SOME PROBLEMS IN TOPOLOGY AND LATTICE THEORY

A STUDY OF CLOSURE AND FUZZY CLOSURE SPACES WITH REFERENCE TO H-CLOSEDNESS AND CONVEXITY

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Declaration

I here by declare that this thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person except where due reference is made in the text of the thesis.

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Certificate

Certified that the work reported in this thesis is based on the bonafied work done by Smt. Bloomy Joseph, under my guidance in the Department of Mathematics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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Chapter 0

Introduction

This thesis is concerned with two aspects—one, study of closure spaces with reference to H-closedness and two, study of fuzzy closure spaces with reference to convexity.

0.1 Closure spaces

E. Čech introduced the concept of Čech closure space. (In this thesis we denote Čech closure space as closure space for convenience). Čech closure spaces, is a generalisation of the concept of topological spaces. Eduard Čech, J. Novak, R. Fric and many others have earlier studied this concept and many topological concepts were extended to the Čech closure spaces.

The concept of a topological space is generally introduced in terms of the axioms for the open sets. However alternate methods to describe a topology in the set X are often used in terms of neighbourhood systems, the family of closed sets, the closure operator, the interior operator etc. Of these, the closure operator was axiomatised by Kuratowski and he associated a topology from a closure space by taking closed sets as sets A such that $cl\ A = A$, where $cl\ A$ becomes the topological closure of a subset A of X. In Čech's approch the condition $ccA = cA$ among Kuratowski axioms need not hold for every subset A of X (Here cA denotes the Čech closure of A in X); when this condition is also true, c is called a topological closure operator. Čech closure space is also called A-space by C. Calude-M. Malitza [C;M]. For them a Čech space is obtained by removing $c(A \cup B) = cA \cup cB$ and introducing $A \subset B \Rightarrow cA \subset cB$ into the axioms of an A-space. However considering universal acceptability we call the former Čech closure spaces and the latter monotone spaces.

The ideas about the concepts of a continuous mapping and of a set endowed with continuous operations (compositions) play a fundamental role in general mathematical analysis. Analogous to the notion of the continuity, we consider the morphisms throughout this thesis. Čech described continuity in closure spaces by means of neighbourhoods, nets etc. Koutnik studied the convergence on non Hausdorff closure space [KO-1]. He studied more about sequential convergence structure in [KO-2, KO-3]. Mashour and Ghanim in 1982 defined [M;G-1] C-almost continuous as a function $f : X \to Y$, where X and Y are closure spaces and is said to be a C-almost continuous if for each $x \in X$ and each $V \subset Y$ with $f(x) \in V^{\circ}$, there is

$U \subset X$ such that $x \in U^{\circ}$ and $f(U) \subset (c(U))^{\circ}$. They also studied some results related to this concept. D. R. Andrew and E. K. Whittlesy [A;W] and James Chew [CHE] studied about closure continuity.

Separation axioms in closure spaces have different implications than the corresponding axioms in topological spaces. According to Čech, a closure space is said to be separated [CE] if any two distinct points are separated by distinct neighbourhoods. Separation properties in closure spaces have been studied by various authors. D. N. Roth and J. W. Carlson studied [R;C] a number of separation properties in closure spaces. They showed that Čech closure operator on a finite set can be represented by a zero-one reflexive matrix. A number of separation properties were studied for finite spaces and characterised in terms of the matrix that represents the closure operator. Separation properties that carry over to the underlying topology were also studied. W. J. Thron studied [T] some separation properties in closure spaces. He defined a space as regular if $x \notin c(A)$ (A is any subset of X) implies that there exist $D, E \subset X$, $D \cap E = \phi$ such that $x \notin c(X - D)$, $A \cap c(X - E) = \phi$. K. C. Chattopadhyay and W. J. Thron studied [CH;T] some separation properties of extensions and obtained some results on the above. In [S] T. A. Sunitha studied higher separation properties in closure spaces.

For topological spaces compactness can be expressed in a number of different ways. However for closure spaces some of these statements are not equivalent. Čech defined [CE] the term compactness for a closure space (X, c) if every proper filter of sets on X has a cluster point in X. He described the fundamental properties of compact closure spaces. He noted that for a closure space (X, c) to be compact it is necessary and sufficient that every interior cover $\mathscr{V}$ (an analogue of an open cover in topological space) of (X, c) has a finite subcover. Chattopadhyay [CH] defined a compact space as a closure space (X, c) if and only if $[G_c^+(x) = x \in X]$ is cover of $\Omega(X)$. He denoted by $\Omega(X)$, a set of ultrafilters on X, by $\mathscr{G}$, a grill on X the $\mathscr{G}^+ = [\mathscr{U} : \mathscr{U} \in \Omega(X), \mathscr{U} \subset \mathscr{G}]$. W. J. Thron mentioned [T] types of compactness. According to him a closure space (X, c) is called linkage (F-linkage) compact if every linked (F-linked) grill on X converges. A grill $\mathscr{G}$ is called linked grill if $A, B \in G \Rightarrow c(A) \cap c(B) = \phi$, F-linked grill if $A_1, A_2 \cdots A_n \in G \Rightarrow \cap[c(A_k)] = \phi$. Some weak forms of compactness like almost c-compactness were introduced and some of its properties were studied by A. S. Mashour and M. N. Ghanim [M;G-1]. Compactness and linkage compactness were defined by K. C. Chattopadhyay [CH].

Čech defined [CE] and developed some properties of connected spaces. According to him a subset A of a closure space X is said to be connected in X if A is not the union of two non-empty semi-separated subsets of X, that is $A = A_1 \cup A_2$, $(cA_1 \cap A_2) \cup (A_1 \cap cA_2) = \phi$ implies $A_1 = \phi$ or $A_2 = \phi$. The concept of connectedness which was defined by Čech in closure spaces precisely coincides with connectedness in the associated topological spaces.

K. C. Chattopadhyay and W. J. Thron [CH;T] were the first persons, who studied the general extension theory of G_0 closure spaces. They studied some special closure operators and considered the case when an extension is topological and also compact. The underlying structure of each nearness space is topological space. The underlying structure of each semi nearness space is a Čech closure space. D. N. Roth and J. W. Carlson showed [R;C] that finitely generated Čech closure spaces are a natural generalisation of finite Čech closure spaces. K. C. Chattopadhyay developed [CH] an extension theory of arbitrary closure spaces which are in general supposed to satisfy no separation axioms. He introduced the concept of regular extensions of closure spaces and satisfied this concept in detail.

Though much work has been done in topological spaces and in Čech spaces, there are still many problems not attempted. In the first part of this thesis we have made an attempt in this direction.

0.2 Fuzzy closure spaces

The basic concept of a fuzzy set was introduced by L. A. Zadeh in 1965 [Z]. It has become important with application in almost all areas of Mathematics, of which one is closure space. A fuzzy set 'A' in a set X is characterised by a membership function μ_A from X to the unit interval $[0, 1]$. Fuzzy set theory is a generalisation of abstract set theory. If A is an ordinary subset of

X, its characteristic function is a special case of a fuzzy set. Zadeh took the closed unit interval $[0, 1]$ as the membership set. J. A. Goguen [G] considered different ordered structures for the membership set. He considered a fuzzy subset as a generalized characteristic function. Thus the ordinary set theory is a special case of the fuzzy set theory where the membership set is $\{0, 1\}$. Goguen suggested that a complete and distributive lattice would be a minimum structure for the membership set.

The theory of fuzzy sets deals with subsets A of a set X, where the transition between full membership and non-membership is gradual. The grade of membership 'one' is assigned to those objects that fully and completely belong to A, while zero is assigned to objects that do not belong to A at all. The more an object X belongs to A, the closer to one is its grade of membership $\mu_A(x)$. The fuzzy set A' defined by $\mu_{A'}(x) = 1 - \mu_A(x)$ is called the complement of the fuzzy set A. Several mathematicians have applied the theory of fuzzy sets to various branches of pure mathematics also, resulting in the development of new areas like, fuzzy topology, fuzzy groups, fuzzy closure space etc.

It was C. L. Chang [C] who defined fuzzy topology for the first time in 1968. According to Chang, a family T if fuzzy sets in X is called a fuzzy topology for X, if

(i) $\phi, X \in T$

(ii) if $A, B \in T$ then $A \cap B \in T$

(iii) if $A_i \in T$ for each $i \in I$, then $\cup A_i \in T$.

Then the pair (X, T) is called a fuzzy topological space or fts in short. The elements of T are called open sets and their complements are called closed sets.

In 1976 R. Lowen [LO-1] has given another definition for a fuzzy topology by taking the set of constant function instead of ϕ and X in axiom (i) of Chang's definition.

The theory of closure space is based on the set operation of union, intersection and complementation. Fuzzy sets do have the same kind of operations. T. P. Johnson [J] and many others studied fuzzy closure spaces. Fuzzy closure space is a generalization of fuzzy topological space. C. L. Chang [C] was the first to define a fuzzy topology. Since then an extensive study of fuzzy topological spaces has been carried out by many researchers. Many mathematicians, while developing fuzzy topology have used different lattice structures for the membership set. R. Lowen [LO-1] modified the definition of fuzzy topology given by C. L. Chang and obtained a fuzzy version of Tychonoff theorem, but he lost the concept that fuzzy topology generalizes topology.

In this thesis we are following Chang's definition rather than Lowen's definition. For other details of fuzzy topological spaces like product and quotient spaces, we refer to C. K. Wong [WO-1].

In the second part of this thesis we have made an attempt to study some problems in fuzzy closure spaces.

0.3 H-closedness

An extension of a topological space X is a space that contains X as a dense subspace. The construction of extensions are of various sorts—compactifications, real compactifications, H-closed extensions—has long been a major area of study in general topology. The most common method of constructing an extension of a space is to let the "new points" of the extension be ultrafilters on certain lattices associated with the space. Examples of such lattices are the lattice of open sets, the lattice of zero-sets and the lattice of clopen sets.

A less well-known construction in general topology is the "absolute" of a space. Associated with each Hausdorff space X is an extremely disconnected zero-dimensional Hausdorff space EX, called the Iliadis absolute of X, and a perfect, irreducible, θ-continuous surjection from EX onto X. A detailed discussion of the importance of the absolute in the study of topology and its applications were studied by Jack R. Porter and Grant Woods [P;W]. What concerns us here is that in most constructions of the absolute the points of EX are certain ultrafilters on lattices associated with X. Thus extensions and absolutes, allthough conceptually very different, are constructed using similar tools.

One of the reason for studying extensions is the possibility of shifting a problem concerning a space X to a problem concerning an extension Y of X where Y is a "nicer" space than X and the "shifted" problem can be solved. Thus an important goal in extension theory is to generate "nice" extensions of a fixed space X. H-closed extensions are one of the nice extensions. We are not attempting extension theory in this thesis, we were motivated and study H-closedness and related ideas because of this.

Here we study H-closedness in closure spaces. It is well known that a topological space (X, τ) is H-closed if X is closed in every Hausdorff space containing X as a subspace. (In fact, "H-closed" is an abbrevation for "Hausdorff-closed"-closed in Hausdorff space).

In [P;W] characterisation of H-closed spaces is available in the following manner.

For a space (X, τ), the following are equivalent:

(1) X is H-closed

(2) for every open cover of X, there is a finite subfamily whose union is dense in X.

(3) every open filter on X has nonvoid adherence and

(4) every open ultrafilter on X converges.

In this thesis we apply the above characterisation to closure space.

Absolutely closedness or H-closedness was first introduced in 1929 by Alexandroff and Urysohn. Here we study some properties of H-closedness in closure spaces. Also we prove some properties of H-closedness in monotone spaces.

0.4 Fuzzy convexity spaces

The study of convex sets is a branch of geometry, analysis and linear algebra that has numerous connections with other areas of mathematics. Though convex sets are defined in various settings, the most useful definition is based on a notion of betweenness. When X is a space in which such a notion is defined, a subset C of X is called convex provided that for any two points x and y of C, C includes all the points between x and y. For example in a linear space, a set C is said to be convex if $\lambda x + (1 - \lambda)y \in C$, for every $x, y \in C$ and $\lambda \in [0, 1]$.

The theory of convexity can be sorted into two kinds. One deals with concrete convexity and the other that deals with abstract convexity. In concrete situations it was considered by R. T. Rockfellar [ROC], Kelly [K], Weiss [WE], S. R. Lay [L] and many others. In abstract convexity theory a convexity space was introduced by F. W. Levi in 1951 [LE]. He defined a convexity space as a pair $(X, \mathscr{L})$ consisting of a set X and a family $\mathscr{L}$ of subsets of X called convex sets satisfying the condition,

(i) $\phi, X \in \mathscr{L}$

(ii) If $A_i \in \mathscr{L}$, for each $i \in I$, then $\bigcap_{i \in I} A_i \in \mathscr{L}$.

The convexity space introduced by Levi was further developed by many authors like D. C. Kay and E. W. Womble [K;W], R. E. Jamison-Waldner [J;W], G. Sierksma [S], M. Van de Vel [V] etc. In addition to the above conditions (i) and (ii) if $\cup A_i \in \mathscr{L}$ whenever $A_i \in \mathscr{L}$ and A_i's are totally ordered by inclusion, then $(X, \mathscr{L})$ is called an aligned space which was introduced by R. E. Jamison-Waldner [J;W].

In abstract situations the notion of a topological convexity structure has been introduced by R. E. Jamison-Waldner in 1974. A triple $(X, \mathscr{L}, \tau)$ consisting of a set X, a topology τ and convexity $\mathscr{L}$ on X is called a topological convexity structure, provided the topology τ is compatible with the convexity $\mathscr{L}$. Now a topology τ is compatible with a convexity $\mathscr{L}$, if all polytopes of $\mathscr{L}$ are closed in (X, τ). R. E. Jameson-Waldner [J;W] also introduced the concept of local convexity.

The notion of convexity can be generalized to fuzzy subsets of a set X. L. A. Zadeh introduced the concept of a convex fuzzy set in 1965. A fuzzy subset 'A' of X is convex if and only if for every $x_1, x_2 \in X$ and $\lambda \in [0, 1]$

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

or equivalently a fuzzy set A is convex if and only if the ordinary set

$A_d = \{x \in X | \mu_A(x) \geq d\}$ is convex for each $d > 0$ and $d \in [0, 1]$.

In concrete situations the concept of a convex fuzzy set was initiated by M. D. Weiss [WE], A. K. Katsaras and D. B. Liu [K;L], R. Lowen [LO-2]. M. D. Weiss [WE] considered a convex fuzzy set in a vector space over real or complex numbers in 1975.

In 1977 Katsaras and D. B. Liu [K;L] applied the concept of a fuzzy set to the elementary theory of vector spaces and topological vector spaces. They have also considered convex fuzzy sets. In 1980, R. Lowen applied the theory of fuzzy sets to some elementary known results of convex sets. For the definition of convex fuzzy sets in vector spaces we refer to A. K. Katsaras and D. B. Liu [K;L].

M. V. Rosa [ROS-1] attempted to develop a fuzzy convexity theory in topological spaces. So in this thesis we attempt to develop fuzzy convexity theory in closure spaces.

0.5 Summary of the thesis

Chapter-1

This chapter is a study of H-closedness in closure and monotone spaces. Here we have four sections. In section 1, we introduce c-denseness, c-adherence and cH-closedness in closure spaces and study some of their properties, which are analogous to the corresponding notions in topological space. Here we prove results like 'If a closure space (X, c) is cH-closed then it is H-closed in associated topological space', but converse of this result is not true. And we prove the converse by means of an example. Also we prove results like 'A closure space (X, c) is H-closed if and only if every open cover $\mathscr{V}$ of (X, c) has a finite subfamily whose union is c-dense in X.'; If 'A closure space (X, c) is cH-closed then every open cover $\mathscr{V}$ of (X, c) has a finite subfamily whose union is dense in X' etc. In section 2, we discuss the inheritance by subsets of properties like H-closed and cH-closed. In this section we prove results like 'If (X, c) be cH-closed and $U \subseteq X$ be open. Then $cl(U)$ is H-closed'; 'If (X, c) be cH-closed and $U \subseteq X$ be open. Then cU is cH-closed' etc.

In section 3, we introduce m-denseness, m-adherence and mH-closedness in monotone space and study some of their properties. This section is similar to section 1 in which closure space is replaced with monotone space. In section 4, we discuss the inheritance by subsets of properties like H-closed and mH-closed. This section is similar to those done in closure spaces.

