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Convex extendable trees

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Abstract

The concept of convex extendability is introduced to answer the problem of finding the smallest distance convex simple graph containing a given tree. A problem of similar type with respect to minimal path convexity is also discussed. © 1999 Elsevier Science B.V. All rights reserved

Keywords: Geodesic convexity; Minimal path convexity; Distance convex simple graphs; Convex extendable trees

1. Introduction

We consider only finite, simple, undirected graphs G of order p and size q. Let us denote, $\mathrm{diam}(G) = \max\{d(u,v),\ \ u,v \in V(G)\}$, $N(u)$ the neighborhood of $u = \{v\colon d(u,v) = 1\}$, $N[u] = \{u\} \cup N(u)$, $N_k(u) = \{v\colon d(u,v) = k\}$ for $k = 1,2,\ldots,\mathrm{diam}(G)$, and $C(G)$ the center of G. The definitions and terms not mentioned here are from [2].

Of concern in this paper are the geodesic convexity and minimal path convexity defined for the vertex set of a connected graph. In a connected graph G with its intrinsic metric d, Mulder [9] has defined the interval between u and v as

$$I(u,v) = \{x\colon x \text{ is on a shortest } u - v \text{ path}\}.$$

$A \subseteq V(G)$ is geodesic convex (d-convex) if $I(u,v) \subseteq A$ for every u and v in A. Several aspects of geodesic convexity in graphs have been discussed by Soltan [11], Hebbare [8], Rao and Hebbare [10], Mulder [9] and Van de Vel [12].

$A \subseteq V(G)$ is minimal path convex (m-convex) if $I(u,v) = \{x\colon x$ is on a chordless $u - v$ path$\} \subseteq A$ for any u and v in A. Separation properties, evaluation of convex

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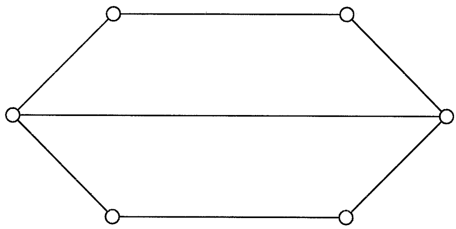

Fig. 1.

invariants, etc. have been studied by Farber and Jamison [6], Duchet [4,5], Bandelt [1] and many others.

In an attempt to classify graphs according to the number of non-trivial convex sets, Hebbare [7] called the empty set, singleton of vertices, set of vertices inducing a complete subgraph and $V(G)$ as trivial convex sets and defined a graph to be (k,ω)-convex if it has k non-trivial convex sets and its clique number, the size of the largest clique, is ω. $(0,2)$-convex graphs were called distance convex simple (d.c.s.) and m-convex simple (m.c.s.) [3] under geodesic convexity and m-convexity, respectively. An in-depth study of d.c.s. graphs have been made in [8,10] especially under planarity.

$K_{m,n}$ for $m,n>1$ are d.c.s. and any d.c.s. graph is m.c.s. The graph G of Fig. 1 is an m.c.s. graph which is not d.c.s.

In this paper we first consider a problem posed in [8]. 'Describe the smallest d.c.s. graph containing a given tree of order at least four'. This problem motivates the definition of a convex extendable tree and we prove that, any tree of order atmost nine is so, this bound is sharp, trees of diameter three, five and trees of diameter four whose central vertex has even degree are also convex extendable. An analogous problem for m.c.s. graph is also discussed.

2. Convex extendable trees

The following properties of d.c.s. graphs are of interest to us.

Theorem 1 (Hebbare [8]). *A distance convex simple graph is planar if and only if* $q=2p-4$.

Theorem 2 (Hebbare [8]). *A connected planar graph of order at least four is distance convex simple if and only if for each vertex u of degree at least three, there is a unique vertex* u' *such that* $N(u)=N(u')$.

Two such vertices u and u' are called partners.

Problem (Hebbare [8]). Describe the smallest distance convex simple graph containing a given tree of order at least four.

