# ON FINITE MIXTURE OF PARETO AND BETA DISTRIBUTIONS

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# Certificate

Certified that the thesis entitled 'On Finite Mixture of Pareto and Beta Distributions' is a bona fide record of work done by Mrs. Maya T. Nair under my guidance, in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included in anywhere previously for the award of any degree or title.

Cochin-22
19-02-2007

Dr.P.G.Sankaran
Research Guide.

# DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

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# Chapter 1

# INTRODUCTION AND REVIEW OF LITERATURE

## 1.1 Introduction

Finite mixtures of distributions provide an important tool in modelling a wide range of observed phenomena, which do not normally yield to modelling through classical distributions like normal, gamma, Poisson, binomial, etc., on account of their heterogeneous nature and inherent complexity. In a finite mixture model, the distribution of random quantity of interest is modelled as a mixture of a finite number of component distributions in varying proportions. A mixture model is, thus, able to model quite complex situations through an appropriate choice of its components to represent accurately the local areas of support of the true distribution. It can handle situations where a single parametric family is unable to provide a satisfactory model for local variation in the observed data. The flexibility and high degree of accuracy of finite mixture models have been the main reason for their successful applications in a wide range of fields in the biological, physical and social sciences.

The concept of finite mixture distribution was pioneered by Newcomb (1886) as a model for outliers. However, the credit for the introduction of statistical modelling using finite mixtures of distributions goes to Pearson (1894) while applying the technique in an analysis of crab morphometry data provided by Weldon (1892, 1893). The data was a set of measurements on the ratio of forehead to body length for 1000 crabs. A plot of the data showed that they were skewed to the right. Weldon (1893) suggested that the reason for this skewness might be that the sample contained representatives of two types of crab but when the data were collected no such differentiation had been recorded. Thus, Pearson (1894) proposed that the distribution of the measurements might be modelled by a weighted sum of two normal distributions, with the two weights being the proportion of the crabs of each type. Mathematically, Pearson suggested the distribution for the measurement on the crabs was of the form

$$f(x) = pN(x;\mu_1,\sigma_1) + (1-p)N(x;\mu_2,\sigma_2), \qquad (1.1)$$

where $p$ is the proportion of a type of crab for which the ratio of forehead to body length has mean $\mu_1$ and standard deviation $\sigma_1$ and $(1-p)$ is the proportion of a type of crab for which the ratio of forehead to body length has mean $\mu_2$ and standard deviation $\sigma_2$. In (1.1), $N(x;\mu_i,\sigma_i)$ is the normal density given by

$$f_i(x) = \frac{1}{\sqrt{2\pi}\sigma_i}\exp[-\frac{1}{2\sigma_i^2}(x-\mu_i)^2], \qquad -\infty < x < \infty,$$

$$-\infty < \mu_i < \infty, \quad \sigma_i > 0, \quad i = 1,2.$$

The density (1.1) will be bimodal if the two component distributions are widely separated or will simply display a degree of skewness when the separation of the components is not so great. To model the distribution (1.1) to a set of data, the parameters $p,\mu_1,\mu_2,\sigma_1$ and $\sigma_2$ has to be estimated. Pearson (1894) obtained the moments-based estimates of the five parameters of the mixture of normal distribution as a solution of a ninth degree polynomial which was a task, at the time so computationally demanding. Various attempts were made over the ensüing years to simplify Pearson's (1894) moments based approach to fitting of a normal mixture model. The use of mixture of normal distributions (1.1) to model the different species of crab motivated extensive use of finite mixture distributions in other applied sciences.

## 1.2 Definition

Let $X$ be a random variable with a family of probability distributions $\{g(\mathbf{x};\underline{\theta});\underline{\theta}\in\Theta\}$, where the parameter space $\Theta$ is a subset of $R^m$, the $m^{th}$ dimensional Eucledian space. Let $G(\underline{\theta})$ be a cumulative distribution function of $\underline{\theta}$. Then the probability density function $f(x)$ defined by

$$f(x) = \int_{\Theta} g(x;\underline{\theta})dG(\underline{\theta}), \tag{1.2}$$

is called a general mixture density function. In (1.2), $G(.)$ is called the mixing distribution.

When $G(.)$ is discrete and assigns positive probability to only a finite number of points $\theta_i$ $(i = 1,2,...,k)$, the density (1.2) can be written in the form

$$f(x,\theta) = p_1 f_1(x,\theta) + p_2 f_2(x,\theta) + ... + p_k f_k(x,\theta) \tag{1.3}$$

where

$$f_i(x,\theta) \geq 0, \quad \int f_i(x,\theta)dx = 1, \quad p_i \geq 0 \quad (i = 1,2,...,k) \text{ and } p_1 + p_2 + ... + p_k = 1.$$

Then (1.3) is called a finite mixture of densities. The constants $p_i$'s are called mixing weights and $f_i$'s are called component densities.

## 1.3 Identifiability

The concept of identifiability plays a vital role in the analysis of the finite mixture models. A mixture is identifiable if there exists a one-to-one correspondence between the mixing distribution and the resulting mixture. Mixtures that are not identifiable cannot be expressed uniquely as functions of component and mixing distributions. For example, the finite mixture of uniform distributions is not identifiable, since the mixture density $f(x)$ can be represented in two different forms

$$f(x) = \frac{1}{3}U(-1,1) + \frac{2}{3}U(-2,2)$$

and

$$f(x) = \frac{1}{2}U(-2,1) + \frac{1}{2}U(-1,2).$$

where $U(a,b)$ is a uniform distribution over the interval $(a,b)$.

### 1.3.1 Definition

Consider the mixture density function given in (1.2). Let $\Lambda$ be the class of all mixing distributions $G$ and $\zeta$ be the corresponding class of mixtures. Define $Q:\Lambda \rightarrow \zeta$ by $Q(G) = f$. The class $\Lambda$ and equivalently the family $\zeta$ is said to be identifiable with respect to the family $\{f(x;\underline{\theta});\underline{\theta}\in \Theta\}$ of probability distributions if the mapping $Q$ is a one-to-one mapping between $\Lambda$ and $\zeta$. That is, for each element in one class we can identify one corresponding element in the other class. A class $\zeta$ of finite mixtures of densities is said to be identifiable if and only if, for all $f(x,\underline{\theta})$ belonging to $\zeta$, the equality of two representations

$$\sum_{i=1}^{k} p_i f_i(x,\theta_i) = \sum_{i=1}^{k'} p_i' f_i(x,\theta_i')$$

implies that $k = k'$, $p_i = p_i'$ and $\theta_i = \theta_i'$, $i = 1,2,...,k$.

The importance of identifiability is whether or not the distribution $G$ can be uniquely determined from observations. The inference procedures on the mixture distributions can be meaningfully discussed only if the family of mixture distributions is identifiable.

Teicher (1960, 1961, 1963, and 1967) introduced the concept of identifiability and developed a theory to identify mixtures. Teicher (1960) showed that a finite mixture of Poisson distributions is identifiable, where as mixtures of binomial distributions are not identifiable if

$$N < 2k - 1,$$

where $N$ is the common number of trials in the component binomial distributions and $k$ is the number of components in the mixture. Yakowitz and Spragins (1968) showed that finite mixtures of negative binomial component distributions are identifiable. Titterington et.al. (1985) has given a lucid account of the concept of identifiability of mixtures. They pointed out that most of the finite mixtures of continuous densities are identifiable; an exception is a mixture of uniform densities. Some discussion on identifiability of finite mixtures is given in Patil and Bildikar (1966), McLachlan and

Basford (1988) and Maritz and Levin (1989). Prakasa Rao (1992) and Lindsay (1995) have given a nice review on this topic. Recently, Al-Hussaini et.al. (2000) showed that a finite mixture of Gompertz densities is identifiable.

## 1.4 Estimation of parameters

Over the years, a number of methods have been suggested for estimating the parameters in a finite mixture model. Pearson (1894), for example, applied the method of moments, which needs to find the roots of a ninth degree polynomial to derive the estimates of the five parameters involved in the model (1.1). Cohen (1967) subsequently developed an iterative method for solving the same problem that only requires solving cubic polynomials.

The method of moments was shown to be inferior to maximum likelihood estimation (M.L.E) in a mixture of two normal distributions by Fryer and Robertson (1972) and Tan and Chang (1972). Indeed, the method of moments does not guarantee any sort of optimality of solutions but was initially useful in certain situations where the M.L.E solutions were intractable. However, M.L.E became popular for general mixture problems with the availability of digital computers and the development of the expectation maximization (E.M) algorithm by Dempster et.al. (1977).

Redner and Walker (1984) pointed out some difficulties with M.L.E for finite mixtures. First, the likelihood function cannot be assumed generally to have an upper bound, creating the possibility of divergence. Second, there are often many sub-optimal local minima of the likelihood function. McLachlan and Basford (1988) pointed out that these difficulties make the performance of the E.M algorithm very sensitive to the starting value of the model parameters. The E.M algorithm does not have the ability to escape sub-optimal local minima and can diverge if initialized close to a singularity in the likelihood function. Furthermore, the convergence is generally slow and this is exacerbated by a poor initialization.

Another method for the estimation of parameters of the mixture models is Bayes estimation, where the likelihood of the data is combined with prior belief about the parameters to draw an inference. Use of prior information gives an advantage for Bayes method over M.L.E since an inference can be made even with a small number of data points. M.L.E solutions, especially for models with many parameters, become ill-posed when the data set is small. However, in certain situations, Bayes estimation of parameters of mixture distributions currently suffers from the lengthy computations required to perform the inference, as integrals must be performed over a potentially unbounded multidimensional space. Further, the posterior inference can rarely be sampled directly, simplifying conjugate priors rarely exist, and in many situations there are no sufficient statistics to simplify the analysis.

## 1.5 Estimation of number of components

In the analysis of finite mixture models, a question that remains is how to estimate $k$, the number of components in the mixture. A likelihood ratio test is a natural candidate for testing, say $k = k_0$ against $k = k_1 (k_1 > k_0)$, but this is well known to have problems in the context of finite mixture densities (see Everitt (1996)); nevertheless, the test can still often be useful as an informal indicator of number of components.

A variety of other techniques to estimate the number of components in a mixture is described in McLachlan and Peel (2000). The problem is essentially equivalent to that of determining the number of clusters in the samples. (see Everitt et. al. (2001)).

## 1.6 Fields of applications

Finite mixture models are being used extensively for statistical analysis in many real life situations. For example, in the study of the distribution of length of fish, the sample of measurements consist of observations either from more than one species or from more than one age group of one species. The biologist, however, would have no way of classifying the items of the sample according to different age group. Thus the situation becomes a mixture distribution, the mixing being over a parameter depending on the unobservable variate age. The graph on measurements on length of fish will show multimodality if the fish stock contains populations from different years's spawning.

In reliability theory, the mixture distributions are used for the analysis of the failure times of a sample of items of coherent laser used in telecommunication network. In an experiment, one hundred and three laser devices were operated at a temperature of 70 degree Celsius until all had failed. The experiment was run longer than one year before all the devices had failed, because most of the devices were extremely reliable. The sample thus consists of two distinct populations, one with a very short mean life and one with a much longer mean life. This can be considered as an example of a mixture of two exponential distributions with probability density function of the form

$$f(x;\lambda_1,\lambda_2)=\frac{p}{\lambda_1}\exp[\frac{-x}{\lambda_1}]+\frac{(1-p)}{\lambda_2}\exp[\frac{-x}{\lambda_2}] \quad ,0<x<\infty$$

where $0<p<1$ and $\lambda_i>0$ , $i=1,2$.

The above model will be useful to predict how long all manufactured lasers should be life tested to assure that the final product contained no device from the infant mortality population.

There are several areas in which mixture distributions are being applied more frequently. Chronologically, a representative cross-section of the field of applications include the study on distributions of evening temperatures (Charlier and Wicksell, 1924), mice death times (Muench, 1936), comet frequencies (Schilling, 1947), chromosome association (Skellam, 1948), lifetime of valves (Davis, 1952; Everitt and Hand, 1981), plankton frequencies (Cassie, 1962), plant heights (Tanaka, 1962), response times (Cox,1966), death notice frequencies (Hasselblad, 1969), pike lengths (Macdonald, 1971), gaps in traffic (Ashton, 1971), Pollen grains (Usinger, 1975), crop concentrations (Peters and Coberly, 1976), clinical test scores (Symons, 1981), crime frequencies (Harris, 1983), mixed stock fishery composition (Miller, 1987), philatelic mixtures (Izenmann and Sommer,1988), and task completion (Desmond and Chapman, 1993). Apart from these applications, mixture models are useful in robustness studies (Hyrenius, 1950; Tan, 1980), cluster analysis and latent structure models (Fielding, 1977; Symons, 1981 and McLachlan and Basford, 1988), approximating other distributions (Dalal, 1978), random variate generation (Peterson and Kronmal, 1982), kernel based density estimation (Titterington, 1983), analyzing outliers (Barnett and Lewis, 1984), modelling prior densities (Diaconis and Ylvisaker, 1985) and artificial neural networks (Ripley, 1994). Finite mixture densities are also useful in medical research to model age of onset of schizophrenia (Levine, 1981; McLachlan, 1987; McLachlan and Peel, 2000 and Everitt, 2003). Finite mixture distributions are useful to model variation of mortality rates between geographical areas (Betemps and Buncher, 1993), to model survival data (McGiffin et.al., 1993; McLachlan and McGiffin, 1994 and McLachlan and Peel, 2000) and to identify regions of brain activation in functional magnetic resonance imaging (Bullmore et.al., 1996; Everitt and Bullmore, 1999 and Everitt, 1998).

In the present study, the role of finite mixture of Pareto and finite mixture of beta distributions in the context of reliability and income analysis is examined.

## 1.7 Mixture distributions in reliability analysis

Various parametric models are used in the analysis of lifetime data and in problems related to the modelling of ageing or failure processes. Among univariate models, a few particular distributions occupy a central role because of their demonstrated usefulness in a wide range of situations. The exponential distribution appeared suitable for modelling the lifetimes of various types of manufactured items (see Davis, 1952; Epstein and Sobel, 1953). Later, Weibull and lognormal distributions were employed as popular models for lifetimes of manufactured items (see Kao, 1959; Nelson, 1972; Nelson and Hahn, 1972; Whittemore and Altschuler, 1976).

In lifetime data analysis, the population of lifetimes can be decomposed into subpopulations, based on lifetimes of units in different production periods, with differences in designs, made up of different raw materials etc. In such situations, it is usual to model the data using a finite mixture of distributions. Accordingly, Mendenhall and Hader (1958), considered the finite mixture of two exponential distributions for the analysis of failure times for ARC-1 VHF communication transmitter-receivers of a single commercial airline. Cox (1959) has analyzed the data on failure times using a mixture of exponential models by classifying the data on failure times into two subpopulations depending on whether the cause of failure was identified or not. Kao (1959) used finite mixture of two Weibull components for life testing of electron tubes. The finite mixtures of inverse

Gaussian distributions in the context of reliability were studied by Ahmad (1982), Amoh (1983) and Al-Hussaini and Ahmad (1984). Characterizations using reliability concepts of finite mixture models are discussed in Ahmad (1996). Al-Hussaini and Osman (1997) obtained the median of finite mixture of $k$ components. Al-Hussaini (1999) used the Bayesian method to predict observations under a mixture of two exponential components model. Later, Al-Hussaini et.al. (2000) studied the finite mixture of two-component Gompertz densities as a lifetime model. Bayesian predictive densities for finite mixture models based on order statistics have been discussed by Al-Hussaini (2001). Block et.al. (2003) studied the monotonic behaviour of the hazard rate of finite mixture of distributions. Jaheen (2003) obtained Bayesian prediction under a mixture of two-component Gompertz lifetime model. Recently, Cross (2004) has developed efficient tools for reliability analysis using finite mixture distributions. The finite mixture of distributions can also play a useful role in modelling time to failure of a system in the competing risk situations (see Crowder, 2001). For more properties and applications of finite mixture models in reliability theory, one could refer to Al-Hussaini and Sultan (2001).

## 1.8 Basic concepts in reliability

The term reliability is used to denote the probability of a device, component, material or structure, performing its intended function satisfactorily, for a given length of time in an environment for which it is designed. Even though the above definition of reliability is explained with reference to the failure behaviour or length of life of equipment, it is equally applicable in the analysis of any duration variable that describes a well defined population subject to decrementation due to the operation of forces of attrition over time. Accordingly, the concepts and tools in reliability have found applications in many areas of study such as biology, medicine, engineering, economics, epidemology and demography. It is evident that most of the difficulties in reliability modelling can be substantially reduced by understanding certain concepts associated with the failure process. In the present section, we discuss such basic concepts that are useful to model lifetime data.

