RELIABILITY CONCEPTS IN QUANTILE-BASED ANALYSIS OF LIFETIME DATA

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CERTIFICATE

Certified that the thesis entitled **'Reliability Concepts in Quantile-based Analysis of Lifetime Data'** is a bonafide record of works done by Shri. Vineshkumar B. under my guidance in the Department of Statistics, Cochin University of Science and Technology, Cochin-22, Kerala, India and that no part of it has been included anywhere previously for the award of any degree or title.

Kochi-22 **Dr. N. Unnikrishnan Nair**

30th May 2012

DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Kochi- 22 **Vineshkumar B.**

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Contents

Page No.

Chapter 1 Introduction **1**

Chapter 2 Basic concepts and review of literature **11**

2.1 Quantile functions 11

2.1.1 Definitions and properties 11

2.1.2 Quantile versions of some important concepts 14

2.1.3 Percentiles 16

2.1.4 Moments 17

2.1.5 Order statistics 17

2.1.6 L-moments 20

2.2 Reliability concepts 23

2.2.1 Hazard rate function 23

2.2.2 Mean residual function 24

2.2.3 Variance residual life function 25

2.2.4 Percentile residual life function 27

2.3 Reliability concepts in reversed time 29

2.3.1 Reversed hazard rate 29

2.3.2 Reversed mean residual life 30

2.3.3 Reversed variance residual life 31

2.4 Quantile-based reliability concepts 31

2.4.1 Hazard quantile function 32

2.4.2 Mean residual quantile function 32
2.4.3 Residual variance quantile function 33
2.4.4 Quantile forms of some other reliability concepts 34
2.5 Total time on test transform 36
2.5.1 Relationships of TTT with reliability functions 37
2.6 Some distributions defined by quantile functions 38
2.6.1 Lambda distributions 39
2.6.2 Generalized lambda distributions 40
2.6.3 Generalized Tukey lambda distributions 46
2.6.4 Power-Pareto distributions 51
2.6.5 van-Staden & Loots model 56
2.7 Q-Q plot 62
2.8 Order relations 63
2.8.1 Usual stochastic order 63
2.8.2 Dispersive ordering 63
2.8.3 Convex order 64
2.8.4 Transform orders 65
2.8.5 The monotone convex and monotone concave orders 65
2.8.6 TTT order and excess wealth order 65
2.8.7 Reliability orders 66

Chapter 3 Quantile function models 68
3.1 Introduction 68
3.2 Lambda distributions 69
3.2.1 Generalized lambda distribution 69

3.2.2 Generalized Tukey lambda distribution 75
3.3 Power-Pareto distribution 80
3.4 van-Staden & Loots model 84
3.5 Govindarajulu distribution 89
3.6 Modelling lifetimes by quantile functions using Parzen's score function and tail exponent function 107

Chapter 4 Ageing concepts 123
4.1 Introduction 123
4.2 Ageing concepts based on hazard quantile function 124
4.3 Concepts based on residual function 131
4.4 Concepts based on survival function 139

Chapter 5 Total time on test transform of order *n* 150
5.1 Introduction 150
5.2 Definition and properties of TTT-*n* 151
5.3 Characterizations 153
5.4 Characterization of ageing concepts 157
5.5 Order relations 164

Chapter 6 L-moments of residual life 173
6.1 Introduction 173
6.2 Definition and properties 173
6.3 L-moments of reversed residual life 187
6.4 Characterizations 189
6.5 Applications 197

Chapter 7 Reversed percentile residual life and related concepts **202**

7.1 Introduction 202

7.2 Definition and properties 203

7.3 Models 209

7.4 Classification of distributions 212

Chapter 8 Conclusions and future works **218**

References **223**

Chapter 1

Introduction

A probability distribution can be specified either in terms of the distribution function $F(x)$ or by the quantile function defined by

$$Q(u) = \inf\left[x \middle| F(x) \geq u\right], \quad 0 \leq u \leq 1 .$$

Both distribution function and quantile function convey the same information about the distribution with different interpretations. In the existing literature of statistical analysis, the concepts and methodologies based on distribution function are more popular. However, there are many distinct properties for quantile functions that are not shared by the distribution functions, which make the former attractive in certain practical situations. For inference purposes, statistics based on quantiles are often more robust than those based on moments in distribution function approach. In many cases, quantile functions provide a much simpler straightforward analysis and in some cases like characterizations, the solutions exist only in terms of quantile functions that are not invertible to distribution functions.

Researchers have used the quantile-based measures in various applications of statistics even before the nineteenth century. The Belgian scientist Quetelet (1846) initiated the use of inter-quartile range as a quantile-based measure for statistical analysis. Subsequently, researchers have focused on different applications of quantiles such as representation of distributions by quantile functions, use of different measures like median, quartiles and inter-quartile range, estimation procedures based on sample quartiles, studying large sample behaviour and limiting distribution of quantile-based statistics, etc.. For example see Galton (1883, 1889). Hastings et al. (1947) have introduced a family of distributions by a quantile function. This was a major achievement, which led to the development of many quantile-based families of distributions in the later period. The family of distribution by Hastings et al. (1947) was later refined by Tukey (1962) to form a symmetric distribution, which paved the way for many extensions in subsequent years. These include various forms of quantile functions discussed in Ramberg and Schmeiser (1974) , Ramberg (1975), Ramberg et al. (1979) , Freimer et al. (1988), Gilchrist (2000) and Tarsitano (2004) in the name of lambda distributions. The turning point in the development of the quantile function is the paper by Parzen (1979) in which he emphasized the representation of a distribution in terms of a quantile function and its role in data modelling. These were enriched by further works by Parzen (1991, 1997, 2004) in different areas. Gilchrist (2000) systematically presented various properties of quantile function and its use in statistical modelling.

In reliability studies, the distribution function $F(x)$, the associated survival function $\bar{F}(x)=1-F(x)$ and the probability density function $f(x)$ along with various other characteristics such as failure rate, mean, percentiles and higher moments of residual life, etc.., are used for understanding how the failure time data arises in practice. Some researchers like Parzen (1979), Friemer et al. (1988) and Gilchrist (2000) have indicated the scope of using quantile functions in reliability theory. These require conversion of various existing concepts and methodologies in terms of quantile functions.

A systematic study on the application of quantile functions in reliability studies has been carried out by Nair and Sankaran (2009), in which they have defined commonly used reliability measures in terms of quantile function, and various relationships connecting them were derived. They have also analyzed a quantile function model discussed in Hankin and Lee (2006) in the context of reliability analysis. Our present work extends these basic ideas to develop the necessary theoretical framework for the analysis of lifetime data using quantile functions. This new approach provides alternative methodology and new models that have desirable properties. In this thesis we study more aspects on quantile-based reliability analysis such as identifying quantile functions that can be used in lifetime modelling, deriving new families of quantile functions using various properties of reliability functions and related measures, and proposing new measures based on quantile functions that can be used for various applications in reliability analysis.

The work in this context, presented in the rest of the current thesis is organized into eight chapters. After this introductory chapter, in Chapter 2 we give a brief review of the background materials needed for deliberations in the subsequent chapters. In Chapter 2, we present the definition and the properties of quantile function, quantile functions of some important concepts such as residual function, score function and tail exponent function defined in Parzen (1979), Gini's mean difference, etc.. Subsequently, the definitions of various measures such as moments, percentiles, etc. are also given in terms of quantile function. We express the distribution and expectation of order statistics in terms of quantile function as the concept order statistics have implications in reliability analysis. The definition of L-moments, which are alternative to conventional moments and proved to have several advantages over usual moments, different reliability measures based on distribution function, their equivalent definitions in terms quantile function, the total time on test transform (TTT), and various order relations are presented. We explain the Q-Q plot, a useful tool to check whether the given quantile function is valid for the data situation under consideration. As a topic of considerable interest in modelling, we review various lambda distributions such as lambda distributions by Ramberg and Schmeiser (1974), Freimer et al. (1988), the power Pareto distribution discussed in Hankin and Lee (2006) and a model by van-Staden and Loots (2009).

One of the objectives of quantile-based reliability analysis is to make use of quantile functions as models in lifetime data analysis. Representation of reliability characteristics through quantile functions permits the use of various lambda distributions, hitherto not considered as lifetime models. The lambda distributions are particularly useful, when the physical characteristics that govern the failure pattern in a specific problem are unknown to choose a particular distribution function. This is because, there are members of lambda families that can either exactly or approximately represent most of the continuous distributions by a judicious choice of its parameters. In Chapter 3, we discuss the reliability characteristics of some lambda distributions and other quantile function models, and demonstrate their applicability in lifetime data analysis. The distributions considered in this chapter are lambda distributions by Ramberg and Schmeiser (1974) and Freimer et al. (1988), the power Pareto distribution discussed in Hankin and Lee (2006), a four-parameter model derived in van-Staden and Loots (2009) and the Govindarajulu distribution proposed by Govindarajulu (1977). As order statistics have implications in reliability analysis, the distributions and the expectations of order statistics are also derived in the case of distributions mentioned above. To ascertain the adequacy of these distributions in lifetime modelling we have shown that they represent various real data situations.

Other than the lambda distributions, we also discuss the Govindarajulu model as it is introduced by Govindarajulu (1977) as a lifetime model and demonstrated its potential use in reliability studies through real data. We undertake a detailed study of the model and demonstrate that being a simple model with only two parameters it has competing features in terms of model parsimony with regard to other competing models. This is ascertained by comparing the distribution with some known models in the analysis of a real lifetime data.

In addition to the analysis of existing quantile functions, we present a method for developing quantile functions with monotone as well as non- monotone hazard quantile function using the properties of the score functions and tail exponent function, first suggested by Parzen (1979). Our study is motivated by the fact that the functions have nice relationship with the hazard quantile function. Further the monotonic behaviour of these functions implies those of the hazard quantile functions through some simple identities. The quantile functions hence derived represent flexible family of distributions that contains tractable and intractable form of $F(x)$. The reliability properties of the distributions are studied, and applications of the distributions in lifetime data analysis are ascertained by fitting the distributions to real data.

The concept of ageing plays a critical role in reliability analysis. Concepts of ageing describe how a component or a system improves or deteriorate with age. Many classes of life distributions are categorized or defined in the literature according to their ageing properties. 'No ageing' means that the age of a component has no effect on the distribution of the residual lifetime of the component. 'Positive ageing' describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. On the other hand, 'negative ageing' has an opposite effect on the residual lifetime. Most of the ageing concepts exist in the literature are described on the basis of measures defined in terms of the distribution function. We will see from the discussions in Chapter 3 that many quantile functions can be utilized in the lifetime data analysis. Since most of them do not possess tractable forms of their distribution functions, the existing definitions based on distribution function are not adequate. Thus, as a follow up to quantile-based analysis, in Chapter 4, we introduce the ageing concepts in terms of quantile functions to facilitate a quantile-based analysis. We also illustrate various ageing concepts in the case of quantile functions. Various ageing concepts we have considered in Chapter 4 are increasing (decreasing) –IHR(DHR), hazard rate increasing (decreasing) average hazard rate- IHRA (DHRA), new better than used in hazard rate (NBUHR), increasing hazard rate of order 2 (IHR(2)), new better than used in hazard rate average (NBUHRA) and IHRA*t_0, and their duals, decreasing (increasing) mean residual life –DMRL (IMRL), net decreasing (increasing) mean residual life (NDMRL (NIMRL)), decreasing (increasing) variance residual life DVRL (IVRL), decreasing (increasing) renewal mean residual life, decreasing $\alpha-$percentile residual life (DPRL-α) and new better than used with respect to the $\alpha-$ percentile residual life (NBUP-α) and their duals, new better (worse) than used, NBU (NWU) and those generated from it like NBUE, HNBUE, etc..

The total time on test transforms (TTT) is a widely accepted statistical tool, which has applications in different fields such as reliability analysis, econometrics, stochastic modelling, tail ordering, ordering of distributions, etc.. In Chapter 5, we study a generalization of TTT, named TTT of order n (TTT-n) by an iteration of the definition of TTT. We will show that TTT-n is a quantile function of a random variable, say X_n.We derive various identities connecting the hazard quantile function, mean residual quantile function and the density quantile function of the base random variable X and the transformed random variable X_n. These relations enable characterization of distributions of X and X_n. We present several theorems in this context. One property of the generalized transform is that the distribution with constant or decreasing hazard quantile function tends to become a distribution with increasing hazard quantile function as the process of iteration continues with positive n. This fact is exploited to suggest a simple mechanism to derive bathtub hazard quantile function distributions. In the last section of Chapter 5 we discuss some order relations connecting the baseline and transformed distributions. We also define a new order relation known as TTT-n order and its implications are studied.

L-moments are alternative to conventional moments, and like the conventional moments, L-moments can be used to provide summary measures of probability distributions, to identify distributions and to fit models to data. It has been proved theoretically and emperially that the L-moments have several advantages over conventional moments. In reliability analysis, residual life function and related measures are good indicators in describing ageing patterns of a distribution, and these are being used in other disciplines also. Note that most popular measures of residual life that are discussed in the literature are based on ordinary moments, for example the mean of residual life, variance of residual life, etc.. Considering the advantages of L-moments over ordinary moments, it is worthy to study the measures of residual life based on L-moments. In Chapter 6, we investigate the properties of first two L-moments of residual life and their relevance in various aspects of reliability analysis.

The second L-moment of the residual life is half the mean difference of the residual life. Thus we can treat the second L-moment of residual life as a measure of variation and alternative to variance residual quantile function. We derive the relationship of the second L-moment of residual life with popular reliability functions and study its reliability implications. We analyze the relative merits of the second L-moment of residual life over the well known measure of variation, the variance residual quantile function. We show with examples that these two functions may not exhibit same kind of monotonic behaviour. We also consider the implications between mean residual quantile function and the second L-moment of residual life. The expressions of L-moments of reversed residual life and their relationships with reliability measures are also derived in the chapter. We present some characterization theorems employing the reliability concepts discussed above that can help the identification of the underlying lifetime distribution. In the last section we point out some applications of the derived measures in reliability analysis and economics.

The median residual life function and its generalization, the percentile residual function has been evolved as alternative measures to overcome the shortcomings of mean residual life. Schmittlein and Morrison (1981) pointed out the advantages of median residual life function over the mean residual life function. The general version of median residual life function, originally introduced by Haines and Singpurwalla (1974), is the α^{th} percentile residual life of the lifetime variable X. Theoretically there is analogy in the works relating to residual and reversed residual life functions, the properties and models relating to them differ substantially to merit the study of the latter. The relevance of various existing concepts in reversed time and the enormous literature on percentile residual lifetime motivate us to study the properties of the reversed version of the percentile residual life function (RPRL) in Chapter 7.

We discuss some properties of RPRL in Chapter 7. To begin with, the problem of characterizing the distribution function by the functional form of RPRL is studied. We demonstrate through an example that the RPRL for a given α does not determine the distribution uniquely. Thus we are lead to the search for some general conditions under which the distribution function is determined uniquely and we seek for the conditions for two distributions to have the same RPRL for a given α.

We derived a relationship of RPRL has with reversed hazard rate (RHR) as to deduce further features of RPRL. For many of the standard lifetime models like exponential, Weibull, Pareto, etc., which have simple forms for the hazard rate, the expression for RHR is more complicated. Even for such models with simple forms for failure rate, it is difficult to deduce properties of RHR from them. Hence it is desirable to have models that have simple functional forms for RHR. In Chapter 7, we also discuss a general method for obtaining such models.

The fact that RHR (RPRL) is non-increasing (non-decreasing) on the entire positive real line leaves little scope for classification or identification of life distributions on the basis of their monotonicity as with the cases of ordinary hazard rate function and percentile residual life. To resolve this problem we compare the growth rates of RHR (RPRL) to classify the distributions and several examples are given.

Finally, Chapter 8 summarizes major conclusions of the present study and discusses future work that originates from the present study to be carried out in this area.

Chapter 2

Basic concepts and review of literature

2.1 Quantile functions

In this section we present the definition and properties of quantile function, quantile functions of some widely used concepts, expressions for different measures such as moments, percentiles, L-moments in terms of quantile function, and distribution and moments of order statistics in terms of quantile function. All these concepts are needed for the discussions in the subsequent chapters.

2.1.1 Definitions and properties

In this section, the definition and basic properties of the quantile function are presented.

Definition 2.1 Let X be a real valued random variable with distribution function $F(x)$ which is continuous from right. Then quantile function $Q(u)$ of X is defined as

$$Q(u)=F^{-1}(u)=\inf\{x:F(x)\geq u\},\ 0\leq u\leq 1. \tag{2.1}$$

For $-\infty<x<\infty$ and $0<u<1$,

$$F(x)\geq u \text{ if and only if } Q(u)\leq x.$$

Thus if there exists an x such that $F(x)=u$, then $F(Q(u))=u$ and $Q(u)$ is the smallest value of x satisfying $F(x)=u$. When $F(x)$ is continuous,

$$Q(u)=\inf\{x : F(x)=u\}$$

and when $F(x)$ is strictly increasing too, $Q(u)$ is the unique value of x such that $F(x)=u$. In this case we can easily write the quantile function by solving $F(x)=u$ for x in terms of u.

Some other important properties of quantile function are:

1. For a general distribution function, from the definition of $Q(u)$ we have

   (i) $Q(u)$ is non-decreasing on $(0,1)$ with $Q(F(x))\leq x$ for all $-\infty<x<\infty$ for which $0<F(x)<1$.

   (ii) $F(Q(u))\geq u$ for any $0<u<1$.

   (iii) $Q(u)$ is continuous from the left, i.e., $Q(u^{-})=Q(u)$.

   (iv) $Q(u^{+})=\inf\{x : F(x)>u\}$ so that $Q(u)$ has limits from above.

   (v) Any jumps of $F(x)$ are flat points of $Q(u)$ and flat points of $F(x)$ are jumps of $Q(u)$.

2. Since for a uniform random variable U over $(0,1)$

   $$P\{Q(U)\leq x\}=P\{U\leq F(x)\}$$
   $$=F(x),$$

   $Q(U)$ and X are identically distributed.

3. The distribution $-Q(1-u)$ is the reflection of the distribution $Q(u)$ in the line $x=0$.
4. If $Q_1(u)$ and $Q_2(u)$ are two quantile functions then $Q_1(u)+Q_2(u)$ is also a quantile function.
5. The product of two positive quantile functions is also a quantile function.
6. If $T(x)$ is a non-decreasing function of x, then $T(Q(u))$ is a quantile function. On the other hand if $T(x)$ is non-increasing, then $T(Q(1-u))$ is also a quantile function.
7. If $Q(u)$ is the quantile function of X with continuous distribution function $F(x)$ and $T(u)$ is a non-decreasing function satisfying the boundary conditions $T(0)=0$ and $T(1)=1$, then $Q(T(u))$ is a quantile function of a random variable with the same support as X.

For further details on the properties of the quantile function we refer to Gilchrist (2000).

Definition 2.2 If $f(x)$ is the probability density function of X, then $f(Q(u))$ is called the density quantile function. The derivative of $Q(u)$,

$$q(u)=Q'(u)$$

is known as the quantile density function of X. The prime ' denotes differentiation. Differentiating $F(Q(u))=u$, we have

$$q(u)f(Q(u))=1. \tag{2.2}$$

2.1.2 Quantile versions of some important concepts

In this section we give quantile versions of some important general concepts, for use in the sequel.

The concept of residual life is of special interest in reliability theory. The remaining life associated with a lifetime random variable X is the random variable

$$X_t = (X - t \mid X > t).$$

The survival function of X_t is given by

$$\overline{F}_t(x) = P\{X_t > x\} = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \tag{2.3}$$

where $\overline{F}(x) = P\{X > x\} = 1 - F(x)$. Thus

$$F_t(x) = \frac{F(x+t) - F(t)}{1 - F(t)}. \tag{2.4}$$

Let $F(t) = u_0$, $F(x+t) = v$ and $F_t(x) = u$. Then

$$x + t = Q(v),\ x = Q_1(u),\ \text{say.}$$

We have

$$Q_1(u) = Q(v) - Q(u_0)$$

and from (2.4)

$$u(1 - u_0) = v - u_0$$

or

$$v = u_0 + (1 - u_0)u.$$

Thus the quantile function of X_t becomes

$$Q_1(u) = Q(u_0 + (1 - u_0)u) - Q(u_0). \tag{2.5}$$

Another useful function given in terms of quantile function is the score function defined by Parzen (1979) as

$$J(u) = -(fQ)'(u) = \frac{-f'Q(u)}{fQ(u)} = \frac{q'(u)}{q^2(u)}. \tag{2.6}$$

He used this function to classify probability distributions according to the heaviness of their right tail by constructing the tail exponent function as

$$\alpha(1-u) = (1-u)\, J(u)\, q(u). \tag{2.7}$$

The score function defined in (2.7) is equivalent to the Glaser's function

$$\eta(x) = \frac{-f'(x)}{f(x)}, \tag{2.8}$$

which was used to develop criteria to distinguish increasing, decreasing, bathtub, upside down bathtub hazard rates. The score function and tail exponent function will be used in Chapter 3 to construct quantile functions as models of lifetimes with monotone and non-monotone hazard functions.

A popular measure that has received wide attention in economics is the Gini's mean difference defined as

$$\begin{aligned}\Delta &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\, f(x)\, f(y)\, dx\, dy \\ &= 2\int_{-\infty}^{\infty} F(x)\big(1-F(x)\big)dx. \end{aligned} \tag{2.9}$$

Setting $F(x) = u$, we have

$$\Delta = 2\int_0^1 u(1-u)\, q(u)\, du \tag{2.10}$$

$$= 2\int_0^1 (2u-1)\, Q(u)\, du. \tag{2.11}$$

The equivalent expression in (2.11) is the result of integrating (2.10) by parts. One may use (2.10) or (2.11) depending on whether $q(u)$ or $Q(u)$ is specified.

2.1.3 Percentiles

The specification of a distribution through its quantile function takes away the need to describe a distribution through the moments. Alternative measures in terms of the quantiles that reduces the shortcomings of the moment based ones can be thought of. Here we list out most commonly used measures of location, scale, skewness and kurtosis based on percentiles.

Median is a measure of location, defined by

$$M = Q(0.5). \tag{2.12}$$

To measure dispersion, the inter-quantile range

$$IQR = Q_3 - Q_1, \tag{2.13}$$

where $Q_3 = Q(0.75)$ and $Q_1 = Q(0.25)$ can be used.

Skewness is measured by the Galton's coefficient

$$S = \frac{Q_1 + Q_3 - 2M}{Q_3 - Q_1}. \tag{2.14}$$

It can be seen that the Galton coefficient of skewness lies between -1 and $+1$, and the extreme positive skewness occurs when $Q_1 \rightarrow M$ and the extreme negative skewness attains when $Q_3 \rightarrow M$. When distribution is symmetric, $M = \frac{Q_1 + Q_3}{2}$ and hence $S = 0$.

A kurtosis measure based on percentiles is the Moor's kurtosis defined by

$$T = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{IQR}. \tag{2.15}$$

This measure is proposed by Moors (1988) based on the fact that kurtosis can be large when the probability mass is concentrated at the mean or at the tail.

As we need only substitute the appropriate value of u, the calculation of all the above coefficients are very simple when the form $Q(u)$ is given. Thus the method of percentiles for estimation by matching the sample and population percentiles has been considered by many researchers as a simple and effective method.

2.1.4 Moments

The r^{th} conventional moment

$$\mu_r^1 = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx \tag{2.16}$$

can be expressed in terms of quantile function by

$$\mu_r' = \int_0^1 (Q(u))^r\, du. \tag{2.17}$$

In particular the mean is

$$\mu = \int_0^1 Q(u) du$$

$$= \int_0^1 (1-u) q(u) du. \tag{2.18}$$

Higher moments to describe spread, skewness and kurtosis in terms of quantiles are given by

$$\text{Variance, } \sigma^2 = E(X-\mu)^2 = \int_0^1 (Q(u)-\mu)^2\, du, \tag{2.19}$$

$$\mu_3 = E(X-\mu)^3 = \int_0^1 (Q(u)-\mu)^3\, du \tag{2.20}$$

and

$$\mu_4 = E(X-\mu)^4 = \int_0^1 (Q(u)-\mu)^4\, du. \tag{2.21}$$

2.1.5 Order statistics

In life testing experiments, suppose n items are put on test and the random variable of interest is their failure times. The failure times $X_1, X_2, ..., X_n$ of the n items constitute a random sample of size n from the population with the distribution function $F(x)$. The random variables $X_{1:n}, X_{2:n}, ..., X_{n:n}$ correspond to the ordered sample values x_i of $X_i, i = 1, 2, ..., n$ and are referred to as order statistics, where $X_{1:n} = \min_{1 \le i \le n} X_i$ and $X_{n:n} = \max_{1 \le i \le n} X_i$. The distribution of r^{th} order statistic,

$$F_r(x) = P\{X_{r:n} \le x\}$$

$$= \sum_{k=r}^{n} \binom{n}{k} (F(x))^k (1 - F(x))^{n-k}. \tag{2.22}$$

In particular the distributions of $X_{1:n}$ and $X_{n:n}$ are

$$F_1(x) = 1 - (1 - F(x))^n \tag{2.23}$$

and

$$F_n(x) = (F(x))^n. \tag{2.24}$$

To derive the quantile form of the above distributions recall the definitions of beta function,

$$B(m,n) = \int_0^1 t^{m-1} (1-t)^{n-1} dt; \quad m,n > 0$$

and the incomplete beta function ratio

$$I_x(m,n) = \frac{B_x(m,n)}{B(m,n)},$$

where

$$B_x(m,n) = \int_0^x t^{m-1} (1-t)^{n-1} dt.$$

The well known relationship between upper tail of the binomial distribution and the incomplete beta function ratio is

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = I_p(r, n-r+1). \tag{2.25}$$

From (2.22) and (2.25), we have

$$u_r = I_u\left(r,\ n-r+1\right), \tag{2.26}$$

where $u_r = F_r(x)$ and $F(x) = u$. Thus

$$Q_r(u_r) = Q\left(I_{u_r}^{\ -1}\left(r, n-r+1\right)\right). \tag{2.27}$$

Since $u = I_{u_r}^{\ -1}\left(r, n-r+1\right)$. In (2.27), the symbol I^{-1} represents the inverse of the beta function I. The order statistics $X_{1:n}$ and $X_{n:n}$ have simple forms for their quantile functions given by

$$Q_1(u_1) = Q\left(1-(1-u_1)^{1/n}\right) \tag{2.28}$$

and

$$Q_n(u_n) = Q\left(u_n^{\ 1/n}\right). \tag{2.29}$$

The probability density function of $X_{r:n}$ is

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x)\left(1-F(x)\right)^{n-r} f(x)$$

and hence

$$\begin{aligned} \mu_{r:n} &= E\left(X_{r:n}\right) \\ &= \int x f_r(x)\,dx \\ &= \frac{n!}{(r-1)!(n-r)!}\int_0^1 u^{r-1}(1-u)^{n-r}\, Q\ (u)\,du. \end{aligned} \tag{2.30}$$

The concepts of order statistics have implications in reliability analysis and life testing experiments. Consider a system consisting of n components whose lifetimes $X_1, X_2, ..., X_n$ are independently and identically distributed. The system is said to have a series structure if it functions only when all the components are functioning. When the components constitute a series system, $X_{1:n}$ represents the lifetime of the system. When the system functions, if and only if at least one of the components function, we have a parallel system. Thus $X_{n:n}$ is the lifetime of the parallel system formed by the components. For a k-out-of-n system, which functions if and only if at least k of the components function, the lifetime is obviously $X_{n-k+1:n}$.

2.1.6 L-moments

In the Sections 2.1.3 and 2.1.4, we have discussed the moments and percentiles that are capable for summarizing probability distributions. In this section, we consider the L-moments, which are the competing alternatives to the conventional moments. A unified theory and a systematic study of L-moments have been presented by Hosking (1990). His subsequent works, for example, Hosking (1992, 1996, 2006) and Hosking and Wallis (1997) have made detailed studies on the properties of L-moments, its application in summarizing and identifying probability distributions, estimation techniques based on L-moments, characterizations of distributions by L-moments and the comparison between the conventional moments and L-moments in analyzing measures of distributional shapes.

The r^{th} L-moment of the random variable X is defined as

$$L_r = r^{-1}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(X_{r-k:r}\right),\ r = 1,2,... \tag{2.31}$$

Using the expression $E\left(X_{r:n}\right)$ given in (2.30), we have

$$L_r = r^{-1}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} \frac{r!}{k!(r-k-1)!}\int_0^1 u^{r-k-1}(1-u)^k Q(u)du.$$

Expanding $(1-u)^k$ in powers of u using binomial theorem and combining powers of u,

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} u^k Q(u) du. \tag{2.32}$$

In particular, the first four L-moments are

$$L_1 = \int_0^1 Q(u) du = \mu, \tag{2.33}$$

$$L_2 = \int_0^1 (2u-1) Q(u) du, \tag{2.34}$$

$$L_3 = \int_0^1 (6u^2 - 6u + 1) Q(u) du \tag{2.35}$$

and

$$L_4 = \int_0^1 (20u^3 - 30u^2 + 12u - 1) Q(u) du. \tag{2.36}$$

The equivalent formulae in terms of quantile density function are

$$L_1 = \int_0^1 (1-u) q(u) du, \tag{2.37}$$

$$L_2 = \int_0^1 (u - u^2) q(u) du, \tag{2.38}$$

$$L_3 = \int_0^1 (3u^2 - 2u^3 - u) q(u) du, \tag{2.39}$$

and

$$L_4 = \int_0^1 (u - 6u^2 + 10u^3 - 5u^4) q(u) du. \tag{2.40}$$

Like the conventional moments, L-moments can also be used to summarize the characteristics of probability distributions, to identify distributions and to fit models to data. L-moments are capable of characterizing a wider range of distributions compared to the conventional moments. A distribution may be specified by its L-moments, even if some of its conventional moments do not exist (Hosking (1990)). The L-moments have generally lower sampling variances and robust against outliers.

The L-moments exist whenever $E(X)$ is finite, where as for many distributions additional restrictions are required for the conventional moments to be finite. A distribution whose mean exists is characterized by $\{L_r, r=1, 2, ...\}$. However, any set that contains all L-moments except one is not sufficient to characterize a distribution. See Hosking (1996, 2006). Using (2.11), we can write, $L_2 = \frac{1}{2}\Delta$; therefore L_2 is a measure of spread. Thus the first (being the mean) and second L-moments provide measures of location and spread. Yitzhaki (2003), in his comparative study on the relative merits of the variance and the mean difference concludes that the mean difference is more informative than the variance in deriving properties of distributions that depart from normality. He also compares the algebraic structure of variance and Δ, and examines the relative superiority of the later from the point of view of the stochastic dominance, exchangeability and stratification.

As part of the standardization of higher L-moments $L_r, r \geq 3$, define the L-moments ratio by

$$\tau_r = \frac{L_r}{L_2}, \quad r = 3,4,... \tag{2.41}$$

For a non-degenerate random variable X with finite mean, $|\tau_r|<1$. Thus τ_r are dimensionless and bounded. Analogous to the coefficient of variation, the L-coefficient of variation is defined by

$$\tau_2 \equiv \frac{L_2}{L_1}. \tag{2.42}$$

In particular from (2.41) for $r = 3,4$, we get L-skewness

$$\tau_3 = \frac{L_3}{L_2} \tag{2.43}$$

and the L-kurtosis

$$\tau_4 = \frac{L_4}{L_2} \ . \tag{2.44}$$

The range of τ_3 is $(-1, 1)$ and that of τ_4 is

$$\frac{1}{4}\left(5\tau_3^{\ 2} - 1\right) \le \tau_4 < 1.$$

For more details we refer to Hosking (1990) and Jones (2004).

2.2 Reliability concepts

The notion of reliability, in the statistical sense, is the probability that an equipment or unit will perform the required function under the conditions specified for its operations for a given period of time. The primary concern in reliability theory is to understand the patterns in which failures occur, for different mechanisms and under varying operating environments, as a function of its age. This is accomplished by identifying the probability distribution of the lifetime represented by a nonnegative random variable X. Accordingly, several concepts have been developed that help in evaluating the effect of age, based on the distribution function of the lifetime random variable X and the residual life. In this section, we recall from Lai and Xie (2006) the definitions and properties of the reliability measures, that are essential to our discussions in the subsequent chapters.

2.2.1 Hazard rate function

Let X be a continuous nonnegative random variable with distribution function $F(x)$. The hazard rate of X is defined as

$$h(x) = \lim_{\delta \to 0} \frac{P\{x \le X \le x + \delta \mid X > x\}}{\delta} . \tag{2.45}$$

For small δ, $\delta\, h(x)$ is approximately the conditional probability of failure in the next small interval of time δ, given no failure has occurred $(0,x]$. When $F(x)$ is absolutely continuous with pdf $f(x)$, hazard rate is given by

$$h(x)=\frac{f(x)}{\bar{F}(x)}=\frac{-d\log\bar{F}(x)}{dx} \tag{2.46}$$

for all x for which $\bar{F}(x)>0$. From (2.46), on integration, we get

$$\bar{F}(x)=\exp\left\{-\int_0^x h(t)dt\right\}. \tag{2.47}$$

The equation (2.47) is used to characterize life distributions in terms of the functional term of $h(x)$, that could be postulated from the physical properties of the failure rate patterns.

2.2.2 Mean residual function

Mean residual function is a well known measure, which has been widely used in the fields of reliability, statistics, survival analysis and insurance. In section (2.1.2) we have defined the residual life function as

$$X_t=X-t\,|\,X>t$$

and the survival function of X_t was given as

$$\bar{F}_t(x)=\frac{\bar{F}(x+t)}{\bar{F}(t)}.$$

The expected value of X_t is called the mean residual life function (MRL), which is denoted by $m(t)$. If $E(X)<\infty$, then

$$m(t)=E(X-t\,|\,X>t)$$

$$=\frac{1}{\bar{F}(t)}\int_t^\infty \bar{F}(x)dx, \tag{2.48}$$

for all t for which $\bar{F}(t)>0$. Clearly $m(0)=\mu=E(X)$.

The function $m(t)$ determines the distribution of X uniquely by virtue of the formula

$$\bar{F}(t)=\frac{\mu}{m(t)}\exp\left\{-\int_0^t \frac{dx}{m(x)}\right\}. \tag{2.49}$$

When $m(t)$ is differentiable, using (2.46) it can easily be shown that

$$m'(t)=m(t)h(t)-1$$

or

$$h(t)=\frac{1+m'(t)}{m(t)}. \tag{2.50}$$

A compact review of the mean residual life function is given in Lai and Xie (2006). The references therein help to get the progress in the study of mean residual life in such contexts as characterizations, analysis of life distributions, monotonic properties, applications in ageing and orderings, etc..

2.2.3 Variance residual life function

Variance residual life is another concept which has been developed in recent years. For the lifetime random variable X with $E\left(X^2\right)<\infty$, the variance residual life function (VRL) is defined as

$$\begin{aligned}\sigma^2(x)&=V(X-x \mid X>x)\\&=V(X \mid X>x)\\&=E\left((X-x)^2 \mid X>x\right)-m^2(x)\\&=\frac{1}{\bar{F}(x)}\int_x^\infty (t-x)^2 f(t)\,dt-m^2(x),\end{aligned} \tag{2.51}$$

when the pdf $f(x)$ of X exists. Integrating the right side of (2.51) and simplifying we get

$$\sigma^2(x)=\frac{2}{\bar{F}(x)}\int_x^\infty\int_x^\infty \bar{F}(t)\,dt\,dx-m^2(x),\ \forall x\geq 0. \tag{2.52}$$

To know about the earlier development of VRL, we refer to Launer (1984), Gupta (1987) and Gupta et al. (1987). The monotonicity of the function has been studied by these authors. One important implication of the VRL is that it appears in the expression of the variance of the estimator of mean residual function. Gupta and Kirmani (2000) considered characterizations using VRL (see also Gupta (2006) and Gupta et al. (1987)). Gupta (2006) established that

$$\frac{d}{dx}\sigma^2(x)=h(x)\left(\sigma^2(x)-m^2(x)\right). \tag{2.53}$$

From (2.47), it follows that

$$\bar{F}(x)=\exp\left\{-\int_0^x\frac{\frac{d}{dt}\sigma^2(t)}{\sigma^2(t)-m^2(t)}dt\right\}. \tag{2.54}$$

The implication of (2.54) is that both $\sigma^2(x)$ and $m(x)$ are required to retrieve $F(x)$. The ratio

$$C(x)=\frac{\sigma(x)}{m(x)} \tag{2.55}$$

is called coefficient of variation of residual life. Gupta (2006) showed that

$$\frac{d}{dx}\sigma^2(x)=m(x)\left(1+m'(x)\left(C^2(x)-1\right)\right). \tag{2.56}$$

Unlike the hazard rate and mean residual life functions, there is no direct formula that expresses $F(x)$ in terms of VRL only. This fact motivated many authors to work on the characterization of specific distributions or families by the functional form of VRL. See Dallas (1981), Adatia et al. (1991), Koicheva (1993), Ghittany et al. (1995), Navarro et al. (1998) and El-Arishi (2005). Most of these results obtained are subsumed in the general formula given in Nair and Sudheesh (2010).

2.2.4 Percentile residual life function

The median residual life function and its generalization, the percentile residual function has been evolved as alternative measures to overcome the shortcomings of mean residual life. Schmittlein and Morrison (1981) pointed out the advantages of median residual life function over the mean residual life function. The general version of median residual life function, originally introduced by Haines and Singpurwalla (1974), is the α^{th} percentile residual life of the lifetime variable X. The α^{th} percentile residual life of X is denoted by $p_\alpha(t)$ and is defined by

$$\begin{aligned} p_\alpha(t) &= F_t^{-1}(\alpha) \\ &= \inf\left\{x \mid F_t(x) \geq \alpha\right\} \\ &= F^{-1}\left(1-(1-\alpha)\bar{F}(t)\right)-t. \end{aligned} \tag{2.57}$$

$p_\alpha(t)$ is interpreted as the age that will be survived on the average by $(1-\alpha)\%$ of units that have lived beyond age x. It is clear that $p_\alpha(t)$ is solution of the fundamental equation

$$F\left(p_\alpha(t)+t\right)=1-(1-\alpha)\bar{F}(t), \quad t \geq 0. \tag{2.58}$$

The importance of percentile residual life function has been revealed through the works of Arnold and Brocket (1983), Joe and Proschan (1984), Gupta and Langford (1984), Joe (1985), Csorgo and Csorgo (1987), Csorgo and Viharos (1992), Schmittlein and Morrison

(1981), Song and Cho (1995), Lillo (2005) and Franco-Perieira et al. (2010).

Gupta and Langford (1984) established that a single percentile residual life function does not characterize a life distribution and provided a comprehensive solution to the problem. They identified (2.58) as a particular case of the Schroder's functional equation

$$R(\phi(t)) = u R(t), \quad 0 \le t < \infty, \tag{2.59}$$

$0 < u < 1$ and $\phi(t)$ is a continuous and strictly increasing function on $(0,\infty)$ satisfying $\phi(t) > t$ for all t. The general solution of (2.59) is

$$R(t) = R_0(t) K(\log R_0(t)), \tag{2.60}$$

where $K(.)$ is a periodic function with period $-\log u$ and $R_0(t)$ is a particular solution of (2.59) which is positive, continuous and strictly decreasing such that $R(0) = 1$. Thus there is no unique solution to (2.58).

Song and Cho (1995) have shown that if $F(t)$ is continuous, strictly increasing and for $0 < \alpha, \beta < 1$, $\frac{\log(1-\alpha)}{\log(1-\beta)}$ is irrational, then $F(t)$ is uniquely determined by $p_\alpha(t)$ and $p_\beta(t)$. This is the corrected form of the result given in Arnold and Brocket (1983). Recently Lin (2009) has established some general results.

In Chapter 7, we discuss the properties of the percentile residual life function in reversed time and characterization problems using it. Its relationships with other reliability concepts are also discussed.

2.3 Reliability concepts in reversed time

In Section 2.2 we have described some commonly used reliability concepts that conditioned on the event $X \geq x$. Recently in a parallel theoretical framework some concepts which are conditioned on $X \leq x$ have been successfully employed in various applications. They are generally called functions in reversed time. These functions help to analyse the behaviour of the lifetime random variable X given that failure has occurred in $[0,x]$. In this section, definitions and properties of such functions are given.

2.3.1 Reversed hazard rate

For the lifetime random variable X, the reversed hazard rate is defined by,

$$\lambda(x) = \lim_{\Delta \to 0} \frac{p\{x - \Delta < X \leq x \mid X \leq x\}}{\Delta}. \tag{2.61}$$

When pdf $f(x)$ of X exists, (2.61) can be written as

$$\lambda(x) = \frac{f(x)}{F(x)} = \frac{d}{dx} \log F(x), \tag{2.62}$$

for all x for which $F(x) > 0$. From (2.62), we have

$$\Delta \lambda(x) = P\{x - \Delta < X \leq x \mid X \leq x\}.$$

That is, $\Delta \lambda(x)$ is approximately the probability that the system with lifetime X which has survived up to $x - \Delta$ will fail in the next small interval of time Δ given that it will not survive age x.

The function $\lambda(x)$ has been first introduced by Keilson and Sumitha (1982). Later many researchers have used it in various situations such as estimation and modelling of left-censored data, stochastic orderings, and characterization of distributions and in evolving repair and maintenance strategies. Block et al. (1998) have shown that there does not exist a nonnegative random variable having increasing or constant reversed hazard rate function. The function $\lambda(x)$ determines the distribution uniquely through the inversion formula

$$F(x) = \exp\left\{-\int_x^\infty \lambda(t)dt\right\}. \tag{2.63}$$

When X has support $(a, b), -\infty \le a < b \le \infty$, then the $\lambda(x)$ has relationships with hazard rate $h(x)$ as

$$\lambda_{-X}(x) = h_X(-x). \tag{2.64}$$

2.3.2 Reversed mean residual life

The random variable $(x - X \mid X \le x)$ is called the inactivity time or reversed residual life of X. It represents the time elapsed since the failure of a unit given that its lifetime is atmost x. The distribution function of the reversed residual life is

$$_xF(t) = \frac{F(x) - F(x-t)}{F(x)} \tag{2.65}$$

with density function

$$_xf(t) = \frac{F(x-t)}{F(x)}. \tag{2.66}$$

The reversed mean residual life of X is denoted by $r(x)$ and is defined as

$$r(x) = E(x - X \mid X \le x) \tag{2.67}$$

$$= \frac{1}{F(x)}\int_0^x t f(x-t)dt$$

$$= \frac{1}{F(x)}\int_0^x F(t)dt \quad . \tag{2.68}$$

The relationship between $r(x)$ and $\lambda(x)$ is given by

$$\lambda(x) = \frac{1 - r'(x)}{r(x)}$$

and hence using (2.63) we have

$$F(x) = \exp\left\{-\int_x^\infty \frac{1 - r'(t)}{r(t)} dt\right\}. \tag{2.69}$$

2.3.3 Reversed variance residual life

The reversed variance residual life function of X is given by

$$\begin{aligned} V(x) &= V(x - X \mid X \le x) \\ &= E\left[(x - X)^2 \mid X \le x\right] - r^2(x) \\ &= 2\int_0^x \int_0^u F(t)\, dt\, du - r^2(x). \end{aligned} \tag{2.70}$$

For some properties of $V(x)$ we refer to Kundu and Nanda (2010).

2.4 Quantile-based reliability concepts

In the previous two sections, we have described reliability concepts and their properties, which are based on the distribution function. There exist several quantile functions that cannot be inverted to obtain the corresponding distribution functions. These distributions sometimes seem useful in analyzing the lifetime data. See Chapter 3 for some such models and its applications in reliability analysis. In view of this, we need the definitions and properties of these functions in terms of quantile functions. In this section, we give these definitions of reliability concepts. Main source of this section is Nair and Sankaran (2009). Here we assume that $F(x)$ is continuous and strictly increasing.

2.4.1 Hazard quantile function

Setting $x = Q(u)$ in equation (2.46) and then using the relationship (2.2), we have the definition of the hazard quantile function as

$$H(u) = h(Q(u)) = [(1-u)q(u)]^{-1}. \tag{2.71}$$

We interpret $H(u)$ as the conditional probability of failure of a unit in the next small interval of time given the survival of unit at $100(1-u)\%$ point of distribution. From (2.71)

$$q(u) = [(1-u)H(u)]^{-1} \tag{2.72}$$

and

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{2.73}$$

Thus $H(u)$ uniquely determines the quantile function $Q(u)$.

2.4.2 Mean residual quantile function

When X has a density f, we can write the mean residual function $m(t)$ as

$$m(t) = \frac{\int_t^\infty x f(x)dx}{\bar{F}(t)} - t. \tag{2.74}$$

Letting $t = Q(u)$, we have the quantile version of $m(t)$, the mean residual quantile function is given by

$$\begin{aligned} M(u) &= m(Q(u)) \\ &= (1-u)^{-1} \int_u^1 [Q(p) - Q(u)]dp. \end{aligned} \tag{2.75}$$

In terms of quantile density function, (2.75) can be written as

$$M(u)=(1-u)^{-1}\int_u^1 (1-p)\, q(p)dp\,. \tag{2.76}$$

Notice that $M(u)$ is the mean remaining life beyond the $100(1-u)\%$ of the distribution. From (2.76) we have

$$M(u)=(1-u)^{-1}\int_u^1 \left(H(p)\right)^{-1} dp\,. \tag{2.77}$$

Differentiating (2.77), we get

$$\left(H(u)\right)^{-1}=M(u)-(1-u)M'(u). \tag{2.78}$$

The distribution is uniquely determined by $M(u)$ through the formulas

$$Q(u)=\mu-M(u)+\int_0^u (1-p)^{-1} M(p)dp \tag{2.79}$$

and

$$q(u)=(1-u)^{-1} M(u)-M'(u). \tag{2.80}$$

2.4.3 Residual variance quantile function

The quantile form of variance residual function, the residual variance quantile function is defined as

$$V(u)=(1-u)^{-1}\int_u^1 Q^2(p)dp-\left(M(u)+Q(u)\right)^2. \tag{2.81}$$

The above expression can easily be obtained by letting $x=Q(u)$ in (2.51). Nair and Sankaran (2009) derived the relationship between $M(u)$ and $V(u)$ as

$$M^2(u)=V(u)-(1-u)V'(u) \tag{2.82}$$

or

$$V(u)=(1-u)^{-1}\int_u^1 M^2(p)dp. \tag{2.83}$$

Since $M(u)$ characterizes the distribution, from (2.82) and (2.83) it follows that $V(u)$ also characterizes the distribution.

