# STUDIES IN FUZZY COMMUTATIVE ALGEBRA

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Paul Isaac

# *Certificate*

This is to certify that thesis entitled **'Studies in Fuzzy Commutative Algebra'** is a bonafide record of the research work carried out by Mr. Paul Isaac under our supervision in the Department of Mathematics, Cochin University of Science and Technology. The result embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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# *Declaration*

I hereby declare that the work presented in this thesis entitled **'Studies in Fuzzy Commutative Algebra'** is based on the original work done by me under the supervision of Dr. R. S. Chakravarti and Prof. T. Thrivikraman, in the Department of Mathematics, Cochin University of Science and Technology, Kochi-22, Kerala; and no part thereof has been presented for the award of any other degree or diploma.

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# Chapter 1

# INTRODUCTION

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## 1. 1 History and Development.

A crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups - members and non members. A sharp unambiguous distinction exists between the members and nonmembers of the class or category represented by the crisp set. Many of the collections and categories we commonly employ, however, do not exhibit this characteristic. Instead their boundaries seem vague, and the transition from member to nonmember appears gradual rather than abrupt. Thus fuzzy set introduces vagueness by eliminating the sharp boundary dividing members of the class from nonmembers. Real situations are very often not crisp and deterministic and they can not be described precisely. Such situations in our real life which are characterized by vagueness or imprecision can not be answered just in yes or no. Lotfi A. Zadeh [76] in 1965 introduced the notion of a fuzzy set to describe vagueness mathematically in its very abstractness and tried to solve such problems by giving a certain grade of membership to each member of a given set. This in fact laid the foundation of fuzzy set theory. Zadeh has defined a fuzzy set as a generalisation of characteristic function of a set wherein the degree of membership of an element is more general than merely "yes" or "no". A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is compatible with the concept represented by the fuzzy set. The membership grades are very often represented by real number values ranging in the closed interval between 0 and 1. The nearer the value of an element to unity, the higher the grade of its membership. The term Fuzzy in the sense used here seems to have been first introduced by Zadeh [75] in 1962. In that paper Zadeh called for a mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions. This paper was followed in 1965 by the technical exposition of just such a mathematics now termed the "Theory of Fuzzy Sets"

Although the range of values between 0 and 1, both inclusive, is the most commonly used, for representing membership grades, an arbitrary set with some natural total/partial ordering can in fact be used. Elements of this set are not required to be numbers as long as the ordering among them can be interpreted as representing various strengths of membership degree. Thus the membership set can be any set that is at least partially ordered and the most frequently used membership set is a lattice. J. A. Goguen [14] in 1967 introduced the notion of a fuzzy set with a lattice as the membership set. Fuzzy sets defined with a lattice as the membership set are called $L$-fuzzy sets or $L$-sets, where $L$ is intended as an abbreviation for lattice.

The fuzzy set theory - a theory of graded concepts, a theory in which every thing is a matter of degree - can be considered as a generalisation of the classical set theory. Because of this, the fuzzy set theory has a wider scope of applicability than classical set theory in solving various problems. Since the inception of the theory of fuzzy sets, applications of this theory have mushroomed. Applications appear in computer science, artificial intelligence, decision analysis, information science, system science, control engineering, expert systems, pattern recognition, management science, operations research and robotics. Theoretical mathematics has also been touched by the concept of fuzziness.

Roughly speaking fuzzy set theory in the last three decades has developed along two lines.

(1) As a formal theory which became developed by generalising (fuzzifying) the original ideas and concepts in classical mathematical areas such as algebra, graph theory, topology and so on.

(2) As a very powerful modeling language, that can cope with a large fraction of uncertainties of real life situations. Because of its generality it can be well adapted to different circumstances and contexts.

Fuzzy set theory, a developing subject in Mathematics is making inroads into different disciplines of pure mathematics also. Among various branches of pure mathematics, algebra was one of the first few subjects where the notion of fuzzy set was applied. The first paper on fuzzy groups was published by A. Rosenfeld [62] in 1971, in which the concepts of fuzzy subgroupoid and fuzzy subgroups were introduced. In 1979 J.M. Anthony and H. Sherwood [1] redefined fuzzy subgroup under the so called triangular norm function and studied some results of Rosenfeld under this new notion. After a considerable period of time W. J. Liu [41] opened the way towards the development of fuzzy algebraic structures by introducing the notions of fuzzy normal subgroup, fuzzy subring and the product of fuzzy sets. Liu [42] introduced the notion of a fuzzy ideal of a ring. N. Kuroki [38] demonstrated the utility of the notion of the fuzzy set in the more general setting of semigroups. The concepts of fuzzy fields and fuzzy linear spaces were introduced by S. Nanda [54]. Ever since A. Rosenfeld introduced fuzzy sets in the realm of group theory, many researchers have been involved in extending the notions of abstract algebra to the broader framework of fuzzy setting. J. N. Mordeson, D. S. Malik, M. M. Zahedi, M. Das, M. K Chakraborty, B. B. Makamba, V. Murali, A. K. Katsaras, D. B. Liu, M. Asaad, P.S. Das, N. P. Mukherjee, P. Bhattacharya, F. I. Sidky, M. A. Mishref, and M. Akhul, T. K. Mukherjee, M. K. Sen, V. N. Dixit, N. Ajmal, R. Kumar are a few among the others who contributed a lot to the theory of fuzzy algebraic structures. As a consequence, a number of concepts have been formulated and explored.

The concept of fuzzy modules and $L$-modules were introduced by Negoita and Ralescu [56] and Mashinchi and Zahedi [47] respectively. Subsequently they were further studied by Golan [15], Muganda [51], Pan [58, 59, 60, 61], Zahedi and Ameri [78, 79, 80, 81]. The notion of free fuzzy modules was introduced by Muganda [51] in 1993 as an extension of free modules in the fuzzy context. In

1994 Zahedi and Ameri [80] introduced the concept of fuzzy exact sequences in the category of fuzzy modules and in 1995 they introduced the concepts of fuzzy projective and injective modules [81]. Tremendous and rapid growth of fuzzy algebraic concepts resulted in a vast literature. The book of Mordeson and Malik [49] gives an account of all these up to 1998. However many concepts are yet to be "fuzzified" The main objective of this thesis is to extend some basic concepts and results in module theory to the fuzzy setting.

## 1.2 Summary of the Thesis.

The thesis contains five chapters.

In this chapter we are giving the history and development of the subject, the summary of the thesis and the prerequisites including some basic definitions and results in fuzzy set theory which are required in the subsequent chapters.

In the second chapter after the introduction, in the second section we give the basic concepts of an $L$-module and give some definitions and results in this area contributed by pioneers in this field. In the third section we give some theory related to the $L$-submodules of a quotient module. In the next section we prove some results regarding direct sums of $L$-modules which include:

- If $L$ is regular and if $\mu, \eta, \nu \in L(M)$, (where $L(M)$ denotes set of all $L$-submodules of a module $M$) are such that $\mu = \eta \oplus \nu$, then $\mu^* = \eta^* \oplus \nu^*$ with a counter example to show that the converse need not be true. Here $\mu^*$, $\eta^*$ and $\nu^*$ respectively denote supports of $\mu$, $\eta$ and $\nu$.

- If $L$ satisfies the complete distributive property, $\mu_i$, $(i \in I)$ and $\lambda$ are elements in $L(M)$ such that $\sum_{i\in I} \mu_i$ is a direct sum $\bigoplus_{i\in I} \mu_i$; and if $\lambda \cap \sum_{i\in I} \mu_i = 1_{\{0\}}$, then $\lambda + \sum_{i\in I} \mu_i$ is a direct sum $\lambda \oplus (\bigoplus_{i\in I} \mu_i)$.

In the third chapter the first section is the introduction and in the second and third sections the concepts of simple and semisimple $L$-modules respectively are introduced and explored. In this chapter we prove some interesting results which include:

- Results analogous to the results "every submodule of a semisimple module is semisimple", "a semisimple module contains a simple submodule" in crisp theory.

We also prove that

- If $L$ is regular, then $M$ is simple if and only if $1_M$ is a simple left $L$-module.
- If $M$ is a left module over a ring $R$, then $M$ is semisimple if and only if $1_M$ is a semisimple left $L$-module.
- If $\mu \in L(M)$ is a semisimple $L$-module, then $\mu_a^>$ is a semisimple submodule of $M$ $\forall\ a \neq 0 \in L$.

Finally we get some equivalent conditions for $\mu \in L(M)$ to be semisimple.

In the fourth chapter the section after the introduction contains the concept of exact sequences of $L$-modules. Recall that a sequence of $R$-modules and $R$-module homomorphisms $\ldots \to A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \to \ldots$ is said to be exact at $A_i$ if Im $(f_i)$ = Ker$(f_{i+1})$. In this section we extend this concept to the fuzzy setting and prove some results in this direction. The following are some of the results we prove in this section.

- Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of $R$-modules exact at $B$ and let $\mu \in L(A)$, $\eta \in L(B)$, $\nu \in L(C)$. Then the sequence $\mu \xrightarrow{f} \eta \xrightarrow{g} \nu$ of $L$-modules is exact at $\eta$ only if $\mu^* \xrightarrow{f'} \eta^* \xrightarrow{g'} \nu^*$ is a sequence of $R$-modules exact at $\eta^*$, where $f'$ and $g'$ are restrictions of $f$ and $g$ to $\mu^*$ and $\eta^*$ respectively.
- Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence exact at $B$ and let $\mu \in L(A)$, $\eta \in L(B)$, $\nu \in L(C)$ be such that $\mu \xrightarrow{f} \eta \xrightarrow{g} \nu$ is a sequence of $L$-modules exact at $\eta$. Then $f(\mu_a^{>}) \subseteq$ Ker $g$ $\forall$ $a \in L$.

Also we define weakly isomorphic exact sequences of $L$-modules and get the conditions under which the exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$ is weakly isomorphic to the given exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$. We also get the conditions for the exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ to be weakly isomorphic to the exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$.

In the next section of this chapter we define split exact sequences of $L$-modules and establish a relation between semisimple $L$-modules and split exact sequences of $L$-modules.

In the fifth chapter after the introduction, in the second section we introduce the concept of projective $L$-modules and prove results in this context, some of which are:

- Every free $L$-module is a projective $L$-module.
- Let $P$ be a projective module and $\mu \in L(P)$ be a projective $L$-module. If $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is a short exact sequence of $R$-modules and $\eta \in L(A)$, $\nu \in L(B)$ are such that $0 \to \eta \xrightarrow{f} \nu \xrightarrow{g} \mu \to 0$ is a short exact sequence of $L$-modules, then $\eta \oplus \mu$ is weakly isomorphic to $\nu$. That is $\eta \oplus \mu \simeq \nu$.
- A projective $L$-module is the fuzzy direct summand of a free $L$-module.
- $\underset{i \in I}{\oplus} \mu_i$ is projective only if $\mu_i$ is projective $\forall\ i$.

In the next section we introduce the notion of injective $L$-modules and prove results regarding this concept, some of which are:

- Let $J$ be an injective module and $\mu \in L(J)$ be an injective $L$-module. If $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of $R$-modules and $\nu \in L(B)$, $\eta \in L(C)$ are such that $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is a short exact sequence of $L$-modules, then $\nu$ is weakly isomorphic to $\mu \oplus \eta$. That is $\nu \simeq \mu \oplus \eta$.

- Let $Q_\alpha$ $(\alpha\in I)$ be injective $R$-modules and $\mu_\alpha\in L(Q_\alpha)$ $(\alpha\in I)$ be $L$-modules. Then $\bigoplus_{\alpha\in I}\mu_\alpha\in L(\bigoplus_{\alpha\in I}Q_\alpha)$ is injective if and only if $\mu_\alpha$ is injective $\forall\ \alpha\in I$.

In the last section we define the concept of essential $L$-submodules of an $L$-module and prove that:

- If $L$ is regular, $\mu\in L(M)$, $1_{\{0\}}\neq\eta\subseteq\mu$ ; then $\eta\in L(M)$ is an essential $L$-submodule of $\mu$ if and only if for each $0\neq x\in M$, with $\mu(x)>0$, there exists an $r\in R$ such that $rx\neq 0$ and $\eta(rx)>0$.
- If $\eta$, $\nu$, $\mu\in L(M)$ satisfy $\eta\subseteq\nu\subseteq\mu$. Then $\eta\subseteq_e\mu$ if and only if $\eta\subseteq_e\nu$ and $\nu\subseteq_e\mu$. ( $\subseteq_e$ means 'is an essential submodule of'.)
- Let $\eta_1$, $\eta_2$, $\mu_1$, $\mu_2\in L(M)$. If $\eta_1\subseteq_e\mu_1$ and $\eta_2\subseteq_e\mu_2$, then $\eta_1\cap\eta_2\subseteq_e\mu_1\cap\mu_2$.
- Let $L$ be regular $\eta$, $\mu\in L(M)$ where $\eta\subseteq\mu$. Let $f$: $A\to M$ be a module homomorphism such that $f(\nu)\subseteq\mu$ where $\nu\in L(A)$. If $\eta\subseteq_e\mu$, then $f^{-1}(\eta)\subseteq_e\nu$.
- Let $L$ be regular and $\eta_1$, $\eta_2$, $\mu_1$, $\mu_2\in L(M)$ be such that $\eta_i\subseteq_e\mu_i$ ; $i=1, 2$. If $\eta_1\cap\eta_2=1_{\{0\}}$, then $\mu_1\cap\mu_2=1_{\{0\}}$ and $\eta_1\oplus\eta_2\subseteq_e\mu_1\oplus\mu_2$.

The thesis ends with a conclusion of the work done and further scope of the study.

Some of the results contained in this thesis have been presented in seminars/communicated to journals as given below :

(1). On $L$-modules, Proceedings of the National Conference on Mathematical Modeling, March 14-16, 2002; Baselius College, Kottayam, Kerala, India, 123-134.

(2). Simple and Semisimple $L$-modules, The Journal of Fuzzy Mathematics, 12(4); 2004 (to appear).

(3). On Semisimple $L$-Modules, Proceedings of the National Seminar on Graph Theory and Fuzzy Mathematics, August 28-30, 2003; Catholicate College, Pathanamthitta, Kerala, India, 81-90.

(4). Exact sequences of $L$-modules (communicated).

(5). Semisimple $L$-modules and Split Exact Sequences of $L$-modules (communicated).

(6). Free $L$-modules and Projective $L$-modules, Proceedings of the National Seminar on Fuzzy Mathematics and Applications, January 08-10, 2004; Payyannur College, Payyannur, Kerala, India.

(7). On Projective $L$-modules (to appear in Iranian Journal of Fuzzy Systems.).

(8). On Injective $L$- Modules (communicated).

(9). Essential $L$-Submodules of an $L$-Module (communicated).

## 1.3 Some Basic Definitions and Results of Fuzzy Set Theory.

In this section unless otherwise stated, $L(\vee, \wedge, 1, 0)$ represents a complete distributive lattice with maximal element '1' and minimal element '0'; '$\vee$' denotes the supremum and '$\wedge$' the infimum in $L$. $L$ is said to be regular if

$a \wedge b \neq 0 \ \forall \ a, b \in L$ such that $a \neq 0 \neq b$. The closed interval [0, 1] together with the operations 'min', 'max', and '$\leq$' form a complete distributive lattice. '$\subseteq$' denotes the inclusion and '$\subset$' the strict inclusion.

**1.3.1 Definition [49]:**

By an $L$-subset of $X$, we mean a mapping from $X$ into $L$. The set of all $L$-subsets of $X$ is called the $L$-power set of $X$ and is denoted by $L^X$. In particular, when $L$ is [0, 1], the $L$-subsets of $X$ are called fuzzy subsets.

**1.3.2 Definition [49]:**

Let $Y \subseteq X$ and $a \in L$. We define $a_Y \in L^X$ as follows.

$$a_Y(x) = \begin{cases} a & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

For $x \in X$, $a \in L$ the $L$-subset which takes the value $a$ at $x$ and 0 elsewhere is denoted by $a_{\{x\}}$. That is $a_{\{x\}}(y) = \begin{cases} a & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$

**1.3.3 Definition [49]:**

Let $\mu, \nu \in L^X$. If $\mu(x) \leq \nu(x) \ \forall \ x \in X$, then we say that $\mu$ is contained in $\nu$ (or $\nu$ contains $\mu$) and we write $\mu \subseteq \nu$ (or $\nu \supseteq \mu$). If $\mu \subseteq \nu$ and $\mu \neq \nu$, then $\mu$ is said to be properly contained in $\nu$ (or $\nu$ properly contains $\mu$) and we write $\mu \subset \nu$.

**1.3.4 Definition [49]:**

Let $\mu, \nu \in L^X$. Define $\mu \cup \nu$ and $\mu \cap \nu$ in $L^X$ as follows: $\forall \ x \in X$

$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$ and

$(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$.

Then $\mu \cup \nu$ and $\mu \cap \nu$ are called the union and intersection of $\mu$ and $\nu$ respectively.

**1.3.5 Definition [49]:**

For $\mu \in L^X$, we define the following.

(i) $\{\mu(x) : x \in X\}$ is called the image of $\mu$, and is denoted by $\mu(X)$ or $\mathrm{Im}(\mu)$;

(ii) $\mu^* = \{x \in X : \mu(x) > 0\}$; called the support of $\mu$.;

(iii) For $a \in L$, $\mu_a = \{x \in X : \mu(x) \geq a\}$; called the $a$- cut or $a$-level subset of $\mu$ and $\mu_a^> = \{x \in X : \mu(x) > a\}$; called the strict $a$-cut or strict $a$- level subset of $\mu$.

It is easy to verify that for any $\mu$, $\nu \in L^X$,

(i) $\mu \subseteq \nu$, $a \in L \Rightarrow \mu_a \subseteq \nu_a$,

(ii) $a \leq b$; $a, b \in L \Rightarrow \mu_b \subseteq \mu_a$,

(iii) $\mu = \nu \Leftrightarrow \mu_a = \nu_a$, $\forall\, a \in L$

The next theorem gives some basic properties of cuts and its proof is straightforward.

**1.3.6 Theorem [49]:**

Suppose $\{\mu_i \mid i \in I\} \subseteq L^X$ where $I$ denotes an arbitrary nonempty index set. Then for any $a \in L$,

(1) $\bigcup_{i \in I} (\mu_i)_a \subseteq (\bigcup_{i \in I} \mu_i)_a$

(2) $\bigcap_{i \in I} (\mu_i)_a = (\bigcap_{i \in I} \mu_i)_a$

Moreover, when $L$ is a finite chain, we have equality in (1). ■

**1.3.7 Definition [49]:**

Let $f$ be a mapping from $X$ into $Y$, and let $\mu \in L^X$ and $\nu \in L^Y$. The $L$-subsets $f(\mu) \in L^Y$ and $f^{-1}(\nu) \in L^X$, defined by $\forall\, y \in Y$,

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) : x \in X, f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and $\forall\, x \in X$, $\quad f^{-1}(\nu)(x) = \nu(f(x))$,

are called, respectively, the image of $\mu$ under $f$ and the pre-image of $\nu$ under $f$.