$K_{2,n}$ is such a graph for $K_{1,n}$. For a tree T which is not a star, let V_1 and V_2 be the bipartition of $V(T)$ with $|V_1|=m$, $|V_2|=n$, then $K_{m,n}$ is a d.c.s. graph containing a tree isomorphic to T. However, to find the smallest d.c.s. graph we note by Theorem 1 that, for any d.c.s. graph $q \geqslant 2p-4$ and the lower bound is attained if and only if it is planar. So, for a given tree T if there exists a planar d.c.s. graph containing T as a spanning subgraph, then that will be the smallest d.c.s. graph containing T. This observation motivates,

Definition 1. A tree T is convex extendable if it is a spanning tree of a distance convex simple graph.

From the remarks made above, it is clear that $K_{1,n}$ is convex non-extendable. Hence, we consider only trees which are not stars.

Definition 2 (*Buckley and Harary* [2]) The sequential join $G_1+G_2+\cdots+G_n$ of the graphs $G_1, G_2, \ldots, G_n$ is the graph obtained by joining all the vertices of G_i to all the vertices of G_{i+1} for $i=1,2,\ldots,n-1$. The tree $S_{m,n} \simeq \bar{K}_m + K_1 + K_1 + \bar{K}_n$ is called a double star.

We shall now describe an operation frequently used in this paper. Let u and v be non-adjacent vertices of G. Join u to all the vertices in $N(v)$ and v to all the vertices in $N(u)$. The resulting graph is denoted by $G \star (u,v)$ and in this graph $N(u)=N(v)$.

Remark 2.1. If G is planar, $u, v \in V(G)$, $uv \notin E(G)$, for any $w_1, w_2 \in N(u) \cup N(v)$, $w_1 \notin N(u) \cap N(v), \{u,v\}$ is a $w_1 - w_2$ separator and if G can be embedded so that $u, v, N(u)$ and $N(v)$ are all in the same face, then $G \star (u,v)$ is planar. Also, if u and v are partners, then $G \star (u,v) \simeq G$.

Lemma 3. *Any path of length at least four is convex extendable.*

Proof. Let P be a path of at least four, $C(P)$ be its center and let $u \in C(P)$. Then $N_i(u)$ consists of two non-adjacent vertices for $i=1,2,\ldots,r-1$ and $N_r(u)$ is either a pair of non-adjacent vertices or a singleton according as $C(P) \simeq K_1$ or K_2, where r is the radius of P. Now the graph

$$G = \langle u \rangle + \langle N(u) \rangle + \langle N_2(u) \rangle + \cdots + \langle N_r(u) \rangle$$

is a planar d.c.s. graph containing P. □

Theorem 4. *Any tree of order at most nine is convex extendable.*

Proof. If T is a path then it is convex extendable by Lemma 3. Suppose that T is not a path. Let u be a vertex of T such that $d(u) \geqslant 3$ and let $N(u) = \{a_1, b_1, \ldots, a_n\}$, $n \geqslant 3$.

Case I: $N_3(u) = \phi$. Assume that $d(a_1) = \min\{d(a_i)\colon a_i \in N(u)\}$. Choose $u' \in N_2(u)$ such that $N_2(u) \cap N(a_1) \backslash \{u'\} = \phi$. Construct $G \simeq T \star (u, u') \star (a_1, a_2) \star \cdots \star (a_{n-1}, a_n)$ if n is even and $G \simeq T \star (u, u') \star (a_2, a_3) \star \cdots \star (a_{n-1}, a_n)$ if n is odd. Using Theorem 2 and Remark 1, it follows that G is a planar d.c.s. graph which contains T.

Case II: $N_3(u) \neq \phi$. Choose $u' \in N_2(u)$ such that $d(u') = \max\{d(v)\colon v \in N_2(u)\}$ and let $N = N(u) \cup N(u') = \{v_1, v_2, \ldots, v_m\}$. Note that, $m > 3$. Since $|V(T)| \leqslant 9$, $N(v_i) - \{u, u'\} = \phi$ for at least one value of i.