### 1.8.1 Reliability function

Let $X$ be a non-negative random variable defined on a probability space $(\Omega, A, P)$ with distribution function $F(x) = P(X \leq x)$. In the reliability analysis, $X$ generally represents the length of life of a device, measured in units of time and the function,

$$S(x) = P(X > x), \tag{1.4}$$

is called the reliability (survival) function. $S(x)$ gives the probability that the device will operate without failure for a time $x$.

### 1.8.2 Hazard rate

Let $L = \inf\{x; F(x) = 1\}$. Then for $x < L$, the hazard rate $h(x)$ of $X$, is defined as

$$h(x) = \lim_{\delta x \to 0^+} \frac{P\{x < X \leq x + \delta x \mid X > x\}}{\delta x}. \tag{1.5}$$

When $f(x)$ is the probability density function of $X$, (1.5) reduces to

$$h(x) = \frac{f(x)}{S(x)},$$

$$= \frac{d}{dx}(-\log S(x)). \tag{1.6}$$

$h(x)$ measures instantaneous rate of failure or death at time $x$, given that an individual survives up to time $x$. In extreme-value theory, $h(x)$ is known as the intensity rate and its reciprocal is termed as Mills ratio in economics. For various applications of the hazard rate, one could refer to Mann et.al. (1974), Kalbfleisch and Prentice (1980), Elandt-Johnson and Johnson (1980), Nelson (1982) and Lawless (2003).

When $X$ is non-negative and has absolutely continuous distribution function, (1.6) provides that

$$S(x) = \exp[-\int_0^x h(t)dt]. \tag{1.7}$$

Examination of (1.7) indicates that $h(x)$ is a non-negative function with $\int_0^u h(t)dt < \infty$, for some $u > 0$ and $\int_0^\infty h(t)dt = \infty$.

In view of (1.7), $h(x)$ determines the distribution uniquely. It is shown that the constancy of $h(x)$ is a characteristic property of the exponential model (Galambos and Kotz (1978). Mukherjee and Roy (1986) has established that for a non-negative random variable $X$ in the support of the set of non-negative real numbers, hazard rate of the form

$$h(x) = \frac{1}{ax+b} \tag{1.8}$$

characterizes

(i) the exponential distribution with survival function

$$S(x) = \exp[-\lambda x], x \geq 0, \lambda > 0. \tag{1.9}$$

(ii) the Pareto II distribution with survival function

$$S(x) = (\frac{\alpha}{x+\alpha})^{\beta}, x \geq 0, \beta > 1, 0 < \alpha < \infty. \tag{1.10}$$

(iii) the beta distribution with survival function

$$S(x) = (1-\frac{x}{R})^{c}, 0 < x < R, c > 1. \tag{1.11}$$

according as $a = 0, a > 0$ and $a < 0$.

### 1.8.3 Mean residual life

The mean residual life (M.R.L), known as expectation of life in actuarial studies, was reintroduced in the reliability context by Knight in 1959 (see, Kupka and Loo, 1989). When $X$ is defined on the real line with $E(X^{+}) < \infty$, the Borel-measurable function $r(x)$ given by

$$r(x) = E[X - x \mid X \geq x], \tag{1.12}$$

for all $x$ such that $P[X \geq x] > 0$, is called the M.R.L function of X. $r(x)$ represents the average lifetime remaining to a component, which has already survived up to time $x$. When X is non-negative with $E(X) < \infty$ and $F(x)$ is absolutely continuous with respect to Lebesgue measure, (1.12) becomes

$$r(x) = \frac{1}{S(x)} \int_x^\infty S(t)dt . \tag{1.13}$$

Further, for every x in (0, L),

$$h(x) = \frac{1 + r'(x)}{r(x)} \tag{1.14}$$

and

$$S(x) = \frac{r(0)}{r(x)} \exp[-\int_0^x \frac{dt}{r(t)}] . \tag{1.15}$$

From equation (1.15), it follows that a M.R.L function uniquely determine a distribution and therefore, modelling can be done through an appropriate functional form for $r(x)$. A set of necessary and sufficient conditions for $r(x)$ to be an M.R.L, given by Swartz (1973) is that

(i) $r(x) \geq 0$,

(ii) $r(0) = E(X)$,

(iii) $r'(x) \geq -1$

and

(iv) $\int_0^\infty \frac{dx}{r(x)}$ diverges.

Cox (1972) has established that the M.R.L function is a constant for the exponential distribution. Mukherjee and Roy (1986) observed that a relation of the form

$$r(x)h(x) = k \tag{1.16}$$

where k is a constant, holds if and only if $X$ follows the exponential distribution (1.9) when $k=1$, the Pareto II distribution (1.10) when $k>1$ and the beta distribution (1.11) when $0<k<1$. In view of (1.15), Hitha (1991) has observed that a linear M.R.L of the form

$$r(x)=ax+b \tag{1.17}$$

characterizes the exponential distribution (1.9) if $a=0$, the Pareto II distribution (1.10) if $a>0$ and the beta distribution (1.11) if $a<0$.

### 1.9 Reliability and hazard rate for finite mixture models

Let $X$ be a non-negative random variable with probability density function (p.d.f), $f(x)$ given by (1.3).The reliability function $S(x)$ corresponding to the finite mixture of $k$ components (1.3) is given by

$$S(x)=p_1S_1(x)+p_2S_2(x)+...+p_kS_k(x). \tag{1.18}$$

where $S_j(x)$ is the reliability function corresponding to the $j^{th}$ component in the mixture, $j=1,2,...,k$. For simplicity, we consider the case of finite mixture model (1.3) with two components. The hazard rate function and mean residual life function of a mixture may be written in terms of the hazard rate functions and mean residual life functions of the two components as follows,

$$h(x)=A(x)h_1(x)+(1-A(x))h_2(x), \tag{1.19}$$

and

$$r(x)=A(x)r_1(x)+(1-A(x))r_2(x), \tag{1.20}$$

where $h_j(x)$ and $r_j(x)$ are hazard rate functions and M.R.L functions of $j^{th}$ component, $j=1,2$; and $h(x)$ and $r(x)$ are hazard rate function and M.R.L function of the mixture and

$$A(x)=\frac{pS_1(x)}{pS_1(x)+(1-p)S_2(x)}$$

with $S_j(x)$ is the reliability function of the $j^{th}$ component, $j=1,2$.

On differentiation, the identities (1.19) and (1.20) give

$$h'(x) = A(x)h_1'(x) + (1 - A(x))h_2'(x) - A(x)(1 - A(x)(h_1(x) - h_2(x))^2 \quad (1.21)$$

and

$$r'(x) = A(x)r_1'(x) + (1 - A(x))r_2'(x) - A(x)(1 - A(x)(r_1(x) - r_2(x))^2. \quad (1.22)$$

where prime denote the derivative with respect to $x$.

From (1.21), it follows that if $h_j'(x) < 0$, for all $x$, $(j = 1,2)$, then $h'(x) < 0$, for all $x$. Therefore, a mixture with decreasing hazard rate components has decreasing hazard rate. However, if the components have increasing hazard rates, their mixture need not have increasing hazard rate. For more details, one could refer to Al-Hussaini and Sultan (2001).

## 1.10 Censoring

In reliability studies, censored data often occurs due to many reasons. In many practical situations, it is desirable to discontinue the study prior to failure of all items in the sample, the resulting data are called incomplete data. The data are incomplete in the sense that we do not know the exact time to failure of the unfailed items, but only known that their failure times are greater than the recorded study time. Then we have two types of observations in the sample, actual failure times and bounds on failure times. Data of this type are called censored data. Various types of censoring are possible in life testing experiments and the type which depends on the criterion used to determine when to conclude experimentation.

### 1.10.1 Type I censoring

Censoring that occurs as a function of time is called type I censoring. In this case, the period of study terminated at a particular point of time and the data recorded are lifetimes of items that failed prior to that time. More specifically, data of this type are called right censored. If all items are put on test or put in service at the same time and the observation period is the same for all items (that is, observation stops at a fixed time $T$ ), the data are singly censored. (more precisely, singly right censored). Data for which unfailed items have been operating for variable amounts of time are called multiply censored (multiply right censored).

Data may also be censored on the left. Left censored data result when it is only known that failures have occurred prior to some time. In lifetime data analysis, right censoring is very common, but left censoring is fairly rare.

### 1.10.2 Type II censoring

Type I censoring may provide limited amount of information if the study period is too short. An alternative is to continue observation until a predetermined number $r(< n)$ of failures have occurred. This also results incomplete data, since the lifetimes of the remaining $n-r$ items are not observed. This type of censoring is called type II censoring. An extension of this type of censoring is progressive type II censoring. Under this censoring scheme, batches of items are removed from test as in type II censoring in two or more stages, as follows: $n$ items are initially put on test. After $r_1$ failures have occurred, an additional $n_1 - r_1$ items are removed from test. Testing is continued on the remaining $n-n_1$ until an additional $r_2$ failures are observed, at which time an additional $n_2 - r_2$ items are removed, with testing continuing on the remaining $n-n_1-n_2$ items, and so on, through $k$ stages $(k > 1)$.

### 1.10.3 Other types of censoring

Various other combinations of censoring mechanisms are sometimes used. For example, it may be desirable to observe a reasonable number of failures and limited test time. In this case, a combination of type I and II censoring may be used; that is testing continues until either $r$ failures occur or test duration $T$ is reached, whichever comes first.

## 1.11 Stress-strength models

The word "stress" has acquired a special meaning to a modern world, as all of us are continuously under stress and not always have the strength to over come it. The stress-strength relationship is nowadays studied in many branches of sciences and social sciences such as psychology, medicine, pedagogy, pharmaceutics and engineering.

In the context of reliability, the stress- strength model describes the life of the component, which is having a random strength $X$ that is subjected to a random stress $Y$. The component fails at the instant the stress applied to it exceeds the strength and the component works properly if there is no other cause of failure whenever $Y < X$.

Thus the system reliability for stress-strength model

$$R = P(Y < X)$$

$$= \int_0^\infty \int_0^x g(y) f(x) dy dx \qquad (1.23)$$

is an important measure of component reliability, where $f(x)$ and $g(x)$ are the probability density functions of $X$ and $Y$ respectively. For example, if $Y$ represents the maximum chamber pressure generated by ignition of a solid propellant and $X$ represents the strength of the rocket chamber, then $R$ is the probability of the successful firing of the rocket. The receptor in the human eye operates only if it is stimulated by a source where magnitude $X$ is greater than a random lower threshold $Y$ for the eye. In this case, $R$ is the probability that the receptor operates. For more details on stress-strength models, see Kotz, et.al. (2003).

### 1.11.1 Estimation of system reliability

One of the major problems in the analysis of stress-strength models is the estimation of the stress-strength reliability $R = P[Y < X]$. The point and interval estimation of $R$ in the non parametric setup was considered by Birnbaum (1956), Birnbaum and McCarty (1958), Owen et.al. (1964) and Govindarajulu (1967, 1968). The problem of estimation of $R$, with different parametric models for $X$ and $Y$ was discussed in Kelly et.al. (1976), Tong (1974), Church and Harris (1970), Downton (1973), Woodward and Kelley (1977), Beg and Singh (1979), Tong (1977), Enis and Geissel (1971), Ferguson (1973), Hollander and Korwar (1976) and Bhattacharyya and Johnson (1974).

The estimation problem when $X$ and $Y$ have joint bivariate exponential distribution was considered by Awad et.al. (1981), Jana (1994), Jana and

Roy (1994) and Hanagal (1995, 1997b). Jeevanand (1997) and Hanagal (1997a) studied the estimation of reliability under bivariate Pareto stress-strength model. Jeevanand and Nair (1994) considered the estimation of stress-strength reliability from exponential samples containing spurious observations. Al-Hussaini et.al. (1997) studied the role of finite mixture of lognormal components in a stress-strength model. The estimation of $R$ when the stress and strength follows finite mixture of inverse Gaussian distributions was developed by Akman et.al. (1999). Inference for stress-strength models based on Wienman multivariate exponential samples were obtained by Cramer (2001).Stress-strength model for skew-normal distributions was obtained by Gupta and Brown (2001). For more details, we refer to Johnson (1988) and Kotz et.al. (2003). Recently, Nadarajah (2004) considered a class of Laplace distributions and derive the stress-strength reliability for this class of distributions.

## 1.12 Mixture models in income analysis

Modelling of income data for studying the patterns and causes of prevalent distributional inequalities is of vital importance in socio-economic policy-making. This has been a very live area of research since the variegation and dynamism of income distributions allow as well as demand experimenting with new models and theories for purpose of increasing accuracy and interpretational capabilities. Statistical analysis of personal income distributions originated with the formulation of the famous Pareto's income law in 1897 which states that the number of persons in a population whose incomes exceed '$x$' can be approximated by a curve of the form $kx^{-\alpha}$, where $k$ is any real number and $\alpha$ is a positive real number. Accordingly, this distribution is known as Pareto type I (classical Pareto) and it is widely accepted as an income model. The law proposed by Pareto holds good only for the upper tail of the income distribution. Gibrat (1931) suggested the two-parameter lognormal distribution, further examined by Aitchinson and Brown (1969), for the size distribution of income. Champernowne (1953) proposed a Markov chain model for income distribution. Lomax (1954) introduced Pareto type II distribution that has come to stay as a very viable model in income studies. The beta distribution of first kind was used for modelling income data by Thurow (1970). Salem and Mount (1974) proposed gamma distribution and Bartels and van Metelel (1975) suggested Weibull to model the income data. For various models useful in income analysis, one could refer to Kleiber and Kotz (2003).

In income analysis, population of income of individuals can be divided into income from different sources. Further, any set of income data with earners in different categories will be well described as a finite mixture of distributions. Accordingly, Flachaire and Nunez (2004) proposed an explanatory model to study the structure of the income distribution based on mixture models. Reed and Jorgensen (2004) suggested a mixture of lognormal and Pareto distributions for modelling size distributions. However, this mixture function is still unimodal and cannot catch bi-modality. Paap and van Dijk (1998) considered a mixture of two distributions to model the world income distribution. They used different choices of distributions-truncated normal, lognormal, gamma and Weibull- and they selected the best mixture model based on a criterion of goodness -of- fit tests.

## 1.13 Basic economic concepts

The problem of modelling income data as well as that of measurement of income inequality has a history of about two hundred years and has been attracting a lot of researchers in economics, statistics, and sociology etc. As is customary in most statistical analysis, the extend of variation in incomes is represented in terms of certain summary measures. Thus a measure of income inequality is designed to provide an index that can abridge the variations prevailing among the individuals in a group. For a detailed study on various measures of income inequality, we refer to Kakwani (1980), Anand (1983) and Arnold (1987).

Recently concepts and ideas in reliability theory have been extensively used to study measures of income inequality. Chandra and Singpurwalla (1981) established certain relationships among concepts in reliability theory and in economics. These aspects were further investigated by Klefsjo (1984). Later Bhattacharjee (1988, 1993) investigated the role of anti-ageing distributions in reliability theory as reflecting certain features of skewness and heavy tails, typical for income data. One of the fundamental concepts in the measurement of income inequality is Lorenz curve.

### 1.13 .1 Lorenz curve

The Lorenz curve is an important tool for the measurement of income inequality. Lorenz (1905) defined this concept for finite populations as a function $L(p)$ on $[0,1]$, such that for fixed $p$, $L(p)$ represents the proportion of the total income in the population accounted for by the $100p\%$ poorest individuals in the population.

For a non-negative random variable $X$ with distribution function $F(x)$ and finite mean $\mu$, the Lorenz curve $L(p)$ is then defined in terms of the following two parametric equations in $x$ (see Kendall and Stuart (1958)) namely

$$p = F(x) = \int_0^x f(t)dt \qquad (1.24)$$

and

$$L(p) = \frac{1}{\mu}\int_0^x tf(t)dt \qquad (1.25)$$

where $f(t)$ is the p.d.f of $X$ . $L(p)$ determined by (1.25) is called 'the standard Lorenz curve'. $L(p)$ can be viewed as the proportional share of the total income of the population receiving an income less than or equal to $x$. It follows from (1.25) that the Lorenz curve is the first moment distribution function of $F(x)$. One can easily see that

(i) $L(0)=0$ , $L(1)=1$, $L(p)$ is continuous and strictly increasing on $(0,1)$, as

$$L'(p) = \frac{1}{\mu}x, \text{ which is greater than zero.}$$

(ii) $L(p)$ is twice differentiable and is strictly convex on $(0,1)$ as

$$L''(p) = \frac{1}{\mu f(x)} > 0.$$

Gastwirth (1971) relaxed the assumption that the distribution function $F(x)$ is absolutely continuous and defined the Lorenz curve $L(p)$ by

$$L(p)=\frac{1}{\mu}\int_0^p Q(t)dt\,,\quad 0\le p\le 1 \tag{1.26}$$

where

$$Q(x)=\inf\{x:F(x)\ge x\}$$

is the quantile function. When $F(x)$ is absolutely continuous, $Q(x)$ is the inverse function of $F(x)$ and (1.26) is the solution for $L(p)$ obtained from (1.24) and (1.25).