It is not very difficult to see the following:

**1.3.8 Theorem [49]:**

Let $f$ be a mapping from $X$ into $Y$. Then the following assertions hold:

(i) For $\{\mu_i \mid i \in I\} \subseteq L^X$, where $I$ an arbitrary nonempty index set,

$f(\bigcup_{i \in I} \mu_i) = \bigcup_{i \in I} f(\mu_i)$ and hence $\mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2)$ $\forall \mu_1, \mu_2 \in L^X$.

(ii) For $\{\nu_j \mid j \in J\} \subseteq L^Y$, where $J$ an arbitrary nonempty index set,

$f^{-1}(\bigcup_{j \in J} \nu_j) = \bigcup_{j \in J} f^{-1}(\nu_j)$ and $f^{-1}(\bigcap_{j \in J} \nu_j) = \bigcap_{j \in J} f^{-1}(\nu_j)$

and hence $\nu_1 \subseteq \nu_2 \Rightarrow f^{-1}(\nu_1) \subseteq f^{-1}(\nu_2)$ $\forall \nu_1, \nu_2 \in L^Y$.

(iii) $f^{-1}(f(\mu)) \supseteq \mu$ $\forall \mu \in L^X$. In particular if $f$ is an injection, then $f^{-1}(f(\mu)) = \mu$ $\forall\, \mu \in L^X$.

(iv) $f(f^{-1}(\nu)) \subseteq \nu \quad \forall \nu \in L^Y$. In particular if $f$ is a surjection, then $f(f^{-1}(\nu)) = \nu$ $\forall \nu \in L^Y$.

(v) $f(\mu) \subseteq \nu \Leftrightarrow \mu \subseteq f^{-1}(\nu) \quad \forall \mu \in L^X$ and $\forall \nu \in L^Y$. ■

In this chapter in the last section we have given the basic concepts and results in fuzzy set theory which are essential for the study in fuzzy commutative algebra. In the next chapter we introduce the concept of $L$-modules and give some necessary definitions and results which are required in the subsequent chapters.

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# Chapter 2

# $L$-MODULES



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* Some results of this chapter have appeared in the Proceedings of the National Conference on Mathematical Modelling held at Baselius College, Kottayam, Kerala; March 14-16, 2002.

## 2.1 Introduction.

It is well known that the central concept of the axiomatic development of linear algebra is that of a vector space over a field. The concept of a module is an immediate generalisation of a vector space obtained by replacing the underlying field by a ring. The concept of a module seems to have made its first appearance in algebra in algebraic number theory-in studying subsets of rings of algebraic numbers closed under addition and multiplication by elements of a specified ring. They became important with the development of homological algebra in the 1940's and 1950's.

Fuzzy set theory in the last three decades has developed in one way as a formal theory by fuzzifying the original ideas and concepts in classical mathematical areas such as algebra, graph theory, topology and so on. Among various branches of pure and applied mathematics algebra was one of the first few subjects where the notion of fuzzy set was applied. The concept of fuzzy modules and *L*-modules were introduced by Negoita and Ralescu [56] and Mashinchi and Zahedi [47] respectively. Subsequently they were further studied by Golan [15], Muganda [51], Pan [58, 59, 60, 61], Zahedi and Ameri [78, 79, 80, 81],. Tremendous and rapid growth of fuzzy algebraic concepts resulted in a vast literature. The book of Mordeson and Malik [49] gives an account of all these up to 1998.

In this chapter we quote from [49] basic definitions like $L$-modules, quotients and direct sums of $L$-modules and theorems which are essential for our study. Also some new related theorems of interest are stated and proved.

Let $R$ be a ring with unity. A left $R$-module is an additive group $M$ together with an operation ' . ' from $R \times M$ into $M$ such that

(1) $(r + s).\, x = r.\, x + s.\, x$

(2) $r.\, (x + y) = r.\, x + r.\, y$

(3) $r.\, (s.\, x) = (rs).x$

(4) $1.\, x = x$

$\forall\ r, s \in R;\ x, y \in M$. We write $r\,x$ for $r.\, x$ . Similarly we define a right $R$-module. If $R$ is commutative, we do not distinguish between a left and a right $R$-module and simply call it an $R$-module. If $R$ is a field then obviously an $R$-module is a vector space.

Throughout this thesis, unless otherwise stated, $L(\vee, \wedge, 1, 0)$ represents a complete distributive lattice with maximal element '1' and minimal element '0'; $R$ a ring with unity '1' and $M$ a left module over $R$. '$\vee$' denotes the supremum and '$\wedge$' the infimum in $L$. '$\subseteq$' denotes the inclusion and '$\subset$' the strict inclusion. The set of all $L$- subsets of $M$ is denoted by $L^M$.

## 2.2 Basic Concepts.

In this section, we consider some operations of $L$-subsets of a module induced by the operations in the module. We then give the definition of an $L$-module and quote some related theorems.

**2.2.1 Definition [49]:**

For $\mu, \nu \in L^M$, we define $\mu + \nu$, $-\mu \in L^M$ as follows.

$$(\mu + \nu)(x) = \vee\{\mu(y) \wedge \nu(z) : y, z \in M,\ y + z = x\}$$

and $\quad -\mu(x) = \mu(-x)$

$\forall\, x \in M$. Then $\mu + \nu$ is called the sum of $\mu$ and $\nu$, and $-\mu$ the negative of $\mu$.

**2.2.2 Definition [49]:**

Let $\mu_i \in L^M$, $i \in I$ be a family of $L$-subsets of $M$. Then we define

$$\sum_{i\in I} \mu_i(x) = \vee\left\{\bigwedge_{i\in I} \mu_i(x_i) : x_i \in M, i \in I, \sum_{i\in I} x_i = x\right\}$$

where in the expression $x = \sum_{i\in I} x_i$, at most finitely many $x_i$'s are $\neq 0$. $\sum_{i\in I} \mu_i$ is called the weak sum of the $\mu_i$'s.

**2.2.3 Definition [49]:**

Let $r \in R$ and $\mu \in L^M$. Define $r\mu \in L^M$ as $(r\mu)(x) = \vee\{\mu(y) : y \in M,\ ry = x\}$ $\forall\, x \in M$. Then $r\mu$ is called the product of $r$ and $\mu$.

**2.2.4 Theorem [49]:**

Suppose $M$ and $N$ are left $R$-modules and $f : M \to N$ is a module homomorphism. Let $r, s \in R$ and $\mu, \nu \in L^M$. Then

(i) $f(\mu + \nu) = f(\mu) + f(\nu)$

(ii) $f(r\mu) = r f(\mu)$

(iii) $f(r\mu + s\nu) = r f(\mu) + s f(\nu)$.

**2.2.5 Definition [49]:**

Let $\mu \in L^M$. Then $\mu$ is said to be an $L$- submodule of $M$ if

(i) $\mu(0) = 1$

(ii) $\mu(x + y) \geq \mu(x) \wedge \mu(y) \quad \forall\, x, y \in M$

(iii) $\mu(rx) \geq \mu(x) \quad \forall\, r \in R, \ \forall\, x \in M$

**Note:** By saying $\mu$ is a left $L$-module we mean $\mu$ is an $L$-submodule of some left module $M$ over a ring $R$. The set of all $L$-submodules of $M$ is denoted by $L(M)$.

**2.2.6 Definition [49]:**

Let $\mu \in L^R$. Then $\mu$ is said to be a left $L$-ideal of $R$ if

(i) $\mu(0) = 1$

(ii) $\mu(x + y) \geq \mu(x) \wedge \mu(y) \quad \forall\, x, y \in R$

(iii) $\mu(xy) \geq \mu(y) \quad \forall\, x, y \in R$

Similarly a right $L$-ideal of $R$ is defined by replacing (iii) with

(iii)' $\mu(xy) \geq \mu(x) \quad \forall\, x, y \in R$

**Note:** Clearly $\mu$ is a left $L$-ideal of $R$ if and only if $\mu$ is an $L$-submodule of the left module ${}_RR$.

**2.2.7 Example:**

Let $R$ be a ring. Consider $M = R^2 = \{(p, q) : p, q \in R\}$. Then $M$ is a module over $R$ with respect to the usual operations. Define $\mu : M \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{2} & \text{if } x = (p,0);\ p \neq 0 \\ 0 & \text{if } x = (p,q);\ q \neq 0 \end{cases}$$

Then $\mu$ is an $L$-module where $L$ is $[0,1]$.

**2.2.8 Theorem [49]:**

Let $\mu \in L^M$. Then $\mu \in L(M)$ if and only if $\mu$ satisfies the following conditions:

(i) $\mu(0) = 1$

(ii) $\mu(rx + sy) \geq \mu(x) \wedge \mu(y)\ \ \forall\ r, s \in R$ and $x, y \in M$.

**2.2.9 Theorem [49]:**

Let $\mu \in L^M$. Then $\mu \in L(M)$ if and only if $\mu$ satisfies the following conditions:

(i) $1_{\{0\}} \subseteq \mu$

(ii) $r\mu + s\mu \subseteq \mu\ \ \forall\ r, s \in R$

**2.2.10 Theorem [49]:**

Let $\mu, \nu \in L(M)$, then $\mu + \nu \in L(M)$. Moreover if $\mu_i \in L(M)$, $(i \in I)$ then $\sum_{i \in I} \mu_i \in L(M)$.

**2.2.11 Theorem:**

Let $\mu \in L^M$. Then $\mu \in L(M)$ if and only if $\mu_a$ is a submodule of $M$ $\forall\, a \in L$, where $\mu_a = \{x \in X : \mu(x) \geq a\}$, the $a$-level subset of $\mu$.

**Proof:**

Let $\mu \in L(M)$. Then for $a \in L$, consider $\mu_a$. Then clearly $\mu(0) = 1 \geq a$. Therefore $0 \in \mu_a$.

Also $\quad x, y \in \mu_a \quad \Rightarrow \mu(x) \geq a,\ \mu(y) \geq a$

$$\Rightarrow \mu(x + y) \geq \mu(x) \wedge \mu(y) \geq a \wedge a = a$$

$$\Rightarrow x + y \in \mu_a$$

and $\quad x \in \mu_a \quad \Rightarrow \mu(x) \geq a$

$$\Rightarrow \mu(rx) \geq \mu(x) \geq a$$

$$\Rightarrow rx \in \mu_a$$

Thus $\mu_a$ is a submodule of $M$.

Conversely suppose that $\mu_a$ is a submodule of $M$ $\forall\, a \in L$. Then $\mu_a$ contains 0, $\forall\, a \in L$, and in particular for $a = 1$. Therefore $\mu(0) = 1$. Also for $x, y \in M$, let $\mu(x) = p$, $\mu(y) = q$. Consider $\mu_r$ where $r = p \wedge q$.

Then $\quad x, y \in \mu_r \quad \Rightarrow x + y \in \mu_r$

$$\Rightarrow \mu(x + y) \geq r = p \wedge q = \mu(x) \wedge \mu(y)$$

So $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ $\forall\, x, y \in M$.

Now for $x \in M$, let $\mu(x) = a$. Then $x \in \mu_a$ and so $rx \in \mu_a$, which implies $\mu(rx) \geq a = \mu(x)$.

Thus $\mu(rx) \geq \mu(x)\ \forall\ r \in R,\ \forall\ x \in M.$

Therefore $\mu \in L(M)$. This completes the proof of the theorem. ■

**2.2.12 Theorem:**

If $L$ is regular and if $\mu \in L(M)$, then $\mu^*$ is a submodule of $M$ where $\mu^* = \{x \in X : \mu(x) > 0\}$, the support of $\mu$.

**Proof:**

We have,

$$\mu(0) = 1 \Rightarrow 0 \in \mu^*$$

Also $x, y \in \mu^* \Rightarrow \mu(x) > 0, \mu(y) > 0$

$$\Rightarrow \mu(x + y) \geq \mu(x) \wedge \mu(y) > 0 \qquad \text{( since } L \text{ is regular)}$$

$$\Rightarrow x + y \in \mu^*.$$

And, $x \in \mu^* \Rightarrow \mu(x) > 0$

$$\Rightarrow \mu(rx) \geq \mu(x) > 0,\ \forall\ r \in \mathrm{R}$$

$$\Rightarrow rx \in \mu^*.$$

Therefore $\mu^*$ is a submodule of $M$. ■

**Note:** If $L$ is not regular, then $\mu^*$ need not be a submodule of $M$ as we see in the following example.

**2.2.13 Example:**

Suppose $L$ is the lattice having four elements 0, 1, $a$ & $b$ where $a \vee b = 1$ and $a \wedge b = 0$. Consider $M = R^2 = \{(p, q) : p, q \in R\}$; $R$ a ring. Then $M$ is a module over $R$ with respect to the usual operations. Define $\mu : M \rightarrow L$ by:

$$\mu(x)=\begin{cases}1 & \text{if } x=(0,0)\\ a & \text{if } x=(p,0);\ \ p\neq 0\\ b & \text{if } x=(0,q);\ \ q\neq 0\\ 0 & \text{if } x=(p,q);\ \ p\neq 0 \text{ and } q\neq 0\end{cases}$$

Then clearly $\mu \in L(M)$. But $\mu^* = M - \{(p, q) : p \neq 0 \text{ and } q \neq 0\} = (R, 0) \cup (0, R)$ which is not a submodule of $M$.

**2.2.14 Theorem:**

Let $\mu, \eta \in L(M)$. Then $(\mu + \eta)^* \subseteq \mu^* + \eta^*$. If $L$ is regular then $(\mu + \eta)^* = \mu^* + \eta^*$.

**Proof:**

We have, $x \in (\mu + \eta)^* \Rightarrow (\mu + \eta)(x) > 0$

$\Rightarrow \vee\{\mu(y) \wedge \eta(z) : y, z \in M;\ \ x = y + z\} > 0$

$\Rightarrow \mu(y) \wedge \eta(z) > 0$ for some $y, z \in M;\ x = y + z$

$\Rightarrow \mu(y) > 0$ and $\eta(z) > 0$ for some $y, z \in M; x = y + z$

$\Rightarrow y \in \mu^*$ and $z \in \eta^*$ for some $y, z \in M\ ; x = y + z$

$\Rightarrow x \in \mu^* + \eta^*$

Therefore $(\mu + \eta)^* \subseteq \mu^* + \eta^*$.

Conversely if $L$ is regular,

$x \in \mu^* + \eta^* \Rightarrow x = y + z$ for some $y \in \mu^*$, $z \in \eta^*$

$\Rightarrow \mu(y) > 0,\ \eta(z) > 0$

$\Rightarrow \mu(y) \wedge \eta(z) > 0$ (since $L$ is regular)

$\Rightarrow (\mu + \eta)(x) > 0$ (since $x = y + z$)

$$\Rightarrow x \in (\mu + \eta)^*$$

Therefore $\mu^* + \eta^* \subseteq (\mu + \eta)^*$.

Hence $(\mu + \eta)^* = \mu^* + \eta^*$ if $L$ is regular. This completes the proof. ■

**Note:** The regularity of $L$ is essential for equality in the above theorem. For example consider the following:

**2.2.15 Example:**

Let $M$ and $L$ be as in example 2.2.13. Then $L$ is not regular. Define $\mu, \eta \in L^M$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ a & \text{if } x = (p,0);\ p \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ b & \text{if } x = (0,q);\ q \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then obviously $\mu, \eta \in L(M)$; and $\mu^* + \eta^* = M$. But $(\mu + \eta)^* \neq M$

## 2.3 $L$-Submodules of Quotient Modules.

In this section we give the definition of an $L$-submodule of a quotient module and the definition of the quotient $\nu/\mu$ of an $L$-module $\nu$ with respect to an $L$-submodule $\mu$ of $\nu$, which are available in the literature, and quote some theorems in connection with this.

**2.3.1 Theorem [49]:**

Let $\mu \in L(M)$ and let $A$ be a submodule of $M$. Define $\xi \in L^{M/A}$ as follows:

$$\xi(x + A) = \vee\{\mu(y) : y \in x + A\}$$

$\forall\ x \in M$, where $M/A$ denotes the quotient module of $M$ with respect to $A$. Then $\xi \in L(M/A)$.

**2.3.2 Definition [49]:**

Let $\mu, \ \nu \in L(M)$ be such that $\mu \subseteq \nu$ and assume $L$ is regular. Then obviously $\mu^{*}$ and $\nu^{*}$ are submodules of $M$ and $\mu^{*} \subseteq \nu^{*}$. Thus $\mu^{*}$ is a submodule of $\nu^{*}$. Moreover $\nu|_{\nu^{*}} \in L(\nu^{*})$. Now define $\xi \in L^{\nu^{*}/\mu^{*}}$ as follows:

$$\xi(x + \mu^{*}) = \vee\{\nu(y) : y \in x + \mu^{*}\} \ \ \forall\, x \in \nu^{*}.$$

Then $\xi \in L(\nu^{*}/\mu^{*})$ and is called the quotient of $\nu$ with respect to $\mu$ and is written as $\nu/\mu$.

**2.3.3 Theorem [49]:**

Let $\mu \in L(M)$ and assume that $N$ is also a left $R$-module and $f : M \rightarrow N$ is a homomorphism. Then $f(\mu) \in L(N)$.

**2.3.4 Definition [49]:**

Let $M$ and $N$ be left $R$-modules and let $\mu \in L(M)$, $\nu \in L(N)$.

(1) A homomorphism $f$ of $M$ onto $N$ is called a weak homomorphism of $\mu$ into $\nu$ if $f(\mu) \subseteq \nu$. If $f$ is a weak homomorphism of $\mu$ into $\nu$, then we say that $\mu$ is weakly homomorphic to $\nu$ and we write $\mu \sim \nu$.

(2) An isomorphism $f$ of $M$ onto $N$ is called a weak isomorphism of $\mu$ into $\nu$ if $f(\mu) \subseteq \nu$. If $f$ is a weak isomorphism of $\mu$ into $\nu$, then we say that $\mu$ is weakly isomorphic to $\nu$ and we write $\mu \simeq \nu$.

(3) A homomorphism $f$ of $M$ onto $N$ is called a homomorphism of $\mu$ onto $\nu$ if $f(\mu) = \nu$. If $f$ is a homomorphism of $\mu$ onto $\nu$, then we say that $\mu$ is homomorphic to $\nu$ and we write $\mu \approx \nu$.

(4) An isomorphism $f$ of $M$ onto $N$ is called an isomorphism of $\mu$ onto $\nu$ if $f(\mu) = \nu$. If $f$ is an isomorphism of $\mu$ onto $\nu$, then we say that $\mu$ is isomorphic to $\nu$ and we write $\mu \cong \nu$.

**2.3.5 Theorem [49]:**

Let $\mu, \nu \in L(M)$ be such that $\mu \subseteq \nu$ and assume $L$ is regular. Then $\nu|_{\nu^*} \approx \nu/\mu$. ■

**2.3.6 Theorem [49]:**

Let $\nu \in L(M)$ and assume that $N$ is also a left $R$-module and $\xi \in L(N)$ is such that $\nu \approx \xi$. Suppose that $L$ is regular. Then there exists $\mu \in L(M)$ such that $\mu \subseteq \nu$ and $\nu/\mu \cong \xi|_{\xi^*}$. ■

**2.3.7 Theorem [49]:**

Let $\mu, \nu \in L(M)$ and assume that $L$ is regular. Then $\nu/(\mu \cap \nu) \simeq (\mu + \nu)/\mu$. ■

## 2.4 Direct Sum of $L$-Modules.

In this section we recall the definition of the direct sum of $L$-submodules of a module $M$ which is a straight forward generalisation of that concept in crisp theory to the fuzzy setting. Also we prove some nice results in this context.