Chapter-2

In this chapter we study the implicational relationships between various types of morphisms between closure spaces and between monotone spaces with respect to associated topological spaces. Here we have four sections.

In section 1, we discuss the continuity of functions between closure spaces, which is analogous to the θ-continuity of functions between topological spaces given in [P;W]. Here we define $c\theta$-continuous, $\theta c'$-continuous and $c\theta c'$-continuous. And by means of this definition we prove results like 'If f is $\theta c'$-continuous at x_0 then f is θ-continuous at x_0'; If f is $c\theta c'$-continuous at x_0 then f is $c\theta$-continuous at x_0' etc. Section 2 is similar to section 1 in which closure spaces is replaced with monotone spaces.

In section 3 we discuss the mapping between closure spaces with respect to asociated topological spaces and prove some results by means of these mappings. Here we also find the relations between continuity and different types of morphisms. Like 'a mapping $f : (X, c) \to (Y, c')$ be a cl-cl' morphism, then it is c-cl' morphism'. But the converse is not true, which is proved with the help of an example. Section 4 is similar to section 3 in which closure spaces is replaced with monotone spaces.

Chapter-3

In this chapter we study some properties of closure spaces and product closure spaces. Here we have two sections. In section 1 we find some properties of mapping into product closure spaces with respect to associated topological spaces. The properties are proved using the result proved in [CE] namely, 'a mapping f of a space X into the product space $X = \Pi X_a$ is continuous 'if and only if the mapping $\pi_a \circ f$ is continuous for each a' and the conditions c-c' morphism implies c-cl' morphism and cl-cl' implies c-cl' morphism from chapter two. Also we find some separation properties in product closure spaces.

In section 2 we discuss some separation properties involving zero sets, like 'a c-Hausdorff space X is c-completely regular if and only if the family $Z_1(X) = \{Z_1(f) : f \in C_1(X)\}$ is a base for the closed sets in the associated topology of X.

Chapter-4

This chapter is a study of fuzzy closure spaces (*fcs*). Here we introduce denseness in fuzzy closure spaces and also introduce the concept of various types of mappings between fuzzy closure spaces and prove some results based on these. And the chapter consists of two sections.

In section 1 we introduce denseness in fuzzy closure spaces known as fuzzy c-denseness and prove some results in fuzzy closure spaces using denseness property. Here we prove results like 'a fuzzy subset 'A' of a fuzzy closure spaces (X, c) is c-dense in X if and only if for every nonempty open subset B of X, $A \cap B \neq \phi$. In section 2 we discuss mapping between fuzzy closure spaces (fcs) with respect to associated fuzzy topological spaces (fts) and prove some results. Here we define fuzzy c-c' morphism, fuzzy cl-c' morphism fuzzy c-cl' morphism, fuzzy cl-cl' morphism where f be mapping from a closure space (X, c) to closure space (Y, c') and cl, cl' be closure operators in the respective associated fuzzy topological spaces. Here we prove results like 'If f is fuzzy cl-c' morphism from a fuzzy closure spaces (X, c) to fuzzy closure spaces (Y, c') then f is fuzzy cl-cl' morphism.'

Chapter-5

In this chapter we define fuzzy closure fuzzy convexity spaces. And we prove some properties of fuzzy closure fuzzy convexity spaces. Here we have three sections. In section 1 we consider a fuzzy closure together with a fuzzy convexity on the same underlying set and introduce fuzzy closure fuzzy convexity spaces. Also we introduce the subspace and product of fuzzy closure fuzzy convexity spaces.

In section 2 we study locally fuzzy closure fuzzy convexity spaces. Here we prove results like 'any subspace of a locally $(fc)(fco)s$ is a locally $(fc)(fco)s$.' In section 3 we introduce the separation axioms in fuzzy closure fuzzy convexity spaces. Here we define $FCNS_0$, $FCNS_1$, $FCNS_2$, Pseudo $FCNS_3$, $FCNS_3$, semi $FCNS_4$, $FCNS_4$ where $FCNS$ stands for 'Fuzzy closure neighbourhood separation' and prove results like, a nonempty product is $FCNS_i$, if each factor is $FCNS_i$ for every $i = 0, 1, 2$ and supporting examples for $FCNS_2 \Rightarrow FCNS_1 \Rightarrow FCNS_0$ and $FCNS_3 \Rightarrow$ Pseudo $FCNS_3$.

Chapter-6

In this chapter we introduce fuzzy closured convexity space and find the relationship between fuzzy topological convexity spaces and fuzzy closured convexity spaces. Also we find the relationship between fuzzy closure fuzzy convexity spaces and fuzzy topological fuzzy convexity spaces. Here we have two sections.

In section 1 we define fuzzy closured convexity space and some of its properties. Here we prove the result 'A fuzzy topological convexity space $(X, \mathscr{L}, T)$ is fuzzy closured convexity space $(X, \mathscr{L}, c)$. Here T is the associated fuzzy topology of the fuzzy closure space (X, c)'. With the help of an example we prove the fact that $(fc)(fco)s$ is not an $fc - cos$. Also we find the relationship between fuzzy topological convexity spaces and fuzzy closured convexity spaces. In section 2 we find the relationship between fuzzy topological fuzzy convexity spaces and fuzzy closure fuzzy convexity spaces. Here we prove the related results like, 'any subspace of a FNS_i space is FNS_i and hence $FCNS_i$ for all $i = 0, 1, 2$ '.

0.6 Preliminary definitions and results used in the thesis

For details refer [P;W, CE, C, ROS-1, S, J].

Definition 0.6.1. A topological space (X, τ) is H-closed if X is closed in every Hausdorff space containing X as a subspace.

Definition 0.6.2. A function c from a power set of X to itself is called a closure operation for X, provided that the following conditions are satisfied.

(i) $c\phi = \phi$

(ii) $A \subset cA$ for every $A \subset X$

(iii) $c(A \cup B) = cA \cup cB$ for every $A, B \subset X$.

A structure (X, c) where X is a set and c is a closure operation for X will be called closure space or Čech space. Let us consider the following conditions.

(iv) $A \subset B \Rightarrow cA \subset cB$ for every $A, B \subset X$

(v) for every family $\{A_i\}_{i\in I}$ of subsets of X,

$$c(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} c(A_i).$$

(vi) $c(cA) = cA$ for every $A \subseteq X$.

The structure (X, c) where c has the properties (i), (ii) and (iv) is called a monotone space [C;M]. A Čech space which satisfies the condition (vi) is called Kuratowski (topological) space [C;M]. A Čech space (Kuratowski) space is total if the condition (v) holds [C;M].

Definition 0.6.3. A subset A of a closure space (X, c) will be called closed if $cA = A$ and open if its complement is closed, that is, if $c(X\setminus A) = X\setminus A$. And the set of open sets of (X, c) is said to constitute the associated topology of the closure space.

Definition 0.6.4. An interior operator Int_c on X is a function from the power set of X to itself such that for each $A \subset X$, $\text{Int}_c A = X \backslash c(X \backslash A)$. The set $\text{Int}_c A$ is called the interior of A in (X, c). Also A is called a neighbourhood of x if $x \in \text{Int}_c A$; A is an open neighbourhood if A is also open that is if $\text{Int}_c A = A$.

Definition 0.6.5. Let (X, c) be a closure space and $Y \subset X$. The closure c' on Y is defined as $c'A = Y \cap cA$ for every $A \subset Y$. The closure space (Y, c') is called a subspace of (X, c).

Definition 0.6.6. Let Y be a subspace of a closure space X.

(a) If A is closed (open) in X then $Y \cap A$ is closed (open) in Y

(b) If Y is closed (open) in X and A is closed (open) in Y then A is closed (open) in X.

Definition 0.6.7. Let X be a set and m be a monotone operator on X and take the collection $\{c_\alpha : c_\alpha$ is a closure operator coarser than m *i.e.*, $mA \subseteq c_\alpha A$ for all $A \subseteq X\}$. Then the associated closure operator c is defined by $cA = \cap c_\alpha A$ for all $A \subseteq X$.

Result 0.6.8. For each monotone operator there is a uniquely associated closure operator.

Definition 0.6.9. A neighbourhood of a subset A of a monotone space is any subset U of X such that $A \subseteq X \backslash m(X \backslash U)$. By a neighbourhood of a point x of X we mean a neighbourhood of the one point set $\{x\}$.

Definition 0.6.10. Let (X, m) be a monotone space and $Y \subseteq X$. The monotone operator m_Y on Y is defined as $m_Y A = Y \cap m(A)$ for all $A \subseteq Y$. Then m' is called the relativisation of m to Y and the space (Y, m_Y) is called the subspace of (X, m).

Definition 0.6.11. Let (X, c) and (Y, c') be two closure spaces and cl, cl' be the Kuratowski closure operations in the respective associated topologies. A map $f : X \to Y$ is said to be a,

(i) $c - c'$ morphism if $f(cA) \subseteq c'f(A)$ for all $A \subseteq X$

(ii) $c - cl'$ morphism if $f(cA) \subseteq cl'f(A)$ for all $A \subseteq X$

(iii) $cl - cl'$ morphism or continuous map if $f(cl\ A) \subseteq cl'f(A)$ for all $A \subseteq X$.

Definition 0.6.12. Let (X, m) and (Y, m') be two monotone spaces and cl, cl' be the Kuratowski closure operations in the respective associated topologies. A map $f : X \to Y$ is said to be a

(i) $m - m'$ morphism if $f(mA) \subseteq m'f(A)$ for all $A \subseteq X$

(ii) $m - cl'$ morphism if $f(mA) \subseteq cl'f(A)$ for all $A \subseteq X$

(iii) $cl - cl'$ morphism or a continuous map if $f(cl\ A) \subseteq cl'f(A)$ for all $A \subseteq X$.

Definition 0.6.13. Let $\{(X_a, c_a) : a \in A\}$ be a family of closure spaces, X be the product of the family $\{X_a\}$ that is, $X = \Pi X_a$ of underlying sets, π_a be the projection of X onto X_a for each a, then the product closure c is the coarest closure on the product of underlying sets such that all the projections are continuous.

Definition 0.6.14. A closure space (X, c) is said to be Hausdorff if for any two distinct points, there exist neighbourhoods U of x and V of y such that $U \cap V = \phi$.

Definition 0.6.15. A closure space (X, c) is said to be completely regular, if for every point x and a closed set A not containing x, there exist a $c - cl$ morphism $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in A$.

Definition 0.6.16. (i) $C(X) = \{f : X \to R \ / \ f$ is continuous from (X, t) to R with usual topology, here 't' is any topology on $X\}$.

(ii) $C^*(X) = \{f \in C(X) | f$ is bounded$\}$.

(iii) For $f \in C(X)$, $Z(f) = \{x \in X | f(x) = 0\}$ zero set of f and $Z(X) = \{Z(f) | f \in C(X)\}$.

Definition 0.6.17. Let A and B be fuzzy sets in a set X. Then

(i) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ for all $x \in X$

(ii) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in X$

(iii) $C = A \cup B \Leftrightarrow \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

(iv) $D = A \cap B \Leftrightarrow \mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$

(v) $E = A' \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$ for all $x \in X$.

For any family $\{A_i\}_{i \in I}$ of fuzzy sets in X, we define intersection

$\cap_{i\in I} A_i$ and the union $\cup_{i\in I} A_i$ respectively by,

$$\mu_{\underset{i\in I}{\cap A_i}}(x) = \inf_{i\in I} \mu_{A_i}(x) \quad \text{and} \quad \mu_{\underset{i\in I}{\cup A_i}}(x) = \sup_{i\in I} \mu_{A_i}(x) \text{ for all } x \in X.$$

The symbol ϕ will be used to denote the empty set such that $\mu_\phi(x) = 0$ for all $x \in X$. For X, we have by definition $\mu_X(x) = 1$ for all $x \in X$.

Definition 0.6.18. Let f be a mapping from a set X to a set Y. If A is a fuzzy set in X, then the fuzzy set $f(A)$ in Y is defined by,

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

If B is a fuzzy set in Y, then the fuzzy set $f^{-1}(B)$ in X is defined by $\mu_{f^{-1}(B)}(x) = \mu_B f(x)$.

Definition 0.6.19. A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions,

(i) $\phi, X \in T$

(ii) If $A, B \in T$, then $A \cap B \in T$

(iii) If $A_i \in T$ for each $i \in I$, then $\cup_{i\in I} A_i \in T$.

The pair (X, T) is called a fuzzy topological space (or *fts* in short). Every member of T is called a T-open fuzzy set (or simply an open fuzzy set). A fuzzy set is T-closed (or simply closed) iff its complement is T-

open.

As in general topology, the indiscrete fuzzy topology contains ϕ and X while the discrete fuzzy topology contains all fuzzy sets.

Definition 0.6.20. A fuzzy point P in X is a fuzzy set with membership function

$$\mu_P(x) = \begin{cases} \lambda & \text{for } x = x_0 \\ 0 & \text{otherwise.} \end{cases}$$

where $0 < \lambda \leq 1$. P is said to have support x_0 and value λ and we write $P = x_{0\lambda}$.

Two fuzzy points are said to be distinct if their supports are distinct, when $\lambda = 1$, P is called a fuzzy singleton.

Definition 0.6.21. A Čech fuzzy closure operator on a set X is a function from I^X to I^X satisfying the following three axioms.

(i) $c\phi = \phi$

(ii) $A \subseteq c(A)$ for all fuzzy subsets A of X

(iii) $c(A \cup B) = c(A) \cup c(B)$ for all fuzzy subsets A and B of X.

Here (X, c) is called a fuzzy closure space (or fcs in short). If $c(cA) = c(A)$ for all fuzzy subsets A of X then fuzzy closure space (X, c) is said to be a fuzzy topological space.

Definition 0.6.22. A fuzzy subset A in fuzzy closure space (X, c) is said to be fuzzy closed if $cA = A$ and fuzzy open if its complement is fuzzy closed.

Definition 0.6.23. A fuzzy closure operator c on X is said to be coarser than another fuzzy closure operator c' on the same set X if $c'(A) \subseteq c(A)$ for each fuzzy subset A of X.

Definition 0.6.24. If $(X.c)$ be a *fcs* then we denote the associated fuzzy topology on X by $\delta = \{A' | c(A) = A$ where A is a fuzzy subset of X $\}$ members of δ are the open sets of *fcs* (X, c) and their complements are the closed sets.

Definition 0.6.25. A fuzzy subset 'A' in a *fts* (X, δ) is said to be fuzzy dense if $\overline{A} = X$, where $\overline{A} = \cap\{B | B \supseteq A$ and B is closed in $(X, \delta)\}$.

Definition 0.6.26. Let X be any set. A fuzzy alignment on X is a family $\mathscr{L}$ of fuzzy sets in X which satisfies the following conditions,

(i) $\phi, X \in \mathscr{L}$

(ii) If $A_i \in \mathscr{L}$ for each $i \in I$ then $\cap_{i \in I} A_i \in \mathscr{L}$

(iii) If $A_i \in \mathscr{L}$ for each $i \in I$ and if A_i's are totally ordered by inclusion then $\cup_{i \in I} A_i \in \mathscr{L}$.

The pair $(X. \mathscr{L})$ is called a fuzzy aligned space or a fuzzy convexity space or $fco\,s$ in short. Every member of $\mathscr{L}$ is called a convex fuzzy set.

Definition 0.6.27. From axioms (i) and (ii) in definition 0.6.26 we have that for any subset S of X there is a smallest convex fuzzy set $\mathscr{L}(S)$ which contains S and is called the convex hull of the fuzzy set S.

That is $\mathscr{L}(S) = \cap\{K \in \mathscr{L} | S \subseteq K\}$

Definition 0.6.28. Let $(X, \mathscr{L})$ be a $fcos$. A collection $\mathscr{C}$ of fuzzy subsets of X generates $\mathscr{L}$ if $\mathscr{C} \subseteq \mathscr{L}$ and $\mathscr{L}$ is the smallest fuzzy alignment containing $\mathscr{C}$.

Definition 0.6.29. Let $(X, \mathscr{L})$ be an $fcos$ and M a crisp subset of X. Then a fuzzy alignment on M is given by $\mathscr{L}_M = \{L \cap M | L \in \mathscr{L}\}$. Then the pair $(M, \mathscr{L}_M)$ is a fuzzy subspace of $(X, \mathscr{L})$.