Chakrabarthy (1982) pointed out that the analysis and criticism of Lorenz curve constitute a major part of the growing literature on inequality , its measurement and interpretation and stated that Lorenz curve has remained the most popular and powerful tool in the analysis of size distribution of income, both empirical and theoretical. For various applications of the Lorenz curve, one may refer to Chatterjee and Bhattacharya (1974), Gastwirth (1972), Kakwani and Podder (1976), Goldie (1977) and Moothathu (1985a, 1985b and 1990).

## 1.14 Present study

The generalized Pareto family, constituting the three distributions, namely, exponential, Pareto II, and beta are extensively employed in modelling of lifetime data as it possess linear mean residual life functions. For the exponential distribution, the mean residual life function is constant. The mean residual life function is increasing (decreasing) for Pareto II (beta) distributions. These three distributions are widely employed in life testing situations and in income analysis.

As mentioned earlier, the situation of heterogeneity is inherent in most of the observed data in reliability and income analysis. For example, in reliability analysis, the population of lifetimes can be decomposed into subpopulations, based on lifetimes of units in different production periods, with differences in designs, made up of different raw materials etc. In the case of income data with earners in different categories or of different sizes makes the data heterogeneous. The heterogeneity prevents modelling the data using standard univariate distributions. As finite mixture models are typically used in describing heterogeneous populations, they have become very popular models in such situations and they offer realistic interpretations of the mechanisms that generated the data.

Accordingly, finite mixture of distributions becomes useful tools for the modelling and analysis of data in the context of reliability and income analysis.

In literature, considerable work has been devoted to study the properties and applications of finite mixture of exponential distribution in the context of reliability analysis (see Mendenhall and Hader, 1958; Nassar, 1988; Nassar and Mahmoud, 1985 and Al-Hussaini, 1999). Despite the importance and flexibility of Pareto II distribution in modelling and analysis of life time and income data, a comprehensive study on finite mixture of Pareto II distribution has not been considered so far. In many practical situations, the observed lifetimes of an item varies over only a finite range. Further, beta distribution is employed for modelling lifetime data of systems having increasing hazard rate pattern. Accordingly, beta distribution is considered to be useful model in the context of reliability analysis. However, a systematic study on finite mixture of beta distributions in the context of reliability and income analysis is not yet carried out.

Motivated by these facts, the present work aims to study the role of finite mixture of Pareto II and finite mixture of beta distributions in the context of reliability and income analysis.

The present work is organized into seven chapters. After this introductory chapter, in Chapter 2, we study the properties of finite mixture of Pareto II distributions in the context of reliability analysis. We develop estimation of parameters of the finite mixture of Pareto II distributions using different methods for complete as well as censored samples. Simulation studies are carried out to assess the performance of the estimators. We, then, illustrate the method with a real life data on survival times of leukemia patients.

In Chapter 3, we study various properties of the finite mixture of beta distributions in the context of reliability analysis. We discuss estimation of parameters of the finite mixture of beta distributions using different methods such as maximum likelihood, Bayes, method of moments and maximum product of spacing. We carried out a simulation study to observe finite sample properties of the estimates and their robustness. We, then, apply the procedure to a real life data on survival times of cancer patients.

Chapter 4 discusses the Bayesian predictive densities when the underlying distribution is assumed to have a finite mixture of Pareto II distributions. Based on type I censored samples, prediction bounds for the $k$-th future order statistics and survival functions are obtained. The prior belief of the experimenter is measured by Jeffrey's invariant prior and also by a proper conjugate prior which was suggested by Al-Hussaini (1999). We then consider the real life data on survival times of leukemia patients and we estimate the reliability and prediction bounds using Bayesian predictive density function. The last part of this chapter deals with the Bayesian predictive density function of finite mixture of beta distributions. We illustrate the procedure for a real life data on survival times of cancer patients.

The analysis of stress-strength models using finite mixture of distributions is discussed in Chapter 5. The maximum likelihood estimation of the stress-strength reliability for complete as well as censored samples are obtained in which we assume the distribution of strength as finite mixture of Pareto II and the distribution of stress as different models such as exponential, Pareto II and finite mixture of Pareto II. Simulation studies are carried out to assess the performance of the estimators. The method of estimation of stress-strength reliability is illustrated using a real life data. Chapter 5 concludes with the discussion of the same for the finite mixture of beta distributions.

The role of finite mixture of Pareto II distributions and finite mixture of beta distributions in income analysis is studied in Chapter 6. We derive the maximum likelihood estimate of the Lorenz curve based on complete as well as type I censored samples of finite mixture of Pareto II distributions. We, also, develop the maximum likelihood estimate of Lorenz curve for the finite mixture of beta distributions. We carried out a simulation study to assess the performance of the estimates. Finally, we illustrate the method for a real data on household expenditures of men and women on four commodity groups.

Finally, Chapter 7 summarizes the major conclusions of the present study.

# Chapter 2

# FINITE MIXTURE OF PARETO II DISTRIBUTIONS

## 2.1 Introduction

In literature, considerable work has been devoted to study the properties and applications of Pareto distribution, in the context of reliability analysis. Mixture models, as discussed in Chapter 1, provide additional flexibility in modelling and analysis of lifetime from a heterogeneous population. Despite its importance and flexibility in modelling and analysis of lifetime data, a comprehensive study on finite mixture of Pareto II distributions has not been considered so far. Motivated by these facts, in the present chapter, we discuss the statistical analysis of finite mixture of Pareto II distributions in the context of reliability theory.

We give the definition and properties of the finite mixture of Pareto distributions in Section 2.2. In Section 2.3, we study the reliability characteristics of the model. We show that a finite mixture of two Pareto II densities is identifiable in Section 2.4. Sections 2.5 to 2.7, discuss various estimation procedures for complete as well as for censored data. In Section 2.8, simulation studies are carried out to assess the performance of the estimators. In Section 2.9, we illustrate the method for a real data on survival times of leukaemia patients and finally, in Section 2.10, we give the conclusion of the chapter.

## 2.2 Definition and properties

Let $X$ be a non-negative random variable admitting an absolutely continuous distribution function $F(x)$ with respect to a Lebesgue measure in the support $R^+$. Assume that the probability density function (p.d.f.) of $X$, $f(x)$ exists. A finite mixture of Pareto II distribution with $k$ components can be represented in the form

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + \ldots + p_k f_k(x) \qquad (2.1)$$

where

$$p_i > 0, i = 1,2,\ldots,k \,; \sum_{i=1}^{k} p_i = 1$$

and

$$f_i(x) = a_i b_i (1 + a_i x)^{-(b_i+1)}, \; x > 0, \; a_i > 0, \; b_i > 0, \; (i = 1,2,\ldots,k).$$

As a special case of (2.1) with $k = 2$, we have,

$$f(x) = p_1 a_1 b_1 (1 + a_1 x)^{-(b_1+1)} + (1 - p_1) a_2 b_2 (1 + a_2 x)^{-(b_2+1)}, \qquad (2.2)$$

$$x > 0, \; a_i > 0, \; b_i > 0, \; (i = 1,2).$$

Throughout this chapter, we consider a finite mixture of two Pareto II distributions. The results can easily be extended to the general set up.

The $r^{th}$ moment of the distribution (2.2) is given by

$$E_f(X^r) = \int_0^\infty x^r [p_1 a_1 b_1 (1 + a_1 x)^{-(b_1+1)} + (1 - p_1) a_2 b_2 (1 + a_2 x)^{-(b_2+1)}] dx$$

$$= \frac{p_1 \Gamma(r+1)\Gamma(b_1 - r)}{a_1^r \Gamma b_1} + \frac{(1 - p_1)\Gamma(r+1)\Gamma(b_2 - r)}{a_2^r \Gamma b_2}. \qquad (2.3)$$

where $\Gamma p = \int_0^\infty e^{-x} x^{p-1} dx$, is the gamma function.

When $r = 1$, (2.3) reduces to the mean of $X$, which is equal to

$$E_f(X) = \frac{p_1}{a_1(b_1-1)} + \frac{(1-p_1)}{a_2(b_2-1)}, \quad 0<p_1<1,\ b_i>1,\ a_i>0,\ i=1,2 \tag{2.4}$$

When $r=2$, (2.3) becomes

$$E_f(X^2) = \frac{2p_1}{a_1^2(b_1-1)(b_1-2)} + \frac{2(1-p_1)}{a_2^2(b_2-1)(b_2-2)}.$$

Thus the variance of (2.2) is given by

$$V_f(x) = \frac{p_1(-2+2b_1+2p_1-b_1p_1)}{a_1^2(b_1-1)^2(b_1-2)} + \frac{(1-p_1)(b_2+(b_2-2)p_1)}{a_2^2(b_2-1)^2(b_2-2)} - \frac{2p_1(1-p_1)}{a_1a_2(b_1-1)(b_2-1)}. \tag{2.5}$$

## 2.3 Reliability characteristics

Let $X$ represents the lifetime of a component with survival function $S(x)$. Then the survival function of the model (2.2) is obtained as

$$S(x) = p_1(1+a_1x)^{-b_1} + (1-p_1)(1+a_2x)^{-b_2},\ x>0,\ a_i>0,\ b_i>0,\ (i=1,2),\ 0<p_1<1. \tag{2.6}$$

For the model (2.2), the hazard rate $h(x)$ and the mean residual life function $r(x)$ are given by

$$h(x) = \frac{p_1a_1b_1(1+a_1x)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}}{p_1(1+a_1x)^{-b_1} + (1-p_1)(1+a_2x)^{-b_2}} \tag{2.7}$$

and

$$r(x) = \frac{p_1a_2(b_2-1)(1+a_1x)^{-(b_1-1)} + (1-p_1)a_1(b_1-1)(1+a_2x)^{-(b_2-1)}}{a_1a_2[p_1(1+a_1x)^{-b_1} + (1-p_1)(1+a_2x)^{-b_2}](b_1-1)(b_2-1)}. \tag{2.8}$$

From (2.7) and (2.8), we obtain an identity connecting $r(x)$ and $h(x)$ as

$$r(x) - \frac{h(x)(1+a_1x)(1+a_2x)}{a_1a_2(b_1-1)(b_2-1)}\left[\frac{a_2(b_2-1)-a_1(b_1-1)+a_1a_2(b_2-b_1)x}{a_1b_1-a_2b_2-a_1a_2(b_2-b_1)x}\right]$$

$$= \frac{a_1^2b_1(b_1-1)(1+a_2x)^2 - a_2^2b_2(b_2-1)(1+a_1x)^2}{a_1a_2(b_1-1)(b_2-1)[a_1b_1-a_2b_2-a_1a_2(b_2-b_1)x]}. \tag{2.9}$$

Figures 2.1 to 2.8 show the behaviour of hazard rate and the mean residual life function for different parameters of the model (2.2). From the figures, it follows that $h(x)$ is decreasing in $x$ while $r(x)$ is increasing in $x$.

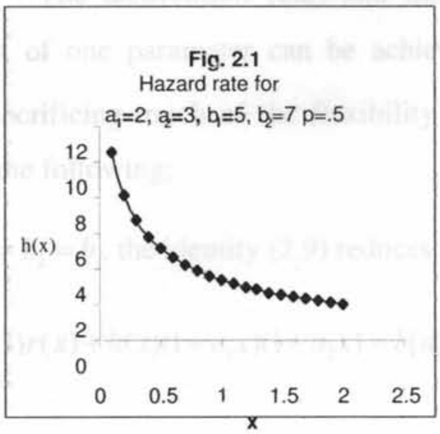



Fig. 2.1 Hazard rate for $a_1=2, a_2=3, b_1=5, b_2=7$ p=.5

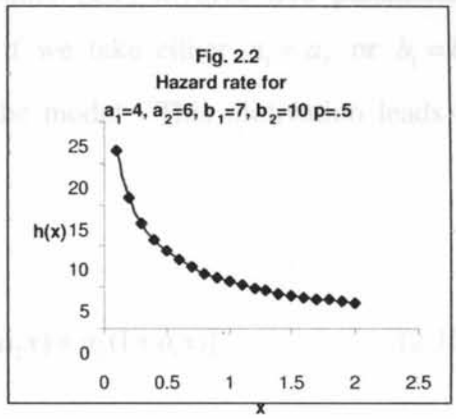



Fig. 2.2 Hazard rate for $a_1=4, a_2=6, b_1=7, b_2=10$ p=.5

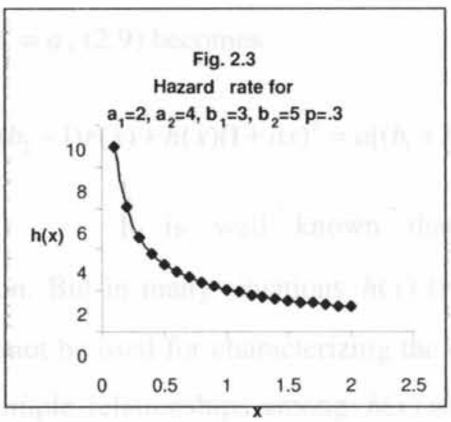



Fig. 2.3 Hazard rate for $a_1=2, a_2=4, b_1=3, b_2=5$ p=.3

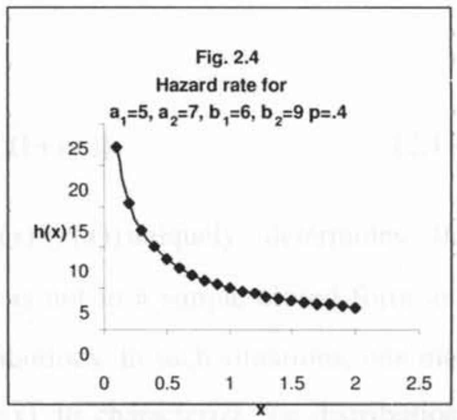



Fig. 2.4 Hazard rate for $a_1=5, a_2=7, b_1=6, b_2=9$ p=.4

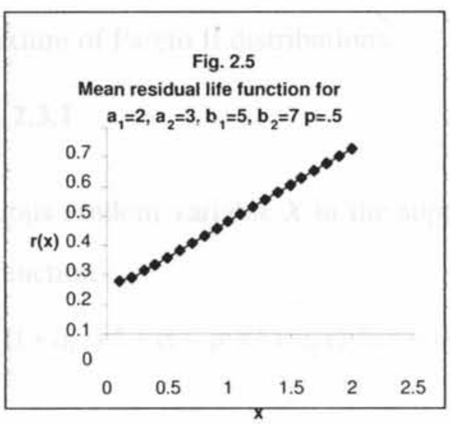



Fig. 2.5 Mean residual life function for $a_1=2, a_2=3, b_1=5, b_2=7$ p=.5

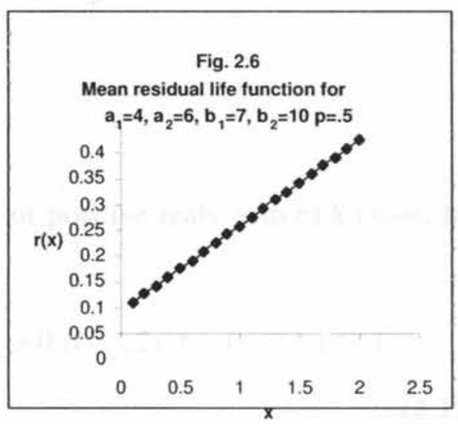



Fig. 2.6 Mean residual life function for $a_1=4, a_2=6, b_1=7, b_2=10$ p=.5

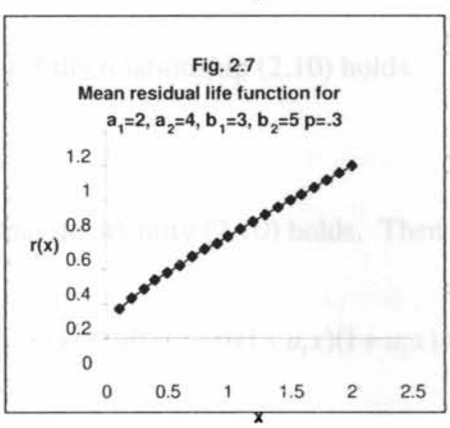



Fig. 2.7 Mean residual life function for $a_1=2, a_2=4, b_1=3, b_2=5$ p=.3

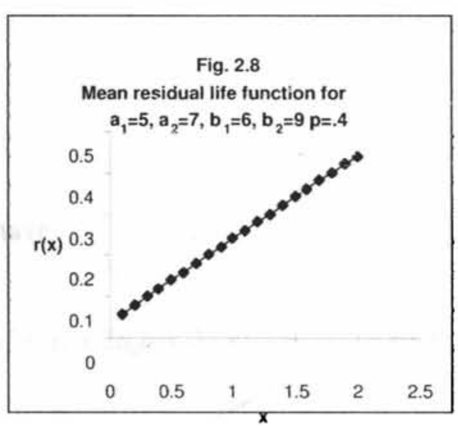



Fig. 2.8 Mean residual life function for $a_1=5, a_2=7, b_1=6, b_2=9$ p=.4

The distribution (2.2) and the identity (2.9) involve five parameters. Reduction of one parameter can be achieved if we take either $a_1 = a_2$ or $b_1 = b_2$ without sacrificing much of the flexibility of the model. This motivation leads to consider the following;

When $b_1 = b_2 = b$, the identity (2.9) reduces to

$$a_1 a_2 b(b-1)r(x) + h(x)(1+a_1 x)(1+a_2 x) = b[a_1(1+a_2 x) + a_2(1+a_1 x)] \tag{2.10}$$

and

for $a_1 = a_2 = a$, (2.9) becomes

$$a^2(b_1-1)(b_2-1)r(x) + h(x)(1+ax)^2 = a[(b_1+b_2-1)(1+ax)]. \tag{2.11}$$

It is well known that $h(x)$ $(r(x))$ uniquely determines the distribution. But in many situations $h(x)$ $(r(x))$ is not in a simple closed form and hence cannot be used for characterizing the distributions. In such situations, one may explore simple relationships among $h(x)$ and $r(x)$ to characterize the distributions (see Nair and Sankaran, 1991). Motivated by this, we prove a characterization result for the mixture of Pareto II distributions.