**2.4.1 Definition [49]:**

Let $\mu, \eta, \nu \in L(M)$. Then $\mu$ is said to be the direct sum of $\eta$ and $\nu$ if

(i) $\mu = \eta + \nu$

(ii) $\eta \cap \nu = 1_{\{0\}}$

In this case we write $\mu = \eta \oplus \nu$.

**2.4.2 Definition:**

Let $\mu_i \in L(M)$, $\forall\ i \in I$. Then we say that $\mu$ is the direct sum of $\{\mu_i : i \in I\}$, denoted $\underset{i \in I}{\oplus} \mu_i$, if

(i) $\mu = \sum_{i \in I} \mu_i$

(ii) $\mu_j \cap \sum_{i \in I - \{j\}} \mu_i = 1_{\{0\}}$

**2.4.3 Example:**

Let $M = R^2 = \{(p, q) : p,\ q \in R\}$ where $R$ is any ring and let $L = [0, 1]$.

Define $\mu$, $\eta$, $\nu$ in $L^M$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{2} & \text{if } x = (p,0),\ p \neq 0 \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

$$\eta(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{2} & \text{if } x = (p,0),\ p \neq 0 \\ 0 & \text{if } x = (p,q),\ q \neq 0 \end{cases}$$

and $$\nu(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{4} & \text{if } x = (0,q),\ q \neq 0 \\ 0 & \text{if } x = (p,q),\ p \neq 0 \end{cases}$$

It is a matter of routine verification to see that $\mu$, $\eta$, $\nu \in L(M)$; $\mu = \eta + \nu$ and $\eta \cap \nu = 1_{\{0\}}$. Therefore $\mu = \eta \oplus \nu$.

**2.4.4 Theorem:**

Suppose $L$ is regular. If $\mu$, $\eta$, $\nu \in L(M)$ are such that $\mu = \eta \oplus \nu$, then $\mu^* = \eta^* \oplus \nu^*$.

**Proof:**

We have, $x \in \mu^* \quad \Rightarrow \mu(x) > 0$

$\Rightarrow \vee\{\eta(y) \wedge \nu(z) : y, z \in M, \ y + z = x\} > 0$

$\Rightarrow \eta(y) \wedge \nu(z) > 0$ for some $y$, $z \in M$, with $y + z = x$

$\Rightarrow \exists \ y, \ z \in M$ such that $\eta(y) > 0$, $\nu(z) > 0$ and $y + z = x$

$\Rightarrow \exists \ y \in \eta^*, \ z \in \nu^*$ such that $y + z = x$;

Thus $x \in \mu^* \ \Rightarrow \ \exists \ y \in \eta^*, \ z \in \nu^*$ such that $y + z = x$. From this it follows that $\mu^* = \eta^* + \nu^*$.

Also, $x \in \eta^* \cap \nu^* \quad \Rightarrow \eta(x) > 0, \nu(x) > 0$

$\Rightarrow \eta(x) \wedge \nu(x) > 0$ (since $L$ is regular)

$\Rightarrow \eta(x) \wedge \nu(x) = 1$ (since $\eta \cap \nu = 1_{\{0\}}$)

$\Rightarrow x = 0$

Hence $\eta^* \cap \nu^* = \{0\}$ and so $\mu^* = \eta^* \oplus \nu^*$. ∎

**Note:** The converse of the above theorem need not be true as we see in the following example.

**2.4.5 Example:**

Let $M = R^2 = \{(p, q) : p, q \in R\}$ where $R$ is any ring. And let $L = [0, 1]$. Define $\mu$, $\eta$, $\nu \in L^M$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{2} & \text{if } x = (p,0),\ p \neq 0 \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

$$\eta(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{2} & \text{if } x = (p,0),\ p \neq 0 \\ 0 & \text{if } x = (p,q),\ q \neq 0 \end{cases}$$

and $$\nu(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{4} & \text{if } x = (0,q),\ q \neq 0 \\ 0 & \text{if } x = (p,q),\ p \neq 0 \end{cases}$$

Then $\mu$, $\eta$, $\nu \in L(M)$; $\mu^* = M = R^2$; $\eta^* = (R, 0)$ and $\nu^* = (0, R)$. Obviously $\mu^* = \eta^* \oplus \nu^*$.

But $$(\eta + \nu)(x) = \begin{cases} 1 & \text{if } x = (0,0) \\ \frac{1}{2} & \text{if } x = (p,0),\ p \neq 0 \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

Thus $\mu^* = \eta^* \oplus \nu^*$, but not even $\mu = \eta + \nu$.

**Note:** The above theorem doesn't hold if $L$ is not regular; because if $L$ is not regular, then $\mu^*$ need not be a submodule of $M$ even if $\mu \in L(M)$.

### 2.4.6 Theorem:

Suppose $L$ satisfies the complete distributive property. Suppose $\mu_i$, $(i \in I)$ and $\lambda$ are elements in $L(M)$, where $\sum_{i\in I}\mu_i$ is a direct sum $\bigoplus_{i\in I}\mu_i$; and suppose $\lambda \cap (\sum_{i\in I}\mu_i) = 1_{\{0\}}$. Then $\lambda + (\sum_{i\in I}\mu_i)$ is a direct sum $\lambda \oplus (\bigoplus_{i\in I}\mu_i)$.

**Proof:**

Given $\sum_{i\in I}\mu_i$ is a direct sum and $\lambda \cap \sum_{i\in I}\mu_i = 1_{\{0\}}$. Let $I_j$ denotes the set I - $\{j\}$.

Then we have,

$$\left(\mu_j \cap \left(\lambda + \sum_{i\in I_j}\mu_i\right)\right)(x)$$

$$= \mu_j(x) \wedge \left(\lambda + \sum_{i\in I_j}\mu_i\right)(x)$$

$$= \mu_j(x) \wedge \vee\left\{\left(\lambda(x_\lambda) \wedge \underset{i\in I_j}{\wedge}\mu_i(x_i)\right) : x = x_\lambda + \sum_{i\in I_j}x_i;\ x_\lambda,\ x_i \in M;\ i \in I_j\right\}$$

$$= \left\{\mu_j(x_\lambda + \sum_{i\in I_j}x_i) \wedge \vee\left(\lambda(x_\lambda) \wedge \underset{i\in I_j}{\wedge}\mu_i(x_i)\right) : \right.$$

$$\left. x = x_\lambda + \sum_{i\in I_j}x_i;\ x_\lambda,\ x_i \in M;\ i \in I_j\right\}$$

$$= \left(\vee\left\{\mu_j(x_\lambda) \wedge \underset{i\in I_j}{\wedge}\mu_j(x_i) : x = x_\lambda + \sum_{i\in I_j}x_i;\ x_\lambda,\ x_i \in M;\ i \in I_j\right\}\right) \wedge$$

$$\left(\vee\left\{\left(\lambda(x_\lambda) \wedge \underset{i\in I_j}{\wedge}\mu_i(x_i)\right) : x = x_\lambda + \sum_{i\in I_j}x_i;\ x_\lambda,\ x_i \in M;\ i \in I_j\right\}\right)$$

$$= \vee \left\{ \left( \mu_j(x_\lambda) \wedge \underset{i \in I_j}{\wedge} \mu_j(x_i) \right) \wedge \left( \lambda(x_\lambda) \wedge \underset{i \in I_j}{\wedge} \mu_i(x_i) \right) \right.$$

$$\left. : x = x_\lambda + \sum_{i \in I_j} x_i; \ \ x_\lambda, x_i \in M; \ i \in I_j \right\}$$

$$= \vee \left\{ \mu_j(x_\lambda) \wedge \lambda(x_\lambda) \wedge \underset{i \in I_j}{\wedge} \left( \mu_j(x_i) \wedge \mu_i(x_i) \right) \right.$$

$$\left. : x = x_\lambda + \sum_{i \in I_j} x_i; \ \ x_\lambda, x_i \in M; \ i \in I_j \right\}$$

$$= \begin{cases} 1 & \text{if } x_\lambda = 0, \ x_i = 0 \ \forall \ i \in I_j \\ 0 & \text{if } x_\lambda \neq 0 \text{ or } x_i \neq 0 \text{ for some } i \in I_j \end{cases}$$

(since $\mu_j \cap \lambda = 1_{\{0\}}$ and $\mu_j \cap \mu_i = 1_{\{0\}}$ $\forall$ $i \in I_j$)

$$= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Thus $\mu_j \cap \left( \lambda + \sum_{i \in I_j} \mu_i \right) = 1_{\{0\}}$ $\forall$ $j \in I$ and therefore $\lambda + \sum_{i \in I} \mu_i$ is a direct sum $\lambda \oplus (\underset{i \in I}{\oplus} \mu_i)$. This completes the proof. ■

In this chapter we have given the basic concepts regarding *L*-modules which are required for the further development of the theory given in the subsequent chapters. In the next chapter we introduce the notion of simple and semisimple *L*-modules and study some properties.

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# Chapter 3

# SIMPLE AND SEMISIMPLE

# $L$-MODULES

**3.1 Introduction**

**3.2 Simple $L$-Modules**

**3.3 Semisimple $L$-Modules**

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* Some results of this chapter will appear in a paper accepted for publication by the Journal of Fuzzy Mathematics.

** Some other results of this chapter have appeared in the Proceedings of the National Seminar on Graph Theory and Fuzzy Mathematics held at Catholicate College, Pathanamthitta, Kerala; August 28-30, 2003.

## 3.1 Introduction.

The concepts of simple and semisimple modules form an important area of study in the theory of $R$-modules. Recall that a left module $M$ over a ring $R$ is said to be simple if it does not contain any submodule other than 0 and $M$, and if $M \neq 0$. A left module $M$ is said to be semisimple if each of its proper submodules is a direct summand of $M$ and there are several other equivalent definitions in the literature. In this chapter we extend these notions to the fuzzy setting and investigate some properties.

## 3.2 Simple $L$-Modules.

In this section we introduce the concept of simple $L$-modules and prove that if $L$ is regular, then $M$ is simple if and only if $1_M$ is a simple left $L$-module.

### 3.2.1 Definition:

Let $\mu \in L(M)$ be a left $L$-module. Then $\lambda \in L^M$ is said to be an $L$-submodule of $\mu$ if $\lambda$ itself is a left $L$-module such that $\lambda \subseteq \mu$. That is if

(i) $\lambda(0) = 1$

(ii) $\lambda(x+y) \geq \lambda(x) \wedge \lambda(y) \ \forall\, x, y \in M$

(iii) $\lambda(rx) \geq \lambda(x) \ \forall\, r \in R, \forall\, x \in M$

(iv) $\lambda(x) \leq \mu(x) \ \forall\, x \in M$

### 3.2.2 Definition:

Let $\mu : M \to L$ be a left $L$-module. Then a left $L$-module $\eta : M \to L$ is said to be a strictly proper $L$-submodule of $\mu$ if $\eta \subseteq \mu$, $\eta \neq 1_{\{0\}}$, $\eta(x) = \mu(x)$ $\forall\, x$ for which $\eta(x) > 0$ and $\eta^* \subset \mu^*$; and $\eta : M \to L$ is said to be a proper $L$-submodule of $\mu$ if $\eta \subseteq \mu$, $\eta \neq 1_{\{0\}}$, $\eta^* \subset \mu^*$.

**3.2.3 Definition:**

$\mu \in L(M)$ is said to be a simple left $L$-module if $\mu$ has no proper $L$-submodules.

**3.2.4 Example:**

Let $D$ be a division ring. Let $R = M_n(D)$ be the set of all $n \times n$ matrices with entries in $D$. Let $R_i = \{A \in R : j^{\text{th}}$ column of $A$ is $\bar{0}$, for $j \neq i\}$. Then $R_i$ is a left $R$- module.

For $i = 1, 2, 3\ldots, n$; define $\mu_i : R \to [0, 1]$ as

$$\mu_i(A) = \begin{cases} 1 & \text{if } A = 0 \\ \dfrac{1}{2^i} & \text{if } A \in R_i - \{0\} \\ 0 & \text{if } A \notin R_i \end{cases}$$

Then $\mu_i$; $i = 1, 2, 3\ldots, n$ are simple left $L$-modules.

**3.2.5 Theorem:**

Suppose $L$ is regular. Then $M$ is simple if and only if $1_M$ is a simple left $L$-module.

**Proof:**

Suppose $M$ is simple. Then $M$ has no proper submodules. If possible let $1_M$ be not a simple left $L$-module. Then $1_M$ has a proper left $L$-submodule say $\mu$

such that $\mu \neq 1_{\{0\}}$, $\mu^* \subset 1_M^* = M$. Since $\mu \in L(M)$, and since $L$ is regular, $\mu^*$ is a submodule of $M$ and $\mu^* \neq \{0\}$, $\mu^* \neq M$. That is $\mu^*$ is a proper submodule of $M$. This contradicts the fact that $M$ is simple.

Conversely suppose that $1_M$ is a simple left $L$-module. If possible assume that $M$ is not simple. Let $N$ be a proper submodule of $M$. Then $N \neq \{0\}, N \neq M$. Define $\mu : M \to L$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases}$$

Then $\mu \in L(M)$; $\mu \subseteq 1_M$, $\mu \neq 1_{\{0\}}$ or $1_M$ and $\mu^* \subset M = 1_M^*$. Hence $\mu$ is a proper $L$-submodule of $1_M$ which is a contradiction. ■

## 3.3 Semisimple $L$-Modules.

Now we introduce the notion of semisimple $L$-modules and prove the fuzzy analogues of the theorems 'every submodule of a semisimple module is semisimple' and 'every semisimple module contains a simple submodule' in the crisp case. We also prove some other theorems which are relevant in the fuzzy setting.

### 3.3.1 Definition:

Let $\mu \in L(M)$. Then $\mu$ is said to be a semisimple left $L$-module if whenever $\lambda$ is a strictly proper $L$-submodule of $\mu$, there exists a strictly proper $L$-submodule $\eta$ of $\mu$ such that $\mu = \lambda \oplus \eta$.

That is if $\lambda$ is a proper $L$-submodule of $\mu$ such that $\lambda(x) = \mu(x)$ $\forall$ $x$ for which $\lambda(x) > 0$; then there exists a proper $L$-submodule $\eta$ of $\mu$ satisfying $\eta(x) = \mu(x)$ $\forall$ $x$ for which $\eta(x) > 0$, such that $\mu = \lambda \oplus \eta$.

**3.3.2 Example:**

Let $D$ be a division ring. Consider $R = M_3(D) = \{3\times3$ matrices over $D\}$, which is a ring with unity with respect to the addition and multiplication of matrices. Let $R_i = \{A \in R : j^{th}$ column of $A$ is $\bar{0}$, for $j \neq i\}$. Then $R_i$ is a simple left module over $R$ for $i = 1, 2, 3$ and ${}_RR$ is a semisimple left module.

Define $\mu : R \to [0, 1]$ by

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq 0 \\ \frac{1}{2} & \text{if } A \in R_1 - \{0\} \\ \frac{1}{3} & \text{if } A \in R_1 + R_2 - \{R_1\} \\ \frac{1}{4} & \text{if } A \in R_1 + R_2 + R_3 - \{R_1 + R_2\} \end{cases}$$

Then $\mu$ is a semisimple left $L$-module.

**3.3.3 Theorem:**

Let $M$ be a left module over a ring $R$. Then $M$ is semisimple if and only if $1_M$ is a semisimple left $L$-module.

**Proof :**

Suppose $M$ is semisimple. To prove that $1_M$ is a semisimple left $L$-module. Let $\mu$ be a strictly proper $L$-submodule of $1_M$. To show that there exists a strictly proper $L$-module $\eta \in L(M)$ such that $1_M = \mu \oplus \eta$.

For this let $S = \{x \in M : \mu(x) = 1\}$. Then obviously $S$ is a submodule of $M$ ; $S \neq 0, S \neq M$. Therefore since $M$ is semisimple, $S$ is a direct summand of $M$. Hence we can write $M = S \oplus T$ for some submodule $T$ of $M$. Now define $\eta$ : $M \to L$ by,

$$\eta(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{if } x \notin T \end{cases}$$

Then $\eta \in L(M)$. Further $\eta(x) = 1_M(x)$ $\forall$ $x$ for which $\eta(x) > 0$. Now $(\mu + \eta)(x)$ $= \vee\{\mu(y) \wedge \eta(z) : y, z \in M, y + z = x\}$. Since $M = S \oplus T$, $x \in M$ can be uniquely expressed as $x = s + t$, where $s \in S$ and $t \in T$. Thus $x = s + t$; where $\mu(s) = 1$, $\eta(t) = 1$. Therefore $(\mu + \eta)(x) = 1$ $\forall$ $x \in M$. Thus we get $\mu + \eta = 1_M$. Also, since $S \cap T = \{0\}$, we get $\mu \cap \eta = 1_{\{0\}}$ and hence $1_M = \mu \oplus \eta$. This proves the first part.

Conversely suppose that $1_M$ is a semisimple left $L$-module. To prove that $M$ is semisimple. For this let $S$ be any proper submodule of $M$. To prove that $S$ is a direct summand of $M$. Define $\mu \in L^M$ by,

$$\mu(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Then clearly $\mu \subseteq L(M)$ and $\mu$ is a strictly proper $L$-submodule of $1_M$. Since $1_M$ is semisimple, $1_M = \mu \oplus \eta$ for some strictly proper $L$-submodule $\eta$ of $1_M$. Take $T = \{x \in M : \eta(x) = 1\}$. Then $T$ is a submodule of $M$. We show that $M = \mathrm{S} \oplus \mathrm{T}$. For all $x \in M$, we have,

$$1 = 1_M(x) = (\mu + \eta)(x) = \vee\{\mu(y) \wedge \eta(z) : y + z = x; \ y, z \in M\}$$

which implies $\mu(y) = \eta(z) = 1$ for some $y, z \in M$, where $y + z = x$. (since $\mu(y) =$ 1 or 0 and $\eta(z) = 1$ or 0). Thus if $x \in M$ then $x = y + z$ for some $y \in S$, $z \in T$. So $M = S + T$. Also, since $\mu \cap \eta = 1_{\{0\}}$, we get $S \cap T = \{0\}$. Therefore M = S ⊕ T. This completes the proof. ■

**3.3.4 Theorem:**

Let $\mu \in L(M)$ be a semisimple left $L$-module. Then $\mu_a^>$ is a semisimple submodule of $M$ $\forall\, a \neq 0 \in L$.