In quantile terminology, the co-efficient of variation is defined as

$$c(u)=\frac{V(u)}{M^2(u)}. \tag{2.84}$$

Using (2.82), (2.84) become

$$\frac{1}{c(u)}=1-\frac{(1-u)V'(u)}{V(u)}$$

or

$$\frac{d}{du}\log V(u)=(1-u)^{-1}\left[1-(c(u))^{-2}\right]. \tag{2.85}$$

2.4.4 Quantile forms of some other reliability concepts

In this section, we present quantile forms of some more reliability concepts, which are useful for the discussions in the subsequent chapters. These definitions are listed below.

(a) Percentile residual quantile function:-

This is the quantile form of percentile residual life function given in (2.57). It is denoted by $P_\alpha(u)$ and is defined by

$$P_\alpha(u)=p_\alpha(Q(u))=Q[1-(1-\alpha)(1-u)]-Q(u). \tag{2.86}$$

(b) Reversed hazard quantile function:-

It is denoted by $\Lambda(u)$ and is defined as

$$\Lambda(u)=\lambda(Q(u))=(uq(u))^{-1}. \tag{2.87}$$

$\Lambda(u)$ determines the distribution through the formula

$$Q(u)=\int_0^u (p\Lambda(p))^{-1}\,dp. \tag{2.88}$$

and $\Lambda(u)$ related with $H(u)$ as

$$H(u)=(1-u)^{-1}u\Lambda(u). \tag{2.89}$$

(c) Reversed mean residual quantile function:-

The reversed mean residual life function defined in (2.67) can be translated to the quantile terminology to get reversed mean residual quantile function, which is denoted as $R(u)$ and has the form

$$\begin{aligned} R(u) &= r(Q(u)) \\ &= u^{-1}\int_0^u (Q(u)-Q(p))dp \\ &= u^{-1}\int_0^u p\,q(p)dp. \end{aligned} \tag{2.90}$$

Nair and Sankaran (2009) derived the following relationships connecting $R(u)$ with other functions

$$Q(u) = R(u) + \int_0^u p^{-1}R(p)dp, \tag{2.91}$$

$$(\Lambda(u))^{-1} = R(u) + uR'(u), \tag{2.92}$$

$$R(u) = u^{-1}\int_0^u (\Lambda(p))^{-1} dp, \tag{2.93}$$

and

$$((1-u)M(u)) = \mu + uR(u) - Q(u). \tag{2.94}$$

(d) Reversed variance residual quantile function:-

Quantile form of the variance residual function is denoted by $D(u)$ and is defined as

$$D(u) = u^{-1}\int_0^u Q^2(p)dp - (Q(u)-R(u))^2 \tag{2.95}$$

and has relationship with $R(u)$ as

$$R^2(u) = D(u) + uD'(u) \tag{2.96}$$

or

$$D(u) = u^{-1}\int_0^u R^2(p)dp, \tag{2.97}$$

where $D'(u)$ is the derivative of $D(u)$ with respect to u.

2.5 Total time on test transform

The total time on test transforms (TTT) is a widely accepted statistical tool, which has applications in different fields such as reliability analysis, econometrics, stochastic modelling, tail ordering, ordering of distributions, etc.. When several units are tested to determine their life lengths for a specified period of time, some of the units fail while others may survive the period of the test. The sum of all observed and incomplete life lengths is the total time on test statistic. As the number of items becomes infinite, the limit of this statistic is called the total time on test transform (TTT). This quantile-based concept was first studied in the early seventies (See Barlow and Doksum (1972) and Barlow et al. (1972)). A major part of the existing literature on TTT is concerned with reliability analysis that include characterization of ageing properties, model identification, tests of hypothesis, age replacement policies in maintenance, ordering of life distributions and defining new classes of life distributions. See Bergman and Klefsjö (1984), Bartoszewicz (1995), Haupt and Schabe (1997), Kochar et al. (2002), Li and Zuo (2000), Ahmad et al. (2005), Li and Shaked (2007) and the references therein for further details. The works in recent years manifest the increased interest of the researchers in the properties and applications of TTT.

The total time on test transform of a nonnegative random variable X is defines as

$$H_F^{-1}(u) = \int_0^{F^{-1}(u)} \bar{F}(t)\,dt \tag{2.98}$$

is called the total time on test transform. When expressed in terms of quantile functions, it is given by

$$T(u) = \int_0^u (1-p)q(p)dp. \tag{2.99}$$

We call $\phi(u) = \dfrac{T(u)}{\mu}$ as the scaled TTT transform.

Further properties of this quantile version and its generalization will be discussed in detail in Chapter 5.

2.5.1 Relationships of TTT with reliability functions

From (2.71) and (2.99), we have the relationship between the hazard quantile function and TTT as

$$T'(u) = \frac{1}{H(u)}. \tag{2.100}$$

Again from (2.18) and (2.99),

$$T(u) = \mu - \int_u^1 (1-p)q(p)dp = \mu - (1-u)M(u)$$

or

$$M(u) = \frac{\mu - T(u)}{1-u}. \tag{2.101}$$

The equation (2.101) gives the relationship between TTT and mean residual quantile function. We have, from (2.83),

$$V(u) = \frac{1}{1-u}\int_u^1 M^2(p)dp$$

and hence

$$V(u) = \frac{1}{1-u}\int_u^1 \left(\frac{\mu - T(p)}{1-p}\right)^2 dp \tag{2.102}$$

or

$$T(u) = \mu - (1-u)\left[(1-u)V'(u) - V(u)\right]^{1/2}. \tag{2.103}$$

With regard to functions in reversed time,

$$T(u) = Q(u) - \int_0^u p\,q(p)\,dp$$

or

$$u\,q(u) = \frac{d}{du}\big[Q(u) - T(u)\big].$$

Therefore

$$\Lambda(u) = \left[\frac{d}{du}\big(Q(u) - T(u)\big)\right]^{-1}. \qquad (2.104)$$

The reversed mean residual quantile function given in (2.90) satisfies

$$\begin{aligned} u\,R(u) &= \int_0^u p\,q(p)\,dp \\ &= Q(u) - T(u). \end{aligned}$$

Hence

$$T(u) = Q(u) - u\,R(u). \qquad (2.105)$$

Finally, we can connect TTT with reversed variance residual quantile function as

$$\begin{aligned} D(u) &= \frac{1}{u}\int_0^u R^2(p)\,dp \\ &= \frac{1}{u}\int_0^u \frac{Q(p) - T(p)}{p^2}\,dp. \end{aligned} \qquad (2.106)$$

The above relationships will be made use of in the discussions in Chapter 5.

2.6 Some distributions defined by quantile functions

There are many distributions that can be specified by quantile functions. Some of them are obtained by direct inversion of the distribution function. Others are defined in terms of the quantile function only, as there is no analytically tractable distribution function for them. In this section, we discuss some distributions belonging to the latter category used in the sequel. It may be noticed that the primary focus of the present thesis is such distributions and their analysis in the context of reliability theory.

2.6.1 Lambda distributions

The base of the development of lambda distributions is the work of Hastings et al. (1947), who introduced a family of distributions by a quantile function. During the past sixty years considerable efforts were made to generalize this family of distributions and the refined model by Tukey (1962) and to study their new applications and inference procedures. These generalized versions have been used for the modelling and analysis in different fields such as inventory control (Silver (1977)), logistic regression (Pregibon (1980)), meteorology (Osturk and Dale (1982)), survival analysis (Lefante Jr (1987)), queuing theory (Robinson and Chan (2003)), random variate generation and goodness of fit tests (Cao and Lugosi (2005)), fatigue studies (Bigerelle et al. (2005)), process control (Fournier et al. (2006)), biochemistry (Ramos Fernandez et al. (2008)) and economics (Haritha et al. (2008)). In this work, we try to point out the role of lambda distributions and other quantile functions in reliability analysis.

The Tukey lambda distribution with quantile function

$$Q(u)=\frac{u^{\lambda}-(1-u)^{\lambda}}{\lambda},\ 0\le u\le 1,\ \lambda\ne 0 \tag{2.107}$$

is the basic model from which all other generalizations originated. When $\lambda\to 0$,

$$Q(u)=\log\left(\frac{u}{1-u}\right),$$

which is the quantile function of the logistic distribution. For $\lambda = 1$ and $\lambda = 2$, (2.107) becomes the uniform over $(-1, 1)$ and $\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$ respectively. The density functions are U shaped for $1 < \lambda < 2$ and unimodal for $\lambda < 1$ or $\lambda < 2$. Being symmetric and having range for negative values of X, it has limited use in reliability modelling.

2.6.2 Generalized lambda distribution (GLD)

Ramberg and Schmeiser (1974) generalized the Tukey lambda distribution to a four-parameter distribution specified by quantile function

$$Q(u) = \lambda_1 + \frac{1}{\lambda_2}\left(u^{\lambda_3} - (1-u)^{\lambda_4}\right), \quad 0 \le u \le 1, \tag{2.108}$$

where λ_1 is a location parameter, λ_2 is a scale parameter and λ_3 and λ_4 are shape parameters. The distribution (2.108) is called the generalized lambda distribution. This is the most discussed member of the family of lambda distributions because of its versatility and special properties. The distribution takes a wide range of values for X for the different choices of parameters as shown in Table 2.1. In these range of parameters, the quantile function provides a valid distribution. In all regions λ_1 takes real values.

The quantile function (2.108) is a valid distribution also for values in $(-1 < \lambda_3 < 0,\ \lambda_3 > 0)$ for which

$$\frac{(1-\lambda_3)^{1-\lambda_3}}{(\lambda_4 - \lambda_3)^{\lambda_3 - \lambda_4}}(\lambda_4 - 1)^{\lambda_4 - 1} < \frac{-\lambda_3}{\lambda_4}$$

and for values in $(\lambda_3 > 0,\ -1 < \lambda_4 < 0)$ for which

$$\frac{(1-\lambda_4)^{1-\lambda_4}}{(\lambda_3-\lambda_4)^{\lambda_3-\lambda_4}}(\lambda_3-1)^{\lambda_3-1} < \frac{-\lambda_4}{\lambda_3}.$$

See Karian and Dudewicz (2000) for details.

Table 2.1-Ranges of generalized lambda distribution.

Region	Supports
1. $\lambda_2 < 0,\ \lambda_3 < -1,\ \lambda_4 > 1$	$\left(-\infty, \lambda_1 + \frac{1}{\lambda_2}\right)$
2. $\lambda_2 < 0,\ \lambda_3 > 1,\ \lambda_4 < -1$	$\left(\lambda_1 - \frac{1}{\lambda_2}, \infty\right)$
3. $\lambda_2 > 0,\ \lambda_3 > 1,\ \lambda_4 > 0$	$\left(\lambda_1 - \frac{1}{\lambda_2}, \lambda_1 + \frac{1}{\lambda_2}\right)$
4. $\lambda_2 > 0,\ \lambda_3 = 0,\ \lambda_4 > 0$	$\left(\lambda_1, \lambda_1 + \frac{1}{\lambda_2}\right)$
5. $\lambda_2 > 0,\ \lambda_3 > 0,\ \lambda_4 = 0$	$\left(\lambda_1 - \frac{1}{\lambda_2}, \lambda_2\right)$
6. $\lambda_2 < 0,\ \lambda_3 < 0,\ \lambda_4 < 0$	$(-\infty, \infty)$
7. $\lambda_2 < 0,\ \lambda_3 = 0,\ \lambda_4 < 0$	(λ_1, ∞)
8. $\lambda_2 < 0,\ \lambda_3 < 0,\ \lambda_4 = 0$	$(-\infty, \lambda_1)$

A constraint on the parameters for the quantile function to be a lifetime distribution is $Q(0) = \lambda_1 - \frac{1}{\lambda_2} \geq 0$. The quantile density function has the expression

$$q(u) = \frac{1}{\lambda_2}\left(\lambda_3 u^{\lambda_3-1} + \lambda_4 (1-u)^{\lambda_4-1}\right) \tag{2.109}$$

and the density quantile function is

$$f(Q(u)) = \lambda_2\left[\lambda_3 u^{\lambda_3-1} + \lambda_4 (1-u)^{\lambda_4-1}\right]^{-1}, \tag{2.110}$$

which should be nonnegative for (2.108) to be a proper distribution. The distribution has a wide variety of shapes for the density function as shown in figure (2.1).

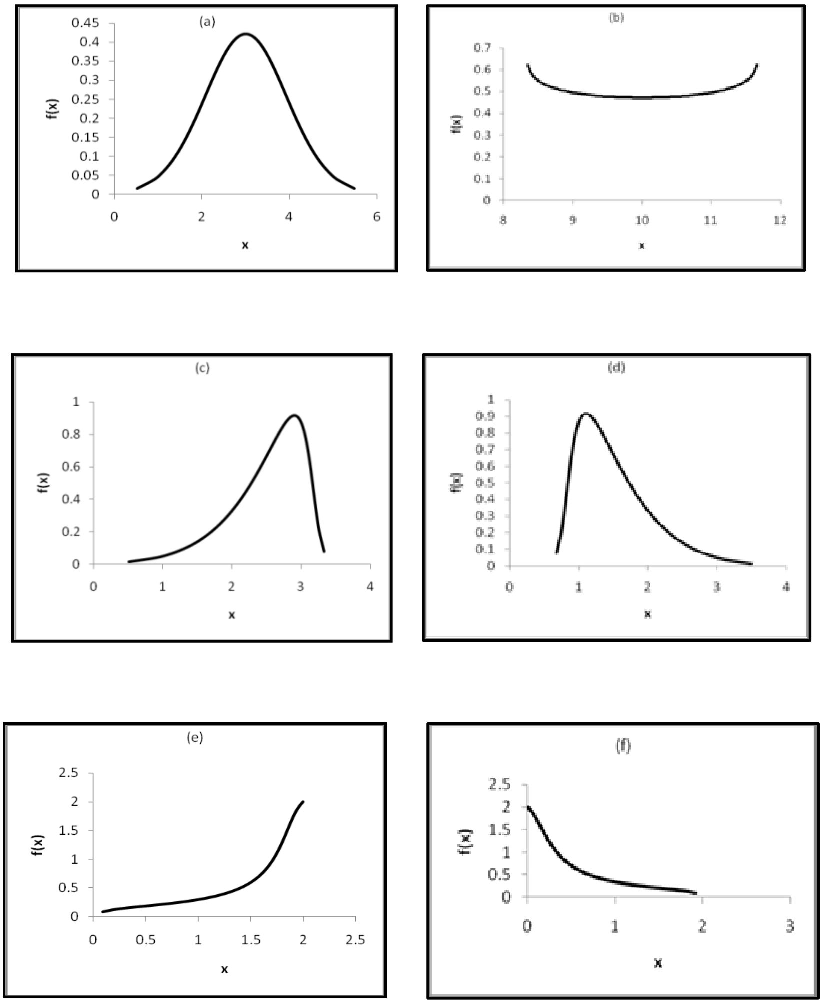


Figure 2.1 - Density plots of the GLD (Ramberg and Shreimer model) when $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are (a)$(1, 0.2, 0.13, 0.13)$, (b)$(1, 0.6, 1.5, -1.5)$, (c)$(1, 0.2, 0.13, 0.013)$, (d)$(1, 0.2, 0.013, 0.13)$, (e)$(1, 1, 0.5, 4)$, (f) $(1, 1, 3, 0.5)$.

2.6.2.1 Moments of Generalized Lambda Distribution

The expression for r^{th} ordinary moment from equation (2.17) is given by

$$E\left(X^r\right)=\int_0^1\left(\lambda_1+\frac{u^{\lambda_3}-(1-u)^{\lambda_4}}{\lambda_2}\right)^r du$$

In particular, the first moment, the mean is

$$E(X)=\mu=\int_0^1\left(\lambda_1+\frac{u^{\lambda_3}-(1-u)^{\lambda_4}}{\lambda_2}\right)du$$

$$=\lambda_1+\frac{1}{\lambda_2}\left(\frac{1}{\lambda_3+1}-\frac{1}{\lambda_4+1}\right) \qquad (2.111)$$

Ramberg and Schmeiser (1974) showed that when $\lambda_1=0$, the r^{th} moment when it exists has the expression

$$E\left(X^r\right)=\lambda_2^{-r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i B\left(\lambda_3(r-i)+1,\, \lambda_4^{\,i}+1\right),$$

where $B(.)$ denotes the beta function. Since the arguments of the beta function should be nonnegative, the r^{th} moment exists only when $\frac{-1}{r}<\min(\lambda_3,\lambda_4)$. A detailed study of the skewness and kurtosis for different values of λ_3 and λ_4 is given in Karian and Dudewicz (2000).

2.6.2.2 Percentiles

The basic characteristics of the distribution can also be expressed in terms of the percentiles. Using (2.12) through (2.15), the median

$$M=Q(0.5)=\lambda_1+\frac{1}{\lambda_2}\left(\left(\frac{1}{2}\right)^{\lambda_3}-\left(\frac{1}{2}\right)^{\lambda_4}\right), \qquad (2.112)$$

the inter-quartile range

$$IQR=Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)=\frac{1}{\lambda_2}\left(\frac{3\lambda_3-1}{4^{\lambda_3}}+\frac{3\lambda_4-1}{4^{\lambda_4}}\right), \qquad (2.113)$$

Galton's measures of skewness

$$S=\frac{Q\left(\frac{1}{4}\right)+Q\left(\frac{3}{4}\right)-2M}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$$

$$=\frac{4^{-\lambda_3}\left(3^{\lambda_3}-2^{\lambda_3+1}-1\right)-4^{\lambda_4}\left(1+3^{\lambda_4}-2^{\lambda_4+1}\right)}{\frac{3^{\lambda_3-1}}{4^{\lambda_3}}+\frac{3\lambda_4-1}{4^{\lambda_4}}} \tag{2.114}$$

and the Moor's measures of kurtosis

$$T=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{IQR}.$$

$$=\frac{8^{-\lambda_3}\left(1+3^{\lambda_3}+5^{\lambda_3}+7^{\lambda_3}\right)-8^{-\lambda_4}\left(1+3^{\lambda_4}+5^{\lambda_4}+7^{\lambda_4}\right)}{4^{-\lambda_3}\left(3^{\lambda_3}-1\right)+4^{-\lambda_4}\left(3^{\lambda_4}-1\right)} \tag{2.115}$$

2.6.2.3 L-moments

Using the equations (2.33) through (2.36), Asquith (2007) has obtained simple expressions for L-moments of the distribution, which are given by

$$L_1=\int_0^1 Q(u)du$$

$$=\int_0^1\left[\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}-(1-u)^{\lambda_4}\right)\right]du$$

$$=\lambda_1+\frac{1}{\lambda_2}\left(\frac{1}{\lambda_3+1}-\frac{1}{\lambda_4+1}\right) \tag{2.116}$$

$$L_2=\int_0^1(2u-1)Q(u)du$$

$$=\int_0^1(2u-1)\left[\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}-(1-u)^{\lambda_4}\right)\right]du$$

$$= \frac{1}{\lambda_2}\left(\frac{\lambda_3}{(\lambda_3+1)(\lambda_3+2)} + \frac{\lambda_4}{(\lambda_4+1)(\lambda_4+2)}\right) \tag{2.117}$$

$$L_3 = \int_0^1 (6u^2 - 6u + 1)Q(u)du$$

$$= \int_0^1 (6u^2 - 6u + 1)\left[\lambda_1 + \frac{1}{\lambda_2}\left(u^{\lambda_3} - (1-u)^{\lambda_4}\right)\right]du$$

$$= \frac{1}{\lambda_2}\left(\frac{\lambda_3(\lambda_3-1)}{(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)} - \frac{\lambda_4(\lambda_4-1)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}\right) \tag{2.118}$$

and

$$L_4 = \int_0^1 (20u^3 - 30u^2 + 12u - 1)Q(u)du$$

$$= \int_0^1 (20u^3 - 30u^2 + 12u - 1)\left[\lambda_1 + \frac{1}{\lambda_2}\left(u^{\lambda_3} - (1-u)^{\lambda_4}\right)\right]du$$

$$= \frac{1}{\lambda_1}\left(\frac{\lambda_3(\lambda_3-1)(\lambda_3-2)}{(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)} + \frac{\lambda_4(\lambda_4-1)(\lambda_4-2)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}\right) \tag{2.119}$$

The L-skewness and L-kurtosis have the expression

$$\tau_3 = \frac{L_3}{L_2} = \frac{\lambda_3(\lambda_3-1)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3) - \lambda_4(\lambda_4-1)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)}{\lambda_3(\lambda_3+3)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3) + \lambda_4(\lambda_4+3)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)}$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{\lambda_3(\lambda_3-1)(\lambda_3-2)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4) + \lambda_4(\lambda_4-1)(\lambda_4-2)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)}{\lambda_3(\lambda_3+3)(\lambda_3+4)(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4) - \lambda_4(\lambda_4+3)(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)}$$

All the L-moments exist for every $\lambda_3,\ \lambda_4 > -1$. It may be noted that the conventional moments require more restricted condition that $\lambda_3, \lambda_4 > \frac{-1}{4}$ for the evaluation of Pearson's skewness β_1 and kurtosis β_2. Thus L-skewness and L-kurtosis permit a larger range of values in the parameter space.

2.6.3 Generalized Tukey lambda distribution

We have seen in the last section that the generalized lambda distribution by Ramberg and Schmeiser (1974) has a limitation that the distribution is not valid in certain regions of the parameter space. Freimer et al. (1988) introduced a modified generalized lambda distribution which is well defined for the values of the parameters over the entire two dimensional space. This distribution is specified by the quantile function

$$Q(u) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right),\ 0 \le u \le 1. \tag{2.120}$$

The quantile density function is

$$q(u) = \frac{1}{\lambda_2}\left(u^{\lambda_3 - 1} + (1-u)^{\lambda_4 - 1}\right). \tag{2.121}$$

Since our interest in (2.120) is as a life distribution, we should have

$$Q(0) = \lambda_1 - \frac{1}{\lambda_2 \lambda_3} \ge 0.$$

In this case the support becomes $\left(\lambda_1 - \frac{1}{\lambda_2\lambda_3},\ \lambda_1 + \frac{1}{\lambda_2\lambda_4}\right)$ whenever $\lambda_3 > \lambda_4 > 0$ and $\left(\lambda_1 - \frac{1}{\lambda_2\lambda_3}, \infty\right)$ if $\lambda_3 > 0$ and $\lambda_4 \le 0$. This is a critical point to be verified when the distribution is used to model data pertaining to nonnegative random variables. The exponential distribution is a particular case of the family as $\lambda_3 \to \infty$ and $\lambda_4 \to 0$. Considerable richness is seen in the density shapes, there being members that are unimodal, U-shaped, J-shaped and monotone, which are symmetric or skew with short, median and long tails, see Figure 2.2.

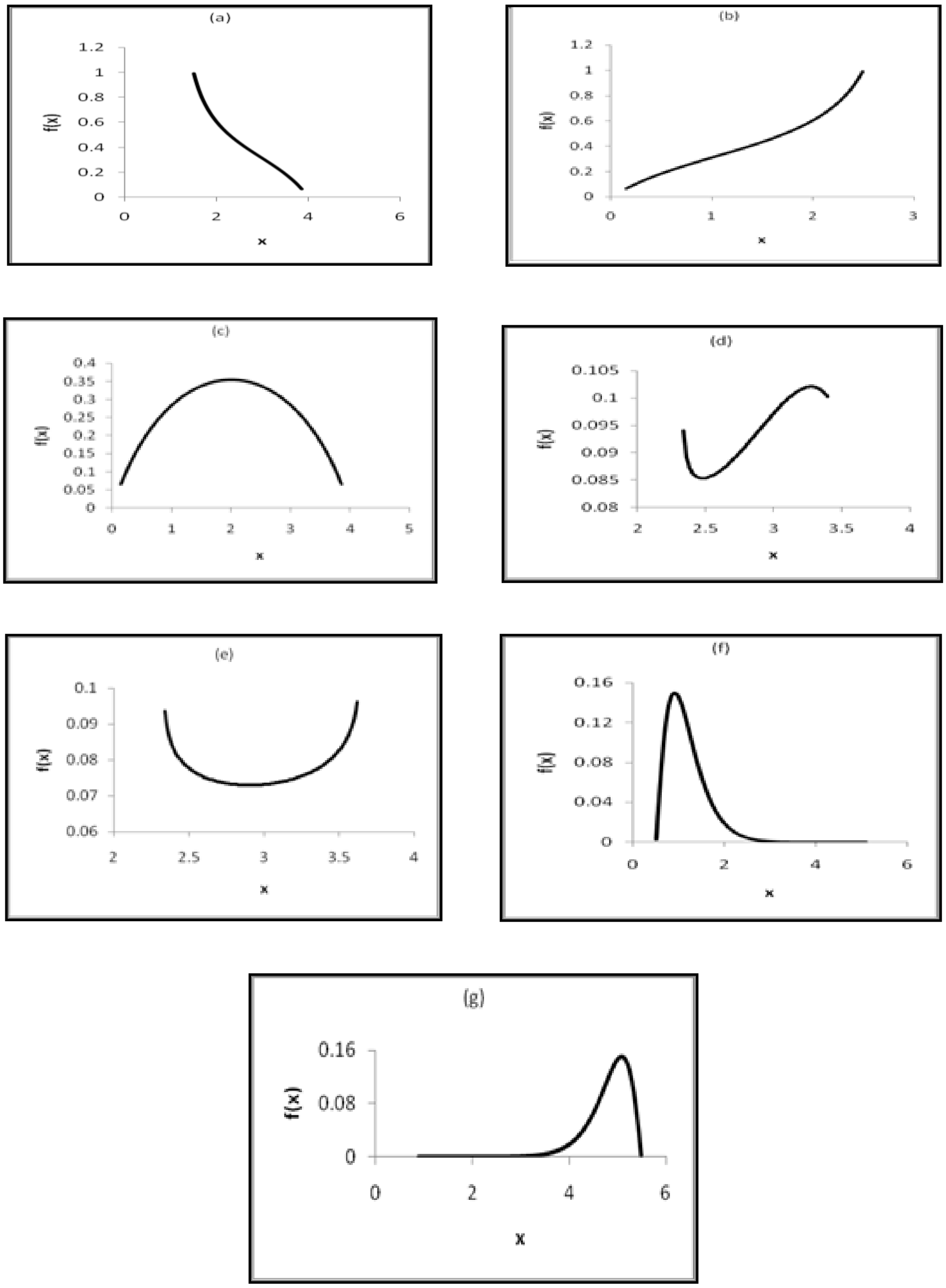


Figure 2.2- Density plots of the GLD (Freimer model) when $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are (a) $(2, 1, 2, 0.5)$, (b) $(2, 1, 0.5, 2)$, (c) $(2, 1, 0.5, 0.5)$, (d) $(3, 1, 1.5, 2.5)$, (e) $(3, 1, 1.5, 1.6)$, (f) $(1, 1, 2, 0.1)$, (g) $(5, 1, 0.1, 2)$.

2.6.3.1 Moments

Using (2.17), the first four moments of the distribution are obtained as

$$\mu = \int_0^1 Q(u)du$$

$$= \int_0^1 \left[\lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right)\right]du$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left[\frac{1}{\lambda_3+1} - \frac{1}{\lambda_4+1}\right], \qquad (2.122)$$

$$\mu_2' = \int_0^1 \left(Q(u)\right)^2 du$$

$$= \int_0^1 \left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right)\right)^2 du$$

$$= \frac{1}{\lambda_2^2}\left[\frac{1}{\lambda_3^2(2\lambda_3+1)} - \frac{1}{\lambda_4^2(2\lambda_4+1)} - \frac{2}{\lambda_3\lambda_4}B(\lambda_3+1,\lambda_4+1)\right] \qquad (2.123)$$

$$\mu_3' = \int_0^1 \left(Q(u)\right)^3 du$$

$$= \int_0^1 \left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right)\right)^3 du$$

$$= \frac{1}{\lambda_2^3}\left[\begin{array}{r}\frac{1}{\lambda_3^3(3\lambda_3+1)} - \frac{1}{\lambda_4^3(3\lambda_4+1)} - \frac{3}{\lambda_3^2\lambda_4}B(2\lambda_3+1,\lambda_4+1) \\ + \frac{3}{\lambda_3\lambda_4^2}B(\lambda_3+1,2\lambda_4+1)\end{array}\right]$$

(2.124)

and

$$\mu_4' = \int_0^1 (Q(u))^4 du$$

$$= \int_0^1 \left(\lambda_1 + \frac{1}{\lambda_2} \left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4} \right) \right)^4 du$$

$$= \frac{1}{\lambda_2^4} \left[\frac{1}{\lambda_3^4 (4\lambda_3 + 1)} + \frac{1}{\lambda_4^3 (3\lambda_4 + 1)} + \frac{6}{\lambda_3^2 \lambda_4^2} B(2\lambda_3 + 1, 2\lambda_4 + 1) \right.$$
$$\left. - \frac{4}{\lambda_3^2 \lambda_4} B(3\lambda_3 + 1, \lambda_4 + 1) + \frac{4}{\lambda_3 \lambda_4^3} B(\lambda_3 + 1, 3\lambda_4 + 1) \right]$$

(2.125)

In order to have a finite moment of order k, it is necessary that $\min(\lambda_3, \lambda_4) > -1/k$. An elaborate discussion on the skewness and kurtosis is carried out in Freimer et al. (1988).

2.6.3.2 Percentiles

The measures of location, spread, skewness and kurtosis based on percentiles are derived using the equations (2.12) through (2.15) as

$$M = Q\left(\frac{1}{2}\right) = \lambda_1 + \frac{1}{\lambda_2} \left[\frac{\left(\frac{1}{2}\right)^{\lambda_3} - 1}{\lambda_3} - \frac{\left(\frac{1}{3}\right)^{\lambda_4} - 1}{\lambda_4} \right], \tag{2.126}$$

$$IQR = Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)$$
$$= \frac{1}{2\lambda_2} \left[\frac{\left(\frac{3}{4}\right)^{\lambda_3} - \left(\frac{1}{4}\right)^{\lambda_3}}{\lambda_3} - \frac{\left(\frac{3}{4}\right)^{\lambda_4} - \left(\frac{1}{4}\right)^{\lambda_4}}{\lambda_4} \right], \tag{2.127}$$

$$S=\frac{Q\left(\frac{1}{4}\right)+Q\left(\frac{3}{4}\right)-2M}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$$

$$=\frac{\lambda_4\left(\left(\frac{3}{4}\right)^{\lambda_3}-2\left(\frac{1}{2}\right)^{\lambda_3}+\left(\frac{1}{4}\right)^{\lambda_3}\right)-\lambda_3\left(\left(\frac{3}{4}\right)^{\lambda_4}-2\left(\frac{1}{2}\right)^{\lambda_4}+\left(\frac{1}{4}\right)^{\lambda_4}\right)}{\lambda_4\left(\left(\frac{3}{4}\right)^{\lambda_3}-\left(\frac{1}{4}\right)^{\lambda_3}\right)+\lambda_3\left(\left(\frac{3}{4}\right)^{\lambda_4}-\left(\frac{1}{4}\right)^{\lambda_4}\right)} \tag{2.128}$$

and

$$T=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{IQR}$$

$$=\frac{\lambda_4\left[\left(\frac{7}{8}\right)^{\lambda_3}-\left(\frac{5}{8}\right)^{\lambda_3}+\left(\frac{3}{8}\right)^{\lambda_3}-\left(\frac{1}{8}\right)^{\lambda_3}\right]-\lambda_3\left[\left(\frac{7}{8}\right)^{\lambda_4}-\left(\frac{5}{8}\right)^{\lambda_4}+\left(\frac{3}{8}\right)^{\lambda_4}-\left(\frac{1}{8}\right)^{\lambda_4}\right]}{\lambda_4\left[\left(\frac{3}{4}\right)^{\lambda_3}-\left(\frac{1}{4}\right)^{\lambda_3}\right]+\lambda_3\left[\left(\frac{3}{4}\right)^{\lambda 4}-\left(\frac{1}{4}\right)^{\lambda_4}\right]}.$$

(2.129)

2.6.3.3 L-moments

The L-moments of the distribution have simple forms. From the equations (2.33) through (2.36) the first four L-moments are

$$L_1=\mu=\lambda_1-\frac{1}{\lambda_2}\left[\frac{1}{\lambda_3+1}-\frac{1}{\lambda_4+1}\right], \tag{2.130}$$

$$L_2=\int_0^1(2u-1)Q(u)du$$

$$=\int_0^1(2u-1)\left[\lambda_1+\frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}-1}{\lambda_3}-\frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right)\right]du$$

$$=\frac{1}{\lambda_2}\left[\frac{1}{(\lambda_3+1)(\lambda_3+2)}-\frac{1}{(\lambda_4+1)(\lambda_4+2)}\right], \tag{2.131}$$

$$L_3 = \int_0^1 (6u^2 - 6u + 1)Q(u)du$$

$$= \int_0^1 (6u^2 - 6u + 1)\left[\lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right)\right]du$$

$$= \frac{1}{\lambda_2}\left[\frac{\lambda_3 - 1}{(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)} - \frac{\lambda_4 - 1}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}\right] \quad (2.132)$$

and

$$L_4 = \int_0^1 (20u^3 - 30u^2 + 12u - 1)Q(u)du$$

$$= \int_0^1 (20u^3 - 30u^2 + 12u - 1)\left[\lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right)\right]du$$

$$= \frac{1}{\lambda_2}\left[\frac{(\lambda_3 - 1)(\lambda_3 - 2)}{(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)} - \frac{(\lambda_4 - 1)(\lambda_4 - 2)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}\right]. \quad (2.133)$$

2.6.4 Power-Pareto distribution

This is a flexible distribution with nonnegative support, obtained by multiplying together the quantile functions of the power and Pareto distributions (see Gilchrist 2000). A detailed study on the properties such as tail behaviour, shape of density functions, skewness and kurtosis, approximations with other well known distributions are carried out in Hankin and Lee (2006). The distribution has the quantile function

$$Q(u) = Cu^{\lambda_1}(1-u)^{-\lambda_2}, \quad 0 \le u \le 1, \quad C, \lambda_1, \lambda_2 \ge 0. \quad (2.134)$$

The function in (2.134) is the product of power distribution with quantile function

$$Q_1(u) = \alpha u^{\lambda_1}, \alpha, \lambda_1 > 0, \ 0 \le u \le 1$$

and the Pareto distribution with quantile function

$$Q_2(u) = \sigma(1-u)^{-\lambda_2}, \ \sigma, \ \lambda_2 > 0, \quad 0 \le u \le 1.$$

This is possible that one (but not both) of λ_1 and λ_2 may be zero. The quantile density function of the distribution is

$$q(u) = Cu^{\lambda_1}(1-u)^{-\lambda_2}\left(\frac{\lambda_1}{u}+\frac{\lambda_2}{1-u}\right) \tag{2.135}$$

and accordingly, the density quantile function can be written as

$$f(Q(u)) = \left[Cu^{\lambda_1}(1-u)^{-\lambda_2}\left(\frac{\lambda_1}{u}+\frac{\lambda_2}{1-u}\right)\right]^{-1}.$$

In (2.134) C is the scale parameter, λ_1 and λ_2 are shape parameters with λ_1 controlling the left tail and λ_2, the right tail. See Hankin and Lee (2006) for more details. The shapes of density functions for different choices of parameters are given in Figure 2.3.

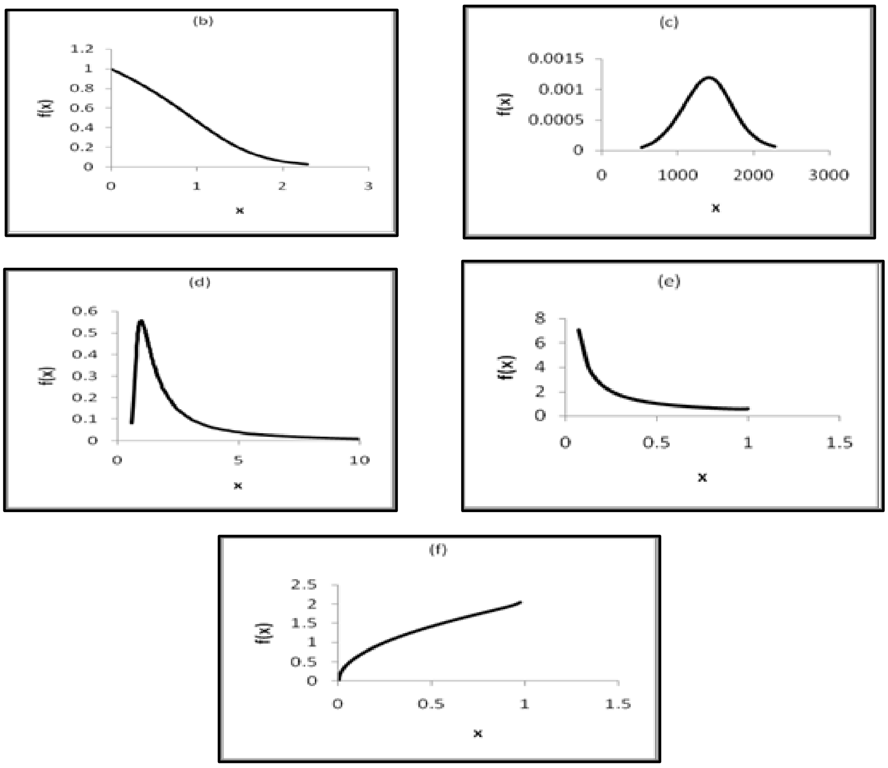


Figure 2.3- Density plots Power-Pareto distribution when $(C, \lambda_1, \lambda_2)$ are (a) $(1, 0.5, 0.01)$, (b) $(1, 1, 0.2)$, (c) $(1500, 0.2, 0.1)$, (d) $(1, 0.1, 1)$, (e) $(1, 0.5, 0.001)$, (f) $(1, 2, 0.001)$.

2.6.4.1 Moments of the distribution

From the equation (2.17), the r^{th} ordinary moment of the distribution is given by the expression

$$\begin{aligned}\mu_r' &= \int_0^1 \left(Q(p)\right)^r dp \\ &= \int_0^1 \left(Cu^{\lambda_1}(1-u)^{-\lambda_2}\right)^r du \\ &= C^r B\left(1+r\lambda_1, 1-r\lambda_2\right).\end{aligned}$$

Note that the r^{th} moment exists only when $\lambda_2 < \frac{1}{r}$. The mean and variance are

$$\begin{aligned}\mu &= CB\left(1+\lambda_1, 1-\lambda_2\right) \\ &= C\frac{\Gamma(1+\lambda_1)\,\Gamma(1-\lambda_2)}{\Gamma(2+\lambda_1-\lambda_2)}\end{aligned}$$

and

$$\begin{aligned}\sigma^2 &= \mu_2' - \mu^2 \\ &= C^2\left[\frac{\Gamma(1+2\lambda_1)\,\Gamma(1-2\lambda_2)}{\Gamma(2+2\lambda_1-2\lambda_2)} - \left(\frac{\Gamma(1+\lambda_1)\,\Gamma(1-\lambda_2)}{\Gamma(2+\lambda_1-\lambda_2)}\right)^2\right].\end{aligned}$$

A detailed study on skewness and kurtosis of the distribution has been made in Hankin and Lee (2006). The measures of skewness and kurtosis based on the ordinary moments exist over the range $\lambda_1 > 0,\ 0 \le \lambda_2 < \frac{1}{4}$. Minimum skewness is -2 attained at $\lambda_2 = 0$ and minimum kurtosis is attained at $\lambda_2 = 0$. Both the measures increase with respect to λ_1 and λ_2. The range of possible kurtosis values increases with increasing skewness. They pointed out that the distribution is more suitable for positively skewed data and the possible skewness and kurtosis ranges for the gamma, weibull and log-normal distributions lie entirely within the range of power-Pareto distribution.

2.6.4.2 Percentiles

We have seen that the usual characteristics based on the conventional moments have restrictions due to the nonexistence of the moments throughout the parameter space. The measures based on percentiles are good alternatives to those based on moments. They have simple closed expressions, and can be used in situations where the characteristics based on ordinary moments fail. The usual measures based on percentiles using equations (2.12) through (2.15) are

$$\text{median, } M = Q\left(\frac{1}{2}\right) = C\left(\frac{1}{2}\right)^{\lambda_1-\lambda_2},$$

the inter-quartile range

$$IQR = Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) = C\left(\frac{1}{4}\right)^{\lambda_1-\lambda_2}\left(3^{\lambda_1} - 3^{-\lambda_2}\right),$$

the Bowley coefficient of skewness

$$\begin{aligned} S &= \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2M}{IQR} \\ &= \frac{3^{\lambda_1} + 3^{-\lambda_2} - 2^{\lambda_2-\lambda_1+1}}{3^{\lambda_1} - 3^{-\lambda_2}} \\ &= 1 - \frac{2^{\lambda_2-\lambda_1+1}}{3^{\lambda_1} - 3^{-\lambda_2}} \end{aligned}$$

and the measure of kurtosis based on percentiles

$$\begin{aligned} T &= \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{IQR} \\ &= \frac{2^{\lambda_2-\lambda_1}\left(7^{\lambda_1} - 5^{\lambda_1}3^{-\lambda_2} + 3^{\lambda_1}5^{-\lambda_2} - 7^{-\lambda_2}\right)}{3^{\lambda_1} - 3^{-\lambda_2}}. \end{aligned}$$

2.6.4.3 L-moments

The first four L-moments of the distribution obtained using equations (2.33) through (2.36) are

$$L_1 = \mu = CB(\lambda_1 + 1, 1 - \lambda_2) \tag{2.136}$$

$$\begin{aligned} L_2 &= \int_0^1 (2u-1)\left(Cu^{\lambda_1}(1-u)^{-\lambda_2}\right)du \\ &= C\left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2 + 2}\right)B(\lambda_1 + 1, 1 - \lambda_2) \end{aligned} \tag{2.137}$$

$$L_3 = \int_0^1 (6u^2 - 6u + 1)Q(u)du$$

$$= \int_0^1 (6u^2 - 6u + 1)\left(Cu^{\lambda_1}(1-u)^{-\lambda_2}\right)du$$

$$= C\ \frac{4\lambda_1\lambda_2 - \lambda_1 + \lambda_2 + \lambda_1^{\ 2} + \lambda_2^{\ 2}}{(\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_2 + 3)}B(\lambda_1 + 1, 1 - \lambda_2) \tag{2.138}$$

$$L_4 = \int_0^1 (20u^3 - 30u^2 + 12u - 1)Q(u)du$$

$$= \int_0^1 (20u^3 - 30u^2 + 12u - 1)\left(Cu^{\lambda_1}(1-u)^{-\lambda_2}\right)du$$

$$= C\ \frac{(\lambda_1 + \lambda_2)\left(\lambda_1^{\ 2} + \lambda_2^{\ 2} + 8\lambda_1\lambda_2 - 3(\lambda_1 - \lambda_2) + 2\right)}{(\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_2 + 3)(\lambda_1 - \lambda_2 + 4)}B(\lambda_1 + 1, 1 - \lambda_2). \tag{2.139}$$

L-moments of all order exist when $\lambda_2 < 1$. Thus the utility of L-moments is better than that of ordinary moments, as in the case of the latter choices of the parameter λ_2 shrinks as the order of moments increases. The L-skewness and L-kurtosis have simple expressions

$$\tau_3 = \frac{L_3}{L_2} = \frac{\lambda_1^{\ 2} - \lambda_1 + 4\lambda_1\lambda_2 + \lambda_2 + \lambda_2^{\ 2}}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2 + 3)}$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{\lambda_1^{\ 2} + \lambda_2^{\ 2} + 8\lambda_1\lambda_2 - 3(\lambda_1 - \lambda_2) + 2}{(\lambda_1 - \lambda_2 + 3)(\lambda_1 - \lambda_2 + 4)}.$$

2.6.5 van-Staden & Loots model

van-Staden & Loots (2009) have proposed a new four-parameter distribution as a parameterization of the GLD. Without proposing a new estimation technique, they derived this parameterization to overcome the difficulties persist in the estimation procedures of the generalized lambda distribution. They generated the new model by considering the generalized Pareto model with quantile function

$$Q_1(u) = \begin{cases} \frac{-1}{\lambda_4}\left((1-u)^{\lambda_4} - 1\right), & \lambda_4 \neq 0 \\ -\ln(1-u) & , \lambda_4 = 0 \end{cases}$$

and its reflexion

$$Q_2(u) = \begin{cases} \frac{1}{\lambda_4}\left(u^{\lambda_4} - 1\right), & \lambda_4 \neq 0 \\ \log u & , \lambda_4 = 0. \end{cases}$$

Taking the weighted sum of the above quantile function lead to the generation of the new model by introducing the location parameter λ_1. The quantile function of the model is

$$Q(u) = \lambda_1 + \lambda_2\left[(1-\lambda_3)\frac{u^{\lambda_4} - 1}{\lambda_4} - \lambda_3\frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right], \ \lambda_2 > 0 . \qquad (2.140)$$

When $\lambda_4 = 0$, (2.140) becomes the quantile function of the skew-logistic distribution,

$$Q(u) = (1-\lambda_3)\ln u + \lambda_3\left(-\ln(1-u)\right)$$

and becomes a uniform distribution when $\lambda_4 = 1$ and also when $\lambda_3 = \frac{1}{2}$ and $\lambda_4 = 2$. The support of the distribution is as follows:

Table 2.2 – Ranges of Van-Staden & Loots model.

Ranges of parameters	Supports
$\lambda_3 = 0,\ \lambda_4 = 0$	$(-\infty, \lambda_1)$
$\lambda_3 = 0,\ \lambda_4 > 0$	$\left(\lambda_1 - \frac{\lambda_2}{\lambda_4}, \lambda_1\right)$
$0 < \lambda_3 < 1,\ \lambda_4 \le 0$	$(-\infty, \infty)$
$0 < \lambda_3 < 1,\ \lambda_4 > 0$	$\left[\lambda_1 - \frac{(1-\lambda_3)\lambda_2}{\lambda_4}, \lambda_1 + \frac{\lambda_3\lambda_2}{\lambda_4}\right]$
$\lambda_3 = 1,\ \lambda_4 \le 0$	$[\lambda_1, \infty)$
$\lambda_3 = 1,\ \lambda_4 > 0$	$\left[\lambda_1, \lambda_1 + \frac{\lambda_2}{\lambda_4}\right]$

The condition for the model to be a life distribution is $\lambda_1 - \frac{\lambda_2(1-\lambda_3)}{\lambda_4} \ge 0$.