Subcase 1: $N[u] \cup N[u'] = V(T)$. Then $T \star \langle u, u' \rangle \simeq K_{2, p-2}$ is such a planar d.c.s. graph.

Subcase 2: $N[u] \cup N[u'] \neq V(T)$, but $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) = V(T)$.

Without loss of generality, assume that $N(v_1) \backslash \{u, u'\} = \phi$. Then, the required graph is $T \star (u, u') \star (v_1, v_2) \star (v_3, v_4) \star \cdots \star (v_{m-1, v_m})$ if m is even and $T \star (u, u') \star (v_2, v_3) \star \cdots \star (v_{m-1}, v_m)$ if m is odd.

Subcase 3: $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \neq V(T)$ but $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \cup (\bigcup_{i=1}^{m} N_2(v_i)) = V(T)$.

Here, note that $N(v_i) \backslash \{u, u'\} \neq \phi$ for atmost two values of i say 1 and 2. Let $w_1 \in N(v_i) - \{u, u'\}$ be such that $d(w_1) \geqslant 2$. Since $|V(T)| \leqslant 9$, $d(w_1)$ cannot exceed three. If $d(w_1) = 3$, by the choice of u', $w_1 \in N_4(u)$ in T and let $u - v_2 - u' - v_1 - w_1$, be the $u - w_1$ path in T (that is $v_1 \in N(u)$ and $v_2 \in N(u')$).

Now $G \simeq T \star (u, w_1) \star (v_1, v_2)$ is the required planar d.c.s. graph.

If $d(w_1) = 2$, let $w_2 \in N(w_1) - \{v\}_1$, then $T \star (u, u') \star (w_2, v_1) \star (v_2, v_3)$ is the required graph.

Subcase 4: $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \cup (\bigcup_{i=1}^{m} N_2(v_i)) \neq V(T)$. Then $N[u] \cup N[u'] \cup (\bigcup_{i=1}^{m} N(v_i)) \cup (\bigcup_{i=1}^{m} N_2(v_i)) \cup (\bigcup_{i=1}^{m} N_3(v_i)) = V(T)$.

Note that, $N(v_i) - \{u, u'\} \neq \phi$ for only one value of i. There is only one vertex w_1 in it and there are two vertices w_2 and w_3 such that $w_1 w_2$ and $w_2 w_3 \in E(T)$. Then, $T \star (u, u') \star (v_1, w_2)$ is the required graph. Since $|V(T)| \leqslant 9$, the proof is complete. $\square$

Theorem 5. *The following classes of trees are convex extendable*:

(a) *Trees of diameter three.*

(b) *Trees of diameter four whose central vertex has even degree.*

(c) *Trees of diameter five.*

Proof

(a) Since T is of diameter three, $T \simeq S_{m,n}$ for some $m, n > 0$. Let $C(T) = \{c_1, c_2\}$, $V(\bar{K}_m) = \{a_1, a_2, \ldots, a_m\}$ and $v(\bar{K}_n) = \{b_1, b_2, \ldots, b_n\}$. Then $T \star (b_1, c_1) \star (a_1, c_2)$ is a planar d.c.s. graph containing T as a spanning tree.

(b) Let $\mathrm{diam}(T) = 4$ and the central vertex c has even degree. Let $N(c) = \{a_1, a_2, \ldots, a_n\}$ and $c' \in N_2(c)$. Then $T \star (c, c') \star (a_1, a_2) \star (a_3, a_4) \star \cdots \star (a_{n-1}, a_n)$ is the required graph.

(c) Proof is on similar lines. $\square$

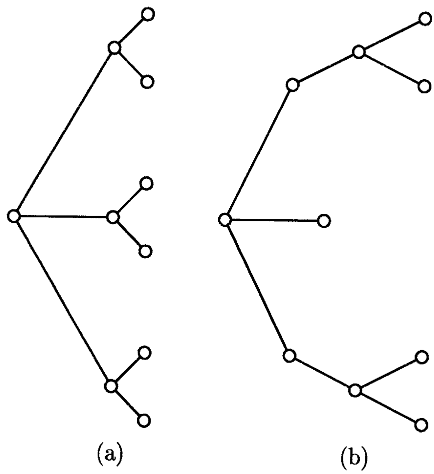

Fig. 2. Convex non-extendable trees of order 10.