**Theorem 2.3.1**

A continuous random variable $X$ in the support of positive reals with $E(X) < \infty$, has survival function

$$S(x) = p_1(1+a_1 x)^{-b} + (1-p_1)(1+a_2 x)^{-b}, x > 0,\ a_i > 0\ (i = 1,2), b > 0,\ 0 < p_1 < 1. \tag{2.12}$$

if and only if the relationship (2.10) holds.

**Proof.**

Suppose that the identity (2.10) holds. Then we have

$$a_1 a_2 b(b-1)\int_x^{\infty} S(t)dt + f(x)(1+a_1 x)(1+a_2 x) = b[a_1 + a_2 + 2a_1 a_2 x]S(x). \tag{2.13}$$

Differentiating (2.13) twice with respect to $x$, we get

$$a_1a_2(b+1)(b+2)f(x)+(b+2)f'(x)[a_1(1+a_2x)+a_2(1+a_1x)]+f''(x)(1+a_1x)(1+a_2x)=0 \qquad (2.14)$$

where $f'(x)$ and $f''(x)$ respectively denote the first and the second derivatives of $f(x)$ with respect to $x$. The solution of the differential equation (2.14) is obtained as

$$S(x)=c_1(1+a_1x)^{-b}+c_2(1+a_2x)^{-b}. \qquad (2.15)$$

Since $S(0)=1$, we have $c_2=1-c_1$, which gives (2.15) as required in (2.12).The proof of the converse part is direct.

**Theorem 2.3.2**

The relationship (2.11) characterizes a mixture of Pareto II distribution with survival function

$$S(x)=p_1(1+ax)^{-b_1}+(1-p_1)(1+ax)^{-b_2}$$

For the proof, see Abraham and Nair (2001).

## 2.4 Identifiability

As mentioned in Chapter 1, lack of identifiability is common even for finite mixtures. In the following, we show that a finite mixture of two Pareto II densities is identifiable.

**Theorem 2.4.1**

Finite mixture of Pareto II densities is identifiable.

**Proof:**

From Teicher (1961), it follows that a finite mixture of $k$ exponential component densities is identifiable. If $Z$ follows exponential with parameter $\lambda$ and $Y=\frac{e^{z}-1}{a}$, then $Y$ follows Pareto II with density

$$g(y)=a\lambda(1+ay)^{-(\lambda+1)}.$$

Since the transformation $Y = \frac{e^{z} - 1}{a}$ is one to one and onto, a finite mixture of Pareto II with component densities $g_i(y) = a_i \lambda_i (1 + a_i y)^{-(\lambda_i + 1)}$ , $i = 1, 2$ is identifiable. The rest of the proof is similar to the one given for finite mixture of Gompertz densities (see Al-Hussaini et.al. 2000).

## 2.5 Estimation of parameters

In this section, we discuss the estimation of parameters $a_1, a_2, b_1, b_2$ and $p_1$ of the model (2.2), using different methods for the complete as well as censored samples.

## 2.6 Complete sample set up

### 2.6.1 Estimation of parameters when the observations belonging to each subpopulation are known

We consider the situation when there are only two subpopulations with mixing proportions $p_1$ and $(1 - p_1)$ and $f_1(x)$ and $f_2(x)$ are Pareto II densities with parameters $(a_1, b_1)$ and $(a_2, b_2)$ respectively. We consider the case when items that fail can be classified and can be attributed to the appropriate subpopulations.

Then the likelihood of the sample is given by (see Sinha (1986))

$$L(a_1, a_2, b_1, b_2, p_1 \mid \underline{x})$$

$$= \frac{n!}{n_1!, n_2!} p_1^{n_1} (1 - p_1)^{n_2} a_1^{n_1} b_1^{n_1} a_2^{n_2} b_2^{n_2} \prod_{j=1}^{n_1} (1 + a_1 x_{1j})^{-(b_1 + 1)} \prod_{j=1}^{n_2} (1 + a_2 x_{2j})^{-(b_2 + 1)}. \qquad (2.16)$$

where $x_{ij}$ denote the failure time of the $j$-th unit belonging to the $i$-th subpopulation, $j = 1, 2, ..., n_i$, $i = 1, 2$ and the observed data $\underline{x} = \{x_{11}, x_{12}, ..., x_{1n_1}; x_{21}, x_{22}, ..., x_{2n_2}\}$.

**(a) Maximum likelihood estimation (M.L.E).**

Maximization of log-likelihood function of (2.16) with respect to the parameters yields the following equations,

$$\frac{n_1}{a_1}-\frac{(b_1+1)}{a_1}\sum_{j=1}^{n_1}(1+\frac{1}{a_1x_{1j}})^{-1}=0, \tag{2.17}$$

$$\frac{n_2}{a_2}-\frac{(b_2+1)}{a_2}\sum_{j=1}^{n_2}(1+\frac{1}{a_2x_{2j}})^{-1}=0, \tag{2.18}$$

$$\frac{n_1}{b_1}-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})=0, \tag{2.19}$$

$$\frac{n_2}{b_2}-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})=0, \tag{2.20}$$

and

$$\frac{n_1}{p_1}-\frac{n_2}{(1-p_1)}=0. \tag{2.21}$$

The equation (2.21) implies

$$\hat{p}_1=\frac{n_1}{n_1+n_2}.$$

The solution of the equations (2.17) to (2.20) provides the estimates of the parameters $a_1, a_2, b_1$ and $b_2$. Standard numerical methods like Newton-Raphson method can be used to solve above equations.

To obtain variance of the estimates, we need to compute the observed information matrix $I(a_1, a_2, b_1, b_2, p_1)$.

The second derivative of $\log L(a_1, a_2, b_1, b_2, p_1 \mid \underline{x})$ leads to the following

$$\frac{\partial^2 \log L}{\partial a_1^2}=-\left[\frac{n_1}{a_1^2}+\frac{(b_1+1)}{a_1}\sum_{j=1}^{n_1}\frac{1}{x_{1j}a_1^2}(1+\frac{1}{a_1x_{1j}})^{-2}-\frac{(b_1+1)}{a_1^2}\sum_{j=1}^{n_1}(1+\frac{1}{a_1x_{1j}})^{-1}\right],$$

$$\frac{\partial^2 \log L}{\partial a_1 \partial b_1}=\frac{\partial^2 \log L}{\partial b_1 \partial a_1}=-\left[\frac{1}{a_1}\sum_{j=1}^{n_1}(1+\frac{1}{a_1x_{1j}})^{-1}\right],$$

$$\frac{\partial^2 \log L}{\partial a_2^{\,2}} = -\left[\frac{n_2}{a_2^{\,2}} + \frac{(b_2+1)}{a_2}\sum_{j=1}^{n_2}\frac{1}{x_{2j}a_2^{\,2}}(1+\frac{1}{a_2x_{2j}})^{-2} - \frac{(b_2+1)}{a_2^{\,2}}\sum_{j=1}^{n_2}(1+\frac{1}{a_2x_{2j}})^{-1}\right],$$

$$\frac{\partial^2 \log L}{\partial a_2 \partial b_2} = \frac{\partial^2 \log L}{\partial b_2 \partial a_2} = -\left[\frac{1}{a_2}\sum_{j=1}^{n_2}(1+\frac{1}{a_2x_{2j}})^{-1}\right],$$

$$\frac{\partial^2 \log L}{\partial b_1^{\,2}} = -\frac{n_1}{b_1^{\,2}},$$

$$\frac{\partial^2 \log L}{\partial b_2^{\,2}} = -\frac{n_2}{b_2^{\,2}},$$

$$\frac{\partial^2 \log L}{\partial p_1^{\,2}} = -\frac{(n_1+n_2)}{p_1^{\,2}},$$

and all other second derivatives are zero.

The observed information matrix is thus

$$I(a_1,a_2,b_1,b_2,p_1) = \begin{pmatrix} -\frac{\partial^2 \log L}{\partial a_1^{\,2}} & -\frac{\partial^2 \log L}{\partial a_1 \partial a_2} & -\frac{\partial^2 \log L}{\partial a_1 \partial b_1} & -\frac{\partial^2 \log L}{\partial a_1 \partial b_2} & -\frac{\partial^2 \log L}{\partial a_1 \partial p_1} \\ \\ \cdot \\ \\ -\frac{\partial^2 \log L}{\partial p_1 \partial a_1} & -\frac{\partial^2 \log L}{\partial p_1 \partial a_2} & -\frac{\partial^2 \log L}{\partial p_1 \partial b_1} & -\frac{\partial^2 \log L}{\partial p_1 \partial b_2} & -\frac{\partial^2 \log L}{\partial p_1^{\,2}} \end{pmatrix}$$

For large $n$, the joint distribution of the estimates $\hat{a}_1,\hat{a}_2,\hat{b}_1,\hat{b}_2$ and $\hat{p}_1$ is approximately multivariate normal with mean $(a_1,a_2,b_1,b_2,p_1)$ and covariance matrix $I^{-1}(\hat{a}_1,\hat{a}_2,\hat{b}_1,\hat{b}_2,\hat{p}_1)$, which is the inverse of $I(\hat{a}_1,\hat{a}_2,\hat{b}_1,\hat{b}_2,\hat{p}_1)$

**(b) Bayes estimation.**

We assume that the parameters $a_1, a_2, b_1, b_2$ and $p_1$ are independent. Using Jeffrey's invariant prior for $a_1, a_2, b_1$ and $b_2$ and uniform prior in the interval (0, 1) for $p_1$, the joint prior distribution of $a_1, a_2, b_1, b_2$ and $p_1$ is given by

$$g(a_1, a_2, b_1, b_2, p_1) \;\alpha\; \frac{1}{a_1 a_2 b_1 b_2}. \tag{2.22}$$

From (2.16) and (2.22), the joint posterior distribution of $a_1, a_2, b_1, b_2$ and $p_1$ is given by

$$\Pi(a_1, a_2, b_1, b_2, p_1 \mid \underline{x}) \;\alpha\; p_1^{n_1}(1-p_1)^{n_2} a_1^{n_1-1} a_2^{n_2-1} b_1^{n_1-1} b_2^{n_2-1}$$

$$\exp[-(b_1+1)\sum_{j=1}^{n_1}\log(1+a_1 x_{1j})]\exp[-(b_2+1)\sum_{j=1}^{n_2}\log(1+a_2 x_{2j})]. \tag{2.23}$$

From (2.23), we obtain the marginal posterior distributions of $a_1, a_2, b_1, b_2$ and $p_1$ as

$$\Pi_1(a_1 \mid \underline{x}) = C\frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1 x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1 x_{1j})]^{n_1}}\int_0^\infty \frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2 x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2 x_{2j})]^{n_2}}da_2, \tag{2.24}$$

$$\Pi_2(a_2 \mid \underline{x}) = C\frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2 x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2 x_{2j})]^{n_2}}\int_0^\infty \frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1 x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1 x_{1j})]^{n_1}}da_1, \tag{2.25}$$

$$\Pi_3(b_1 \mid \underline{x}) = \frac{C}{\Gamma(n_1)} b_1^{n_1-1}\int_0^\infty a_1^{n_1-1}\exp[-(b_1+1)(\sum_{j=1}^{n_1}\log(1+a_1 x_{1j})]da_1$$

$$\int_0^\infty \frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2 x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2 x_{2j})]^{n_2}}da_2, \tag{2.26}$$

$$\Pi_4(b_2 \mid \underline{x}) = \frac{C}{\Gamma(n_2)} b_2^{n_2-1} \int_0^\infty a_2^{n_2-1} \exp[-(b_2+1)(\sum_{j=1}^{n_2} \log(1+a_2x_{2j}))] da_2$$

$$\int_0^\infty \frac{a_1^{n_1-1} \exp[-\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]^{n_1}} da_1 , \qquad (2.27)$$

and

$$\Pi_5(p_1 \mid \underline{x}) = \frac{C}{B(n_1+1+n_2+1)} p_1^{n_1}(1-p_1)^{n_2} \int_0^\infty \frac{a_1^{n_1-1} \exp[-\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]^{n_1}} da_1$$

$$\int_0^\infty \frac{a_2^{n_2-1} \exp[-\sum_{j=1}^{n_2} \log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2} \log(1+a_2x_{2j})]^{n_2}} da_2 . \qquad (2.28)$$

where

$$C^{-1} = \int_0^\infty \frac{a_1^{n_1-1} \exp[-\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]^{n_1}} da_1 \int_0^\infty \frac{a_2^{n_2-1} \exp[-\sum_{j=1}^{n_2} \log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2} \log(1+a_2x_{2j})]^{n_2}} da_2 \qquad (2.29)$$

and

$B(p,q)$ is the beta function of the first kind.

Under squared error loss function, we obtain the Bayes estimators of $a_1, a_2, b_1, b_2$ and $p_1$ as

$$a_1^* = C \int_0^\infty \frac{a_1^{n_1} \exp[-\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1} \log(1+a_1x_{1j})]^{n_1}} da_1 \int_0^\infty \frac{a_2^{n_2-1} \exp[-\sum_{j=1}^{n_2} \log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2} \log(1+a_2x_{2j})]^{n_2}} da_2 , \qquad (2.30)$$

$$a_2^* = C\int_0^\infty \frac{a_2^{n_2}\exp[-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]^{n_2}}da_2\int_0^\infty \frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]^{n_1}}da_1, \qquad (2.31)$$

$$b_1^* = Cn_1\int_0^\infty \frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]^{n_1+1}}da_1\int_0^\infty \frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]^{n_2}}da_2, \qquad (2.32)$$

$$b_2^* = Cn_2\int_0^\infty \frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]^{n_2+1}}da_2\int_0^\infty \frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]^{n_1}}da_1, \qquad (2.33)$$

and

$$p_1^* = \frac{B(n_1+2,n_2+1)}{B(n_1+1,n_2+1)}$$

$$= \frac{n_1+1}{n_1+n_2+2}\ . \qquad (2.34)$$

Bayes estimators of $a_1, a_2, b_1$ and $b_2$ will be obtained by numerical integration procedure.

To obtain the performance of the estimators, we calculate the posterior variance of the estimates. We first compute

$$E(a_1^2 \mid \underline{x}) = C\int_0^\infty \frac{a_1^{n_1+1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]^{n_1}}da_1\int_0^\infty \frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]^{n_2}}da_2, \qquad (2.35)$$

$$E(a_2^2 \mid \underline{x}) = C\int_0^\infty \frac{a_2^{n_2+1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]^{n_2}}da_2\int_0^\infty \frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]^{n_1}}da_1, \qquad (2.36)$$

$$E(b_1^2 \mid \underline{x}) = Cn_1(n_1+1)\int_0^\infty \frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]^{n_1+2}}da_1 \int_0^\infty \frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]^{n_2}}da_2, \tag{2.37}$$

$$E(b_2^2 \mid \underline{x}) = Cn_2(n_2+1)\int_0^\infty \frac{a_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]}{[\sum_{j=1}^{n_2}\log(1+a_2x_{2j})]^{n_2+2}}da_2 \int_0^\infty \frac{a_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]}{[\sum_{j=1}^{n_1}\log(1+a_1x_{1j})]^{n_1}}da_1, \tag{2.38}$$

and

$$E(p_1^2 \mid \underline{x}) = \frac{B(n_1+3, n_2+1)}{B(n_1+1, n_2+1)}. \tag{2.39}$$

Then, the posterior variance of $a_1$ is obtained from (2.30) and (2.35) as,

$$V(a_1 \mid \underline{x}) = E(a_1^2 \mid \underline{x}) - [E(a_1 \mid \underline{x})]^2.$$

Similarly, we obtain the posterior variances $V(a_2 \mid \underline{x}), V(b_1 \mid \underline{x})$, $V(b_2 \mid \underline{x})$ and $V(p_1 \mid \underline{x})$ from (2.31), (2.32), (2.33), (2.34), (2.36), (2.37), (2.38) and (2.39).