**Proof :**

Given $\mu \in L(M)$ is semisimple. To prove that $\mu_a^>$ is a semisimple submodule of $M$ $\forall\, a \neq 0 \in L$. Assume $a \neq 0$. Let $A$ be a submodule of $\mu_a^>$. To show that $A$ is a direct summand of $\mu_a^> = \{x \in M : \mu(x) > a\}$. Define $\eta \in L^M$ by

$$\eta(x) = \begin{cases} \mu(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then clearly $\eta \in L(M)$ and $\eta$ is a strictly proper $L$-submodule of $\mu$ such that $\eta_a^> = A$. Since $\mu$ is semisimple and $\eta$ is a strictly proper $L$-submodule of $\mu$, there exists a strictly proper $L$-submodule $\nu$ of $\mu$ such that $\mu = \eta \oplus \nu$. Then $\nu(x) = \mu(x)$ $\forall\, x$ for which $\nu(x) > 0$. Take $B = \nu_a^> = \{x \in M : \nu(x) > a\}$. We prove that $\mu_a^> = A \oplus B$. That is we prove that $\mu_a^> = \eta_a^> \oplus \nu_a^>$.

For: $x \in \mu_a^> \Rightarrow \mu(x) > a$

$$\Rightarrow (\eta \oplus \nu)(x) > a$$

$$\Rightarrow \vee\{\eta(y) \wedge \nu(z) : y, z \in M;\ y + z = x\} > a$$

$\Rightarrow \exists\, y, z \in M$ with $y + z = x$ such that $\eta(y) \wedge \nu(z) > a$

$\Rightarrow \exists\, y, z \in M$ with $y + z = x$ such that $\eta(y) > a$ and $\nu(z) > a$

Thus $x \in \mu_a^{>} \Rightarrow \exists\; y, z \in M$ with $y + z = x$ such that $y \in \eta_a^{>}$ and $z \in \nu_a^{>}$.

Hence $\mu_a^{>} = \eta_a^{>} + \nu_a^{>}$.

Also, $x \in \eta_a^{>} \cap \nu_a^{>} \Rightarrow x \in \eta_a^{>},\ x \in \nu_a^{>}$

$\Rightarrow \eta(x) > a,\ \nu(x) > a$

$\Rightarrow (\eta \cap \nu)(x) = \eta(x) \wedge \nu(x) \geq a > 0$

$\Rightarrow x = 0$ (since $\eta \oplus \nu$ is a direct sum)

Thus $\eta_a^{>} \cap \nu_a^{>} = \{0\}$. Hence $\mu_a^{>} = \eta_a^{>} \oplus \nu_a^{>} = A \oplus B$. That is $A$ is a direct summand of $\mu_a^{>}$. So $\mu_a^{>}$ is a semisimple submodule of $M$ $\forall\, a \neq 0 \in L$. ■

**Note :** In the above theorem, if $L$ is regular, $\mu_a^{>}$ is semisimple even if $a = 0$. That is $\mu^{*}$ is a semisimple submodule of $M$, if $L$ is regular.

**3.3.5 Theorem:**

Suppose $L$ satisfies the complete distributive property. Then every strictly proper $L$-submodule of a semisimple left $L$-module is semisimple.

**Proof :**

Let $\mu$ be a given semisimple left $L$-module and $\lambda$ be a strictly proper $L$-submodule of $\mu$. To show that $\lambda$ is a semisimple left $L$-module. For this let $\eta$ be a strictly proper $L$-submodule of $\lambda$. Since $\lambda$ is a strictly proper $L$-submodule of $\mu$ we see that $\eta$ is a strictly proper $L$-submodule of $\mu$. Since $\mu$ is semisimple there exists a strictly proper $L$-submodule $\delta$ of $\mu$ such that $\mu = \eta \oplus \delta$.

Now we prove that $\lambda \cap (\eta \oplus \delta) = (\lambda \cap \eta) \oplus (\lambda \cap \delta)$. We have

$$[\lambda \cap (\eta + \delta)](x)$$

$$= \lambda(x) \wedge (\eta + \delta)(x)$$

$$= \lambda(x) \wedge (\vee\{\eta(y) \wedge \delta(z) : y, z \in M;\ y + z = x\})$$

$$= (\vee\{\lambda(y) \wedge \lambda(z) : y, z \in M;\ y + z = x\}) \wedge (\vee\{\eta(y) \wedge \delta(z) :$$

$$y, z \in M;\ y + z = x\})$$

(Since $\lambda$ being an $L$- module, for $x = y + z$, $\lambda(x) = \lambda(y+z) \geq \lambda(y) \wedge \lambda(z)$; and equality is attained for $x = x + 0$ or $x = 0 + x$)

$$= \vee\{(\lambda(y) \wedge \lambda(z)) \wedge (\eta(y) \wedge \delta(z)) : y, z \in M;\ y + z = x\}$$

$$= \vee\{(\lambda(y) \wedge \eta(y)) \wedge (\lambda(z) \wedge \delta(z)) : y, z \in M;\ y + z = x\}$$

$$= \vee\{(\lambda \cap \eta)(y) \wedge (\lambda \cap \delta)(z) : y, z \in M;\ y + z = x\}$$

$$= [(\lambda \cap \eta) + (\lambda \cap \delta)](x)$$

Thus $\lambda \cap (\eta + \delta) = (\lambda \cap \eta) + (\lambda \cap \delta) = \eta + (\lambda \cap \delta)$ (since $\eta \subseteq \lambda$)

Now $(\eta \cap (\lambda \cap \delta))(x) = (\eta \cap (\delta \cap \lambda))(x)$

$$= ((\eta \cap \delta) \cap \lambda)(x)$$

$$= 1_{\{0\}}(x) \wedge \lambda(x)$$

$$= 1_{\{0\}}(x)$$

Thus $\eta \cap (\lambda \cap \delta) = 1_{\{0\}}$. Hence $\eta + (\lambda \cap \delta) = \eta \oplus (\lambda \cap \delta)$.

Therefore $\lambda \cap (\eta \oplus \delta) = \eta \oplus (\lambda \cap \delta)$. So we get $\lambda = \lambda \cap \mu = \lambda \cap (\eta \oplus \delta) = \eta \oplus (\lambda \cap \delta)$. Obviously $\lambda \cap \delta$ is a strictly proper $L$-submodule of $\lambda$. Therefore $\lambda$ is a semisimple $L$-module. This completes the proof of theorem. ∎

**3.3.6 Theorem:**

Suppose $L$ is regular. Let $\mu \in L(M)$ be a semisimple left $L$-module. Then $\mu$ contains a simple left $L$-module.

**Proof :**

Given that $\mu \in L(M)$ is semisimple. Then for $a \in L$, $\mu_a^>$ is a semisimple submodule of $M$. Therefore $\mu_a^>$ contains a simple submodule say $A$. That is $A$ has no proper submodule.

Define $\eta : M \to L$ by

$$\eta(x) = \begin{cases} \mu(x) & \text{if } x \in A \subseteq \mu_a^> \\ 0 & \text{if } x \notin A \end{cases}$$

We claim that $\eta$ is a simple left $L$-module. If not $\eta$ has a proper $L$-submodule $\nu$; $\nu \neq 1_{\{0\}}$, $\nu^* \subset \eta^* \subseteq A$. Thus $\{0\} \subset \nu^* \subset A$. But $\nu^* = \{x \in M : \nu(x) > 0\}$ is clearly a submodule of $A$. (since $L$ is regular and $\nu \in L(M)$). Thus $\nu^*$ is a proper sub-module of $A$ which is a contradiction. Hence $\eta$ is a simple left $L$-module. ■

For a left $R$-module ${}_RM$ the equivalence of the following three properties is well known in crisp theory.

(1) $M$ is semisimple.

(2) $M$ is the sum of a family of simple submodules.

(3) $M$ is the direct sum of a family of simple submodules.

Similar to this result we have the following theorem in the fuzzy case.

**3.3.7 Theorem:**

Let $L$ be a complete distributive lattice and let $\mu \in L(M)$ be a left $L$-module. Then the following are equivalent.

(1) $\mu$ is semisimple.

(2) $\mu$ is the sum of a family of strictly proper simple $L$-submodules $\mu_i$, ( $i \in I$ ) of $\mu$.

(3) $\mu$ is the direct sum of a family of strictly proper simple $L$-submodules $\mu_j$, ( $j \in J$ ) of $\mu$.

**Proof :**

(1) $\Rightarrow$ (2). Suppose $\mu \in L(M)$ is semisimple. Let $\lambda$ be the sum of all strictly proper simple $L$-submodules $\mu_i$, ($i \in I$) of $\mu$ , where $\mu_i(x) = \mu(x)$ $\forall x$ for which $\mu_i(x) > 0$, ( $i \in I$ ). Then clearly $\lambda$ is a strictly proper $L$-submodule of $\mu$ such that $\lambda(x) = \mu(x)$ $\forall$ $x$ for which $\lambda(x) > 0$. Therefore there exists a strictly proper $L$-submodule $\eta$ of $\mu$ such that $\mu = \lambda \oplus \eta$. We claim that $\eta = 1_{\{0\}}$ so that $\mu = \lambda$. If not, being an $L$- submodule of $\mu$ which is strictly proper, $\eta$ is semisimple and so $\eta$ contains a simple $L$- submodule say $\delta$. Moreover we can choose $\delta$ such that $\delta(x) = \eta(x)$ $\forall$ $x$ for which $\delta(x) > 0$, and so $\delta(x) = \mu(x)$ $\forall$ $x$ for which $\delta(x) > 0$ (since $\eta$ is a strictly proper submodule of $\mu$). Then $\delta \neq 1_{\{0\}}$, $\delta \subseteq \eta$ and $\delta^* \subset \eta^*$. Also being a strictly proper simple $L$-submodule of $\mu$ such that $\delta(x) = \mu(x)$ $\forall$ $x$ for which $\delta(x) > 0$, we get $\delta \subseteq \lambda$. Thus we get $\delta \subseteq \lambda \cap \eta$ which in turn implies that $\delta = 1_{\{0\}}$. This is a contradiction. Hence $\eta = 1_{\{0\}}$ and so $\mu = \lambda$.

<u>(2) ⇒ (1)</u>. Conversely let $\mu$ be the sum of a family of strictly proper simple $L$-submodules $\mu_i$ ( $i \in I$ ) of $\mu$ say $\mu = \sum_{i \in I} \mu_i$ where for $i \in I$, $\mu_i(x) = \mu(x)$ $\forall\, x$ for which $\mu_i(x) > 0$. To show that $\mu$ is a semisimple left $L$-module. That is to show that corresponding to any strictly proper $L$-submodule $\lambda$ of $\mu$ there exists a strictly proper $L$-submodule $\eta$ of $\mu$ such that $\mu = \lambda \oplus \eta$.

Let $\lambda$ be a strictly proper $L$-submodule of $\mu$. Consider subsets $J \subseteq I$ with the properties

(i) $\sum_{j \in J} \mu_j$ is a direct sum $\oplus_{j \in J} \mu_j$

(ii) $\lambda \cap \sum_{j \in J} \mu_j = 1_{\{0\}}$

Consider the family $F$ of all such $J$'s with respect to ordinary inclusion. $F \neq \phi$ as it contains the empty set. By Zorn's lemma there exists a maximal element in $F$. Take such a maximal $J$. For this $J$, let $\mu' = \lambda + \sum_{j \in J} \mu_j = \lambda \oplus (\oplus_{j \in J} \mu_j)$. Then $\mu'$ is such that $\mu'(x) = \mu(x)$ $\forall\, x$ for which $\mu'(x) > 0$. Now we show that $\mu' = \mu$. For this we prove that $\mu_i \subseteq \mu'$ $\forall\, i \in I$. Suppose not. Then $\mu_i \not\subset \mu'$ for some $i$. Consider $\mu' \cap \mu_i$ for this $i$. It is an $L$-submodule of $\mu_i$. Since $\mu_i$ is simple we have $\mu' \cap \mu_i = 1_{\{0\}}$ or $(\mu' \cap \mu_i)^* = \mu_i^*$. Therefore $\mu' \cap \mu_i = 1_{\{0\}}$ or $\mu_i$ (since $L$ is regular, if $(\mu' \cap \mu_i)(x) > 0$ then both $\mu'(x)$, $\mu_i(x) > 0$; and then $\mu_i(x) = \mu(x) = \mu'(x)$). Since $\mu_i \not\subset \mu'$ we get $\mu' \cap \mu_i = 1_{\{0\}}$. Therefore $\mu' + \mu_i$ is a direct sum $\mu' \oplus \mu_i = \lambda \oplus (\oplus_{j \in J} \mu_j) \oplus \mu_i$. This contradicts the maximality of $J$. Therefore $\mu_i \subseteq \mu'$

$\forall\ i \in I$. This implies $\mu = \sum_{i \in I} \mu_i \subseteq \mu'$. That is $\mu \subseteq \mu'$. Clearly $\mu' \subseteq \mu$. Hence $\mu = \mu' = \lambda \oplus \sum_{j \in J} \mu_j$. Thus there exists a strictly proper $L$-submodule $\eta = \sum_{j \in J} \mu_j$ of $\mu$, where $\eta(x) = \mu(x)\ \ \forall\ x$ for which $\eta(x) > 0$, such that $\mu = \lambda \oplus \eta$. Therefore $\mu$ is semisimple.

(2) $\Rightarrow$ (3). Suppose $\mu \in L(M)$ is the sum of a family of strictly proper simple $L$-submodules $\mu_i$, $(i \in I)$ of $\mu$ where $\mu_i(x) = \mu(x)\ \ \forall\ x$ for which $\mu_i(x) > 0$. To show that $\mu$ is the direct sum of a family of such simple $L$- submodules.

Consider $\mu = \sum_{i \in I} \mu_i$ where $\mu_i$'s are strictly proper simple $L$-submodules of $\mu$ such that $\mu_i(x) = \mu(x)\ \ \forall\ x$ for which $\mu_i(x) > 0$. Consider the family $F = \{J \subseteq I :$ $\sum_{j \in J} \mu_j$ is a direct sum$\}$ with respect to the ordinary inclusion. Then $F$ contains a maximal element $J$. Then as in the proof of (2) $\Rightarrow$ (1) it is easy to see that $\mu = \bigoplus_{j \in J} \mu_j$.

(3) $\Rightarrow$ (2). This is obvious. ■

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# Chapter 4

# EXACT SEQUENCES OF $L$-MODULES

## 4.1 Introduction.

The concepts of exact sequences and split exact sequences of $R$-modules form an important area of study in module theory. Zahedi and Ameri [80] introduced the notion of fuzzy exact sequences in the category of fuzzy modules. According to them, a sequence $\ldots \rightarrow \mu_{i-1_{A_{i-1}}} \xrightarrow{\tilde{f}_{i-1}} \mu_{i_{A_i}} \xrightarrow{\tilde{f}_i} \mu_{i+1_{A_{i+1}}} \rightarrow \ldots$ of $R$-fuzzy module homomorphisms is said to be fuzzy exact if Im $\tilde{f}_{i-1}$ = Ker $\tilde{f}_i$ for all $i$, where Im $\tilde{f}_{i-1}$ and Ker $\tilde{f}_i$ mean $\mu_i|_{\text{Im } f_{i-1}}$ and $\mu_i|_{\text{Ker } f_i}$ respectively. In this chapter, as an extension of the concept of exact sequences of $R$-modules in classical module theory to the fuzzy setting, we give a more general definition and prove some interesting results in this context

## 4.2 Exact Sequences of $L$-modules.

From the theory of $R$-modules recall that a sequence of $R$-modules and $R$-module homomorphisms $\ldots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \ldots$ is said to be exact at $A_i$ if Im $(f_i)$ = Ker $(f_{i+1})$; and the sequence is said to be exact if it is exact at each $A_i$. In this section we extend this notion to the fuzzy setting and prove some results.

### 4.2.1 Definition:

Let $A_i$; $i \in Z$ be $R$-modules and let $\mu_i \in L(A_i)$. Suppose that $\ldots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \ldots$ is an exact sequence of $R$-modules.

Then the sequence $\ldots \xrightarrow{f_{i-1}} \mu_{i-1} \xrightarrow{f_i} \mu_i \xrightarrow{f_{i+1}} \mu_{i+1} \xrightarrow{f_{i+2}} \ldots$ of $L$-modules is said to be exact if, for all $i \in Z$,

(i) $f_{i+1}(\mu_i) \subseteq \mu_{i+1}$ and

(ii) $f_i(\mu_{i-1})(x) > 0$ if $x \in \operatorname{Ker} f_{i+1}$ ; and $f_i(\mu_{i-1})(x) = 0$ if $x \notin \operatorname{Ker} f_{i+1}$ .

**Remark:** From here onwards the above situation in the definition will be mentioned by saying that

$$\begin{array}{c} \ldots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \ldots \\ \ldots \xrightarrow{f_{i-1}} \mu_{i-1} \xrightarrow{f_i} \mu_i \xrightarrow{f_{i+1}} \mu_{i+1} \xrightarrow{f_{i+2}} \ldots \end{array}$$

is an exact sequence of $L$-modules.

Recall from chapter 2 that if $L$ is regular and if $\mu, \eta \in L(M)$ are such that $\mu + \eta$ is a direct sum of $L$-modules, then $\mu^* + \eta^*$ is a direct sum of $R$-modules.

**4.2.2 Theorem:**

Let $L$ be a regular lattice. Let $\mu, \eta \in L(M)$ be such that $\mu \oplus \eta$ is a direct sum of $L$-modules so that $\mu^* \oplus \eta^*$ is a direct sum of $R$-modules. Then the sequence $0 \to \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \to 0$ is exact, considering $\mu, \eta$ as $\mu \in L(\mu^*)$, $\eta \in L(\eta^*)$.

**Proof:**

Note that the sequence $0 \to \mu^* \xrightarrow{i} \mu^* \oplus \eta^* \xrightarrow{\pi} \eta^* \to 0$ is an exact sequence of $R$- modules where ' $i$ ' and ' $\pi$ ' are respectively the canonical injection and projection. We have to prove that the sequence $0 \to \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \to 0$ is an exact sequence of $L$-modules.

Let $x \in \mu^* + \eta^*$. Then $i(\mu)(x) = \vee\{\mu(t) : t \in \mu^*, \ i(t) = x\}$

$$= \begin{cases} \mu(x) & \text{if } x \in \mu^* \\ 0 & \text{if } x \notin \mu^* \end{cases} \qquad \ldots. \ (1)$$

Also, $(\mu + \eta)(x) = \vee\{\mu(y) \wedge \eta(z) : \ y, z \in M, \ y + z = x\}$

$$= \mu(x) \ \text{ if } x \in \mu^* \qquad (2)$$

(Note that $\mu \oplus \eta$ is a direct sum. If $x = y + z$ with $x \in \mu^*$, then the only possibility is $x = x + 0$ or $x = y + z$; $y, z \in \mu^*$. But in the second case $\eta(z) = 0$.)

It follows from (1) and (2) that $i(\mu) \subseteq \mu + \eta$.

For $x \in \eta^*$, $(\pi(\mu + \eta))(x) = \vee\{(\mu + \eta)(t) : \ t \in \mu^* + \eta^*; \ \pi(t) = x\}$

$= \vee\{(\mu + \eta)(r + x) : \ r \in \mu^*\}$

(since $\pi : \mu^* \oplus \eta^* \to \eta^*$ is the projection.)