This gives members with both finite and infinite support, depending upon whether λ_4 is positive or negative. The quantile density function of the distribution is

$$q(u) = \lambda_2\left[(1-\lambda_3)u^{\lambda_4 - 1} + \lambda_3(1-u)^{\lambda_4 - 1}\right]. \tag{2.141}$$

The density quantile function has the expression

$$f(Q(u)) = \frac{1}{\lambda_2}\left[(1-\lambda_3)u^{\lambda_4 - 1} + \lambda_3(1-u)^{\lambda_4 - 1}\right]^{-1}.$$

See Figure 2.4 for the shapes of the density function for some selected set of parameter values.

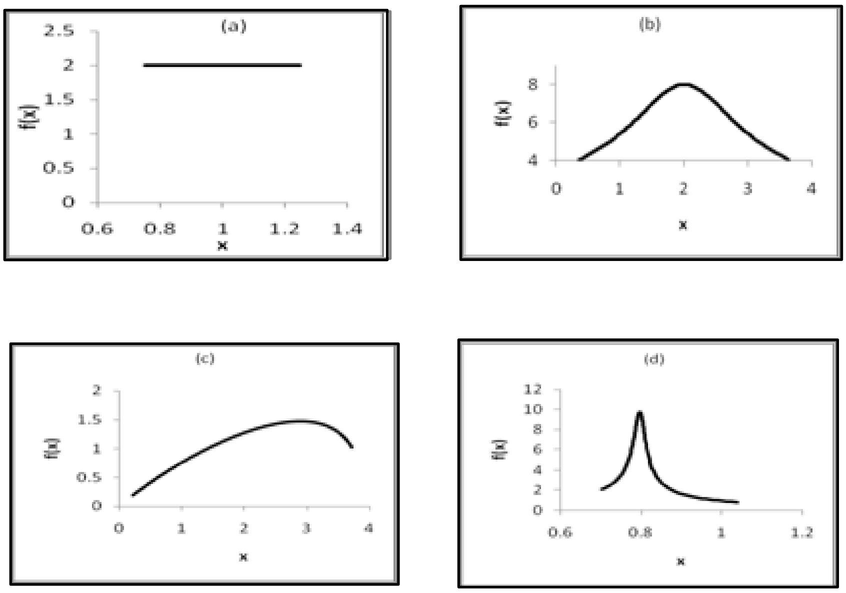


Figure 2.4- Density plots of the GLD proposed by Staden and Loots (2009) when $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are (a) $(1, 1, 0.5, 2)$, (b) $(2, 1, 0.5, 3)$, (c) $(3, 2, 0.25, 0.5)$ and (d) $(1, 2, 0.1, -1)$.

2.6.5.1 Moments

The r^{th} ordinary moment of the distribution is given by

$$E\left(X^r\right) = \int_0^1 \left(Q(u)\right)^r du.$$

In particular the mean

$$\begin{aligned}
\mu &= \int_0^1 Q(u)du \\
&= \int_0^1 \left(\lambda_1 + \lambda_2 \left[(1-\lambda_3)\frac{u^{\lambda_4}-1}{\lambda_4} - \lambda_3 \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right]\right) du \\
&= \lambda_1 - \frac{\lambda_2}{1+\lambda_4}(1-2\lambda_3)
\end{aligned}$$

and the variance

$$\sigma^2 = \int_0^1 \left(Q(u)\right)^2 du - \mu^2$$

$$= \int_0^1 \left(\lambda_1 + \lambda_2 \left[(1-\lambda_3) \frac{u^{\lambda_4} - 1}{\lambda_4} - \lambda_3 \frac{(1-u)^{\lambda_4} - 1}{\lambda_4} \right] \right)^2 du - \mu^2$$

$$= \frac{\lambda_2^2}{(1+\lambda_4)^2} \left[\frac{\lambda_3^2 + (1-\lambda_3)^2}{1+2\lambda_4} - \frac{2\lambda_3(1-\lambda_3)}{\lambda_3} \left((1+\lambda_4)^2 B(1+\lambda_3, 1+\lambda_4) - 1 \right) \right].$$

The higher order ordinary moments of the distribution have lengthy complicated forms, contains special functions, which make skewness-kurtosis analysis of the distribution using ordinary moments tedious.

2.6.5.2 L-moments

The L-moments of the distribution have closed forms, which can be used as a better alternative to the ordinary moments. L-moments of all order exist when $\lambda_4 > -1$. The general expression for the r^{th} L-moment for $r = 3, 4, \ldots$ is derived in van-Staden and Loots (2009). The L-moments are

$$L_1 = \mu$$

$$L_2 = \frac{\lambda_2}{(\lambda_4 + 1)(\lambda_4 + 2)}$$

and

$$L_r = \lambda_2 (1 - 2\lambda_3)^S \frac{\prod_{i=1}^{r-2} (\lambda_4 - i)}{\prod_{i=1}^{r} (\lambda_4 + i)},$$

where

$$S = \begin{cases} 1, \, r \text{ is odd} \\ 0, \, r \text{ is even} \end{cases}$$

In particular

$$L_3 = \frac{\lambda_2 (1 - 2\lambda_3)(\lambda_4 - 1)}{(\lambda_4 + 1)(\lambda_4 + 2)(\lambda_4 + 3)}$$

and

$$L_4 = \frac{\lambda_2(\lambda_4 - 1)(\lambda_4 - 2)}{(\lambda_4 + 1)(\lambda_4 + 2)(\lambda_4 + 3)(\lambda_4 + 4)}.$$

Hence the L-skewness and L-kurtosis are

$$\tau_3 = \frac{L_3}{L_2} = \frac{(\lambda_4 - 1)(1 - 2\lambda_3)}{(\lambda_4 + 3)}$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{(\lambda_4 - 1)(\lambda_4 - 2)}{(\lambda_4 + 3)(\lambda_4 + 4)}.$$

van-Staden and Loots (2009) pointed out that as in the case of lambda distributions discussed earlier, there is no unique (λ_3, λ_4) pair for a given value of (τ_3, τ_4). When $\lambda_3 = \frac{-1}{2}$, the distribution is symmetric. L-skewness covers the permissible span $(-1, 1)$ and kurtosis is independent of λ_3 with a minimum attained at $\lambda_4 = \sqrt{6} - 1$.

2.6.5.3 Percentiles

Different percentile measures of the distribution are obtained from equations (2.12) through (2.15).

$$\text{Median, } M = Q\left(\frac{1}{2}\right)$$

$$= \lambda_1 + \frac{\lambda_2(1 - 2\lambda_3)}{\lambda_4}\left(\left(\frac{1}{2}\right)^{\lambda_4} - 1\right),$$

$$IQR = Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) = \frac{\lambda_2(3^{\lambda_4} - 1)}{\lambda_4 4^{\lambda_4}},$$

$$S = \frac{Q\left(\frac{1}{4}\right) + Q\left(\frac{3}{4}\right) - 2M}{IQR} = \frac{(1 - 2\lambda_3)(1 + 3^{\lambda_4} - 2^{\lambda_4 + 1})}{3^{\lambda_4} - 1}$$

and

$$T = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{IQR}$$
$$= \frac{(1-2\lambda_3)2^{\lambda_4}\left(7^{\lambda_4} + 5^{\lambda_4} + 3^{\lambda_4} + 1\right)}{3^{\lambda_4} - 1}.$$

2.6.5.4 Estimation

The method of ordinary moments to estimate the parameters is difficult to use as the third and fourth moments have complicated expressions. The simple closed forms of the L-moments allow estimating the parameters easily. van-Staden and Loots (2009) have derived estimates of the parameters based on L-moments. Using the sample L-kurtosis t_4, the estimate of λ_4 is

$$\hat{\lambda}_4 = \frac{3 + 7t_4 \pm \sqrt{t_4^{\,2} + 98t_4 + 1}}{2(1 - t_4)}. \tag{2.142}$$

$\hat{\lambda}_4$ can be utilized to estimate λ_3 by

$$\hat{\lambda}_3 = \begin{cases} \frac{1}{2}\left[1 - \frac{t_3\left(\hat{\lambda}_4 + 3\right)}{\hat{\lambda}_4 - 1}\right], & \hat{\lambda}_4 \neq 1 \\ \frac{1}{2}, & \hat{\lambda}_4 = 1, \end{cases} \tag{2.143}$$

where t_3 is the sample L-skewness. λ_1 and λ_2 can be estimated using $\hat{\lambda}_3$ and $\hat{\lambda}_4$ by

$$\hat{\lambda}_2 = l_2\left(\hat{\lambda}_4 + 1\right)\left(\hat{\lambda}_4 + 2\right) \tag{2.144}$$

and

$$\hat{\lambda}_1 = l_1 + \frac{\hat{\lambda}_2\left(1 - 2\hat{\lambda}_3\right)}{\hat{\lambda}_4 + 1}, \tag{2.145}$$

where l_1 and l_2 are the first two sample L-moments.

The method of percentiles can also be used to estimate the parameters. The ratio of the Moor's measure and Galton's measure is the function of λ_4 alone, using their sample counterparts we can estimate λ_4. Substituting the estimate of λ_4 in Moor's measure, the estimate of λ_3 can easily be derived using the sample counterparts of Moor's measure. These two estimates can be utilized to estimate the other two parameters with the expressions of the median and IQR along with their sample counterparts.

2.7 Q-Q Plot

Once a candidate distribution is chosen to represent the data and its fit is obtained by some method, one has to see whether the distribution is valid for the data situation under consideration. Though there are many methods of validation, one that is relevant to modelling with quantiles is the Q-Q plot. The Q-Q plot is the graph of $\left(Q(u_r), x_{r:n}\right), r = 1, 2, ..., n$ and $u_r = \dfrac{r-0.5}{n}$. Here $x_{r:n}$ denotes the r^{th} ordered observation, when observations are arranged in ascending order of magnitude. For application purposes we replace $Q(u_r)$ with the fitted quantile function. In the ideal case, the graph should show a straight line that bisects the axes of coordinates. However, since the sample is random and the fitted values of $Q(u)$ are points lying approximately along the line specified above, it can be taken as indication of a satisfactory model. This method will be used in the subsequent chapters when the sample size is small enough to create sufficient number of classes to employ other conventional tests or when there are other difficulties in carrying all the latter.

2.8 Order relations

There is considerable growth in the study of stochastic orders in recent years. Different order relations have been developed using measures in connection with many fields such as reliability, economics, queuing theory, survival analysis, insurance, operations research, etc.. Details of order relations and related results are well documented in Shaked and Shanthikumar (2007). In the present section, we give definitions of some order relations that are useful for the discussions in the subsequent chapters. Equivalent conditions, implications, etc. of the given definitions are available in Shaked and Shanthikumar (2007).

2.8.1 Usual stochastic order

Let X and Y be two random variables with distribution functions $F_X(x)$ and $F_Y(x)$, and quantile functions $Q_X(u)$ and $Q_Y(u)$ respectively. We say that X is smaller than Y in usual stochastic order, denoted by $X \le_{st} Y$ if and only if

$$\bar{F}_X(x) \le \bar{F}_Y(x), \quad \forall x$$

or equivalently

$$Q_X(u) \le Q_Y(u), \qquad \forall\, u \in (0,1).$$

2.8.2 Dispersive ordering

With the above notations, if

$$Q_X(v) - Q_X(u) \le Q_Y(v) - Q_Y(u), \qquad 0 < u \le v < 1,$$

then X is said to be smaller than Y in the dispersive order, and denoted by $X \le_{disp} Y$. As this ordering consider the difference between two quantiles, the comparison corresponds to the variability of X and Y .

2.8.3 Convex order

(a) For two random variables X and Y if $E\left(\phi(X)\right) \le E\left(\phi(Y)\right)$ for all convex functions $\phi : R \to R,$ provided the expectations exist, then X is said to be smaller than Y in the convex order. It is denoted by $X \le_c Y$. In terms of quantile function, $X \le_c Y$ if and only if

$$\int_0^u Q_X(p)dp \ge \int_0^u Q_Y(p)dp$$

or

$$\int_u^1 Q_X(p)dp \le \int_u^1 Q_Y(p)dp.$$

(b) Some other orders are defined using the concept of convex ordering. For example, if

$$\left[X - E(X)\right] \le_c \left[Y - E(Y)\right],$$

the order is called dilation order, which is denoted by $X \le_{dil} Y$. $X \le_{dil} Y$ if and only if

$$\frac{1}{1-u}\int_u^1 \left[Q_X(p) - Q_Y(p)\right]dp \le \int_0^1 \left[Q_X(p) - Q_Y(p)\right]dp.$$

(c) We say that X is less than Y in Lorenz order if

$$\frac{X}{E(X)} \le_c \frac{Y}{E(Y)},$$

and is denoted by $X \le_{lorenz} Y$. The Lorenz curve corresponding to a nonnegative random variable X is defined as

$$L_X(u) = \frac{1}{\mu}\int_0^u Q(p)dp.$$

Let $L_X(u)$ and $L_Y(u)$ be the Lorenz curves of X and Y. Then $X \le_{lorenz} Y$ if and only if

$$L_X(u) \ge L_Y(u), \ \forall \ u \in (0,1).$$

2.8.4 Transform orders

(a) We say that X is smaller than Y in the convex transform order, denoted by $X \leq_{cx} Y$, if $G^{-1}F(x)$ is convex in x on the support of F.

(b) Another order in this category is the star order. The random variable X is smaller than Y in the star order if $G^{-1}F(x)$ is star-shaped, that is $\frac{G^{-1}F(x)}{x}$ increases for $x \geq 0$. This order is denoted by $X \leq_{*} Y$. In quantile form, $X \leq_{*} Y$ if and only if

$$\frac{Q_Y(u)}{Q_X(u)} \text{ is increasing for } u \in (0,1).$$

(c) We say that X is smaller than Y in the superadditive order, written as $X \leq_{su} Y$, where $\leq_{su}$ stands for the superadditive order if $G^{-1}F(x)$ is super additive in x. That is,

$$G^{-1}F(x+y) \geq G^{-1}F(x) + G^{-1}F(y),$$

for all $x \geq 0$ and $y \geq 0$.

2.8.5 The monotone convex and monotone concave orders

Two random variables X and Y are such that $E\left(\phi(X)\right) \leq E\left(\phi(Y)\right)$ for all increasing convex (concave) functions $\phi : R \to R$, provided the expectations exist, then X is smaller than Y in the increasing convex (concave) order, denoted by $X \leq_{icx} Y$ ($X \leq_{icv} Y$). Similarly decreasing convex (concave) order may be defined for decreasing convex (concave) functions ϕ (denoted by $X \leq_{dcx} (\leq_{dcv}) Y$).

2.8.6 TTT order and excess wealth order

Recall the definition of total time on test transform given in (2.98), given by

$$H_F^{-1}(u) = \int_0^{F^{-1}(u)} F(t)\,dt.$$

Another transform that is closely related to the TTT is the excess wealth transform, defined as

$$W_X(u) = \int_{F^{-1}(u)}^{\infty} \bar{F}(x)dx \quad 0 \le u \le 1.$$

Notice that

$$W_X(u) = \mu - H_F^{-1}(u).$$

For two random variables X and Y with distribution function $F(x)$ and $G(x)$, X is said to be smaller than Y in TTT order (denoted by $X \le_{ttt} Y$) if

$$\int_0^{F^{-1}(u)} \bar{F}(x)dx \le \int_0^{G^{-1}(u)} \bar{G}(x)dx \text{ for all } u \in (0,1).$$

If $W_X(u) \le W_Y(u)$, then we say that X is smaller than Y in excess wealth order (which is also called the right spread order), denoted by $X \le_{EW} Y$.

2.8.7 Reliability orders

The stochastic orders defined so far has no explicit connection with reliability concepts considered in Sections 2.2 and 2.3. We now consider some partial orders based on reliability concepts that enable the comparison of lifetime distribution.

(i) If X and Y are lifetime random variables with absolutely continuous distribution functions, we say that X is smaller than Y in hazard rate order, denoted by $X \le_{hr} Y$ if

$$h_X(t) \ge h_Y(t),$$

for all t, where $h_X(t)$ and $h_Y(t)$ are the hazard rates of X and Y respectively.

(ii) We say that X is smaller than Y in mean residual life, denoted by $X \leq_{mrl} Y$, if

$$M_X(t) \leq M_Y(t),$$

for all $t > 0$, where $M_X(t)$ and $M_Y(t)$ are mean residual functions of X and Y respectively.

(iii) Another stochastic order that involves the mean residual life is the harmonic mean residual order. The random variable X is said to be smaller than Y in harmonically mean residual life orders, denoted by $X \leq_{hmrl} Y$ if and only if

$$\left[\frac{1}{x}\int_0^x \frac{dt}{m_X(t)}\right]^{-1} \leq \left[\frac{1}{x}\int_0^x \frac{dt}{m_Y(t)}\right]^{-1}.$$

(iv) Recall the definitions of variance residual life function and its quantile version given in Section 2.2.3 and 2.4.3. We say that the random variable X is smaller than Y in variance residual life denoted by $X \leq_{vrl} Y$ if

$$\sigma_X^{\,2}(x) \leq \sigma_Y^{\,2}(x) \text{ for all } x > 0.$$

There exist many other orders in the categories discussed above. We have given only those concepts that will be used in the forthcoming chapters. A general discussion of various stochastic orders and their relationships are available in Shaked and Shanthikumar (2007).

Chapter 3

Quantile function models

3.1 Introduction

We pointed out in Chapter 1 that one of the objectives of quantile-based reliability analysis is to make use of quantile functions as models in lifetime data analysis. In the present chapter, we discuss the characteristics of some quantile function models and demonstrate their applicability in lifetime data analysis. The basic characteristics of the models such as moments, percentiles, L-moments and measures of skewness and kurtosis, etc. were discussed in Chapter 2. Also presented here are the commonly used reliability measures in terms of quantile function for each model, which are utilized latter to describe reliability properties and ageing pattern of the models.

The distributions considered are the generalized lambda distribution of Ramberg and Schmeiser, the generalized Tukey family of Freimer, Kollia, Mudholkar and Lin, the four-parameter distribution of van-Staden and Loots, the power Pareto model, reviewed in Chapter 2 and the Govindarajulu's model proposed by Govindarajulu (1977). A major objective of this chapter is to introduce the above distributions as alternatives to the conventional distributions employed in lifetime data analysis. Their adequacy to represent real life situations are examined in the light of various data sets. We have given special consideration to Govindarajulu model as it is a simple model with only two parameters and has competing features in terms of model parsimony. Although introduced earlier, its properties and applications in the context of reliability theory do not appear to have been discussed in the literature. In the last major section of this chapter, we introduce a new technique to derive new classes of life distributions using the properties of Parzen's score function and tail exponent function.

3.2 Lambda distributions *

3.2.1 Generalized lambda distributions

When used as a lifetime distribution, the generalized lambda family specified by the quantile function

$$Q(u)=\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}-(1-u)^{\lambda_4}\right),\ \ 0\le u\le 1 \tag{3.1}$$

should be such that the parameters satisfy the condition $\lambda_1-\frac{1}{\lambda_2}\ge 0$. Thus from Table 2.1 the supports of the distribution for different ranges of λ_2, λ_3 and λ_4 become $\left(\lambda_1-\frac{1}{\lambda_2},\infty\right)$, $\left(\lambda_1-\frac{1}{\lambda_2},\lambda_1+\frac{1}{\lambda_2}\right)$, $\left(\lambda_1,\lambda_1+\frac{1}{\lambda_2}\right)$, $\lambda_1>0$, $\left(\lambda_1-\frac{1}{\lambda_2},\lambda_2\right)$, $\lambda_2>0$ and (λ_1,∞), $\lambda_1>0$. Hence the life distributions have members with finite and infinite supports. We present below some of the properties that have not been discussed in literature.

3.2.1.1 Order Statistics

We have discussed the distribution and moments of order statistics in Section 2.1.5. The moments of order statistics of generalized lambda distribution have closed forms. Recall from Equation 2.30 that

**some parts of the materials in this section have appeared in Nair and Vineshkumar (2010), Journal of Statistical Planning and Inference – see reference no. 93.*

the expectation of the of r^{th} order statistic of a random sample of size n from a population is

$$E\left(X_{r:n}\right)=\frac{n!}{(r-1)!(n-r)!}\int_0^1 u^{r-1}\left(1-u\right)^{n-r}Q\left(u\right)du.$$

In the case of generalized lambda distribution, the above expression takes the form

$$\begin{aligned}E\left(X_{r:n}\right)&=\frac{n!}{(r-1)!(n-r)!}\int_0^1 u^{r-1}\left(1-u\right)^{n-r}\left(\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}-\left(1-u\right)^{\lambda_4}\right)\right)du\\&=\lambda_1+\frac{1}{\lambda_2}\frac{\Gamma(\lambda_3+r)}{\Gamma(r)}\frac{\Gamma(n+1)}{\Gamma(\lambda_3+n+1)}+\frac{1}{\lambda_2}\frac{\Gamma(n+\lambda_4-r+1)\,\Gamma(n+1)}{\Gamma(n+\lambda_4+1)\,\Gamma(n-r)}.\end{aligned}\tag{3.2}$$

In particular, the expectation of first and n^{th} order statistics are given by

$$E\left(X_{1:n}\right)=\lambda_1+\frac{n!}{\lambda_2\left(\lambda_3+1\right)_{(n)}}-\frac{n!}{\lambda_2\left(\lambda_4+n\right)}\tag{3.3}$$

and

$$E\left(X_{n:n}\right)=\lambda_1+\frac{n!}{\lambda_2\left(\lambda_3+n\right)}-\frac{n!}{\lambda_2\left(\lambda_4+1\right)_{(n)}},\tag{3.4}$$

where $\left(n\right)_{(r)}=n\left(n+1\right)...\left(n+r-1\right)$ is the ascending factorial function. Using (2.28) and (2.29), we can obtain the distribution of the order statistics $X_{1:n}$ and $X_{n:n}$ as

$$\begin{aligned}Q_1\left(u\right)&=Q\left(1-\left(1-u_1\right)^{1/n}\right)\\&=\lambda_1+\frac{1}{\lambda_2}\left(\left(1-\left(1-u\right)^{\frac{1}{n}}\right)^{\lambda_3}-\left(1-u\right)^{\frac{\lambda_4}{n}}\right)\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}Q_n\left(u\right)&=Q\left(u^{1/n}\right).\\&=\lambda_1+\frac{1}{\lambda_2}\left(u^{\frac{\lambda_3}{n}}-\left(1-u^{\frac{1}{n}}\right)^{\lambda_4}\right).\end{aligned}\tag{3.6}$$

Recall that the implications of the order statistics in reliability modelling have been pointed out in Section 2.1.5. Accordingly the expectations in equations (3.2) through (3.4) provide the expected life of the *r*-out-of-*n* system, series system and parallel system respectively. Further (3.5) and (3.6) provide nice functional forms for further manipulations.

3.2.1.2 Reliability functions

We can make use of the definitions of quantile-based concepts given in Section 2.4 to obtain the reliability functions of the distribution. The hazard quantile function has the simple form

$$\begin{aligned} H(u) &= \left((1-u)q(u)\right)^{-1} \\ &= \frac{\lambda_2}{(1-u)\left(\lambda_3 u^{\lambda_3-1} + \lambda_4 (1-u)^{\lambda_4-1}\right)}. \end{aligned} \tag{3.7}$$

The analysis of $H(u)$ that reveals the potential of (3.7) in describing the physical characteristics of ageing will be taken up in Chapter 4.

From equation (2.76), the mean residual quantile function has the expression

$$\begin{aligned} M(u) &= \frac{1}{1-u}\int_u^1 (1-p)\, q(p)\,dp \\ &= \frac{1}{1-u}\int_u^1 (1-p)\frac{1}{\lambda_2}\left(\lambda_3 p^{\lambda_3-1} + \lambda_4 (1-p)^{\lambda_4-1}\right)dp \\ &= \frac{1}{\lambda_2(1-u)}\left(\frac{\lambda_4}{\lambda_4+1}(1-u)^{\lambda_4+1} - \frac{1}{\lambda_3+1}\left(1-\lambda_3 u^{\lambda_3+1}\right) - u^{\lambda_3}\right). \end{aligned}$$

The variance residual quantile function is obtained using the equation (2.81) as

$$V(u) = \frac{1}{1-u}\int_u^1 Q^2(p)\,dp - \left(\frac{1}{1-u}\int_u^1 Q(p)\,dp\right)^2$$

$$= A_1(u) - A_2^2(u),$$

where

$$A_1(u) = \frac{1}{1-u}\int_u^1 \left[\lambda_1 + \frac{1}{\lambda_2}\left(p^{\lambda_3} - (1-p)^{\lambda_4}\right)\right]^2 dp$$

$$= \frac{1}{\lambda_2^{\,2}(1-u)}\left(\frac{1-u^{2\lambda_3+1}}{2\lambda_3+1} + \frac{(1-u)^{2\lambda_4+1}}{2\lambda_4+1} - 2B_{1-u}(\lambda_3+1,\, \lambda_4+1)\right),$$

$$A_2(u) = \frac{1}{1-u}\int_u^1 \left[\lambda_1 + \frac{1}{\lambda_2}\left(p^{\lambda_3} - (1-p)^{\lambda_4}\right)\right] dp$$

$$= \frac{1}{\lambda_2(1-u)}\left[\frac{1-u^{\lambda_3+1}}{\lambda_3+1} - \frac{(1-u)^{\lambda_4+1}}{\lambda_4+1}\right],$$

and $B_{1-u}(m,n) = \int_u^1 t^{m-1}(1-t)^{n-1}\,dt$ is the incomplete beta function.

The generalized lambda distribution is determined as

$$a - \frac{d}{du}(1-u)\mu(u),$$

where $\mu(u)$ is the quantile form of the vitality function $E(X|X > x)$, which has the expression $\mu(u) = \frac{1}{1-u}\int_u^1 Q(p)dp$ when it satisfies

$$(1-u)\mu(u) = a + b\left(\frac{1-u^c}{c} - \frac{(1-u)^d}{d}\right),$$

for real a, b, c and d for which $Q(0) \geq 0$.

From the equation (2.86), the α^{th} percentile residual quantile function of the distribution has the expression

$$P_\alpha(u) = Q[1-(1-\alpha)(1-u)] - Q(u)$$

$$= \frac{1}{\lambda_2}\left[\left(u + \alpha(1-u)\right)^{\lambda_3} - u^{\lambda_3} + (1-u)^{\lambda_4}\left(1-(1-\alpha)^{\lambda_4}\right)\right].$$

Various functions in reversed time, following the notations in Chapter 2 are the reversed hazard quantile function derived using (2.87).

$$\begin{aligned}\Lambda(u) &= \left(u\,q(u)\right)^{-1}\\ &= \left[u\frac{1}{\lambda_2}\left(\lambda_3 p^{\lambda_3-1}+\lambda_4(1-p)^{\lambda_4-1}\right)\right]^{-1}\\ &= \lambda_2\left[\lambda_3 u^{\lambda_3}+\lambda_4 u(1-u)^{\lambda_4-1}\right]^{-1},\end{aligned}$$

from the equation (2.91), the reversed mean residual quantile function

$$\begin{aligned}R(u) &= u^{-1}\int_0^u p\,q(p)\,dp\\ &= u^{-1}\int_0^u p\left(\lambda_3 p^{\lambda_3-1}+\lambda_4(1-p)^{\lambda_4-1}\right)dp\\ &= \frac{1}{\lambda_2 u}\left[\frac{\lambda_3}{\lambda_3+1}u^{\lambda_3+1}+\frac{1}{\lambda_4+1}\left(1-(1-u)^{\lambda_4}(1+\lambda_4 u)\right)\right]\end{aligned}$$

and the reversed variance residual quantile function obtained using equation (2.95)

$$\begin{aligned}D(u) &= \frac{1}{u}\int_0^u Q^2(p)\,dp-\left(\frac{1}{u}\int_0^u Q(p)\,dp\right)^2\\ &= B_1(u)-B_2^2(u),\end{aligned}$$

where

$$B_1(u)=\frac{1}{\lambda_2^2 u}\left[\frac{u^{2\lambda_3+1}}{2\lambda_3+1}-\frac{(1-u)^{2\lambda_4+1}-1}{2\lambda_4+1}-2B_u(2\lambda_3+1,\lambda_4+1)\right]$$

and

$$B_2(u)=\frac{1}{\lambda_2 u}\left[\frac{u^{\lambda_3+1}}{\lambda_3+1}-\frac{(1-u)^{\lambda_4+1}-1}{\lambda_4+1}\right].$$

3.2.1.3 Application to lifetime data

To ascertain the application of the distribution in lifetime modelling, we fitted the model to the data on the time of the first external leakage of 32 centrifugal pumps cited in Lai and Xie (2006). We used the method of L-moments for estimating the parameters by equating the first four L-moments of the distribution with the sample counterparts. The sample L-moments are calculated using the formula

$$l_i = \frac{1}{n}\sum_{k=0}^{i-1}\frac{(-1)^{i-k}(i+k)!}{k!(i-k)!}\sum_{j=1}^{n}\frac{(j-1)...(j-k)}{(n-1)(n-2)...(n-k)}X_{k:n},\ i=1,2,3,4. \tag{3.8}$$

that give

$$l_1 = 5024.7187,\ l_2 = 1644.6764,\ l_3 = 201.5913 \text{ and } l_4 = 110.1990.$$

The equations of estimation are

$$\lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3+1} - \frac{1}{\lambda_4+1}\right) = 5024.7187,$$

$$\frac{1}{\lambda_2}\left(\frac{\lambda_3}{(\lambda_3+1)(\lambda_3+2)} + \frac{\lambda_4}{(\lambda_4+1)(\lambda_4+2)}\right) = 1644.6764,$$

$$\frac{1}{\lambda_2}\left(\frac{\lambda_3(\lambda_3-1)}{(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)} - \frac{\lambda_4(\lambda_4-1)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}\right) = 201.5913,$$

$$\frac{1}{\lambda_1}\left(\frac{\lambda_3(\lambda_3-1)(\lambda_3-2)}{(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)} + \frac{\lambda_4(\lambda_4-1)(\lambda_4-2)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}\right) = 110.1990.$$

The expressions in the left hand side of the above equations are given in (2.116) through (2.119). The above nonlinear equations in $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are solved using the software package Mathematica. There are multiple solutions to the equations. Of these solutions, the sets that do not conform to the condition $\lambda_1 - \frac{1}{\lambda_2} \geq 0$ and those that do not lie in the prescribed parameter space were discarded. From among the legitimate parameter values, one that provided best fit on the basis of the Q-Q plots were selected as the final estimate. This leads to the estimates

$$\hat{\lambda}_1 = 6242.88,\ \hat{\lambda}_2 = 0.000175,\ \hat{\lambda}_3 = 5.17987,\ \hat{\lambda}_4 = 1.67014.$$

We utilize the Q-Q plot explained in Section 2.7 to ascertain the goodness of fit. The Q-Q plot associated with the above solution shows that the generalized lambda distribution with the above designated parameter values provides a reasonable visual validation of the distribution to the data. See Figure 3.1. Further analysis of the data using the model can be carried out with the aid of the formula for various functions associated with the lifetime.

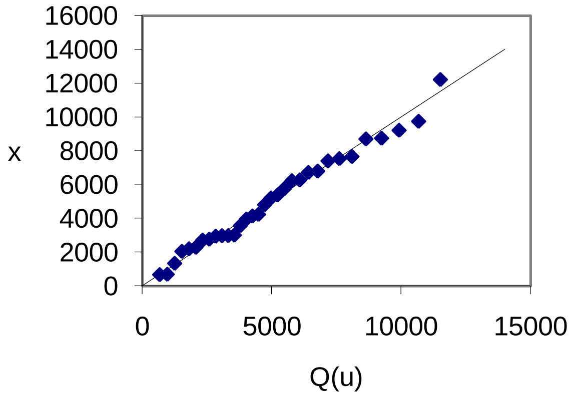


Figure 3.1 - Q-Q Plot

3.2.2 Generalized Tukey lambda distribution

Now we consider the generalized Tukey lambda distribution discussed in Section 2.6.3, specified by the quantile function

$$Q(u) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right), \; 0 \le u \le 1. \tag{3.9}$$

The quantile density function is

$$q(u) = \frac{1}{\lambda_2}\left(u^{\lambda_3 - 1} + (1-u)^{\lambda_4 - 1}\right). \tag{3.10}$$

We have pointed out in Section 2.6.3 that to be a life distribution we should have

$$Q(0) = \lambda_1 - \frac{1}{\lambda_2 \lambda_3} \ge 0$$

and in this case the support becomes $\left(\lambda_1 - \frac{1}{\lambda_2\lambda_3}, \lambda_1 + \frac{1}{\lambda_2\lambda_4}\right)$ whenever $\lambda_3 > \lambda_4 > 0$ and $\left(\lambda_1 - \frac{1}{\lambda_2\lambda_3}, \infty\right)$ if $\lambda_3 > 0$ and $\lambda_4 \le 0$. This is a critical point to be verified when the distribution is used to model data pertaining to nonnegative random variables. In the following sections, we study the relevance of the distribution in reliability analysis. Commonly used reliability measures of the distribution that are capable to analyze the reliability properties of the distribution are derived and adequacy of the distribution in real life situations is examined. Such a study does not appear to have been discussed in literature.

3.2.2.1 Order statistics

Recall the expressions of distribution and expectation of order statistics given Section 2.1.5. From equation (2.30), the expected value of the r^{th} order statistic $X_{r:n}$ is

$$\begin{aligned}
\mu_{r:n} &= E\left(X_{r:n}\right) \\
&= \frac{n!}{(r-1)!(n-r)!}\int_0^1 u^{r-1}(1-u)^{n-r}\left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right)\right)du \\
&= \lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{1}{\lambda_2\lambda_3}\frac{\Gamma(\lambda_3+r)}{\Gamma(n+\lambda_3+1)}\frac{n!}{r!} - \frac{1}{\lambda_2\lambda_4}\frac{n!}{(n-r)!}\frac{\Gamma(n+\lambda_3-r+1)}{\Gamma(n+\lambda_4+1)}.
\end{aligned}$$

Specifically for $r = 1$ and $r = n$

$$E\left(X_{1:n}\right) = \lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{n!}{\lambda_2\left(\lambda_3\right)_{(n+1)}} - \frac{n}{\lambda_2\lambda_4\left(\lambda_4+n\right)},$$

and

$$E\left(X_{n:n}\right) = \lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} + \frac{n}{\lambda_2\lambda_3\left(\lambda_3+n\right)} - \frac{n!}{\lambda_2\left(\lambda_4\right)_{(n+1)}}.$$

The distributions of $X_{1:n}$ and $X_{n:n}$, using the equations (2.28) and (2.29) are specified by

$$\begin{aligned} Q_1(u) &= Q\left(1-(1-u)^{1/n}\right) \\ &= \lambda_1 + \frac{1}{\lambda_2}\left[\frac{\left[1-\left(1-u^{1/n}\right)\right]^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4/n}-1}{\lambda_4}\right] \end{aligned}$$

and

$$\begin{aligned} Q_n(u) &= Q\left(u^{1/n}\right) \\ &= \lambda_1 + \frac{1}{\lambda_2}\left[\frac{u^{\lambda_3/n}-1}{\lambda_3} - \frac{\left(1-u^{1/n}\right)^{\lambda_4}-1}{\lambda_4}\right]. \end{aligned}$$

Recalling the use of the concepts of order statistics in reliability analysis, which we have pointed out in Section 2.1.5, we can conclude that all the expressions derived above have importance.

3.2.2.2 Reliability functions

Various reliability functions of the model have closed form algebraic expressions except for the variance which contains the beta function. From the quantile density function

$$q(u) = \frac{1}{\lambda_2}\left(u^{\lambda_3-1} + (1-u)^{\lambda_4-1}\right),$$

the hazard quantile function is obtained using (2.71) as

$$\begin{aligned} H(u) &= \left((1-u)q(u)\right)^{-1} \\ &= \left[(1-u)\left(\frac{1}{\lambda_2}\left(u^{\lambda_3-1} + (1-u)^{\lambda_4-1}\right)\right)\right]^{-1} \\ &= \lambda_2\left[(1-u)^{\lambda_4} + (1-u)u^{\lambda_3-1}\right]. \end{aligned}$$

From (2.76), the mean residual quantile function simplifies to

$$M(u)=\frac{1}{1-u}\int_u^1 (1-p)\,q(p)dp$$
$$=\frac{1}{1-u}\int_u^1 (1-p)\frac{1}{\lambda_2}\left(\left(p^{\lambda_3-1}+(1-p)^{\lambda_4-1}\right)\right)dp$$
$$=\frac{(1-u)^{\lambda_4}}{\lambda_2(\lambda_4+1)}+\frac{1-u^{\lambda_3+1}}{\lambda_2\lambda_3(1+\lambda_3)(1-u)}-\frac{(1-u)^{\lambda_3-1}}{\lambda_2\lambda_3}.$$

The variance residual quantile function is

$$V(u)=\frac{1}{1-u}\int_u^1 Q^2(p)dp-\left(\frac{1}{1-u}\int_u^1 Q(p)dp\right)^2$$
$$=A_1(u)-A_2^{\,2}(u),$$

where

$$A_1(u)=\frac{1}{1-u}\int_u^1\left[\lambda_1+\frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}-1}{\lambda_3}-\frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right)\right]^2 dp$$
$$=\frac{1-u^{2\lambda_3+1}}{\lambda_2^{\,2}(2\lambda_3+1)(1-u)}+\frac{(1-u)^{2\lambda_4}}{\lambda_2\lambda_4(2\lambda_4+1)}-\frac{2B_{1-u}(\lambda_4+1,\lambda_3+1)}{\lambda_2^{\,2}\lambda_3\lambda_4(1-u)}$$

and

$$A_2(u)=\frac{1}{1-u}\int_u^1\left[\lambda_1+\frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}-1}{\lambda_3}-\frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right)\right]dp$$
$$=\frac{1-u^{\lambda_3+1}}{\lambda_2\lambda_3(1+\lambda_3)(1-u)}-\frac{(1-u)^{\lambda_4+1}}{\lambda_2\lambda_4(\lambda_4+1)}.$$

Using the equation (2.86), percentile residual quantile function becomes

$$P_\alpha(u)=Q[1-(1-\alpha)(1-u)]-Q(u)$$
$$=\frac{1}{\lambda_2}\left[(1-(1-\alpha)(1-u))^{\lambda_3}+(1-u)^{\lambda_4}\left(1-(1-\alpha)^{\lambda_4}\right)-u^{\lambda_3}\right].$$

Expressions for functions in reversed time are obtained using equations (2.87), (2.90) and (2.95). The reversed hazard quantile function

$$\begin{aligned}\Lambda(u) &= \left(u\,q(u)\right)^{-1}\\ &= \left[u\left(\frac{1}{\lambda_2}\left(u^{\lambda_3-1}+(1-u)^{\lambda_4-1}\right)\right)\right]^{-1}\\ &= \left[\frac{u}{\lambda_2}\left(u^{\lambda_3-1}+(1-u)^{\lambda_4-1}\right)\right]^{-1},\end{aligned}$$

the reversed mean residual quantile function

$$\begin{aligned}R(u) &= u^{-1}\int_0^u p\,q(p)\,dp\\ &= u^{-1}\int_0^u p\left(\frac{1}{\lambda_2}\left(p^{\lambda_3-1}+(1-p)^{\lambda_4-1}\right)\right)dp\\ &= \frac{1}{\lambda_2}\left[\frac{u^{\lambda_3}}{\lambda_3+1}-\frac{(1-u)^{\lambda_4}}{\lambda_4+1}-\frac{(1-u)^{\lambda_4+1}}{\lambda_4(\lambda_4+1)u}+\frac{1}{\lambda_4(\lambda_4+1)u}\right]\end{aligned}$$

and the reversed variance residual quantile function

$$\begin{aligned}D(u) = D(u) &= \frac{1}{u}\int_0^u Q^2(p)\,dp-\left(\frac{1}{u}\int_0^u Q(p)\,dp\right)^2\\ &= B_1(u)-B_2^{\,2}(u),\end{aligned}$$

with

$$\begin{aligned}B_1(u) &= \frac{1}{u}\int_0^u\left[\frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}-1}{\lambda_3}-\frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right)\right]dp\\ &= \frac{u^{2\lambda_3}}{\lambda_2^{\,2}\lambda_3^{\,2}(2\lambda_3+1)}+\frac{(1-u)^{2\lambda_4+1}-1}{\lambda_2^{\,2}\lambda_4^{\,2}(2\lambda_4+1)u}-\frac{2B(\lambda_3+1,\lambda_4+1)}{u\lambda_2^{\,2}\lambda_3\lambda_4}\end{aligned}$$

and

$$\begin{aligned}B_2(u) &= \frac{1}{u}\int_0^u\left[\frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}-1}{\lambda_3}-\frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right)\right]dp\\ &= \frac{u^{\lambda_3}}{\lambda_2\lambda_3(\lambda_3+1)}-\frac{(1-u)^{\lambda_4+1}-1}{\lambda_2\lambda_3(\lambda_4+1)u}.\end{aligned}$$

3.2.2.3 Application to real data

We fitted the distribution to the observed lifetimes of 100 strips of aluminium coupon (omitting the last observations to extract equal frequencies) in Birnbaum and Saunders (1958) by estimating the parameters using the method of L-moments. The estimates are obtained by equating the expressions of equations (2.130) through (2.133) with the corresponding sample moments in equation (3.8).

For the data considered here $l_1 = 1391.79$, $l_2 = 215.6837374$, $l_3 = 3.570321583$ and $l_4 = 20.767677$. The estimates of the parameters are selected from the set of solutions as explained in Section 3.2.1.3. The estimates hence obtained are

$$\hat{\lambda}_1 = 1382.18,\ \hat{\lambda}_2 = 0.0033,\ \hat{\lambda}_3 = 0.2706 \text{ and } \hat{\lambda}_4 = 0.2211.$$

The data arranged in ascending order of magnitude were divided into 10 groups each containing 10 observations. Using the expression $Q(p)$ for $p = \frac{1}{10}, \frac{2}{10}, \ldots$ and the fact that if U has uniform distribution on $[0,1]$, then X and $Q(u)$ have identical distributions, the observed frequencies were found as $10, 10, 9, 12, 8, 11, 10, 8, 12$ and 10. The resulting $\chi^2 = 1.8$, does not reject the lambda distribution for the given data.

3.3 Power-Pareto distribution

Recalling from equation (2.134), the power-Pareto distribution is specified by the quantile function

$$Q(u) = Cu^{\lambda_1}(1-u)^{-\lambda_2}, \quad 0 \le u \le 1, \quad C, \lambda_1, \lambda_2 \ge 0. \tag{3.11}$$

The quantile density function corresponding to (3.11) is

$$q(u) = Cu^{\lambda_1}(1-u)^{-\lambda_2}\left(\frac{\lambda_1}{u} + \frac{\lambda_2}{1-u}\right) \tag{3.12}$$

Although Gilchrist (2000) was the first to introduce this distribution and later Hankin and Lee (2006) have studied various properties of (3.11), several aspects of this distribution still remains to be explored. We state a few new results that are relevant to the present study.

3.3.1 Order statistics

The expectation of r^{th} order statistic has simple form, which is derived using (2.30) and is expressed in terms of beta function by

$$\begin{aligned}\mu_{r:n} &= E\left(X_{r:n}\right)\\ &= \frac{n!}{(r-1)!(n-r)!}\int_0^1 u^{r-1}(1-u)^{n-r}\left(Cu^{\lambda_1}(1-u)^{-\lambda_2}\right)du\\ &= C\,\frac{B(\lambda_1+r,n-\lambda_2-r)}{B(r,n-r+1)},\ n>\lambda_2+r,\ r=1,2,...\end{aligned}$$

The quantile functions of the 1^{st} and n^{th} order statistics are given by

$$\begin{aligned}Q_1(u) &= Q\left(1-(1-u)^{1/n}\right)\\ &= C\,\frac{\left[1-(1-u)^{\frac{1}{n}}\right]^{\lambda_1}}{(1-u)^{\frac{n}{\lambda_2}}}\end{aligned}$$

and

$$\begin{aligned}Q_n(u) &= Q\left(u^{1/n}\right)\\ &= C\,\frac{u^{\frac{\lambda_1}{n}}}{\left(1-u^{\frac{1}{n}}\right)^{\lambda_2}}.\end{aligned}$$

In views of the discussions in Section 2.1.5, the above concepts seem useful in lifetime modelling with power-Pareto distribution.

3.3.2 Reliability functions

Most commonly used reliability functions of the distribution are the hazard quantile function defined in (2.71)

$$H(u) = \left((1-u)q(u)\right)^{-1}$$
$$= \left(Cu^{\lambda_1}(1-u)^{-\lambda_2+1}\left(\frac{\lambda_1}{u}+\frac{\lambda_2}{1-u}\right)\right)^{-1},$$

the mean residual quantile function given in (2.76)

$$M(u) = \frac{1}{1-u}\int_u^1 (1-p)\, q(p)\,dp$$
$$= \frac{1}{1-u}\int_u^1 (1-p)Cp^{\lambda_1}(1-p)^{-\lambda_2}\left(\frac{\lambda_1}{p}+\frac{\lambda_2}{1-p}\right)dp$$
$$= \frac{1}{1-u}B_{1-u}(1+\lambda_1, 1-\lambda_2) - \frac{Cu^{\lambda_1}}{(1-u)^{\lambda_2}},$$

the reversed hazard quantile function (using equation (2.87))

$$\Lambda(u) = \left(uq(u)\right)^{-1}$$
$$= \left[u\left(Cu^{\lambda_1}(1-u)^{-\lambda_2}\left(\frac{\lambda_1}{u}+\frac{\lambda_2}{1-u}\right)\right)\right]^{-1}$$
$$= \left[Cu^{\lambda_1+1}(1-u)^{-\lambda_2}\left(\frac{\lambda_1}{u}-\frac{\lambda_2}{1-u}\right)\right]^{-1},$$

and the reversed mean residual quantile function obtained using the equation (2.90)

$$R(u) = u^{-1}\int_0^u pq(p)\,dp$$
$$= u^{-1}\int_0^u p\left(Cp^{\lambda_1}(1-p)^{-\lambda_2}\left(\frac{\lambda_1}{p}+\frac{\lambda_2}{1-p}\right)\right)dp$$
$$= Cu^{\lambda_1}(1-u)^{-\lambda_2} - \frac{1}{u}B_u(\lambda_1+1, 1-\lambda_2).$$

Note that the reliability functions except hazard quantile function and reversed hazard quantile function contain special functions. Nair and Sankaran (2009) have studied the reliability properties of the distribution and analyzed the nature of the hazard quantile function. They pointed out that $H(u)$ decreases when $\lambda_1(1-4\lambda_2)+4\lambda_2^{\,2} \le 0$ or

$\lambda_1 = 0$ for all u or when $\lambda_1(1-4\lambda_2)+4\lambda_2^2 > 0$ for all u outside the interval (β,α), where α and β are the admissible roots of the equation

$$g(u) = (\lambda_1 - \lambda_2)^2 u^2 + (\lambda_1 - 2\lambda_1^2 + 2\lambda_1\lambda_2)u + \lambda_1(\lambda_1 - 1) = 0, \text{ with } \alpha > \beta,$$

where $g(u)$ is the term that determines the sign of $H'(u)$.