Note 0.6.30. The convex hull operator on M is given by $\mathscr{L}_M(S) = \mathscr{L}(S) \cap M$ for fuzzy subset S of M.

Definition 0.6.31. Let $(X, \mathscr{L}_1)$ and $(Y, \mathscr{L}_2)$ be two fuzzy convexity spaces and let $f : X \rightarrow Y$. Then f is said to be

(i) a fuzzy convexity preserving function (FCP function) if for each convex fuzzy set K in Y, $f^{-1}(K)$ is a convex fuzzy set in X

(ii) a fuzzy convex to convex function (FCC function) if for each convex fuzzy set K in X, $f(K)$ is a convex fuzzy set in Y.

Definition 0.6.32. Let $(X_\alpha, \mathscr{L}_\alpha)_{\alpha \in I}$ be a family of fuzzy convexity spaces. Let $X = \prod_{\alpha \in I} X_\alpha$ be the product space and let π_α: $X \rightarrow X_\alpha$ be the projection map. Then X can be equipped with the fuzzy alignment $\mathscr{L}$ generated by the convex fuzzy sets of the form $\{\pi_\alpha^{-1}(C_\alpha) | C_\alpha \in \mathscr{L}_\alpha; \alpha \in I\}$. Then

$\mathscr{L}$ is called the product fuzzy alignment for X and $(X, \mathscr{L})$ is called the product fuzzy convexity space.

Chapter 1

H-Closedness in Closure and Monotone Spaces

Introduction

In this chapter we introduce the concept of H-closedness in closure and monotone spaces and investigate its properties. Also we study the relations to the H-closedness of the associated topological space.

Absolutely closedness or H-closedness was introduced in 1929 by Alexandroff and Urysohn (for the definition and details, see [P;W]). E. Čech defined closure spaces (cf. [CE]) and T. A. Sunitha [S] discussed relations between closure spaces and the associated topological spaces. In [C;M], C. Calude and M. Malitza defined a different notion of Čech spaces which are now called monotone spaces and in [S], T. A. Sunitha discussed relations between monotone spaces, associated closure spaces and associated topological spaces. These motivated the study of H-closedness in closure and monotone spaces.

In section 1.1, we introduce denseness, adherence and H-closedness in closure spaces and call them respectively c-denseness, c-adherence and cH-closedness analogous to the corresponding notions in topological spaces. Here we study some of their properties; also we obtain relations between these properties and similar properties in the associated topological spaces.

In section 1.2, we discuss the inheritance in closure spaces, by subsets of properties like H-closed and cH-closed. Section 1.3 and section 1.4 are similar to section 1.1 and section 1.2 respectively in which closure space is replaced with monotone space. Here we also find the relationship between monotone space, associated closure space and the associated topological space with respect to the properties like H-closed and cH-closed.

1.1 c-denseness, c-adherence and cH-closedness

In this section we introduce the concept of denseness, adherence and H-closedness in closure space and they are called c-denseness, c-adherence and cH-closedness respectively. Also we study some of their properties

Definition 1.1.1. A set 'A' in a closure space (X, c) is said to be c-dense in X if $cA = X$.

Result 1.1.2. *If $cA = X$ then $\text{cl}\, A = X$ for any $A \subseteq X$; that is, a set $A \subseteq X$ is c-dense in X implies A is dense in X.*

Definition 1.1.3. Let (X, c) be a closure space, $\mathscr{F}$ be a filter on X then the set $\cap\{cF : F \in \mathscr{F}\}$ is called the c-adherence of $\mathscr{F}$ and is denoted by $a^c(\mathscr{F})$.

Note 1.1.4. $a^c(\mathscr{F}) \subseteq a(\mathscr{F})$, the adherence of $\mathscr{F}$ where $a(\mathscr{F}) = \cap\{\text{cl}\, F : F \in \mathscr{F}\}$ where $\text{cl}\, F$ is the closure of F in the associated topological space.

Definition 1.1.5. A closure space (X, c) is said to be cH-closed if every open filter on X has non-void c-adherence.

Note 1.1.6. The above definition is analogous to the characterisation of H-closedness as given in [P;W], namely a topological space (X, τ) is H-closed if and only if every open filter on X has a nonempty adherence. We say that (X, c) is H-closed if (X, τ) is H-closed where τ is the associated topology for c.

Proposition 1.1.7. *If a closure space (X, c) is cH-closed, then it is H-closed (by this we mean that the associated topological space is H-closed).*

Proof. We know that [P;W] a topological space (X, τ) is H-closed if and only if every open filter has non-void adherence, that is, if and only if for any open filler $\mathscr{F}$, $\cap\{\text{cl}\, F : F \in \mathscr{F}\} \neq \phi$. Given (X, c) is cH-closed. So if $\mathscr{F}$ is any open filler then $\cap\{cF : F \in \mathscr{F}\} \neq \phi$. So $\cap\{\text{cl}\, F : F \in \mathscr{F}\} \neq \phi$ since $cF \subseteq \text{cl}\, F$. Hence (X, c) is H-closed. □

Proposition 1.1.8. *A closure space (X, c) is H-closed if and only if every open cover $\mathscr{V}$ of (X, c) has a finite subfamily whose union is c-dense in X.*

Proof. We know that [P;W], a topological space (X, τ) is H-closed, if and only if every open cover $\mathscr{V}$ of (X, τ) has a finite subfamily whose union is dense in X. Let $\mathscr{V}$ be an open cover of X such that for each finite set $A \subseteq \mathscr{V}$, $X \neq c(\cup A)$. Let $\mathscr{I} = \{U : U$ open and $U \subseteq X \backslash c(\cup A)$ for some finite set $A \subseteq \mathscr{V}\}$. Clearly $\mathscr{I}$ is nonempty and $\mathscr{I}$ is an open filter on X and $a(\mathscr{I}) = \cap\{\text{cl}\, U : U \in \mathscr{I}\}$

$$\subseteq \cap\{\text{cl}\,(X\backslash c(\cup A)) : A \subseteq \mathscr{V} \text{ is finite}\}$$
$$\subseteq \cap\{\text{cl}\,(X\backslash cV) : V \in \mathscr{V}\}$$
$$\subseteq X\backslash \cup(\mathscr{V}) = \phi \text{ since } \cup(\mathscr{V}) = X.$$

Thus (X, c) is not H-closed.

Conversely, c-dense implies dense. Hence (X, c) is H-closed. □

Proposition 1.1.9. *If a closure space (X, c) is cH-closed then every open cover $\mathscr{V}$ of (X, c) has a finite subfamily whose union is dense in X.*

Proof. cH-closed implies H-closed (by proposition 1.1.7) and by the proposition 1.1.8 we have the result. □

Note 1.1.10. H-closed does not imply cH-closed.

Eg:- Let $X = N$, the set of natural numbers define $cA = A \cup (A - 1)$ for

$A \subseteq N$ where $A-1 = \{x-1 | x \in A\}$ and $1-1=0$ is not considered. Then $\{N, \{2,3,4,\ldots\}, \{3,4,5,\ldots\}, \ldots\}$ is an open filter base for the associated topology. Then (X, c) is H-closed and not cH-closed. Thus converse of the proposition 1.1.9 is not true.

Proposition 1.1.11. *If a closure space (X, c) is cH-closed then every open cover $\mathscr{V}$ of (X, c) has a finite subfamily whose union is c-dense in X.*

Proof. cH-closed implies H-closed and by proposition 1.1.8 we have the result. □

Note 1.1.12. Converse of the above proposition is not true in general, since H-closed does not imply cH-closed by note 1.1.10.

1.2 Inheritance of properties in cH-closed spaces

In this section, we discuss the inheritance by subsets of properties like H-closed and cH-closed.

Proposition 1.2.1. *Let (X, c) be cH-closed and $U \subseteq X$ be open. Then $cl\,(U)$ is cH-closed.*

Proof. Let U be open in (X, c) and $\mathscr{F}$ be an open filter on $\mathrm{cl}\,(U) = A$. Then $\{F \cap U : F \in \mathscr{F}\}$ is an open filter base on X. Let $G = \{W \subseteq X : W$ is open in X and $W \supseteq F \cap U$ for some $F \in \mathscr{F}\}$. Then G is an open filter on X. Given (X, c) is cH-closed. So $\phi \neq a^c_X(G) \subseteq \cap\{c_X(F \cap U) :$

$F \in \mathscr{F}\} = \cap\{c_A(F \cap U) : F \in \mathscr{F}\} \subseteq \cap\{c_A(F) : F \in \mathscr{F}\} = a^c_A(\mathscr{F})$.
Thus cl (U) is cH-closed. □

Corollary 1.2.2. *Let (X, c) be cH-closed and $U \subseteq X$ be open. Then cl (U) is H-closed.*

Proof. By Proposition 1.2.1 cl U is cH-closed. But by 1.1.7 cH-closed implies H-closed. Thus the corollary. □

Remark 1.2.3. Let (X, c) be H-closed and $U \subseteq X$ be open. Then cl (U) need not be cH-closed.

Proposition 1.2.4. *Let (X, c) be cH-closed and $U \subseteq X$ be open. Then cU is cH-closed.*

Proof. Proof is exactly similar to that of Proposition 1.2.1 with cl (U) replaced with $c(U)$. □

Proposition 1.2.5. *Let (X, c) be cH-closed and $U \subseteq X$ be open. Then $c(U)$ is H-closed.*

Proof. By proposition 1.2.4, $c(U)$ is cH-closed. And we know cH-closed implies H-closed. Hence the result. □

Remark 1.2.6. Let (X, c) be H-closed and $U \subseteq X$ be open. Then cU need not be cH-closed.

Proposition 1.2.7. *If (Y, c') is a $c'H$-closed subspace of a Hausdorff closure space (X, c) where $c' = c$ restricted to Y, then Y is closed in (X, c).*

Proof. Given (Y, c') is $c'H$-closed implies (Y, c') is H-closed. Thus Y is closed in X. □

1.3 m-denseness, m-adherence and mH-closedness

In this section we introduce the concept of denseness, adherence and closedness in monotone space and they are called m-denseness, m-adherence and mH-closedness respectively. Also we investigate their properties and relationship between them and the same in the associated closure space and the associated topological space.

Definition 1.3.1. A set A in a monotone space (X, m) is said to be m-dense in X if $mA = X$.

Remark 1.3.2. [S] If (X, m) is a monotone space and cl is the closure operation in the associated topological space. Then cl $\leq m$, that is, cl $A \supseteq mA$ for all $A \subseteq X$.

Result 1.3.3. If $mA = X$ then cl $A = X$ where $A \subseteq X$, that is, a set $A \subseteq X$ is m-dense in X implies A is dense in X.

Proof. Trivial using remark 1.3.2. □

Definition 1.3.4. Let (X, m) be a monotone space, $\mathscr{F}$ be a filter on X, then the set $\cap\{mF : F \in \mathscr{F}\}$ called m-adherence of $\mathscr{F}$ and is denoted by $a^m(\mathscr{F})$.

Note 1.3.5. Clearly $a^m(\mathscr{F}) \subseteq a(\mathscr{F})$ where $a(\mathscr{F}) = \cap\{\text{cl}\, F : F \in \mathscr{F}\}$ is the adherence of $\mathscr{F}$ in the associated topology.

Definition 1.3.6. A monotone space (X, m) is said to be mH-closed if every filter on X of open sets in the associated topology has non void m-adherence.

Note 1.3.7. The above definition is motivated by the characterisation of H-closedness given in [P;W], namely a topological space (X, τ) is H-closed if and only if every open filter on X has a nonempty adherence and the definition of H-closedness in closure spaces given in section 1.1. Here we say that (X, m) is cH-closed if it is H-closed in the associated closure space that is cH-closed in (X, c) where c is the associated closure operator, and we say (X, m) is H-closed if it is H-closed in the associated topology.

Proposition 1.3.8. *If a monotone space (X, m) is mH-closed then it is cH-closed and H-closed.*

Proof. Given (X, m) is mH-closed. So if $\mathscr{F}$ is any open filter then $\cap\{mF : F \in \mathscr{F}\} \neq \phi$. Thus $\cap\{cF : F \in \mathscr{F}\} \neq \phi$. Hence (X, m) is cH-closed. Also we have cH-closed implies H-closed. Thus we have the proposition. □

Proposition 1.3.9. *A monotone space (X, m) is H-closed if and only if every open cover $\mathscr{V}$ of (X, m) has a finite subfamily where union is m-dense in X.*

Proof. We know that (cf. [P;W]), a topological space (X, τ) is H-closed if and only if every open cover $\mathscr{V}$ of (X, τ) has a finite subfamily whose union is dense in X. Here given that (X, m) is H-closed. That is, (X, t) is H-closed where t is the associated topology of (X, m). We have to prove that every open cover $\mathscr{V}$ of (X, m) has a finite subfamily whose union is m-dense in X. For that we assume the contrary. Let $\mathscr{V}$ be an open cover of X, such that for each finite set $A \subseteq \mathscr{V}$, $X \neq m(\cup A)$. Let $\mathscr{I} = \{U : U$ is nonempty and open and $U \subseteq X \backslash m(\cup A)$ for some finite set $A \subseteq \mathscr{V}\}$. Clearly $\mathscr{I}$ is an open filter on X and adherence of $\mathscr{I}$, that is, $a(\mathscr{I}) = \cap\{\text{cl}\, U : u \in \mathscr{I}\} \subseteq \cap\{\text{cl}\,(X \backslash m(\cup A)) \colon A \subseteq \mathscr{V} \text{ is finite}\}$ $\subseteq \cap\{\text{cl}\,(X \backslash mV) : V \in \mathscr{V}\} \subseteq X \backslash \cup \mathscr{V} = \phi$ since $\cup \mathscr{V} = X$. Thus (X, m) is not H-closed. □

Converse follows from the fact that m-dense implies dense and by the characterisation of H-closedness in topological space.

Proposition 1.3.10. *If a monotone space (X, m) is mH-closed then every open cover $\mathscr{V}$ of (X, m) has a finite subfamily where union is m-dense in X and hence c-dense and dense in X.*

Proof. We know that mH-closed implies H-closed and by the proposition 1.3.9 we have the result. □

1.4 Inheritance of properties in mH-closed spaces

In this section we discuss the inheritance by subsets of properties H-closed, cH-closed and mH-closed.

Proposition 1.4.1. *Let (X, m) be mH-closed and $U \subseteq X$ be open then $cl(U)$ is mH-closed and hence cH-closed and H-closed.*

Proof. We have to prove $\mathrm{cl}\, U$ is mH-closed, that is, to prove $a^m(\mathscr{I}) \neq \phi$ with respect to $\mathrm{cl}\, U$, where $\mathscr{I}$ is an open filter on $\mathrm{cl}\, U$. Let U be open in (X, m) and $\mathscr{I}$ be an open filter on $\mathrm{cl}\, U = A$. Then $\{F \cap U : F \in \mathscr{I}\}$ is an open filter base on X. Let $G = \{W \subseteq X : W$ is open in X and $W \supseteq F \cap U$ for some $F \in \mathscr{I}\}$. Then G is an open filter on X. Given (X, m) is mH-closed. So $\phi \neq a^m_X(G) \subseteq \cap\{m_X(F \cap U) : F \in \mathscr{I}\} = \cap\{m_A(F \cap U) : F \in \mathscr{I}\} \subseteq \cap m_A(F) : F \in \mathscr{I}\} = a_m(\mathscr{I})$. Hence the proposition. □

Proposition 1.4.2. *Let (X, m) be mH-closed and $U \subseteq X$ be open then cU is mH-closed and hence cH-closed and H-closed.*

Proof. Proof is exactly similar to that of proposition 1.4.1 with $\mathrm{cl}\, U$ replaced with cU. □

Proposition 1.4.3. *If (Y, m_Y) is a $m_Y H$-closed subspace of a Hausdorff monotone space (X, m), then Y is closed in (X, m).*

Proof. Given (Y, m_Y) is mH-closed then (Y, m_Y) is H-closed. Thus Y is closed in X, since X is given to be Hausdorff. Thus the proposition. □

1.5 Conclusion

In this chapter we defined denseness, adherence and H-closedness in closure and monotone spaces. Using these definitions we prove Kuratowski closure of any open set in monotone space (or closure space) is H-closed in monotone space (or closure space) and hence H-closed in associated topological space of (X, m).