$= \vee\{\mu(r) \wedge \eta(x) : \ r \in \mu^*\}$

$= \eta(x)$ (since $\mu(r) = 1$ with $r = 0$)

Hence $\pi(\mu + \eta) = \eta$. Now by (1), $i(\mu)(x) = \begin{cases} \mu(x) > 0 & \text{if } x \in \mu^* = \operatorname{Ker} \pi \\ 0 & \text{if } x \notin \mu^* = \operatorname{Ker} \pi \end{cases}$

Therefore $0 \to \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \to 0$ is an exact sequence of $L$-modules. ■

**Remark:** Note that in the above theorem, for convenience, we have denoted the $L$-module $1_{\{0\}} \in L(M)$ by 0.

Also if $0 \longrightarrow A \xrightarrow{f} B$ is a sequence of $R$-modules and $\mu \in L(A)$, $\eta \in L(B)$, then it is easy to see that $0 \longrightarrow \mu \xrightarrow{f} \eta$ is an exact sequence of $L$-modules if and only if $0 \longrightarrow A \xrightarrow{f} B$ is an exact sequence of $R$- modules, if and only if $f$ is injective.

**4.2.3 Definition:**

Let $L$ be a regular lattice. Let $\mu, \eta \in L(M)$ be such that $\mu \oplus \eta$ is a direct sum of $L$-modules. Then the exact sequence $0 \rightarrow \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \rightarrow 0$ of $L$- modules is called a split exact sequence of $L$- modules.

Now we obtain a necessary condition for a given sequence $\mu \xrightarrow{f} \eta \xrightarrow{g} \nu$ to be exact at $\eta$.

**4.2.4 Theorem:**

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of $R$-modules exact at $B$ and let $\mu \in L(A)$, $\eta \in L(B)$, $\nu \in L(C)$. Then the sequence $\mu \xrightarrow{f} \eta \xrightarrow{g} \nu$ of $L$-modules is exact at $\eta$ only if $\mu^* \xrightarrow{f'} \eta^* \xrightarrow{g'} \nu^*$ is a sequence of $R$-modules exact at $\eta^*$, where $f'$ and $g'$ are restrictions of $f$ and $g$ to $\mu^*$ and $\eta^*$ respectively.

**Proof:**

Suppose the sequence $\mu \xrightarrow{f} \eta \xrightarrow{g} \nu$ is exact at $\eta$. Then by definition $f(\mu) \subseteq \eta$, $g(\eta) \subseteq \nu$ and $f(\mu)(x) > 0$ if $x \in \text{Ker } g$ ; $f(\mu)(x) = 0$ if $x \notin \text{Ker } g$. (That is $(f(\mu))^* = \text{Ker } g$) . Now consider the sequence $\mu^* \xrightarrow{f'} \eta^* \xrightarrow{g'} \nu^*$ We claim that this sequence is exact at $\eta^*$.

For : $x \in (f(\mu))^* \Leftrightarrow f(\mu)(x) > 0$

$$\Leftrightarrow \vee\{\mu(t) : f(t) = x,\ t \in A\} > 0$$

$$\Leftrightarrow \exists\ t \in A \text{ such that } x = f(t),\ \mu(t) > 0$$

$$\Leftrightarrow x \in f(\mu^*)$$

Thus we get $(f(\mu))^* = f(\mu^*)$. Similarly we get $(g(\eta))^* = g(\eta^*)$. Therefore $f'(\mu^*)$ $= f(\mu^*) = (f(\mu))^* \subseteq \eta^*$ as $f(\mu) \subseteq \eta$. Similarly $g'(\eta^*) = (g(\eta))^* \subseteq \nu^*$. Now since $(f(\mu))^*$ = Ker $g$ it follows that $f'(\mu^*)$ = Ker $g'$. Thus the sequence $\mu^* \xrightarrow{f'} \eta^* \xrightarrow{g'} \nu^*$ is exact at $\eta^*$ which completes the proof of the theorem. ■

**Remark:** The converse of the above theorem is not true. That is the sequence $\mu^* \longrightarrow \eta^* \longrightarrow \nu^*$ is exact at $\eta^*$ doesn't imply that the sequence $\mu \longrightarrow \eta \longrightarrow \nu$ is exact at $\eta$.

**4.2.5 Example:**

Let $M$ be an $R$-module, $A$ and $B$ be submodules of $M$ such that $A \oplus B$ is a direct sum. Let $L = [0, 1]$. Define $\mu \in L(A)$, $\eta \in L(B)$, $\nu \in L(A \oplus B)$ as follows.

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{1}{2} & \text{if } x = A \setminus \{0\} \end{cases}$$

$$\eta(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{1}{3} & \text{if } x = B \setminus \{0\} \end{cases}$$

$$\nu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{1}{4} & \text{if } x = A \oplus B \setminus \{0\} \end{cases}$$

Then $\mu^* = A$, $\eta^* = B$ and $\nu^* = A \oplus B$. Obviously $A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B$ is exact at $A \oplus B$ .That is $\mu^* \xrightarrow{i} \nu^* \xrightarrow{\pi} \eta^*$ is exact at $\nu^*$. Now $i(\mu)(x) = \vee\{\mu(t) : t \in A,$ $i(t) = x\} = \mu(x)$ (with $t = x \in A$). That is $i(\mu) = \mu$ and clearly $\mu \not\subset \nu$. Therefore the sequence $\mu \xrightarrow{i} \nu \xrightarrow{\pi} \eta$ is not an exact sequence of $L$-modules.

**4.2.6 Theorem:**

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence exact at $B$ and let $\mu \in L(A)$, $\eta \in L(B)$, $\nu \in L(C)$ be such that $\mu \xrightarrow{f} \eta \xrightarrow{g} \nu$ is a sequence of $L$-modules exact at $\eta$. Then $f(\mu_a^>) \subseteq \text{Ker } g \ \forall \ a \in L$.

**Proof:**

For $a \in L$, consider the strict $a$-level subsets $\mu_a^>$, $\eta_a^>$ and $\nu_a^>$.

Then $\quad x \in f(\mu_a^>) \Rightarrow \exists \ t$ such that $x = f(t)$, $\mu(t) > a$

$$\Rightarrow \vee \{\mu(t) : x = f(t)\} > a$$

Therefore it follows that $f(\mu)(x) > a$ and hence $x \in (f(\mu))_a^>$. Thus we get $f(\mu_a^>) \subseteq (f(\mu))_a^>$. Similarly we get $g(\eta_a^>) \subseteq (g(\eta))_a^>$.

Now $x \in f(\mu_a^>) \subseteq (f(\mu))_a^> \Rightarrow f(\mu)(x) > a$.

Therefore it follows that $f(\mu)(x) > 0$. Hence we get $x \in \text{Ker } g$. (since by definition $f(\mu)(x) > 0$ if and only if $x \in \text{Ker } g$). Thus $f(\mu_a^>) \subseteq \text{Ker } g \ \forall \ a \in L$. ■

**4.2.7 Definition:**

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of $R$-modules. Let $\mu \in L(A)$, $\eta \in L(B)$, and $\nu \in L(C)$. Then an exact sequence of $L$-modules of the form $0 \to \mu \xrightarrow{f} \eta \xrightarrow{g} \nu \to 0$ is called a short exact sequence of $L$-modules.

Extending the concept of isomorphism between short exact sequences of $R$-modules in classical module theory to the fuzzy setting, we define isomorphism and weak isomorphism between short exact sequences of $L$-modules and obtain some sufficient conditions under which the exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$ is weakly isomorphic to the exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$. Also we get another set of sufficient conditions under which the exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ is weakly isomorphic to the exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$.

Recall that two short exact sequences of $R$-modules are said to be isomorphic if there is a commutative diagram of module homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \psi & & \downarrow \xi & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

such that $\phi$, $\psi$, $\xi$ are isomorphisms.

**4.2.8 Definition:**

Let $\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow \psi & & \downarrow \xi & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$ be two isomorphic short exact sequences of $R$-modules with the given isomorphisms. Let $\mu \in L(A)$, $\nu \in L(B)$, $\eta \in L(C)$, $\mu' \in L(A')$, $\nu' \in L(B')$ and $\eta' \in L(C')$ be such that

$$0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0 \qquad (1)$$

and $$0 \to \mu' \xrightarrow{f'} \nu' \xrightarrow{g'} \eta' \to 0 \qquad (2)$$

are two short exact sequences of $L$-modules. Then the sequence (1) is said to be weakly isomorphic to the sequence (2) if $\varphi(\mu) \subseteq \mu'$, $\psi(\nu) \subseteq \nu'$, and $\xi(\eta) \subseteq \eta'$.

The sequence (1) is said to be isomorphic to the sequence (2) if $\varphi(\mu) = \mu'$, $\psi(\nu) = \nu'$, and $\xi(\eta) = \eta'$.

**4.2.9 Definition:**

Let $A$ and $B$ be two $R$-modules; $\mu \in L(A)$, $\eta \in L(B)$. Consider the direct sum $A \oplus B$. We extend the definition of $\mu$ and $\eta$ to $A \oplus B$ to get $\mu'$ and $\eta'$ in $L(A \oplus B)$ as follows.

$$\mu'(x) = \begin{cases} \mu(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{i.e. } \mu'(a,b) = \begin{cases} \mu(a) & \text{if } b = 0 \\ 0 & \text{if } b \neq 0 \end{cases} \quad \text{for } (a, b) \in A \oplus B$$

$$\text{and} \quad \eta'(x) = \begin{cases} \eta(x) & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases} \quad \text{i.e. } \eta'(a,b) = \begin{cases} \eta(b) & \text{if } a = 0 \\ 0 & \text{if } a \neq 0 \end{cases} \quad \text{for } (a, b) \in A \oplus B$$

Then $\mu', \eta' \in L(A \oplus B)$. Moreover $(\mu' \cap \eta')(x) = \mu'(x) \wedge \eta'(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$.

Hence $\mu' + \eta'$ is in fact a direct sum and we denote it by $\mu \oplus \eta$.

**Remark:** Note that

$$\begin{aligned} (\mu + \eta)(a, b) &= (\mu' + \eta')(a, b) \\ &= \vee\{\mu'(a_1, b_1) \wedge \eta'(a_2, b_2) : (a_1, b_1), (a_2, b_2) \in A \oplus B; \\ &\qquad (a_1, b_1) + (a_2, b_2) = (a, b)\} \\ &= \mu'(a, 0) \wedge \eta'(0, b) \\ &= \mu(a) \wedge \eta(b). \end{aligned}$$

If $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ is a short exact sequence of $R$-module homomorphisms, the equivalence of the following conditions is well known in crisp module theory.

(i) There is an $R$-module homomorphism $h : A_2 \to B$ with $g \circ h = I_{A_2}$

(ii) There is an $R$-module homomorphism $k : B \to A_1$ with $k \circ f = I_{A_1}$

(iii) The given sequence is isomorphic to the direct sum short exact sequence

$0 \to A_1 \xrightarrow{i} A_1 \oplus A_2 \xrightarrow{\pi} A_2 \to 0$.

Related to this we have the following theorems in fuzzy module theory.

**4.2.10 Theorem:**

Let $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ be a short exact sequence of $R$-modules and let $\mu_1 \in L(A_1)$, $\mu_2 \in L(A_2)$, $\eta \in L(B)$ be such that $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ is a short exact sequence of $L$-modules. If there is an $R$-module homomorphism $h : A_2 \to B$ with $g \circ h = I_{A_2}$ such that $h(\mu_2) \subseteq \eta$, then the short exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$ is weakly isomorphic to the given short seque-nce $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$. In particular $\mu_1 \oplus \mu_2 \simeq \eta$.

**Proof :**

We have by definition, $f(\mu_1) \subseteq \eta$, $g(\eta) \subseteq \mu_2$ and $f(\mu_1)(x) > 0$ if $x \in \text{Ker}\, g$; $f(\mu_1)(x) = 0$ if $x \notin \text{Ker}\, g$. Also it is given that $h(\mu_2) \subseteq \eta$. Now we consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{i} & A_1 \oplus A_2 & \xrightarrow{\pi} & A_2 & \longrightarrow & 0 \\
0 & \longrightarrow & \mu_1 & \xrightarrow{i} & \mu_1 \oplus \mu_2 & \xrightarrow{\pi} & \mu_2 & \longrightarrow & 0 \\
 & & \downarrow I_{A_1} & & \downarrow \phi & & \downarrow I_{A_2} & & \\
0 & \longrightarrow & \mu_1 & \xrightarrow{f} & \eta & \xrightarrow{g} & \mu_2 & \longrightarrow & 0 \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0
\end{array}$$

where $\phi : A_1 \oplus A_2 \to B$ is defined by $\phi(a_1, a_2) = f(a_1) + h(a_2)$. Then $\phi$ is a module homomorphism. Moreover $\phi \circ i = f \circ I_{A_1}$ and $g \circ \phi = I_{A_2} \circ \pi$. Since $I_{A_1}$ and $I_{A_2}$ (identity maps) are isomorphisms $\phi$ is also an isomorphism (by short five lemma for exact sequences of $R$-modules) and so $B$ is isomorphic to $A_1 \oplus A_2$, and the sequences $0 \to A_1 \xrightarrow{i} A_1 \oplus A_2 \xrightarrow{\pi} A_2 \to 0$ and $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ are isomorphic short exact sequences of $R$-modules. Obviously $I_{A_1}(\mu_1) = \mu_1$ and $I_{A_2}(\mu_2) = \mu_2$.

Now let $x = \phi(a_1', a_2') \in B$ be arbitrary where $a_1' \in A_1$, $a_2' \in A_2$. Then we get

$(\phi(\mu_1 \oplus \mu_2))(x)$

$$= \vee\{(\mu_1 \oplus \mu_2)(t_1, t_2) : (t_1, t_2) \in A_1 \oplus A_2;\ \phi(t_1, t_2) = x\}$$

$$= \vee\{(\mu_1)(t_1) \wedge \mu_2(t_2) : t_1 \in A_1,\ t_2 \in A_2;\ f(t_1) + h(t_2) = \phi(a_1', a_2')\}$$

$$= \vee\{(\mu_1)(t_1) \wedge \mu_2(t_2) : t_1 \in A_1,\ t_2 \in A_2;\ f(t_1) + h(t_2) = f(a_1') + h(a_2')\}$$

$$= \vee\{(\mu_1)(t_1) \wedge \mu_2(t_2) : t_1 \in A_1,\ t_2 \in A_2;$$

$$f(t_1) = f(a_1'),\ h(t_2) = h(a_2')\} \quad \ldots \quad (1)$$

(since $A_1 \cong \phi(A_1) = f(A_1)$ ; $A_2 \cong \phi(A_2) = h(A_2)$, we get $B \cong A_1 \oplus A_2$

$\cong f(A_1) \oplus h(A_2))$

Also since $f(\mu_1) \subseteq \eta$ and $h(\mu_2) \subseteq \eta$ it follows that

$$\vee\{\mu_1(t_1) : t_1 \in A_1;\ f(t_1) = f(a_1')\} \leq \eta(f(a_1')) \quad \ldots \quad (2)$$

and $$\vee\{\mu_2(t_2) : t_2 \in A_2;\ h(t_2) = h(a_2')\} \leq \eta(h(a_2')) \quad \ldots \quad (3)$$

Since $\eta$ is an $L$-module, from (2) and (3) we get

$$(\vee\{\mu_1(t_1) : t_1 \in A_1;\ f(t_1) = f(a_1')\}) \wedge (\vee\{\mu_2(t_2) : t_2 \in A_2;\ h(t_2) = h(a_2')\})$$

$$\leq \eta(f(a_1') + h(a_2')) = \eta(\phi(a_1', a_2')) = \eta(x)$$

From this using the complete distributive property of $L$ we get

$$\vee\{\mu_1(t_1) \wedge \mu_2(t_2) : t_1 \in A_1,\ t_2 \in A_2;\ f(t_1) = f(a_1'),\ h(t_2) = h(a_2')\} \leq \eta(x).$$

Therefore from (1) we get $(\phi(\mu_1 \oplus \mu_2))(x) \leq \eta(x)\ \forall\ x \in B$. Thus $\phi(\mu_1 \oplus \mu_2) \subseteq \eta$ and hence by definition, the short exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$ is weakly isomorphic (with identity maps on $\mu_1$ and $\mu_2$) to the given exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ and $\mu_1 \oplus \mu_2 \simeq \eta$. ■

**4.2.11 Theorem:**

Let $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ be a short exact sequence of $R$-modules and let $\mu_1 \in L(A_1)$, $\mu_2 \in L(A_2)$, $\eta \in L(B)$ be such that $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ is a short exact sequence of $L$-modules. If there is an $R$-module homomorphism $k : B \to A_1$ with $k \circ f = I_{A_1}$ such that $k(\eta) \subseteq \mu_1$, then the given short exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ is weakly isomorphic to the short exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$. In particular $\eta \simeq \mu_1 \oplus \mu_2$.

**Proof:**

We have $f(\mu_1) \subseteq \eta$, $g(\eta) \subseteq \mu_2$, $k(\eta) \subseteq \mu_1$ and we have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \\
0 & \longrightarrow & \mu_1 & \xrightarrow{f} & \eta & \xrightarrow{g} & \mu_2 & \longrightarrow & 0 \\
 & & \downarrow I_{A_1} & & \downarrow \psi & & \downarrow I_{A_2} & & \\
0 & \longrightarrow & \mu_1 & \xrightarrow{i} & \mu_1 \oplus \mu_2 & \xrightarrow{\pi} & \mu_2 & \longrightarrow & 0 \\
0 & \longrightarrow & A_1 & \xrightarrow{i} & A_1 \oplus A_2 & \xrightarrow{\pi} & A_2 & \longrightarrow & 0
\end{array}$$

where $\psi : B \to A_1 \oplus A_2$ is defined by $\psi(b) = (k(b), g(b))$. Then $\psi$ is a module homomorphism. Moreover $i \circ I_{A_1} = \psi \circ f$ and $\pi \circ \psi = I_{A_2} \circ g$. Since $I_{A_1}$ and $I_{A_2}$ are isomorphisms $\psi$ is also an isomorphism (by short five lemma for exact sequences of $R$-modules) and so $B$ is isomorphic to $A_1 \oplus A_2$, and the sequences $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ and $0 \to A_1 \xrightarrow{i} A_1 \oplus A_2 \xrightarrow{\pi} A_2 \to 0$ are isomorphic short exact sequences of $R$-modules. Also $I_{A_1}(\mu_1) = \mu_1$ and $I_{A_2}(\mu_2) = \mu_2$.