When α is the only admissible root of $g(u) = 0$, $H(u)$ decreases whenever $u < \alpha$ *or* $u > \alpha$ and $\lambda_1(1-4\lambda_2)+4\lambda_2^2 > 0$. $H(u)$ is increasing for all u in (β,α) whenever $\lambda_1(1-4\lambda_2)+4\lambda_2^2 > 0$. There is no value of u for which $H(u)$ is monotonic in other case. They have also established some characterization results using the relationships between the reliability functions. Note that when $\lambda_1 = \lambda_2 = \lambda$, $Q(u)$ is the quantile function of log-logistic distribution.

3.3.3 Application to lifetime data

To ascertain the adaptability of the distribution to real lifetime data, we examined its adequacy to the times to failure of 20 electric carts reported in Zimmer et al. (1998) by the method of L-moments. The estimates were found from the solution of the equations obtained by equating the population and sample moments. These equations are

$$CB(\lambda_1+1, 1-\lambda_2) = 1400.188,$$

$$C\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2+2}\right)B(\lambda_1+1, 1-\lambda_2) = 219.8109$$

$$C\,\frac{4\lambda_1\lambda_2-\lambda_1+\lambda_2+\lambda_1^2+\lambda_2^2}{(\lambda_1-\lambda_2+2)(\lambda_1-\lambda_2+3)}B(\lambda_1+1, 1-\lambda_2) = 5.4559$$

The estimates thus obtained are

$$\hat{C} = 1530.53,\ \hat{\lambda}_1 = 0.234621,\ \hat{\lambda}_2 = 0.09669.$$

The goodness of fit is tested by the Q-Q plot, since the number of observations is small to accommodate the chi-square test. The Figure 3.2 shows that the distribution gives a reasonable fit to the data. The example vindicates the scope of the distribution to use as a lifetime model.

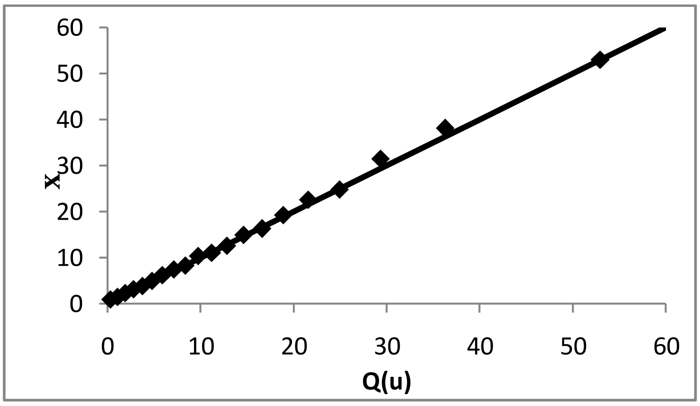


Figure 3.2- Q-Q plot

3.4 van-Staden & Loots model

Recall the van-Staden & Loots model discussed in Section (2.6.5). The distribution has the quantile function

$$Q(u) = \lambda_1 + \lambda_2 \left[(1-\lambda_3) \frac{u^{\lambda_4} - 1}{\lambda_4} - \lambda_3 \frac{(1-u)^{\lambda_4} - 1}{\lambda_4} \right], \; \lambda_2 > 0 \tag{3.13}$$

and quantile density function

$$q(u) = \lambda_2 \left[(1-\lambda_3) u^{\lambda_4 - 1} + \lambda_3 (1-u)^{\lambda_4 - 1} \right] \tag{3.14}$$

The condition for the model to be a life distribution is $\lambda_1 - \frac{\lambda_2 (1-\lambda_3)}{\lambda_4} \geq 0$.

This gives members with both finite and infinite support, depending upon whether λ_4 is positive or negative.

3.4.1 Order statistics

The 1^{st} and n^{th} order statistics of the distribution have quantile functions (obtained using equations (2.28) and (2.29))

$$\begin{aligned} Q_1(u) &= Q\left(1-(1-u)^{\frac{1}{n}}\right) \\ &= \lambda_1+\lambda_2\left[\frac{1-\lambda_3}{\lambda_4}\left(\left(1-(1-u)^{\frac{1}{n}}\right)^{\lambda_4}-1\right)-\frac{\lambda_3}{\lambda_4}\left((1-u)^{\frac{1}{\lambda_4}}-1\right)\right] \end{aligned}$$

and

$$\begin{aligned} Q_n(u) &= Q\left(u^{\frac{1}{n}}\right) \\ &= \lambda_1+\lambda_2\left[\frac{1-\lambda_3}{\lambda_4}\left(u^{\frac{1}{n}}-1\right)-\frac{\lambda_3}{\lambda_4}\left(\left(1-u^{\frac{1}{n}}\right)^{\lambda_4}-1\right)\right]. \end{aligned}$$

From (2.30), the expectation of the r^{th} order statistic is

$$\begin{aligned} E(X_{r:n}) &= \frac{n!}{(r-1)!(n-r)!}\int_0^1 u^{r-1}(1-u)^{n-r}\left(\lambda_1+\lambda_2\left[(1-\lambda_3)\frac{u^{\lambda_4}-1}{\lambda_4}-\lambda_3\frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right]\right)du \\ &= \lambda_1+\frac{\lambda_2}{\lambda_4}\left[\frac{(1-\lambda_3)\Gamma(\lambda_4+1)}{\Gamma(r)}-\frac{\lambda_3\Gamma(n+\lambda_4-r+1)}{\Gamma(n-r+1)}\right]\frac{n!}{\Gamma(\lambda_4+n+1)}+\frac{\lambda_2}{\lambda_4}(2\lambda_3-1). \end{aligned}$$

In particular, the expectations of first and n^{th} order statistics are

$$E(X_{1:n})=\lambda_1+\frac{\lambda_2}{\lambda_4}\left[(1-\lambda_3)\Gamma(\lambda_4+1)-\frac{\lambda_3\Gamma(n+\lambda_4)}{\Gamma(n)}\right]\frac{n!}{\Gamma(\lambda_4+n+1)}+\frac{\lambda_2}{\lambda_4}(2\lambda_3-1)$$

and

$$E(X_{n:n})=\lambda_1+\frac{\lambda_2}{\lambda_4}\left[\frac{(1-\lambda_3)\Gamma(\lambda_4+1)}{\Gamma(n)}-\lambda_3\Gamma(\lambda_4+1)\right]\frac{n!}{\Gamma(\lambda_4+n+1)}+\frac{\lambda_2}{\lambda_4}(2\lambda_3-1).$$

The last three formulae can be used to find the expectations of life of an r-out-of-n system, a series system and a parallel system of components whose lifetimes are assumed to follow van-Staden and Loots distribution.

3.4.2 Reliability functions

The hazard quantile function obtained using (2.71) has the expression

$$H(u) = \left((1-u)q(u)\right)^{-1}$$

$$= \left((1-u)\lambda_2\left[(1-\lambda_3)u^{\lambda_4-1} + \lambda_3(1-u)^{\lambda_4-1}\right]\right)^{-1}$$

$$= \lambda_2^{-1}\left[(1-\lambda_3)u^{\lambda_4-1}(1-u) + \lambda_3(1-u)^{\lambda_4}\right]^{-1}.$$

Its derivative is

$$H'(u) = \frac{\lambda_2\left[\lambda_3\lambda_4(1-u)^{\lambda_4-1} + (1-\lambda_3)u^{\lambda_4-2}(\lambda_4 u + 1 - \lambda_4)\right]}{\left(H(u)\right)^2}.$$

When $0 < \lambda_4 < 1$, $H'(u) > 0$, means $H(u)$ is increasing. The hazard quantile function $H(u)$ possesses different shapes. See appendix of Chapter 4.

From (2.76), the mean residual quantile function

$$M(u) = \frac{1}{1-u}\int_u^1 (1-p)\, q(p)\, dp$$

$$= \frac{1}{1-u}\int_u^1 (1-p)\lambda_2\left[(1-\lambda_3)p^{\lambda_4-1} + \lambda_3(1-p)^{\lambda_4-1}\right]dp$$

$$= \lambda_2\left[\frac{1-\lambda_3}{\lambda_2(1-u)}\left(\frac{1+\lambda_4 u^{\lambda_4+1}}{\lambda_4+1} - u^{\lambda_4}\right) + \frac{\lambda_3}{\lambda_4+1}(1-u)^{\lambda_4}\right].$$

The reversed hazard quantile function derived from (2.87) is

$$\begin{aligned}\Lambda(u) &= \left(uq(u)\right)^{-1}\\ &= \left[\lambda_2\left((1-\lambda_3)u^{\lambda_4}+\lambda_3 u(1-u)^{\lambda_4-1}\right)\right]^{-1}\end{aligned}$$

and from (2.90) reversed mean residual quantile function is

$$\begin{aligned}R(u) &= \frac{1}{u}\int_0^u pq(p)dp\\ &= \frac{\lambda_2}{u}\left[(1-\lambda_3)\frac{u^{\lambda_4+1}}{\lambda_4+1}+\frac{\lambda_3}{\lambda_4(\lambda_4+1)}\left(1-(1-u)^{\lambda_4}(1+\lambda_4 u)\right)\right]\\ &= \frac{\lambda_2}{\lambda_4(\lambda_4+1)}\left[(1-\lambda_3)\lambda_4 u^{\lambda_4}+\lambda_3 u^{-1}\left(1-(1-u)^{\lambda_4}(1+\lambda_4 u)\right)\right].\end{aligned}$$

Using (2.86), the percentile residual quantile life function has the expression

$$\begin{aligned}P_\alpha(u) &= Q[1-(1-\alpha)(1-u)]-Q(u)\\ &= \frac{\lambda_2}{\lambda_4}\Big[(1-\lambda_3)\left\{(1-(1-\alpha)(1-u))^{\lambda_4}-u^{\lambda_4}\right\}\\ &\qquad -\lambda_3\left\{1-(1-(1-\alpha)(1-u))^{\lambda_4}-(1-u)^{\lambda_4}\right\}\Big].\end{aligned}$$

3.4.3 Application to real data

We fitted the distribution to the data on lifetimes (in cycles) of 20 sodium sulphur batteries given in Lai and Xie (2006) (page 348). We used the L-moment estimates given in Section 2.6.5.4 for estimating the parameters of the distribution. The first four sample L-moments of the data are

$l_1 = 995.9,\ l_2 = 470.2684, l_3 = 145.2$ and $l_4 = 51.6703$.

The two sets of estimates of the parameters obtained using expressions of L-moment estimators given in equations (2.142) through (2.145) are

1. $\hat{\lambda}_1 = -254.569$, $\hat{\lambda}_2 = 1224.41$, $\hat{\lambda}_3 = 1.10731$, $\hat{\lambda}_4 = 0.18927$ and

2. $\hat{\lambda}_1 = 3026.65$, $\hat{\lambda}_2 = 14342.3$, $\hat{\lambda}_3 = 0.142828$, $\hat{\lambda}_4 = 4.04509$.

For the first set of estimates, we have the good fit (see the Q-Q plot given in Figure 3.3) and the parameter values satisfy the condition for the model to be a life distribution. While the second set does not satisfy the condition and hence discarded from further consideration.

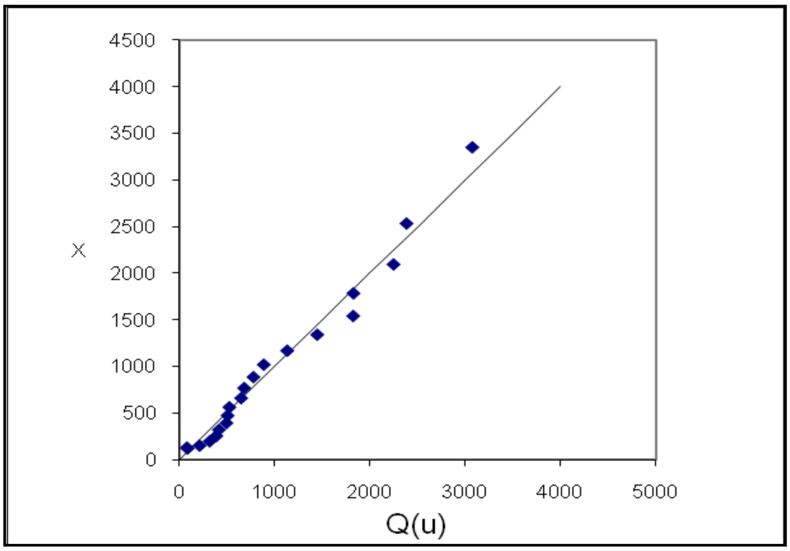


Figure 3.3- Q-Q plot

3.5 Govindarajulu distribution **

The quantile models discussed in the previous sections like various forms of generalized lambda distributions contain at least four parameters. Although highly flexible in nature, the application of these models becomes difficult due to various theoretical and computational problems in the estimation of the parameters. Govindarajulu (1977) introduced the distribution specified by

$$Q(u) = \theta + \sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right), \quad \theta, \sigma, \beta > 0, \ \ 0 \le u \le 1 \tag{3.15}$$

and demonstrated its potential as a lifetime model by fitting it to the data on the failure times of a set of 25 refrigerators which were run to destruction under advanced stress conditions. However, other than proposing the model, Govindarajulu (1977) did not investigate the various characteristics of the distribution as a general model as well as its role in reliability analysis. Accordingly in this section we carry out a detailed study of the model.

The support of the distribution (3.15) is $\left(Q(0), Q(1)\right) = \left(\theta, \theta+\sigma\right)$. Since we treat (3.15) as a lifetime model, θ is set to be zero, then the Govindarajulu distribution has only two parameters and simple quantile functional form

$$Q(u) = \sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right)$$

The quantile density function is

$$q(u) = \sigma\beta(\beta+1)u^{\beta-1}(1-u). \tag{3.16}$$

The distribution function or density function of X cannot be expressed in closed form by solving (3.15) and has to be evaluated numerically. Thus no analytical manipulation of the properties of X based on the

*** Part of the discussion in this section has appeared in Nair, Sankaran and Vineshkumar (2012), Communications in Statistics- Theory and Methods (see reference no 88).*

distribution function is possible in this case. However in view of equations (2.2) and from (3.16), we can write

$$f(x)=\left[\sigma\beta(\beta+1)\right]^{-1}F^{1-\beta}(x)\left(1-F(x)\right)^{-1}. \tag{3.17}$$

Hence the Govindarajulu model belongs to the class of distributions defined in Jones (2007) and inherits the general properties discussed therein. From the derivative of $q(u)$,

$$q'(u)=\sigma\beta(\beta+1)u^{\beta-2}\left((\beta-1)-\beta u\right),$$

we conclude that the quantile density is monotone decreasing for $\beta\leq 1$ and $q'(u)=0$ gives $u=\beta^{-1}(\beta-1)$. Thus there is an antimode at $\beta^{-1}(\beta-1)$ when $\beta>1$. Figure (3.4) shows the shapes of density function $f(x)$ at $\beta=\frac{1}{2}$, 2 and 3 respectively.

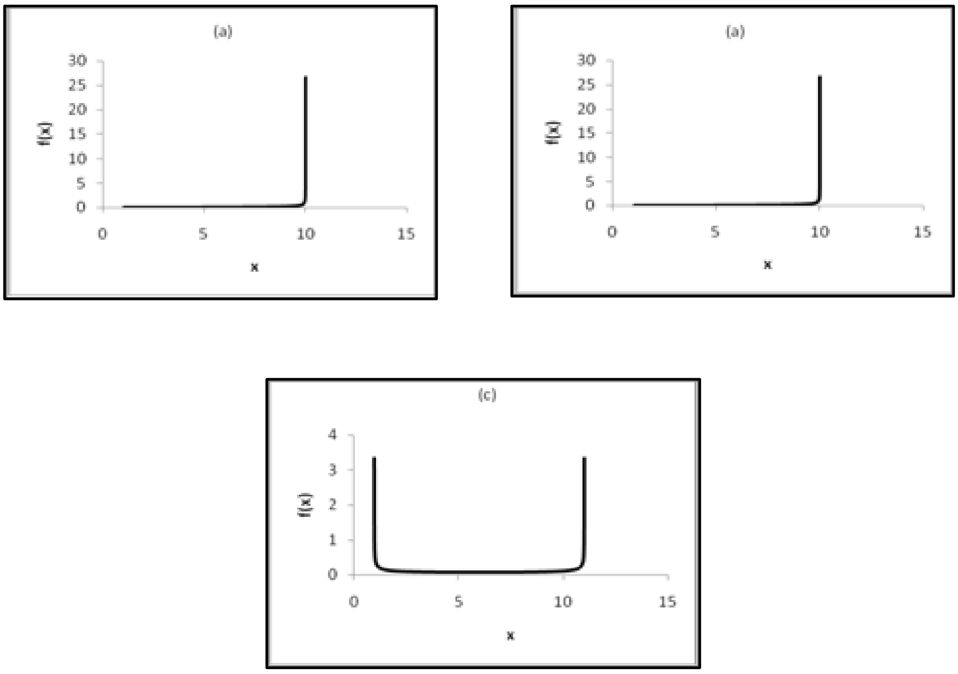


Figure 3.4- Density plots of Govindarajulu distribution when (a) $\beta=3$, (b) $\beta=0.5$ and (c) $\beta=2$

3.5.1 Moments

Recall the expressions of moments given in (2.17) through (2.21). The r^{th} conventional moment is given by

$$\begin{aligned} E\left(X^r\right) &= \int_0^1 \left(Q(p)\right)^r dp \\ &= \int_0^1 \left(\sigma\left((\beta+1)p^{\beta} - \beta p^{\beta+1}\right)\right)^r dp \\ &= \sigma \sum_{j=0}^{r} (-1)^j \binom{r}{j} (\beta+1)^{r-j} \beta^j \frac{1}{r\beta + j + 1} \end{aligned}$$

and in particular, mean,

$$\mu = 2\sigma(\beta+2)^{-1}$$

and variance

$$\mu_2 = \mu_2' - \mu^2 = \frac{\beta^2(5\beta+7)\sigma^2}{(2\beta+1)(2\beta+3)(\beta+2)^2}.$$

The third and fourth ordinary moments are

$$\begin{aligned} E\left(X^3\right) &= \int_0^1 \left(Q(p)\right)^3 dp \\ &= \int_0^1 \left(\sigma\left((\beta+1)p^{\beta} - \beta p^{\beta+1}\right)\right)^3 dp \\ &= \frac{8 + 30\beta + 26\beta^2\sigma}{8 + 42\beta + 63\beta^2 + 27\beta^3} \end{aligned}$$

and

$$\begin{aligned} E\left(X^4\right) &= \int_0^1 \left(Q(p)\right)^4 dp \\ &= \int_0^1 \left(\sigma\left((\beta+1)p^{\beta} - \beta p^{\beta+1}\right)\right)^4 dp \\ &= \frac{15 + 92\beta + 174\beta^2 + 103\beta^3\sigma}{15 + 122\beta + 328\beta^2 + 352\beta^3 + 128\beta^4}. \end{aligned}$$

The measures of skewness and kurtosis based on ordinary moments have lengthy expressions that make further analysis difficult.

3.5.2 Percentiles

The location and dispersion measures using the percentiles have been obtained using the equations (2.12) and (2.13). These measures are

$$\text{Median, M} = Q\left(\frac{1}{2}\right) = \sigma 2^{-(\beta+1)}(\beta+2)$$

and the inter-quartile range

$$\begin{aligned} IQR &= Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) \\ &= \sigma\, 4^{-(\beta+1)}\left[3^{\beta}(\beta+4) - (3\beta+4)\right]. \end{aligned}$$

The measures of skewness and kurtosis given in (2.14) and (2.15) have the forms

$$\begin{aligned} S &= \frac{Q_1 + Q_3 - 2M}{Q_3 - Q_1} \\ &= \frac{\sigma\left[(\beta+1)\dfrac{3^{\beta}+1}{4^{\beta}} - \beta\dfrac{3^{\beta+1}+1}{4^{\beta+1}} - \dfrac{\beta+2}{2^{\beta}}\right]}{IQR} \end{aligned}$$

and

$$\begin{aligned} T &= \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{IQR} \\ &= \frac{\sigma\left[(\beta+1)\dfrac{7^{\beta} - 5^{\beta} + 3^{\beta} - 1}{8^{\beta}} - \beta\dfrac{7^{\beta+1} - 5^{\beta+1} + 3^{\beta+1} - 1}{8^{\beta+1}}\right]}{IQR}. \end{aligned}$$

3.5.3 L-moments

Compared to the conventional descriptive measures discussed in the previous sections, the L-moments have compact expressions. The first four L-moments derived using equations (2.33) and (2.36) are

$$L_1 = \mu,$$

$$L_2 = \int_0^1 (2u-1)Q(u)du = \int_0^1 (2u-1)\sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right)du$$
$$= 2\beta\sigma\left[(\beta+2)(\beta+3)\right]^{-1}$$

$$L_3 = \int_0^1 (6u^2 - 6u + 1)Q(u)du = \int_0^1 \left(6u^2 - 6u + 1\right)\sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right)du$$
$$= 2\sigma\beta(\beta-2)\left[(\beta+2)(\beta+3)(\beta+4)\right]^{-1}$$

and

$$L_4 = \int_0^1 (20u^3 - 30u^2 + 12u - 1)Q(u)du$$
$$= \int_0^1 (20u^3 - 30u^2 + 12u - 1)\sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right)du$$
$$= \sigma\left(2\beta^3 - 12\beta^2 + 10\beta\right)\left[(\beta+2)(\beta+3)(\beta+4)(\beta+5)\right]^{-1}.$$

The L-coefficient of variation, analogous to the coefficient of variation based on ordinary moments defined in (2.42) is given by

$$\tau_2 = \frac{L_2}{L_1} = \frac{\beta}{\beta+3}.$$

To measure the skewness of the distribution we use the L-coefficient of skewness given in (2.43)

$$\tau_3 = \frac{L_3}{L_2} = \frac{\beta-2}{\beta+4}.$$

Being an increasing function of β, the limits for τ_3 are obtained as $\beta \to 0$ and $\beta \to \infty$. Thus τ_3 lies between $\left(\frac{-1}{2}, 1\right)$. Hence the distribution has negatively skewed, symmetric (at $\beta = 2$ when $\tau_3 = 0$) and positively skewed members. From (2.44), the L-coefficient of kurtosis is

$$\tau_4 = \frac{L_4}{L_2} = \frac{(\beta-5)(\beta-1)}{(\beta+4)(\beta+5)},$$

which is non-monotone decreasing initially, reaches its lower value in the symmetric case and then increases to unity.

3.5.4 Order statistics

A particularly attractive property of the Govindarajulu distribution is the simple form for the expected values of the order statistics. If $X_{r:n}$ denotes the r^{th} order statistic in a random sample of size n, the density function of $X_{r:n}$ is

$$
\begin{aligned}
f_r(x) &= \frac{1}{B(r,n-r+1)} f(x) F^{r-1}(x) \big(1-F(x)\big)^{n-r} \\
&= \frac{1}{\sigma\beta(\beta+1)\, B(r,n-r+1)} F^{r-\beta}(x) \big(1-F(x)\big)^{n-r-1}
\end{aligned}
$$

by virtue of (3.17). Hence

$$
\begin{aligned}
E\big(X_{r:n}\big) &= \frac{1}{B(r,n-r+1)} \int_0^{\sigma} x F^{r-\beta}(x) \big(1-F(x)\big)^{n-r-1} dx \\
&= \frac{1}{B(r,n-r+1)} \int_0^1 Q(u) u^{r-\beta} (1-u)^{n-r-1} q(u) du \\
&= \frac{1}{B(r,n-r+1)} \int_0^1 \Big[u^{r-\beta} (1-u)^{n-r-1} \sigma \Big((\beta+1) u^{\beta} - \beta u^{\beta+1} \Big) \\
&\qquad \sigma\beta(\beta+1) u^{\beta-1} (1-u) \Big] du \\
&= \frac{\sigma n!\, \Gamma(\beta+r)}{\Gamma(r)\, \Gamma(n+\beta+2)} \big[(n+1)(\beta+1) - \beta(r-1) \big] \ .
\end{aligned}
$$

In particular,

$$
E\big(X_{1:n}\big) = \frac{\sigma(n+1)!}{(\beta+1)\ \dots\ (\beta+n+1)}
$$

and

$$
E\big(X_{n:n}\big) = \frac{n(n+2\beta+1)\sigma}{(n+\beta)\ (\beta+n+1)} \,.
$$

Further using (2.28) and (2.29), $X_{1:n}$ and $X_{n:n}$ have quantile functions

$$Q_1(u) = Q\left(1-(1-u)^{\frac{1}{n}}\right)$$

$$= \sigma\left(1-(1-u)^{\frac{1}{n}}\right)^{\beta}\left(1+\beta(1-u)^{\frac{1}{n}}\right)$$

and

$$Q_n(u) = Q\left(u^{1/n}\right) = \sigma\left[(\beta+1)u^{\frac{\beta}{n}} - \beta u^{\frac{\beta+1}{n}}\right].$$

3.5.5 Reliability functions

Reliability function of the Govindarajulu distribution has tractable forms. The hazard quantile function obtained using (2.71) is given by

$$H(u) = \left[(1-u)q(u)\right]^{-1}$$

$$= \left[\sigma\beta(\beta+1)u^{\beta-1}(1-u)^2\right]^{-1}, \quad 0<u<1. \tag{3.18}$$

For $\beta > 1$, $H(u) \to \infty$ as $u \to 0$ or 1 and for $\beta < 1$, $H(u) \to 0$ as $u \to 0$. When $\beta = 1$, $Q(u)$ is invertible to obtain

$$F(x) = \left(1-\frac{x}{\sigma}\right)^{\frac{1}{2}}, \; 0 \le x \le \sigma,$$

the rescaled beta distribution function, where

$$H(u) = \left(2(1-u)^2\sigma\right)^{-1}.$$

Life distributions are classified according to the behaviour of their hazard quantile functions. To study the reliability properties of the Govindarajulu distribution we need the following notions

(a) A lifetime random variable X is increasing (decreasing) hazard quantile, IHR (DHR) if and only if its hazard quantile function satisfies $H'(u) \ge (\le) 0$ for $0<u<1$.

(b) Further, X is said to have a bathtub-shaped (upside down bathtub-shaped) hazard quantile function if $H'(u)<(>)0$ for u in $(0,u_0)$, $H'(u_0)=0$ and $H'(u)>(<)0$ in $(u_0,1)$. We call u_0 as a change point. These concepts are further explained in Chapter 4.

Differentiating (3.18)

$$H'(u)=\frac{\beta u+\beta-1}{\sigma\beta(\beta+1)u^{\beta}(1-u)^{3}},$$

so that $H(u)$ is increasing for $\beta<1$ and for $\beta>1$, $H(u)$ decreases in the interval $\left(0,\frac{\beta-1}{\beta+1}\right)$ reaches a minimum at $u=\frac{\beta-1}{\beta+1}$ and then increases in $\left(\frac{\beta-1}{\beta+1},1\right)$. We conclude that X is IHR for $\beta\le 1$ and bathtub-shaped for $\beta>1$ with change point at $u=\frac{\beta-1}{\beta+1}$. See also Govindarajulu (1977) for the same conclusion in terms of the hazard rate.

Recalling the equation (2.76), the mean residual quantile function has the expression

$$\begin{aligned}M(u)&=(1-u)^{-1}\int_u^1(1-p)q(p)dp\\&=(1-u)^{-1}\int_u^1(1-p)\sigma\beta(\beta+1)u^{\beta-1}(1-p)dp\\&=\sigma\left[2-(\beta+1)(\beta+2)u^{\beta}+\beta(\beta+2)u^{\beta+1}-\beta(\beta+1)u^{\beta+2}\right]\left[(\beta+2)(1-u)\right]^{-1}.\end{aligned}\tag{3.19}$$

The behaviour of the function $M(u)$ and the second L-moment of residual life are discussed in Chapter 6.

In the expression for the quantile function, the parameter β is influential in controlling the left tail and therefore the concepts in time conditioned on $X \le x$ are of significance. Three important concepts in this context are the reversed hazard quantile function given in (2.87)

$$\begin{aligned}\Lambda(u) &= \left(uq(u)\right)^{-1} \\ &= \left(\beta(\beta+1)\sigma u^{\beta}(1-u)\right)^{-1},\end{aligned}$$

the reversed mean residual quantile function given in (2.90)

$$\begin{aligned}R(u) &= u^{-1}\int_0^u pq(p)dp \\ &= u^{-1}\int_0^u p\sigma\beta\left(\beta+1\right)p^{\beta-1}\left(1-p\right)dp \\ &= \frac{\beta\sigma}{\beta+2}u^{\beta}\left(\beta+2-(\beta+1)u\right)\end{aligned}$$

and from equation (2.97), the reversed variance residual quantile function

$$\begin{aligned}D(u) &= u^{-1}\int_0^u R^2(p)dp \\ &= \frac{\sigma^2\beta^2u^{2\beta}}{(\beta+2)^2}\left[\frac{(\beta+1)^2}{2\beta+3}u^2-(\beta+2)u+\frac{(\beta+2)^2}{2\beta+1}\right].\end{aligned}$$

Further note that the product

$$R(u)\Lambda(u) = \frac{(\beta+1)^{-1}-(\beta+2)^{-1}u}{1-u}, \tag{3.20}$$

which is a bilinear function in u. Expressions of quantile forms of some other measures useful in the modelling and analysis of lifetime data are total time on test transform (TTT) given in (2.99)

$$T(u) = \int_0^u (1-p)q(p)dp = \sigma\left[\beta+1-2\beta u+\frac{\beta(\beta+1)}{\beta+2}u^2\right]u^{\beta}$$

and the percentile residual quantile function given in (2.81)

$$\begin{aligned}P_{\alpha}(u) &= Q\left(1-(1-\alpha)(1-u)\right)-Q(u) \\ &= \sigma\left[\left(1-(1-\alpha)(1-u)\right)^{\beta}\left(1+\beta(1-\alpha)(1-u)\right)-u^{\beta}\left(\beta+1-\beta u\right)\right].\end{aligned}$$

Characterization problems by the relation between the reversed hazard rate and reversed mean residual life function in the distribution function approach are discussed in the literature, e.g. Chandra and Roy (2001). So far no characterization in terms of quantile-based functions appears to have been proposed. Motivated by (3.20), we prove the following result.

Theorem 3.1

For a nonnegative random variable X, *the relationship*

$$R(u)\,\Lambda(u)=\frac{a+bu}{1-u} \tag{3.21}$$

holds for all $0<u<1$ *if and only if*

$$Q(u)=K\left(\frac{a}{1-a}u^{\frac{1}{a}-1}-au^{\frac{1}{a}}\right), \tag{3.22}$$

provided that a *and* b *are real numbers satisfying* $\frac{1}{a}+\frac{1}{b}=-1.$

Proof: Assume that (3.21). Then

$$\left[\frac{1}{u}\int_0^u pq(p)dp\right]\left[uq(u)\right]^{-1}=\frac{a+bu}{1-u} \tag{3.23}$$

Equation (3.23) simplifies to

$$\frac{uq(u)}{\int_0^u pq(p)}=\frac{1-u}{u(a+bu)}$$

$$=\frac{1}{au}-\frac{a+b}{a(a+bu)}.$$

Integrating

$$\log\int_0^u pq(p)dp=\frac{1}{a}\log u-\frac{a+b}{ab}\log(a+bu)+\log K$$

or

$$\int_0^u pq(p)dp=Ku^{\frac{1}{a}}(a+bu),\ \text{on using } \frac{1}{a}+\frac{1}{b}=-1\,.$$

Hence

$$uq(u) = K\left[u^{\frac{1}{a}}b + \frac{a+bu}{a}u^{\frac{1}{a}-1}\right]$$

or

$$q(u) = Ku^{\frac{1}{a}-2}(1-u)\,.$$

Integrating the last expression over $(0, u)$ and noting $Q(0) = 0$, we have (3.22). The converse part follows from the equations

$$\Lambda(u) = \left[Ku^{\frac{1}{a}-1}(1-u)\right]^{-1}$$

and

$$R(u) = \frac{Kau^{\frac{1}{a}-1}(a+1-u)}{a+1}\,.$$

Remark 3.1 The Govindarajulu distribution is verified when $a = (1+\beta)^{-1}$. The condition on a and b assumed in the theorem can be relaxed to real a and b, to provide a more general family. Another characterization result using the relationship between first and second L-moments of reversed residual life will be discussed in Chapter 6.

3.5.6 Estimation and application to lifetime data

The conventional methods of estimation like matching regular moments, percentiles, or the method of least squares, usually applied for quantile functions work out quite easily for the model. The simple algebraic expressions of the L-moments derived in above admit the applicability of the L-moment method for estimating the parameters of the Govindarajulu distribution.

We carried out a study to learn whether the three matching methods viz, method of moments, percentiles and L-moments generate fair estimates, and to study which method behave better. Thousand samples of sizes $n = 25, 50 \text{ and } 100$ for two sets of parameter values $\sigma = 1, \beta = 2$ and $\sigma = 5, \beta = 0.5$ were generated using the result that if U has uniform distribution over $(0,1)$ then $Q(u)$ and X have the same distribution. The method of moment estimates are obtained by equating the mean and variance of the population with those of the samples. We matched 25th and 75th percentiles to get the percentile method of estimates. The equations used to obtain the L-moment estimators are

$$\frac{2\sigma}{\beta+2} = l_1$$

and

$$\frac{2\sigma\beta}{(\beta+2)(\beta+3)} = l_2,$$

where the left hand side of the equation corresponds to first two L-moments given in Section 3.5.3 and the right hand side corresponds to the sample L-moments. The MSEs calculated are summarized in Table.3.1. These indicate that the method of moments and L-moments give better estimates compared to the method of percentiles. For the parameter β, the former methods give estimates with almost same MSE, while for the parameter σ the method of L-moment give lower MSE. Note that for smaller sample sizes, the MSE is comparatively high and decreases as sample size increases. Based on this limited empirical study carried out here, we conclude that the method of L-moments gives better estimates in comparison with other two methods.

In Govindarajulu (1977), it is already demonstrated that the model can be a lifetime model. More importantly it can also provide good approximations to many other lifetime model specified in terms of distribution functions. We illustrate this by appealing to a real data situation.

Table 3.1- MSE of Estimates

Method of Estimation			Moment	Percentile	L-moment
MSE of Estimates when	$n=25$	$\sigma=1$	0.005234	0.020553	0.004971
		$\beta=2$	0.352435	0.381133	0.370841
		$\sigma=5$	0.022991	0.016421	0.011447
		$\beta=0.5$	0.024805	0.030126	0.023722
	$n=50$	$\sigma=1$	0.002185	0.008364	0.001927
		$\beta=2$	0.134422	0.182411	0.137226
		$\sigma=5$	0.011068	0.005743	0.004987
		$\beta=0.5$	0.010098	0.014061	0.009285
	$n=100$	$\sigma=1$	0.001051	0.004537	0.000895
		$\beta=2$	0.063119	0.095118	0.063487
		$\sigma=5$	0.005035	0.002888	0.002187
		$\beta=0.5$	0.00482	0.009029	0.004836

We consider the data on the failure times of 50 devices (Aarset Data) arranged in order of magnitude cited in Lai and Xie (2006). In view of the above empirical study, it is proposed to use the L-moment method for the analysis. The parameters of the distribution, estimated by the method of L-moments are

$$\hat{\sigma}=93.463 \text{ and } \hat{\beta}=2.0915.$$

Dividing the data into 5 groups of 10 observations each and taking $u_i=\frac{i}{5},\ i=1,2,\dots,5$ the corresponding x values of the random variable were computed using (3.15) (taking $\theta = 0$) with the estimates quoted above. The observed frequencies in the 5 classes were 11, 8, 8, 13 and 10 against the expected frequency of 10 in each class (U has uniform distribution over $(0,1)$ then $Q(u)$ and X have the same distribution). Thus the chi-square value of 1.8 realized here does not reject the hypothesis that the model follows Govindarajulu distribution. With the model adequacy verified, we can reanalyze the data with reference to the various reliability aspects mentioned in the previous sections. We note that the hazard quantile function (see Figure 3.5) is bathtub-shaped with change point at $u = 0.3531$

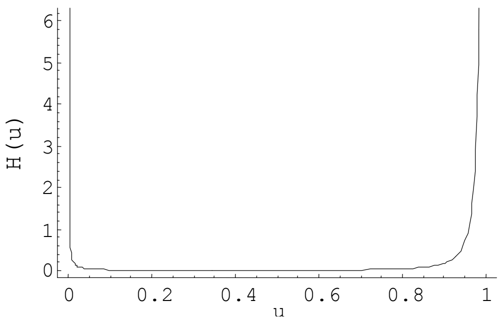


Figure 3.5- Shape of $H(u)$

3.5.7 Comparison with other models

We compare the Govindarajulu distribution with some other distributions in literature that have bathtub-shaped failure rate function. Xie et al. (2002) introduced the modified Weibull extension with survival function

$$\bar{F}(x) = \exp\left[\lambda\alpha\left(1 - \exp\left(\frac{x}{\alpha}\right)^{\gamma}\right)\right], \quad x \geq 0, \quad \lambda, \alpha, \gamma > 0\,.$$

The failure rate is

$$h(x) = \lambda\gamma\left(\frac{x}{\beta}\right)^{\gamma-1} \exp\left[-\left(\frac{x}{\alpha}\right)^{\gamma}\right],$$

which is bathtub-shaped for $\gamma < 1$. For the Aarset data considered in the previous section the estimates of the parameters (Xie et al. (2002)) are

$$\hat{\alpha} = 110.0909,\ \hat{\gamma} = 0.8408,\ \hat{\lambda} = 0.0141.$$

With above value of $\hat{\gamma}$, the failure rate is bathtub-shaped with change point $x_0 = 15.211$. Later Lai et al. (2003) proposed another modification to the Weibull distribution by suggesting the model

$$\bar{F}(x) = \exp\left[-\theta x^{\alpha} e^{\lambda x}\right],$$

with

$$h(x) = \theta(\alpha + \lambda x)x^{\alpha-1}e^{\lambda x}.$$

The turning point of $h(x)$ is

$$x = \frac{\sqrt{\alpha} - \alpha}{\lambda}, \quad 0 < \alpha < 1.$$

Using the probability plot they estimated the parameters using the same data as

$$\left(\hat{\theta}\ \hat{\lambda}, \hat{\alpha}\right) = (0.0876, 0.389, 0.01512).$$

These estimates were further refined in Bebbington et al. (2008) with

$$\left(\hat{\theta}\ \hat{\lambda}, \hat{\alpha}\right) = (0.0624, 0.3548, 0.0233),$$

which also gave a satisfactory fit and bathtub failure rate. The transformation model introduced by Mudholkar et al. (2009) is represented by

$$\bar{F}(x) = \exp\left[-\left(\frac{-1}{\sigma}\,\frac{x}{1-\theta x}\right)^{\alpha}\right],\ 0 < x < \frac{1}{\theta},$$

with failure rate

$$h(x) = \frac{\alpha}{\sigma^{\alpha}}\left(\frac{x}{1-\theta x}\right)^{\alpha+1}\frac{1}{x^2}.$$

They found that for the Aarset data with

$$\left(\hat{\sigma}, \hat{\alpha}, \hat{\theta}\right) = (354.4160, 0.3850, 0.0116)$$

provide a fit that is better than some of the above models giving bathtub shape.

As mentioned earlier, a more general version of the Govindarajulu distribution is the Jones (2007) family specified by the quantile density function

$$q(u) = Ku^{-\alpha}(1-u)^{-\beta}, \ 0 < u < 1.$$

Being a three parameter family with α controlling the left tail also, it is natural to expect that it should describe the data at least as good as the member, the Govindarajulu distribution. The method of L-moments was used to estimate the parameters by matching the first three moments

$$L_1 = K\,B\left(1-\alpha, 2-\beta\right),$$

$$L_2 = K\,B\left(2-\alpha, 2-\beta\right) = \frac{1-\alpha}{3-\alpha-\beta} L_1$$

and

$$\begin{aligned} L_3 &= K\left[2B\left(3-\alpha, 2-\beta\right) - B\left(2-\alpha, 2-\beta\right)\right] \\ &= \frac{(\beta-\alpha)(1-\alpha)}{(4-\alpha-\beta)(3-\alpha-\beta)} L_1, \end{aligned}$$

with their sample counterparts from the Aarset data

$$l_1 = 45.686, \ l_2 = 18.7672 \ \text{and} \ l_3 = -1.00173.$$

The estimates are

$$\hat{\alpha} = \text{-}2.46032, \ \hat{\beta} = \text{-}2.96332, \ \hat{K} = 7571.68\,.$$

In the chi-square goodness of fit test, $\chi^2 = 0.2$ offers a very close fit, better than that of Govindarajulu model. Since the parameter values are less than unity, from Jones (2007), the corresponding distribution is the complementary beta with density function

$$f(u) = \frac{B(1-\alpha,1-\beta)}{\left[I_u(1-\alpha,1-\beta)\right]^{\alpha}\left[1-I_u(1-\alpha,1-\beta)\right]^{\beta}},$$

where $I_u(1-\alpha,1-\beta)$ is the inverse of the incomplete beta function ratio. It is obvious from the form of the density that the general properties and reliability functions have complicated expressions and requires more involvement theoretically and computationally in application than the Govindarajulu model.

All the models discussed above give satisfactory fits to the same data set and also a bathtub-shaped failure rate. The only difference between the models is that they give different change points. The change points estimated from the data for the different models are given in Table 3.4. Thus for analyzing Aarset data, Govindarajulu distribution is preferable as its reliability aspects are easier to analyze in comparison with that of other models.

Table 3.4 -Change points of the failure rate

Model	**Change Point (in u)**
Xie et al. (2002)	0.58
Lai et al. (2003)	0.22
Bebbington et al. (2008)	0.26
Mudholkar et al. (2009)	0.11
Govindarajulu	0.35
Jones (2007)	0.38

Usually the χ^2 value and the Akaike information criterion are employed to evaluate the performance of a model among competing ones. The number of parameters involved in the model has a marked influence in both cases. Since Govindarajulu distribution has only two parameters as against three other models, a comparison appears less objective. We have considered some of the two parameter distributions like those of Mukherjee and Islam (1983), Chen (2000), etc. for comparison with our distribution. But we could not find real data that give satisfactory fit to such models as well as the Govindarajulu distribution to make a comparison among two parameter models.

In the above sections, we have considered different lambda distributions and the Govindarajulu distribution, derived their basic properties and checked their applications to lifetime data analysis by fitting them to real data. Tractable expressions of reliability functions and their properties reveal that the distributions can be utilized in lifetime data modelling when one wishes a quantile-based analysis. In the next section we consider a new technique of deriving life distributions using Parzen's score function and tail exponent function.

3.6 Modelling lifetimes by quantile functions using Parzen's score function and tail exponent function ***

Construction of life distributions with non-monotonic hazard rates have been a fertile topic of research in reliability analysis and allied fields during the last five decades. Various methods of construction of bathtub-shaped models include, postulating forms of hazard rates and deriving the corresponding survival functions, considering mixtures of distributions, models based on convex functions, models arising from series systems and distributions derived from physical properties of failure patterns. A survey of different models along with their characteristics is available in Rajarshi and Rajarshi (1988), Lai and Xie (2006) and Bebbington et al. (2007). A method for constructing bathtub distribution using a general version of TTT will be discussed in Chapter 5.

One common feature of all these approaches is that the distribution of failure times is represented by the distribution function. In the previous discussions we have seen that lifetime models can also be described in terms of quantile functions. In this section we introduce a method for developing quantile functions with monotone as well as non-monotone hazard rates using the properties of the score functions and tail exponent function, first suggested by Parzen (1979) in connection with the study of heaviness of probability distributions. Our study is motivated by the fact that the functions have nice relationship with the hazard quantile function. Further the monotonic behaviour of the functions implies those of the hazard quantile functions through some simple inequalities.

****The discussions in this section have appeared in Nair, Sankaran and Vineshkumar (2012),* ***Statistics*** *(see reference no 89).*

3.6.1 Reliability properties of the Parzen's functions

Consider a nonnegative random variable X with absolutely continuous distribution function $F(x)$, survival function $\bar{F}(x)$ and probability density function $f(x)$. Recall the definitions and properties of $Q(u)$ of X given in Chapter 2. In Section 2.1.2, we have given definitions of Parzen's score function and tail exponent function. We also point out the equivalency of score function with the quantile form of Glaser's function. For the sake of convenience we recall the definitions of score function and tail exponent function here. The Parzen's tail exponent function is given by

$$\alpha(1-u) = (1-u)J(u)q(u), \tag{3.24}$$

where $J(u)$ is called the score function defined as

$$J(u) = -\left(fQ(u)\right)' = \frac{q'(u)}{q^2(u)}, \tag{3.25}$$

which is the quantile form of Glaser's function $\eta(x)$, given in (2.8). Note that using the definition of hazard quantile function $H(u)$ in (2.71),

$$J(u) = -\frac{d}{du}\frac{1}{q(u)} = -\frac{d}{du}(1-u)H(u) \tag{3.26}$$

or

$$(1-u)H'(u) = H(u) - J(u). \tag{3.27}$$

Thus X has increasing (decreasing) hazard quantile function as $H(u) \geq (\leq) J(u)$ for all u. Further change points of non-monotonic $H(u)$ are zeroes of $H(u) - J(u)$. Geometrically for increasing (decreasing), the $H(u)$ curve lies above (below) $J(u)$ and for bathtub (upside-down bathtub), $H(u)$ crosses $J(u)$ from below (above). By the equivalence of $\eta(x)$ and $J(u)$, all results in Gupta and Warren (2001) can be reformulated in terms of quantiles. For instance their main result in Theorem 4.1 translates as follows.