Chapter 2

Mappings between Closure and Monotone Spaces

Introduction

In this chapter we study the implicational relationships between various types of morphisms between closure spaces and between monotone spaces with respect to associated topological spaces.

In [CE] E. Čech defined closure spaces and continuity between closure spaces, in [C;M] C. Calude and M. Malitza defined a different notion of Čech spaces which are now called monotone spaces and in [P;W] is defined θ-continuity of functions between topological space. These motivated the study of continuity of functions between closure and monotone spaces. In [S] T. A. Sunitha discussed morphisms from one closure space into another, and this motivated the further study in mappings.

In section 2.1 we discuss the continuity of functions between closure spaces, this is analogous to the θ-continuity of functions between topological spaces given in [P;W]. Section 2.2 is similar to section 2.1 in which closure spaces is replaced with monotone spaces. In section 2.3 we discuss the mappings between closure spaces with respect to the associated topological spaces and prove some results. Also we find relations between continuity and different types of morphisms. Section 2.4 is similar to section 2.3 in which closure spaces is replaced with monotone spaces.

2.1 Continuity of functions between closure spaces

Here we discuss the continuity of functions between closure spaces, which is analogous to θ-continuity of functions between topological spaces given in [P;W], namely (X, τ) and (Y, τ') be topological spaces and $f : X \to Y$ be any map and let $x_0 \in X$ and f is θ-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\text{cl}_X U) \subseteq \text{cl}_Y(V)$. And the set of all θ-continuous functions from X to Y is denoted by θ-$c(X, Y)$.

Definition 2.1.1. Let (X, c), (Y, c') be two closure spaces and $f : X \to Y$ be any map and let $x_0 \in X$,

1. (i) f is θ-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\text{cl}_X U) \subseteq \text{cl}_Y(V)$.

(ii) f is $c\theta$-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(cU) \subseteq \text{cl}_Y(V)$.

(iii) f is $\theta c'$-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\text{cl}_X U) \subseteq c'(V)$.

(iv) f is $c\theta c'$-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(cU) \subseteq c'(V)$.

2. f is θ-continuous if f is θ-continuous at each point of X. Similarly $c\theta$-continuous, $\theta c'$-continuous and $c\theta c'$-continuous.

Proposition 2.1.2. *If f is $\theta c'$-continuous at x_0 then f is θ-continuous at x_0.*

Proof. We have given f is $\theta c'$-continuous at x_0. So $f[\text{cl}_X U] \subseteq c'(V)$. But $c'(V) \subseteq \text{cl}_Y(V)$. That is, $f(\text{cl}_X U) \subseteq \text{cl}_Y(V)$. Hence f is θ-continuous at x_0, where U and V are neighbourhoods of x_0 and $f(x_0)$ respectively. □

Note 2.1.3. Converse of the above proposition is not true.

Proposition 2.1.4. *If f is $c\theta c'$-continuous at x_0 then f is $c\theta$-continuous at x_0.*

Proof. Given f is $c\theta c'$-continuous at x_0. So for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(cU) \subseteq c'(V)$. But $c'(V) \subseteq \text{cl}_Y V$. That is, $f(cU) \subseteq \text{cl}_Y(V)$. Hence f is $c\theta$-continuous at x_0. □

Note 2.1.5. Converse of the above proposition is not true.

Proposition 2.1.6. *If f is $\theta c'$-continuous at x_0 then f is $c\theta$-continuous at x_0.*

Proof. Given f is $\theta c'$-continuous at x_0. So for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\mathrm{cl}_X U) \subseteq c'(V)$. But $c'(V) \subseteq \mathrm{cl}_Y(V)$. That is, $f(\mathrm{cl}_X U) \subseteq \mathrm{cl}_Y(V)$. Also $cU \subseteq \mathrm{cl}_X U$, hence $f(cU) \subseteq f(\mathrm{cl}_X U)$. Thus $f(cU) \subseteq \mathrm{cl}_Y(V)$, that is f is $c\theta$-continuous at x_0. □

Note 2.1.7. Converse of the above proposition is not true.

Proposition 2.1.8. *If f is θ-continuous at x_0 then f is $c\theta$-continuous at x_0.*

Proof. We have, for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\mathrm{cl}_X U) \subseteq \mathrm{cl}_Y(V)$ by the definition of θ-continuity of f at x_0. Also $c(U) \subseteq \mathrm{cl}_X U$, hence $f(cU) \subseteq f(\mathrm{cl}_X U)$. Thus we get $f(cU) \subseteq \mathrm{cl}_Y(V)$, that is f is $c\theta$-continuous at x_0. □

Note 2.1.9. Converse of the above proposition is not true.

Proposition 2.1.10. *If f is $\theta c'$-continuous at x_0 then f is $c\theta c'$-continuous at x_0.*

Proof. Given f is $\theta c'$-continuous at x_0. So for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f[\mathrm{cl}_X U] \subseteq c'(V)$. Also $cU \subseteq \mathrm{cl}_X U$, so $f(cU) \subseteq f(\mathrm{cl}_X U)$. Hence we get $f(cU) \subseteq c'(V)$, that is f is $c\theta c'$-continuous at x_0. □

2.2 Continuity of functions between monotone spaces

In this section we discuss the continuity of functions between monotone spaces, which is analogous to the θ-continuity of functions between topological spaces given in [P;W]. Here we also find the relationship between continuity of functions between closure spaces and between monotone spaces.

Definition 2.2.1. Let (X, m), (Y, m') be two monotone spaces and $f : X \to Y$ be any map and let $x_0 \in X$,

1. (i) f is θ-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\text{cl}_X U) \subseteq \text{cl}_Y(V)$ by [P;W]

   (ii) f is $m\theta$-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(mU) \subseteq \text{cl}_Y(V)$.

   (iii) f is $\theta m'$-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\text{cl}_X U) \subseteq m'(V)$.

   (iv) f is $m\theta m'$-continuous at x_0 if for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(mU) \subseteq m'(V)$.

2. f is θ-continuous if f is θ-continuous at each point of X. Similarly $m\theta$-continuous, $\theta m'$-continuous and $m\theta m'$-continuous.

Proposition 2.2.2. *If f is $\theta m'$-continuous at x_0 then f is θ-continuous at x_0.*

Proof. Given f is $\theta m'$-continuous at x_0. So for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\text{cl}_X U) \subseteq m'(V)$. But $m'(V) \subseteq \text{cl}_Y(V)$. Hence $f(\text{cl}_X U) \subseteq \text{cl}_Y(V)$,that is f is θ-continuous at x_0. $\square$

Note 2.2.3. Converse of the above proposition is not true.

Proposition 2.2.4. *If f is $m\theta m'$-continuous at x_0 then f is $m\theta$-continuous at x_0.*

Proof. Given f is $m\theta m'$-continuous at x_0. So for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(mU) \subseteq m'(V)$. But $m'(V) \subseteq \text{cl}_Y V$, that is, $f(mU) \subseteq \text{cl}_Y(V)$. Hence f is $m\theta$-continuous at x_0. $\square$

Note 2.2.5. Converse of the above proposition is not true.

Proposition 2.2.6. *If f is $\theta m'$-continuous at x_0 then f is $m\theta$-continuous at x_0.*

Proof. We have, for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\text{cl}_X U) \subseteq m'(V)$ by the definition $\theta m'$-continuity of f at x_0. Also $m(U) \subseteq \text{cl}_X U$, hence $f(mU) \subseteq f(\text{cl}_X U)$. Thus $f(mU) \subseteq m'(V) \subseteq \text{cl}_Y(V)$, that is f is $m\theta$-continuous at x_0. $\square$

Note 2.2.7. Converse of the above proposition is not true.

Proposition 2.2.8. *If f is θ-continuous at x_0 then f is $m\theta$-continuous at x_0.*

Proof. Given f is θ-continuous at x_0. So $f(\mathrm{cl}_X U) \subseteq \mathrm{cl}_Y(V)$. But $f(mU) \subseteq \mathrm{cl}_X(V)$. Thus $f(mU) \subseteq \mathrm{cl}_Y(V)$, that is f is $m\theta$-continuous at x_0. Here U and V are neighbourhoods of x_0 and $f(x_0)$ respectively. □

Note 2.2.9. Converse of the above proposition is not true.

Proposition 2.2.10. *If f is $\theta m'$-continuous at x_0 then f is $m\theta m'$-continuous at x_0.*

Proof. We have, for each open neighbourhood V of $f(x_0)$ there is an open neighbourhood U of x_0 such that $f(\mathrm{cl}_X U) \subseteq m'(V)$ by the definition of $\theta m'$-continuity of f at x_0. Also $mU \subseteq \mathrm{cl}_X U$, so $f(mU) \subseteq f(\mathrm{cl}_X U)$. Hence we get $f(mU) \subseteq m'(V)$, That is f is $m\theta m'$-continuous at x_0. □

2.3 Mappings between closure spaces

In this section we discuss the mappings between closure spaces.

Definition 2.3.1. A map $f : (X, c) \to (Y, c')$ is said to be $\mathrm{cl} - c'$ morphism if $f(\mathrm{cl}\, A) \subseteq c' f(A)$ for all $A \subseteq X$. Here cl is the closure operator on the associated topology of (X, c).

Result 2.3.2. *Let $f : (X, c) \to (Y, c')$ be a $\mathrm{cl} - \mathrm{cl}'$ morphism. Then it is $c - \mathrm{cl}'$ morphism, that is, continuity implies $c - \mathrm{cl}'$ morphism.*

Proof. Given f is $\mathrm{cl} - \mathrm{cl}'$ morphism, that is, $f(\mathrm{cl}\, A) \subseteq \mathrm{cl}' f(A)$ for all $A \subseteq X$. But $cA \subseteq \mathrm{cl}\, A$ so $f(cA) \subseteq f(\mathrm{cl}\, A)$. Hence we have $f(cA) \subseteq \mathrm{cl}' f(A)$, that is f is a $c - \mathrm{cl}'$ morphism. Hence the result. □

Note 2.3.3. Converse of the result 2.3.2 is not true in general

Eg:- Let $X = N$, the set of all natural numbers. For $A \subseteq N$ define $cA = A \cup (A-1)$ where $A - 1 = \{x - 1 | x \in A\}$. (When $1 \in A$ and $x = 1$ we do not consider $x - 1$). The corresponding associated topological space (X, t) has closed sets $\{N, \phi, \{1\}, \{1, 2\}, \{1, 2, 3\}, \cdots\}$ and consider $Y = N \cup \{w\}$ where w is some element not in N. $cA = A \cup (A - 1)$ with the understanding that $w - 1 = w$ and $A - 1 = \{x - 1 | x \in A$ and when $1 \in A$ and $x = 1$ we do not consider $x - 1\}$.

Now define a map $f : X \to Y$ as $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ w & \text{if } x = 1 \end{cases}$. Clearly f is $c - \text{cl}'$ morphism but not $\text{cl} - \text{cl}'$ morphism. For, let $A = \{5, 10, 15, \ldots\}$; $f(A) = \{5, 10, 15, \ldots\}$; $cA = \{4, 5, 9, 10, \ldots\}$; $\text{cl}' f(A) = N$; $\text{cl}\, A = N$. Thus $f(\text{cl}\, A) \not\subseteq \text{cl}' f(A)$ but $f(cA) \subseteq \text{cl}' f(A)$.

Result 2.3.4. *Let $f : (X, c) \to (Y, c')$ be a cl $- c'$ morphism. Then it is continuous, that is, it is a cl $-$ cl$'$ morphism.*

Proof. Given f is $\text{cl} - c'$ morphism. That is $f(\text{cl}\, A) \subseteq c' f(A)$ for all $A \subseteq X$. But $c' f(A) \subseteq \text{cl}' f(A)$. Hence we have $f(\text{cl}\, A) \subseteq \text{cl}' f(A)$, that is f is continuous. □

Note 2.3.5. In above example (Note 2.3.3), let $g : X \to Y$ be $g(x) = x$ for all x. Then g is $\text{cl} - \text{cl}'$ morphism but not $\text{cl} - c'$ morphism. For if, $A = \{3\}$; $g(A) = \{3\}$; $cl(A) = \{1, 2, 3\}$; $c' g(A) = \{2, 3\}$; $g(\text{cl}\, A) = \{1, 2, 3\}$.

Remark 2.3.6. Using result 2.3.2 and 2.3.4 we have if $f : (X, c) \to (Y, c')$ be a $\mathrm{cl} - c'$ morphism then f is continuous and hence f is $c - \mathrm{cl}'$ morphism.

Result 2.3.7. *Let $f : (X, c) \to (X, c')$ and $c' \leq c$ then f is $c - cl'$ morphism if f is cl-morphism.*

Proof. It is easy to see that if $c' \leq c$ and f is cl-morphism then f is $\mathrm{cl} - \mathrm{cl}'$ morphism (See [S]). But by result (2.3.2), if f is $\mathrm{cl} - \mathrm{cl}'$ morphism then f is $c - \mathrm{cl}'$ morphism. Thus the result. □

Result 2.3.8. *Let c, c' be closure operators on X. Then c is finer than the associated closure operation cl' (and hence finer than c') if and only if the identity mapping from (X, c) onto (X, c') is $c - cl'$ morphism.*

Proof. Suppose $c \geq \mathrm{cl}'$, that is, $c(A) \subseteq \mathrm{cl}'(A)$ for all $A \subseteq X$. Now $f(cA) = cA \subseteq \mathrm{cl}'(A) = \mathrm{cl}' f(A)$ where f is identity map. That is $f(cA) \subseteq \mathrm{cl}' f(A)$. Thus f is $c - \mathrm{cl}'$ morphism.

Conversely, suppose $f(cA) \subseteq \mathrm{cl}' f(A)$, that is, $cA \subseteq \mathrm{cl}'(A)$ for all $A \subseteq X$, where f is the identity map. Thus $c \geq \mathrm{cl}'$, that is, c is finer than the associated closure operation cl'. □

Result 2.3.9. *Let $f : (X, c) \to (Y, c')$ be a $c - cl'$ morphism and $g : (Y, c') \to (Z, c'')$ is continuous then their composite mapping $g \circ f : (X, c) \to (Z, c'')$ is $c - cl''$ morphism.*

Proof. Given f is $c - \mathrm{cl}'$ morphism, that is, $f(cA) \subseteq \mathrm{cl}' f(A)$ for all $A \subseteq X$. To prove $g \circ f$ is $c - \mathrm{cl}''$ morphism. For that, $(g \circ f)(cA) = g(f(cA)) \subseteq$

$g(\mathrm{cl}'f(A)) \subseteq \mathrm{cl}''g(f(A))$, since f is $c - \mathrm{cl}'$ morphism and g is continuous. Thus, $(g \circ f)(cA) \subseteq \mathrm{cl}''g(f(A))$. That is $(g \circ f)$ is $c - \mathrm{cl}''$ morphism. □

Remark 2.3.10. The conclusion in the above result is true if g is $\mathrm{cl}' - c''$ morphism.

2.4 Mappings between monotone spaces

To each monotone space, we can associate uniquely a Čech closure space and there by a topological space [S]. In this section we discuss the mappings between monotone spaces.

Definition 2.4.1. Let (X, m) and (Y, m') be two monotone spaces then a map $f : (X, m) \to (Y, m')$ is said to be $\mathrm{cl} - m'$ morphism if $f(\mathrm{cl}\, A) \subseteq m'f(A)$ for all $A \subseteq X$, where cl is the closure operator on the associated topology of (X, m).

Result 2.4.2. *Let $f : (X, m) \to (Y, m')$ be a cl − cl′ morphism. Then it is m − cl′ morphism, that is, continuity implies m − cl′ morphism.*

Proof. Proof is exactly similar to the proof of result 2.3.2 in which 'c' is replaced with m. □

Result 2.4.3. *Let $f : (X, m) \to (Y, m')$ be a cl − m' morphism. Then it is continuous, that is, it is cl − cl′ morphism.*

Proof. Proof is exactly similar to the proof of result 2.3.4 in which 'c' is replaced with m. □

Remark 2.4.4. Using result 2.4.2 and 2.4.3 we have if $f : (X, m) \to (Y, m')$ be a $\mathrm{cl} - m'$ morphism then f is continuous and hence f is $m - \mathrm{cl}'$ morphism.

Note 2.4.5. Let 'm' be a monotone operator on a set X and $f : (X, m) \to (X, m)$ be a mapping then $m - m$ morphism need not imply $m - cl$ morphism or $\mathrm{cl} - m$ morphism or cl – cl morphism and vice versa.