Now, for $(a_1, a_2) \in A_1 \oplus A_2$, we get

$$\psi(\eta)(a_1, a_2) = \vee\{\eta(b) : b \in B;\ k(b) = a_1, g(b)) = a_2\} \quad \ldots \quad (1)$$

Also since $k(\eta) \subseteq \mu_1$ and $g(\eta) \subseteq \mu_2$ we get

$$\vee\{\eta(b) : b \in B;\ k(b) = a_1\} \leq \mu_1(a_1) \qquad (2)$$

and $$\vee\{\eta(b) : b \in B;\ g(b) = a_2\} \leq \mu_2(a_2) \qquad (3)$$

From (2) and (3) we deduce that,

$$\vee\{\eta(b) : b \in B;\ k(b) = a_1,\ g(b)) = a_2\} \leq \mu_1(a_1) \wedge \mu_2(a_2)$$

Hence from (1) it follows that, $\psi(\eta)(a_1, a_2) \leq \mu_1(a_1) \wedge \mu_2(a_2) \quad \ldots \quad (4)$

Also we get,

$$(\mu_1 \oplus \mu_2)(a_1, a_2) = \vee\{\mu_1(x_1, x_2) \wedge \mu_2(y_1, y_2) : (x_1, x_2), (y_1, y_2) \in A_1 \oplus A_2;$$
$$(x_1, x_2) + (y_1, y_2) = (a_1, a_2)\}$$

$$= \mu_1(a_1) \wedge \mu_2(a_2) \tag{5}$$

Now from (4) and (5) we see that $\psi(\eta) \subseteq \mu_1 \oplus \mu_2$. Thus the given short sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ is weakly isomorphic (with identity maps on $\mu_1$ and $\mu_2$) to the short exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$. In particular $\eta \simeq \mu_1 \oplus \mu_2$. This proves the theorem. ■

We have defined the concepts of exact sequences and split exact sequences of $L$-modules. We obtained some results which are extensions of results available in classical module theory. In the last two theorems we gave some sufficient conditions for the exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$ to be weakly isomorphic to the exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ and for the exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ to be weakly isomorphic to the exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$. In the crisp case the conditions given in these two theorems are equivalent and become necessary also.

## 4.3 Semisimple $L$-modules and Split Exact Sequence of $L$-modules.

In this section we establish a relation between semisimple $L$-modules and split exact sequence of $L$-modules.

### 4.3.1 Theorem:

Let $L$ be a regular lattice, $A$ and $B$ be two left $R$-modules and $\mu \in L(A)$, $\eta \in L(B)$ where $\mu^* = A$. Then the sequence

$$0 \to A \xrightarrow{i} A \oplus B \xrightarrow{\pi} B \to 0$$
$$0 \to \mu \longrightarrow \mu \oplus \eta \longrightarrow \eta \to 0$$

is an exact sequence of $L$-modules.

**Proof:**

We have to prove that $0 \rightarrow \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \rightarrow 0$ is exact.

Let $(a, b) \in A \oplus B$. Then $i(\mu)(a, b) = \vee\{\mu(t) : t \in A,\ i(t) = (a, b)\}$

$$= \begin{cases} \mu(a) & \text{if } b = 0 \\ 0 & \text{if } b \neq 0 \end{cases} \qquad \ldots \quad (1)$$

Also, $(\mu + \eta)(a, b) = \mu(a) \wedge \eta(b)$

$$= \mu(a) \ \text{ if } b = 0 \qquad \ldots \quad (2)$$

From (1) and (2) we get $i(\mu)(a, b) \subseteq (\mu+\eta)(a, b) \quad \forall (a, b) \in A \oplus B$

Therefore $i(\mu) \subseteq \mu + \eta$. (3)

For $x \in B$, $(\pi(\mu + \eta))(x) = \vee\{(\mu + \eta)(a, b) : (a, b) \in A \oplus B;\ \pi(a, b) = x\}$

$= \vee\{(\mu + \eta)(a, x) : a \in A\}$

(since $\pi : A \oplus B \rightarrow B$ is the projection.)

$= \vee\{\mu(a) \wedge \eta(x) : a \in A\}$

$= \eta(x)$ (since $\mu(a) = 1$ with $a = 0$)

Thus we get $\pi(\mu + \eta)(x) = \eta(x) \ \forall x \in B$. Hence $\pi(\mu + \eta) = \eta$. ... (4)

Now, since $\mu^* = A$, it follows from (1) that

$i(\mu)(a, b) > 0$ if $b = 0$, that is if $(a, b) \in \text{Ker } \pi$

and $i(\mu)(a, b) = 0$ if $b \neq 0$, that is if $(a, b) \notin \text{Ker } \pi$. ... (5)

From (3), (4) and (5) we see that $0 \rightarrow \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \rightarrow 0$ is an exact sequence of $L$-modules. ∎

**4.3.2 Definition:**

Let $A$ and $B$ be two left $R$-modules; let $\eta \in L(A)$, $\nu \in L(B)$ and $\mu \in L(A \oplus B)$. Then a short exact sequence of $L$-modules of the form

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{i} & A \oplus B & \xrightarrow{\pi} & B & \to 0 \\ 0 \to & \eta & \longrightarrow & \mu & \longrightarrow & \nu & \to 0 \end{array}$$

is said to be a split exact sequence if $\mu = \eta \oplus \nu$.

**4.3.3 Definition:**

Let $A$, $B$ and $C$ be left $R$-modules and let $\eta \in L(A)$, $\mu \in L(B)$, $\nu \in L(C)$. Then a short exact sequence of $L$-modules of the form

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \\ 0 \to & \eta & \longrightarrow & \mu & \longrightarrow & \nu & \to 0 \end{array}$$

is said to be a split exact sequence if $B = A \oplus C$ and $\mu = \eta \oplus \nu$ so that the given sequence is isomorphic to the short exact sequence

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{i} & A \oplus C & \xrightarrow{\pi} & C & \to 0 \\ 0 \to & \eta & \longrightarrow & \eta \oplus \nu & \longrightarrow & \nu & \to 0 \end{array}$$

**4.3.4 Theorem:**

Suppose $L$ is regular. Then all short exact sequences of $L$-modules split if and only if all $L$-modules are semisimple.

**Proof:**

Assume all short exact sequences of $L$-modules are split exact sequences. Let $M$ be a semisimple $R$-module and let $\mu \in L(M)$. To show that $\mu$ is semisimple. That is to show that if $\eta \in L(M)$ is such that $\eta \subseteq \mu$, $\eta \neq 1_{\{0\}}$, $\eta(x) = \mu(x)$ $\forall$ $x$

for which $\eta(x) > 0$, $\eta^* \subset \mu^*$, then there exists a $\lambda \in L(M)$ such that $\lambda \subseteq \mu$, $\lambda \neq 1_{\{0\}}$ , $\lambda(x) = \mu(x)$ $\forall\, x$ for which $\lambda(x) > 0$, $\lambda^* \subset \mu^*$ satisfying $\mu = \eta \oplus \lambda$.

Since $\eta^* \subset \mu^*$ we have the short exact sequence of $R$-modules $0 \to \eta^* \xrightarrow{i} \mu^* \xrightarrow{\pi} \dfrac{\mu^*}{\eta^*} \to 0$. We consider the $L$-modules $\eta \in L(\eta^*)$, $\mu \in L(\mu^*)$ and $\dfrac{\mu}{\eta} \in L\left(\dfrac{\mu^*}{\eta^*}\right)$. We claim that the sequence

$$0 \to \eta^* \xrightarrow{i} \mu^* \xrightarrow{\pi} \frac{\mu^*}{\eta^*} \to 0$$

$$0 \to \eta \longrightarrow \mu \longrightarrow \frac{\mu}{\eta} \to 0$$

of $L$-modules is exact. For:

(i) $\quad i(\eta)(x) = \vee\{\eta(t) : t \in \eta^*,\ i(t) = x\}$

$$= \begin{cases} \eta(x) & \text{if } x \in \eta^* \\ 0 & \text{if } x \notin \eta^* \end{cases}$$

Since $\eta \subseteq \mu$ we get $i(\eta) \subseteq \mu$.

(ii) $\quad (\pi(\mu))(x + \eta^*) = \vee\{\mu(t) : t \in \mu^*;\ \pi(t) = x + \eta^*\}$

$$= \vee\{\mu(t) : t \in \mu^*;\ t + \eta^* = x + \eta^*\}$$

$$= \vee\{\mu(t) : t \in \mu^*;\ t \in x + \eta^*\}$$

$$= \frac{\mu}{\eta}(x + \eta^*) \qquad \text{(see the definition 2.3.2)}$$

Thus $(\pi(\mu))(x + \eta^*) = \dfrac{\mu}{\eta}(x + \eta^*)$ and so $\pi(\mu) = \dfrac{\mu}{\eta}$.

and (iii) since $i(\eta)(x) = \begin{cases} \eta(x) & \text{if } x \in \eta^* \\ 0 & \text{if } x \notin \eta^* \end{cases}$ we see that

$$i(\eta)(x) > 0 \text{ if } x \in \operatorname{Ker} \pi \text{ and } i(\eta)(x) = 0 \text{ if } x \notin \operatorname{Ker} \pi.$$

Since all exact sequences of $L$-modules are split exact sequences we get $\mu^* = \eta^* \oplus \frac{\mu^*}{\eta^*}$ and $\mu = \eta \oplus \frac{\mu}{\eta}$ where $\eta \in L(\eta^*)$ and $\frac{\mu}{\eta} \in L\left(\frac{\mu^*}{\eta^*}\right)$. Now $\frac{\mu^*}{\eta^*}$ can be considered as a submodule say $N$ of $\mu^*$ and hence of $M$ and $\frac{\mu}{\eta} \in L\left(\frac{\mu^*}{\eta^*}\right) = L(N)$ can be extended to $M$ by taking $\frac{\mu}{\eta}(x) = 0 \ \forall \ x \notin N$ and thus $\frac{\mu}{\eta}$ can be considered as an $L$-submodule of $M$. Also note that

$$\begin{aligned}
\left(\frac{\mu}{\eta}\right)^* &= \{x + \eta^* \in \frac{\mu^*}{\eta^*} : \frac{\mu}{\eta}(x + \eta^*) > 0\} \\
&= \{x + \eta^* \in \frac{\mu^*}{\eta^*} : (\vee \{\mu(t) : t \in x + \eta^*\}) > 0\} \\
&= \{x + \eta^* \in \frac{\mu^*}{\eta^*} : \exists \ t \in x + \eta^* \text{ with } \mu(t) > 0\} \\
&= \frac{\mu^*}{\eta^*}
\end{aligned}$$

Thus $\mu^* = \eta^* \oplus N$ where $\mu^*$, $\eta^*$ and $N$ are all submodules of $M$, and since $\mu = \eta \oplus \frac{\mu}{\eta}$ it follows that $\frac{\mu}{\eta}(x) = \mu(x) \ \forall \ x \in N = \frac{\mu^*}{\eta^*} = \left(\frac{\mu}{\eta}\right)^*$. Thus there exists a strictly proper $L$-submodule $\frac{\mu}{\eta}$ of $\mu$ such that $\frac{\mu}{\eta}(x) = \mu(x)\ \forall\ x \in \left(\frac{\mu}{\eta}\right)^*$ and $\mu = \eta \oplus \frac{\mu}{\eta}$. Therefore $\mu$ is semisimple.

Conversely suppose that all $L$-modules are semisimple. Consider the short exact sequence of $L$-modules

$$\begin{array}{l} 0 \to A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \to 0 \\ 0 \to \eta^* \xrightarrow{f|_{\eta^*}} \mu^* \xrightarrow{g|_{\mu^*}} \nu^* \to 0 \\ 0 \to \eta \longrightarrow \mu \longrightarrow \nu \to 0 \end{array}$$

To show that this sequence splits. For this it is enough to show that $\mu = \eta \oplus \nu$. Note that since $B$ is semisimple we have $B=A\oplus C$ and so the short exact sequence $0\to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is isomorphic to $0 \to A \xrightarrow{i} A\oplus C \xrightarrow{\pi} C \to 0$ So we consider the sequence

$$\begin{array}{l} 0 \to A \xrightarrow{\ i\ } A\oplus C \xrightarrow{\ \pi\ } C \to 0 \\ 0 \to \eta^* \xrightarrow{i|_{\eta^*}} \mu^* \xrightarrow{\pi|_{\mu^*}} \nu^* \to 0 \\ 0 \to \eta \longrightarrow \mu \longrightarrow \nu \to 0 \end{array}$$

Obviously $\eta = i(\eta) \subseteq \mu$. Therefore since $\mu$ is semisimple $\mu = \eta \oplus \beta$ for some strictly proper $L$-submodule $\beta$ of $\mu$. Since $L$ is regular we get $\mu^* = \eta^* \oplus \beta^*$. Also since a submodule of a semisimple $R$-module is semisimple we see that $\mu^*$ is semisimple and so we get $\mu^* = \eta^* \oplus \nu^*$. Hence $\beta^* \cong \frac{\mu^*}{\eta^*} \cong \nu^*$. So $\beta$ can be considered as an $L$-submodule of $C$ and we can define $\beta$ to be $\nu$ so that $\mu = \eta \oplus \nu$. This completes the proof of the theorem. ■

The main focus of this chapter was to introduce the concept of exact sequences of $L$-modules by fuzzifying the concepts in crisp theory. We established a relation between semisimple $L$-modules and split exact sequences of $L$-modules. In the next chapter we introduce the concepts of projective and injective $L$-modules and study some properties.

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# Chapter 5

# PROJECTIVE AND INJECTIVE $L$-MODULES



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* Some results of this chapter will appear in a paper accepted for publication by the Iranian Journal of Fuzzy Systems.

** Some other results of this chapter have been presented in the National Seminar on Fuzzy Mathematics and Applications held at Payyannur College, Payyannur, Kerala; January 08-10, 2004.

## 5.1 Introduction.

The notion of free modules, projective modules, injective modules and the like form an important area of study in commutative algebra. Recall that an $R$-module $F$ is a free module on the set $X$ with respect to the function $i : X \to F$ if for every module $A$ over $R$ and for any map $k : X \to A$ there exists a unique homomorphism $h : F \to A$ such that $k = h \circ i$. Also an $R$-module $P$ is projective if for every epimorphism of $R$-modules $g : A \to B$ and for every $R$-module homomorphism $f : P \to B$ there exists an $R$-module homomorphism $h : P \to A$ such that $g \circ h = f$. Free modules are most like vector spaces. The concept of a projective module is a generalisation of that of a free module. From crisp module theory it is well known that every free module is a projective module and an arbitrary projective module (which need not be free) has some of the same properties as a free module. Injectivity is the dual notion to projectivity in crisp theory.

The notion of a free fuzzy module was introduced by Muganda [51] in 1993 which is later generalised to that of a free $L$-module [49]. Zahedi and Ameri [81] introduced the concepts of fuzzy projective and injective modules in 1995. In this chapter we give an alternate definition each for projective $L$-modules and injective $L$-modules, and prove some related results. Also we introduce the concept of essential $L$-submodules of an $L$-module with some related results.

## 5.2 Projective $L$-Modules.

In this section we give an alternate definition for projective $L$-modules and prove that every free $L$-module is a projective $L$-module. Also we prove that if $\mu \in L(P)$ is a projective $L$-module, and if $0 \to \eta \xrightarrow{f} \nu \xrightarrow{g} \mu \to 0$ is a short exact sequence of $L$-modules then $\eta \oplus \mu \simeq \nu$. Further it is proved that if $\mu \in L(P)$ is a projective $L$-module then $\mu$ is a fuzzy direct summand of a free $L$-module.

**5.2.1 Definition [49]:**

Let $F$ be a free module over $R$ on the set $X$ with respect to the function $i : X \to F$. Let $\beta$ be an $L$-subset of $X$. Let $\mu \in L(F)$. Then $\mu$ is said to be free with respect to $\beta$ if $i(\beta) = \mu$ on $i(X)$ and for every module $A$ over $R$, and $\eta \in L(A)$ with $k : X \to A$ and $k(\beta) = \eta$ on $k(X)$, there exists a unique homomorphism $h : F \to A$ such that $k = h \circ i$ and $h(\mu) \subseteq \eta$.

**5.2.2 Definition:**

Let $\mu$ be a fuzzy set in a set $S$. Then $\mu$ is said to have the supremum property if for each subset $A \subseteq S$, there exists $y$ in $A$ such that $\vee\{\mu(x) : x \in A\} = \mu(y)$.

**5.2.3 Definition:**

Let $P$ be a projective $R$-module and let $\mu \in L(P)$. Then $\mu$ is said to be a projective $L$-submodule of $P$ if for every epimorphism of $R$-modules $g : A \to B$,

$\eta \in L(A)$ with supremum property, $\nu \in L(B)$ with $g(\eta) = \nu$ on $g(A)$, and for every $R$-module homomorphism $f : P \to B$ with $f(\mu) = \nu$ on $f(P)$, there exists an $R$-module homomorphism $h : P \to A$ such that $g \circ h = f$ and $h(\mu) \subseteq \eta$.

It is well known from classical module theory that every free $R$-module is a projective $R$-module. The same is also true in the case of $L$-modules as we see in the following theorem.

**5.2.4 Theorem:**

Every free $L$-module is a projective $L$-module.

**Proof:**

Suppose $F$ is free on $X$ with respect to the function $i : X \to F$ and let $\mu \in L(F)$ be free with respect to $\beta \in L^X$. We have to show that $\mu$ is a projective $L$-module. Consider the epimorphism of $R$-modules $g : A \to B$. Let $\eta \in L(A)$ satisfies the supremum property, $\nu \in L(B)$ and $g(\eta) = \nu$ on $g(A)$. Also let $f : F \to B$ be a homomorphism such that $f(\mu) = \nu$ on $f(F)$. We show that there exists an $R$-module homomorphism $h : F \to A$ such that $g \circ h = f$ and $h(\mu) \subseteq \eta$.

Now for $x \in X$, $i(x) \in F$, $f(i(x)) \in B$. Since $g$ is onto there exists $a \in A$ such that $g(a) = f(i(x))$. Since $\eta \in L(A)$ satisfies the supremum property there exists $a_x$ in $A$ such that $g(a_x) = f(i(x))$ and $\eta(a_x) = \vee\{\eta(a) : a \in A,\ g(a) = f(i(x))\}$ and if $f(i(x)) = f(i(y))$, we choose $a_x = a_y$. Now consider the map $k : X \to A$ defined by $k(x) = a_x$. Since $F$ is free on $X$, this extends to an $R$-module homomorphism $h : F \to A$ such that $h \circ i = k$. Thus we have $(h \circ i)(x) = k(x) = a_x\ \forall\ x \in X$ which implies $(g \circ h \circ i)(x) = g(a_x) = f(i(x))\ \forall\ x \in X$. Since $F$ is free on $X$ it follows that $g \circ h = f$.

It remains to prove that $h(\mu) \subseteq \eta$. Since $\mu$ is a free $L$-module with respect to $\beta$, we have $i(\beta) = \mu$ on $i(X)$. Also we have $f(\mu) = \nu$ on $f(F)$ and $g(\eta) = \nu$ on $g(A)$. If we show that $k(\beta) = \eta$ on $k(X)$, then by the definition of a free $L$-module it follows that $h(\mu) \subseteq \eta$. Therefore we need only prove that $k(\beta) = \eta$ on $k(X)$.