**Remark 2.1**

One can use other types of priors for the analysis. However, when we use conjugate priors for $p_1, a_1$, $a_2$, $b_1$ and $b_2$, the estimation procedure will become more complex.

### 2.6.2 Estimation of parameters when the observations belonging to each subpopulation are unknown

We consider the situation when there are only two subpopulations with mixing proportions $p_1$ and $(1-p_1)$ and $f_1(x)$ and $f_2(x)$ are Pareto II densities with parameters $(a_1, b_1)$ and $(a_2, b_2)$ respectively. In this case, we cannot assign a unit to a particular subpopulation and hence the likelihood of the sample is given by

$$L(a_1, a_2, b_1, b_2, p_1 \mid \underline{x}) = \prod_{j=1}^{n}[p_1a_1b_1(1+a_1x_j)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x_j)^{-(b_2+1)}]. \tag{2.40}$$

where $x_j$ denote the failure time of the $j$-th unit in a sample $(x_1, x_2, ..., x_n)$ of size $n$.

**(a) Maximum likelihood estimation (M.L.E).**

Maximization of log-likelihood function of (2.40) with respect to the parameters yields the following equations,

$$\sum_{j=1}^{n} \frac{p_1 b_1 (1 - a_1 b_1 x_j)(1 + a_1 x_j)^{-(b_1+2)}}{p_1 a_1 b_1 (1 + a_1 x_j)^{-(b_1+1)} + (1 - p_1) a_2 b_2 (1 + a_2 x_j)^{-(b_2+1)}} = 0, \tag{2.41}$$

$$\sum_{j=1}^{n} \frac{(1 - p_1) b_2 (1 - a_2 b_2 x_j)(1 + a_2 x_j)^{-(b_2+2)}}{p_1 a_1 b_1 (1 + a_1 x_j)^{-(b_1+1)} + (1 - p_1) a_2 b_2 (1 + a_2 x_j)^{-(b_2+1)}} = 0, \tag{2.42}$$

$$\sum_{j=1}^{n} \frac{p_1 a_1 (1 - b_1 \log(1 + a_1 x_j))(1 + a_1 x_j)^{-(b_1+1)}}{p_1 a_1 b_1 (1 + a_1 x_j)^{-(b_1+1)} + (1 - p_1) a_2 b_2 (1 + a_2 x_j)^{-(b_2+1)}} = 0, \tag{2.43}$$

$$\sum_{j=1}^{n} \frac{(1 - p_1) a_2 (1 - b_2 \log(1 + a_2 x_j))(1 + a_2 x_j)^{-(b_2+1)}}{p_1 a_1 b_1 (1 + a_1 x_j)^{-(b_1+1)} + (1 - p_1) a_2 b_2 (1 + a_2 x_j)^{-(b_2+1)}} = 0, \tag{2.44}$$

and

$$\sum_{j=1}^{n} \frac{a_1 b_1 (1 + a_1 x_j)^{-(b_1+1)} - a_2 b_2 (1 + a_2 x_j)^{-(b_2+1)}}{p_1 a_1 b_1 (1 + a_1 x_j)^{-(b_1+1)} + (1 - p_1) a_2 b_2 (1 + a_2 x_j)^{-(b_2+1)}} = 0. \tag{2.45}$$

The solutions of above equations provide the estimates of the parameters $a_1, a_2, b_1, b_2$ and $p_1$. Standard numerical methods like Newton Raphson method can be used to solve above equations.

**(b) Bayes estimation.**

We assume that the joint prior density of $a_1, a_2, b_1, b_2$ and $p_1$ is (2.22). From (2.22) and (2.40), the joint posterior distribution of $a_1, a_2, b_1, b_2$ and $p_1$ is obtained as

$$\prod(a_1,a_2,b_1,b_2,p_1 \mid \underline{x}) \propto [\prod_{i=1}^{n}(p_1 a_1 b_1(1+a_1x_i)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2}. \tag{2.46}$$

We, then, obtain the marginal posterior distributions of $a_1, a_2, b_1, b_2$ and $p_1$ as

$$\Pi_1(a_1 \mid \underline{x}) = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1 a_1 b_1(1+a_1x_i)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2}dp_1db_1db_2da_2, \tag{2.47}$$

$$\Pi_2(a_2 \mid \underline{x}) = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1 a_1 b_1(1+a_1x_i)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2}dp_1db_1db_2da_1, \tag{2.48}$$

$$\Pi_3(b_1 \mid \underline{x}) = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1 a_1 b_1(1+a_1x_i)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2}dp_1db_2da_2da_1, \tag{2.49}$$

$$\Pi_4(b_2 \mid \underline{x}) = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1 a_1 b_1(1+a_1x_i)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2}dp_1db_1da_2da_1, \tag{2.50}$$

and

$$\Pi_5(p_1 \mid \underline{x}) = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty [\prod_{i=1}^{n}(p_1 a_1 b_1(1+a_1x_i)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2}db_1db_2da_2da_1. \tag{2.51}$$

where

$$D^{-1} = \int_0^1\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty [\prod_{i=1}^{n}(p_1a_1b_1(1+a_1x_i)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2} db_1db_2da_2da_1dp_1 \ . \qquad (2.52)$$

Then the Bayes estimates of the parameters, under squared error loss function, are given as

$$a_1^* = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1a_1b_1(1+a_1x_i)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_2b_1b_2} dp_1db_1db_2da_2da_1 , \qquad (2.53)$$

$$a_2^* = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1a_1b_1(1+a_1x_i)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1b_1b_2} dp_1db_1db_2da_1da_2 , \qquad (2.54)$$

$$b_1^* = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1a_1b_1(1+a_1x_i)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_2} dp_1db_2da_2da_1db_1 \ , \qquad (2.55)$$

$$b_2^* = D\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{n}(p_1a_1b_1(1+a_1x_i)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1} dp_1db_1da_2da_1db_2 \ , \qquad (2.56)$$

and

$$p_1^* = D\int_0^1\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty p_1 [\prod_{i=1}^{n}(p_1a_1b_1(1+a_1x_i)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]$$

$$\frac{1}{a_1a_2b_1b_2} db_1db_2da_2da_1dp_1 . \qquad (2.57)$$

**(c) Method of moments.**

Of the classical methods, one that is simple for the estimation of parameters is the method of moments.

Let $X_1, X_2 \ldots, X_n$ be a random sample from the model (2.2). Let $m_1', m_2', m_3', m_4'$ and $m_5'$ denote the sample raw moments. Now we equate the sample moments to the population moments, which provide the following equations,

$$\frac{p_1}{a_1(b_1-1)} + \frac{(1-p_1)}{a_2(b_2-1)} = m_1', \tag{2.58}$$

$$\frac{2p_1}{a_1^2(b_1-1)(b_1-2)} + \frac{2(1-p_1)}{a_2^2(b_2-1)(b_2-2)} = m_2', \tag{2.59}$$

$$\frac{6p_1}{a_1^3(b_1-1)(b_1-2)(b_1-3)} + \frac{6(1-p_1)}{a_2^3(b_2-1)(b_2-2)(b_2-3)} = m_3', \tag{2.60}$$

$$\frac{24p_1}{a_1^4(b_1-1)(b_1-2)(b_1-3)(b_1-4)} + \frac{24(1-p_1)}{a_2^4(b_2-1)(b_2-2)(b_2-3)(b_2-4)} = m_4', \tag{2.61}$$

and

$$\frac{120p_1}{a_1^5(b_1-1)(b_1-2)(b_1-3)(b_1-4)(b_1-5)} + \frac{120(1-p_1)}{a_2^5(b_2-1)(b_2-2)(b_2-3)(b_2-4)(b_2-5)} = m_5'. \tag{2.62}$$

The estimates of $a_1, a_2, b_1, b_2$ and $p_1$ can be obtained by solving the above five equations. An important difficulty which may arise in this process is the multiple solutions of (2.58), (2.59), (2.60), (2.61) and (2.62). In such situations we choose those values, which minimize the standard error. Pearson (1894) has recommended choosing the set of estimates which is in closest agreement to the actual value. This is not possible in the real world situation since we do not know the actual values.

**(d) Maximum product of spacing.**

In most applications, the parameters in the mixture model are estimated by the method of maximum likelihood (M.L.E). It is well known, however, that the maximum likelihood estimation can be very sensitive to outliers in the data. In such situations, we obtain the estimate of parameters $a_1, a_2, b_1, b_2$ and $p_1$ using the maximum product of spacing method introduced by Cheng and Amin (1983). In the presence of outliers, this approach is superior in comparison with the M.L.E. The estimates by this method are obtained by maximizing the geometric mean of the spacing

$$G=\frac{1}{n}\sum_{i=1}^{n}\log[F(x_i)-F(x_{i-1})].$$

For the model (2.2), $G$ will be equal to

$$G=\frac{1}{n}\sum_{i=1}^{n}\log\{p_1[(1+a_1x_{i-1})^{-b_1}-(1+a_1x_i)^{-b_1}]+(1-p_1)[(1+a_2x_{i-1})^{-b_2}-(1+a_2x_i)^{-b_2}]\}. \tag{2.63}$$

where $x_i$ and $x_{i-1}$ denote the failure times of the $i$-th and $(i-1)$-th unit in a sample $(x_1,x_2,...,x_n)$ of size $n$.

Differentiating (2.63) with respect to $a_1,a_2,b_1,b_2$ and $p_1$ and equating to zero, we have

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p_1b_1[(1+a_1x_i)^{-(b_1+1)}x_i-(1+a_1x_{i-1})^{-(b_1+1)}x_{i-1}]}{p_1[(1+a_1x_{i-1})^{-b_1}-(1+a_1x_i)^{-b_1}]+(1-p_1)[(1+a_2x_{i-1})^{-b_2}-(1+a_2x_i)^{-b_2}]}=0, \tag{2.64}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(1-p_1)b_2[(1+a_2x_i)^{-(b_2+1)}x_i-(1+a_2x_{i-1})^{-(b_2+1)}x_{i-1}]}{p_1[(1+a_1x_{i-1})^{-b_1}-(1+a_1x_i)^{-b_1}]+(1-p_1)[(1+a_2x_{i-1})^{-b_2}-(1+a_2x_i)^{-b_2}]}=0, \tag{2.65}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p_1[(1+a_1x_i)^{-b_1}\log(1+a_1x_i)-(1+a_1x_{i-1})^{-b_1}\log(1+a_1x_{i-1})]}{p_1[(1+a_1x_{i-1})^{-b_1}-(1+a_1x_i)^{-b_1}]+(1-p_1)[(1+a_2x_{i-1})^{-b_2}-(1+a_2x_i)^{-b_2}]}=0, \tag{2.66}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(1-p_1)[(1+a_2x_i)^{-b_2}\log(1+a_2x_i)-(1+a_2x_{i-1})^{-b_2}\log(1+a_2x_{i-1})]}{p_1[(1+a_1x_{i-1})^{-b_1}-(1+a_1x_i)^{-b_1}]+(1-p_1)[(1+a_2x_{i-1})^{-b_2}-(1+a_2x_i)^{-b_2}]}=0, \tag{2.67}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\frac{[(1+a_1x_{i-1})^{-b_1}-(1+a_1x_i)^{-b_1}+(1+a_2x_i)^{-b_2}-(1+a_2x_{i-1})^{-b_2}]}{p_1[(1+a_1x_{i-1})^{-b_1}-(1+a_1x_i)^{-b_1}]+(1-p_1)[(1+a_2x_{i-1})^{-b_2}-(1+a_2x_i)^{-b_2}]}=0. \tag{2.68}$$

The solution of above equations provides the estimates of the parameters $a_1,a_2,b_1,b_2$ and $p_1$.

## 2.7 Censored set up

In this situation, we discuss the estimation of parameters of mixture of Pareto II model (2.2) for the censored data.

**2.7.1 Estimation based on type I censored samples when the observations belonging to each subpopulation are known**

Consider the situation, when there are only two subpopulations with mixing proportions $p_1$ and $(1-p_1)$ and $f_1(x)$ and $f_2(x)$ are respective Pareto II densities with parameters $(a_1,b_1)$ and $(a_2,b_2)$. Consider a life testing experiment with $n$ items and the experiment is terminated after a preassigned time $T$ hours have elapsed. Suppose that $r$ units have failed during the interval $(0,T)$ of which $r_1$ units from the first subpopulation and $r_2$ units from the second subpopulation with $r=r_1+r_2$ and $(n-r)$ units are still functioning. Let $t_{ij}$ denote the failure time of the $j^{th}$ unit belonging to the $i^{th}$ subpopulation, $t_{ij} \le T$, $j=1,2,...,r_i$; $i=1,2$ and let $\beta_i = a_i T$ and $x_{ij} = \dfrac{t_{ij}}{T}$ and observed data $\underline{x} = \{x_{11},x_{12},...,x_{1r_1};x_{21},x_{22},...,x_{2r_2}\}$. Now we discuss the estimation of parameters using M.L.E. and Bayes technique.

**(a) Maximum likelihood estimation (M.L.E).**

The likelihood function based on a type I censored sample (see Mendenhall and Hader (1958), Sinha (1986)) is given by,

$$L(\beta_1,\beta_2,b_1,b_2,p_1 \mid \underline{x}) = \frac{n!}{r_1!r_2!(n-r)!} p_1^{r_1}(1-p_1)^{r_2} \prod_{j=1}^{r_1} f_1(x_{1j}) \prod_{j=1}^{r_2} f_2(x_{2j}) [S(T)]^{(n-r)}. \quad (2.69)$$

For the model (2.2), (2.69) becomes

$$L(\beta_1,\beta_2,b_1,b_2,p_1 \mid \underline{x}) = \frac{n!}{r_1!r_2!(n-r)!} \frac{p_1^{r_1}(1-p_1)^{r_2}}{T^r} \prod_{j=1}^{r_1}\left\{b_1\beta_1(1+\beta_1 x_{1j})^{-(b_1+1)}\right\}$$

$$\prod_{j=1}^{r_2}\left\{b_2\beta_2(1+\beta_2 x_{2j})^{-(b_2+1)}\right\}\{p_1(1+\beta_1)^{-b_1}+(1-p_1)(1+\beta_2)^{-b_2}\}^{(n-r)} \quad (2.70)$$

Maximization of log-likelihood function of (2.70) with respect to the model parameters yields the following equations,

$$\frac{r_1}{\beta_1} - \frac{(b_1+1)}{\beta_1}\sum_{j=1}^{r_1}\left(1+\frac{1}{\beta_1 x_{1j}}\right)^{-1} - \frac{(n-r)p_1 b_1(1+\beta_1)^{-(b_1+1)}}{[p_1(1+\beta_1)^{-b_1}+(1-p_1)(1+\beta_2)^{-b_2}]} = 0, \quad (2.71)$$

$$\frac{r_2}{\beta_2}-\frac{(b_2+1)}{\beta_2}\sum_{j=1}^{r_2}\left(1+\frac{1}{\beta_2 x_{2j}}\right)^{-1}-\frac{(n-r)(1-p_1)b_2(1+\beta_2)^{-(b_2+1)}}{[p_1(1+\beta_1)^{-b_1}+(1-p_1)(1+\beta_2)^{-b_2}]}=0, \tag{2.72}$$

$$\frac{r_1}{b_1}-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})-\frac{(n-r)p_1(1+\beta_1)^{-b_1}\log(1+\beta_1)}{[p_1(1+\beta_1)^{-b_1}+(1-p_1)(1+\beta_2)^{-b_2}]}=0, \tag{2.73}$$

$$\frac{r_2}{b_2}-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})-\frac{(n-r)(1-p_1)(1+\beta_2)^{-b_2}\log(1+\beta_2)}{[p_1(1+\beta_1)^{-b_1}+(1-p_1)(1+\beta_2)^{-b_2}]}=0, \tag{2.74}$$

and

$$\frac{r_1}{p_1}-\frac{r_2}{(1-p_1)}+\frac{(n-r)[(1+\beta_1)^{-b_1}-(1+\beta_2)^{-b_2}]}{[p_1(1+\beta_1)^{-b_1}+(1-p_1)(1+\beta_2)^{-b_2}]}=0. \tag{2.75}$$

The solution of the five equations (2.71) to (2.75), using numerical iteration method, yields the maximum likelihood estimate (M.L.E) of $\beta_1,\beta_2,b_1,b_2$ and $p_1$ and thus we obtain the maximum likelihood estimate of $a_1,a_2,b_1,b_2$ and $p_1$.