Eg:- Let $X = \{a, b, c\}$ and m be defined on X such that $m\{a\} = \{a\}$; $m\{b\} = \{b, c\}$; $m\{c\} = \{c\}$, $m\{a, b\} = m\{b, c\} = m\{a, c\} = mX = X$; $m\phi = \phi$. Then m is a monotone operator. Then the associated closure operator 'c' is defined by $cA = \cap c_\alpha A$ for all $A \subseteq X$ where c_α is a closure operator coarser than m, that is, $mA \subseteq c_\alpha A$ for all $A \subseteq X$. Thus, $c\phi = \phi$; $c\{a\} = \{a, b\}$; $c\{b\} = \{b, c\}$; $c\{c\} = \{a, c\}$; $c\{a, b\} = c\{b, c\} = c\{a, c\} = cX = X$.

Now let f be a map from (X, m) into (X, m) defined in such a way that $f(a) = b$, $f(b) = c$, $f(c) = a$. Clearly f is $c - c$ morphism, c – cl morphism, m – cl morphism cl – cl morphism but not $m - m$ morphism or cl – m morphism or cl – c morphism.

Result 2.4.6. *Let m, m' be two monotone operators on X. Then m is finer than the associated closure operator cl' and hence finer than m' if and only* *if the identity mapping (X, m) onto (X, m') is $m - cl'$ morphism.*

Proof. Proof is exactly similar to the proof of result 2.3.8 in which 'c' is replaced with m. □

Result 2.4.7. *Let $f : (X, m) \to (Y, m')$ be a $m - cl'$ morphism and $g : (Y, m') \to (Z, m'')$ is continuous then their composite mapping $g \circ f : (X, m) \to (Z, m'')$ is $m - cl''$ morphism.*

Proof. Given f is $m - \text{cl}'$ morphism, that is, $f(mA) \subseteq \text{cl}' f(A)$ for all $A \subseteq X$. To prove $g \circ f$ is $m - \text{cl}''$ morphism. For that, $(g \circ f)(mA) = g(f(mA)) \subseteq g(\text{cl}' f(A)) \subseteq \text{cl}'' g(f(A))$, since f is $m - \text{cl}'$ morphism and g is continuous. Thus, $(g \circ f)(mA) \subseteq \text{cl}''(g \circ f)(A)$, that is, $g \circ f$ is $m - \text{cl}''$ morphism. □

Remark 2.4.8. The above result is true if g is $\text{cl}' - m''$ morphism.

2.5 Conclusion

In this chapter we introduced the concept of θ-continuity of functions between closure spaces and between monotone spaces. Here we also mentioned morphisms between closure spaces (or between monotone spaces).

Chapter 3

Properties of Product Closure Spaces

Introduction

In this chapter we study some properties of closure spaces and product closure spaces. Also we discuss some separation properties.

In [CE] E. Čech defined closure spaces, product of closure spaces and their associated topological spaces. In [S] T. A. Sunitha discussed morphisms between closure spaces and some separation properties in closure spaces. These motivated the study of the properties of product closure spaces.

In section 3.1 we find some properties of mappings into product closure spaces with respect to associated topological spaces [S]. The properties are proved using the result proved in [CE] namely, 'a mapping f of a space X into the product-space $X = \Pi X_a$ is continuous if and only if the mapping $\pi_a \circ f$ is continuous for each a'. Also we find some separation properties in product closure spaces.

In section 3.2 we discuss some separation properties involving zero sets. In this section we say a closure space (X, c) is c-completely regular if it is completely regular in closure space, c-Hausdorff if it is Hausdorff in closure space. Also (X, c) is completely regular if it is completely regular in the associated topological space. Similarly (X, c) is Hausdorff if it is Hausdorff in the associated topological space.

3.1 Some properties in product closure spaces

In this section we find some properties of product closure spaces. For this we use properties of mapping between product closure spaces with respect to associated topological spaces.

Remark 3.1.1. It is known from [CE] that a mapping f of a closure space with closure operator c into the product closure space with closure operator c' is $c - c'$ morphism if and only if the mapping $\pi_a \circ f$ is $c - c'$ morphism for each π_a (in our terminology).

Note 3.1.2. Using the above remark and the following conditions,

(i) $c - c'$ morphism $\Rightarrow c - \mathrm{cl}'$ morphism (Trivial using definitions)

(ii) continuity $\Rightarrow c - \mathrm{cl}'$ morphism (Result 2.3.2)

(where $f : (X, c) \to (Y, c')$ and $\mathrm{cl}\,, \mathrm{cl}'$ are closure operators on associated topological space of (X, c), (Y, c') respectively) we have the following results.

Result 3.1.3. *(i) A mapping f of a closure space with closure operator c into the product closure space with closure operator c' is $c - cl'$ morphism if the mapping $\pi_a \circ f$ is $c - c'$ morphism for each a where π_a is the projection function.*

(ii) If f is $c - c'$ morphism then the mapping $\pi_a \circ f$ is $c - cl'$ morphism for each a.

(iii) If the mapping $\pi_a \circ f$ is continuous for each a then f is $c - cl'$ morphism.

(iv) If f is continuous then the mapping $\pi_a \circ f$ is $c - cl'$ morphism for each a.

Remark 3.1.4. It is known that [W] an arbitrary product space is completely regular if and only if each factor space is completely regular and we know completely regular implies c-completely regular [S]. Hence we have the following results.

Result 3.1.5. *An arbitrary product space is c-completely regular if each factor space is completely regular.*

Proof. Given each factor space is completely regular and by remark (3.1.4) we have, the arbitrary product space is completely regular. Also by [S] completely regular implies c-completely regular. Thus arbitrary product space is c-completely regular. □

Result 3.1.6. *An arbitrary product space is completely regular then each factor space is c-completely regular.*

Proof. Given arbitrary product space is completely regular, so by remark (3.1.4) we have each factor space is completely regular. By [S] completely regular implies c-completely regular. Thus each factor space is c-completely regular. ☐

Result 3.1.7. *Every subspace of a c-Hausdorff space is c-Hausdorff.*

Proof. Let A be a subspace of a c-Hausdorff space (X, c). Let x and y be two distinct points of A. Since (X, c) is c-Hausdorff by [S] there exist disjoint neighbourhoods U and V containing x and y respectively. Then $U \cap A$ and $V \cap A$ are disjoint neighbourhoods of x and y in A. Thus A is c-Hausdorff. ☐

Remark 3.1.8. It is known that [W] a non-empty product space is Hausdorff if and only if each factor space is Hausdorff and we know Hausdorff implies c-Hausdorff [S]. Hence we have the following results.

Result 3.1.9. *A non-empty product space is c-Hausdorff if each factor space is Hausdorff.*

Proof. Given each factor space is Hausdorff and by above remark (3.1.8) we have, product space is Hausdorff. Also by [S] Hausdorff implies c-Hausdorff. Thus product space is c-Hausdorff. ☐

Result 3.1.10. *If a non-empty product space is Hausdorff then each factor space is c-Hausdorff.*

Proof. Given product space is Hausdorff and by remark (3.1.8) we have each factor space is Hausdorff. Also by [S] Hausdorff implies c-Hausdorff. Hence each factor space is c-Hausdorff. □

Remark 3.1.11. It is known that [W] a non-empty product space is regular if and only if each factor space is regular and we know regular implies c-regular [S]. Hence we have the following results.

Result 3.1.12. *(i) A non-empty product space is c-regular if each factor space is regular*

(ii) If a non-empty product space is regular then each factor space is c-regular.

Proof. Proof is trivial using remark 3.1.11. □

3.2 Some separation properties in closure spaces

In this section we discuss some separation properties involving zero set.

Notation 3.2.1. We denote $C_1(X) = \{f : X \to R | f$ is $c - \mathrm{cl}$ morphism where c is the closure operator on X and cl is the Kuratowski closure operator on $R\}$ and the zero set of X is denoted by $Z_1(X) = \{Z_1(f) : f \in C_1(X)\}$ and $Z_1(f) = \{x \in X | f(x) = 0\}$.

Theorem 3.2.2. *A c-Hausdorff space X is c-completely regular if and only if the family $Z_1(X) = \{Z_1(f) : f \in C_1(X)\}$ is a base for the closed sets in the associated topology of X.*

Proof. Suppose X is c-completely regular. Then whenever F is closed set and $x \in F'$ there exist $f \in C_1(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$. Then $Z_1(f) \supseteq F$ and $x \notin Z_1(f)$. Thus $Z_1(X)$ is a base. ☐

Conversely, suppose $Z_1(X)$ is a base. So if F is closed set and $x \in F'$ then there exist $g \in C_1(X)$ such that $Z_1(g) \supseteq F$ and $x \notin Z_1(g)$. Let $r = g(x)$, then $r \neq 0$ and let $f = gr^{-1} \in C_1(X)$ and $f(x) = 1$, $f(F) = \{0\}$so that Hausdorff space X is completely regular. But by [S] completely regular implies c-completely regular. Hence the theorem.

Remark 3.2.3. It is known that [CHA] 'A Hausdorff space X is completely regular if and only if $\{Z(f) | f \in C^*(X)\}$ forms a base for the closed set. Also continuity implies $c - \mathrm{cl}'$ morphism by 2.3.2.

Remark 3.2.4. A Hausdorff space X is completely regular then $Z_1(X) = \{Z_1(f) : f \in C_1(X)\}$ is a base for the closed sets in the associated topology.

Proof. Trivial using remark (3.2.2). ☐

3.3 Conclusion

In this chapter we found some properties in product closure space using morphisms. Also we find some properties of complete regularity in closure space.

Chapter 4

Fuzzy Closure Spaces

Introduction

Fuzzy topology was introduced by C. L. Chang (1968) Čech closure space (or simply a closure space if there is no closure for confusion) is a generalisation of the concept of topological space. In 1985 A. S. Mashour and M. H. Ghanim [M;G-1] defined Čech fuzzy closure spaces. In chapter 1 we defined denseness in closure spaces and some of their properties and this motivated the study of denseness in fuzzy closure space. In [S], T. A. Sunitha introduced mappings between closure spaces and in chapter 2 we found relationship of mappings between closure spaces with respect to associated topological spaces. These motivated the study of mappings between fuzzy closure spaces (or *fcs* in short).

In this chapter we introduce denseness in fuzzy closure space and also introduce the concept of various types of mappings between fuzzy closure spaces and prove some results based on these.

In section 4.1, we introduce denseness in fuzzy closure space known as fuzzy c-denseness and prove some results in fuzzy closure space using denseness property.

In section 4.2, we introduce mappings between fuzzy closure spaces with respect to associated fuzzy topological spaces and prove some results.

4.1 Denseness in fuzzy closure spaces

In this section we discuss denseness in fuzzy closure space which we call fuzzy c-denseness and also find some of its properties. Also we discuss fuzzy adherence, fuzzy c-adherence, fuzzy H-closed and fuzzy cH-closed in fuzzy closure spaces.

Definition 4.1.1. A fuzzy set 'A' in a fcs (X, c) is said to be fuzzy c-dense if $cA = X$.

Result 4.1.2. *Fuzzy c-dense in fuzzy closure space implies fuzzy denseness in associated fuzzy topological space (or fts in short).*

Proof. Let A be fuzzy c-dense in fcs (X, c) then $cA = X$. But $\overline{A} \supseteq cA$, therefore $\overline{A} \supseteq X$. Thus we get $\overline{A} = X$. Hence fuzzy c-dense implies fuzzy dense in associated fts. □

Result 4.1.3. *A fuzzy subset 'A' of a fcs (X, c) is fuzzy c-dense in X if and only if for every non-empty open subset B of X, $A \cap B \neq \phi$.*

Proof. Suppose A is fuzzy c-dense in fcs (X, c) and B is a non-empty open subset of X. If $A \cap B = \phi$ then $A \subseteq X \backslash B$, hence $cA \subseteq c(X \backslash B) = X \backslash B$. But $cA = X$, that is, $X \subseteq X \backslash B$ not possible. So $A \cap B \neq \phi$.

Conversely, assume that A meets every non-empty open subset of X. This is possible only when $cA = X$. Thus the result. □

Definition 4.1.4. [Y;M] Let (X, δ) be fts. $\mathscr{I}$ be a fuzzy filter on X, then the set $\cap\{\overline{F} : F \in \mathscr{I}\}$ is called the adherence of fuzzy filter $\mathscr{I}$ and is denoted by $a(\mathscr{I})$.

Definition 4.1.5. Let (X, c) be a fcs. $\mathscr{I}$ be a fuzzy filter on X, then the set $\cap\{cF : F \in \mathscr{I}\}$ is called the c-adherence of fuzzy filter $\mathscr{I}$ and is denoted by $a^c(\mathscr{I})$.

Definition 4.1.6. A fts (X, δ) is said to be fuzzy H-closed if adherence of every open filter on X is non-void.

Definition 4.1.7. A fcs (X, c) is said to be fuzzy cH-closed if c-adherence of every open fuzzy filter on X is non-void.

Result 4.1.8. *If a fcs (X, c) is fuzzy cH-closed then it is fuzzy H-closed.*

Proof. Given fcs (X, c) is fuzzy cH-closed, that is, $\cap\{cF : F \in \mathscr{I}\} \neq \phi$. But $\overline{F} \supseteq cF$ implies $\cap\{\overline{F} : F \in \mathscr{I}\} \neq \phi$. Hence (X, c) is fuzzy H-closed. □

Result 4.1.9. *Let Y be a subspace of a fcs (X, c) and if F is closed (open) in X then $\chi_Y \cap F$ is closed (open) in Y (where χ_Y is the characteristic function on Y).*

Proof. Given F is closed. To prove $\chi_Y \cap F$ is closed. F is closed implies $cF = F$. To prove $c(\chi_Y \cap F) = \chi_Y \cap F$. By definition of fcs, we have $\chi_Y \cap F \subseteq c(\chi_Y \cap F)$. But χ_Y is the characteristic function so we get $\chi_Y \cap F \supseteq c(\chi_Y \cap F)$. Hence the result. □

4.2 Mappings between fuzzy closure spaces

In this section we discuss mappings between fuzzy closure spaces and prove some results.

Definition 4.2.1. Let (X, c), (Y, c') be fcs and cl, cl$'$ be closure operator in the respective associated fuzzy topological space (or fts in short), then a map $f : X \to Y$ is said to be,

(i) fuzzy $c - c'$ morphism if $f(cA) \subseteq c'f(A)$ for all fuzzy subsets A of X

(ii) fuzzy cl $- c'$ morphism if $f(\overline{A}) \subseteq c'f(A)$ for all fuzzy subsets A of X

(iii) fuzzy $c -$ cl$'$ morphism if $f(cA) \subseteq \overline{f(A)}$ for all fuzzy subsets A of X

(iv) fuzzy cl $-$ cl$'$ morphism or fuzzy continuous if $f(\overline{A}) \subseteq \overline{f(A)}$ for all fuzzy subsets A of X.

Result 4.2.2. *Let X be a set, c and c' denote fuzzy closure operators on X. If f is fuzzy $c - c$ morphism and c' is coarser closure operator on X, then*

f is fuzzy $c - c'$ morphism.

Proof. Given f is fuzzy $c - c$ morphism, that is, $f(cA) \subseteq cf(A)$ for all fuzzy subset A of X. But $cf(A) \subseteq c'f(A)$ since c' is coarser than c. Thus 'f' is fuzzy $c - c'$ morphism. □

Remark 4.2.3. In result 4.2.2, if f is fuzzy $c - c'$ morphism and c is coarser than c' then f is fuzzy $c - c$ morphism.