Consider the restriction of the map $g : A \to B$ to the subset $k(X)$ of $A$. For convenience we denote this restriction by $g$ itself. Then $g$ is a one-one mapping from $k(X)$ onto $f(i(X))$. Therefore we can consider $g^{-1}: f(i(X)) \to k(X)$. Then obviously $k = g^{-1} \circ f \circ i$. Since $g(\eta) = \nu$ on $g(A)$ we get $g(\eta) = \nu$ on $g(k(X)) = f(i(X))$. Thus for $a_x \in k(X)$, we get $\nu(g(a_x)) = g(\eta)(g(a_x)) = \vee\{\eta(a): g(a) = g(a_x), a \in A\} = \eta(a_x)$. So we have $g^{-1}(\nu)(a_x) = \eta(a_x)\ \forall\ a_x \in k(X)$. Thus on $k(X)$, $\eta = g^{-1}(\nu) = g^{-1}(f(\mu)) = g^{-1}(f(i(\beta)) = (g^{-1} \circ f \circ i)(\beta)$. That is $\eta = k(\beta)$ on $k(X)$. This completes the proof of the theorem. ■

It is well known in classical module theory that an $R$-module $P$ is projective if and only if every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ splits so that $B \cong A \oplus P$. Also we know that $P$ is projective if and only if there exists a free $R$-module $F$ and an $R$-module $K$ such that $F \cong K \oplus P$. Analogous to these, in the case of $L$-modules we have the following theorems.

**5.2.5 Theorem:**

Let $P$ be a projective module and $\mu \in L(P)$ be a projective $L$-module. If $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is a short exact sequence of $R$-modules and $\eta \in L(A)$, $\nu \in L(B)$ are such that $0 \to \eta \xrightarrow{f} \nu \xrightarrow{g} \mu \to 0$ is a short exact sequence of $L$-modules then $\eta \oplus \mu$ is weakly isomorphic to $\nu$. That is $\eta \oplus \mu \simeq \nu$.

**Proof:**

Given $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is a short exact sequence of $R$-modules. Consider the diagram:

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "I_P"] & \\ B \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

Since $P$ is projective there exists a module homomorphism $h : P \to B$ such that $g \circ h = I_P$. Therefore the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ splits and $B \cong A \oplus P$ and the exact sequence $0 \to A \xrightarrow{i} A \oplus P \xrightarrow{\pi} P \to 0$ is isomorphic to the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$.

Now since $\mu \in L(P)$ is a projective $L$-module, from the definition we get $h(\mu) \subseteq \nu$. Thus there exists a homomorphism $h : P \to B$ such that $g \circ h = I_P$ and $h(\mu) \subseteq \nu$. Then by the theorem 3.2.10 we get $0 \to \eta \xrightarrow{i} \eta \oplus \mu \xrightarrow{\pi} \mu \to 0$ is weakly isomorphic to $0 \to \eta \xrightarrow{f} \nu \xrightarrow{g} \mu \to 0$ and $\eta \oplus \mu \simeq \nu$. ∎

### 5.2.6 Theorem:

Let $P$ be a projective module and $\mu \in L(P)$ be a projective $L$-module. Then there exists a free $R$-module $F$ and a free $L$-module $\xi \in L(F)$ such that $\xi = \sigma \oplus \mu$ for some $L$-module $\sigma$.

**Proof:**

Since $P$ is projective, there exists a free $R$-module $F$ and an epimorphism $g : F \to P$ such that $0 \to \mathrm{Ker}\, g \xrightarrow{\subseteq} F \xrightarrow{g} P \to 0$ is split exact so that $F \cong \mathrm{Ker}\, g \oplus P$. Suppose $F$ is free on $B$ and consider the diagram:

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "I_P"] & \\ F \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

Since $P$ is projective there exists an $R$-module homomorphism $h : P \to F$ such that $g \circ h = I_P$. Define $\xi \in L(F)$ by $\xi = g^{-1}(\mu)$. We will show that $\xi$ is a free $L$-submodule of $F$. Obviously $g(\xi)(x) = \mu(x)$ $\forall\, x \in g(F) = P$. That is $g(\xi) = \mu$ on $g(F) = P$. Thus we have $I_P(\mu) = \mu$ on $I_P(P) = P$ and $g(\xi) = \mu$ on $g(F) = P$. Therefore since $\mu \in L(P)$ is projective, $h : P \to F$ is such that $g \circ h = I_P$ and $h(\mu) \subseteq \xi = g^{-1}(\mu)$. Take $\beta = i^{-1}(\xi)$ so that $i(\beta) = \xi$ on $i(B)$. Let $Y$ be any $R$-module and $\eta \in L(Y)$, and let $k : B \to Y$ be a given map. Since $F$ is free on $B$, there exists an $R$-module homomorphism $h' : F \to Y$ such that $h' \circ i = k$. If $\eta \in$

$L(Y)$ is such that $k(\beta) = \eta$ on $k(B)$, then we have to show that $h'(\xi) \subseteq \eta$. But obviously $\eta = k(\beta) = k(i^{-1}(\xi)) = (h' \circ i)(i^{-1}(\xi)) = h'(\xi)$. Therefore $\xi$ is a free $L$-submodule of $F$.

$$\begin{tikzcd} B \arrow[d, "k"'] \arrow[dr, "i"] & & P \arrow[dl, "h"'] \arrow[d, "I_P"] \\ Y & F \arrow[l, "h'"] \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

Now it remains to show that $\xi = \sigma \oplus \mu$ for some $L$-module $\sigma$. We have $F \cong \operatorname{Ker} g \oplus P$. Define $\sigma \in L(F)$ by

$$\sigma(x) = \begin{cases} \xi(x) & \text{if } x \in \operatorname{Ker} g \\ 0 & \text{if } x \notin \operatorname{Ker} g \end{cases}$$

Also we can extend the $\mu \in L(P)$ to $\mu \in L(F)$ by defining $\mu(x) = 0$ for $x \notin P$. Then for all $x \in F$, we have:

$$(\sigma + \mu)(x) = \vee\{\sigma(y) \wedge \mu(z) : y, z \in F,\ y + z = x\}$$

$$= \vee\{\sigma(y) \wedge \mu(z) :\ y \in \operatorname{Ker} g,\ z \in P;\ \ y + z = x\}$$

$$= \vee\{\xi(y) \wedge \xi(z) :\ y \in \operatorname{Ker} g,\ z \in P;\ \ y + z = x\}$$

(since $g$: $F \cong \operatorname{Ker} g \oplus P \to P$ is onto it can be considered as the projection map and so we get $\xi(z) = \mu(z)$ on $P$.)

$$= \vee\{\xi(y + z) :\ y \in \operatorname{Ker} g,\ z \in P\ ;\ y + z = x\}$$

(since $\xi$ is an $L$-module)

$= \xi(x).$

$$\text{Also } (\sigma \cap \mu)(x) = \sigma(x) \wedge \mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Therefore $\sigma + \mu$ is a direct sum. Thus we get $\xi = \sigma \oplus \mu$. ∎

**5.2.7 Theorem:**

Let $P$ be an $R$-module and $\mu \in L(P)$. Let $F$ be a free $R$-module and $K$ be an $R$-module such that $F = K \oplus P$. If $\xi \in L(F)$ is a free $L$-module such that $\xi = \sigma \oplus \mu$ for some $\sigma \in L(K)$, then $\mu$ is a projective $L$-module.

**Proof:**

Consider the diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ A \arrow[r, "g"'] & B \arrow[r] & 0 \end{tikzcd}$$

Let $\eta \in L(A)$ satisfies the supremum property, $\nu \in L(B)$, $g(\eta) = \nu$ on $g(A)$, $f(\mu) = \nu$ on $f(P)$. Since $F \cong K \oplus P$, we have the canonical maps $i : P \to F \cong K \oplus P$ (injection) and $\pi : F \cong K \oplus P \to P$ (projection). Since $F$ is free it is projective and therefore there exists an $R$-module homomorphism $h' : F \to A$ such that $g \circ h' = f \circ \pi$. Consider $h = h' \circ i : P \to A$. Then $g \circ h = g \circ h' \circ i = (f \circ \pi) \circ i = f \circ (\pi \circ i) = f \circ I_P = f$. Therefore $P$ is projective. Now since $\xi \in L(F)$ is free it is projective. Since $\eta \in L(A)$ satisfies the supremum property, $\nu \in L(B)$, $g(\eta) = \nu$ on $g(A)$,

$f(\mu) = \nu$ on $f(P)$ and since $\xi = \sigma \oplus \mu$ ; $F = K \oplus P$ we get $(f \circ \pi)(\xi) = \nu$ on $(f \circ \pi)(F) = f(P)$. For:

$$\begin{tikzcd}
 & F \arrow[d, shift right, "\pi"'] \arrow[ddl, "h'"'] & \\
 & P \arrow[u, shift right, "i"'] \arrow[dl, "h=h'\circ i"'] \arrow[d, "f"] & \\
A \arrow[r, "g"'] & B \arrow[r] & 0
\end{tikzcd}$$

Given $b \in (f \circ \pi)(F) = f(P)$ we have,

$$\begin{aligned}
(f \circ \pi)(\xi)(b) &= \vee\{\xi(y) : y \in F ;\ (f \circ \pi)(y) = b\} \\
&= \vee\{(\sigma \oplus \mu)(k+p) : k \in K,\ p \in P ;\ (f \circ \pi)(k+p) = b\} \\
&= \vee\{(\sigma(k) \wedge \mu(p) : k \in K,\ p \in P ;\ f(p) = b\} \\
&= \vee\{(\sigma(0) \wedge \mu(p) : p \in P ;\ f(p) = b\} \\
&= \vee\{\mu(p) : p \in P ;\ f(p) = b\} \\
&= f(\mu)(b) \\
&= \nu(b)
\end{aligned}$$

Thus $(f \circ \pi)(\xi)(b) = \nu(b)\ \forall\ b \in (f \circ \pi)(F) = f(P)$. Therefore since $\xi \in L(F)$ is projective we get $h'(\xi) \subseteq \eta$ where $g \circ h' = f \circ \pi$. Now to prove that $\mu \in L(P)$ is projective we need only prove that $h(\mu) \subseteq \eta$.

Now $\quad h(\mu)(a) = \vee\{\mu(x) : x \in P ;\ h(x) = a\}$

$$= \vee\{\mu(x) : x \in P ;\ (h' \circ i)(x) = a\}$$

$$= \vee\{\xi(i(x)) : x \in P;\ \ h'(i(x)) = a\}$$

$$\le \vee\{\xi(y) : y \in F;\ \ h'(y) = a\} \ = \ h'(\xi)(a)$$

$$\le \eta(a)$$

Thus we have $h(\mu)(a) \le \eta(a)$ $\forall\ a \in A$. Therefore $\mu \in L(P)$ is projective. ■

**5.2.8 Corollary:**

Let $P_i$ $(i \in I)$ be a projective $R$-module and $\mu_i \in L(P_i)$ $\forall\ i \in I$. Then $\bigoplus\limits_{i \in I} \mu_i$ is projective only if $\mu_i$ is projective $\forall\ i$.

**Proof:**

In the above proof we used only the fact that $\xi$ is a projective $L$-module and therefore the result follows by replacing $\xi$ with $\bigoplus\limits_{i \in I} \mu_i$, $\sigma$ with $\sum\limits_{i \in I \setminus \{j\}} \mu_i$ and $\mu$ with $\mu_j$ in the above proof. ■

## 5.3 Injective $L$-Modules.

Injectivity is the dual notion to projectivity in crisp theory. An $R$-module $J$ is said to be injective if for any pair of $R$-modules $A$, $B$ ; for any monomorphism $g : A \rightarrow B$ and for any $R$-module homomorphism $f : A \rightarrow J$, we have that there exists an $R$-module homomorphism $h : B \rightarrow J$ such that $h \circ g = f$. Zahedi and Ameri [81] introduced the concept of fuzzy injective modules in 1995. In this section we give an alternate definition for injective $L$-modules and prove that a direct sum of $L$-modules is injective if and only if each $L$-summand is injective. Also we prove that if $\mu \in L(J)$ is an injective $L$-module, and if $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is a short exact sequence of $L$-modules then $\nu \simeq \mu \oplus \eta$.

### 5.3.1 Definition:

Let $J$ be an injective $R$-module and let $\mu \in L(J)$. Then $\mu$ is said to be an injective $L$-submodule of $J$ if for $R$-modules $A$, $B$ and $\eta \in L(A)$, $\nu \in L(B)$, $g$ any monomorphism from $A$ to $B$ such that $g(\eta) = \nu$ on $g(A)$, and $f: A \to J$ any $R$-module homomorphism such that $f(\eta) = \mu$ on $f(A)$, we have that there exists an $R$-module homomorphism $h: B \to J$ such that $h \circ g = f$ and $h(\nu) \subseteq \mu$.

From the crisp module theory we know that an $R$-module $J$ is injective if and only if every short exact sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits so that $B \cong J \oplus C$. We have an analogous result in the case of $L$-modules also.

### 5.3.2 Theorem:

Let $J$ be an injective module and $\mu \in L(J)$ be an injective $L$-module. If $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of $R$-modules and $\nu \in L(B)$ and $\eta \in L(C)$ are such that $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is a short exact sequence of $L$-modules, then $\nu$ is weakly isomorphic to $\mu \oplus \eta$. That is $\nu \simeq \mu \oplus \eta$.

**Proof:**

Given $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of $R$-modules. Consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "f"] \arrow[d, "I_J"'] & B \\ & J & \end{tikzcd}$$

Since $J$ is injective there exists a module homomorphism $h : B \to J$ such that $h \circ f = I_J$.

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "f"] \arrow[d, "I_J"'] & B \arrow[dl, "h"] \\ & J & \end{tikzcd}$$

Therefore the short exact sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits and $B \cong J \oplus C$ and the exact sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is isomorphic to the exact sequence $0 \to J \xrightarrow{i} J \oplus C \xrightarrow{\pi} C \to 0$. Now since $\mu \in L(J)$ is an injective $L$-module, from the definition we get $h(\nu) \subseteq \mu$. Thus there exists a homomorphism $h : B \to J$ such that $h \circ f = I_J$ and $h(\nu) \subseteq \mu$. Then by the theorem 3.2.11 we get that the exact sequence $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is weakly isomorphic to the exact sequence $0 \to \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \to 0$ and in particular $\nu \simeq \mu \oplus \eta$. ∎

In the crisp theory we have the theorem: 'A direct sum of modules is injective if and only if each summand is injective'. The same is also true in the fuzzy case.

**5.3.3 Theorem:**

Let $Q_\alpha$ ($\alpha \in I$) be an injective $R$-module and $\mu_\alpha \in L(Q_\alpha)$ $\forall$ $\alpha \in I$. Then $\underset{\alpha\in I}{\oplus} \mu_\alpha \in L(\underset{\alpha\in I}{\oplus} Q_\alpha)$ is injective if and only if $\mu_\alpha$ is injective $\forall$ $\alpha \in I$.

**Proof:**

We know from theory of modules that $\underset{\alpha\in I}{\oplus} Q_\alpha$ is injective if and only if $Q_\alpha$ is injective $\forall \alpha \in I$. Also we know that $\underset{\alpha\in I}{\oplus} \mu_\alpha \in L(\underset{\alpha\in I}{\oplus} Q_\alpha)$. Suppose $\underset{\alpha\in I}{\oplus} \mu_\alpha$ is an injective $L$-submodule of $\underset{\alpha\in I}{\oplus} Q_\alpha$. To prove that $\mu_\alpha$ is injective $\forall$ $\alpha \in I$. Let $A$, $B$ be $R$-modules, $\eta \in L(A)$, $\nu \in L(B)$; $g$ any monomorphism from $A$ to $B$ such that $g(\eta) = \nu$ on $g(A)$. For $\alpha \in I$ if $f_\alpha : A \to Q_\alpha$ is any $R$-module homomorphism such that $f_\alpha(\eta) = \mu_\alpha$ on $f_\alpha(A)$, then we have to show that there exists an $R$-module homomorphism $h_\alpha : B \to Q_\alpha$ such that $h_\alpha \circ g = f_\alpha$ and $h_\alpha(\nu) \subseteq \mu_\alpha$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "g"] \arrow[dr, "f_\alpha"'] & B \arrow[d, "h_\alpha"] \\
& & Q_\alpha \arrow[d, shift right, "i_\alpha"'] \\
& & \oplus Q_\alpha \arrow[u, shift right, "\pi_\alpha"']
\end{tikzcd}$$

Given $\underset{\alpha\in I}{\oplus} Q_\alpha$ is injective. Let $i_\alpha : Q_\alpha \to \underset{\alpha\in I}{\oplus} Q_\alpha$ and $\pi_\alpha : \underset{\alpha\in I}{\oplus} Q_\alpha \to Q_\alpha$ be respectively the canonical injection and projection. Consider $i_\alpha \circ f_\alpha : A \to \underset{\alpha\in I}{\oplus} Q_\alpha$.

First of all we show that $(i_\alpha \circ f_\alpha)(\eta) = \underset{\alpha\in I}{\oplus} \mu_\alpha$ on $(i_\alpha \circ f_\alpha)(A)$. For:

We have $(i_\alpha \circ f_\alpha)(\eta) \in L(\underset{\alpha\in I}{\oplus} Q_\alpha)$ and if $x = \oplus\, x_\alpha \in (i_\alpha \circ f_\alpha)(A) \subseteq \underset{\alpha\in I}{\oplus} Q_\alpha$, where $x_\alpha \in Q_\alpha$ $(\alpha \in I)$, then $x = (i_\alpha \circ f_\alpha)(a)$ for some $a \in A$. That is $x = i_\alpha(f_\alpha(a))$ where $f_\alpha(a) \in Q_\alpha$.

$$
\begin{aligned}
\text{Then}\quad (i_\alpha \circ f_\alpha)(\eta)(x) &= \vee\{\eta(a') : a' \in A\,;\ (i_\alpha \circ f_\alpha)(a') = x\}\\
&= \vee\{\eta(a') : a' \in A\,;\ (i_\alpha \circ f_\alpha)(a') = (i_\alpha \circ f_\alpha)(a)\}\\
&= \vee\{\eta(a') : a' \in A\,;\ i_\alpha(f_\alpha(a')) = i_\alpha(f_\alpha(a))\}\\
&= \vee\{\eta(a') : a' \in A\,;\ f_\alpha(a') = f_\alpha(a)\} \qquad \ldots \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
\text{Also}\quad \underset{\alpha\in I}{\oplus} \mu_\alpha(x) &= \vee\{\wedge \mu_\alpha(x_\alpha) : x = \sum_{\alpha\in I} x_\alpha\}\\
&= \mu_\alpha(f_\alpha(a)) \qquad \text{(since supremum is attained for the direct sum decomposition } x = 0 + 0 + \ldots + 0 + f_\alpha(a) + 0 + \ldots + 0.)\\
&= f_\alpha(\eta)(f_\alpha(a))\\
&= \vee\{\eta(a') : a' \in A;\ f_\alpha(a') = f_\alpha(a)\} \qquad (2)
\end{aligned}
$$

From (1) and (2) we get $(i_\alpha \circ f_\alpha)(\eta)(x) = \underset{\alpha\in I}{\oplus} \mu_\alpha(x)\ \forall\, x \in (i_\alpha \circ f_\alpha)(A)$.