Theorem 3.2

Let $q(u)$ *be twice differentiable on* $(0,1)$. *If* $J'(u)$ *has* n *zeros* $0 = u_0 < u_1 < u_2 < \ldots < u_n < 1$, *there exists atmost one zero of* $H'(u)$ *in the closed interval* $[u_{k-1}, u_k], k = 1, 2, \ldots, n$.

From 3.26, we have

$$q(u) = \left[\int_u^1 J(p)dp\right]^{-1}.$$

Thus $J(u)$ uniquely determines the distribution of X (provided $f(\infty) = 0$). Further there exist simple relationships between $J(u)$ and $H(u)$ that characterize many life distributions. See the following example.

Example 3.1 Consider the generalized Pareto model

$$Q(u) = ba^{-1}\left[(1-u)^{-a/a+1} - 1\right], a > -1, b > 0.$$

Then

$$J(u) = \frac{2a+1}{(1-u)^{-a/a+1}}$$

and

$$H(u) = \frac{a+1}{b}\frac{1}{(1-u)^{-a/a+1}}.$$

Thus we have the relationship of the form

$$J(u) = CH(u). \tag{3.28}$$

Using (3.25) and the expressions for $H(u)$ given in (2.71), we can easily show that (3.28) characterizes generalized Pareto distribution (which includes exponential, Pareto and rescaled beta distribution). Thus a method of construction can be envisaged through a relationship between $J(u)$ and $H(u)$ in which $H(u) > (<, =) J(u)$ can result in distributions that are increasing (decreasing, bathtub or upside-down bathtub).

To examine the reliability properties of tail exponent function, we have from (3.24) that

$$\alpha(1-u) = \frac{J(u)}{H(u)} \ . \tag{3.29}$$

Thus $\alpha(1-u) \leq (\geq) 1$ provides distributions possessing increasing (decreasing) hazard quantile function and $\alpha(1-u) = 1$ for some u_0 and $\alpha(1-u) \leq (\geq) 1$ in $(0, u_0)$ and $\alpha(1-u) \geq (\leq) 1$ in $(u_0, 1)$ leads to upside-down bathtub or decreasing (bathtub or increasing). These observations are employed in Section 3.6.2 to construct life distributions with desirable properties. The function $\alpha(1-u)$ is also useful in comparing life distributions, in terms of ageing concepts, with the aid of stochastic orders. A brief account of some results in this connection is given below.

Let X and Y be lifetime random variables with distribution functions F and G respectively. We have from section 2.6.4, when the support of X is an interval, we say that X is smaller than Y in convex transform order denoted by $X \leq_{cx} Y$, if $G^{-1}F(x)$ is convex on the support of X. This means that

$$\frac{d}{dx} G^{-1} F(x) = \frac{f(x)}{g G^{-1} F(x)}$$

is increasing. Converting to quantile function we see that $X \leq_{cx} Y$ if and only if $\frac{q_Y(u)}{q_X(u)}$ is increasing or equivalently

$$\frac{q'_Y(u)}{q_Y(u)} \geq \frac{q'_X(u)}{q_X(u)} \ . \tag{3.30}$$

Alzaid and Al-Osh (1999) have proved (also evident from the above) that

$$X \leq_{cx} Y \Leftrightarrow \alpha_F(1-u) \leq \alpha_G(1-u).$$

In terms of the hazard quantile function $\frac{q_Y(u)}{q_X(u)}$ is increasing is equivalent to increasing $\frac{H_X(u)}{H_Y(u)}$. When $Q_X(0) = Q_Y(0)$ and Q_Y is strictly increasing we say that X is less IHR than Y if and only if $H_X(u)/H_Y(u)$ is increasing for $u \in [0, 1]$. Then

$$\alpha_X(1-u) \leq \alpha_Y(1-u) \Leftrightarrow X \leq_{IHR} Y \ .$$

The IHR order implies the IHRA and NBU orders as well as the DMRL and NBUE orders discussed in Shaked and Shantikumar (2007). Our inequality $\alpha_X(1-u) \leq \alpha_Y(1-u)$ gives a simple sufficient condition to verify the validity of all these orders. For several implications of $X \leq_{cx} Y$ with other stochastic orders we refer to Shaked and Shantikumar (2007).

3.6.2 Models based on score function

In this section we utilize the relationship between $H(u)$ and $J(u)$ in deriving life distributions in terms of quantile functions that have monotone and non-monotone hazard quantile functions.

Theorem 3.3

The relationship

$$J(u) = AH(u) + B \tag{3.31}$$

is satisfied for all u *and real constants* A *and* B *if and only if the distribution has quantile function*

$$Q(u)=\begin{cases}\log\left(1+\frac{B}{1-A}\right)^{1/B}\left[c+\frac{B}{1-A}(1-u)^{1-A}\right]^{-1/B}, & c\le 1,\ A\ne 1\\ -B^{-1}\log\left(\frac{c}{c+B\log(1-u)}\right), & c>0,\ A=1\end{cases}. \tag{3.32}$$

Proof: Assuming (3.31), we have

$$\frac{q'(u)}{q^2(u)}-\frac{A}{(1-u)q(u)}=B.$$

This is a Bernoulli differential equation which is solved by reducing it to a linear equation after setting $Y=\frac{1}{q(u)}$. The solution is

$$\left(q(u)\right)^{-1}=\begin{cases}\frac{B(1-u)}{1-A}+c(1-u)^{A},\ A\ne 1\\ (1-u)\left(c+B\log(1-u)\right),\ A=1.\end{cases}$$

On integrating $q(u)$ from 0 to u, we have (3.32). The quantile density function is

$$q(u)=\begin{cases}(1-u)^{-1}\left(\frac{B}{1-A}+c(1-u)^{A-1}\right)^{-1},\ A\ne 1\\ (1-u)^{-1}\left(c+B\log(1-u)\right)^{-1},\ A=1.\end{cases} \tag{3.33}$$

Calculating $H(u)$ and $J(u)$ from (3.33), we have

$$J(u)=\begin{cases}\frac{B}{1-A}+cA(1-u)^{A-1},\ A\ne 1\\ B+c+B\log(1-u),\ A=1\end{cases} \tag{3.34}$$

and

$$H(u)=\begin{cases}\frac{B}{1-A}+c(1-u)^{A-1},\ A\ne 1\\ c+B\log(1-u),\ A=1\end{cases} \tag{3.35}$$

which verifies (3.31). This completes the proof.

The quantile function (3.32) represents a flexible family of distributions that contains tractable form of $F(x)$ in common use as well as other that require the use of quantile-based analysis. Some of the known members are:

(i) When $A=1, B=0,$

$$q(u)=\frac{c}{1-u}.$$

Therefore X is exponential with mean c^{-1}.

(ii) When $A<1, B=0,$

$$q(u)=\left[c(1-u)^{A-2}\right]^{-1},$$

$$Q(u)=\int_0^u\left[c(1-u)^{A}\right]^{-1}du$$
$$=\frac{1}{c(1-A)}\left[1-(1-u)^{1-A}\right],$$

which is the quantile function corresponding to the rescaled beta distribution

$$\bar{F}(x)=\left(1-\frac{x}{R}\right)^{d}, 0<x<R, R, d>0$$

with $R=\left[c(1-A)\right]^{-1}$, $d=(1-A)^{-1}$. The distribution has reciprocal linear hazard rate which is decreasing.

(iii) For $A>1, B=0,$ we have the Pareto model having

$$\bar{F}(x)=\left(1+\frac{x}{\alpha}\right)^{\beta}, x>0,\ \alpha, \beta>0$$

with $\alpha=c(A-1)^{-1}$, $\beta=(A-1)^{-1}$. The distribution has increasing reciprocal linear hazard rate.

(iv) The half logistic distribution with survival function, we get

$$\bar{F}(x) = 2\left(1 + e^{x/\sigma}\right)^{-1}, x > 0, \ \sigma > 0,$$

when $A = 2, B = -\sigma^{-1}$ and $c = -\sigma^{-1}$. The corresponding hazard rate is increasing. The relevance of the distribution in reliability studies is discussed in Balakrishnan (1992).

(v) The Gompertz law

$$\bar{F}(x) = \exp\left\{-B^{-1}\left(1 - e^{-Bx}\right)\right\}$$

for $A = 1, c = 1, B < 0$ and the negative Gompertz for $A = 1, c = 1, B > 0$. The properties and hazard rate behaviour of these two distributions are evident from the discussions given below.

From (3.35)

$$H'(u) = \begin{cases} -c(A-1)(1-u)^{A-2}, & A \neq 1 \\ \dfrac{-B}{1-u}, & A = 1 \end{cases}. \tag{3.36}$$

Thus in general, the hazard quantile functions of (3.32) are monotonic. When $A = 1$ and $B > 0$, $H(u)$ is decreasing and the case $A = 1, B < 0$ gives increasing hazard quantile function. They are increasing whenever $A < 1, \ c < 0$ or $A > 1, \ c > 0$. The distributional characteristics of the model for $A = 1$, namely Gompertz law, are described in Marshall and Olkin (2007). When $A \neq 1$, using (2.2) density function of X has the form

$$f(x) = \bar{F}(x)\left[\frac{B}{1-A} + c\bar{F}^{A-1}(x)\right], \ \ x > 0 \tag{3.37}$$

and

$$f'(x) = 0 \Rightarrow \bar{F}(x) = \left[\frac{-B}{cA(1-A)}\right]^{1/A-1}.$$

Thus the densities are either monotone or unimodal for $A > 1$ with the mode at

$$1-u=\bar{F}(x)=\left[\frac{-B}{cA(1-A)}\right]^{1/A-1}.$$

Thus various percentiles and there from the median and inter-quartile range (IQR), as measures of location and dispersion are readily obtained, using the expressions given in (2.12) through (2.15). One can also obtain the L-moments and descriptive measures based on them using (3.32).

Our second illustration is a family of distributions that has non-monotone hazard quantile function for some of its members and monotone function for others.

Theorem 3.4

The functions $J(u)$ and $H(u)$ are such that

$$J(u)=\left(A+\frac{B}{cu}\right)H(u) \tag{3.38}$$

for all u if and only if

$$q(u)=Ku^{\alpha}(1-u)^{-(A+\alpha)}, \tag{3.39}$$

where $\alpha=Bc^{-1}$ and A are real constants, and K is the normalizing constant.

Proof: Equation (3.38) is equivalent to

$$\frac{q'(u)}{q^2(u)}=\left(A+\frac{B}{cu}\right)\left[(1-u)q(u)\right]^{-1}$$

or

$$\frac{q'(u)}{q(u)}=(1-u)^{-1}\left(A+\frac{B}{cu}\right).$$

Integrating we have

$$\log q(u)=-A\log(1-u)+\frac{B}{c}\left[\log u-\log(1-u)\right]+\log K$$

or

$$\log q(u) = \log\left[Ku^{B/c}(1-u)^{-\left(A+B/c\right)}\right]$$

leads to (3.39). For the distribution (3.39), we have

$$H(u) = K^{-1}u^{-\alpha}(1-u)^{A+\alpha-1}$$

and

$$J(u) = K^{-1}u^{-\alpha-1}(1-u)^{A+\alpha-1}(Au+\alpha).$$

From these expressions we can easily verify (3.38), which completes the proof.

The family of distributions given by (3.39) nests several well known distributions. Of these the models of interest in reliability include

(i) The exponential ($\alpha = 0$, $A = 1$), Pareto ($\alpha = 0$, $A < 1$) and rescaled beta($\alpha = 0$, $A > 1$) mentioned in the Theorem 4.2

(ii) When $A-2, \alpha = -\lambda - 1$,

$$q(u) = \lambda(\lambda+1)u^{-\lambda-1}(1-u)^{\lambda-1},$$

which is the quantile density function of log logistic distribution specified by

$$\bar{F}(x) = \frac{x^{\frac{1}{\lambda}}}{\alpha^{\frac{1}{\lambda}} + x^{\frac{1}{\lambda}}}, \quad x > 0, \ \alpha, \lambda > 0.$$

The reliability aspects of the distribution are discussed in Gupta et al. (1999). Further

$$J(u) = \frac{2u+\lambda-1}{u}H(u),$$

shows that X is upside-down bathtub in this case, with change point $u = 1-\lambda$.

(iii) Setting $\alpha = \beta - 1$ and $A = -\beta$

$$q(u) = Ku^{\beta-1}(1-u)$$

and $K = \sigma\beta(\beta+1)$ gives

$$Q(u) = \theta + \sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right),$$

the Govindarajulu distribution discussed in Chapter 3.

(iv) In terms of distribution function (3.39) has the form.

$$f(x) = K\left[F(x)\right]^{-\alpha}\left[1-F(x)\right]^{A+\alpha}. \tag{3.40}$$

We further note that if $Q(u)$ satisfies (3.38), then $KQ(u)$ also satisfies the equation for any $K > 0$. Thus equation (3.40) belongs to the class of distribution defined by the relationship between their density and distributions of Jones (2007). We refer to Jones (2007) for the distributional properties. Among the members of this class, when $\alpha > -1, A + \alpha < 1$, we have the complementary beta distribution studied in detail in Jones (2002). Since the reliability properties are not mentioned in the above papers, we note that the hazard quantile function is

$$H(u) = K^{-1}(1-u)^{A+\alpha-1}u^{-\alpha}.$$

Now

$$J(u) - H(u) = K^{-1}u^{-\alpha-1}(1-u)^{A+\alpha-1}\left[(A-1)u + \alpha\right];$$

Therefore we conclude that X has increasing hazard quantile function for $\alpha < 0, A < 1$, decreasing for $A > 1, \alpha > 0$, bathtub for $\alpha > 0, A < 1$ and upside-down bathtub for $\alpha < 0, A > 1$.

Theorem 3.5

The relationship

$$J(u) = \left[A + B\left(\log(1-u)\right)^{-1}\right]H(u) \tag{3.41}$$

is satisfied for all u *and real* A *and* B *if and only if*

$$q(u) = K(1-u)^{-A}\left(-\log(1-u)\right)^{-B}. \tag{3.42}$$

Proof: Equation (3.41) is the same as

$$\frac{q'(u)}{q(u)} = \frac{A}{1-u} + \frac{B}{(1-u)\log(1-u)}$$

Integrating, we have (3.42). To prove the converse, applying logarithmic differentiation on (3.1), we get

$$\frac{q'(u)}{q(u)} = \frac{A}{1-u} + \frac{B}{(1-u)\log(1-u)},$$

which is equivalent to (3.41).

The quantile function corresponding to (3.42) when $A, B < 1$ is

$$Q(u) = K(1-A)^{B-1}\, I\left(1-B, \log(1-u)^{A-1}\right),$$

where $I(a,x) = \int_0^x e^{-t}t^{a-1}dt$ is the incomplete gamma function. The density function of X is

$$f(x) = C\left(1-F(x)\right)^{A}\left(\log\left(1-F(x)\right)\right)^{B}, \quad x > 0 \tag{3.43}$$

Some known special cases of (3.42) are

(i) The Weibull distribution, corresponding to $A = 1,\ B = \frac{\lambda-1}{\lambda}$,with shape parameter λ and scale parameter $\sigma = K\lambda$.The exponential and Rayleigh distributions are further special cases.

(ii) Uniform, when $A = 0, B = 0$.

(iii) Pareto $(A > 1, B = 0)$ and rescaled beta $(A < 1, B = 0)$.

The hazard quantile function is

$$H(u) = K(1-u)^{A-`1}\left(-\log(1-u)\right)^{B}$$

$$H'(u) = K(1-u)^{A-`2}\left(-\log(1-u)\right)^{B-1}\left[(1-A)\left(\left(-\log(1-u)\right)+B(1-u)\right)\right],$$

which shows that

X is IHR whenever $A \le 1,\ B > 0$ or $A < 1,\ B = 0$

X is DHR whenever $A \ge 1,\ B < 0$ or $A > 1,\ B = 0$

X is bathtub whenever $A < 1,\ B < 0$

X is upside down bathtub whenever $A > 1,\ B > 0$ and

X is exponential if $A = 1,\ B = 0$.

From (3.43) we see that the distributions are either unimodal or monotonic, with mode at $u_0 = 1 - \exp\left\{\frac{-B}{A}\right\}$. The summary measures of the distribution can be obtained from the L- moments. Using (2.37) through (2.40), the first four L- moments are

$$L_1 = \int_0^1 (1-u)q(u)\,du = \frac{K\Gamma(1-B)}{(2-A)^{1-B}},$$

$$L_2 = \int_0^1 \left(u^2-u\right)q(u)\,du = L_1\left[\left(\frac{2-A}{3-A}\right)^{1-B} - 1\right],$$

$$L_3 = \int_0^1 \left(2u^3 - 3u^2 + u\right)q(u)\,du$$

$$= L_1\left[3\left(\frac{2-A}{3-A}\right)^{1-B} - 2\left(\frac{1-A}{4-A}\right)^{1-B} - 1\right]$$

and

$$L_4 = \int_0^1 \left(5u^4 - 10u^3 + 6u^2 - u\right)q(u)\,du$$

$$= L_1\left[5\left(\frac{5-A}{2-A}\right)^{1-B} - 10\left(\frac{4-A}{2-A}\right)^{1-B} + 6\left(\frac{3-A}{2-A}\right)^{1-B} - 1\right].$$

Thus the measure of skewness is $\tau_3 = \frac{L_3}{L_2}$ and the measure of kurtosis is $\tau_4 = \frac{L_4}{L_2}$, which are more reliable than the usual moment based measure and needs only the existence of L_1 as the condition for their finiteness .

3.6.4 Application to real data

The fact that all the three models presented in the last two sections contain as special cases several life distributions considered in literature. This points out the applicability of the models in a wide range of situations. Further all the models can represent different data situations either exactly or approximately show that they are flexibile. In this section we demonstrate the adequacy of two representative models in real life situations. Since our objective is confined only to verification of the adequacy of the models to real data, an extensive data analysis or refined methods of estimation comparing desirable properties is not attempted.

Our first analysis is based on model (3.39) against the data on survival times in days from a clinical trial on gastric carcinoma involving 90 patients quoted in (Kleinbaum (1996), p.296). The survival times alone is considered as a single set in fitting the distribution. For estimating the parameters, the method of percentiles is used by equating the sample and population percentiles at $u = 0.25, 0.50$ and 0.75. The population percentiles are derived from

$$Q(u) = \int_0^u K p^{\alpha} (1-p)^{-(A+\alpha)} dp$$

at the above u values. We obtain the estimates of the parameters as

$$\hat{\alpha} = -0.3128, \quad \hat{A} = 1.7692 \text{ and } \hat{K} = 296.267,$$

based on which the observed and expected frequencies are exhibited in Table 3.5.

Table 3.5 Observed and expected frequencies (Oi and Ei) for the Gastric Carcinoma data

interval	0 –111.77	111.77 –197	197 –289.42	289.42 –401	401 –550.38	550.38 –782	782 –1265.53	> 1265.53
O_i	13	10	9	15	14	10	14	10
E_i	12	12	12	12	12	12	12	11

The χ^2 value of 3.14 does not reject the hypothesis that (3.39) is appropriate for the data. The hazard quantile function for the above parameter values has upside-down bathtub shape as seen from the Figure 3.9.

Secondly we have considered the model (3.42) and the data on failure times of 50 devices given in (Lai & Xie 2006, p.353). Again the method of percentiles was used with the choice of $u = 0.25, 0.50$ and 0.75. Notice that in this case

$$Q(u) = \int_0^u K(1-p)^A\left(-\log(1-p)\right)^{-B} dp.$$

The estimates were found to be

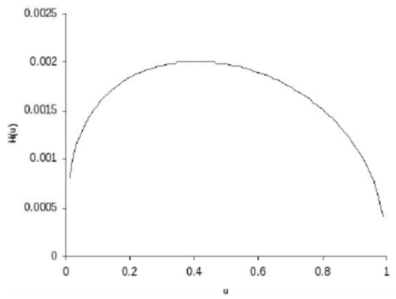
A=-1.8224, B= -1.2576 and k=875.927.

Table 3.6. Observed and expected frequencies for the failure time data given in Lai & Xie (2006).

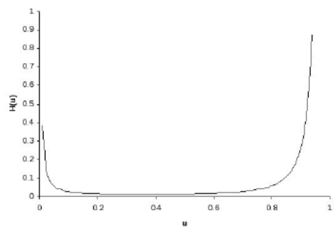
interval	0 –3.184	3.184 –13.5	13.5 –29.48	29.48 –48.5	48.5 –67.47	67.47 –83.25	> 83.25
O_i	9	4	6	6	9	6	8
E_i	6	6	6	6	6	6	14

The χ^2 value of 4.809 does not reject the model. The hazard quantile function for the above parameter values has bathtub shape as seen from Figure 3.9.

These two data sets have been analyzed earlier by means of distributions that do not form any of the exact particular cases that were discussed in connection with (3.39) and (3.42). This and our experiments with other data sets reveals that the quantile functions introduced here have the scope of approximating other life distributions that are not particular cases of the four models. An extensive study of the four models regarding their distributional properties as well as approximations they can offer to other acclaimed models will be presented as a future work.



Plot of estimates of $H(u)$ for the data on gastric carcinoma



Plot of estimates of $H(u)$ for the Aarset data

Figure 3.9

Chapter 4

Ageing Concepts*

4.1 Introduction

The notion of ageing plays an important role in reliability analysis and in identifying life distributions. Ageing represents the phenomenon by which the residual life of a unit is affected by its age in some probabilistic sense. Most of the ageing concepts exist in the literature are described on the basis of measures defined in terms of the distribution function. It is seen in Chapter 3 that many quantile functions can be utilized for the lifetime data analysis. When one wishes to analyze the ageing properties of such models, the existing definitions based on distribution function are not adequate. Thus, as a follow up to quantile-based analysis, in the present chapter, we review the existing definitions and express them in terms of quantile functions to facilitate a quantile-based analysis. The definitions and the properties of the basic ageing classes using the distribution function have been taken from Lai and Xie (2006) and the references for others are given in the text at the appropriate places. The ageing concepts are studied in three broad heads, those based on hazard quantile function, residual quantile functions and quantile functions. We also illustrate the various ageing concepts in the case of quantile functions.

**Part of the contents of this chapter has been published in Nair and Vineshkumar (2011), Statistics and Probability Letters (see reference no 92).*

The objectives of the work in the present chapter are manifold. Firstly it enables understanding of the failure mechanism of the unit under observation, through a distribution modelled by an appropriate quantile function. Secondly various concepts generate classes of life distributions, so that the identification of the model can be limited to that particular class. Lastly we have a new methodology that paves way for different kinds of analysis.

4.2 Ageing concepts based on hazard quantile function

Recalling the notations introduced in Chapter 2, a random variable X or its distribution function represented by $F(x)$ is said to be increasing hazard rate (IHR) (decreasing hazard rate (DHR) if the hazard function $h(x)$ is increasing (decreasing). In terms of survival function, F is IHR (DHR) if and only if for all t the survival function of the residual life,

$$\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}$$

is decreasing in t for all $x \geq 0$. The following proposition describes this concept based on quantile function. Since we are seeking equivalent conditions as in the conventional definition, the same names for the various concepts will be retained for the ageing classes under the quantile approach also.

Proposition 4.1

The random variable X has increasing hazard quantile function (IHR) (decreasing hazard quantile function (DHR)) if and only if any of the following equivalent conditions hold

(i) $H(u_2) \geq (\leq) H(u_1)$ for all $u_2 \geq u_1$, $0 \leq u_1, u_2 < 1$

(ii) $Q\big(v+(1-v)u\big) - Q(v)$ *is decreasing (increasing) function of* v

(iii) $H'(u) \geq (\leq) 0$, *provided* $H(u)$ *is differentiable.*

Proof: (i) follows from the condition $h(x_2) \geq (\leq) h(x_1)$ for all $x_2 \geq x_1$ for IHR (DHR), by setting $x_i = Q(u_i), i = 1, 2.$ To prove (ii) we first note that X is IHR (DHR) if and only if the survival function

$$\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}$$

of the residual life $X_t = \left(X - t \middle| X > t\right)$ is decreasing in t. The quantile function of X_t is $Q\left(v + (1-v)u\right) - Q(v).$ Assuming (ii),

$Q\left(v + (1-v)u\right) - Q(v)$ is decreasing in v

$$\begin{aligned}
&\Leftrightarrow q\left(v + (1-v)u\right)(1-u) \leq q(v) \\
&\Leftrightarrow \frac{1}{q(v)} \leq \frac{1}{(1-u)q\left(v + (1-v)u\right)} \\
&\Leftrightarrow \frac{1}{(1-v)q(v)} \leq \frac{1}{\left(1 - \left(v + (1-v)u\right)\right)q\left(v + (1-v)u\right)} \\
&\Leftrightarrow H(v) \leq H\left(v + (1-v)u\right) \\
&\Leftrightarrow X \text{ is IHR, by (i)}
\end{aligned}$$

Condition (iii) is obvious from

$$\frac{d}{dx}h(x) \geq 0 \Leftrightarrow \frac{d}{dQ(u)}h\left(Q(u)\right) \geq 0 \Leftrightarrow \frac{H'(u)}{q(u)} \geq 0$$

and $q(u) > 0$ since $Q(u)$ is an increasing function.

Remark 4.1: When $H'(u) = 0$ for all u, X is exponential.

Remark 4.2: The distribution F is bathtub failure rate distribution (BT) (upside down bathtub failure rate distribution (UBT)) if and only if $h'(x) < (>) 0$ for x in $(0, x_0)$, $h'(x_0) = 0$ and $h'(x) > (<) 0$ in (x_0, ∞). In terms of quantile function, if $H'(u) < (>) 0$ in $[0, u_0)$, $H'(u_0) = 0$ and

$H'(u) > (<) 0$ in $(u_0, 1)$, we say that the hazard quantile function is bathtub-shaped (BT) (upside-down bathtub-shaped (UBT)).

Example 4.1 From (3.7), the hazard quantile function of the lambda distribution by Ramberg and Schmeiser (1974) has the simple expression

$$H(u) = \lambda_2 (1-u)^{-1} \left[\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1} \right]^{-1}.$$

Its derivative is

$$H'(u) = \frac{\lambda_2 \left[\lambda_3 u^{\lambda_3 - 2} (\lambda_3 u - \lambda_3 + 1) + \lambda_4^2 (1-u)^{\lambda_4 - 1} \right]}{\left[\lambda_3 u^{\lambda_3 - 1} (1-u) + \lambda_4 (1-u)^{\lambda_4} \right]^2},$$

the sign of which depends on

$$g(u) = \lambda_2 \left[\lambda_4^2 (1-u)^{\lambda_4 - 1} + \lambda_3 u^{\lambda_3 - 2} (\lambda_3 u + 1 - \lambda_3) \right].$$

The distribution accommodates increasing, decreasing, BT and UBT shaped hazard quantile functions. To illustrate this we consider some special cases. When $\lambda_3 = 0$, the distribution is IHR if $\lambda_2 > 0$ and DHR if $\lambda_2 < 0$ subject to $\lambda_1 - \frac{1}{\lambda_2} > 0$. Setting $\lambda_4 = 0$,

$$g(u) = \lambda_2 \left[\lambda_3 u^{\lambda_3 - 2} (\lambda_3 u + 1 - \lambda_3) \right].$$

In this case $H(u)$ is increasing for points in $(\lambda_2 > 0,\ 0 < \lambda_3 < 1)$ and BT for values in $(\lambda_2 > 0,\ \lambda_3 > 1)$ with change point $u_0 = \frac{\lambda_3 - 1}{\lambda_3}$. Finally let $\lambda_3 = 2,\ \lambda_4 = 1$,

$$g(u) = \lambda_2 (4u - 1),$$

so that when $\lambda_2 > 0$, $g(u) = 0$ at $u = \frac{1}{4}$ and the distribution is UBT with change point at $u_0 = \frac{1}{4}$. In Figure 4.1 the shapes of hazard quantile function for some selected values of parameters are presented. The shapes of hazard quantile function for arbitrary choice of parameter values of other quantile models are given in the appendix of this chapter.

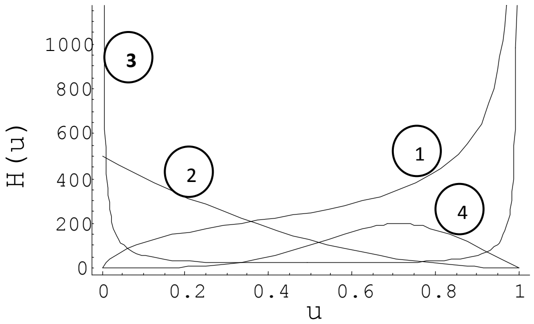


Figure 4.1-Shapes of hazard quantile function when (1) $\lambda_1 = 1$, $\lambda_2 = 100$, $\lambda_3 = 0.05$, $\lambda_4 = 0.5$, (2) $\lambda_1 = 0$, $\lambda_2 = -1000$, $\lambda_3 = 0, \lambda_4 = -2$ (3) $\lambda_1 = 1$, $\lambda_2 = 10$, $\lambda_3 = 2$, $\lambda_4 = 0$, (4) $\lambda_1 = 0$, $\lambda_2 = -1000$, $\lambda_3 = -2$, $\lambda_4 = -1$.

Another basic concept is increasing (decreasing) average hazard rate- IHRA (DHRA) defined by the condition $\frac{1}{x}\int_0^x h(t)dt$ is increasing (decreasing) in x. Equivalently the distribution is said to be IHRA (DHRA) if $-\left(\frac{1}{x}\right)\log \bar{F}(x)$ is increasing (decreasing) in $x \geq 0$. In this connection we have the following proposition.

Proposition 4.2

We say that X *is IHRA (DHRA) if and only if any one of the following conditions are satisfied*

(i) $\dfrac{\int_0^u H(p)q(p)dp}{\int_0^u q(p)dp} = \dfrac{1}{Q(u)}\int_0^u H(p)q(p)dp = \dfrac{-\log(1-u)}{Q(u)}$ *is increasing (decreasing)*

(ii) $H(u) \geq (\leq) \dfrac{-\log(1-u)}{Q(u)}$.

Proof: Recall that X is IHRA if and only if $\frac{1}{x}\int_0^x h(t)dt$ is increasing. Now

$$\frac{1}{x}\int_0^x h(t)dt \text{ is increasing } \Leftrightarrow \frac{1}{Q(u)}\int_0^u H(p)q(p)dp \text{ is increasing},$$

by setting $x = Q(u)$ and $t = Q(p)$, completes the proof of (i).

To prove (ii),

$$\frac{1}{Q(u)}\int_0^u H(p)q(p)dp \text{ is increasing} \Leftrightarrow Q(u)H(u)q(u)\text{-}q(u)\int_0^u H(p)q(p)dp \geq 0$$

$$\Leftrightarrow Q(u)H(u) - \int_0^u (1-p)^{-1}dp \geq 0$$

$$\Leftrightarrow H(u) \geq -\frac{1}{Q(u)}\log(1-u).$$

Remark 4.3 In the quantile formula IHRA takes into consideration a weighted average of $H(p)$ with weight $\frac{q(p)}{Q(u)}$ for p in $(0, u)$.

Some other notions which are less frequently used in analysis are new better than used in hazard rate (NBUHR), increasing hazard rate of order 2 (IHR(2)), new better than used in hazard rate average (NBUHRA) and IHRA*t_0, and their duals. A lifetime X is

(i) NBUHR if and only if $h(0) \leq h(x)$ for all x (Loh (1984)).

(ii) NBUHRA if and only if $h(0) \leq \frac{1}{x}\int_0^x h(t)dt, \quad x > 0$

(Loh (1984)).

(iii) IHR (2) if and only if $\int_0^x \overline{F}_s(t)dt \geq \int_0^x \overline{F}_u(t)dt$

$for\ all\ x \geq 0,\ u \geq s$ (Deshpande et al. (1986)).

(iv) IHRA*x_0 if and only if for all $x \geq x_0$,

$\frac{1}{x}\int_0^x h(t)dt \leq \frac{1}{x_0}\int_0^{x_0} h(t)dt$ (Li and Li (1998)).

From the above definitions the following results are straight forward.

Proposition 4.3

(i) *X is NBUHR (NWUHR) if and only if*

$H(0) \le (\ge) H(u),\ 0 \le u \le 1$

(ii) *X is NBUHRA (NWUHRA) if and only if*

$$H(0) \le (\ge) \frac{\int_0^u H(p)q(p)dp}{\int_0^u q(p)dp},\ \textit{for all } u.$$

(iii) *X is IHR(2) if and only if*

$$\int_0^u \left[Q\left(t+(1-t)v\right)-Q(t)\right]q(v)dv \ge \int_0^u \left[Q\left(s+(1-s)v\right)-Q(s)\right]q(v)dv$$

for all $u \ge 0$ and $t \ge s$.

(iv) *X is IHRA* u_0 if and only if*

$$\frac{\int_0^u H(p)q(p)dp}{Q(u)} \le \frac{\int_0^{u_0} H(p)q(p)dp}{Q(u_0)} \text{ for all } u \ge u_0.$$

Example 4.2 It has been of interest in reliability theory to find distributions whose hazard rates $h(x)$ have simple functional forms like linear, quadratic, reciprocal linear, etc. in x. In a similar fashion we seek distributions for which hazard quantile function is linear, that is

$$H(u) = a + bu.$$

From the representation (equation (2.73))

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)},$$

we have

$$Q(u) = \frac{1}{a+b}\log\frac{a+bu}{1-u} + C. \tag{4.1}$$

Setting $u=0$, the constant C is determined as $C=-(a+b)^{-1}\log a$. Thus the distribution of X with linear $H(u)$ is

$$Q(u)=\log\left(\frac{a+bu}{a(1-u)}\right)^{\frac{1}{a+b}},\ 0\le u\le 1,\ a>0 \tag{4.2}$$

with quantile density function

$$q(u)=\left[(1-u)(a+bu)\right]^{-1}. \tag{4.3}$$

In fact (4.2) represents a family of distributions. When $a>0,\ b=0$

$$Q(u)=-\frac{1}{a}\log(1-u),$$

which is exponential with mean a. When $a=b>0$

$$Q(u)=\frac{1}{2a}\log\frac{1+u}{1-u}$$

corresponds to the half logistic distribution.

Taking $a>0,\ b=-ap,\ 0<p<1,$

$$Q(u)=\frac{1}{\lambda}\log\frac{1-pu}{1-u},\quad \lambda=a(1-p)$$

is the quantile function of the exponential-geometric distribution of Adamidis et al. (2005). Notice that the family is IHR, IHRA and NBUHR for $b>0$.

Example 4.3 For the Govindarajulu model discussed in Section 3.5, as illustrated in Section 3.5.5, we conclude that X is IHR for $\beta\le 1$ and bathtub-shaped for $\beta>1$ with change point at $u=\dfrac{\beta-1}{\beta+1}$.

4.3 Concepts based on residual function

First we discuss the concepts based on the mean of the residual life. Recall the definitions given in (2.48) and (2.75). Based on the distribution functions, F is said to be in decreasing (increasing) mean residual life –DMRL (IMRL) class if $m(x)$ is a decreasing (increasing) function in $x>0$. That is, $m(s)\geq(\leq)m(t)$ for $0\leq s\leq t$. Equivalent conditions of this ageing concept in terms of quantile function are given in the following proposition.

Proposition 4.4

We say that X is decreasing mean residual quantile function (DMRL) (increasing mean residual quantile function (IMRL)) if and only if one of the following equivalent conditions hold

(i) $M(u_1)\leq(\geq)M(u_2),\ u_1\geq u_2$

(ii) $\int_0^1\left\{Q\left(u+(1-u)p\right)-Q(u)\right\}dp$ *is decreasing (increasing) in* u

(iii) $M'(u)\leq(\geq)0$

(iv) $M(u)\leq(\geq)\dfrac{1}{H(u)}$

Proof: To prove (i), from the condition

$$X \text{ is DMRL} \Rightarrow m(s)\geq(\leq)m(t) \text{ for } 0\leq s\leq t$$

$$\Rightarrow M(u_1)\leq M(u_2),\ u_1\geq u_2$$

by setting $t=Q(u_1)$ and $s=Q(u_2)$.

We have from equation (2.75)

$$M(u)=(1-u)^{-1}\int_u^1\left[Q(p)-Q(u)\right]dp.$$

Substituting $t=u+(1-u)p$, we have

$$\int_0^1\left[Q\left(u+(1-u)p\right)-Q(u)\right]dp=M(u).$$

Now assume (i), which means

$$M(u) \text{ is decreasing} \Leftrightarrow \int_0^1\left[Q\left(u+(1-u)p\right)-Q(u)\right]dp \text{ is decreasing}$$

$$\Rightarrow \text{(ii)}.$$

Clearly (iii) $\Leftrightarrow$ (ii). We have from (2.77)

$$M(u) = (1-u)^{-1} \int_u^1 (H(p))^{-1} dp.$$

Differentiating

$$(1-u)M'(u) - M(u) = -\frac{1}{H(u)}$$

or

$$M'(u) = \frac{1}{(1-u)}\left(M(u) - \frac{1}{H(u)}\right).$$

Now

$$\begin{aligned} M'(u) \le 0 &\Leftrightarrow \frac{1}{(1-u)}\left(M(u) - \frac{1}{H(u)}\right) \le 0 \\ &\Leftrightarrow M(u) \le \frac{1}{H(u)}. \end{aligned}$$

Hence (iii) $\Leftrightarrow$ (iv), which completes the proof.

Example 4.4 For the linear hazard quantile function distribution (4.2)

$$\begin{aligned} M(u) &= \frac{1}{1-u}\int_u^1 (Q(p) - Q(u))dp \\ &= \frac{1}{b(1-u)} \log \frac{a+b}{a+bu}. \end{aligned}$$

Note that

$$\frac{1}{b(1-u)} \log \frac{a+b}{a+bu} \le \frac{a+b}{(a+bu)b(1-u)} \le \frac{1}{a+bu} = \frac{1}{H(u)}$$

for $b < 0$ and hence X is DMRL by (iv) of Proposition 4.4.

Four other ageing properties involving mean residual life are net decreasing (increasing) mean residual life (NDMRL (NIMRL)) defined by $m(x) \le (\ge) m(0)$, used better (worse) than aged (UBA (UWA)) defined by (Alzaid (1994))

$$\overline{F}_t(x) \geq (\leq) \exp\left[-\frac{x}{m(\infty)}\right],\ m(\infty) < \infty,$$

used better (worse) than aged in expectation, UBAE (UWAE) (Alzaid (1994)) satisfying

$$m(x) \geq (\leq) m(\infty)$$

and decreasing (increasing) mean residual life in harmonic average, DMRLHA (IMRLHA) that satisfies the condition (Deshpande et al. (1986))

$$\left[\frac{1}{x}\int_0^x \frac{dt}{m(t)}\right]^{-1} \text{ decreasing (increasing) in } x.$$

In the following proposition, the above definitions are expressed based on quantile functions.

Proposition 4.5

A lifetime random variable X with $M(1) = \lim_{u \to 1^-} M(u) < \infty$ is

(i) *net decreasing (increasing) mean residual function, NDMRL (IDMRL) if and only if $M(u) \leq (\geq) M(0)$*

(ii) *UBA (UWA) if and only if*

$$Q\big(u + (1-u)v\big) - Q(u) \geq (\leq) -\frac{1}{M(1)}\log(1-u) \text{ for all}$$

$0 \leq u,\ v < 1$.

(iii) *UBAE (UWAE) if and only if $M(u) \geq (\leq) M(1)$ for all $0 < u < 1$*

(iv) *DMRLHA (IMRLHA) if and only if* $\dfrac{\int_0^u \frac{q(p)}{M(p)}dp}{\int_0^u q(p)dp}$ *is (decreasing) in u.*

Proof: To prove (i), we know that X is NDMRL if and only if

$$m(x) \le m(0).$$

Setting $x = Q(u)$ and noting $Q(0) = 0$, we have X is NDMRL if and only if

$$M(u) \le M(0),$$

which proves the assertion (i). The random variable X is UBA if and only if

$$\overline{F}_t(x) \ge \exp\left[-\frac{x}{m(\infty)}\right],\ m(\infty) < \infty.$$

In the above inequality, left side is the survival function of $X_t = X - t | X > t$ and right side is the survival function of the exponential distribution with mean $m(\infty) < \infty$. Recalling from Section 2.1.2, the quantile function of X_t is

$$Q_1(u) = Q\big(v + (1-v)u\big) - Q(v)$$

by setting $F(t) = v$ and $F_t(x) = u$. The quantile function of the exponential distribution mentioned above is

$$Q_E(u) = \frac{-1}{M(1)} \log(1-u),$$

where $m(\infty) = \lim_{u \to 1^-} M(u) = M(1)$. Thus X is UBA if and only if

$$1 - \left[Q\big(v + (1-v)u\big) - Q(v)\right] \le 1 - \frac{-1}{M(1)} \log(1-u)$$

or

$$\left[Q\big(v + (1-v)u\big) - Q(v)\right] \ge \frac{-1}{M(1)} \log(1-u).$$

The proof of (iii) is straight forward from the definition of UBAE. To prove (iv), note that

$$X \text{ is DMRLHA} \Rightarrow \left[\frac{1}{x}\int_0^x \frac{dt}{m(t)}\right]^{-1} \text{ is decreasing}.$$

Setting $x = Q(u)$, the above condition is equivalent to

$$\left[\frac{1}{Q(u)}\int_0^u \frac{1}{M(p)}q(p)dp\right]^{-1} \text{ is decreasing}$$

$$\Rightarrow \left[\frac{\int_0^u \frac{1}{M(p)}q(p)dp}{\int_0^u q(p)dp}\right]^{-1} \quad \text{is decreasing}$$

$$\Rightarrow \frac{\int_0^u \frac{1}{M(p)}q(p)dp}{\int_0^u q(p)dp} \text{ is increasing.}$$

This completes the proof.

Other ageing concept based on residual quantile function is in connection with the variance of residual function (VRL) discussed in Section 2.2.3 and its quantile version given in Section 2.4.3. The random variable X has decreasing (increasing) variance residual life, abbreviated as DVRL (IVRL) if and only if $\sigma^2(x)$ is decreasing (increasing) in x. Recall the quantile-based definitions of VRL given in (2.70) through (2.72) and the coefficient of variation in (2.73). We have the following proposition for DVRL (IVRL) in terms of quantile function.

Proposition 4.6

The following conditions are equivalent

(i) X *is DVRL (IVRL)*

(ii) $C(u) \le (\ge) 1, \ where \ C^2(u) = \dfrac{V(u)}{M^2(u)}$ (4.4)

Proof: Assume that (i) is true. To prove (ii) we have

$$V(u) = \frac{1}{1-u}\int_u^1 M^2(p)dp.$$

Differentiating

$$V(u)-(1-u)V'(u)=M^2(u)$$

or

$$(1-u)V'(u)=V(u)-M^2(u),$$

since X is DVRL,

$$V'(u)\le 0 \Leftrightarrow V(u)-M^2(u)\le 0$$
$$\Leftrightarrow C(u)\le 1,$$

which proves the assertion.

Another version of mean residual life is obtained by considering the mean of the asymptotic distribution of residual life given survival beyond age x, called the renewal mean residual life (RMRL) (Nair and Sankaran (2010)) defined as

$$m_R(x)=\frac{\int_x^{\infty}(t-x)\overline{F}(t)dt}{\int_x^{\infty}\overline{F}(t)dt}.$$

Setting $x=Q(u)$, we have the renewal mean residual quantile function

$$M_R(u)=\frac{\int_u^1\big(Q(p)-Q(u)\big)(1-p)q(p)dp}{\int_u^1(1-p)q(p)dp}. \tag{4.5}$$

A lifetime variable X is decreasing (increasing) RMRL, DRMRL (IRMRL) if and only if $m_R(x)$ or equivalently $M_R(u)$ is decreasing (increasing) in x (u).

Proposition 4.7

X, DRMRL (IRMRL) $\Leftrightarrow M_R(u)\le(\ge)M(u)$ for all u.

Proof: We have

$$M_R(u) = \frac{\int_u^1 \big(Q(p)-Q(u)\big)(1-p)q(p)dp}{\int_u^1 (1-p)q(p)dp}$$

or

$$M_R(u)\int_u^1 (1-p)q(p)dp = \int_u^1 \big(Q(p)-Q(u)\big)(1-p)q(p)dp\,.$$

Differentiating, we get

$$-(1-u)q(u)M_R(u) + M_R'(u)(1-u)M(u) = -q(u)(1-u)M(u)$$

or

$$M(u) = \frac{M_R(u)q(u)}{q(u)+M_R'(u)}$$

Thus X is DRMRL (IRMRL) $\Leftrightarrow M_R(u) \le (\ge) M(u)$.

Various researchers have used percentile residual life (PRL) defined in Section 2.2.4 to define ageing classes. Important ageing concepts based on PRL are decreasing $\alpha-$percentile residual life (DPRL-α) and new better than used with respect to the $\alpha-$percentile residual life (NBUP-α) and their duals. We say that F is DPRL-α if $p_\alpha(t)$ is decreasing in t and NBUP-α if $p_\alpha(0) \ge p_\alpha(t)$ for all t. For the earlier development of these ageing classes we refer to Haines and Singpurwalla (1974) and Joe and Prochan (1984). Recently Franco-Pereira et al. (2010) have proved the equivalence of the following conditions for F to be DPRL-α.

(i) $p_\alpha(t)$ is decreasing

(ii) $(1-\alpha)f(t) \le f\left[\overline{F}^{-1}\left((1-\alpha)\overline{F}(t)\right)\right]$

(iii) $(1-\alpha)f\left(F^{-1}(u)\right) \le f\left(\overline{F}^{-1}\left((1-\alpha)u\right)\right), \quad 0<u<1$

(iv) $h(t) \le h\left(t + p_\alpha(t)\right), \ t \in (0, T)$.