Result 4.2.4. *If f is fuzzy cl $- c'$ morphism from a fcs (X, c) to fcs (Y, c') then f is fuzzy continuous.*

Proof. Given f is fuzzy cl $- c'$ morphism. Hence for any fuzzy subset A of X, we have $f(\overline{A}) \subseteq c'f(A)$. But by definition $c'(f(A)) \subseteq \overline{f(A)}$. Thus '$f$' is fuzzy continuous. □

Result 4.2.5. *If f is fuzzy $c - c'$ morphism from a fcs (X, c) to fcs (Y, c') then f is fuzzy $c - cl'$ morphism.*

Proof. Given f is fuzzy $c - c'$ morphism. Hence for any fuzzy subset A of X, we have $f(cA) \subseteq c'f(A)$. But by definition $c'(f(A)) \subseteq \overline{f(A)}$. Thus $f(cA) \subseteq \overline{f(A)}$. Hence the result. □

Result 4.2.6. *If f is fuzzy cl $- c'$ morphism from a fcs (X, c) to (Y, c') then f is fuzzy $c - cl'$ morphism.*

Proof. Given f is fuzzy cl $- c'$ morphism. Hence for any fuzzy subset A of X, we have $f(\overline{A}) \subseteq c'f(A)$. But $c'(f(A)) \subseteq \overline{f(A)}$ and $f(cA) \subseteq f(\overline{A})$. Hence $f(cA) \subseteq \overline{f(A)}$. □

Result 4.2.7. *If f is fuzzy continuous from a fcs (X, c) to (Y, c') then f is fuzzy $c - cl'$ morphism.*

Proof. Given f is fuzzy continuous. Hence for any fuzzy subset A of X, we have $f(cA) \subseteq c'f(A)$. But $c'(f(A)) \subseteq \text{cl}'f(A)$. Hence we get, $f(cA) \subseteq \text{cl}'f(A)$. Thus the result. □

Result 4.2.8. *Let (X, c), (Y, c') and (Z, c'') be three fcs. If $f : X \to Y$ is fuzzy $c - cl'$ morphism and $g : Y \to Z$ be fuzzy continuous then their composite*

$g \circ f : X \to Z$ is fuzzy $c - cl''$ morphism.

Proof. To prove $g \circ f$ is fuzzy $c - \text{cl}''$ morphism. Given f is fuzzy $c - \text{cl}'$ morphism, that is $f(cA) \subseteq \text{cl}'f(A)$ for all fuzzy subset A of X.
Now, $(g \circ f)(cA) = g(f(cA)) \subseteq g(\text{cl}'f(A)) \subseteq \text{cl}''g(f(A))$. Since g is fuzzy continuous. Thus $(g \circ f)(cA) \subseteq \text{cl}''g(f(A))$, that is $(g \circ f)$ is fuzzy $c - \text{cl}''$ morphism. □

Remark 4.2.9. The above conclusion is true also when g is fuzzy $\text{cl}' - c''$ morphism.

4.3 Conclusion

In this chapter we introduced fuzzy H-closedness in fuzzy topological spaces and fuzzy H-closedness in fuzzy closure spaces. Also we discussed mappings between fuzzy closure spaces.

Chapter 5

Fuzzy Closure Fuzzy Convexity Spaces

Introduction

In this chapter we introduce the notion of fuzzy closure fuzzy convexity spaces or $(fc)(fco)s$ in short. And we prove some properties of $(fc)(fco)s$. In [CE] E. Čech defined closure space, in [S] T. A. Sunitha discussed mappings between closure spaces and in [ROS-1] M. V. Rosa introduced Fuzzy topology fuzzy convexity spaces. These motivated the study of fuzzy closure fuzzy convexity spaces and their properties.

The study of convex sets is a branch of geometry, analysis and linear algebra that has numerous connections with other areas of mathematics. The theory of convexity can be sorted into two kinds. One deals with concrete convexity and the other that deals with abstract convexity. Here we deal with abstract convexity. In abstract convexity theory a convexity space was introduced by F. W. Levi in 1951 [LE]. The convexity space introduced by Levi was further developed by many authors like D. C. Kay and E. W. Womble [K;W], R. E. Jamison-Waldner [J;W], G. Sierksma [SI] and M. Van de Vel [V]. The notion of a topological convexity structure and aligned space has been introduced by R. E. Jamison-Waldner [J;W] in 1974. He also introduced the concept of local convexity. L. A. Zadeh introduced the concept of a convex fuzzy set in 1965. In 1980, R. Lowen applied the theory of fuzzy sets to some elementary known results of convex sets.

In section 5.1 we consider a fuzzy closure together with a fuzzy convexity on the same underlying set and introduce fuzzy closure fuzzy convexity spaces or in short $(fc)(fco)s$. Also we introduce the notions of subspace and products of an $(fc)(fco)s$.

In section 5.2 we study fuzzy local convexity. Here also we study subspaces and products of such spaces.

In section 5.3 we introduce the separation axioms in fuzzy closure fuzzy convexity spaces. The separation involves closed convex fuzzy neighbourhoods. Here we study concepts $FCNS_0$, $FCNS_1$, $FCNS_2$, pseudo $FCNS_3$, $FCNS_3$, semi $FCNS_4$, $FCNS_4$ spaces where $FCNS$ stands for 'Fuzzy closure neighbourhood separation'.

5.1 Fuzzy closure fuzzy convexity spaces

In this section we define fuzzy closure fuzzy convexity spaces and some of its properties.

Definition 5.1.1. A triple $(X, \mathscr{L}, c)$ consisting of a set X, a fuzzy alignment $\mathscr{L}$, and a fuzzy closure 'c' is called a fuzzy closure fuzzy convexity space or $(fc)(fco)s$ in short.

Eg:- $X = \{a, b, d\}$ and c be a fuzzy closure operator defined on X such that

$$c(A)(x) = \begin{cases} A(x) + 1/2 & \text{if } 0 < A(x) \leq 1/2 \\ 0 & \text{if } A(x) = 0 \\ 1 & \text{if } A(x) > 1/2 \end{cases}$$

where 'A' is a fuzzy subset of X.

Now choose fuzzy alignment $\mathscr{L} = \{\phi, X, B\}$ where B is the set,

$$B : \begin{array}{l} a \longrightarrow 0 \\ b \longrightarrow 1 \\ d \longrightarrow 1. \end{array}$$

Then $(X, \mathscr{L}, c)$ is $(fc)(fco)s$.

Definition 5.1.2. Let 'a_λ' be a fuzzy point in an $(fc)(fco)s$ $(X, \mathscr{L}, c)$, then a fuzzy subset U of X is called a fuzzy c-neighbourhood of a_λ if $a_\lambda \in X \backslash c(X \backslash U)$.

Definition 5.1.3. Let $(X, \mathscr{L}, c)$ be a $(fc)(fco)s$. Let Y be an ordinary subset of X. Then a fuzzy closure c_Y on Y is defined as $c_Y(A) = Y \cap cA$ for all A where A is any fuzzy subset on X and a fuzzy convexity space on Y is given by $\mathscr{L}_Y = \{Y \cap L | L \in \mathscr{L}\}$. Then the corresponding triple $(M, \mathscr{L}_Y, c_Y)$ is a subspace of $(X, \mathscr{L}, c)$.

Definition 5.1.4. Let (X, c) be a fuzzy closure space (or fcs in short). Then a collection $\mathscr{V}$ of fuzzy subsets of X is a local base of the fuzzy c-neighbourhood system of a fuzzy subset A of X (or a fuzzy point a_λ in X) iff each $V \in \mathscr{V}$ is a c-neighbourhood of A (or of a_λ) and every fuzzy c-neighbourhood of A (or of a_λ) contains a $V \in \mathscr{V}$.

Result 5.1.5. *If $\mathscr{U}(a_\lambda)$ is a local base at a fuzzy point a_λ then the following assertions are true.*

(i) $\mathscr{U}(a_\lambda) \neq \phi$

(ii) for each $U \in \mathscr{U}(a_\lambda)$, $a_\lambda \in U$

(iii) for each U_1 and U_2 in $\mathscr{U}(a_\lambda)$ there exist a U in $\mathscr{U}(a_\lambda)$ with $U \subset U_1 \cap U_2$.

Proof. Using the definition of fuzzy c-neighbourhood of a fuzzy point, the result is trivial. □

Result 5.1.6. *For each fuzzy point a_λ of a fuzzy closure space (X, c), let $\mathscr{U}(a_\lambda)$ be a collection of fuzzy subsets of X satisfying the three conditions mentioned in (5.1.5). Then there exists exactly one fuzzy closure operation c for X such that, for each a_λ in X, $\mathscr{U}(a_\lambda)$ is a local base at a_λ in (X, c).*

Here c is defined by $cA = \sup a_\lambda$ such that $U \in \mathscr{U}(a_\lambda)$, $U \cap A \neq \phi$ where A is a fuzzy subset of X.

Proof. If $a_\lambda \in X$ and $U \in \mathscr{U}(a_\lambda)$ then U is a fuzzy neighbourhood of a_λ. For, otherwise U is not a fuzzy c-neighbourhood of the point a_λ that is $a_\lambda \in c(X \backslash U)$ so for each $V \in \mathscr{U}(a_\lambda)$, $V \cap (X \backslash U) \neq \phi$ not possible when $V = U$. Next we have to prove that every fuzzy c-neighbourhood W of a_λ contains $U \in \mathscr{U}(a_\lambda)$; otherwise, $U - W = U \cap (X \backslash W) \neq \phi$ for all $U \in \mathscr{U}(a_\lambda)$ then $a_\lambda \in c(X \backslash W)$ that is, W is not a neighbourhood of x. Hence the result. □

Theorem 5.1.7. *A fuzzy point a_λ in fuzzy closure space (X, c) belongs to the closure of a fuzzy set A in X iff each fuzzy c-neighbourhood of a_λ in X intersect A.*

Proof. If a fuzzy c-neighbourhood U of a_λ does not meet A then $a_\lambda \in X \backslash c(A)$ that is, $a_\lambda \notin c(A)$. Conversely if $a_\lambda \notin c(A)$ then $X \backslash A$ is a fuzzy c-neighbourhood of a_λ which does not meet A. Thus the theorem. □

Corollary 5.1.8. *If $\mathscr{U}$ is a local base at a fuzzy point a_λ in a fuzzy closure space (X, c) then $a_\lambda \in c(A)$ iff A is a fuzzy subset of X and each $U \in \mathscr{U}$ intersects A.*

Proof. If $a_\lambda \in c(A)$ then by above theorem 5.1.7 each fuzzy c-neighbourhood of a_λ and hence each member of $\mathscr{U}$ will intersect A. Conversely if each member of $\mathscr{U}$ intersect A then each fuzzy c-neighbourhood of a_λ meet A. So by above theorem $a_\lambda \in c(A)$. □

Proposition 5.1.9. *Let c_1 and c_2 be two fuzzy closure operators for a set X. In order that c_1 should be coarser than c_2 it is necessary and sufficient that for each $a_\lambda \in X$ every fuzzy c_1-neighbourhood of a_λ be a fuzzy c_2-neighbourhood of a_λ.*

Proof. Given c_1 is coarser than c_2 and if $a_\lambda \in X \backslash c_1(X \backslash U)$ where U is the fuzzy neighbourhood of a_λ then $a_\lambda \in X \backslash c_2(X \backslash U)$. Hence every fuzzy c_1-neighbourhood of a_λ is fuzzy c_2-neighbourhood of a_λ. Conversely, suppose each $a_\lambda \in X$ every fuzzy c_1-neighbourhood of a_λ be a fuzzy c_2-neighbourhood of a_λ. Then by theorem 5.1.7 $a_\lambda \in c_2 A$ then $a_\lambda \in c_1 A$ for each $A \subseteq X$ that is, $c_1 A \supseteq c_2 A$ for all fuzzy subset A of X. Thus c_1 is coarser than c_2. □

Corollary 5.1.10. *Let c_1 and c_2 be two fuzzy closure operators for a set X. For each a_λ in X let $\mathscr{U}(a_\lambda)$ and $\mathscr{V}(a_\lambda)$ be local bases at a fuzzy point a_λ in (X, c_1) and (X, c_2) respectively. Then c_1 is coarser than c_2 if and only if for each $a_\lambda \in X$ every element of $\mathscr{U}(a_\lambda)$ contains an element of $\mathscr{V}(a_\lambda)$. In particular $c_1 = c_2$ iff for each $a_\lambda \in X$ every $U \in \mathscr{U}(a_\lambda)$ contains a $V \in \mathscr{V}(a_\lambda)$ and every $V \in \mathscr{V}(a_\lambda)$ contains a $U \in \mathscr{U}(a_\lambda)$.*

Proof. Trivial by theorem 5.1.9. □

Theorem 5.1.11. *For each fuzzy point a_λ of a fuzzy closure space (X, c), let $\mathscr{U}(a_\lambda)$ be a collection of subsets satisfying the three c-neighbourhood condition given in 5.1.5, then there exist exactly one fuzzy closure operation c for X such that for each a_λ in X $\mathscr{U}(a_\lambda)$ is a local base at a_λ in (X, c).*

Proof. By Corollary 5.1.8 we can define the closure operator 'c' in the following way,

$$cA = \sup a_\lambda \text{ such that for all } U \in \mathscr{U}(a_\lambda) \text{ and } U \cap A \neq \phi$$

where A is a fuzzy subset of X. (*)

Now by the corollary 5.1.10 there exist at most one fuzzy closure operator c with local base $\mathscr{U}(a_\lambda)$. Since by 5.1.6 $\mathscr{U}(a_\lambda)$ forms a local base at a_λ of X. Thus for proving the theorem we have to prove c is a fuzzy closure operator. For, $c\phi = \phi$ by (*) hence first axiom satisfied. Also $cA \supseteq A$ for all fuzzy subset A of X by second c-neighbourhood condition. Thus second condition is also satisfied. Next we have to prove only the third axiom that is, $c(A \cup B) = cA \cup cB$ where A and B are fuzzy subsets of X. Let $a_\lambda \in c(A \cup B)$ that is, $a_\lambda \leq c(\max A(x), B(x))$ that is, $a_\lambda \leq c(A(x))$ or $a_\lambda \leq c(B(x))$ that is, $a_\lambda \leq \max(C(A)(x), C(B(x)))$ that is, $a_\lambda \in C(A) \cup C(B)$. Thus $c(A \cup B) \subseteq c(A) \cup c(B)$.

For converse inclusion we retrace the above steps. Thus 3rd axiom is also satisfied. Hence there exist exactly one fuzzy closure operator with

$\mathscr{U}(a_\lambda)$ as local base. □

5.2 Locally fuzzy closure fuzzy convexity spaces

In this section we study locally fuzzy closure fuzzy convexity spaces and some of its properties.

Definition 5.2.1. A $(fc)(fco)s$ $(X, \mathscr{L}, c)$ is said to be locally fuzzy closure fuzzy convex at a fuzzy point a_λ if for every fuzzy c-neighbourhood U of a_λ there is some convex fuzzy c-neighbourhood C of a_λ which is contained in U.
$(X, \mathscr{L}, c)$ is locally fuzzy closure fuzzy convex if it is locally fuzzy closure fuzzy convex at each of its fuzzy points.

Definition 5.2.2. Let (X, c_1) and (Y, c_2) be two fuzzy closure spaces. A function $f : X \to Y$ is said to be fuzzy open if whenever A is open fuzzy subset of X, $f(A)$ is open fuzzy subset of Y.

Proposition 5.2.3. *An FCC [ROC], f-open f-morphism (mappings between fuzzy closure spaces given in chapter 4) image of locally $(fc)(fco)s$ is a (fc) $(fco)s$.*

Proof. Let $f : (X, \mathscr{L}_1, c_1) \to (Y, \mathscr{L}_2, c_2)$ be a FCC, f-open fuzzy morphism onto map. Let a_λ be a fuzzy point in Y. Then we can find a point 'b' in X such that $f(b) = a$. Then clearly $f(b_\lambda) = a_\lambda$. Let U be a fuzzy neighbourhood of a_λ in Y then $f^{-1}(U)$ is a fuzzy neighbourhood of b_λ in X. Since X is locally $(fc)(fco)s$ there exist an $\mathscr{L}_1$ – convex fuzzy c-neighbourhood C [A fuzzy c-neighbourhood C which is a member of aligned space $(X, \mathscr{L}_1)$ is called L_1-convex fuzzy neighbourhood] of b_λ in X such that $b_\lambda \in C \subseteq f^{-1}(U)$ $\therefore f(b_\lambda) \in f(C) \subseteq U$ that is, $a_\lambda \in f(C) \subseteq U$ since f is an FCC, open onto map, f(C) is an L_2-convex fuzzy neighbourhood of a_λ in Y. Hence Y is a locally $(fc)(fco)s$. □

Proposition 5.2.4. *Any subspace of a locally $(fc)(fco)s$ is a locally (fc) $(fco)s$.*

Proof. Let $(X, \mathscr{L}, c)$ be a locally $(fc)(fco)s$. Let $M \subseteq X$ and $(M, \mathscr{L}_M, c_M)$ be the corresponding subspace of $(X, \mathscr{L}, c)$. Let a_λ be a fuzzy point in M and let U be a fuzzy c-neighbourhood of a_λ in M that is, $a_\lambda \in X \backslash c(X \backslash U)$ also $c_m U = M \cap cV$ for some fuzzy subset V of X. Since X is locally $(fc)(fco)s$, there exist a convex fuzzy c-neighbourhood W of a_λ such that $a_\lambda \in W \subseteq cV$. Then $a_\lambda \in cW \cap M \subseteq V \cap M$. Now $cW \cap M$ is a convex fuzzy c-neighbourhood of a_λ in $(M, \mathscr{L}_M, c_M)$ and so M is locally $(fc)(fco)s$. □

Definition 5.2.5. Let $(X, \mathscr{L}, c)$ be a $(fc)(fco)s$ and M a fuzzy subset of X. Then define, $\mathscr{L}_M = \{L \cap M | L \in \mathscr{L}\}$ and $c_M A = \inf(M \cap cV)$ such that $V \supset A$ where V is a fuzzy subset of X and A is fuzzy subset of M then we can say that $(M, \mathscr{L}_M, c_M)$ is a fuzzy closure fuzzy convexity fuzzy subspace of $(X, \mathscr{L}, c)$ in the following sense.