**(b) Bayes estimation.**

The method of Bayes estimation provides accurate estimates in many censored situations (see Sinha, 1998). To discuss the method of Bayes estimation for the model (2.2) based on type I censored data, we assume that $\beta_1,\beta_2,b_1,b_2$ and $p_1$ are independent. Using Jeffrey's invariant prior for $\beta_1,\beta_2,b_1$ and $b_2$ and uniform prior in the interval $(0,1)$ for $p_1$, the joint prior distribution of $\beta_1,\beta_2,b_1,b_2$ and $p_1$ is given by

$$g(\beta_1,\beta_2,b_1,b_2,p_1)\ \alpha\ \frac{1}{\beta_1\beta_2 b_1 b_2}. \tag{2.76}$$

From (2.70) and (2.76), the joint posterior distribution of $\beta_1,\beta_2,b_1,b_2$ and $p_1$ is obtained as

$$\Pi(\beta_1,\beta_2,b_1,b_2,p_1\mid\underline{x})\ \alpha\ \sum_{k=0}^{n-r}\binom{n-r}{k}p_1^{\,n-r_2-k}(1-p_1)^{r_2+k}\beta_1^{r_1-1}\beta_2^{r_2-1}b_1^{r_1-1}b_2^{r_2-1}$$

$$\exp[-(b_1+1)\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]\exp[-(b_2+1)\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})](1+\beta_1)^{-b_1(n-r-k)}(1+\beta_2)^{-b_2 k}. \tag{2.77}$$

From equation (2.77), we obtain the marginal posterior distributions of $\beta_1, \beta_2, b_1, b_2$ and $p_1$ as

$$\Pi_1(\beta_1 \mid \underline{x}) = E\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\frac{\beta_1^{r_1-1}\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]+(n-r-k)\log(1+\beta_1)]^{r_1}}$$

$$\int_0^\infty \frac{\beta_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2}}d\beta_2, \qquad (2.78)$$

$$\Pi_2(\beta_2 \mid \underline{x}) = E\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\frac{\beta_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2}}$$

$$\int_0^\infty \frac{\beta_1^{r_1-1}\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]+(n-r-k)\log(1+\beta_1)]^{r_1}}d\beta_1, \qquad (2.79)$$

$$\Pi_3(b_1 \mid \underline{x}) = \frac{E}{\Gamma r_1}\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)b_1^{r_1-1}$$

$$\int_0^\infty \beta_1^{r_1-1}\exp[-b_1(\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})+(n-r-k)\log(1+\beta_1))]\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]d\beta_1$$

$$\int_0^\infty \frac{\beta_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2}}d\beta_2, \qquad (2.80)$$

$$\Pi_4(b_2 \mid \underline{x}) = \frac{E}{\Gamma r_2} \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1) b_2^{r_2-1}$$

$$\int_0^\infty \beta_2^{r_2-1} \exp[-b_2(\sum_{j=1}^{r_2} \log(1+\beta_2 x_{2j}) + k\log(1+\beta_2))] \exp[-\sum_{j=1}^{r_2} \log(1+\beta_2 x_{2j})] d\beta_2$$

$$\int_0^\infty \frac{\beta_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1} \log(1+\beta_1 x_{1j})] + (n-r-k)\log(1+\beta_1)]^{r_1}} d\beta_1, \tag{2.81}$$

and

$$\Pi_5(p_1 \mid \underline{x}) = E \sum_{k=0}^{n-r} \binom{n-r}{k} p_1^{n-r_2-k} (1-p_1)^{r_2+k} \int_0^\infty \frac{\beta_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1} \log(1+\beta_1 x_{1j})] + (n-r-k)\log(1+\beta_1)]^{r_1}} d\beta_1$$

$$\int_0^\infty \frac{\beta_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2} \log(1+\beta_2 x_{2j})] + k\log(1+\beta_2)]^{r_2}} d\beta_2. \tag{2.82}$$

where

$$E^{-1} = \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1) \int_0^\infty \frac{\beta_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2} \log(1+\beta_2 x_{2j})] + k\log(1+\beta_2)]^{r_2}} d\beta_2$$

$$\int_0^\infty \frac{\beta_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1} \log(1+\beta_1 x_{1j})] + (n-r-k)\log(1+\beta_1)]^{r_1}} d\beta_1. \tag{2.83}$$

with $B(p,q)$ is the beta function of the first kind.

Under squared error loss function, we obtain the Bayes estimators of $\beta_1, \beta_2, b_1, b_2$ and $p_1$ as

$$\beta_1^* = E\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\int_0^\infty \frac{\beta_1^{r_1}\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]+(n-r-k)\log(1+\beta_1)]^{r_1}}d\beta_1$$

$$\int_0^\infty \frac{\beta_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2}}d\beta_2, \qquad (2.84)$$

$$\beta_2^* = E\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\int_0^\infty \frac{\beta_2^{r_2}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2}}d\beta_2$$

$$\int_0^\infty \frac{\beta_1^{r_1-1}\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]+(n-r-k)\log(1+\beta_1)]^{r_1}}d\beta_1, \qquad (2.85)$$

$$b_1^* = Er_1\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\int_0^\infty \frac{\beta_1^{r_1-1}\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]+(n-r-k)\log(1+\beta_1)]^{r_1+1}}d\beta_1$$

$$\int_0^\infty \frac{\beta_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2}}d\beta_2, \qquad (2.86)$$

$$b_2^* = Er_2\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\int_0^\infty \frac{\beta_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2+1}}d\beta_2$$

$$\int_0^\infty \frac{\beta_1^{r_1-1}\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]+(n-r-k)\log(1+\beta_1)]^{r_1}}d\beta_1, \qquad (2.87)$$

and

$$p_1^* = E\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+2,r_2+k+1)\int_0^\infty \frac{\beta_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]}{[\sum_{j=1}^{r_2}\log(1+\beta_2 x_{2j})]+k\log(1+\beta_2)]^{r_2}}d\beta_2$$

$$\int_0^\infty \frac{\beta_1^{r_1-1}\exp[-\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]}{[\sum_{j=1}^{r_1}\log(1+\beta_1 x_{1j})]+(n-r-k)\log(1+\beta_1)]^{r_1}}d\beta_1. \qquad (2.88)$$

Bayes estimators of $\beta_1,\beta_2,b_1,b_2$ and $p_1$ and hence that of $a_1,a_2,b_1,b_2$ and $p_1$ will obtained by numerical integration procedure.

**Remark 2.2**

As mentioned earlier, one can use other types of priors for the analysis, which may yield complex situations.

### 2.7.2 Estimation based on type I censored samples when the observations belonging to each subpopulation are unknown

#### (a) Maximum likelihood estimation (M.L.E)

In this section, we develop maximum likelihood estimator of parameters under type I censoring. Suppose there are $n$ sample units $(x_1,x_2,...,x_n)$ from the mixture model (2.2) and the lifetime of the units is censored by the time $T$ and $x_i$ denote the failure time of the $i$-th unit in a sample of size $n$. The likelihood function based on a type I censored sample (see Lawless, 2003) is given by

$$L=\prod_{i=1}^{n}\{f(x_i)^{\delta_i}S(T)^{1-\delta_i}\}, \qquad (2.89)$$

where $\delta_i=1$ if $x_i\le T$ and $\delta_i=0$ if $x_i>T$.

For the model (2.2), (2.89) becomes,

$L(a_1, a_2, b_1, b_2, p_1 \mid \underline{x}) =$

$$\prod_{i=1}^{n}\{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i} [p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{1-\delta_i}\}. \quad (2.90)$$

Maximization of log-likelihood function of (2.90) with respect to the model parameters yields the following equations,

$$\sum_{i=1}^{n}[\frac{\delta_i [p_1 b_1 (1+a_1 x_i)^{-(b_1+1)} - p_1 b_1 a_1 (b_1+1)(1+a_1 x_i)^{-(b_1+2)} x_i]}{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]} - \frac{(1-\delta_i) p_1 b_1 (1+a_1 T)^{-(b_1+1)} T}{[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]}] = 0, \quad (2.91)$$

$$\sum_{i=1}^{n}[\frac{\delta_i [(1-p_1) b_2 (1+a_2 x_i)^{-(b_2+1)} - (1-p_1) b_2 a_2 (b_2+1)(1+a_2 x_i)^{-(b_2+2)} x_i]}{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]} - \frac{(1-\delta_i)(1-p_1) b_2 (1+a_2 T)^{-(b_2+1)} T}{[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]}] = 0, \quad (2.92)$$

$$\sum_{i=1}^{n}[\{\frac{\delta_i \{p_1 a_1 (1+a_1 x_i)^{-1} [(1+a_1 x_i)^{-b_1} - b_1 (1+a_1 x_i)^{-b_1} \log(1+a_1 x_i)]}{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]}\} + \frac{(1-\delta_i)[-p_1 (1+a_1 T)^{-b_1} \log(1+a_1 T)}{[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]}] = 0, \quad (2.93)$$

$$\sum_{i=1}^{n}[\{\frac{\delta_i \{(1-p_1) a_2 (1+a_2 x_i)^{-1} [(1+a_2 x_i)^{-b_2} - b_2 (1+a_2 x_i)^{-b_2} \log(1+a_2 x_i)]}{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]}\} + \frac{(1-\delta_i)[-(1-p_1)(1+a_2 T)^{-b_2} \log(1+a_2 T)}{[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]}] = 0, \quad (2.94)$$

and

$$\sum_{i=1}^{n}[\frac{\delta_i [a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} - a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]}{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]} + \frac{(1-\delta_i)[(1+a_1 T)^{-b_1} - (1+a_2 T)^{-b_2}]}{[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]}] = 0. \quad (2.95)$$

The solution of the five equations (2.91) to (2.95), using numerical iteration scheme, yields the maximum likelihood estimate (M.L.E) of $a_1, a_2, b_1, b_2$ and $p_1$ .

**(b) Bayes estimation.**

To estimate the parameters, consider the joint prior distribution (2.22). Using (2.90), we then obtain the joint posterior distribution of $a_1, a_2, b_1, b_2$ and $p_1$ as

$$\prod(a_1, a_2, b_1, b_2, p_1 \mid \underline{x}) \;\alpha\; \prod_{i=1}^{n} \{(p_1 a_1 b_1 (1+a_1 x_{(i)})^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_{(i)})^{-(b_2+1)}]^{\delta_i}$$

$$[p_1(1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1 b_2} . \qquad (2.96)$$

We then obtain the marginal posterior distributions as

$$\Pi_1(a_1 \mid \underline{x}) = K \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1(1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1 b_2} dp_1 db_1 db_2 da_2 , \qquad (2.97)$$

$$\Pi_2(a_2 \mid \underline{x}) = K \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1(1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1 b_2} dp_1 db_1 db_2 da_1 , \qquad (2.98)$$

$$\Pi_3(b_1 \mid \underline{x}) = K \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1(1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1 b_2} dp_1 da_2 da_1 db_2 , \qquad (2.99)$$

$$\Pi_4(b_2 \mid \underline{x}) = K \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1(1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1 b_2} dp_1 da_2 da_1 db_1 , \qquad (2.100)$$

and

$$\Pi_5(p_1 \mid \underline{x}) = K \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1 b_2} db_2 db_1 da_2 da_1 . \qquad (2.101)$$

where

$$K^{-1} = \int_0^1\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1 b_2} db_2 db_1 da_2 da_1 dp_1 . \qquad (2.102)$$

Then the Bayes estimates of the parameters, under squared error loss function, are given as

$$a_1^* = K \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_2 b_1 b_2} dp_1 db_1 db_2 da_2 da_1 , \qquad (2.103)$$

$$a_2^* = K \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 b_1 b_2} dp_1 db_1 db_2 da_1 da_2 , \qquad (2.104)$$

$$b_1^* = K \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_2} dp_1 da_2 da_1 db_2 db_1 , \qquad (2.105)$$

$$b_2^* = K \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \prod_{i=1}^{n} \{[p_1 a_1 b_1 (1+a_1 x_i)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1 (1+a_1 T)^{-b_1} + (1-p_1)(1+a_2 T)^{-b_2}]^{(1-\delta_i)}\} \frac{1}{a_1 a_2 b_1} dp_1 da_2 da_1 db_1 db_2 , \qquad (2.106)$$

and

$$p_1^* = K\int_0^1\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty p_1\prod_{i=1}^{n}\{[p_1a_1b_1(1+a_1x_i)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_i)^{-(b_2+1)}]^{\delta_i}$$

$$[p_1(1+a_1T)^{-b_1}+(1-p_1)(1+a_2T)^{-b_2}]^{(1-\delta_i)}\}\frac{1}{a_1a_2b_1b_2}db_2db_1da_2da_1dp_1. \qquad (2.107)$$

Bayes estimators of $a_1, a_2, b_1, b_2$ and $p_1$ will obtained by numerical integration procedure.

**Remark 2.3**

One can use other types of priors for the analysis, which may yield complex situations as in the previous case.

### 2.7.3 Estimation based on type II censored samples

In type II censoring, we observe $r$ smallest lifetimes, $x_{(1)}, x_{(2)}, ..., x_{(r)}$ of $n$ units ($1 \le r \le n$).

**(a) Maximum Likelihood estimation (M.L.E).**

The likelihood function based on a type II censored sample (see Lawless, 2003) is given by

$$L(a_1,a_2,b_1,b_2,p_1 \mid \underline{x}) = \frac{n!}{(n-r)!}[\prod_{i=1}^{r} f(x_{(i)})][S(x_{(r)})]^{(n-r)}. \qquad (2.108)$$

For the model (2.2), (2.108) becomes,

$$L(a_1,a_2,b_1,b_2,p_1 \mid \underline{x}) = \frac{n!}{(n-r)!}[\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})]$$

$$[p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2}]^{(n-r)}. \qquad (2.109)$$

Maximization of log-likelihood function of (2.109) with respect to the parameters yields the following equations.

$$\sum_{i=1}^{r}\left\{\frac{p_1b_1(1-a_1b_1x_{(i)})(1+a_1x_{(i)})^{-(b_1+2)}}{p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}}\right\}$$

$$-\frac{p_1b_1(n-r)x_{(r)}(1+a_1x_{(r)})^{-(b_1+1)}}{p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2}} = 0, \qquad (2.110)$$

$$\sum_{i=1}^{r}\left\{\frac{(1-p_1)b_2(1-a_2b_2x_{(i)})(1+a_2x_{(i)})^{-(b_2+2)}}{p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}}\right\}$$

$$-\frac{(1-p_1)b_2(n-r)x_{(r)}(1+a_2x_{(r)})^{-(b_2+1)}}{p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2}}=0, \qquad (2.111)$$

$$\sum_{i=1}^{r}\left\{\frac{p_1a_1(1-b_1\log(1+a_1x_{(i)}))(1+a_1x_{(i)})^{-(b_1+1)}}{p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}}\right\}$$

$$-\frac{(n-r)p_1(1+a_1x_{(r)})^{-b_1}\log(1+a_1x_{(r)})}{p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2}}=0, \qquad (2.112)$$

$$\sum_{i=1}^{r}\left\{\frac{(1-p_1)a_2(1-b_2\log(1+a_2x_{(i)}))(1+a_2x_{(i)})^{-(b_2+1)}}{p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}}\right\}$$

$$-\frac{(n-r)(1-p_1)(1+a_2x_{(r)})^{-b_2}\log(1+a_2x_{(r)})}{p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2}}=0, \qquad (2.113)$$

and

$$\sum_{i=1}^{r}\left\{\frac{a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}-a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}}{p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}}\right\}$$

$$+\frac{(n-r)[(1+a_1x_{(r)})^{-b_1}-(1+a_2x_{(r)})^{-b_2}]}{p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2}}=0. \qquad (2.114)$$

Solving the equations (2.110) to (2.114) we obtain the M.L.E s of $a_1, a_2, b_1, b_2$ and $p_1$.

**(b) Bayes estimation.**

From (2.22) and (2.109), we obtain the joint posterior distribution of $a_1, a_2, b_1, b_2$ and $p_1$ as

$$\prod(a_1,a_2,b_1,b_2,p_1\mid \underline{x})\ \alpha\ \left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}\ . \qquad (2.115)$$

We then obtain the marginal posterior distributions as,

$$\Pi_1(a_1 \mid \underline{x}) = Q\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}\,dp_1db_1db_2da_2 \text{ , (2.116)}$$

$$\Pi_2(a_2 \mid \underline{x}) = Q\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}\,dp_1db_1db_2da_1 \text{, (2.117)}$$

$$\Pi_3(b_1 \mid \underline{x}) = Q\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}\,dp_1db_2da_2da_1 \text{, (2.118)}$$

$$\Pi_4(b_2 \mid \underline{x}) = Q\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}\,dp_1db_1da_2da_1 \text{, (2.119)}$$

and

$$\Pi_5(p_1 \mid \underline{x}) = Q\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}\,db_1db_2da_2da_1 \text{ . (2.120)}$$

where

$$Q^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}\,dp_1db_1db_2da_2da_1 \text{ . (2.121)}$$

Then the Bayes estimates of the parameters, under squared error loss function, are given as

$$a_1^* = Q\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_2b_1b_2}dp_1db_1db_2da_2da_1\ , \quad (2.122)$$

$$a_2^* = Q\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1b_1b_2}dp_1db_1db_2da_1da_2\, , \quad (2.123)$$

$$b_1^* = Q\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_2}dp_1db_2da_2da_1db_1\ , \quad (2.124)$$

$$b_2^* = Q\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 [\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)}]$$

$$[p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2}]^{(n-r)}\frac{1}{a_1a_2b_1}dp_1db_1da_2da_1db_2\, , \quad (2.125)$$

and

$$p_1^* = Q\int_0^1\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty p_1\left\{\prod_{i=1}^{r}(p_1a_1b_1(1+a_1x_{(i)})^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x_{(i)})^{-(b_2+1)})\right\}$$

$$\left\{(p_1(1+a_1x_{(r)})^{-b_1}+(1-p_1)(1+a_2x_{(r)})^{-b_2})^{(n-r)}\right\}\frac{1}{a_1a_2b_1b_2}db_1db_2da_2da_1dp_1\, . \quad (2.126)$$

Bayes estimators of $a_1, a_2, b_1, b_2$ and $p_1$ will obtained by numerical integration procedure.