Thus we have the following proposition.

Proposition 4.8

If X is DPRL-α, then the following conditions are equivalent.

(i) $P_\alpha(u)$ is decreasing.

(ii) $q(u) \ge (1-\alpha)q\left[1-(1-\alpha)(1-u)\right], \ 0 < u < 1$

(iii) $H(u) \le H\left(1-(1-\alpha)(1-u)\right)$

Proof: Note that

$$
\begin{aligned}
X \text{ is DPRL-}\alpha &\Rightarrow p_\alpha(t) \text{ is decreasing} \\
&\Rightarrow \frac{d}{dt} p_\alpha(t) \le 0 \\
&\Rightarrow \frac{d}{dQ(u)} p_\alpha(Q(u)) \le 0 \\
&\Rightarrow \frac{P'_\alpha(u)}{q(u)} \le 0 \\
&\Rightarrow P_\alpha(u) \ \text{ is } \ \text{decreasing}
\end{aligned}
$$

Assuming (i), we have

$$Q\left(1-(1-\alpha)(1-u)\right) - Q(u) \ \text{ is decreasing}$$

$$\Leftrightarrow q\left(1-(1-\alpha)(1-u)\right)(1-\alpha) - q(u) \le 0$$

Thus (i) $\Rightarrow$ (ii).

Also from (ii)

$$
\begin{aligned}
q(u) &\ge (1-\alpha)q\left[1-(1-\alpha)(1-u)\right] \\
&\Leftrightarrow (1-u)q(u) \ge (1-\alpha)(1-u)q\left[1-(1-\alpha)(1-u)\right] \\
&\Leftrightarrow (1-u)q(u) \ge \left[1-\left(1-(1-\alpha)(1-u)\right)\right]q\left[1-(1-\alpha)(1-u)\right] \\
&\Leftrightarrow H(u) \le H\left[1-(1-\alpha)(1-u)\right]
\end{aligned}
$$

means (ii) $\Leftrightarrow$ (iii), which completes the proof.

In quantile terminology, X is *new better than used with respect to the* $\alpha-$*percentile residual life* (NBUP- α) if $P_\alpha(0) \geq P_\alpha(u)$.

4.4 Concepts based on survival functions

The ageing properties in this class are obtained by comparing survival at different points of time. Most important among them are the new better (worse) than used, NBU (NWU) and those generated from it like NBUE, HNBUE, etc.. We say that X is NBU (NWU) if and only if

$$\overline{F}(x+t) \leq (\geq)\overline{F}(x)\overline{F}(t), \text{ for all } x,\ t>0.$$

Based on this definition we have the following proposition.

Proposition 4.9

A lifetime variable X with quantile function $Q(u)$ is NBU (NWU) if and only if

$$Q(u+v-uv) \leq (\geq) Q(u)+Q(v) \text{ for all } u,\ v. \tag{4.6}$$

Proof: The random variable X is NBU (NWU) if and only if

$$\overline{F}(x+t) \leq (\geq)\overline{F}(x)\overline{F}(t), \text{ for all } x,\ t>0.$$

Setting $x = Q(u)$ and $t = Q(v)$, so that $x+t = Q(u)+Q(v)$. Now

$$\begin{aligned}\overline{F}(x+t) \leq \overline{F}(x)\overline{F}(t) &\Rightarrow 1-F(x+t) \leq (1-u)(1-v)\\ &\Rightarrow F(x+t) \geq u+v-uv\\ &\Rightarrow Q(u)+Q(v) \geq Q(u+v-uv),\end{aligned}$$

as asserted.

The equality sign in (4.6) holds good when

$$Q\big(1-(1-u)(1-v)\big) = Q\big(1-(1-u)\big)+Q\big(1-(1-v)\big)$$

or

$$Q\big(1-(1-u)(1-v)\big) = Q\big(1-(1-u)\big)+Q\big(1-(1-v)\big), \tag{4.7}$$

which reduces to the form

$$Q(1-x_1y_1)=Q(1-x_1)+Q(1-y_1).$$

The last equation transform to the Cauchy functional equation

$$a(xy)=a(x)+a(y)$$

with $a(x)=Q(1-x)$.

The only continuous solution to the above functional equation is

$$a(x)=k\log x\,.$$

Thus

$$Q(u)=k\log(1-u),$$

and for this to be a quantile function we must have $k=-c,\ c>0$. Hence

$$Q(u)=-c\log(1-u),$$

represents the exponential law.

Example 4.5 The power-Pareto law in Section 3.3, specified by the quantile function

$$Q(u)=Cu^{\lambda_1}(1-u)^{-\lambda_2}\quad C,\lambda_1,\lambda_2>0,\ 0<u<1$$

contains both NBU and NWU distributions. An obvious case is when $\lambda_2=0$, the model represents the power distribution, which is NBU. On the other hand when $\lambda_1=0$, $Q_3(u)$ becomes Pareto and hence NWU. In general, by (4.6) the NBU (NWU) cases are sorted out from the inequality

$$(u+v-uv)^{\lambda_1}\le(\ge)u^{\lambda_1}(1-v)^{\lambda_2}+(1-u)^{\lambda_2}v^{\lambda_2}\ \text{ for all }\ u,v\,.$$

For example, when $\lambda_1=\lambda_2=1$, we have log-logistic distribution that is NWU.

There are some generalizations of the NBU concepts in the form of NBU*t_0, NBU- t_0, NBU(2) and NBU(2)-t_0 along with the corresponding dual classes. A distribution is a NBU-t_0 (NWU-t_0) class of life distribution if it satisfies

$$\bar{F}(t_0 + x) \le (\ge) \bar{F}(t_0)\bar{F}(x),\ x \ge 0.$$

See Hollander et al. (1985) for details. We say that, X is NBU*t_0 (NWU*t_0) if

$$\bar{F}(x+y) \le (\ge) \bar{F}(x)\bar{F}(y),\ x \ge 0, 0 < t_0 < y \quad \text{(Li and Li (1998))}.$$

The difference between NBU- t_0 and NBU*t_0 is that in the former t_0 is a fixed time while in the later it extends beyond t_0. A lifetime random variable X is said to be NBU(2) (NWU (2)) if and only if

$$\int_0^x \bar{F}(y)dy \ge (\le) \int_0^x \frac{\bar{F}(t+y)}{\bar{F}(t)}dy$$

for all $t, x \ge 0$. This class is obtained by Deshpande et al. (1986) and more details are available there. Elabatal (2007) studied the extension of NBU(2) class at specific age t_0. The identification of these classes of quantile functions is as follows. The proof follows from Proposition 4.9.

Proposition 4.10

We say that X is

(i) *NBU- u_0 (NWU- u_0) for some $0 \le u_0 < 1$,*

$$Q(u + u_0 - uu_0) \le (\ge) Q(u) + Q(u_0) \ \textit{for all}\ 0 \le u \le 1$$

(ii) *NBU* u_0 (NWU* u_0) if and only if*

$$Q(u + v - uv) \le (\ge) Q(u) + Q(v) \quad \textit{for all} \quad 0 \le u < 1 \quad \textit{and} \quad v \ge u_0$$

(iii) *NBU(2) (NWU (2)) if and only if*

$$\frac{1}{1-v}\int_0^u \left[1 - Q^{-1}(Q(p) + Q(v))\right]q(p)dp \le \int_0^u (1-p)q(p)dp, \textit{ for all } 0 \le u,v \le 1$$

(iv) *NBU(2)- u_0 if and only if*

$$\frac{1}{1-u_0}\int_0^u \left[1 - Q^{-1}\left(Q(p) + Q(u_0)\right)\right]q(p)dp \le \int_0^u (1-p)q(p)dp$$

for some u_0.

An integrated version of the NBU (NWU) definition leads to the NBUC (NWUC) class defined by Cao and Wang (1991). We say that X is NBUC (NWUC) if and only if

$$\int_x^\infty \overline{F}_t(y)dy \le (\ge) \int_x^\infty \overline{F}(y)dy$$

and it extends to NBUC-t_0 for a specific age t_0. The counterparts of these two classes in quantile form are presented in the following proposition.

Proposition 4.11

A lifetime variable X belongs to the ageing class

(i) *NBUC (NWUC) if and only if*

$$\frac{1}{1-v}\int_u^1 \left[1-Q^{-1}\left(Q(p)+Q(v)\right)\right]q(p)dp \le (\ge) \int_u^1 (1-p)q(p)dp$$

(ii) *NBUC- u_0 (NWUC- u_0) if and only if for some u_0 in* $[0,1)$

$$\frac{1}{1-u_0}\int_u^1 \left[1-Q^{-1}\left(Q(p)+Q(u_0)\right)\right]q(p)dp \le \int_u^1 (1-p)q(p)dp$$

Proof: To prove (i), setting $F(t)=v$ and $F(y)=p$, so that $t=Q(v)$, $y=Q(p)$ and $dy=q(p)dp$, the condition

$$\int_x^\infty \overline{F}_t(y)dy \le \int_x^\infty \overline{F}(y)dy$$

or

$$\frac{1}{\overline{F}(t)}\int_x^\infty \overline{F}(y+t)dy \le \int_x^\infty \overline{F}(y)dy$$

is equivalent to

$$\frac{1}{1-v}\int_u^1 \left[1-Q^{-1}\left(Q(p)+Q(v)\right)\right]q(p)dp \le \int_u^1 (1-p)q(p)dp,$$

since $$F\left(y+t\right)=1-F\left(y+t\right)=1-Q^{-1}\left(Q\left(p\right)+Q\left(u\right)\right).$$

The proof of (ii) is similar to that of (i) by taking $v = u_0$, a fixed value in $(0, 1)$.

The (ii) part of Proposition 4.11 brings its relationship with NBU(2) (NWU (2)) and NBUC (NWUC). Finally, we have still larger class called harmonically new better (worse) than used in expectation (HNBUE (HNWUE)), which is defined by

$$\int_x^\infty \overline{F}(t)dt \le \mu \exp\left(-\frac{x}{\mu}\right) \text{ for all } x \ge 0.$$

This leads to the next proposition.

Proposition 4.12

The HNBUE (HNWUE) property holds for X if and only if

(i) $$\int_u^1 (1-p)q(p)dp \le (\ge) \mu e^{-\frac{Q(u)}{\mu}}$$

(ii) $$\frac{\int_0^u \frac{q(p)}{M(p)} dp}{\int_0^u q(p)dp} \ge (\le) \frac{1}{\mu}.$$

Proof: The proof of (i) is straight forward from the definition of HNBUE by setting $F(t) = p$ and $F(x) = u$. To prove the equivalence of (i) and (ii),

$$\text{(ii)} \Leftrightarrow \int_0^u q(p)\left(\frac{1}{1-p}\int_p^1 (1-t)q(t)dt\right)^{-1} dp \ge \frac{Q(u)}{\mu}$$

$$\Leftrightarrow \int_0^u \frac{q(p)(1-p)dp}{\int_p^1 (1-t)q(t)dt} \ge \frac{Q(u)}{\mu}$$

$$\Leftrightarrow -\log\left(\mu - \int_0^u (1-t)q(t)dt\right) \ge \frac{Q(u)}{\mu}$$

(by noting $\int_p^1 (1-t)q(t)dt = \mu - \int_0^p (1-t)q(t)dt$)

$\Leftrightarrow$ (i)

This completes the proof.

The equilibrium distribution specified by the density function (of a random variable Z, say)

$$f_Z(x) = \frac{\overline{F}(x)}{\mu}, \ x > 0. \tag{4.8}$$

plays an important role in evolving new ageing classes and also in proving relationships between various concepts. Equation (4.8) is obtained as the asymptotic distribution of age or residual life (or of forward and backward recurrence times) in renewal theory. The distribution function of Z becomes

$$F_Z(x) = \frac{\int_0^x \overline{F}(t)dt}{\mu}.$$

Setting $x = Q(u)$

$$F_Z\left(Q(u)\right) = \frac{1}{\mu}\int_0^u (1-p)q(p)dp.$$

Note that $T(u) = \int_0^u (1-p)q(p)dp$ is the quantile version of the total time on test transform of X and $\phi(u) = \mu^{-1}T(u)$ is the scaled transform (See Section 2.5 for further details). Hence

$$Q(u) = Q_Z(\phi(u))$$

and

$$Q_Z(u) = Q\left(\phi^{-1}(u)\right), \tag{4.9}$$

give the relationship between the quantile functions of Z and X. Since $T(u)$ is also a quantile function and denoting the random variable corresponding to $T(u)$ as X_T, $\phi(u)$ is the quantile function of $\mu^{-1}X_T$. These results help the analysis of equilibrium distributions in terms of quantile functions. As an example, $\phi(u) = u$ for the exponential distribution and hence X and Z are identically distributed. Also $\phi^{-1}(u)$ is the distribution function of a uniform random variable over $(0, 1)$.

Example 4.6 The linear hazard quantile family of distributions discussed in Example 4.2 has the quantile density function

$$q(u) = \left[(1-u)(a+bu)\right]^{-1}$$

and hence

$$\phi(u) = \frac{\int_0^u (1-p)q(p)dp}{\mu}$$
$$= \frac{\log\left(\frac{a+bu}{a}\right)}{\log\left(\frac{a+b}{a}\right)}.$$

The inverse of $\phi(u)$ is

$$\phi^{-1}(u) = \frac{a}{b}\left(\left(\frac{a+b}{a}\right)^u - 1\right), \quad 0 \le u \le 1,$$

a distribution function on $(0, 1)$. Hence, from (4.9)

$$Q_Z(u) = \log\frac{b}{a}\frac{(a+b)^{u-1}}{a^{u-1}-(a+b)^{u-1}}, \quad 0 \le u \le 1$$

is the quantile function of the equilibrium distribution.

There are few ageing classes that involve Z and its residual life $Z_t = (Z - t | Z > t)$. One is new better (worse) than renewal used (NBRU (NWRU)), which is identical with NBUC (NWUC), discussed earlier. The renewal new is better (worse) than used (RNBU (RNWU)) is defined by (Abouammoh et al. (2000))

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} \le (\ge) \frac{1}{\mu}\int_x^\infty \bar{F}(u)du, \tag{4.10}$$

while its integrated version

$$E(X_t) \le (\ge) E(Z) \tag{4.11}$$

is the renewal new better (worse) than used in expectation (RNBUE (RNWUE)). Further we have renewal new is better than renewal used (RNBRU) and its dual RNWRU are defined by

$$\mu \int_{x+t}^{\infty} \overline{F}(u)du \le (\ge) \int_{x}^{\infty} \overline{F}(u)du \int_{t}^{\infty} \overline{F}(u)du$$

and the corresponding integrated version in renewal new better (worse) than renewal used in expectation (RNBRUE (RNWRUE)), whenever

$$E\left(Z_t\right) \le (\ge) E(Z). \tag{4.13}$$

Alternative expressions for the classes in terms of quantile functions are given below. The proof of this proposition is straight forward by following the steps in the proof of Proposition 4.11.

Proposition 4.13

A lifetime variable X belongs to

(i) *RNBU (RNWU) class if and only if*

$$\frac{1}{1-v}\left[1 - Q^{-1}\left(Q(u) + Q(v)\right)\right] \le (\ge) \frac{1}{\mu} \int_{u}^{1} (1-p)q(p)dp$$

(ii) *RNBRU (RNWRU) class if and only if*

$$\mu \int_{u}^{1} \left[1 - Q^{-1}\left(Q(p) + Q(u)\right)\right] q(p)dp$$
$$\le (\ge) \left(\int_{u}^{1} (1-p)q(p)dp\right)\left(\int_{v}^{1} (1-p)q(p)dp\right)$$

Proof: To prove (i), set $F(t) = v,\ F(x) = u,$ so that

$$F(x+t) = Q^{-1}\left(Q\left(u\right) + Q\left(v\right)\right),$$

and take

$$F(s) = p \Rightarrow \bar{F}\left(s\right)ds = \left(1-p\right)q\left(p\right)dp,$$

the definition of RNBU given in (4.10) becomes

$$\frac{1-Q^{-1}\left(Q(u)+Q(v)\right)}{1-v}\leq\frac{1}{\mu}\int_{u}^{1}(1-p)q(p)dp,$$

as asserted. With the above substitutions the proof of (ii) is direct from (4.12).

Remark 4.4 The expectations in (4.11) and (4.13) are obtained by integrating the quantile functions of the variables X_t, Z and Z_t between 0 and 1.

In conclusion, we have provided the definitions of various ageing classes in terms of quantile functions. These definitions become essential when we deal with life distributions specified by quantile functions, especially when they do not have tractable forms of distribution functions. We have given several examples that illustrate this situation.

Appendix

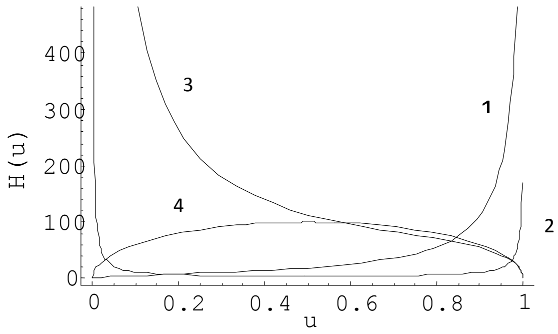


Figure 1-Shapes of hazard quantile function of Power Pareto model when (1) $C = 0.1,\ \lambda_1 = 0.5,\ \lambda_2 = 0.01$ (2) $C = 0.5,\ \lambda_1 = 2,\ \lambda_2 = 0.01$ (3) $C = 0.01,\ \lambda_1 = 2,\ \lambda_2 = 0.5$ (4) $C = 0.01,\ \lambda_1 = 0.5,\ \lambda_2 = 0.5$

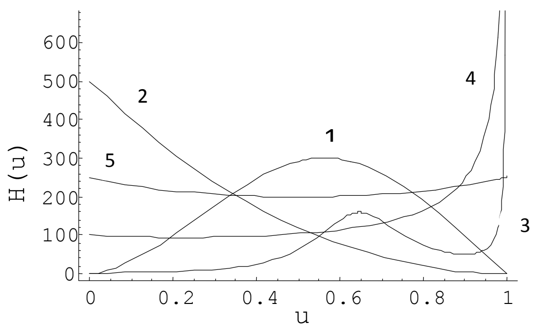


Figure 2-Shapes of hazard quantile function of Freimer et al. model when (1)$\lambda_1 = 0,\ \lambda_2 = 100,\ \lambda_3 = -0.5, \lambda_4 = -0.1$, (2) $\lambda_1 = 0,\ \lambda_2 = 500,\ \lambda_3 = 3, \lambda_4 = 2$ (3)$\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 10, \lambda_4 = 5$, (4) $\lambda_1 = 0,\ \lambda_2 = 100,\ \lambda_3 = 2, \lambda_4 = 0.5$ (5) $\lambda_1 = 0,\ \lambda_2 = 250,\ \lambda_3 = 2, \lambda_4 = 0.001$

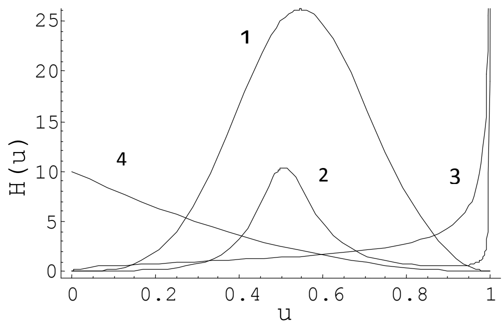


Figure 3-Shapes of hazard quantile function of Staden and Loots model when (1)$\lambda_1 = 0,\ \lambda_2 = 0.01,\ \lambda_3 = 0.5, \lambda_4 = -2$, (2) $\lambda_1 = 0,\ \lambda_2 = 100,\ \lambda_3 = 0.5, \lambda_4 = 10$ (3)$\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 0.6, \lambda_4 = 0.5$, (4) $\lambda_1 = 0,\ \lambda_2 = 0.1,\ \lambda_3 = 1, \lambda_4 = -5$

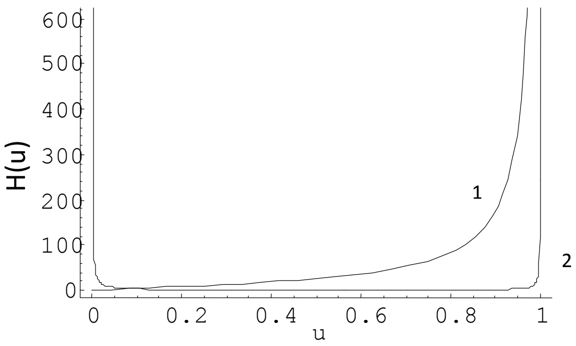


Figure 4-Shapes of hazard quantile function of Govindarajulu model when (1) ß =0.1, (2) ß =2

Chapter 5

Total time on test transform of order *n**

5.1 Introduction

We have discussed the concept of total time on test transform and some of its properties in the context of reliability analysis in the Section 2.5. Although introduced in the early seventies new results and applications in connection with TTT continue to appear in literature. It is seen that from the potential of TTT in different applied fields, its generalizations have been studied by different researchers like Barlow and Doksum (1972).

In the present chapter we study a slightly different version of TTT, named TTT of order n by an iteration of the definition given in (2.99). In the next section we present a quantile-based definition of TTT of order n (TTT- n) and derive some identities connecting the reliability functions of the baseline and transformed distributions. Characterizations of some quantile functions by properties of the n^{th} order transforms, comparison of ageing properties of the initial and transformed distributions, various order relations connection with the n^{th} transform and their applications are discussed in the subsequent sections. It is shown that the

**Part of the materials in this chapter has appeared in Nair, Sankaran and Vineshkumar (2008), Journal of Applied Probability (see reference no. 90)* generalization proposed has potential to wider the area of application than the usual notion of TTT.

5.2 Definition and properties of TTT-*n*

Recall the definition of TTT, given in equation (2.99)

$$T(u)=\int_0^u (1-p)q(p)dp \quad ,$$

and the relationships

$$t(u)=T'(u)=\left(H(u)\right)^{-1}$$

and

$$T(u)=\mu-(1-u)M(u),$$

given in Section 2.5.

Definition 5.1 The TTT of order n of a random variable X is defined as

$$T_n(u)=\int_0^u (1-p)t_{n-1}(p)dp \quad , \qquad n=1,2,... \tag{5.1}$$

with $T_0(u)=Q(u)$ and $t_n(u)=T_n{}'(u)$ provided that

$$\mu_{n-1}=\int_0^1 T_{n-1}(p)dp<\infty.,$$

with $T_1(u)=T(u)$, the usual transform. It may be noted that in (5.1) $T_n(u)$ is also a quantile function. For instance $T_1(u)$ is a quantile function with support $[0,\ \mu]$. It also follows that $t_0(u)=q(u)$.

We denote by X_n the random variable (where $X_0=X$) with quantile function $T_n(u)$, mean μ_n, hazard quantile function $H_n(u)$ and mean residual quantile function $M_n(u)$. Differentiating (5.1) we have

$$t_n(u)=(1-u)t_{n-1}(u)=\left(H_{n-1}(u)\right)^{-1} \tag{5.2}$$

and

$$t_n(u) = (1-u)^n t_0(u) = (1-u)^n q(u) = (1-u)^{n-1}(H(u))^{-1}. \tag{5.3}$$

From (5.2) and (5.3), we have the identity connecting the hazard quantile function of X and X_n as

$$H(u) = (1-u)^n H_n(u), \qquad n = 0, 1, 2, \ldots, \tag{5.4}$$

with $H_0(u) = H(u)$, representing the hazard rate of $X_0 = X$.

Using (2.101) $T_n(u)$ and $M_n(u)$ are related as

$$T_{n+1}(u) = \mu_n - (1-u)M_n(u), \tag{5.5}$$

from which

$$t_{n+1}(u) = M_n(u) - (1-u)M_n'(u). \tag{5.6}$$

This along with $t_{n+1}(u) = (1-u)^n t_1(u)$ and (5.6) specified for n =0 gives the following relationships between the mean residual quantile function of X and X_n

$$M_n(u) - (1-u)M_n'(u) = (1-u)^n \left[M(u) - (1-u)M'(u)\right]. \tag{5.7}$$

Some important life distribution along with the expressions for $Q(u)$, $H(u)$ and $t_n(u)$ are exhibited in Table 5.1 given at the end of the chapter to enable calculation of the above functions for these distributions.

Remark 5.1 Definition 5.1 extends to negative integers as well. For example $Q(u)$ can be considered as the transform of $T_{-1}(u)$, etc.. In this backward recurrence,

$$t_{-n}(u) = (1-u)^{-n} q(u)$$

and

$$H(u) = (1-u)^{-n} H_n(u), \; n = 1, 2, 3, \ldots .$$

Equivalently, one can assume a given distributional form for X_n and revert to the distribution of X.

Remark 5.2 From (5.4), it is clear that the sequence $\langle H_n(u) \rangle$ increases for all positive n and decreases for negative n.Thus the random variable X_n (or the n^{th} order transform) generates a distribution whose hazard rate is larger (smaller) than that of X_{n-1}.

5.3 Characterizations

Various identities connecting the hazard quantile function, mean residual quantile function and the density quantile function of X and X_n enable characterization of distributions of X and X_n. First we note that $T_n(u)$ characterizes the distribution of X. This follows from

$$t_n(u) = (1-u)^n q(u)$$

and

$$Q(u) = \int_0^u (1-p)^{-n} t_n(p)dp.$$

The following theorem tells us how the successive transforms change the distributional properties.

Theorem 5.1

The random variable X_n, $n = 1, 2, 3, \ldots$ *has rescaled beta distribution if and only if* X *is distributed as either exponential, Lomax or rescaled beta.*

Proof: From the expression of quantile density function $t_n(u)$ of exponential distribution in Table 5.1, the n^{th} transform is

$$T_n(u) = \int_0^u t_n(p)\,dp$$
$$= \int_0^u \lambda^{-1}(1-p)^{n-1}\,dp$$
$$= (\lambda n)^{-1}\left[1-(1-u)^n\right],$$

which is rescaled beta with parameter $\left((\lambda n)^{-1}, n^{-1}\right)$. When X is Lomax

$$T_n(u) = \int_0^u t_n(p)\,dp$$
$$= \int_0^u \alpha c^{-1}(1-p)^{\frac{nc-c-1}{c}}\,dp$$
$$= \alpha(nc-1)^{-1}\left[1-(1-u)^{\frac{nc-1}{c}}\right],$$

which is the quantile function of rescaled beta with parameters $\left(\alpha(nc-1)^{-1}, c(nc-1)^{-1}\right)$ in the support of $\left(0,\ \alpha(nc-1)^{-1}\right)$ and similarly when X is rescaled beta with parameter (R, c), X_n has the same distribution in $\left[0,\ R(nc+1)^{-1}\right]$ with parameters $\left[R(nc+1)^{-1}, c(1+n)^{-1}\right]$. This proves the if part.

To prove the converse, we assume that X_n is distributed as rescaled beta so that we can write $T_n(u) = R_n\left[1-(1-u)^{\frac{1}{C_n}-1}\right]$

for some constants $R_n, c_{n.} > 0$. Thus

$$t_n(u) = T_n'(u) = \frac{R_n}{c_n}(1-u)^{\frac{1}{c_n}-1}$$

or

$$(1-u)^n q(u) = \frac{R_n}{c_n}(1-u)^{\frac{1}{c_n}-1} \text{ for all } u.$$

This means that $(1-u)^n$ is a factor of the right side and therefore

$$c_n^{-1} = k_n + n, \text{ for some real } k_n.$$

Hence

$$q(u)=\frac{R_n}{c_n}(1-u)^{\frac{1}{c_n}-n-1}=(k_n+n)R_n(1-u)^{k_n-1}.$$

Since $q(u)$ is independent of n, taking $n=1$ we have

$$Q(u)=\int_0^u q(p)dp=k_1^{-1}R_1(k_1+1)\left(1-(1-u)^{k_1}\right).$$

Hence for $k_1>0$, X follows rescaled beta distribution $\left(0, R_1k_1^{-1}(k_1+1)\right)$, Lomax law for $-1<k_1<0$. Finally taking limit as $k_1\to 0$ by applying L-Hospital's rule, we have

$$Q(u)\to R_1\left(-\log(1-u)\right),$$

which represents the exponential law.

Remark 5.3 The transformed random variable $X_n, n=-1,-2,\ldots$ has Lomax distribution if and only if X is distributed as either exponential or beta or Lomax. The negative transforms are as mentioned in Remark 5.1. The proof is similar as that of Theorem 5.1.

Our next characterization is by a relationship between the mean residual quantile function of X and X_n.

Theorem 5.2

X follows generalized Pareto distribution with quantile function

$$Q(u)=AB^{-1}\left[(1-u)^{-B(B+1)^{-1}}-1\right],\quad B>-1, A>0 \tag{5.8}$$

if and only if for all $n=0, 1, 2,\ldots$ and u

$$M_n(u)=\left(nB+n+1\right)^{-1}(1-u)^n M(u) \tag{5.9}$$

Proof: Assuming (5.9),

$$M_n(u)-(1-u)M_n{}'(u)=\left(nB+n+1\right)^{-1}(1-u)^n\left[M(u)-(1-u)M'(u)+nM(u)\right]$$

and using (5.7)

$$\left(nB+n+1\right)^{-1}\left[M(u)-(1-u)M'(u)+nM(u)\right]=M(u)-(1-u)M'(u),$$

which is simplified to

$$BM(u)=(B+1)(1-u)M'(u)$$

or

$$\frac{M'(u)}{M(u)}=\frac{B}{(B+1)(1-u)}.$$

Integrating

$$M(u)=k(1-u)^{-B/B+1}. \tag{5.10}$$

Substituting (5.10) in the expression (equation (2.79))

$$Q(u)=\mu-M(u)+\int_0^u(1-p)^{-1}M(p)dp$$

and noting that $\mu=M(0)=k$, we have

$$\begin{aligned}Q(u)&=k-k(1-u)^{-B/B+1}+\int_0^u(1-p)^{-1}k(1-p)^{-B/B+1}dp\\&=\frac{k}{B}\left[(1-u)^{-B/B+1}-1\right].\end{aligned}$$

Thus we have quantile function (5.8) with $k=A$. This proves the if part. Next we assume that X has the distribution specified by (5.8), we have

$$q(u)=A(B+1)^{-1}(1-u)^{-B/B+1^{-1}}.$$

Using (5.3) we have

$$\begin{aligned}T_n(u)&=\int_0^u t_n(p)dp\\&=A\left[(B+1)n-B\right](1-u)^{n-\frac{B}{B+1}}.\end{aligned}$$

From (2.75), replacing $Q(u)$ with $T_n(u)$ and simplifying

$$\begin{aligned}M_n(u)&=(1-u)^{-1}\int_u^1\left[T_n(p)-T_n(u)\right]dp\\&=A\left(nB+n+1\right)^{-1}(1-u)^{n-B(B+1)^{-1}}\end{aligned}. \tag{5.11}$$

The quantile function of X in (5.8) verifies

$$M(u) = A(1-u)^{-B(B+1)^{-1}},$$

so that (5.9) holds and proof is completed.

Remark 5.4 Since $(1-u)^n$ is a decreasing function of u and $nB+n+1>0$ for all $n \geq 1$ we see that $M_n(u) < M(u)$ for all u and $n = 1, 2, 3, \ldots$. Thus the process of iteration reduces the mean residual life to the mean residual life of the generalized Pareto law.

Some observations regarding Theorem 5.1 and Remark 5.2 seem to be in order. They reveal that distribution with constant or decreasing (increasing) hazard quantile function tend to become a distribution with increasing (decreasing) hazard quantile function as the process of iteration continues with positive (negative) n. Thus property of the TTT is not visible when the first order TTT alone is observed. We will later show that this behaviour is in general true and that it provides a method of generating new models with the increasing (decreasing) hazard quantile function compared to the original one. Results in Theorem 5.1 and Remark 5.3 further identify two distributions where the monotonicity of hazard quantile function retains the same nature inspite of the iteration. Further the rescaled beta and Lomax distributions play an important role in comparing the ageing properties of X and X_n as the subsequent discussions exemplify.

5.4 Characterization of ageing concepts

As we pointed out in Section 2.5.4, the TTT plays an important role in characterizing ageing concepts. Earlier researchers have characterized popular ageing concepts in terms of TTT. We present some of these results, which seem useful for the discussions in the sequel in Theorem5.3 without giving proof.

Theorem 5.3

A lifetime random variable X is

(i) *IHR (DHR) if and only if the scaled TTT $\phi(u)$ is concave (convex) for $0 \le u \le 1$. (Barlow and Campo (1975))*

(ii) *IHRA (DHRA) if and only if $\frac{\phi(u)}{u}$ is decreasing (increasing) for $0 \le u \le 1$. (Barlow and Campo (1975))*

(iii) *DMRL (IMRL) if and only if $\frac{1-\phi(u)}{1-u}$ is decreasing (increasing) in $[0,1]$. (Klefsjo (1982))*

(iv) *NBUE (NWUE) if and only if $\phi(u) \ge u$ ($\phi(u) \le u$) for $0 \le u \le 1$. (Bergman (1977)).*

(v) *BT (UBT) if $\phi(u)$ has only one reflexion point u_0 such that $o < u_0 < 1$ and it is convex (concave) on $[0, u_0]$ and concave (convex) on $[u_0, 1]$. (Barlow and Campo (1975)).*

Now we consider the ageing properties of the transformed random variable X_n in relation to the baseline random variable X. It was seen in the last section that each iteration of the TTT transform, the hazard quantile function increases for positive n and decreases for negative n. Another important aspect is to ascertain the ageing behaviour in each iteration. Here we prove some general results about the ageing patterns of X_n in relation to X.

Theorem 5.4

(i) *If X is IHR then X_n is IHR for all n.*

(ii) *If* X *is DHR then* X_n *is DHR if* $Q(u) \geq L\left(R, \frac{1}{n}\right)$, *IHR if* $Q(u) \leq L\left(R, \frac{1}{n}\right)$ *and BT if there exists a* u_0 *for which* $Q(u) \geq L\left(R, \frac{1}{n}\right)$ *in* $[0, u_0]$ *and* $Q(u) \leq L\left(R, \frac{1}{n}\right)$ *in* $[u_0, 1]$. *Here* $L(.,.)$ *denotes the quantile function of Lomax distribution in Table 5.1.*

Proof: Since $t_{n+1}(u) = (1-u)^n t_1(u)$, derivative of $t_{n+1}(u)$

$$t'_{n+1}(u) = (1-u)^{n-1}\left[(1-u)t_1'(u) - nt_1(u)\right] . \tag{5.12}$$

Using Theorem (5.3), X is IHR (DHR) if T_1 is concave (convex). Hence from (5.12),

$$\begin{aligned} X \text{ is IHR} &\Rightarrow T_1(u) \text{ is concave} \\ &\Rightarrow t_1(u) \text{ is decreasing} \\ &\Rightarrow t'_{n+1}(u) < 0 \\ &\Rightarrow T_{n+1}(u) \text{ is concave} \\ &\Rightarrow X_n \text{ is IHR.} \end{aligned}$$

Similarly, when X is DHR, $T_1(u)$ is convex and accordingly

$$\begin{aligned} X \text{ is DHR} &\Rightarrow t'_{n+1}(u) \geq 0 \\ &\Rightarrow (1-u)t_1'(u) \geq nt_1(u) \\ &\Rightarrow t_1(u) \geq k(1-u)^{-n} \text{ (on integration)} \\ &\Rightarrow Q(u) \geq L\left(k, \frac{1}{n}\right). \end{aligned}$$

$$\begin{aligned} X_n \text{ is IHR} &\Rightarrow (1-u)t_1'(u) \leq nt_1(u) \\ &\Rightarrow t_1(u) \leq k(1-u)^{-n} \text{ (on integration)} \\ &\Rightarrow Q(u) \leq L\left(k, \frac{1}{n}\right). \end{aligned}$$

The last part of (ii) follows from the above result and part (v) of Theorem 5.3.

Theorem 5.5

(i) If X_n is DHR then X is DHR

(ii) If X_n is IHR then X is IHR if $T_n(u) \le B\left(k(n+1)^{-1}, (n+1)^{-1}\right)$, DHR if $T_n(u) \le B\left(k(n+1)^{-1}, (n+1)^{-1}\right)$ and UBT if there exists a u_0 for which $T_n(u) \le B\left(k(n+1)^{-1}, (n+1)^{-1}\right)$ in $[0, u_0]$ and $T_n(u) \ge B\left(k(n+1)^{-1}, (n+1)^{-1}\right)$ in $[u_0, 1]$. $B(R, c)$ denotes the rescaled beta with parameter (R, c).

Proof: Following the steps of Theorem 5.4, we can prove this result. We have

$$t_1(u) = (1-u)^{-n} t_{n+1}(u)$$

$$t_1'(u) = (1-u)^{-n}\left[n(1-u)^{-1} t_{n+1}(u) + t'_{n+1}(u)\right]. \tag{5.13}$$

$$\begin{aligned} X_n \text{ is DHR} &\Rightarrow t_{n+1}(u) \text{ is increasing} \\ &\Rightarrow t_1'(u) > 0 \\ &\Rightarrow t_1(u) \text{ is convex} \\ &\Rightarrow X \ \text{ is DHR.} \end{aligned}$$

Similarly when X_n is IHR, $T_{n+1}(u)$ is concave and therefore

$$\begin{aligned} X \text{ is IHR (DHR)} &\Rightarrow t_1(u) \ge (\le) 0 \\ &\Rightarrow n(1-u)^{-1} t_{n+1}(u) \ge (\le) - t'_{n+1}(u) \\ &\Rightarrow t_{n+1}(u) \ge (\le) k(1-u)^n \text{ (on integration)} \\ &\Rightarrow T_{n+1}(u) \ge (\le) \frac{k}{n+1}\left[1-(1-u)^{n+1}\right] \text{ (again on integration)} \\ &\Rightarrow T_{n+1}(u) \ge (\le) B\left(k(n+1)^{-1}, (n+1)^{-1}\right). \end{aligned}$$

Again from part (v) of Theorem 5.3, we have the proof of last part of (ii).

Remark 5.5 The importance of theorems 5.4 and 5.5 is that they help the construction of BT and UBT distributions by a simple mechanism. To obtain BT distribution one need to only look DHR distributions for which $t_{n+1}(u)$ has a point of inflexion. Similarly for getting UBT distributions we look for IHR distributions and perform backward recurrence $n<0$ to reach a quantile density function that has an inflexion point. The procedure is illustrated in the following examples.

Example 5.1 The Weibull distribution in Table 5.1 has

$$Q(u)=\sigma\left(-\log(1-u)\right)^{1/\lambda},$$

$$q(u)=\sigma\lambda^{-1}\frac{1}{1-u}\left(-\log(1-u)\right)^{1/\lambda-1}$$

and

$$t_n(u)=\sigma\lambda^{-1}(1-u)^{n-1}\left(-\log(1-u)\right)^{1/\lambda-1}.$$

Now

$$t_n'(u)=\sigma\lambda^{-1}(1-u)^{n-2}\left(-\log(1-u)\right)^{1/\lambda-2}\left[\left(\frac{1}{\lambda}-1\right)+(n-1)\log(1-u)\right].$$

When $\lambda\le 1, T_n(u)$ is convex on $[0,u_0]$ and concave $[u_0,1]$, where

$$u_0=1-\exp\left\{\frac{\lambda-1}{(n-1)\lambda}\right\}.$$

Hence X_n has bathtub-shaped failure rate for $n\ge 1$. With increasing values of n the change point of the failure rate becomes larger so that the range for which X_n is IHR increases. Note also that for $\lambda\ge 1$, $t_n'(u)<0$ and hence X_n is IHR for all n.

Remark 5.5 Haupt and Schabe (1997) proposed a method of constructing bathtub-shaped distribution by choosing a twice differentiable function $\phi(u)$ satisfying $\phi(0)=0$, $\phi(1)=1$ and $0\le\phi(u)\le 1$ with $\phi(u)$ having only one inflexion point u_0 such that it is convex on $[0, u_0]$ and concave on $[u_0, 1]$. Then the solution $F(t)$ of the differential equation

$$\frac{\theta\phi'\left(F(t)\right)dF(t)}{1-F(t)} = dt, \quad \theta = H_F^{-1}(1) > 0 \tag{5.14}$$

is a bathtub-shaped distribution. Converting (5.14) in terms of quantile functions , we have

$$\frac{\theta\phi'(u)}{q(u)} = 1-u$$

or

$$\theta\phi'(u) = (1-u)q(u)\,.$$

Thus the solution is a twice differentiable $T_n(u)$ for some n for which there is a inflexion point, and therefore the results of Haupt and Schabe (1997) are subsumed in Theorem 5.4.

Theorem 5.6

(i) *X is DMRL (decreasing mean residual lifetime) implies that X_n is DMRL.*

(ii) *X_n is IMRL (increasing mean residual lifetime) implies that X is IMRL, when n is negative.*

Proof: To prove (i), recall from Theorem 5.3 that X is DMRL if and only if

$$(1-u)^{-1}\left(1-\mu^{-1}T(u)\right) \text{ is decreasing in } u$$

or alternatively

$$(1-u)^{-1}\left(\mu - T(u)\right) \text{ is decreasing}$$

or by differentiating

$$\mu - T(u) - (1-u)t(u) \le 0\,.$$

Further

$$T_{n+1}(u) = \int_0^u (1-p)^n t(p)dp$$
$$= (1-u)^n T(u) + A(u),$$

where

$$A(u) = n\int_0^u (1-p)^{n-1} T(p)dp > 0, \ \forall u \in (0,1).$$

Now,

$$\begin{aligned}
\mu_n - T_{n+1}(u) - (1-u)t_{n+1}(u) &= \mu_n - (1-u)^n T(u) - A(u) - (1-u)t_{n+1}(u) \\
&\le \mu_n - (1-u)^n T(u) - (1-u)t_{n+1}(u) \\
&\le \mu_0 - T(u) - (1-u)t(u) \\
&= \mu - T(u) - (1-u)t(u) \\
&\le 0.
\end{aligned}$$

Thus X_n is DMRL. The proof of (ii) is similar with negative n.

Theorem 5.7

(i) X *is IHRA* $\Rightarrow$ X_n *is IHRA*

(ii) X_n *is DHRA* $\Rightarrow$ X *is DHRA*, *when n is negative.*

Proof: From Theorem 5.3, X is IHRA if and only if $u^{-1}T(u)$ is decreasing for all u. This means that $t(u) \le u^{-1}T(u)$. We then have

$$\begin{aligned}
t_{n+1} - u^{-1}T_{n+1}(u) &= (1-u)^n t(u) - u^{-1}\left((1-u)^n T(u) - A(u)\right) \\
&\le (1-u)^n \left[t(u) - u^{-1}T(u)\right] \\
&\le t(u) - u^{-1}T(u) \\
&\le 0,
\end{aligned}$$

$\Rightarrow X_n$ is IFRA.

With the similar steps for negative n, we can prove (ii).

Theorem 5.8

(i) X is NBUE $\Rightarrow$ X_n is NBUE

(ii) X_n is NWUE $\Rightarrow$ X is NWUE

Proof: X is NBUE if and only if $\mu^{-1}T(u) > u$ for $u \in [0, 1]$. Hence

$$\begin{aligned} u^{-1}T_n(u) - \mu_n &= u^{-1}\left[(1-u)^n T(u) + A(u)\right] - \mu_n \\ &\geq (1-u)^n \left(u^{-1}T(u) - \mu\right) \\ &\geq 0. \end{aligned}$$

i.e.,

$$\begin{aligned} \left(u^{-1}T_n(u) - \mu\right) \geq 0 &\Rightarrow u^{-1}T_n(u) \geq \mu_n \\ &\Rightarrow \mu_n^{-1}T_n(u) \geq u. \end{aligned}$$

which implies that X_n is NBUE.

From the above theorems it is evident that when X is ageing positively the successive transforms are also ageing positively. It may also be noted that the converse of the above theorems need not be true in view of the characterizations given in Theorem 5.1 and Remark 5.3.

5.6 Order relations

In Section 2.8 we have described the importance of order relations and we have given some well known order relations. In this section we discuss the implications of the results obtained so far in developing some order relations connecting the baseline and transformed distributions. Furthermore, a new partial order based on transforms of order n, which extends some of the existing results, is introduced.

Let X and Y be two nonnegative random variables with finite expectations, distribution functions $F_X(.)$ and $G_Y(.)$, quantile functions $Q_X(u)$ and $Q_Y(u)$, and TTT transform $T(u)$ and $S(u)$ respectively. It is immediate form (5.4) that

$$H(u) = (1-u)^n H_n(u) \leq H_n(u)$$

and therefore, we have $X_n \leq_{hr} X$.

We have seen in Section 2.8.2 that X is smaller than Y in the dispersive order, ($X \leq_{disp} Y$) if and only if

$$Q_X(v)-Q_X(u)\leq Q_Y(v)-Q_Y(u), \quad 0<u\leq v<1,$$

which means that

$$\int_u^v q_X(p)dp \leq \int_u^v q_Y(p)dp,$$

where $q_X(.)$ and $q_Y(.)$ are quantile density functions of X and Y respectively. Setting $Y=X_n$ and $q_Y(u)=t_n(u)=(1-u)^n q_X(u)$, we have $X_n \leq_{disp} X$.

From Section 2.8.4, X is smaller than Y in the convex transform order ($X \leq_{cx} Y$) if $G^{-1}(F(x))$ is convex in the support of F. In terms of quantile density functions, this condition is equivalent to $\dfrac{q_X(u)}{q_Y(u)}$ increasing in u and hence, $X_n \leq_{cx} X$.

In Shaked and Shantikumar (2007), implications of the different orders such as hazard rate order $X \leq_{hr} Y$, usual stochastic order $X \leq_{st} Y$, mean residual life order $X \leq_{mrl} Y$, variance residual life order $X \leq_{vrl} Y$, harmonic mean residual life order $X \leq_{hmrl} Y$, dilation order $X \leq_{dil} Y$, increasing concave order $X \leq_{icv} Y$, star order $X \leq_{*} Y$, super additive order $X \leq_{su} Y$, excess wealth order $X \leq_{ew} Y$, decreasing mean residual life order $X \leq_{dmrl} Y$, NBUE order $X \leq_{nube} Y$, Lorenz order $X \leq_{lorenz} Y$, are discussed. Here we give some of these implications, which are useful to derive the implications of orders connecting X and X_n in the following theorem.