(i) $\phi, M \in \mathscr{L}_M$

(ii) If $A_i \in \mathscr{L}_M$ for each $i \in I$ then $\cap A_i \in \mathscr{L}_M$

(iii) If $A_i \in \mathscr{L}_M$ for each $i \in I$ and if A_i's are totally ordered by inclusion then $UA_i \in \mathscr{L}_M$.

Again,

$$\text{(i) } c_M\phi = \phi \text{ (ii) } A \subseteq c_M A \text{ (iii) } c_M(A \cup B) = c_M A \cup c_M B.$$

Note 5.2.6. Using the above definition and imitating the proof of proposition 5.2.4 we can show that any such subspace of a locally $(fc)(fco)s$ is a locally $(fc)(fco)s$.

Proposition 5.2.7. *A non-empty product space* $\prod_{\alpha\in I}(X_\alpha, \mathscr{L}_\alpha, c_\alpha)$ *is locally* (fc) $(fco)s$ *if and only if each factor is locally* $(fc)(fco)s$.

Proof. Suppose each X_α is locally $(fc)(fco)s$. Let a_λ be a fuzzy point in $X = \Pi X_\alpha$ and consider the basic fuzzy c-neighbourhood, $\pi_{\alpha_1}^{-1}(U_1) \cap \pi_{\alpha_2}^{-1}(U_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(U_n)$ of a_λ in X where π_α is the projection map from X to X_α. Now U_i is a fuzzy c-neighbourhood of $(a_{\alpha_i})_\lambda$ in X_{α_i} for $i = 1, 2, \ldots n$ and since each X_{α_i} is locally $(fc)(fco)s$, so U_i contains a fuzzy convex c-neighbourhood W_i of $(a_{\alpha_i})_\lambda$ *i.e.*, $(a_{\alpha_i})_\lambda \in W_i \subset U_i$. Then $\pi_{\alpha_1}^{-1}(W_1) \cap \pi_{\alpha_2}^{-1}(W_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(W_n)$ is a convex fuzzy c-neighbourhood of a_λ contained in $\pi_{\alpha_1}^{-1}(U_1) \ldots \pi_{\alpha_n}^{-1}(U_n)$. Thus every fuzzy c-neighbourhood a_λ contains a convex fuzzy c-neighbourhood and hence X is locally $(fc)(fco)s$.

Conversely let $(a_\alpha)_\lambda$ be a fuzzy point in X_α. Then we can choose a fuzzy point a_λ in X such that $\pi_\alpha(a_\lambda) = (a_\alpha)_\lambda$. Let U_α be a fuzzy c-neighbourhood of $(a_\alpha)_\lambda$ in X_α. Then $\pi_\alpha^{-1}(U_\alpha)$ is a fuzzy c-neighbourhood of a_λ in X. Since X is locally $(fc)(fco)s$ there exist a convex fuzzy c-neighbourhood W of a_λ contained in $\pi_\alpha^{-1}(U_\alpha)$. Since π_α is FCC [ROC] $\pi_\alpha(C)$ is a convex fuzzy c-neighbourhood of $(a_\alpha)_\lambda$ in X_α contained in U_α. Hence X_α is locally fuzzy convex. □

Definition 5.2.8. Let (X, c) be a fcs. A fuzzy subset U of X is a fuzzy c-neighbourhood or simply fc-neighbourhood (or c-neighbourhood in short) of a subset A of X iff U is a fc-neighbourhood of each point of A. A subset U of X is f-open iff it is fc-neighbourhood of all if its points or equivalently it is a f-neighbourhood of each of itself.

Theorem 5.2.9. *A mapping f of a fuzzy closure space (X, c_1) into another one (Y, c_2) be fuzzy morphism at a point a_λ in c_1 if and only if the inverse image $f^{-1}(V)$ of each fc-neighbourhood of $f(a_\lambda)$ be a fuzzy c-neighbourhood of a_λ or equivalently that for each f-neighbourhood V of $f(a_\lambda)$ there exist a fuzzy c-neighbourhood U of a_λ such that $f(U) \subseteq V$.*

Proof. If $U = f^{-1}(V)$, is not a fuzzy c-neighbourhood of a_λ then by definition. $a_\lambda \in c_1(X \backslash U)$ and $f(a_\lambda) \in c_2(f(X \backslash U)) \subset c_2(Y \backslash V)$ that is, V is not a fuzzy c-neighbourhood of $f(a_\lambda)$ in (Y, c_2). Consequently if V is a fuzzy c-neighbourhood of $f(a_\lambda)$ then $f^{-1}(V)$ is a fuzzy c-neighbourhood of a_λ.

Conversely, if $a_\lambda \in X$, $A \subseteq X$ and $f(a_\lambda) \notin c_2 f(A)$ then $V = Y - f(A)$ is a fuzzy c-neighbourhood of $f(a_\lambda)$ and by hypothesis $f^{-1}(V)$ is a fuzzy c-neighbourhood of a_λ. Thus $f^{-1}(V) \cap A = \phi \Rightarrow a_\lambda \notin c(A)$. It follows that $a_\lambda \in c(A)$ implies $f(a_\lambda) \in c_2 f(A)$. □

Corollary 5.2.10. *A mapping f of a fcs (X, c_1) into a space (Y, c_2) is a fuzzy morphism iff for each a_λ in X, the inverse image of every fuzzy c-neighbourhood of $f(a_\lambda)$ is a fuzzy c-neighbourhood of a_λ or equivalently, every fuzzy c-neighbourhood of $f(a_\lambda)$ contains the image of a fuzzy c-neighbourhood of a_λ.*

Proof. Trivial using theorem 5.2.9. □

Result 5.2.11. *If 'f' is a fuzzy morphism of a fcs (X, c_1) into fcs (Y, c_2) then the inverse image of each f-open (f-closed) subset of X is an f-open (f-closed) subset of Y.*

Proof. If U is f-open in (X, c_1) then U is a fc-neighbourhood of each of its points and by above theorem $f^{-1}(U)$ is a neighbourhood of each of its points that is, $f^{-1}(U)$ is f-open. Similarly we can prove f-closed by taking its complement. □

5.3 Separation axioms in fuzzy closure fuzzy convexity spaces

In this section we study some separation axioms in fuzzy closure fuzzy convexity spaces and some of their basic properties.

Definition 5.3.1. Let $(X, \mathscr{L}, c)$ be a fuzzy closure fuzzy convexity spaces. Then $(X, \mathscr{L}. c)$ is said to be,

(i) $FCNS_0$ if for any two distinct fuzzy points there exists a closed convex fuzzy c-neighbourhood containing one and not containing the other.

(ii) $FCNS_1$, if for any two distinct fuzzy points there exists a closed convex fuzzy c-neighbourhood of each of them not containing the other.

(iii) $FCNS_2$, if for any two distinct fuzzy points there exist a disjoint closed convex fuzzy c-neighbourhood of each of them.

From the above definition it is clear that in $(X, \mathscr{L}, c)$ $FCNS_2 \Rightarrow FCNS_1 \Rightarrow FCNS_0$.

Eg:-

(1) Let $X = N$, the set of natural numbers.

$L = \{\phi, X\} \cup \{\{x\} | x \in X\}$ and the fuzzy closure operator 'c' on X defined by $cA = A$ for all fuzzy subset A of X. Then $(X, \mathscr{L}, c)$ is $FCNS_2$.

(2) Let $X = \{a, b, c\}$.

$L = \left\{\phi, X, \{a\}, \begin{matrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \end{matrix}, a_{1/2}\right\}$ and the fuzzy closure operator 'c' on X

defined by,

$$c\begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix};$$

$$c\begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix};$$

$$c\begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix};$$

$$c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix}; c\phi = \phi \text{ and in all other cases } cA = X$$

where A is fuzzy subset of X. Then $(X, \mathscr{L}, c)$ is $FCNS_0$ but not $FCNS_1$.

(3) Let $X = \{a, b, c\}$.

$L = \left\{\phi, X, \{a\}, \{b, c\}, \begin{matrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \end{matrix}, \begin{matrix} a \longrightarrow 1/2 \\ c \longrightarrow 1 \end{matrix}, a_{1/2}, \{b\}, \{c\}\right\}$ and the fuzzy closure operator 'c' on X defined by,

$$c\begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix};$$

$$c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix};$$

$$c\begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix};$$

$$c\begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix};$$

$$c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{pmatrix};$$

$c\phi = \phi$ and $cA = X$ in all other cases where A is a fuzzy subset of X. Then $(X, \mathscr{L}, c)$ is $FCNS_1$.

Proposition 5.3.2. *Any subspace of an $FCNS_i$ space is $FCNS_i$ for every $i = 0, 1, 2$.*

Proof. Let $i = 2$ and $(X, \mathscr{L}, c)$ be an $FCNS_2$ space and let $(M, \mathscr{L}_M, c_M)$ be a subspace of $(X, \mathscr{L}, c)$. Let a_λ, b_μ be two distinct fuzzy points in M. Then a_λ, b_μ can be considered as distinct fuzzy points in X and X is $FCNS_2$. Therefore there exist disjoint closed convex fuzzy c-neighbourhood U and V for a_λ and b_μ respectively. Then $U \cap M$ and $V \cap M$ are disjoint closed convex fuzzy c-neighbourhood of a_λ and b_μ in $(M, \mathscr{L}_M, c_M)$. $\therefore$

$(M, \mathscr{L}_M, c_M)$ is $FCNS_2$. Similarly $i = 0, 1$. □

Result 5.3.3. *Let (Y, c') be a subspace of a fcs (X, c) then $U \cap Y$ is closed in (Y, c') if U is closed in (X, c).*

Proof. Let U is closed in (X, c). That is $cU = U$, then $c'(U \cap Y) = Y \cap C(U \cap Y) \subseteq Y \cap U$. But (Y, c') is fcs, therefore $c'(U \cap Y) \supset Y \cap U$. Thus $c'(U \cap Y) = U \cap Y$. Hence $U \cap Y$ is closed in (Y, c'). □

Proposition 5.3.4. *A non-empty product is $FCNS_i$ if each factor is $FCNS_i$ for every $i = 0, 1, 2$.*

Proof. Let $i = 2$ and $(X_\alpha, \mathscr{L}_\alpha, c_\alpha)_{\alpha \in I}$ be a family of $FCNS_2$ spaces. Let a_λ, b_μ be two distinct fuzzy points in X, where $(X, \mathscr{L}, c) = \prod_{\alpha \in I}(X_\alpha, \mathscr{L}_\alpha, c_\alpha)$. Then for some α, $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ are distinct fuzzy points in X_α and each X_α is $FCNS_2$, then there exist disjoint closed convex fuzzy c-neighbourhood U_α and V_α in X_α for $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ respectively. Then $U = \pi_\alpha^{-1}(U_\alpha)$ and $V = \pi_\alpha^{-1}(V_\alpha)$ are disjoint closed convex fuzzy c-neighbourhood in X of a_λ and b_μ respectively. There fore $(X, \mathscr{L}, c)$ is $FCNS_2$. Similarly $i = 0, 1$. □

Definition 5.3.5. A $(fc)(fco)s$ $(X, \mathscr{L}, c)$ is pseudo $FCNS_3$ if for each closed convex fuzzy set A in X and a fuzzy point a_λ (not in it) such that the supports of a_λ and A are disjoint, then there exist a closed convex fuzzy c-neighbourhood V of A such that $a_\lambda \notin V$.

Definition 5.3.6. A $(fc)(fco)s$ $(X, \mathscr{L}, c)$ is $FCNS_3$ if for each closed convex fuzzy set A in X and a fuzzy point a_λ (not in it) such that the supports of a_λ and A are disjoint, then there exist a disjoint closed convex fuzzy c-neighbourhood V of a_λ and V of A.

Note 5.3.7. From the above definition it is clear that $FCNS_3 \Rightarrow$ Pseudo $FCNS_3$

Eg:-

1. Let X be any set. $\mathscr{L} = \{\phi, X\} \cup \{\{x\} | x \in X\}$ and the fuzzy closure operator 'c' on X defined by $c(A) = A$ for all fuzzy subset A of X. Then $(X, \mathscr{L}, c)$ is $FCNS_3$.

2. Let $X = \{a, b, c\}$.

$L = \{\phi, X, \{a\}\}$ and the fuzzy closure operator 'c' on X defined by

$$c\begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{pmatrix}; c\begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 0 \\ c \longrightarrow 0 \end{pmatrix}; c\phi = \phi \text{ and}$$

$cA = X$ in all other cases where A is a fuzzy subset of X. Then $(X, \mathscr{L}, c)$ is pseudo $FCNS_3$ but not $FCNS_3$.

Proposition 5.3.8. *A non-empty product of $FCNS_3$ spaces is an $FCNS_3$ space.*

Proof. Let $(X_\alpha, \mathscr{L}_\alpha, c_\alpha)$ be a family of $FCNS_3$ spaces. Let $(X, \mathscr{L}, c) = \prod_{\alpha \in I}(X_\alpha, \mathscr{L}_\alpha, c_\alpha)$. Let a_λ be a fuzzy point in X and A a closed convex fuzzy set in X such that the supports of a_λ and A are disjoint. Let π_α be the projection from X to X_α. Then we can take A as $A = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(U_\alpha)$, where U_α is a closed convex fuzzy set in X_α. Now for some α, the supports of $(a_\alpha)_\lambda$ and U_α are disjoint. Since X_α is $FCNS_3$, there exist closed convex fuzzy neighbourhoods V_α of $(a_\alpha)_\lambda$ and W_α of U_α such that $(a_\alpha)_\lambda \notin W_\alpha$ and V_α and U_α are disjoint. Then $U = \pi_\alpha^{-1}(V_\alpha)$ and $W = \pi_\alpha^{-1}(W_\alpha)$ are closed convex fuzzy c-neighbourhoods of a_λ and A in X respectively. Such that $a_\lambda \notin W$ and U and A are disjoint. □

Proposition 5.3.9. *A non-empty product of pseudo $FCNS_3$ spaces is a pseudo $FCNS_3$ space.*

Proof. Proof is exactly similar to proof of proposition 5.3.8. □

Definition 5.3.10. A $(fc)(fco)s$, $(X, \mathscr{L}, c)$ is semi $FCNS_4$ if for each pair of disjoint closed convex fuzzy sets in X there is a closed convex fuzzy c-neighbourhood U of one of the closed convex fuzzy set such that U and the other are disjoint.

Definition 5.3.11. A $(fc)(fco)s$, $(X, \mathscr{L}, c)$ is $FCNS_4$ if for each pair of disjoint closed convex fuzzy sets A and B in X there is a c-closed convex fuzzy neighbourhoods U of A and V of B respectively, such that U and B are disjoint and A and V are disjoint.

Note 5.3.12. From the above definition it is clear that $FCNS_4 \Rightarrow$ semi $FCNS_4$.

Eg:-

1. Let X be any set. $L = \{\phi, X\} \cup \{\{x\} | x \in X\}$ and the fuzzy closure operator c on X defined by $c(A) = A$ for all fuzzy subset A of X. Then $(X, \mathscr{L}, c)$ is $FCNS_4$.