Remark 2.1.
(i) In (b), if the central vertex has odd degree, the result need not be true (Fig. 2a).
(ii) There exists convex non-extendable trees of diameter six (Fig. 2b).
(iii) A sufficient condition for a tree to be convex non-extendable is that $V(T)$ has a bipartition V_1 and V_2 such that $|V_1|$ is odd and each vertex of V_1 is of degree greater than 2.

3. Minimal path convex simple graphs

If m-convexity is considered, $(0,2)$-convex graphs are called m-convex simple.

Theorem 6 (Changat [3]). *A connected graph G is m-convex simple if and only if G has no non-trivial clique separators.*

We consider a problem similar to the problem discussed in Section 2.

Problem. Find the smallest m.c.s. graph containing a given tree T, $|T|\geqslant 4$.

If $T=K_{1,n}$; $n\geqslant 3$, $K_{2,n}$ is such a graph and its size is $2n$.

Theorem 7. *The size q of the smallest m-convex simple graph containing a tree T $(\neq K_{1,n})$ satisfies, $p-1+m/2\leqslant q\leqslant p+m-2$, where $|V(T)|=p$ and m is the number of pendant vertices.*

Proof. Let u_1 be a pendant vertex of T and v be the vertex adjacent to u_1. Let $u_2,u_3,\ldots,u_k$ be the other pendant vertices adjacent to v. Let $v_1,v_2,\ldots,v_l$ be the pendant

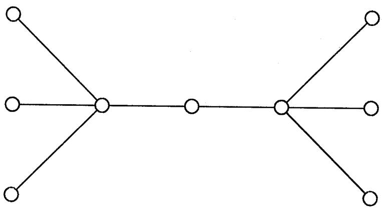

Fig. 3.

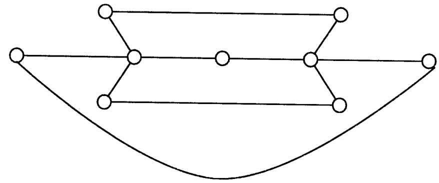

Fig. 4.

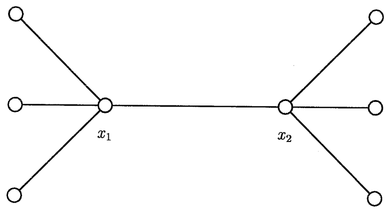

Fig. 5.

vertices other than u'_is. Add edges to T such that $\{u_2, u_3, \dots v_1, v_2, \dots, v_l\}$ induces a tree T' in which $\{u_2, u_3, \dots, u_k\}$ and $\{v_1, v_2, \dots, v_l\}$ is a bipartition. This is possible by taking a spanning tree of $K_{k,l}$. The resulting graph is triangle-free and neither a vertex nor an edge can separate G. So, by Theorem 5, G is an m.c.s. graph and size of G is $p-1+k+l-1=p+m-2$ where m is the number of pendant vertices of T. So, $q \leqslant p+m-2$.

Now, note that m.c.s. graphs are triangle-free blocks and hence all vertices are of degree at least two. Therefore, to make T a block, the degree of pendant vertex is to be increased by atleast one. So, atleast $[m/2]$ edges are to be added and hence $q \geqslant p-1+[m/2] \geqslant p-1+m/2$. $\square$

The following examples illustrate that there are trees attaining both the bounds.

Consider the tree T_1 in Fig. 3. Here $p=9$, $m=6$.

The graph G in Fig. 4 is an m.c.s. of size $q=11=p-1+m/2$ containing T_1.

Consider the tree T_2 of Fig. 5. In T_2, $\{x_1, x_2\}$ is a clique such that $T_2-\{x_1, x_2\}$ is totally disconnected. So, to get an m.c.s. graph atleast five edges are to be added. Hence $q=13=p+m-2$.

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