Theorem 5.9

(i) *If* X *and* Y *are two random variables then*

(a) $X \leq_{hr} Y \Rightarrow X \leq_{st} Y$

(b) $X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$

(c) $X \leq_{mrl} Y \Rightarrow X \leq_{hmrl} Y$.

(ii) *For two random variables* X *and* Y *with finite means if* $X \leq_{disp} Y$ *then* $X \leq_{EW} Y$.

(iii) *If* X *and* Y *be two random variables with left end points of supports* l_X *and* l_Y *such that* $l_X = l_Y = 0$, *then*

$$X \leq_{disp} Y \Rightarrow X \leq_{st} Y.$$

(iv) *When* X *and* Y *are two nonnegative random variables, the following implications hold*

(a) $X \leq_{c} Y \Rightarrow X \leq_{*} Y$

(b) $X \leq_{*} Y \Rightarrow X \leq_{su} Y$

(v) *Let* X *and* Y *be two random variables, each with support of the form* $[0, a), a > 0$. *Then*

(a) $X \leq_{c} Y \Rightarrow X \leq_{dmrl} Y$

(b) $X \leq_{dmrl} Y \Rightarrow X \leq_{nbue} Y$

(c) $X \leq_{nbue} Y \Rightarrow X \leq_{Lorenz} Y$.

Using the results given in Theorem 5.9, we summarize the implications of orders connecting X and X_n in the following diagram.

$$
\begin{array}{ccccccc}
 & & & & X_n \le_{\text{dil}} X & & \\
 & & & & \Uparrow & & \\
 & & X_n \le_{\text{hr}} X & \Rightarrow & X_n \le_{\text{mrl}} X & \Rightarrow & X_n \le_{\text{vrl}} X \\
 & & \Downarrow & & \Downarrow & & \\
X_n \le_{\text{disp}} X & \Rightarrow & X_n \le_{\text{st}} X & \Rightarrow & X_n \le_{\text{hmrl}} X & & \\
\Downarrow & & \Downarrow & & & & \\
X_n \le_{\text{EW}} X & & X_n \le_{\text{icv}} X & & & &
\end{array}
$$

and

$$
\begin{array}{ccccc}
X_n \le_{c} X & \Rightarrow & X_n \le_{*} X & \Rightarrow & X_n \le_{\text{su}} X \\
\Downarrow & & & & \\
X_n \le_{\text{dmrl}} X & \Rightarrow & X_n \le_{\text{nbue}} X & \Rightarrow & X_n \le_{\text{Lorenz}} X.
\end{array}
$$

Definition 5.2 It is said that X is smaller than Y in the TTT transform of order n written as $X \le_{TTT-n} Y$ (or equivalently, $X_n \le_{TTT} Y_n$). If $T_{n+1}(u) \le S_{n+1}(u)$ for all u in $[0,1]$, where $T_n(u)$ and $S_n(u)$ denote the TTT transforms of order n of X and Y, respectively.

First we note that, from the above definition

$$X \le_{TTT-n} Y \Leftrightarrow T_{n+1}(u) \le S_{n+1}(u) \Leftrightarrow X_{n+1} \le_{st} Y_{n+1} \tag{5.15}$$

Hence, all the implications starting from the stochastic order in the chain presented above are implications of the TTT-n order.

Furthermore, the usual TTT order between X and Y satisfies

$$X \le_{TTT} Y \Rightarrow X \le_{TTT-n} Y,$$

as an extension of the result in Shaked and Shantikumar(2007,Theorem 4B.29).

Another order of interest is the NBUE order defined as X is smaller than Y in the NBUE order, $X \le_{nbue} Y$ if

$$\frac{M_X(u)}{M_Y(u)} \leq \frac{E(X)}{E(Y)}, \text{ for } u \in (0,1) \tag{5.16}$$

where $M_X(.)$ and $M_Y(.)$ are the mean residual quantile function of X and Y respectively. An equivalent statement of (5.16) is (using (2.76))

$$\frac{T_1(u)}{E(X)} \geq \frac{S_1(u)}{E(Y)} \tag{5.17}$$

Since $\frac{T_1(u)}{E(X)}$ and $\frac{S_1(u)}{E(Y)}$ are quantile functions of $\frac{X_1}{E(X)}$ and $\frac{Y_1}{E(Y)}$, successive application of (5.17) for $X_2, X_3, \ldots$ in the definition for the NBUE order gives

$$X \leq_{nbue} Y \Leftrightarrow \frac{X_1}{E(X)} \geq_{st} \frac{Y_1}{E(Y)} \Leftrightarrow \frac{X}{E(X)} \geq_{TTT} \frac{Y}{E(Y)}$$

and

$$X_{n-1} \leq_{nbue} Y_{n-1} \Leftrightarrow \frac{X_{n-1}}{E(X_{n-1})} \geq_{TTT} \frac{Y_{n-1}}{E(Y_{n-1})} \Leftrightarrow \frac{X_n}{E(X_{n-1})} \geq_{st} \frac{Y_n}{E(Y_{n-1})}.$$

These results extend Theorem4B.26 of Shaked and Shantikumar (2007).

When X_{n-1} and Y_{n-1} have finite mean and 0 as common left endpoints of their supports, then, for any $\phi(x)$ with $\phi(0)=0$,

$$X \leq_{TTT-n} Y \Rightarrow \phi(X) \leq_{TTT-n} \phi(Y).$$

From (5.18) and (5.20),

$$\frac{X}{E(X)} \geq_{TTT} \frac{Y}{E(Y)} \Rightarrow \frac{X}{E(X)} \geq_{TTT-n} \frac{Y}{E(Y)}.$$

Note that TTT transform of $\frac{X}{E(X)}$ is the scaled TTT transform that is extensively used in many practical applications, including characterization of ageing classes.

An interesting property of the TTT order is that it is preserved under the minima of independent and identically distributed random variables (Kochar et al. (2002)). Following the lines of proof of this result, we prove a similar result for the TTT-n order.

Theorem 5.10

Let $(X_1, X_2, ..., X_n)$ *and* $(Y_1, Y_2, ..., Y_n)$ *be independent copies of two random variables* X *and* Y *that are identically distributed. If* $X \le_{TTT-n} Y$ *then* $V_n \le_{TTT-n} W_n$*, where* $V_n = \min(X_1, X_2, ..., X_n)$ *and* $W_n = \min(Y_1, Y_2, ..., Y_n)$.

Proof: The survival functions of V_n and W_n are $\bar{F}_{V_n}(x) = (\bar{F}_X(x))^n$ and $\bar{F}_{W_n}(x) = (\bar{F}_Y(x))^n$. From Section 2.1.5, if $Q_n(u_n)$ is the quantile function $Q(u)$ of F is related to it as

$$Q(u) = Q_n\left(1-(1-u)^n\right)$$

or

$$q(u) = nq_n\left(1-(1-u)^n\right)(1-u)^{n-1}$$

Similarly, using the symbols of r_n and r for quantile density functions of W_n and Y,

$$r(u) = nr_n\left(1-(1-v)^n\right)(1-v)^{n-1}, \quad v = F_Y(x).$$

Since $X \le_{TTT-n} Y$, we have, using the notation

$$T_{n+1}(u) \le S_{n+1}(u),$$

and, hence,

$$\int_0^u (1-p)^{n+1} q(p)\,dp \le \int_0^u (1-p)^{n+1} r(p)\,dp.$$

Applying lemma A.2(b) of Kochar et al. (2002), which states that for a measure W defined on (a, b) if $\int_a^t dW(x) \geq 0$ for all $t \in (a, b)$ then $\int_a^t g(x)dW(x) \geq 0$, where $g(.)$ is a nonnegative decreasing function defined on (a, b), we have with $W(p) = r(p) - q(p)$ and $g(p) = (1-p)^{n-1}$,

$$\int_0^u (1-p)^{2n} q(p)dp \leq \int_0^u (1-p)^{2n} r(p)dp.$$

Setting $y = 1-(1-p)^n$, we have

$$\int_0^{u_n} (1-y)^{n+1} q_n(y)dy \leq \int_0^{u_n} (1-y)^{n+1} r_n(y)dy,$$

which proves the result.

Another application of the TTT-n order concerns the proportional hazard models that extend the results of Li and Shaked (2007). Let $X(\theta)$ be a random variable of a proportional hazard model with survival function $[F(x)]^\theta$, $\theta > 0$, corresponding to X with models associated with $X(\theta)$ and $Y(\theta)$ are $Q_\theta(u)$ and $R_\theta(u)$ with respective quantile density functions $q_\theta(u)$ and $r_\theta(u)$.

Retaining the previous notation, we can write

$$Q(u) = Q_\theta\left(1-(1-u)^\theta\right)$$

and

$$Q_\theta(u) = Q\left(1-(1-u)^{\frac{1}{\theta}}\right).$$

Theorem 5.11

We have

$$X \leq_{TTT-n} Y \Rightarrow X(\theta) \leq_{TTT-n} Y(\theta), \quad \theta > 1,$$

$$X(\theta) \leq_{TTT-n} Y(\theta) \Rightarrow X \leq_{TTT-n} Y, \quad \theta < 1$$

Proof: Taking $\theta > 1$, $X(\theta) \leq_{TTT-n} Y(\theta)$ is the same as

$$\int_0^v (1-p)^{n+1} q_\theta(p)dp \leq \int_0^v (1-p)^{n+1} r_\theta(p)dp, \tag{5.18}$$

where $v = 1-(1-u)^\theta$. The last inequality reduces to

$$\int_0^{1-(1-u)^{\frac{1}{\theta}}} q\left(1-(1-p)^{\frac{1}{\theta}}\right)\left(1-(1-p)^{\frac{1}{\theta}}\right)^{n+1} Q^{-1}(1-p)^{Q^{-1}-1} dp$$

$$\leq \int_0^{1-(1-u)^{\frac{1}{\theta}}} r\left(1-(1-p)^{\frac{1}{\theta}}\right)\left(1-(1-p)^{\frac{1}{\theta}}\right)^{n+1} Q^{-1}(1-p)^{Q^{-1}-1} dp,$$

which is

$$\int_0^u (1-p)^{n+1} q(p)dp \leq \int_0^u (1-p)^{n+1} r(p)dp. \tag{5.19}$$

Thus, if (5.19) holds then (5.18) applies, which is the first part of the theorem. The case in which $\theta < 1$ is similar.

To conclude, we note that the n^{th} order TTT transform presented here has helped to achieve a more explicit understanding of the effect of transforms on the properties of the baseline distributions. It generates new models that are more IHR or DHR and also BT or UBT from models in common use, and adds more flexibility to model choice by adopting quantile functions that do not convert into simple forms of distribution functions. The reliability properties and order relations extend the existing results and leave scope for new ageing classes. The sample counterpart of TTT-n viz. n^{th} order TTT statistics along with their relationships with the TTT statistic of the original distribution, which is being investigated, can further strengthen the applicability of the theoretical results in the present work.

Table 5.1- Quantile-based functions of life distributions

Distribution	$\bar{F}(x)$	$Q(u)$	$H(u)$	$t_n(u)$
Exponential	$e^{-\lambda x},\ x>0$	$-\lambda^{-1}\log(1-u)$	λ	$\lambda^{-1}(1-u)^{n-1}$
Lomax	$\left(1+\frac{x}{\alpha}\right)^{-c},\ x>0$	$\alpha\left((1-u)^{-1/c}-1\right)$	$c\alpha^{-1}(1-u)^{1/c}$	$\alpha c^{-1}(1-u)^{\frac{nc-c-1}{n}}$
Rescaled beta	$\left(1-\frac{x}{R}\right)^{c},\ 0<x<R$	$R\left(1-\left((1-u)^{1/c}\right)\right)$	$cR^{-1}(1-u)^{-1/c}$	$Rc^{-1}(1-u)^{n+1/c-1}$
Weibull	$\exp\left[-\left(\frac{x}{\sigma}\right)\right]^{\lambda},\ x>0$	$\sigma(-\log(1-u))^{1/\lambda}$	$\lambda\sigma^{-1}(-\log(1-u))^{1-1/\lambda}$	$\sigma\lambda^{-1}(1-u)^{n-1}(-\log(1-u))^{-1/\lambda-1}$
Half-logistic	$(1+c)(1+ce^{\lambda x})^{-1},\ x>0$	$\lambda^{-1}\log\left(\frac{c+u}{c(1-u)}\right)$	$\lambda(1+c)^{-1}(c+u)$	$(1+c)\lambda(c+u)^{-1}(1-u)^{n-1}$
Generalized beta	$(1-x^{\alpha})^{\theta},\ 0\le x\le 1$	$\left(1-(1-u)^{1/\theta}\right)^{1/\alpha}$	$\alpha\theta(1-u)^{-1/\theta}\left(1-(1-u)^{1/\theta}\right)^{1/\alpha-1}$	$(\alpha\theta)^{-1}(1-u)^{n+1/\theta-1}\left(1-(1-u)^{1/\theta}\right)^{1/\alpha-1}$
Lambda	-	$cu^{\lambda_1}(1-u)^{-\lambda_2}$	$\left[cu^{\lambda_1-1}(\lambda_1(1-u)+\lambda_2(u))\right]^{-1}(1-u)^{\lambda_2}$	$(1-u)^{n-\lambda_2-1}cu^{\lambda_1-1}(\lambda_1(1-u)+\lambda_2(u))$
Burr	$(1+x^{\lambda})^{-1},\ x>0$	$u^{\alpha}(1-u)^{-\alpha},\ \alpha=\lambda^{-1}$	$\alpha(1-u)^{-\alpha-1}u^{\alpha-1}$	$\alpha u^{\alpha-1}(1-u)^{n-\alpha-1}$

Chapter 6

L-moments of residual life*

6.1 Introduction

In Section 2.16, we have discussed the basic features of L-moments. We pointed out there that the L-moments are alternative to conventional moments and they have several advantages over the ordinary moments. In reliability analysis, residual life function and related measures are good indicators in describing ageing patterns of a distribution, and these are being used in other disciplines also. Note that most popular measures of residual life that are discussed in the literature are based on ordinary moments, for example the mean of residual life, variance of residual life, etc.. Considering the advantages of L-moments over ordinary moments, it is worthy to study the measures of residual life based on L-moments. In this chapter we investigate the properties of the first two L-moments of residual life and their relevance in various aspects of reliability analysis. This problem does not appear to have been considered in literature.

6.2 Definition and properties

Recall the definition of L-moments of order r from Section 2.1.6, which is given by

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**The discussions in this chapter is based on Nair and Vineshkumar (2010) appeared in the Journal of Statistical Planning and Inference (see reference no. 93)*

$$L_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(X_{r-k:r}\right),$$

$$= \sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}^2 \int_0^\infty x\left(F(x)\right)^{r-k-1}\left(1-F(x)\right)^k f(x)dx\,, \qquad (6.1)$$

where $X_{r:n}$ is the r^{th} order statistic in a sample of size n from $F(x)$ and $f(x)$ is the density function of X. The truncated variable $X_t = \left(X \,|\, X > t\right)$ has survival function $\overline{F}_t(x) = \frac{\overline{F}(x)}{\overline{F}(t)}$ so that (6.1) becomes

$$L_r(t) = \sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}^2 \int_t^\infty x\left(\frac{\overline{F}(t)-\overline{F}(x)}{\overline{F}(t)}\right)^{r-k-1}\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)^k \frac{f(x)}{\overline{F}(t)}dx\,. \qquad (6.2)$$

In particular, setting $r=1$, we have

$$L_1(t) = \frac{1}{\overline{F}(t)}\int_t^\infty xf(x)dx = E\left(X \,|\, X > t\right),$$

which is the vitality function discussed widely in reliability analysis. Since $L_1(t)$ is widely discussed with references to its properties and applications (e.g. Kupka and Loo (1989)), we bestow our attention to the second moment. When $r=2$

$$L_2(t) = \sum_{k=0}^{1}(-1)^k \binom{1}{k}^2 \int_t^\infty x\left(\frac{\overline{F}(t)-\overline{F}(x)}{\overline{F}(t)}\right)^{1-k}\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)^k \frac{f(x)}{\overline{F}(t)}dx$$

$$= \frac{1}{\overline{F}^2(t)}\int_t^\infty \left(\overline{F}(t)-2\overline{F}(x)\right)xf(x)dx$$

$$= \frac{1}{\overline{F}(t)}\int_t^\infty xf(x)dx - \frac{2}{\overline{F}^2(t)}\int_t^\infty x\overline{F}(x)\,f(x)dx$$

$$= L_1(t) - t - \left(\overline{F}(t)\right)^{-2}\int_t^\infty \overline{F}^2(x)dx$$

(applying integration by parts to the second integral)

$$= m(t) - \left(\overline{F}(t)\right)^{-2}\int_t^\infty \overline{F}^2(x)dx, \qquad (6.3)$$

where $m(t)$ is the mean residual function. It follows that $L_2(t) \le m(t)$. However, the equality sign does not hold for any non-degenerate distribution. Thus $L_2(t)$ is strictly less than the mean residual life function.

Differentiating (6.3), we have

$$
\begin{aligned}
L_2'(t) &= m'(t) - \left[\frac{-\left(\bar{F}(t)\right)^4 + 2\left(\bar{F}(t)\right)^2 f(t) \int_t^\infty \left(\bar{F}(x)\right)^2 dx}{\left(\bar{F}(t)\right)^4} \right] \\
&= m'(t) + 1 - \frac{2h(t) \int_t^\infty \left(\bar{F}(x)\right)^2 dx}{\left(\bar{F}(t)\right)^2} \\
&= m(t)h(t) - 2h(t)\left(m(t) + L_2(t)\right) \qquad \text{(using (2.50))} \\
&= h(t)\left(2L_2(t) - m(t)\right). \qquad (6.4)
\end{aligned}
$$

Setting $F(x) = p$ and $F(t) = u$ in (6.2), we get the expression for the r^{th} L-moment residual quantile function of X as

$$\alpha_r(u) = L_r(Q(u)) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k}^2 \int_u^1 \left(\frac{p-u}{1-u}\right)^{r-k-1} \left(\frac{1-p}{1-u}\right)^k \frac{Q(p)}{1-u} dp. \qquad (6.5)$$

In particular, from (6.5)

$$\alpha_1(u) = (1-u)^{-1} \int_u^1 Q(p) dp \qquad (6.6)$$

and

$$
\begin{aligned}
\alpha_2(u) &= \frac{1}{2} \frac{1}{(1-u)^2} \left[2\int_u^1 (p-u)Q(p)dp - 2\int_u^1 (1-p)Q(p)dp \right]. \\
&= (1-u)^{-2} \int_u^1 (2p-u-1)Q(p)dp. \qquad (6.7)
\end{aligned}
$$

Of the last two functions $\alpha_1(u)$ is the quantile form of vitality function. Hence its properties are not pursued further. Note that $\alpha_1(u)$ determines $Q(u)$ through the formula

$$Q(u) = \alpha_1(u) - (1-u)\alpha_1'(u), \tag{6.8}$$

obtained from (6.6). The following theorem establishes the relationship between $\alpha_1(u)$, $\alpha_2(u)$ and $M(u)$, and shows that these functions determine $Q(u)$ uniquely.

Theorem 6.1. *The function* $\alpha_1(u)$, $\alpha_2(u)$ *and* $M(u)$ *determine each other and* $Q(u)$ *uniquely.*

Proof: We have

$$\begin{aligned} M(u) &= \frac{1}{(1-u)}\int_u^1 \big(Q(p)-Q(u)\big)dp \\ &= \alpha_1(u) - Q(u) \\ &= \alpha_1(u) - \big(\alpha_1(u) - (1-u)\alpha_1'(u)\big) \\ &= (1-u)\alpha_1'(u). \end{aligned} \tag{6.9}$$

Differentiating (6.7), we have

$$\begin{aligned} (1-u)^2\alpha_2'(u) - 2(1-u)\alpha_2(u) &= -2uQ(u) - \left[-(u+1)Q(u) + \int_u^1 Q(p)dp\right] \\ &= (1-u)Q(u) - \int_u^1 Q(p)dp \\ &= (1-u)Q(u) - (1-u)\big(M(u)+Q(u)\big) \\ &= -(1-u)M(u) \end{aligned}$$

or

$$M(u) = 2\alpha_2(u) - (1-u)\alpha_2'(u). \tag{6.10}$$

Again from (2.79)

$$Q(u) = \mu - M(u) + \int_0^u (1-p)^{-1}M(p)dp. \tag{6.11}$$

Thus $M(u)$ determines $Q(u)$, and $\alpha_1(u)$ and $\alpha_2(u)$ determine $M(u)$. Also we have from (6.9)

$$\alpha_1(u) = \int_0^u \frac{M(p)}{1-p}dp \tag{6.12}$$

$$= \int_0^u \frac{2\alpha_2(p) - (1-p)\alpha_2'(p)}{1-p} dp. \tag{6.13}$$

Equation (6.12) and (6.13) determines $\alpha_1(u)$ in terms of $M(u)$ and $\alpha_2(u)$, and (6.8) recovers $Q(u)$ from $\alpha_1(u)$. We also have

$$\frac{d}{du}(1-u)^2 \alpha_2(u) = -(1-u)M(u). \tag{6.14}$$

Integrating

$$-(1-u)^2 \alpha_2(u) = -\int_u^1 (1-p)M(p)dp$$

or

$$\begin{aligned} \alpha_2(u) &= (1-u)^{-2} \int_u^1 (1-p)M(p)dp \\ &= (1-u)^{-2} \int_u^1 (1-p)^2 \alpha_1(p) dp, \end{aligned} \tag{6.15}$$

determining $\alpha_2(u)$ from $M(u)$ and $\alpha_1(u)$. Given $\alpha_2(u)$, $M(u)$ can be determined from (6.10) and hence $Q(u)$ from (6.11). Hence the proof.

Remark 6.1. Equation (6.3) is important in deducting the conditions for the monotonic behaviour of $L_2(t)$ when $F(x)$ is used instead of $Q(u)$.

Remark 6.2 Recall the definition of Gini's mean difference given in (2.9) through (2.11). Gini's mean difference of the residual random variable X_t is

$$G(t) = 2\int_t^\infty F_t(x)\overline{F}_t(x)dx .$$

In terms of the quantile functions, this becomes

$$\begin{aligned} \Delta(u) = G(Q(u)) &= 2\int_u^1 \frac{(1-p)(p-u)}{(1-u)^2} Q'(p)dp \\ &= \frac{2}{(1-u)^2} \int_u^1 (p - p^2 - u + up) Q'(p) dp. \end{aligned}$$

Integrating by parts, we have

$$\Delta(u) = \frac{2}{(1-u)^2}\int_u^1 (2p-u-1)Q(p)dp \\ = 2\alpha_2(u). \tag{6.16}$$

Further $\alpha_2(0)$ is half the mean difference of X , which is extensively used as a measure of spread and the latter is an accepted measure of dispersion in the analysis of income and poverty in theoretical and applied economics.

The second L-moment of the conditional distribution of $X|X>t$ is half the mean difference of $X|X>t$. Since the mean difference is location invariant the second L-moment of X_t is same as that of $X_t = X-t|X>t$. Thus we can treat $\alpha_2(u)$ as the second L-moment of residual life, a measure of variation and alternative to variance residual quantile function.

Remark 6.3. Theorem 6.1 shows that the dispersion of the residual life in the sense of mean difference is specified in terms of the mean, by means of equation (6.15).

To derive more reliability implications of $\alpha_2(u)$, we have connected it with some other important reliability functions. Firstly consider the total time on test transform (TTT) defined in (2.99)

$$T(u) = \int_0^u (1-p)Q'(p)dp .$$

Using its relationship with $M(u)$ (equation (2.101))

$$T(u) = \mu-(1-u)M(u)$$

and (6.15), we can easily write

$$\alpha_2(u) = (1-u)^{-2} \int_u^1 (\mu - T(p))\,dp. \tag{6.17}$$

Since $\alpha_2(u)$ is conceived as a measure of dispersion its relationship with the variance residual function is of interest. We have

$$V(u) = (1-u)^{-1} \int_u^1 M^2(p)\,dp$$
$$= (1-u)^{-1} \int_u^1 \left(2\alpha_2(p) - (1-p)\alpha_2'(p)\right)^2 dp.$$

A comparison between $V(u)$ and $\alpha_2(u)$ seems to be in order as they are competing measures of variability in the residual life. The functional form of $\alpha_2(u)$ or equivalently that of the mean difference quantile function (or its reversed form) characterizes the life distribution, and therefore it can be used to identify the distribution. The variance of residual life also characterizes the associated distribution, but unlike $\alpha_2(u)$, there is no simple expression relating $Q(u)$ in terms of $V(u)$ or between $\bar{F}(t)$ and $\sigma^2(t)$. As mentioned in Section 2.1.6, Yitzhaki (2003) has compared the relative merits of variance and mean difference as measures of variability, which is also valid for $V(u)$ and $\alpha_2(u)$. He points out that

(a) the mean difference is more informative than the variance in deriving properties of distributions that depart from normality

(b) mean difference can be used to form necessary conditions for second degree stochastic dominance while variance cannot.

We notice that most of the reliability models are non-normal and second order stochastic dominance is used in defining ageing concepts. In these contexts $\alpha_2(u)$ seems to have preference over variance residual life.

The two functions $V(u)$ and $\alpha_2(u)$ may not exhibit same kind of monotonic behaviour. Even when $V(u)$ increases for larger u, $\alpha_2(u)$ can show a decreasing trend. As an example, consider the distribution with quantile function

$$Q(u) = 4u^3 - 3u^4, \qquad 0 \le u \le 1,$$

which is a particular case of Govindarajulu (1977) model discussed in Chapter 3. In this case, using the expression of $M(u)$ given in (3.19) with $\beta = 3$, we have

$$\begin{aligned} V(u) &= \frac{1}{1-u}\int_u^1 M^2(p)dp \\ &= \frac{1}{175}\left(22 - 6u - 34u^2 - 62u^3 + 50u^4 + 78u^5 + 106u^6 + 9u^7 - 38u^8\right) \end{aligned}$$

giving

$$\frac{dV(u)}{du} = \frac{1}{175}\left(-6 - 68u - 186u^2 + 200u^3 + 390u^4 + 636u^5 + 63u^6 - 324u^7\right),$$

which initially decreases in $(0, u_0)$ and then increases in $(u_{0,}\ 1)$ with a unique change point at $u_0 = 0.554449$. On the other hand

$$\alpha_2(u) = \frac{\left(1-u^2\right)^2}{5}$$

and

$$\alpha_2'(u) = \frac{-4}{5}u\left(1-u^2\right) < 0,$$

showing that $\alpha_2(u)$ is decreasing for all u in $(0,1)$.

There are situations when $\alpha_2(u)$ promises to give better results than residual variance quantile function. We give two such examples that bring out the comparison.

Example 6.1 Let X be distributed as exponential with parameter λ. Then $V(u)=\lambda^{-2}$ and $\alpha_2(u)=(2\lambda)^{-1}$. Five hundred samples were generated from the distribution for each of the values $\lambda=0.5, \lambda=1$ and $\lambda=5$. The parameter λ was estimated by equating the sample and population values of $V(u)$ and $\alpha_2(u)$. We found that $\alpha_2(u)$ gives a better approximation to the model (equivalently estimates of λ with less bias). Also the variance of the estimates of λ is considerably less when we use $\alpha_2(u)$.Table 6.1 contains the number of cases in which each of the functions gave better model, reveal that $\alpha_2(u)$ perform better.

Table 6.1 Comparison of $V(u)$ and $\alpha_2(u)$

Function ↓	Number of cases of better approximation		
	$\lambda=0.5$	$\lambda=1.0$	$\lambda=5.0$
$\alpha_2(u)$	360	349	278
$V(u)$	140	151	222

Example 6.2 Mudholkar and Hutson (1996) analyzed the data on annual flood discharge rates of the Floyd river at James, Iowa using the exponentiated Weibull distribution. In an attempt of modelling the data using power-Pareto distribution

$$Q(u)=Cu^{\lambda_1}(1-u)^{-\lambda_2},\ C, \lambda_1, \lambda_2>0\ ,$$

discussed in Chapter 3, we have classified the 39 observations into 5 classes and estimated the parameters by equating the sample and population L- moments (This method has been discussed in Chapter 3). The estimates thus obtained were

$$\widehat{C}=3495.2, \quad \widehat{\lambda_1}=0.6226, \quad \widehat{\lambda_2}=0.5946\,.$$

The χ^2 value of 2.375 as against the tabulated value 3.84 for one degree of freedom does not reject the power-Pareto model for the data.

In studying the variation, among the two measures $V(u)$ and $\alpha_2(u)$, only $\alpha_2(u)$ can be utilized as the variance residual quantile function (given in Chapter 3) does not exist for the above parameter values since $\lambda_2 > 0.5$. The above discussions reveal some reasonable grounds on which further properties of the second L-moment of residual life can be pursued in the following sections.

Now we consider the implications between mean residual quantile function $M(u)$ and $\alpha_2(u)$. We show with the following examples that $M(u)$ and $\alpha_2(u)$ may or may not possess same monotonicity.

Example 6.3. Consider the modified Tukey-Lambda distribution of Freimer et al. (1988) discussed in Chapter 3. The distribution has

$$M(u) = \frac{1}{\lambda_2}\left[\frac{(1-u)^{\lambda_4}}{\lambda_4+1} - \frac{u^{\lambda_3}}{\lambda_3} + \frac{1-u^{\lambda_3+1}}{(1+\lambda_3)(1-u)}\right]$$

and

$$\alpha_2(u) = \frac{1-u}{\lambda_2\lambda_4} - \frac{2\left(1-u^{\lambda_3+2}\right)}{\lambda_2\lambda_3(\lambda_3+1)(\lambda_3+2)(1-u)^2} + \frac{1-u^{\lambda_4}}{\lambda_2(1+\lambda_4)(2+\lambda_4)} + \frac{(1-u)(1+u^{\lambda_3+1})}{\lambda_2\lambda_3(\lambda_3+1)(1-u)^2}.$$

After taking location and scale parameters $\lambda_1 = 0,\ \lambda_2 = 1$, we have,

$$M(u) = \frac{(1-u)^{\lambda_4}}{\lambda_4} - \frac{u^{\lambda_3}}{\lambda_3} + \frac{1-u^{\lambda_3+1}}{\lambda_3(1+\lambda_3)(1-u)}$$

and

$$\alpha_2(u) = \frac{2\left(1-u^{\lambda_3+2}\right)}{\lambda_3(\lambda_3+2)(1-u)^2} + \frac{(1-u)^{\lambda_4}}{(1+\lambda_4)(2+\lambda_4)} - \frac{(u+1)(1-u^{\lambda_3+1})}{\lambda_3(\lambda_3+1)(1-u)^2}.$$

In this case, both $M(u)$ and $\alpha_2(u)$ can be decreasing (e.g. $\lambda_3 = 2, \lambda_4 = 1$), linear $(\lambda_3 = 1, \lambda_4 = 1)$ or decreasing first and then increasing ($\lambda_3 = 10, \lambda_4 = 5$). However the behaviour of $M(u)$ and $\alpha_2(u)$ need not be similar as in the case of $\lambda_3 = 1, \lambda_4 = -5$ in which case the former is decreasing and increasing, while the latter is decreasing.

In Chapter 3, we fitted the distribution to the aluminium coupon data with parameter values

$$\widehat{\lambda_1} = 1382.18,\ \widehat{\lambda_2} = 0.0033,\ \widehat{\lambda_1} = 0.2706 \text{ and } \widehat{\lambda_4} = 0.2211.$$

The graphs of $M(u)$ and $\alpha_2(u)$ of the fitted distribution given in Figure 6.1 shows that both are decreasing functions of u.

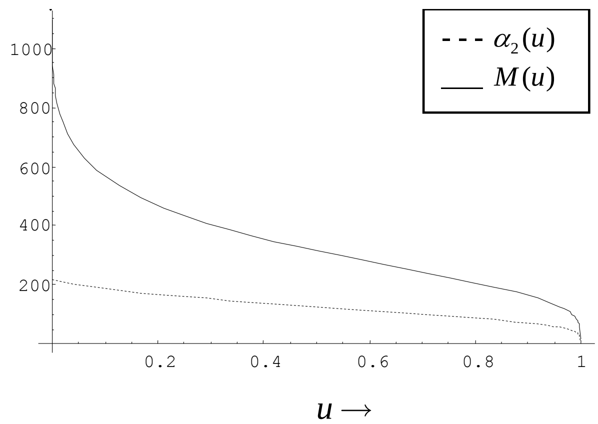

Figure 6.1- $M(u)$ and $\alpha_2(u)$ of the generalised lambda distribution

Example 6.4 The Govindarajulu distribution has (see Section 3.5.5),

$$M(u) = \sigma\left(\frac{2-(\beta+1)(\beta+2)u^{\beta} + 2\beta(\beta+2)u^{\beta+1} - \beta(\beta+1)u^{\beta+2}}{(\beta+2)(1-u)}\right)$$

and

$$\alpha_2(u) = \sigma\left(\frac{2\beta - 2(\beta+3)u + (\beta+2)(\beta+3)u^{\beta+1} - 2\beta(\beta+3)u^{\beta+2} + \beta(\beta+1)u^{\beta+3}}{(\beta+2)(\beta+3)(1-u)^2}\right)$$

we find that $M(u)$ and $\alpha_2(u)$ decrease for $\beta<1$ and when $\beta>1$ both functions either decrease or first increase and then decrease with the change point increasing as β increases. In Govindarajulu (1977), the distribution is fitted to the data on failure times of a set of refrigerator motors, with the estimate of β viz. $\widehat{\beta}=2.94$. Taking $\sigma=1$, for this value of β, $M(u)$ initially increases and then decreases with approximate change point at $u=0.2673$ and $\alpha_2(u)$ decreases for all u. See Figure 6.2.

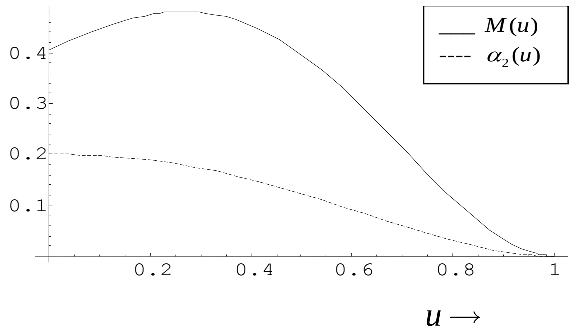

Figure 6.2- $M(u)$ and $\alpha_2(u)$ of Govindarajulu distribution

Example 6.5 In the case of power Pareto distribution described in Chapter 3,

$$M(u)=C(1-u)^{-1}\left(\mathrm{B}_{1-u}(\lambda_1+1,1-\lambda_2)-u^{\lambda_1}(1-u)^{1-\lambda_2}\right)$$

and

$$\alpha_2(u)=C(1-u)^{-2}\left(2\mathrm{B}_{1-u}(\lambda_1+2,1-\lambda_2)-(u+1)\mathrm{B}_{1-u}(\lambda_1+1,1-\lambda_2)\right),$$

where

$$\mathrm{B}_{1-u}\left(p,q\right)=\int_u^1 t^{p-1}(1-t)^{q-1}dt.$$

In general $M(u)$ and $\alpha_2(u)$ possess different patterns of failures, through its functional behaviour, such as, both are increasing (e.g. $C=1,\ \lambda_1=1,\ \lambda_2=0.5$), first increasing and then decreasing (e.g.

$C=1,\ \lambda_1=5,\ \lambda_2=0.1$) and also first decreasing and then increasing (e.g. $C=1,\ \lambda_1=1,\ \lambda_2=0.2$).

For the electric cart data given in Chapter 3, we have found the estimates as

$$\hat{\lambda}_1\text{=0.234621},\ \hat{\lambda}_2\text{=0.0966912},\ \hat{C}=1530.53$$

The nature of $M(u)$ and $\alpha_2(u)$ functions for the data is presented in Figure 6.3.

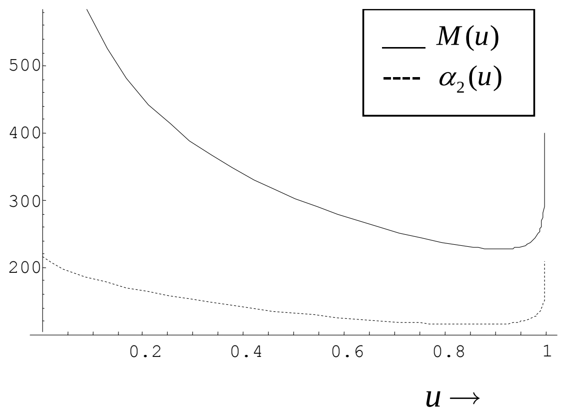

Figure 6.3- $M(u)$ and $\alpha_2(u)$ of power-Pareto distribution

In the above discussions we compared the second L-moment of residual life with the mean and variance of the residual life. The coefficient of variation of residual life defined by

$$C(x)=\frac{\sigma(x)}{m(x)}. \tag{6.18}$$

Encouraging from the properties of coefficient of variation of residual life, here we define the L-coefficient of variation by

$$c(u)=\frac{\alpha_2(u)}{\alpha_1(u)}. \tag{6.19}$$

Gupta and Kirmani (2000) have shown that the coefficient of variation of residual life characterizes the life distribution. We now demonstrate that a similar result exists for the L-coefficient of variation of the residual quantile function defined as $c(u)=\dfrac{\alpha_2(u)}{\alpha_1(u)}$.

Theorem 6.2. *If* $c(u)$ *is differentiable, then*

$$Q(u)=g(u)\exp\left[-\int g(u)\,du\right], \tag{6.20}$$

where

$$g(u)=\frac{(1-u)c'(u)-c(u)+1}{(1-u)\big(1+c(u)\big)}.$$

Proof: From the definition of $c(u)$, (6.6) and (6.7),

$$\int_u^1 (2p-u-1)Q(p)\,dp=(1-u)c(u)\int_u^1 Q(p)\,dp\,.$$

Differentiating and simplifying

$$(1-u)\big(1+c(u)\big)Q(u)=\big[(1-u)c'(u)-c(u)+1\big]\int_u^1 Q(p)\,dp\,,$$

rearranging the terms

$$\frac{Q(u)}{\int_u^1 Q(p)\,dp}=\frac{(1-u)c'(u)-c(u)+1}{(1-u)\big(1+c(u)\big)}.$$

Integrating the above, we get

$$-\log\int_u^1 Q(p)\,dp=\int\frac{(1-u)c'(u)-c(u)+1}{(1-u)\big(1+c(u)\big)},$$

from which (6.20) follows.

Example 6.6 As a simple example, if

$$c(u)=\frac{(1-u)}{3(1+u)} \tag{6.21}$$

then

$$\frac{(1-u)c'(u)-c(u)+1}{(1-u)\big(1+c(u)\big)}=\frac{2u}{1-u^2}$$

and

$$\int \frac{(1-u)c'(u)-c(u)+1}{(1-u)\left(1+c(u)\right)}du = -\log(1-u^2).$$

Applying Theorem 6.20, we have

$$Q(u) = 2u\,,$$

shows that $c(u)$ determine the quantile function of uniform distribution with a change of scale.

In the next section we give the definition and the properties of L-moments of reversed residual life.

6.3 L-moments of reversed residual life

On almost similar lines we can treat the functions related to reversed residual life by considering ${}_tX = \left(X \,|\, X \le t\right)$ whose distribution is ${}_tF(x) = \dfrac{F(x)}{F(t)},\ 0 < x \le t$. Using (6.1) the r^{th} L-moment of ${}_tX$ has the expression

$$B_r(t) = \sum_{k=0}^{r-1}(-1)^r \binom{r-1}{k}^2 \int_0^t x\left(\frac{F(x)}{F(t)}\right)^{r-k-1}\left(1-\frac{F(x)}{F(t)}\right)^k \frac{f(x)}{F(t)}dx. \qquad (6.22)$$

In particular

$$B_1(t) = \int_0^t \frac{xf(x)dx}{F(t)} = E\left(X \,|\, X \le x\right) \qquad (6.23)$$

and

$$B_2(t) = \frac{1}{\overline{F}^2(t)}\int_0^t \left(2F(x)-F(t)\right)xf(x)dx. \qquad (6.24)$$

Setting $u = F(t)$ and $p = F(x)$, we have

$$\beta_1(u) = u^{-1}\int_0^u Q(p)dp \qquad (6.25)$$

and

$$\beta_2(u) = u^{-2}\int_0^u (2p-u)Q(p)dp. \tag{6.26}$$

From (6.25), we have

$$Q(u) = u\beta_1'(u) + \beta_1(u). \tag{6.27}$$

Further from (6.26),

$$\frac{d}{du}u^2\beta_2(u) = \frac{d}{du}\int_0^u (2p-u)Q(p)dp$$

$$\begin{aligned} \Rightarrow u\beta_2'(u) + 2\beta_2(u) &= Q(u) - \frac{1}{u}\int_0^u Q(p)dp \\ &= \frac{1}{u}\int_0^u \left(Q(u) - Q(p)\right)dp \\ &= R(u). \end{aligned} \tag{6.28}$$

Again from (6.28), we have

$$\frac{d}{du}u^2\beta_2(u) = uR(u).$$

Integrating, we have

$$\beta_2(u) = u^{-2}\int_0^u pR(p)dp. \tag{6.29}$$

From (2.91) we have

$$Q(u) = R(u) + \int_0^u p^{-1}R(p)dp. \tag{6.30}$$

Hence using similar arguments in Theorem 6.1, we can conclude that each of $Q(u)$, $R(u)$, $\beta_1(u)$ and $\beta_2(u)$ determine the others uniquely. We have from (2.97), the reversed variance residual quantile function

$$\begin{aligned} D(u) &= u^{-1}\int_0^u R^2(p)dp \\ &= u^{-1}\int_0^u \left(p\beta_2'(p) + 2\beta_2(p)\right)^2 dp. \end{aligned} \tag{6.31}$$

Similar to (6.19), we define the L-coefficient of variation in reversed time as

$$\theta(u) = \frac{\beta_2(u)}{\beta_1(u)}. \tag{6.32}$$

Following the steps of the proof of Theorem 6.2, we can show that $\theta(u)$ determines the distribution up to a change of scale as

$$Q(u) = \frac{u\theta'(u) + \theta(u) + 1}{u(1 - \theta(u))} \exp\left[\int \frac{u\theta'(u) + \theta(u) + 1}{u(1 - \theta(u))} du\right]. \tag{6.33}$$

As an example we can easily show that the power distribution is characterized by a constant value for $\theta(u)$.

6.4 Characterizations

In this section we present some characterization theorems employing the reliability concepts discussed above that can help the identification of the underlying lifetime distribution. Our first result concerns the generalized Pareto distribution with quantile function

$$Q(u) = \frac{b}{a}\left((1-u)^{-\frac{a}{a+1}} - 1\right), \quad a > -1,\ b > 0, \tag{6.34}$$

which is a family consisting of the exponential distribution $(a \to 0)$, rescaled beta $(-1 < a < 0)$ and the Lomax distribution $(a > 0)$. The family is characterized by a linear mean residual life (reciprocal linear hazard rate) function in the conventional reliability analysis.

Theorem 6.3. *Let X be a nonnegative continuous random variable with $E(X) < \infty$. Then X follows the generalized Pareto distribution (6.34) if and only if any one of the following conditions is satisfied for all u in (0, 1).*

(i) $\alpha_2(u) = CM(u), \quad 0 < C < 1$

(ii) $\alpha_2(u) = a_1 \alpha_1(u) + a_2, \quad a_1 > -1,\ a_2 > 0$

(iii) $\alpha_1(u) = AM(u) + B.$

Proof: For the model (6.34), we have

$$\alpha_1(u) = ba^{-1}\left((a+1)(1-u)^{-\frac{a}{a+1}} - 1\right),$$

$$\alpha_2(u) = b(a+1)(a+2)^{-1}(1-u)^{-\frac{a}{(a+1)}}$$

and

$$M(u) = b(1-u)^{-\frac{a}{(a+1)}}.$$

Then we have

$$\alpha_2(u) = \frac{a+1}{a+2}M(u), \tag{6.35}$$

$$\alpha_2(u) = \frac{1}{a+2}\left[a\alpha_1(u) + \frac{b}{a(a+2)}\right] \tag{6.36}$$

and

$$\alpha_1(u) = \frac{a+1}{a}M(u) - \frac{b}{a}. \tag{6.37}$$

Equations (6.35), (6.36) and (6.37) verify the conditions (i), (ii) and (iii). To prove the only if part of (i), we assume (i), then from (6.15)

$$C(1-u)^2 M(u) = \int_u^1 (1-p)M(p)dp.$$

On differentiation,

$$C(1-u)M'(u) = (2C-1)M(u).$$

Rearranging,

$$\frac{M'(u)}{M(u)} = \frac{2C-1}{C}\frac{1}{1-u}.$$

On integration this leads to

$$M(u) = k(1-u)^{\frac{1-2C}{C}}, \quad k = M(0) = \mu.$$

Since $0<C<1$, we can write $C=\dfrac{a+1}{a+2}$ for $a>-1$ and obtain the generalized Pareto distribution. Notice that the exponential (rescaled beta, Lomax) distribution is characterized by $C=\frac{1}{2}\left(<\frac{1}{2},\ >\frac{1}{2}\right)$.

In the case of (ii), it implies

$$(1-u)^{-2}\int_u^1 (2p-u-1)Q(p)dp=(1-u)^{-1}a_1\int_u^1 Q(p)dp+a_2$$

or

$$\int_u^1 (2p-u-1)Q(p)dp=a_1(1-u)\int_u^1 Q(p)dp+a_2(1-u)^2.$$

Differentiating and simplifying

$$(1+a_1)(1-u)Q(u)+2a_2(1-u)=(1-a_1)\int_u^1 Q(p)dp.$$

Again differentiating, we have

$$Q'(u)-\frac{2a_1Q(u)}{(1+a_1)(1-u)}=\frac{2a_2}{(1+a_1)(1-u)}.$$

This is a linear differential equation with integrating factor $(1-u)^{\frac{2a_1}{1+a_1}}$ and hence the solution is

$$(1-u)^{\frac{2a_1}{1+a_1}}Q(u)=-\frac{a_2}{a_1}(1-u)^{\frac{2a_1}{1+a_1}}+C.$$

Setting $u=0$, $C=\dfrac{a_2}{a_1}$ and therefore,

$$Q(u)=\frac{a_2}{a_1}\left((1-u)^{-\frac{2a_1}{1+a_1}}-1\right),$$

which is a generalized Pareto form (the form (6.34) results from the reparametrisation $a_1=\dfrac{a}{a+2}$, $a_2=\dfrac{b}{a+2}$). The result (iii) follows from (i) and (ii) and the proof is completed.