Now since $\underset{\alpha\in I}{\oplus} \mu_\alpha$ is injective we get $i_\alpha \circ f_\alpha : A \to \underset{\alpha\in I}{\oplus} Q_\alpha$ has an extension $k : B \to \underset{\alpha\in I}{\oplus} Q_\alpha$ satisfying $k(\nu) \subseteq \underset{\alpha\in I}{\oplus} \mu_\alpha$. Take $h_\alpha = \pi_\alpha \circ k$. Then $h_\alpha : B \to Q_\alpha$ is an extension of $f_\alpha : A \to Q_\alpha$ satisfying $h_\alpha \circ g = f_\alpha$. It remains to prove that $h_\alpha(\nu) \subseteq \mu_\alpha$.

We have $k(\nu) \subseteq \underset{\alpha\in I}{\oplus} \mu_\alpha$. Therefore $\pi_\alpha(k(\nu)) \subseteq \pi_\alpha(\underset{\alpha\in I}{\oplus} \mu_\alpha)$ . . . (3)

Now for $x_\alpha \in Q_\alpha$,

$$(\pi_\alpha(\underset{\alpha\in I}{\oplus} \mu_\alpha))(x_\alpha) = \vee\{\underset{\alpha\in I}{\oplus} \mu_\alpha(y) : y \in \underset{\alpha\in I}{\oplus} Q_\alpha;\ \pi_\alpha(y) = x_\alpha\}$$

$$= \mu_\alpha(x_\alpha) \qquad \text{(since supremum is attained}$$

$$\text{for } y = (0,\ldots,0, x_\alpha, 0,\ldots,0)\ )$$

Thus $\pi_\alpha(\underset{\alpha\in I}{\oplus} \mu_\alpha) = \mu_\alpha$ and so from (3) we get $(\pi_\alpha \circ k)(\nu) \subseteq \mu_\alpha$. Thus $h_\alpha(\nu) \subseteq \mu_\alpha$ as required.

Conversely suppose that $\mu_\alpha$ is injective $\forall\ \alpha \in I$. To prove that $\underset{\alpha\in I}{\oplus} \mu_\alpha$ is injective.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "\pi_\alpha f"] \arrow[dd, bend right=60, "f"'] & B \arrow[dl, "k_\alpha"] \arrow[ddl, "k"] \\
& Q_\alpha \arrow[d, shift right, "i_\alpha"'] & \\
& \oplus Q_\alpha \arrow[u, shift right, "\pi_\alpha"'] &
\end{tikzcd}$$

Since $Q_\alpha$ is injective $\forall \alpha \in I$ we have $\underset{\alpha\in I}{\oplus} Q_\alpha$ is injective. Let $A$, $B$ be $R$-modules, $\eta \in L(A)$, $\nu \in L(B)$; $g$ any monomorphism from $A$ to $B$ such that $g(\eta) = \nu$ on $g(A)$, and suppose that $f : A \to \underset{\alpha\in I}{\oplus} Q_\alpha$ is a module homomorphism satisfying $f(\eta) = \underset{\alpha\in I}{\oplus} \mu_\alpha$ on $f(A)$. Then $\pi_\alpha \circ f : A \to Q_\alpha$ admits an extension $k_\alpha : B \to Q_\alpha$ such that

$\pi_\alpha \circ f = k_\alpha \circ g$. These homomorphisms $k_\alpha$ give $k : B \to \underset{\alpha \in I}{\oplus} Q_\alpha$ such that $\pi_\alpha \circ k = k_\alpha$ and for each $x \in A$, $(\pi_\alpha \circ k)(g(x)) = k_\alpha(g(x)) = (\pi_\alpha \circ f)(x)$ $\forall \alpha \in I$. Therefore $k(g(x)) = f(x)$ $\forall$ $x \in A$. Therefore $k$ is an extension of $f$ such that $k \circ g = f$. Now if $k_\alpha$ are such that $k_\alpha(\nu) \subseteq \mu_\alpha$, we have to prove that $k(\nu) \subseteq \underset{\alpha \in I}{\oplus} \mu_\alpha$. Since $f(\eta) = \underset{\alpha \in I}{\oplus} \mu_\alpha$ on $f(A)$, we get $(\pi_\alpha \circ f)(\eta) = \pi_\alpha(\underset{\alpha \in I}{\oplus} \mu_\alpha) = \mu_\alpha$ on $(\pi_\alpha \circ f)(A)$.

Now $k_\alpha(\nu) \subseteq \mu_\alpha \; \forall \; \alpha \quad \Rightarrow (\pi_\alpha \circ k)(\nu) \subseteq \mu_\alpha \forall \; \alpha$

$$\Rightarrow \pi_\alpha(k(\nu)) \subseteq \mu_\alpha = \pi_\alpha(\underset{\alpha \in I}{\oplus} \mu_\alpha) \; \forall \; \alpha$$

$$\Rightarrow k(\nu) \subseteq \underset{\alpha \in I}{\oplus} \mu_\alpha$$

This completes the proof of the theorem. ■

## 5.4 Essential $L$-Submodules of an $L$-Module.

From crisp theory we know that an essential submodule of an $R$-module $B$ is any submodule $A$ which has nonzero intersection with every nonzero submodule of $B$. We denote this situation by writing $A \subseteq_e B$, and we also say that $B$ is an essential extension of $A$. In this section we extend the concept of an essential submodule of an $R$-module to the fuzzy setting and prove some results.

### 5.4.1 Definition:

Let $M$ be an $R$-module and $\eta, \mu \in L(M)$ be such that $1_{\{0\}} \neq \eta \subseteq \mu$. Then $\eta$ is called an essential $L$-submodule of $\mu$ if $\eta \cap \nu \neq 1_{\{0\}}$ $\forall \nu \in L(M)$ such that $1_{\{0\}} \neq \nu \subseteq \mu$. We denote this by writing $\eta \subseteq_e \mu$.

**5.4.2 Theorem:**

Let $L$ be regular, $\mu \in L(M)$. Then $1_{\{0\}} \neq \eta \subseteq \mu$ ; $\eta \in L(M)$ is an essential $L$-submodule of $\mu$ if and only if for each $0 \neq x \in M$, with $\mu(x) > 0$, there exists an $r \in R$ such that $rx \neq 0$ and $\eta(rx) > 0$.

**Proof:**

Assume that for each $0 \neq x \in M$, with $\mu(x) > 0$, there exists an $r \in R$ such that $rx \neq 0$ and $\eta(rx) > 0$. Take any $\nu \in L(M)$, $1_{\{0\}} \neq \nu \subseteq \mu$. To show that $\eta \cap \nu \neq 1_{\{0\}}$. Let $x \in M$ be such that $x \neq 0$, $\nu(x) > 0$. Then $\mu(x) \geq \nu(x) > 0$ and therefore there exists an $r \in R$ such that $rx \neq 0$ and $\eta(rx) > 0$. Also $\nu(rx) \geq \nu(x) > 0$. Therefore since $L$ is regular $(\eta \cap \nu)(rx) = \eta(rx) \wedge \nu(rx) > 0$. Thus there exists $rx \neq 0$ such that $(\eta \cap \nu)(rx) > 0$. Therefore $\eta \cap \nu \neq 1_{\{0\}}$.

Conversely suppose that $\eta \subseteq_e \mu$. Let $0 \neq x \in M$ be such that $\mu(x) > 0$. Then $\forall\ r \in R$, $\mu(rx) \geq \mu(x) > 0$. Consider the nonzero submodule $R\,x$ of $M$. Define $\nu \in L^M$ by,

$$\nu(y) = \begin{cases} \mu(y) & \text{if } y \in Rx \\ 0 & \text{otherwise} \end{cases}$$

Obviously $\nu \in L(M)$ and $1_{\{0\}} \neq \nu \subseteq \mu$.. Therefore $\eta \cap \nu \neq 1_{\{0\}}$ and hence there exists $y \neq 0$ satisfying $\eta(y) \wedge \nu(y) > 0$. Thus there exists $y \neq 0$ such that $\eta(y) > 0$ and $\nu(y) > 0$. From this it follows that $y \in R\,x$ and we get that there exists $r \in R$ such that $rx \neq 0$, $\eta(rx) > 0$. This completes the proof of the theorem. ■

**5.4.3 Theorem:**

Let $\eta$, $\nu$, $\mu \in L(M)$ be such that $\eta \subseteq \nu \subseteq \mu$. Then $\eta \subseteq_e \mu$ if and only if $\eta \subseteq_e \nu$ and $\nu \subseteq_e \mu$.

**Proof:**

Assume that $\eta \subseteq_e \mu$. Then $\eta \cap \theta \neq 1_{\{0\}}$ $\forall \theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \mu$. Since $\nu \subseteq \mu$ it follows that $\eta \cap \theta \neq 1_{\{0\}}$ $\forall \theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \nu$. Therefore $\eta \subseteq_e \nu$. Also since $\eta \cap \theta \neq 1_{\{0\}} \forall \theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \mu$, and since $\eta \subseteq \nu$ we get $\nu \cap \theta \neq 1_{\{0\}}$ $\forall \theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \mu$. Hence $\nu \subseteq_e \mu$.

Conversely suppose that $\eta \subseteq_e \nu$ and $\nu \subseteq_e \mu$. To prove that $\eta \subseteq_e \mu$. Since $\nu \subseteq_e \mu$ we have $\nu \cap \theta \neq 1_{\{0\}}$ $\forall \theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \mu$. Then $\nu \cap \theta \in L(M)$ satisfies $1_{\{0\}} \neq \nu \cap \theta \subseteq \nu$ and therefore, since $\eta \subseteq_e \nu$, we get $\eta \cap (\nu \cap \theta) \neq 1_{\{0\}}$. Since $\eta \subseteq \nu$ it follows $\eta \cap \theta \neq 1_{\{0\}}$ $\forall \theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \mu$. Therefore $\eta \subseteq_e \mu$. ■

**5.4.4 Theorem:**

Let $\eta_1$, $\eta_2$, $\mu_1$, $\mu_2 \in L(M)$. If $\eta_1 \subseteq_e \mu_1$ and $\eta_2 \subseteq_e \mu_2$, then $\eta_1 \cap \eta_2 \subseteq_e \mu_1 \cap \mu_2$.

**Proof:**

Let $\theta \in L(M)$ be such that $1_{\{0\}} \neq \theta \subseteq \mu_1 \cap \mu_2 \subseteq \mu_2$. Then since $\eta_2 \subseteq_e \mu_2$, we have $\eta_2 \cap \theta \neq 1_{\{0\}}$. Since $\theta \subseteq \mu_1$, we get $1_{\{0\}} \neq \eta_2 \cap \theta \subseteq \mu_1$. Therefore since $\eta_1 \subseteq_e \mu_1$, we get $\eta_1 \cap (\eta_2 \cap \theta) \neq 1_{\{0\}}$. Thus we get $(\eta_1 \cap \eta_2) \cap \theta \neq 1_{\{0\}}$ $\forall \theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \mu_1 \cap \mu_2$. Hence $\eta_1 \cap \eta_2 \subseteq_e \mu_1 \cap \mu_2$. ■

**5.4.5 Theorem:**

Let $L$ be regular $\eta$, $\mu \in L(M)$ where $\eta \subseteq \mu$. Let $f : A \to M$ be a module homomorphism such that $f(\nu) \subseteq \mu$ where $\nu \in L(A)$. If $\eta \subseteq_e \mu$, then $f^{-1}(\eta) \subseteq_e \nu$.

**Proof:**

We have to prove that $f^{-1}(\eta) \cap \theta \neq 1_{\{0\}}$ $\forall$ $\theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \nu$. That is to show that for given $\theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \nu$, there exists $0 \neq x \in A$ such that $(f^{-1}(\eta) \cap \theta)(x) \neq 0$; that is such that $f^{-1}(\eta)(x) \wedge \theta(x) \neq 0$; that is such that $\eta(f(x)) \wedge \theta(x) \neq 0$.

If $f(\theta) = 1_{\{0\}}$, then $\theta \subseteq f^{-1}(\eta)$. For: $\forall$ $z$ with $f^{-1}(z) \neq \phi$ we have,

$$f(\theta)(z) = 1_{\{0\}}(z) \quad \Rightarrow \vee\{\theta(x) : x \in A,\ f(x) = z\} = \begin{cases} 1 & \text{if } z = 0 \\ 0 & \text{if } z \neq 0 \end{cases}$$

$$\Rightarrow \vee\{\theta(x) : x \in A,\ f(x) = z\} = 0 \text{ if } z \neq 0$$

$$\Rightarrow \{\theta(x) : x \in A,\ f(x) = z\} = \{0\} \text{ if } z \neq 0$$

$$\Rightarrow \theta(x) = 0 \text{ if } f(x) \neq 0$$

Also $\eta(f(x)) = \eta(0) = 1$ if $f(x) = 0$. Therefore $\theta(x) \leq \eta(f(x))$ $\forall$ $x \in A$. Thus in this case $\theta \subseteq f^{-1}(\eta)$ and so we get $f^{-1}(\eta) \cap \theta = \theta \neq 1_{\{0\}}$.

If $f(\theta) \neq 1_{\{0\}}$, to prove that $f^{-1}(\eta) \cap \theta \neq 1_{\{0\}}$ for $\theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \nu$. We have $\theta \subseteq \nu \Rightarrow f(\theta) \subseteq f(\nu) \Rightarrow f(\theta) \subseteq \mu$ as $f(\nu) \subseteq \mu$. Therefore if $f(\theta) \neq 1_{\{0\}}$, since $\eta \subseteq_e \mu$, we get $\eta \cap f(\theta) \neq 1_{\{0\}}$. From this we get $f(\theta)(x) \neq 0$ for some $x \neq 0$. This shows that there exists $y \in A$ with $\theta(y) \neq 0$, where $f(y) = x$. For this $y$

we have, $\eta(f(y)) \wedge f(\theta)f(y) \neq 0$. This implies both $\eta(f(y))$ and $f(\theta)f(y) > 0$. Since $L$ is regular we get $f^{-1}(\eta)(y) \wedge \theta(y) \neq 0$. Hence $f^{-1}(\eta) \cap \theta \neq 1_{\{0\}}$. ■

**5.4.6 Theorem:**

Let $L$ be regular and $\eta_1, \eta_2, \mu_1, \mu_2 \in L(M)$ be such that $\eta_i \subseteq_e \mu_i$; $i=1, 2$. If $\eta_1 \cap \eta_2 = 1_{\{0\}}$, then $\mu_1 \cap \mu_2 = 1_{\{0\}}$ and $\eta_1 \oplus \eta_2 \subseteq_e \mu_1 \oplus \mu_2$.

**Proof:**

First of all we prove that if $\eta \subseteq_e \mu$ then $\eta^* \subseteq_e \mu^*$ and conversely if $L$ is regular, and $\eta^* \subseteq_e \mu^*$ then $\eta \subseteq_e \mu$. Suppose that $\eta \subseteq_e \mu$. Then $\eta \cap \theta \neq 1_{\{0\}}$ $\forall$ $\theta \in L(M)$, $1_{\{0\}} \neq \theta \subseteq \mu$. Let $0 \neq A$ be a submodule of $\mu^*$. Define $\theta \in L^M$ by

$$\theta(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then obviously $1_{\{0\}} \neq \theta \in L(M)$ and therefore $\eta \cap \theta \neq 1_{\{0\}}$. Therefore there exists $0 \neq x \in A$ such that $\eta(x) \wedge \theta(x) \neq 0$. This shows that $\eta^* \cap \theta^* \neq \{0\}$. That is $\eta^* \cap A \neq \{0\}$. Hence $\eta^* \subseteq_e \mu^*$. Conversely suppose that $\eta^* \subseteq_e \mu^*$. We prove that, if $L$ is regular, $\eta \subseteq_e \mu$. For this consider any $1_{\{0\}} \neq \theta \subseteq \mu$ where $\theta \in L(M)$. Then $\theta^* \neq \{0\}$ and $\theta^* \subseteq \mu^*$. Therefore $\eta^* \cap \theta^* \neq \{0\}$.This means that there exists $x \neq 0$ such that $\eta(x) > 0$ and $\theta(x) > 0$. Since $L$ is regular we get $\eta(x) \wedge \theta(x) > 0$. Thus we get $\eta \cap \theta \neq 1_{\{0\}}$. Hence $\eta \subseteq_e \mu$.

Now to prove the theorem, we have $\eta_i \subseteq_e \mu_i$; $i = 1, 2$. Therefore by the above result we get $\eta_i^* \subseteq_e \mu_i^*$; $i = 1, 2$. Since $\eta_1 \cap \eta_2 = 1_{\{0\}}$, the sum $\eta_1 + \eta_2$ is the direct sum $\eta_1 \oplus \eta_2$. Since $\eta_1 \cap \eta_2 \subseteq_e \mu_1 \cap \mu_2$, it follows that $\mu_1 \cap \mu_2 = 1_{\{0\}}$, and so the sum $\mu_1 + \mu_2$ is also the direct sum $\mu_1 \oplus \mu_2$. Therefore since $L$ is regular we have the direct sums of $R$-modules $\eta_1^* \oplus \eta_2^*$ and $\mu_1^* \oplus \mu_2^*$. Since $\eta_i^* \subseteq_e \mu_i^*$, $i = 1, 2$ we get $\eta_1^* \oplus \eta_2^* \subseteq_e \mu_1^* \oplus \mu_2^*$. From this it follows that $(\eta_1 \oplus \eta_2)^* \subseteq_e (\mu_1 \oplus \mu_2)^*$ and so $\eta_1 \oplus \eta_2 \subseteq_e \mu_1 \oplus \mu_2$. ■

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# Conclusion

Since the publication of the classic paper on fuzzy sets by L.A. Zadeh in 1965, the theory of fuzzy mathematics has gained more and more recognition from many researchers in a wide range of scientific fields. Among various branches of pure and applied mathematics, algebra was one of the first few subjects where the notion of fuzzy set was applied. Ever since A. Rosenfeld introduced fuzzy sets in the realm of group theory in 1971, many researchers have been involved in extending the notions of abstract algebra to the broader framework of fuzzy setting. As a result, a number of concepts have been formulated and explored. However many concepts are yet to be 'fuzzified'. The main objective of this thesis was to extend some basic concepts and results in module theory in algebra to the fuzzy setting.

The concepts like simple module, semisimple module and exact sequences of $R$-modules form an important area of study in crisp module theory. In this thesis generalising these concepts to the fuzzy setting we have introduced concepts of 'simple and semisimple $L$-modules' and proved some results which include results analogous to those in crisp case. Also we have defined and studied the concept of 'exact sequences of $L$-modules'.

Further extending the concepts in crisp theory, we have introduced the fuzzy analogues 'projective and injective $L$-modules'. We have proved many results in this context. Further we have defined and explored notion of 'essential $L$-submodules of an $L$-module'. Still there are results in crisp theory related to the topics covered in this thesis which are to be investigated in the fuzzy setting.

There are a lot of ideas still left in algebra, related to the theory of modules, such as the 'injective hull of a module', 'tensor product of modules' etc. for which the fuzzy analogues are not defined and explored.

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