2. Let $X = N$, the set of natural numbers.

$L = X \cup \{Y \subset I^X | Y \subseteq K\}$ and the fuzzy closure operator c on X is defined by,

$$c\begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 2;$$

$$c\begin{pmatrix} 1 \longrightarrow 0 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 0 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 3;$$

$$c\begin{pmatrix} 1 \longrightarrow 1/4 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/4 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 3;$$

$$c\begin{pmatrix} 1 \longrightarrow 1/2 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/2 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 2;$$

$$c\begin{pmatrix} 1 \longrightarrow 2/3 \\ x \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 2/3 \\ x \longrightarrow 1 \end{pmatrix} \text{ for all } x \geq 2;$$

$$c\begin{pmatrix} 1 \longrightarrow 1/2 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/2 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 1 \end{pmatrix} \text{ for all } x \geq 3;$$

$c(x \to 0) = (x \to 0)$ for all x; $c\phi = \phi$ and in all other cases $cA = X$ where A is a fuzzy subset of X.

$\begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \longrightarrow 0 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix}$ are disjoint closed convex fuzzy sets in X. Also $\begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \longrightarrow 1/2 \\ x \longrightarrow 0 \end{pmatrix}$ are disjoint closed convex fuzzy sets. And also there are no other disjoint closed convex fuzzy set in X. Now

$$\begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \subset \begin{pmatrix} 1 \longrightarrow 1/2 \\ x \longrightarrow 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \subset \begin{pmatrix} 1 \longrightarrow 1/3 \\ x \longrightarrow 0 \end{pmatrix} \subset \begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix}$$

where $\begin{pmatrix} 1 \longrightarrow 1/3 \\ x \longrightarrow 0 \end{pmatrix}$ is f-open in (X, c). $\therefore X$ is semi $FCNS_4$. But there is no closed convex fuzzy c-neighbourhood of $\begin{pmatrix} 1 \longrightarrow 0 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix}$ in X and hence X is not $FCNS_4$.

5.4 Conclusion

In this chapter, we have studied fuzzy closure, fuzzy convexity spaces and locally fuzzy closure fuzzy convexity spaces. Also we defined separation axioms in fuzzy closure fuzzy convexity spaces.

Chapter 6

Relationship between $(ft)(fco)s$ and $(fc)(fco)s$

Introduction

In this chapter we wish to introduce fuzzy closured convexity space and find the relationship between fuzzy topological convexity space and fuzzy closured convexity spaces. Also we find the relationship between fuzzy closure fuzzy convexity space and fuzzy topological fuzzy convexity space.

In [ROS-1], M. V. Rosa introduced fuzzy topology fuzzy convexity spaces (or $(ft)(fco)s$ in short) and in chapter 5 we introduced fuzzy closure fuzzy convexity space (or (fc) $(fco)s$ in short). These motivated the study of relationship between $(ft)(fco)s$ and $(fc)(fcos)$. In this chapter we define fuzzy closure convexity spaces (or $fc - cos$ in short) using this definition and fuzzy topological convexity space defined by M. V. Rosa in [ROS-1] we find the relationship between $ft - cos$ and $fc - fco$ spaces.

In section 6.1 we define fuzzy closured convexity space and some of its properties. Also we find the relationship between $ft - cos$ and $fc - cos$.

In section 6.2 we find the relationship between $(ft)(fco)s$ and (fc) $(fco)s$. Here we discuss the relationship between the notions of (ft) $(fco)s$ and $(fc)(fco)s$. And also we compare their properties.

6.1 Fuzzy closured convexity spaces

In this section we define fuzzy closured convexity spaces which is analogous to the definition given by M. V. Rosa in [ROS-1]: "Let X be a set with a fuzzy topology T and a fuzzy convexity $\mathscr{L}$. Then T is said to be compatible with $\mathscr{L}$ if the fuzzy convex hulls of finite fuzzy sets are fuzzy closed in (X, T). Then $(X, \mathscr{L}, T)$ is called a fuzzy topological convexity space (or ft-cos in short)" Also we find the relationship between the notions of $ft - cos$ and $fc - cos$ and also compare their properties.

Definition 6.1.1. Let X be a set with a fuzzy closure c and a fuzzy convexity $\mathscr{L}$. Then 'c' is said to be compatible with $\mathscr{L}$ if the fuzzy convex hulls of finite fuzzy sets are fuzzy closed in (X, c). Then $(X, \mathscr{L}, c)$ is called a fuzzy closured convexity spaces (or $fc - cos$ in short).

Eg:-

Let $X = \{a, b, d\}$; $\mathscr{L} = \left\{ \begin{matrix} & a \longrightarrow 1 & a \longrightarrow 0 & a \longrightarrow 1/4 \\ \phi, X, & b \longrightarrow 1, & b \longrightarrow 0, & b \longrightarrow 1/4 \\ & d \longrightarrow 0 & d \longrightarrow 1 & d \longrightarrow 0 \end{matrix} \right\}$ and the fuzzy closure operator c on X is defined by $c\begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 0 \\ d \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 0 \\ b \longrightarrow 0 \\ d \longrightarrow 1 \end{pmatrix}$;

$c\begin{pmatrix} a \longrightarrow 1/4 \\ b \longrightarrow 1/4 \\ d \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/4 \\ b \longrightarrow 1/4 \\ d \longrightarrow 0 \end{pmatrix}$; $c\begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ d \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ d \longrightarrow 0 \end{pmatrix}$;

$c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1/2 \\ d \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1/2 \\ d \longrightarrow 1 \end{pmatrix}$; $c\begin{pmatrix} a \longrightarrow 1/4 \\ b \longrightarrow 1/4 \\ d \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/4 \\ b \longrightarrow 1/4 \\ d \longrightarrow 1 \end{pmatrix}$;

$c\begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1/2 \\ d \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} a \longrightarrow 1/2 \\ b \longrightarrow 1/2 \\ d \longrightarrow 0 \end{pmatrix}$; $c\phi = \phi$ and $cA = X$ in all other cases where A is fuzzy subset of X. Here $(X, \mathscr{L}, c)$ is $fc - cos$.

Remark 6.1.2. From the definitions of $fc - cos$ and $(fc)(fco)s$ (given in chapter 5) it is clear that $fc - cos$ are $(fc)(fco)s$. But converse is not true in general.

Eg:-

Let $X = N$, the set of natural numbers.

$L = X \cup \{Y \subset I^X | Y \subseteq K\}$ and the fuzzy closure operator c on X is defined by,

$$c\begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 2;$$

$$c\begin{pmatrix} 1 \longrightarrow 0 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 0 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 3;$$

$$c\begin{pmatrix} 1 \longrightarrow 1/4 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/4 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 3;$$

$$c\begin{pmatrix} 1 \longrightarrow 1/2 \\ x \longrightarrow 0 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/2 \\ x \longrightarrow 0 \end{pmatrix} \text{ for all } x \geq 2;$$

$$c\begin{pmatrix} 1 \longrightarrow 2/3 \\ x \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 2/3 \\ x \longrightarrow 1 \end{pmatrix} \text{ for all } x \geq 2;$$

$$c\begin{pmatrix} 1 \longrightarrow 1/2 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 1 \end{pmatrix} = \begin{pmatrix} 1 \longrightarrow 1/2 \\ 2 \longrightarrow 1/4 \\ x \longrightarrow 1 \end{pmatrix} \text{ for all } x \geq 3;$$

$c(x \rightarrow 0) = (x \rightarrow 0)$ for all x; $c\phi = \phi$ and in all other cases $cA = X$ where A is a fuzzy subset of X. Here $(X, \mathscr{L}, c)$ is an $(fc)(fco)s$ but not an $fc - cos$, since

$1 \longrightarrow 1/2$

$2 \longrightarrow 1/2$ is convex but not closed.

$x \longrightarrow 0$

Proposition 6.1.3. *Subspace of an $fc - cos$ is an $fc - cos$.*

Proof. Let $(X, \mathscr{L}, c)$ be a $fc - cos$. Let $(Y, \mathscr{L}_Y, c_Y)$ be a subspace of $(X, \mathscr{L}, c)$ then $(Y, \mathscr{L}_Y, c_Y)$ is a $(fc)(fco)s$. Next we have to show that it is an $fc - cos$. For that let 'A' be the fuzzy convex hull of a fuzzy set generated by a finite fuzzy set with support $\{a_1, a_2, \dots a_k\}$ in Y. If $L \in \mathscr{L}$ be the fuzzy convex hull of $\{a_1, a_2, \dots a_k\}$ in X then $A = Y \cap L$ since A is a convex fuzzy set in Y. Since X is $fc - cos$, L is fuzzy closed in X and hence $A = Y \cap L$ is fuzzy closed in Y. Thus $(Y, \mathscr{L}_Y, c_Y)$ is a $fc - cos$. □

Remark 6.1.4. Let X be a set with a fuzzy topology T and a fuzzy convexity $\mathscr{L}$. Then T is said to be compatible with $\mathscr{L}$, if the fuzzy convex hulls of finite fuzzy sets are fuzzy closed in (X, T). Then $(X, \mathscr{L}, T)$ is called a fuzzy topological convexity space (or $ft - cos$ in short). This definition is given by M. V. Rosa in [ROS-1]. Also by the definition of fuzzy closured convexity space (or $fc - cos$) given in 6.1.1 we can conclude that $ft - cos \Rightarrow fc - cos$, if T is the associated fuzzy topology of the fuzzy closure space (X, c).

Proposition 6.1.5. *Subspace of an $ft - cos$ is $ft - cos$ and hence $fc - cos$ where c having associated fuzzy topology T.*

Proof. In [ROS-1] M. V. Rosa proved that subspace of an $ft-cos$ is $ft-cos$ and by remark 6.1.4 $ft-cos$ implies $fc-cos$. Hence the proposition. □

Proposition 6.1.6. *Quotient of a $ft-cos$ is $ft-cos$ and hence $fc-cos$ if X is finite.*

Proof. In [ROS-1] M. V. Rosa proved that Quotient of a $ft-cos$ is $ft-cos$ and by remark 6.1.4 $ft-cos$ implies $fc-cos$. Hence the proposition. □

6.2 Relationship between $(ft)(fco)s$ and $(fc)(fco)s$

Remark 6.2.1. A triple $(X, \mathscr{L}, T)$ consisting of a set X, a fuzzy alignment $\mathscr{L}$ and fuzzy topology T is called an $(ft)(fco)s$. This definition is given by M. V. Rosa in [ROS-1]. In this definition if 'T' is replaced by 'c', the fuzzy closure operator then the triple $(X, \mathscr{L}, c)$ is called a $(fc)(fco)s$ (given in 5.1.1). Here if T is the associated fuzzy topology of fuzzy closure space (X, c) then we can conclude that, $(ft)(fco)s$ implies $(fc)(fco)s$.

Remark 6.2.2. A fuzzy topological fuzzy convexity space (or $(ft)(fco)s$) $(X, \mathscr{L}, T)$ is said to be

(i) FNS_0, if for any two distinct fuzzy points there exist a closed convex fuzzy neighbourhood containing one and not containing the other.

(ii) FNS_1, if for any two distinct fuzzy points there exist a closed convex fuzzy neighbourhood of each of them not containing the other.

(iii) FNS_2, if for any two distinct fuzzy points there exist a disjoint closed convex fuzzy neighbourhood of each of them.

These definitions were given by M. V. Rosa in [ROS-1] and the analogous definitions of $FCNS_0$, $FCNS_1$ and $FCNS_2$ given in chapter 5 (5.3.1) where T is the associated fuzzy topology of the fuzzy closure space (X, c) then we can conclude that,

(i) $FNS_0 \Rightarrow FCNS_0$

(ii) $FNS_1 \Rightarrow FCNS_1$

(iii) $FNS_2 \Rightarrow FCNS_2$.

Proposition 6.2.3. *Any subspace of an FNS_i space is FNS_i and hence $FCNS_i$ for all $i = 0, 1, 2$.*

Proof. In [ROS-1] M. V. Rosa proved that subspace of a FNS_i space is FNS_i and by remark 6.2.2 we have the proposition. □

Proposition 6.2.4. *A non-empty product space is FNS_i and hence $FCNS_i$ if each factor is FNS_i for all $i = 0, 1, 2$.*

Proof. In [ROS-1] M. V. Rosa proved that a non-empty product is FNS_i if each factor is FNS_i for all $i = 0, 1, 2$ and by remark 6.2.2 we have the proposition. □

Remark 6.2.5. A $(ft)(fco)s$ $(X, \mathscr{L}, T)$ is

(i) pseudo FNS_3, if for each closed convex fuzzy set A in X and a fuzzy point a_λ (not in it) such that the supports of a_λ and A are disjoint, then there exist a closed convex fuzzy neighbourhood V of A such that $a_\lambda \notin V$.

(ii) FNS_3, if for each closed convex fuzzy set A in X and a fuzzy point a_λ (not in it) such that the supports of a_λ and A are disjoint, then there exist a disjoint closed convex fuzzy neighbourhood V of a_λ and V of A.

These definitions were given by M. V. Rosa in [ROS-1] and the analogous definitions pseudo $FCNS_3$, $FCNS_3$ given in chapter 5 (5.3.5 and 5.3.6) where T is the associated fuzzy topology of the fuzzy closure space (X, c) then we can conclude that, pseudo $FNS_3 \Rightarrow$ pseudo $FCNS_3$ and $FNS_3 \Rightarrow FCNS_3$

Proposition 6.2.6. *(1) Fuzzy closed fuzzy convex subspaces of FNS_3 space is FNS_3 and hence $FCNS_3$.*

(2) Fuzzy closed fuzzy convex subspace of a pseudo FNS_3 space is pseudo FNS_3 and hence pseudo $FCNS_3$.

(3) A non-empty product of FNS_3 space is an FNS_3 space and hence $FCNS_3$ space.

(4) A non-empty product of pseudo FNS_3 space is a pseudo FNS_3 space and hence $FCNS_3$ space.

(5) The quotient of an FNS_3, pseudo FNS_3 space is FNS_3, pseudo FNS_3 respectively and hence $FCNS_3$, pseudo $FCNS_3$ respectively if the quotient map is an FCC, F-closed and F-open map.

Proof. First part of the proposition where proved by M. V. Rosa in [ROS-1] and use the remark 6.2.5 we have the propositions. □

Remark 6.2.7. A $(ft)(fco)s$ $(X, \mathscr{L}, T)$ is

(i) semi FNS_4, if for each pair of disjoint closed convex fuzzy sets in X there is a closed convex fuzzy neighbourhood U of one of the closed convex fuzzy set such that U and the other are disjoint.

(ii) semi FNS_4, if for each pair of disjoint closed convex fuzzy sets A and B in X there is a closed convex fuzzy neighbourhood U of A and V of B respectively, such that U and B are disjoint and A and V are disjoint.

These definitions are given by M. V. Rosa in [ROS-1] and the analogous definitions of semi $FCNS_4$, $FCNS_4$ given in chapter 5 (5.3.10 and 5.3.11) where T is the associated fuzzy topology of the fuzzy closure space (X, c). Hence we can conclude that, semi $FNS_4 \Rightarrow$ semi $FCNS_4$ and $FNS_4 \Rightarrow FCNS_4$.

Proposition 6.2.8. *Fuzzy closed fuzzy convex subspace of an FNS_4, semi FNS_4 space is FNS_4, semi FNS_4 respectively and hence $FCNS_4$, semi $FCNS_4$ respectively.*

Proof. First part of the proposition where proved by M. V. Rosa in [ROS-1] and use the remark 6.2.7 we have the proposition. □

Proposition 6.2.9. *The quotient of an FNS_4, semi FNS_4 space is FNS_4, semi FNS_4 respectively and hence $FCNS_4$, semi $FCNS_4$ respectively if the quotient map is an F-closed, FCC and F-open map.*

Proof. In [ROS-1] M. V. Rosa proved that the quotient of an FNS_4, semi

FNS_4 space is FNS_4, semi FNS_4 respectively. And by remark 6.2.7 we have the proposition. □

6.3 Conclusion

In this chapter we defined fuzzy closured convexity spaces and obtained relationship between fuzzy closured convexity spaces and fuzzy topological convexity spaces and between fuzzy closure fuzzy convexity spaces and fuzzy topological fuzzy convexity spaces.

Many of the problems investigated in this thesis are to the extent of introducing different possible directions of study only, much more can be done in each one of them.

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