**Remark 2.4**

One can use other types of priors for the analysis, which may yield complex situations

**Remark 2.5**

When $r=n$, the estimators obtained under type II censoring situation reduces to those under complete sample set up.

## 2.8 Simulation Study

In Section 2.5, we have described different methods for estimation of parameters of the model (2.2) for the complete as well as for type I and type II censored samples. In this section, we carried out simulation studies to assess the performance of the estimates.

For the type I censoring, in which observations belonging to each subpopulation are known, we generate two sets of samples, one from a Pareto II distribution with parameters $a_1$ and $b_1$ and the second from a Pareto II distribution with parameters $a_2$ and $b_2$.

For the type I censoring, in which observations belonging to each subpopulation are unknown and for type II censoring, first we generate two sets of samples, one from a Pareto II distribution with parameters $a_1$ and $b_1$ and the second from a Pareto II distribution with parameters $a_2$ and $b_2$ and then, we obtain samples from a finite mixture of Pareto II distributions using Bernoulli probability $p_1 (0 < p_1 < 1)$.

The estimates of parameters by the method of maximum likelihood, Bayes technique, method of moments and method of maximum product of spacing under complete sample set up are developed for various combinations of $a_1, a_2, b_1, b_2$ and $p_1$. Table 2.1 presents the estimates for $a_1 = 0.05, a_2 = 2, b_1 = b_2 = 5$ and $p_1 = 0.5$. Here, we take $b_1 = b_2 = b$ for the simplicity of calculation of moment estimates and we choose $b = 5$, since in method of moments $b$ should be greater than 4.

The estimates of parameters by the method of maximum likelihood and Bayes technique under type I censoring, in which observations belonging to each subpopulation are known, for $a_1 = .03, a_2 = 3, b_1 = 1, b_2 = 2$ and $p_1 = 0.5$ with various combinations of $n$ (sample size) and $T$ (censoring time) are given in Table 2.2. In Table 2.3, we give the estimates of parameters by maximum likelihood and Bayes technique under type I censoring, in which observations belonging to each subpopulation are unknown, with various combinations of $n$ and $T$ for the same set of parameters. Table 2.4 provide estimators by the maximum likelihood and Bayes under type II censoring for the same set of parameters with various combinations of $n$ and $r$.

The values in brackets provide the variance of the estimates. . In the case of Bayes estimate, we calculate posterior risk. The result obtained under type II censoring is extended to the complete sample by taking $r=n$. Simulation study reported here and other simulations that are not given here, shows that M.L.E. and Bayes methods provide estimates with small bias.. The variance (posterior risk) of the estimates decreases as n increases.

Now, we compare the different estimates to measure the efficiency. For that we define risks improvement (RI) factor, as

$$RI(\%) = \frac{Var(\delta_0) - Var(\delta)}{Var(\delta_0)} \times 100,$$

where $\delta$ is the new estimate and $\delta_0$ is the existing estimate (see Jin and Pal, 1992) and $Var(\delta)$ denotes the variance of $\delta$. The results are presented in Table 2.5

We used the software, Mathematica, to evaluate integrals involved in the expressions. It may be noted that the efficiency of the maximum likelihood estimates depends on starting values in the iterative procedure.

**Table 2.1 Estimates of parameters under complete sample for**

$a_1 = .05, a_2 = 2, b = 5$ **and** $p_1 = 0.5$

| Estimate | M.L.E | Bayes | Method of moments | Maximum product of spacing |
|---|---|---|---|---|
| n=30 | $\hat{a}_1$=.050104 (2.34E-05) | $a_1^*$=.048875 (2.371E-05) | $\tilde{a}_1$=.05178 (.0198) | $\breve{a}_1$=.04912 (.00351) |
| | $\hat{a}_2$=2.1498 (8.47E-06) | $a_2^*$=2.102 (8.83E-06) | $\tilde{a}_2$=1.791 (.0476) | $\breve{a}_2$=2.0923 (.0346) |
| | $\hat{b}$=4.829962 (.025288) | $b^*$=4.9426 (.0408) | $\tilde{b}$=5.2075 (.1209) | $\breve{b}$=4.9231 (.21052) |
| | $\hat{p}_1$=.518161 (.000518) | $p_1^*$=.51815 (.000518) | $\tilde{p}_1$=.4876 (.00178) | $\breve{p}_1$=.49942 (1.9E-10) |
| n=50 | $\hat{a}_1$=.0491342 (1.88E-05) | $a_1^*$=.049001 (1.892E-05) | $\tilde{a}_1$=.05156 (.0132) | $\breve{a}_1$=.04989 (.00218) |
| | $\hat{a}_2$=2.02265 (5.05E-06) | $a_2^*$=2.0208 (5.72E-06) | $\tilde{a}_2$=1.89 (.0219) | $\breve{a}_2$=1.9976 (.001753) |
| | $\hat{b}$=4.918252 (.01636) | $b^*$=4.9474 (.044573) | $\tilde{b}$=5.176 (.1138) | $\breve{b}$=5.017 (.1208) |
| | $\hat{p}_1$=.5271 (.000354) | $p_1^*$=.513624 (.000857) | $\tilde{p}_1$=.4894 (.00112) | $\breve{p}_1$=.50173 (1.3E-12) |
| n=100 | $\hat{a}_1$=.0491522 (1.387E-05) | $a_1^*$=.5101 (1.882E-05) | $\tilde{a}_1$=.0511 (.00987) | $\breve{a}_1$=.0508 (1.1E-03) |
| | $\hat{a}_2$=2.022 (3.8E-06) | $a_2^*$=2.0202 (4.98E-06) | $\tilde{a}_2$=1.9037 (.0187) | $\breve{a}_2$=2.0013 (.000521) |
| | $\hat{b}$=4.929 (.008508) | $b^*$=4.858 (.007085) | $\tilde{b}$=5.1257 (.0732) | $\breve{b}$=5.0078 (.02371) |
| | $\hat{p}_1$=.5136 (.000227) | $p_1^*$=.5255 (.000318) | $\tilde{p}_1$=.50238 (.000179) | $\breve{p}_1$=.500012 (1.1E-13) |

**Table 2.2 Estimates of parameters under type I censoring for $a_1 = .03, a_2 = 3, b_1 = 1, b_2 = 2$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are known.**

| Censoring time | Estimate | n=30 | n=50 | n=100 |
|---|---|---|---|---|
| T=30 | M.L.E | $\hat{a}_1$=.0321006 (3.7E-03) | $\hat{a}_1$=.0310114 (2.8E-03) | $\hat{a}_1$=.0299031 (2.1E-03) |
| | | $\hat{a}_2$=2.87634 (.09324) | $\hat{a}_2$=2.90125 (.03352) | $\hat{a}_2$=3.09768 (.01127) |
| | | $\hat{b}_1$=1.21304 (.1128) | $\hat{b}_1$=1.10124 (.0156) | $\hat{b}_1$=1.0098 (.01376) |
| | | $\hat{b}_2$=2.11321 (.15712) | $\hat{b}_2$=1.973 (.08741) | $\hat{b}_2$=1.99243 (.0127) |
| | | $\hat{p}_1$=.49967 (3.08E-10) | $\hat{p}_1$=.49989 (2.9E-10) | $\hat{p}_1$=.49991 (1.1E-10) |
| | Bayes estimate | $a_1^*$=.0312301 (2.87E-03) | $a_1^*$=.0309141 (1.9E-03) | $a_1^*$=.0302004 (1.27E-04) |
| | | $a_2^*$=2.89137 (.027648) | $a_2^*$=3.10642 (.01139) | $a_2^*$=2.99104 (.01038) |
| | | $b_1^*$=1.13203 (.1428) | $b_1^*$=.9890 (.1014) | $b_1^*$=.99124 (.10014) |
| | | $b_2^*$=2.09913 (.10519) | $b_2^*$=2.0897 (.08431) | $b_2^*$=1.99884 (.0653) |
| | | $p_1^*$=.490578 (2.08E-11) | $p_1^*$=.49952 (1.99E-11) | $p_1^*$=.500124 (1.08E-11) |
| T=100 | M.L.E | $\hat{a}_1$=.031198 (2.1E-03) | $\hat{a}_1$=.030987 (1.97E-03) | $\hat{a}_1$=.0300321 (1.9E-04) |
| | | $\hat{a}_2$=2.9124 (.057894) | $\hat{a}_2$=2.975 (.016809) | $\hat{a}_2$=3.0432 (.00178) |
| | | $\hat{b}_1$=1.1096 (.10742) | $\hat{b}_1$=1.0864 (.09656) | $\hat{b}_1$=1.00031 (.02376) |
| | | $\hat{b}_2$=2.1087 (.1043) | $\hat{b}_2$=2.0965 (.03650) | $\hat{b}_2$=2.0432 (.01368) |
| | | $\hat{p}_1$=.49998 (2.076E-12) | $\hat{p}_1$=.49999 (1.98E-12) | $\hat{p}_1$=.5002 (1.34E-12) |
| | Bayes estimate | $a_1^*$=.03109 (2.01E-03) | $a_1^*$=.03098 (1.8E-03) | $a_1^*$=.030024 (1.01E-04) |
| | | $a_2^*$=2.9521 (.017094) | $a_2^*$=3.10032 (.003456) | $a_2^*$=2.9998 (.001127) |
| | | $b_1^*$=1.0543 (.1021) | $b_1^*$=1.0412 (.0357) | $b_1^*$=1.003 (.009986) |
| | | $b_2^*$=2.0657 (.10042) | $b_2^*$=2.023 (.02341) | $b_2^*$=1.9991 (.0032) |
| | | $p_1^*$=.49999 (2.79E-14) | $p_1^*$=.49999 (1.20E-14) | $p_1^*$=.5001 (1.1E-14) |

**Table 2.3 Estimates of parameters under type I censoring for $a_1 = .03, a_2 = 3, b_1 = 1, b_2 = 2$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are unknown.**

| Censoring time | Estimate | n=30 | n=50 | n=100 |
|---|---|---|---|---|
| T=30 | M.L.E | $\hat{a}_1$=.02789 (2.8E-04) | $\hat{a}_1$=.030932 (2.1E-04) | $\hat{a}_1$=.0299987 (1.7E-04) |
| | | $\hat{a}_2$=2.91132 (5.4E-03) | $\hat{a}_2$=2.92786 (2.9E-03) | $\hat{a}_2$=3.09764 (2.8E-03) |
| | | $\hat{b}_1$=1.1043 (.00279) | $\hat{b}_1$=1.10213 (.001247) | $\hat{b}_1$=1.012 (1.5E-03) |
| | | $\hat{b}_2$=2.0918 (.00176) | $\hat{b}_2$=2.07621 (2.7E-03) | $\hat{b}_2$=2.009 (1.5E-03) |
| | | $\hat{p}_1$=.4998 (3.01E-12) | $\hat{p}_1$=.49999 (2.7E-12) | $\hat{p}_1$=.5001 (2.2E-12) |
| | Bayes estimate | $a_1^*$=.02917 (1.9E-04) | $a_1^*$=.03009 (1.6E-04) | $a_1^*$=.03002 (1.3E-04) |
| | | $a_2^*$=3.1164 (4.7E-03) | $a_2^*$=2.9429 (2.2E-03) | $a_2^*$=3.0917 (1.9E-03) |
| | | $b_1^*$=.9426 (.0017) | $b_1^*$=1.0975 (2.3E-03) | $b_1^*$=1.0013 (1.1E-03) |
| | | $b_2^*$=1.9543 (4.1E-03) | $b_2^*$=1.985 (3.7E-04) | $b_2^*$=2.0019 (2.4E-04) |
| | | $p_1^*$=.4999 (1.8E-13) | $p_1^*$=.500112 (1.5E-13) | $p_1^*$=.5001 (1.3E-13) |
| T=100 | M.L.E | $\hat{a}_1$=.03098 (2.4E-04) | $\hat{a}_1$=.03074 (1.98E-04) | $\hat{a}_1$=.03003 (1.5E-04) |
| | | $\hat{a}_2$=2.9469 (3.6E-03) | $\hat{a}_2$=2.98 (2.7E-03) | $\hat{a}_2$=3.0031 (1.9E-03) |
| | | $\hat{b}_1$=0.9912 (2.3E-03) | $\hat{b}_1$=0.9936 (1.3E-03) | $\hat{b}_1$=1.00021 (5.9E-04) |
| | | $\hat{b}_2$=2.1087 (3.4E-03) | $\hat{b}_2$=2.0965 (2.1E-03) | $\hat{b}_2$=2.0432 (1.1E-03) |
| | | $\hat{p}_1$=.5017 (2.6E-12) | $\hat{p}_1$=.50076 (1.9E-12) | $\hat{p}_1$=.5001 (1.7E-12) |
| | Bayes estimate | $a_1^*$=.03003 (1.9E-04) | $a_1^*$=.0299 (1.4E-04) | $a_1^*$=.03001 (1.3E-04) |
| | | $a_2^*$=3.096 (3.2E-03) | $a_2^*$=3.0863 (2.5E-03) | $a_2^*$=3.0056 (1.2E-03) |
| | | $b_1^*$=1.0023 (7.8E-04) | $b_1^*$=.9989 (5.6E-04) | $b_1^*$=1.0001 (2.1E-04) |
| | | $b_2^*$=2.0657 (2.9E-03) | $b_2^*$=2.023 (1.8E-03) | $b_2^*$=1.9991 (1.02E-03) |
| | | $p_1^*$=.4999 (2.2E-13) | $p_1^*$=.50007 (1.5E-13) | $p_1^*$=.50001 (1.2E-13) |

**Table 2.4 Estimates of parameters under type II censoring for $a_1 = .03, a_2 = 3, b_1 = 1, b_2 = 2$ and $p_1 = 0.5$**

| Estimate | n=30 | | n=50 | | n=100 | |
|---|---|---|---|---|---|---|
| | r=10 | r=30 | r=30 | r=50 | r=60 | r=100 |
| M.L.E | $\hat{a}_1$=.03113 (.000913) | $\hat{a}_1$=.0301 (.000804) | $\hat{a}_1$=.0305 (.000691) | $\hat{a}_1$=.03021 (.0006780) | $\hat{a}_1$=.0301 (.0005478) | $\hat{a}_1$=.02999 (.0000780) |
| | $\hat{a}_2$=3.125 (.09453) | $\hat{a}_2$=3.098 (.05417) | $\hat{a}_2$=3.097 (.00571) | $\hat{a}_2$=3.003 (.002168) | $\hat{a}_2$=2.999 (.00106) | $\hat{a}_2$=3.001 (.00087) |
| | $\hat{b}_1$=1.1103 (.0090037) | $\hat{b}_1$=1.093 (.007396) | $\hat{b}_1$=1.091 (.0020381) | $\hat{b}_1$=1.082 (.002000) | $\hat{b}_1$=1.002 (.001600) | $\hat{b}_1$=1.001 (.0002600) |
| | $\hat{b}_2$=2.135 (.00883956) | $\hat{b}_2$=2.091 (.007023) | $\hat{b}_2$=2.065 (.0065277) | $\hat{b}_2$=2.049 (.0048) | $\hat{b}_2$=1.998 (.001821) | $\hat{b}_2$=1.999 (.000980) |
| | $\hat{p}_1$=.4981 (2.43E-12) | $\hat{p}_1$=.4989 (1.01E-12) | $\hat{p}_1$=.4998 (2.63E-16) | $\hat{p}_1$=.5005 (2.11E-16) | $\hat{p}_1$=.5002 (3.11E-18) | $\hat{p}_1$=.5001 (1.01E-18) |
| Bayes estimate | $a_1^*$=.03123 (1.38E-5) | $a_1^*$=.0304 (1.22E-5) | $a_1^*$=.0287 (1.03E-5) | $a_1^*$=.0298 (1.01E-5) | $a_1^*$=.031 (2.22E-6) | $a_1^*$=.0302 (1.98E-6) |
| | $a_2^*$=3.101 (.09743) | $a_2^*$=3.087 (.0532) | $a_2^*$=3.054 (.0247) | $a_2^*$=2.995 (.02187) | $a_2^*$=2.998 (.00590) | $a_2^*$=2.999 (.00210) |
| | $b_1^*$=1.125 (.0016) | $b_1^*$=1.0987 (.0032) | $b_1^*$=1.085 (.0023) | $b_1^*$=1.0043 (.00191) | $b_1^*$=1.0021 (.00086) | $b_1^*$=1.001 (.00012) |
| | $b_2^*$=2.1054 (.10490) | $b_2^*$=2.045 (.0978) | $b_2^*$=2.014 (.01752) | $b_2^*$=1.997 (.00951) | $b_2^*$=2.004 (.00219) | $b_2^*$=2.001 (.00019) |
| | $p_1^*$=.4976 (2.13E-13) | $p_1^*$=.4987 (1.09E-13) | $p_1^*$=.4998 (1.87E-15) | $p_1^*$=.4999 (1.1E-15) | $p_1^*$=.50021 (2.9E-19) | $p_1^*$=.5001 (2.0E-19) |