Remark 6.3 The relationship between variance residual life and $\alpha_2(u)$ as measures of dispersion is of interest. It is seen from direct calculations that for the generalized Pareto distribution,

$$V(u)=\frac{1+a}{1-a}b^2(1-u)^{-\frac{2a}{a+1}}$$

$$=K\alpha_2^2(u), \quad K=\frac{(a+2)^2}{1-a^2}\ . \tag{6.38}$$

Now we examine whether (6.38) is a characteristic property. Equation (6.38) means that

$$K\alpha_2^2(u)=(1-u)^{-1}\int_u^1 Q^2(p)dp-\left(M(u)+Q(u)\right)^2 .$$

Differentiating and simplifying the resulting expression,

$$2K(1-u)\alpha_2(u)\alpha_2{}'(u)-K\alpha_2{}'(u)=-Q^2(u)-2\left(M(u)+Q(u)\right)\left(M'(u)+Q'(u)\right)(1-u)$$
$$+\left(M(u)+Q(u)\right)^2. \tag{6.39}$$

Since

$$(1-u)\left(M(u)+Q(u)\right)=\int_u^1 Q(p)dp,$$

$$(1-u)\left(M'(u)+Q'(u)\right)-\left(M(u)+Q(u)\right)=-Q(u)$$

or

$$(1-u)\left(M'(u)+Q'(u)\right)=M(u)\,.$$

Substituting in (6.39),

$$K\alpha_2^2(u)-2K(1-u)\alpha_2(u)\alpha_2'(u)=M^2(u)\,.$$

But using (6.10), we have

$$-3K\alpha_2^2(u)+2K\alpha_2(u)M(u)=M^2(u),$$

which can be written by taking $y=\dfrac{\alpha_2(u)}{M(u)}$ as

$$3Ky^2-2Ky+1=0\,, \tag{6.40}$$

The solutions of (6.40) are

$$y=\frac{1}{3}\left(1\pm\left(\frac{K-3}{K}\right)^{\frac{1}{2}}\right).$$

The first solution leads to

$$\alpha_2(u)=\frac{1}{3}\left(1+\left(\frac{K-3}{K}\right)^{\frac{1}{2}}\right)M(u)$$

and as such by (i) of Theorem 6.3, X is distributed as generalized Pareto distribution and with exponential (rescaled beta; Lomax) when $K=4$ $(3\le K<4,\ K>4)$. However the second solution gives

$$\alpha_2(u)=\frac{1}{3}\left(1-\left(\frac{K-3}{K}\right)^{\frac{1}{2}}\right)M(u)<\frac{1}{2}M(u),$$

for all $K\ge 3$ and therefore X is distributed as rescaled beta. As an example when X is exponential $(a=0)$ or rescaled beta with $a=-\frac{4}{5}$ gives $\sigma^2(u)=4\alpha_2^2(u)$. Thus (6.38) is not a characteristic property of the generalized Pareto distribution.

Remark 6.4. $V(u)=\frac{1+a}{1-a}M^2(u),\ a>-1$ characterizes the generalized Pareto distribution. To see this, use the above relationship in

$$V(u)=(1-u)^{-1}\int_u^1 M^2(p)dp,$$

$$\frac{1+a}{1-a}M^2(u)=(1-u)^{-1}\int_u^1 M^2(p)dp.$$

Differentiating

$$\frac{1+a}{1-a}\left[2(1-u)M(u)M'(u)-M^2(u)\right]=-M^2(u),$$

which leads to

$$\frac{M'(u)}{M(u)} = \frac{2a}{(1+a)(1-u)}.$$

Integrating and simplifying

$$M(u) = K(1-u)^{\frac{-2a}{1+a}},$$

the expression for $M(u)$ of the generalized Pareto distribution. This result is proved earlier in Gupta and Kirmani (2004) using the distribution function approach.

The next theorem states the distribution corresponding to the sum of two second L-moment of residual lives (mean residual lives).

Theorem 6.4. I*f* $\alpha_{21}(u)\,(M_1(u))$ *and* $\alpha_{22}(u)\,(M_2(u))$ *are second L-moment (mean residual) quantile function of two random variable* X *and* Y*, then* $\alpha_{21}(u)+\alpha_{22}(u)\ (M_1(u)+M_2(u))$ *is the second L-moment residual (mean residual) quantile function of the distribution with* $Q(u)=Q_1(u)+Q_2(u)$.

Proof: Follows directly from the definitions of $\alpha_2(u)$ and $M(u)$.

Parallel characterizations hold in the case of reversed L-moment quantile functions, where the role of the generalized Pareto distribution is taken by the power distribution. The proof follows the same pattern as in the previous cases and therefore they are omitted.

Theorem 6.5 *Let* X *be distributed as the power distribution with quantile function*

$$Q(u) = au^{\frac{1}{b}}, \quad a,b>0, \ \ 0 \le u \le 1 \tag{6.41}$$

Then for all u,

$$\textit{(i)} \quad \beta_1(u) = C_1 Q(u), \quad 0 < C_1 < 1$$

$$\textit{(ii)} \quad \beta_2(u) = C_2 \beta_1(u), \ \ 0 < C_2 < 1$$

$$\textit{(iii)}\ \beta_2(U) = C_3 R(u),$$

and conversely.

Remark 6.5 $D(u) = K\beta_2^2(u)$ for the power distribution, reduces to the quadratic equations

$$3Kz^2 - 2Kz + 1 = 0,$$

where $z = \frac{\beta_2(u)}{R(u)}$. As before the solutions are

$$\beta_2(u) = \frac{1}{3}\left(1 \pm \left(\frac{K-3}{K}\right)^{\frac{1}{2}}\right) R(u).$$

Since $\frac{\beta_2(u)}{R(u)} < \frac{1}{2}$ one should have $3 < K < 4$ for the first solution and the second solution is valid for all $K > 3$. Hence there is a characterization of the power solution for all $K \geq 4$ and two power distributions result as solutions whenever $3 < K < 4$.

Theorem 6.6 *The identity*

$$\beta_2(u) = \frac{a - bu}{c - au}\beta_1(u) \tag{6.42}$$

is satisfied for all u *and constants* a, b, c *satisfying*

$$a > b,\ c > a, \text{ and } \frac{c-a}{c+a} = \frac{2b}{a-b} \tag{6.43}$$

if and only if X *follows Govindarajulu distribution.*

Proof: When X has the distribution stated in the theorem

$$\beta_1(u) = \sigma u^{\beta}\left[1 - \frac{\beta}{\beta+2}u\right]$$

and

$$\beta_2(u)=\frac{\beta-\dfrac{\beta(\beta+1)}{\beta+3}u}{\beta+2-\beta u}\beta_1(u),$$

showing that form (6.42) and condition (6.43) are met with. Conversely from (6.42) and definitions of the function involved,

$$\int_0^u (2p-u)Q(p)dp=\frac{u(a-bu)}{c-au}\int_0^u Q(p)\,. \tag{6.44}$$

Differentiating (6.44) with respect to u and simplifying

$$\frac{Q(u)}{\int_0^u Q(p)}=\frac{(c-du)(a-2bu)+au(a-bu)+(c-au)^2}{u(c-au)(c-a+u(b-a))}$$

$$=\frac{c+a}{(c-a)u}-\frac{a}{c-au}+\frac{\left(\dfrac{c+a}{c-a}(a-b)-2b\right)}{\left(c-a+(b-a)u\right)}.$$

The last factor vanishes by virtue of (6.43) and hence on integration,

$$\int_0^u Q(p)dp=Ku^{\frac{c+a}{c-a}}(c-au)$$

or

$$Q(u)=Kc\left(\frac{c+a}{c-a}u^{\frac{c+a}{c-a}-1}-\left(\frac{c+a}{c-a}-1\right)u^{\frac{c+a}{C-a}}\right),$$

which is the quantile function of Govindarajulu distribution with parameters $\sigma=Kc$ and $\beta=\dfrac{c+a}{c-a}-1$. This completes the proof.

Remark 6.6 Condition (6.43) can be modified to derive a more general family of distributions satisfying (6.42), but the resulting four-parameter family provides much complicated forms of properties which are not easy for practical use.

6.5 Applications

We have indicated some applications of $\alpha_2(u)$ and $\beta_2(u)$ in modelling lifetime data in the previous sections. In this section we point out some more applications in reliability analysis and also in economics. A detailed study has to be taken up separately.

6.5.1. Reliability:

When conceived as a reliability function the L-moment $\alpha_2(u)$ can also be employed in distinguishing life distributions based on its monotonic behaviour. Since $\alpha_2(u)$ and $\beta_2(u)$ are twice the mean difference, the monotonic behaviour of $\alpha_2(u)$ and $\beta_2(u)$ are those of the corresponding mean differences. Thus we have the following definition of the ageing class based on mean difference in terms of $\alpha_2(u)$ and $\beta_2(u)$.

Definition 6.1: The random variable X is said to be increasing (decreasing) mean difference quantile function – IMDQ (DMDQ) according as $\alpha_2(u)$ is increasing (decreasing). Similarly increasing reversed mean difference quantile function (IRMDQ) and decreasing reversed mean difference quantile function (RDMDQ) are defined with respect to $\beta_2(u)$. Further the mean difference quantile function is first increasing (decreasing) and then decreasing (increasing) with change point at $u = u_0$ will be denoted by IDMDQ (DIMDQ).

Example 6.7 From the expressions of $\alpha_2(u)$ given in the proof of Theorem 6.3, it is clear that the Lomax distribution is IMDQ and the beta distribution is DMDQ. In the Govindarajulu distribution, as it is mentioned earlier that $\alpha_2(u)$ is first increasing and then decreasing for $\beta > 1$, the Govindarajulu distribution is IDMDQ.

The analytic condition for X to be IMDQ or DMDQ is derived from

$$M(u) = 2\alpha_2(u) - (1-u)\alpha_2'(u)$$

as

$$\alpha_2'(u) > 0 \Rightarrow \alpha_2(u) > \frac{1}{2}M(u).$$

Thus X is IMDQ (DMDQ) according as $\alpha_2(u) \geq (\leq) \frac{1}{2}M(u)$. In the case of ID (DI) MDQ, the change point u_0 is obtained from $\alpha_2(u_0) = \frac{1}{2}M(u_0)$. Obviously, the exponential distribution, as in the case of other ageing classes, separates the increasing and decreasing MDQ classes with constant mean difference.

On the other hand, the behaviour of $\beta_2(u)$ results with

$$\beta_2'(u) = u^{-1}\left(R(u) - 2\beta_2(u)\right) \tag{6.45}$$

and hence the turning point of $\beta_2(u)$, if any will be the solution of $R(u) = 2\beta_2(u)$. Looking at a more general equation, $R(u) = C\beta_2(u)$, $C > 0$, we find

$$u^2 R(u) = C\int_0^u pR(p)\,dp, \qquad \text{(using (6.29))}$$

Differentiating and simplifying

$$\frac{R'(u)}{R(u)} = \frac{c-2}{u}.$$

On integration

$$R(u) = Ku^{C-2}.$$

Applying on (6.45)

$$Q(u) = K\frac{C-1}{C-2}u^{C-2},$$

which provides a proper distribution on the positive real line only if $C > 2$. For $C > 2$, $\beta_2'(u) > 0$ implies that for a nonnegative random variable there is no change point for $\beta_2(u)$ and it is an increasing function for all u. Unlike $\alpha_2(u)$, there is limited use for $\beta_2(u)$ in classifying life distributions on the basis of its monotonicity.

6.5.2 Economics

In this section we point out the scope of L-moment of reversed residual life in the field of economics. Let X be the random variable representing the personal incomes in a population. Modelling the distribution of incomes is a traditional problem in which the use of lambda distributions is of recent interest (Tarsitano (2004), Haritha et al. (2008)). One important application of income distributions is the analysis of poverty in a population. This is often accomplished by the choice of a criterion that decides whether an individual is poor and an index which summarizes the amount of poverty in the population under consideration. Taking the poverty line as $X = t$, so that an individual whose income is below t is considered as poor, the well known index, proposed by Sen (1976) is often used in this context. The Sen Index is defined as

$$p(t) = F(t)\big(i(t) + (1 - i(t))g(t)\big), \tag{6.46}$$

where $F(t)$ is interpreted as the headcount ratio,

$$i(t) = 1 - E\left(\frac{X}{t} | X \le t\right), \tag{6.47}$$

is known as the income gap ratio for the poor and

$$g(t)=1-\frac{2}{E(X|X\le t)}\int_0^t y(1-{}_tF(y))_t f(y)dy \qquad (6.48)$$

is the Gini index for poor. In terms of quantile functions,

$$G(u)=g(Q(u))=1-\frac{2}{\beta_1(u)}\left(\int_0^u Q(p)\left(\frac{u-p}{u^2}\right)dp\right). \qquad (6.49)$$

Note that

$$\int_0^u Q(p)dp=u\beta_1(u)$$

and

$$\int_0^u pQ(p)dp=\frac{u^2}{2}\left[\frac{1}{u^2}\int_0^u(2p-u)Q(p)dp+\frac{1}{u}\int_0^u Q(p)dp\right]$$

$$=\frac{u^2}{2}[\beta_1(u)+\beta_2(u)].$$

Thus (6.49) become

$$G(u)=\frac{\beta_2(u)}{\beta_1(u)},$$

the L-coefficient of variation and

$$I(u)=i(Q(u))=1-\left(\frac{\beta_1(u)}{Q(u)}\right).$$

Thus the Sen index (5.46) has the quantile analogue

$$P(u)=u\left(1-\frac{\beta_1(u)}{Q(u)}+\frac{\beta_2(u)}{Q(u)}\right)$$

$$=\frac{u}{Q(u)}(R(u)+\beta_2(u)).$$

Using the above expression we can express $P(u)$ in terms of $\beta_1(u)$ and $\beta_2(u)$ alone by noting $Q(u)=u\beta_1'(u)+\beta_1(u)$ as

$$P(u)=u\left[\frac{u\beta_1'(u)+\beta_2(u)}{u\beta_1'(u)+\beta_1(u)}\right].$$

The above formula becomes handy when quantile functions, whose distributions are not available in closed form, are employed for modelling income data (like the lambda distributions). A detailed discussion in this respect is available in Haritha et al. (2008). Further from (5.29) and (5.30) it is evident that, instead of the distribution, if $R(u)$ can be specified then $P(u)$ can be determined from it.

From the earlier discussion it also evident that there is one to one correspondence between the income gap ratio $I(u)$ (and also the Gini index $G(u)$) and the baseline income distribution of X. The same cannot be said about the correspondence between $Q(u)$ and $P(u)$.

We conclude the present study by noting that the second L-moment of residual life (or equivalently the mean difference) possesses properties similar to the variance residual life. It can be useful in modelling, characterizing and analyzing lifetime data, and the quantile-based approach adds to its applicability to empirical models where the distribution function cannot be expressed in simple analytical form.

Chapter 7

Reversed percentile residual life and related concepts*

7.1 Introduction

In Section 2.2.4, we have given a brief review on percentile residual life function (PRL). A compact review provided there help to know the developments of PRL in different periods. Theoretically there is analogy in the works relating to residual and reversed residual life functions, the properties and models relating to them differ substantially to merit the study of the latter. The relevance of various existing concepts in reversed time and the enormous literature on percentile residual lifetime mentioned above motivate us to study the properties of the reversed version of the percentile residual life function in the present chapter. Such a study along with the relationships that reversed percentile residual life has with other concepts used in this connection, does not appear to have been discussed in literature.

In this chapter we discuss the properties of the reversed percentile residual life function (RPRL) and its relationship with the reversed hazard function (RHR). Some models with simple functional forms for both RHR and RPRL are proposed. A method of distinguishing

**The discussions in this chapter is based on Nair and Vineshkumar (2011) appeared in the Journal of Korean Statistical Society (see reference no. 94)* decreasing (increasing) reversed hazard rates (reversed percentile residual lives) is also presented.

7.2. Definition and properties

Let F be the distribution function of a lifetime random variable (such that $F(0-)=0$) with quantile function $Q(u)$. We have from (2.65) that the reversed residual life ${}_tX=\left(t-X\middle|X\le t\right)$ has distribution function

$$ {}_tF(x)=1-\frac{F(t-x)}{F(t)}, \quad 0\le x\le t. \tag{7.1}$$

Accordingly for $0<\alpha<1$, the α^{th} reversed percentile residual life function of X is defined as

$$\begin{aligned} q_\alpha(t) &= F_t^{-1}(\alpha)=\inf\left(x: {}_tF(x)\ge\alpha\right),\\ &= \inf\left(x:F(t-x)\le(1-\alpha)F(t)\right),\\ &= t-F^{-1}\left((1-\alpha)F(t)\right), \quad 0\le t<Q(1), \end{aligned}\tag{7.2}$$

with $Q(1)=\sup\left(x:F(x)<1\right)$ as the right hand end point of the support of F. From (7.2) we see that the functional equation that solves for $q_\alpha(t)$ is

$$t-q_\alpha(t)=F^{-1}\left((1-\alpha)F(t)\right)$$

or

$$F\left(t-q_\alpha(t)\right)=(1-\alpha)F(t). \tag{7.3}$$

In terms of the quantile function $Q(u)$, the α-th RPRL, as a function of u, is obtained as

$$q_\alpha^*(u)=q_\alpha\left(Q(u)\right)=Q(u)-Q\left((1-\alpha)u\right). \tag{7.4}$$

This definition is quite useful in situations where the quantile functions exist in the simple forms but whose distribution functions do not have closed forms to utilize (7.1) and (7.2). We have seen several such models in Chapter 3 and its application in analyzing lifetime data. As a simple example, for power Pareto distribution given in Section 3.1.3, specified by

$$Q(u) = cu^{\lambda_1}(1-u)^{-\lambda_2}\,,\ c,\lambda_1,\lambda_2 > 0\,,\ 0 < u < 1\,,$$

has

$$q_{\alpha}^{*}(u) = cu^{\lambda_1}\left[(1-u)^{-\lambda_2} - (1-\alpha)^{\lambda_1}\left(1-(1-\alpha)u\right)^{-\lambda_2}\right]. \qquad (7.5)$$

We now discuss some properties of RPRL. First is the problem of characterizing F by the functional form of $q_{\alpha}(t)$. We demonstrate through the following example that the RPRL for a given α does not determine F uniquely. In other words the functional equation (7.3) is satisfied by more than one distribution function for a given $q_{\alpha}(t)$.See the following example.

Example 7.1 Assuming that X follows the power distribution

$$F(t) = t^{a}, \quad 0 \le t \le 1, \quad a > 0.$$

From (7.2), for the choice of $\alpha = 1 - e^{-2\pi a}$

$$q_{\alpha}(t) = \left(1-(1-\alpha)^{\frac{1}{a}}\right)t = \left(1-e^{-2\pi}\right)t.$$

Now consider the distribution specified by

$$G(t) = t^{a}\left(1+\frac{1}{2}\sin\log t\right), \qquad 0 \le t \le 1, \quad a > 0,$$

so that $G(t) \neq F(t)$. Then for the above choice of $q_{\alpha}(t)$ and α,

$$\begin{aligned} G\left(t-q_{\alpha}(t)\right) &= \left(t-q_{\alpha}(t)\right)^{a}\left(1+\frac{1}{2}\sin\log\left(t-q_{\alpha}(t)\right)\right) \\ &= \left(te^{-2\pi}\right)^{a}\left(1+\frac{1}{2}\sin\log\left(te^{-2\pi}\right)\right) \\ &= t^{a}e^{-2\pi a}\left(1+\frac{1}{2}\sin\left(\log t - 2\pi\right)\right) \end{aligned}$$

$$= t^a e^{-2\pi a}\left(1+\frac{1}{2}\sin\log t\right)$$
$$= (1-\alpha)G(t),$$

so that both G and F have the same RPRL satisfying (7.3).

Thus we are lead to the search for some general conditions under which F is determined uniquely. Equation (7.3) is a particular case of Schroder's functional equation

$$S(\phi(t)) = uS(t), \qquad 0 \le t \le \infty, \tag{7.6}$$

discussed in Gupta and Langford (1984), where $0 < u < 1$ and $\phi(t)$ is a continuous and strictly increasing function on $(0, \infty)$ which satisfies $\phi(t) > t$ for all t. The general solution of the equation is

$$S(t) = S_0(t)K(\log S_0(t)),$$

where $K(.)$ is a periodic function with period $-\log u$ and $S_0(.)$ is a particular solution which is continuous and strictly decreasing and satisfies $S_0(0)=1$. In our case, in analogy with (7.6), $\phi(t) = t - q_\alpha(t)$, which does not satisfy the requirement $\phi(t) > t$ for the above solution. Therefore we seek the conditions for two distributions to have the same RPRL for a given α, which is presented in the following theorem.

Theorem 7.1.

Let F and G be two continuous and strictly increasing distribution functions with corresponding RPRL's $q_\alpha(t)$ and $r_\alpha(t)$. Then a necessary and sufficient condition that $q_\alpha(t) = r_\alpha(t)$ for all $t \ge 0$ is that

$$F(t) = G(t).K(-\log G(t)), \tag{7.7}$$

where $K(.)$ is a periodic function with period $-\log(1-\alpha)$, $0 < \alpha < 1$.

Proof: Assume that for a given $0<\alpha<1$, $q_\alpha(t)=r_\alpha(t)$ for all t. Then from (7.2)

$$F^{-1}\big((1-\alpha)F(t)\big)=G^{-1}\big((1-\alpha)G(t)\big). \tag{7.8}$$

Setting $G(t)=u,\ 0<u<1$,

$$(1-\alpha)F\left(G^{-1}(u)\right)=F\left(G^{-1}(1-\alpha)u\right), \quad 0<u<1. \tag{7.9}$$

For the function

$$K(t)=e^t FG^{-1}\left(e^{-t}\right), \tag{7.10}$$

$$G(t)K\left(-\log G(t)\right)=F\left(G^{-1}G(t)\right)=F(t),$$

showing that (7.10) solves (7.7). Further

$$\begin{aligned}K\left(t-\log(1-\alpha)\right)&=e^{t-\log(1-\alpha)}FG^{-1}\left(e^{\log(1-\alpha)-t}\right)\\&=(1-\alpha)^{-1}e^t FG^{-1}\left((1-\alpha)e^{-t}\right)\\&=(1-\alpha)^{-1}e^t(1-\alpha)FG^{-1}\left(e^{-t}\right) \quad \left(\text{by (7.9)}\right)\\&=K(t).\end{aligned}$$

Thus *K (.)* is periodic with period $-\log(1-\alpha)$ and therefore, the condition is necessary. Conversely if (7.7) holds for all *t,*

$$\begin{aligned}FG^{-1}\left((1-\alpha)G(t)\right)&=G\left(G^{-1}(1-\alpha)G(t)\right)K\left(-\log GG^{-1}\left((1-\alpha)G(t)\right)\right)\\&=(1-\alpha)G(t)K\left(-\log(1-\alpha)G(t)\right)\\&=(1-\alpha)G(t)K\left(-\log G(t)\right)\\&=(1-\alpha)F(t).\end{aligned}$$

Since F is strictly increasing,

$$G^{-1}\left((1-\alpha)G(t)\right)=F^{-1}\left((1-\alpha)F(t)\right)$$

and hence $q_\alpha(t)=r_\alpha(t)$ as desired.

Remark 7.1 In equation (7.9), if we set $A(u)=FG^{-1}(u)$, we have

$$A\left((1-\alpha)u\right)=(1-\alpha)A(u),$$

which is a particular case of the Schroder's equation.

In the next tsheorem we seek conditions for the distribution to be determined by two RPRLs.

Theorem 7.2

If F is strictly increasing and continuous and $\frac{\log(1-\alpha)}{\log(1-\beta)}$ is irrational, then F is uniquely determined by the RPRL's $q_\alpha(t)$ and $q_\beta(t)$.

Proof: Since $F(t)$ is a particular solution, another distribution function satisfying (7.7) can be expressed as

$$G(t) = F(t)K\left(-\log F(t)\right).$$

Thus from the condition that $q_\alpha(t)$ and $q_\beta(t)$ are RPRL's we have

$$G(t) = F(t)K_1\left(-\log F(t)\right) = F(t)K_2\left(-\log F(t)\right)$$

where K_1 and K_2 are periodic functions with periods $-\log(1-\alpha)$ and $-\log(1-\beta)$ respectively. The condition of the irrationality of the periods ensures that $G(t) = cF(t)$ where c is a constant. As $t \to Q(1), c = 1$ and hence $G(t) = F(t)$.

Now we look at some more properties of RPRL. For deducing further features of RPRL, we find a relationship it has with RHR. From (7.3),

$$F\left(t - q_\alpha(t)\right) = (1-\alpha)F(t).$$

Now we recall the definition of the reversed hazard rate, $\lambda(x)$ given in Section 2.3.1. We have

$$\lambda(x) = \frac{f(x)}{F(x)} = \frac{d}{dx}\log F(x).$$

Now

$$\int_{t-q_\alpha(u)}^{t} \lambda(x)dx = \log F(t) - \log F\left(t - q_\alpha(t)\right)$$
$$= \log F(t) - \log\left((1-\alpha)F(t)\right)$$
$$= -\log(1-\alpha). \tag{7.11}$$

Differentiating with respect to t, when $q_\alpha(t)$ is differentiable, we have

$$\lambda\left(t - q_\alpha(t)\right)\left(1 - q'_\alpha(t)\right) = \lambda(t)$$

or

$$q'_\alpha(t) = 1 - \frac{\lambda(t)}{\lambda\left(t - q_\alpha(t)\right)}. \tag{7.12}$$

Since

$$(1-\alpha)F(t) < F(t) \Rightarrow F^{-1}\left((1-\alpha)F(t)\right) < t,$$

we conclude that $0 \le q_\alpha(t) \le t$. Hence as t tends to zero from above $q_\alpha(t) = 0$. If we assume that $q_\alpha(t)$ is decreasing we should have $q_\alpha(t) \le q_\alpha(0) = 0$ for all $t > 0$ which is impossible. Thus we have

(a) there is no strictly decreasing RPRL on the whole positive real line;

(b) whenever $\lambda(t)$ is decreasing, $q_\alpha(t)$ is increasing and

(c) the function $q_\alpha(t)$ cannot be a constant on $(0, \infty)$.

To prove (c), we note that the class of distributions with same $q_\alpha(t)$ has the form

$$F(t) = G(t)K(-\log G(t)),$$

where $G(t)$ is a distribution with RPRL $q_\alpha(t)$. Taking

$$G(t) = \exp[c(t-b)], \ -\infty < t < \infty$$

we note that $q_\alpha(t) = -\frac{1}{c}\log(1-\alpha)$, which is a constant. Thus the class of distributions characterized by constant RPRL is

$$F(t) = K(c(b-t))\exp[c(t-b)], \ -\infty < t < b.$$

Hence there is no distribution on $(0,\infty)$ with constant RPRL.

These results show that unlike the percentile residual life functions $q_\alpha(t)$ has limited use in describing ageing classes among various life distributions.

7.3. Models

Equation (7.12) provides a simple identity that relates RPRL with RHR. For many of the standard lifetime models like the exponential, Weibull, Pareto, etc. which have simple forms for the hazard rate, the expression for RHR is more complicated. Even for such models with simple forms for failure rate it is difficult to deduce properties of RHR from them. Hence it is desirable to have models that have simple functional forms for RHR. In the present section, we discuss a general method for obtaining such models from the following theorem.

Theorem 7.3.

For a nonnegative random variable X with hazard rate $h(t)$, *its reciprocal* X^{-1} *has RHR* $\lambda(t)$ *that satisfies*

$$t^2 h(t) = \lambda\left(\frac{1}{t}\right).$$

Proof: Let $G(t)$ be the distribution of X^{-1}. Then

$$F(t) = 1 - G\left(\frac{1}{t}\right).$$

Now

$$\begin{aligned} h(t) &= -\frac{d}{dt}\log(1-F(t)) = -\frac{d}{dt}\log G\left(\frac{1}{t}\right), \\ &= t^{-2}\lambda\left(\frac{1}{t}\right). \end{aligned}$$

Hence the proof.

Remark 7.2. Suppose RHR of $\frac{1}{X}$, $\lambda(t)$ is decreasing. Then

$$\frac{d}{dt}t^{-2}\lambda\left(\frac{1}{t}\right)=\frac{-\lambda'\left(\frac{1}{t}\right)-2t\lambda\left(\frac{1}{t}\right)}{t^4}<0,$$

which means that $h'(t)<0 \Rightarrow h(t)$ is decreasing. It follows that if $\frac{1}{X}$ has increasing RHR, X has decreasing failure rate.

Example 7.2. Consider the form

$$\lambda(t)=\left(at+bt^2\right)^{-1}$$

From (2.63), we have

$$\begin{aligned}F(t)&=\exp\left\{-\int_t^{\infty}\lambda(x)dx\right\}\\&=\exp\left\{-\int_t^{\infty}\left(ax+bx^2\right)^{-1}dx\right\}\\&=\left(\frac{bt}{a+bt}\right)^{\frac{1}{a}},\quad t>0.\end{aligned}$$

For *b<0*, there is no proper distribution function. When *a>0, b>0* we have with appropriate reparametrization,

$$F(t)=\left(\frac{pt}{1+pt}\right)^{c},\quad t>0, p,c>0$$

while for *a<0, b>0*

$$F(t)=\left(\frac{Rt-1}{Rt}\right)^{c},\ \frac{1}{R}<t<\infty, R,c>0$$

and as $a\to 0, b>0$

$$F(t)=e^{-\frac{\lambda}{t}}\quad t,\lambda>0,\ \lambda=b^{-1}. \tag{7.13}$$

The distribution defined in (7.13) is called the reciprocal exponential distribution. The reciprocal random variable X^{-1} has the generalized

Pareto distribution with reciprocal linear failure rate discussed in Lai and Xie (2006).

Example 7.3 Consider the Weibull distribution with survival function

$$\bar{F}(t) = \exp\left\{-\left(\frac{t}{\sigma}\right)^{\lambda}\right\}, \qquad \sigma, \lambda > 0$$

The hazard rate function is

$$h(t) = \frac{\lambda}{\sigma}\left(\frac{t}{\sigma}\right)^{\lambda-1}.$$

Using Theorem 7.3, $Y = \frac{1}{X}$ has reversed hazard rate

$$\begin{aligned}\lambda(t) &= \frac{1}{t^2} h\left(\frac{1}{t}\right)\\ &= \lambda\sigma(\sigma t)^{-\lambda-1}.\end{aligned}$$

Now

$$\begin{aligned}F(t) &= \exp\left\{-\int_t^{\infty} \lambda(x)dx\right\}\\ &= \exp\left\{-\int_t^{\infty} \lambda\sigma(\sigma x)^{-\lambda-1} dx\right\}\\ &= \exp\left\{-\left(\frac{1}{\sigma t}\right)^{\lambda}\right\},\end{aligned}$$

which defines a distribution function for $\sigma, \lambda > 0$. We call this distribution as reciprocal Weibull distribution. The RPRL of Y has the expression

$$q_{\alpha}(t) = \frac{t^{\lambda+1}\sigma^{\lambda}\log(1-\alpha)}{t^{\lambda}\sigma^{\lambda}\log(1-\alpha)-1}.$$

The RHR and RPRL of the above distributions and those of others obtained by the same method from some standard life distributions are given in Table 7.1, at the end of the chapter. The hazard rate properties of these distributions and other reliability aspects are documented in Marshal and Olkin (2007).

7.4 Classification of distributions

The fact that $\lambda(t)$ $(q_{\alpha}(t))$ is non-increasing (non-decreasing) on the entire positive real line leaves little scope for classification or identification of life distributions on the basis of their monotonicity as with the cases of ordinary hazard rate function and percentile residual life. One way of resolving this problem is to compare their growth rates. For the reversed hazard rate we define its growth rate as

$$g(t) = \frac{1}{\lambda(t)} \frac{d\lambda(t)}{dt}, \quad t \geq 0. \tag{7.14}$$

It is easy to see that $g(t)$ determines $\lambda(t)$ up to a constant. Hence the function $g(t)$ is an appropriate quantity to distinguish a suitable model among the class of decreasing reversed hazard rate distributions. From the expressions of $\lambda(t)$ of different distributions given in Table 7.1, we can easily find out the growth rates. For example the power distribution has

$$\lambda(t) = at^{-1}$$

and hence the growth rate

$$g(t) = \frac{-1}{t}.$$

Similarly the growth rate of reciprocal Weibull distribution is

$$g(t) = (-\lambda - 1)t^{-1}.$$

Table 7.2 exhibits the growth rate and behaviour of some important models.

It seems desirable to compare the relative growth rates of one distribution with respect to another distribution to see the extent to which changes are taking place in their reversed hazard rates. Here we study the relative growth rate by comparing the rate of a given distribution with that of the reciprocal exponential. This is motivated by

(i) there is no distribution on $(0, \infty)$ with constant growth rate, which would have been the natural choice if one existed,

(ii) the RHR of reciprocal exponential has a simple form,

(iii) there are many distribution that have growth rate less or more than that of reciprocal exponential and

(iv) $\frac{1}{X}$ has exponential distribution.

Definition 7.1 A life distribution F is said to have higher growth rate – HGR (lower growth rate- LGR) in reversed hazard rate compared to the reciprocal exponential if

$$g_F(t) \geq (\leq) g_{RE}(t) \text{ for all } t > 0,$$

where RE stands for reciprocal exponential distribution.

Using the above definition, we can see that there exist classes of distributions, in the same way as get classes using the monotonicity of failure rates or mean residual life functions.

Example 7.3 Consider the expressions of $g(t)$ presented in Table 7.2. For example $g(t)$ of power distribution is

$$g(t) = \frac{-1}{t},$$

$$g_{RE}(t) = \frac{-1}{2t}.$$

Now

$$\frac{g(t)}{g_{RE}(t)} = \frac{1}{2} < 1,$$

means that the power distribution is LGR. For reciprocal Weibull distribution

$$\frac{g(t)}{g_{RE}(t)} = \frac{\lambda+1}{2},$$

which is >1, $\lambda > 1$ and <1 for $0 < \lambda < 1$. Hence the distribution is HGR for $\lambda > 1$ and LGR for $0 < \lambda < 1$. In the case of reciprocal beta

$$\frac{g(t)}{g_{RE}(t)} = 1 - \frac{1}{2(1-Rt)}.$$

Since $\frac{1}{R} < t < \infty$, $Rt > 1$, $1 - \frac{1}{2(1-Rt)} > 1$, means the distribution is HGR. While the generalized power law is initially HGR and then LGR with a change point at $t = \left(\frac{2}{2-\beta}\right)^{1/\beta}$. To show this note that

$$\frac{g(t)}{g_{RE}(t)} < 1 \Rightarrow \frac{\beta t^{\beta}}{2(t^{\beta}-1)} < 1$$

$$\Rightarrow t > \left(\frac{2}{2-\beta}\right)^{1/\beta}.$$

Behaviour of some other distributions is exhibited in Table 7.2. In a similar manner, we define growth rate for the reversed percentile residual life.

Definition 7.2 F is said to have higher growth rate in reversed percentile life –HGP (lower growth rate-LGP) if

$$a_F(t) \geq (\leq) a_{RE}(t) \quad \text{for all } t \geq 0, \tag{7.15}$$

where

$$a(t) = \frac{1}{P_{\alpha}(t)} \frac{dP_{\alpha}(t)}{dt}. \tag{7.16}$$

Although $a(t)$ determines $q_\alpha(t)$ up to a constant, the latter does not characterize the corresponding distribution.

Example 7.4 The growth rate in reversed percentile residual life of the reciprocal exponential distribution is

$$a_{RE}(t) = \frac{2\lambda - t\log(1-\alpha)}{t\left(\lambda - t\log(1-\alpha)\right)}.$$

For the power distribution the growth rate is

$$a_p(t) = t^{-1}.$$

Now

$$\frac{a_p(t)}{a_{RE}(t)} = \frac{\lambda - t\log(1-\alpha)}{2\lambda - t\log(1-\alpha)}$$
$$= 1 - \frac{\lambda}{2\lambda - t\log(1-\alpha)} < 1$$

Hence power distribution has LGP. On the other hand for the reciprocal beta we have

$$a_B(t) = \frac{2Rt - 1}{t(Rt-1)}.$$

Hence $a_B(t) > a_{RE}(t)$ for all $t < 2\lambda\left(R\lambda + \log\alpha'\right)^{-1}$ and hence for this range of t, reciprocal beta has HGP. Thus the classification is well defined.

Table 7.1 RHR and RPRL of Distributions

Distribution	$F(t)$	RPRL	RHR
Power	$\left(t/b\right)^{a}$, $0\le t\le b$	$\left(1-(1-\alpha)^{1/a}\right)t$	at^{-1}
Reciprocal exponential	$e^{-\lambda/t}$, $t>0$	$t^{2}\log(1-\alpha)\Big/\left(t\log(1-\alpha)-\lambda\right)$	λt^{-2}
Reciprocal beta	$\left(\frac{Rt-1}{Rt}\right)^{c}$, $\frac{1}{R}<t<\infty$	$\left((1-(1-\alpha)^{1/c}\right)t\left(Rt-1\right)$	$\frac{c}{t(Rt-1)}$
Reciprocal Lomax	$\left(\frac{pt}{1+pt}\right)^{c}$, $t>0$	$\left((1-\alpha)^{-1/c}-1\right)t\left(pt+1\right)$	$\frac{c}{t(pt+1)}$
Reciprocal Weibull	$\exp\left[-\left(\frac{1}{\sigma t}\right)^{\lambda}\right]$, $t>0$	$\frac{t^{\lambda+1}\sigma^{\lambda}\log(1-\alpha)}{\left(t^{\lambda}\sigma^{\lambda}\log(1-\alpha)-1\right)}$	$\lambda t^{-\lambda-1}$
Reciprocal Gompertz	$\exp\left[-\theta\left(e^{\lambda/t}-1\right)\right]$, $t>0$	$t-\lambda\left[\log\left(e^{\lambda/t}-\theta^{-1}\log(1-\alpha)\right)\right]^{-1}$	$\theta\lambda e^{-\lambda/t}t^{-2}$
Power exponential	$\left(1-e^{-\lambda t}\right)^{\theta}$, $t>0$	$t+\lambda^{-1}\log\left(1-(1-\alpha)^{1/\theta}\left(1-e^{-\lambda t}\right)\right)$	$\theta\lambda\left(e^{\lambda t}-1\right)$
Burr	$\left(1+t^{-c}\right)^{-k}$, $t>0$	$t\left[1-(1-\alpha)^{1/kc}\left(1+t^{c}\left(1-(1-\alpha)^{1/k}\right)^{-1}\right]$	$kc\left[t\left(1+t^{c}\right)\right]^{-1}$
Generalized Power	$\left(1-t^{-\beta}\right)^{\theta}$, $t>0$	$t-\left[1-(1-\alpha)^{1/\theta}\left(1-t\right)^{-\beta}\right]^{-1/\beta}$	$\beta\theta\left(t^{\beta}-1\right)^{-1}$
Negative Weibull	$\exp\left[-\theta\left(t^{-\beta}-1\right)\right]$, $0<t<1$	$t-\left(t^{-\beta}-\theta^{-1}\log\left(1-\alpha\right)\right)^{-1/\beta}$	$\theta\beta t^{-\beta-1}$

Table 7.2 Growth rate of reversed hazard rate

Distribution	$g(t)$	
Power	$-t^{-1}$	LGR
Reciprocal Exponential	$-2t^{-1}$	HGR/LGR
Reciprocal beta	$\frac{(1-2Rt)}{t(Rt-1)}$	HGR
Reciprocal Lomax	$-\frac{(1+pt)}{t(pt+1)}$	HGR
Reciprocal Weibull	$-(\lambda+1)t^{-1}$	LGR for $0\leq\lambda\leq1$, HGR for $\lambda\geq1$
Power Exponential	$\lambda e^{\lambda t}\left(e^{\lambda t}-1\right)^{-1}$	LGR
Reciprocal Gompertz	$-t^{-1}\left(\lambda t^{-1}+2\right)$	LGR
Generalized Power	$-\beta t^{\beta-1}\left(t^{\beta}-1\right)^{-1}$	LGR for $t\geq\left(\frac{2}{2-\beta}\right)^{1/\beta}$ and HGR $t\leq\left(\frac{2}{2-\beta}\right)^{1/\beta}$
Negative Weibull	$-(\beta+1)t^{-1}$	HGR for $0<\beta\leq1$, LGR $\beta\geq1$

Chapter 8

Conclusions and future works

Reliability analysis is a well established branch of statistics that deals with the statistical study of different aspects of lifetimes of a system of components. As we pointed out earlier that major part of the theory and applications in connection with reliability analysis were discussed based on the measures in terms of distribution function. In the beginning chapters of the thesis, we have described some attractive features of quantile functions and the relevance of its use in reliability analysis. Motivated by the works of Parzen (1979), Freimer et al. (1988) and Gilchrist (2000), who indicated the scope of quantile functions in reliability analysis and as a follow up of the systematic study in this connection by Nair and Sankaran (2009), in the present work we tried to extend their ideas to develop necessary theoretical framework for lifetime data analysis. In Chapter 1, we have given the relevance and scope of the study and a brief outline of the work we have carried out.

Chapter 2 of this thesis is devoted to the presentation of various concepts and their brief reviews, which were useful for the discussions in the subsequent chapters. We have pointed out in Chapter 1 that one of the objectives of the quantile-based reliability analysis is to make use of quantile functions as models in lifetime data analysis. When one wishes to model a lifetime data with a quantile function, so as to carry out a quantile-based analysis, he needs different choices of quantile functions to select an appropriate one. Motivated by this, the objectives of the discussions in Chapter 3 were two-fold. Firstly we have identified some existing quantile functions for lifetime data analysis. Secondly, we have derived new models for use of quantile-based lifetime analysis. The identified distributions were lambda distributions by Ramberg and Schmeiser (1974) and Freimer et al. (1988), the power Pareto distribution, a new model proposed by van-Staden and Loots (2009) and the Govindarajulu distribution. As the Govindarajulu distribution is a simple model with competing features in terms of model parsimony, a detailed study of the distribution in the context of reliability analysis has been carried out. As an effort to fulfill the second objective, we have derived three new families of quantile functions that nest several known models using the properties of Parzen's score function and tail exponent function, and demonstrated their applicability in lifetime data analysis.

In the introduction of Chapter 4, we have pointed out the role of ageing concepts in reliability analysis and in identifying life distributions. As we have mentioned there, all the ageing concepts exist in the literature were defined in terms of measures based on distribution functions. As there exist quantile models useful in the analysis of lifetime data, to describe their ageing properties, quantile-based definitions seem essential. Motivated by this fact, in Chapter 4, we have translated the definitions of most commonly used ageing concepts in terms of quantile function and ageing properties of some quantile functions have been discussed.

In Chapter 5 we have defined a generalization of TTT called TTT of order n (TTT-n), obtained by the iteration of the definition of TTT. It has been shown that TTT-n is also a quantile function of a random variable, say X_n. We have derived various identities connecting the reliability measures of X and X_n. We have made use of some of these identities to derive some characterization results. We have proposed the process of iterations as a method to generate new distributions with different monotonicity for hazard quantile function. We have also considered the ageing properties of X_n in relation to the baseline random variable X. We have proved some general results about the ageing patterns of X_n in connection with that of X. We have proposed a simple mechanism for construction of bathtub or upside-down bathtub distributions. Many results that give the order relations connecting the baseline and transformed random variables were also proved in Chapter 5. A new order relation based on TTT-n has been defined and its preservation property under the minima has been established. As we mentioned in Chapter 5, sample the sample counterpart viz. n^{th} order TTT statistics along with their relationships with the TTT statistic of the original distribution, which is being investigated, can further strengthen the adaptability of the theoretical results in the present work.

In Chapter 6, we have studied the first two L-moments of residual life and their relevance in various applications of reliability analysis. We have shown that the first L-moment of residual function is equivalent to the vitality function, which have been widely discussed in the literature. Relationships of the second L-moment of residual life, $\alpha_2(u)$ with some commonly used reliability measures were derived. We have treated $\alpha_2(u)$ as a measure of variation and studied its merits over the usual variance of residual life (VRL). The monotonicity of $\alpha_2(u)$ in relation with the mean and variance of residual life have also been discussed. We have also defined the L-moments of reversed residual life and studied their properties. In the last section of Chapter 6, we have pointed out some applications of L-moments of residual life in the field of reliability and economics.

In Chapter 7, we have defined percentile residual life in reversed time (RPRL) and derived its relationship with reversed hazard rate (RHR). We have discussed the characterization problem of RPRL and demonstrated with an example that the RPRL for given α does not determine the distribution uniquely. We presented the conditions for RPRL to determine the distributions. We have derived many models with simple form of RHR and RPRL. As RHR (RPRL) is non-increasing (non-decreasing) on the entire positive real line, it leaves little scope for classification or identification of distributions on the basis of their monotonicity. We have used the growth rates of RHR (RPRL) for the classification of distribution and several examples were given.

On the basis of the present study and continuing demand for new models it is felt that the following problems need resolution.

1. Distribution functions by themselves cannot provide adequate models for certain types of data. Identification of more quantile functions and their study in the context of reliability analysis with the aid of quantile-based theory developed so far is required. This can be achieved by using the Parzen's functions, the generalized TTT and specific functional form for quantile-based reliability measures.

2. Researchers have identified many practical difficulties in estimation procedures of various quantile functions such as lambda distributions. Studies are needed to arrive at families of quantile functions which are versatile in answering modelling problems and at the same time with simple methods of estimation.
3. In Chapter 4, we have derived a new model based on the linear hazard quantile function. Likewise one can consider various functional forms of hazard quantile function and other reliability measures to generate new quantile function models, which remains as an open problem.
4. Another future work is the detailed study on the applications of L-moments of residual life in reliability analysis and economics, which we have pointed out in Chapter 6. The study on the properties of higher order L-moments of residual life, and the multivariate extensions of the first and second L-moments of residual lives are also open problems.
5. Ordering of quantile-based reliability functions are essential for comparison of the properties of the underlying quantile function. These orderings are different from the existing ones.

We are currently attempting to resolve some of these problems and hopefully work in this direction is expected to be presented in a future work.

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