mathematical proof of a remark by Arnold (1965) that nonlinear stability criterion for steady state solutions for incompressible equations is never satisfied when three dimensional rather than two dimensional perturbations are considered. A stronger mathematical foundation for Arnold's method and deeper investigation into its applicability are challenging open problems.

* * *

GLOSSARY OF SYMBOLS

X_4	Euclidean 4-dimensional space
F	Function space
$\bar{p}$	Momentum flux
$X_4 \times F$	Tangent bundle
$X_4 \times F^{(1)}$	Jet space
S	3-dimensional submanifold of X_4
dS_α	Oriented element of S
τ	4-dimensional volume
W	Action integral
L	Lagrangian density
$\bar{V}$	Infinitesimal generator
ρ	Density
ϵ	Internal energy
U	Potential of the external forces
Δ	Total variation
∂_α	Total derivative with respect to x^α
δ_α^β	Kronecker delta
ψ_α	Hydromechanical Euler-Lagrange expressions
$\bar{s}$	Entropy flux
s	Specific entropy
V	3-dimensional volume
H	Hamilton density
$\bar{M}$	Momentum vector
∇	Spacial divergence operator

σ	ρs
$e^{\alpha\beta\lambda\mu}$	Ricci's symbol
$\bar{u}$	Velocity vector
$\bar{\omega}$	Vorticity vector
E	Total energy
δ	Local variation
P	Pressure
η	Thermacy
$\frac{D}{Dt}$	Material differentiation operator

* * *

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