**Table 2.5 Risk improvement of the Bayes estimator over the different estimators for the set of parameters $a_1 = .05, a_2 = 2, b = 5$ and $p_1 = 0.5$ in the complete sample set up.**

| Estimate | M.L.E. (%) | Method of moments (%) | Maximum product of spacing (%) |
|---|---|---|---|
| n=30 | 84 | 98 | 98 |
| | 89 | 99 | 93 |
| | 87 | 95 | 96 |
| | 88 | 99 | 95 |
| n=50 | 89 | 96 | 90 |
| | 85 | 97 | 94 |
| | 84 | 98 | 98 |
| | 87 | 96 | 92 |
| n=100 | 80 | 97 | 95 |
| | 87 | 99 | 94 |
| | 85 | 94 | 96 |
| | 86 | 98 | 97 |

## 2.9 Data analysis

For the illustration of the estimation procedure, we consider a data on time to death of two groups of leukaemia patients (see Feigl and Zelen, 1965), which is given in Table 2.6. We then estimate the parameters using M.L.E and Bayes technique. Table 2.7 provides the values of the estimates by M.L.E and Bayes method. We then used the Kolmogorov- Smirnov statistic to test the goodness of fit. The values of the test statistic $D_n$ is given in Table 2.8. The table value at 5 % significance level is 0.233. From the analysis, it concludes that the model (2.2) is a plausible model for the data with parameters given in Table 2.7. Table 2.9 provides the maximum likelihood estimate of survival function at various time points.

**Table 2.6 Survival times of leukaemia patients**

| AG positive patients | AG negative patients |
|---|---|
| 65 | 56 |
| 156 | 65 |
| 100 | 17 |
| 134 | 7 |
| 16 | 16 |
| 108 | 22 |
| 121 | 3 |
| 4 | 4 |
| 39 | 2 |
| 143 | 3 |
| 56 | 8 |
| 26 | 4 |
| 22 | 3 |
| 1 | 30 |
| 1 | 4 |
| 5 | 43 |
| 65 | |

**Table 2.7 Estimates of parameters of survival times of leukaemia patients.**

| M.L.E | Bayes method |
|---|---|
| $\hat{a}_1 = 0.0213968$ | $a_1^* = 0.0194$ |
| $\hat{a}_2 = 0.0387068$ | $a_2^* = 0.0280742$ |
| $\hat{b}_1 = 1.14671$ | $b_1^* = 2.8557$ |
| $\hat{b}_2 = 2.02603$ | $b_2^* = 2.70599$ |
| $\hat{p}_1 = 0.50137$ | $p_1^* = 0.509715$ |

**Table 2.8 Kolmogorov-Smirnov test statistic**

| Max $D_n$ (M.L.E) | Max $D_n$ (Bayes) |
|---|---|
| 0.1019749 | 0.052931 |

**Table 2.9 Maximum likelihood estimate of survival probability at various time points**

| $x$ | 1 | 10 | 50 | 75 | 100 | 120 | 140 |
|---|---|---|---|---|---|---|---|
| $\hat{S}(x)$ | .951048 | .658389 | .273978 | .198853 | .155183 | .131571 | .113936 |

## 2.10 Conclusion

The role of finite mixture of Pareto II distributions in reliability analysis is studied. We proved that finite mixture of two Pareto II distributions is identifiable. In the present chapter, we developed estimates of parameters using different approaches for both complete and censored samples. Maximum likelihood and Bayes methods provide estimates with small bias and less variance (in the case of Bayes, we calculate posterior risk) and the variance of the estimates decreases as n increases. The results derived under type II censoring can be specialized to the complete sample case by taking $r = n$. As expected, the method of moments was shown to be inferior to maximum likelihood estimation (M.L.E), Bayes method and maximum product of spacing (M.P.S). The role of finite mixture of Pareto II distributions is illustrated using a real life data on time to death of two groups of leukaemia patients. The part of this chapter has appeared in Sankaran and Maya (2005).

# Chapter 3

# FINITE MIXTURE OF BETA DISTRIBUTIONS

## 3.1 Introduction

In practical situations, the observed lifetimes of an item varies over only a finite range. Accordingly, Mukherjee and Islam (1983) studied the properties of finite range (beta) model in the context of reliability analysis. The finite mixture of beta distributions is widely used as a lifetime model due to the property that beta distribution has flexible skewed density function over the finite range. Further, beta distribution is widely employed for modelling lifetime data of systems as it has increasing failure rate pattern.

The finite mixtures of beta distributions arise frequently in Bayesian conjugate prior analysis. An example introduced by Diaconis and Ylvisaker (1985) uses finite mixture of beta distributions as a prior for a binomial parameter. They noticed that there is a big difference between spinning a coin on a table and tossing it in the air. The tossing often leads to about an even proportion of heads and tails, while spinning gives an uneven proportion of heads and tails. They justified that the shape of the edge would be a strong determining factor for this bias and used a finite mixture of beta distributions as a reasonable prior.

A systematic study on finite mixture of beta distributions in the context of reliability is not yet carried out. Motivated by this, in the present chapter, we study the properties of finite mixture of finite range (beta) distributions in the context of reliability. Throughout this chapter, we consider a finite mixture of two beta distributions.

In Section 3.2, we give definition and properties of finite mixture of beta distributions. Section 3.3 discusses the important reliability characteristics of the model. Identifiability of finite mixture of beta distribution is discussed in Section 3.4. Estimation of parameters of the model for complete as well as censored case by different techniques is studied in Sections 3.5 to 3.7. Section 3.8 deals with simulation study to investigate the finite sample properties of the estimators. In Section 3.9, we illustrate the role of finite mixture of beta distributions using a real life data. Finally, in Section 3.10, we give the conclusion of the chapter.

## 3.2 Definition and properties

Let $X$ be a non-negative random variable admitting an absolutely continuous distribution function $F(x)$ with respect to a Lebesgue measure. Assume that the probability density function (p.d.f.) of $X$ , $f(x)$ exists. A two-component mixture of beta distribution is given by

$$f(x) = p_1 c_1 d_1 (1 - c_1 x)^{d_1 - 1} + (1 - p_1) c_2 d_2 (1 - c_2 x)^{d_2 - 1}, \tag{3.1}$$

$$0 < x < 1/c_1 ,\ 0 < p_1 < 1 .$$

where $c_2 \geq c_1 > 0$ and $d_1, d_2 > 0$. Notice that the support of the component densities need not be same. For (3.1), the support of the first component is $0 < x < 1/c_1$ and that of the second is $0 < x < 1/c_2$ , where $c_2 \geq c_1$ is assumed, without loss of generality. When $d_1 = d_2 = 1$, (3.1) reduces to a mixture of two uniform distributions.

The $r^{th}$ moment of the distribution (3.1) is given by

$$E_f(X^r) = \frac{p_1 \Gamma(r+1)\Gamma(d_1+1)}{c_1^r \Gamma(d_1+r+1)} + \frac{(1-p_1)\Gamma(r+1)\Gamma(d_2+1)}{c_2^r \Gamma(d_2+r+1)} . \tag{3.2}$$

where $\Gamma p = \int_0^\infty e^{-x} x^{p-1} dx$, is the gamma function.

When $r = 1$, (3.2) reduces to the mean, which is equal to

$$E_f(X) = \frac{p_1}{c_1(d_1+1)} + \frac{(1-p_1)}{c_2(d_2+1)} . \tag{3.3}$$

When $r = 2$, (3.2) becomes

$$E_f(X^2) = \frac{2p_1}{c_1^2(d_1+1)(d_1+2)} + \frac{2(1-p_1)}{c_2^2(d_2+1)(d_2+2)} .$$

Thus the variance of (3.1) is given by

$$V(x)=\frac{p_1(2+2d_1-2p_1-d_1p_1)}{c_1^2(d_1+1)^2(d_1+2)}+\frac{(1-p_1)(d_2-(d_2-2)p_1)}{c_2^2(d_2+1)^2(d_2+2)}-\frac{2p_1(1-p_1)}{c_1c_2(d_1+1)(d_2+1)}. \qquad (3.4)$$

### 3.3 Reliability characteristics

The survival function for the model (3.1) is given by

$$S(x)=p_1(1-c_1x)^{d_1}+(1-p_1)(1-c_2x)^{d_2}\,,\ 0<x<1/c_1\,,\ 0<p_1<1. \qquad (3.5)$$

For the model (3.1), the hazard rate $h(x)$ and the mean residual life function $r(x)$ are given by

$$h(x)=\frac{p_1c_1d_1(1-c_1x)^{d_1-1}+(1-p_1)c_2d_2(1-c_2x)^{d_2-1}}{p_1(1-c_1x)^{d_1}+(1-p_1)(1-c_2x)^{d_2}} \qquad (3.6)$$

and

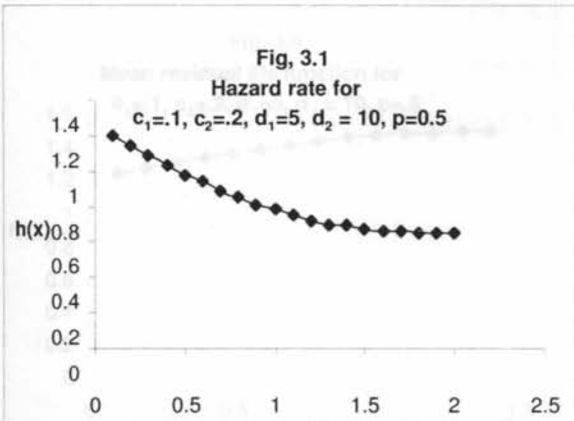
$$r(x)=\frac{p_1c_2(d_2+1)(1-c_1x)^{d_1+1}+(1-p_1)c_1(d_1+1)(1-c_2x)^{d_2+1}}{c_1c_2[p_1(1-c_1x)^{d_1}+(1-p_1)(1-c_2x)^{d_2}](d_1+1)(d_2+1)}\,. \qquad (3.7)$$

From (3.6) and (3.7), we obtain an identity connecting $r(x)$ and $h(x)$ as

$$r(x)-\frac{h(x)(1-c_1x)(1-c_2x)}{c_1c_2(d_1+1)(d_2+1)}\left[\frac{c_2(d_2+1)-c_1(d_1+1)+c_1c_2(d_1-d_2)x}{c_1d_1-c_2d_2-c_1c_2(d_1-d_2)x}\right]$$

$$=\frac{c_1^2d_1(d_1+1)(1-c_2x)^2-c_2^2d_2(d_2+1)(1-c_1x)^2}{c_1c_2(d_1+1)(d_2+1)[c_1d_1-c_2d_2-c_1c_2(d_1-d_2)x]}\,. \qquad (3.8)$$

Figures 3.1 to 3.16 show the behaviour of hazard rate and the mean residual life function for different parameters of the model (3.1). From the graphs, it is easy to see that the behaviour of h(x) and r(x) are depending on the parameters of the model.




Fig, 3.1
Hazard rate for
$c_1$=.1, $c_2$=.2, $d_1$=5, $d_2$ = 10, p=0.5

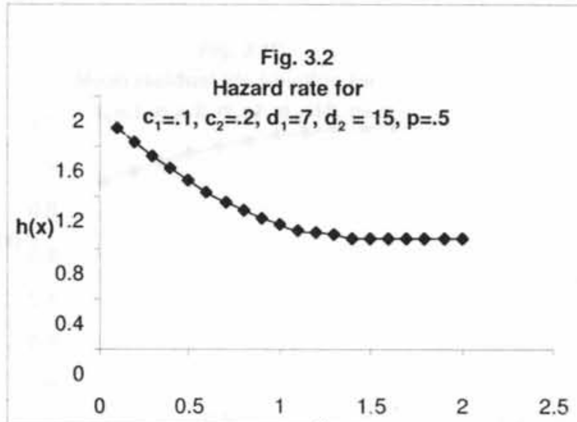



Fig. 3.2
Hazard rate for
$c_1$=.1, $c_2$=.2, $d_1$=7, $d_2$ = 15, p=.5

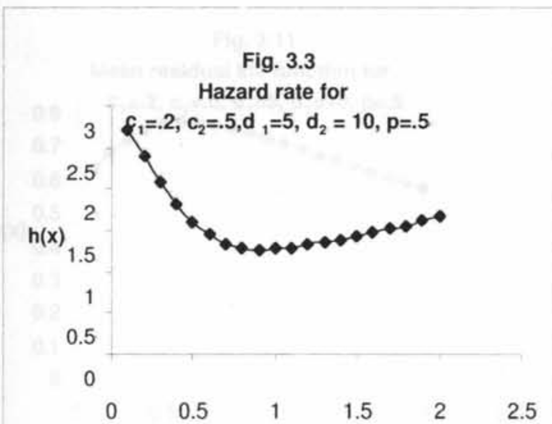



Fig. 3.3
Hazard rate for
$c_1$=.2, $c_2$=.5, $d_1$=5, $d_2$ = 10, p=.5

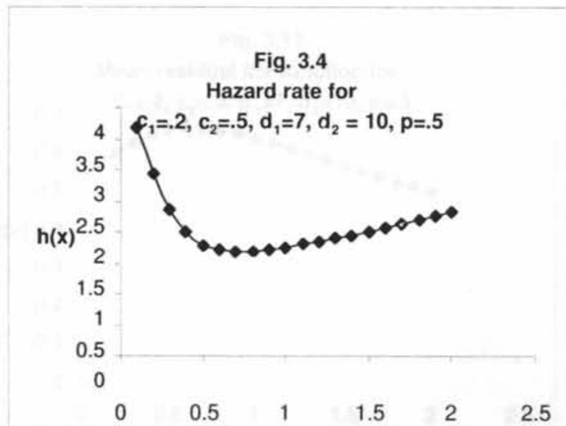



Fig. 3.4
Hazard rate for
$c_1$=.2, $c_2$=.5, $d_1$=7, $d_2$ = 10, p=.5

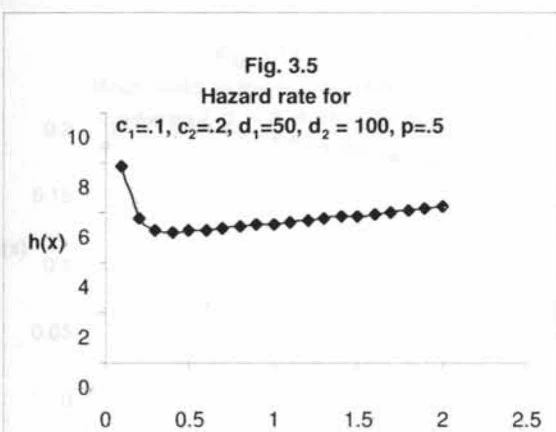



Fig. 3.5
Hazard rate for
$c_1$=.1, $c_2$=.2, $d_1$=50, $d_2$ = 100, p=.5

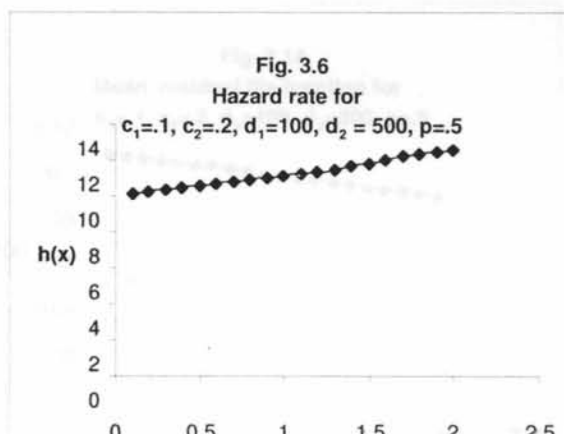



Fig. 3.6
Hazard rate for
$c_1$=.1, $c_2$=.2, $d_1$=100, $d_2$ = 500, p=.5

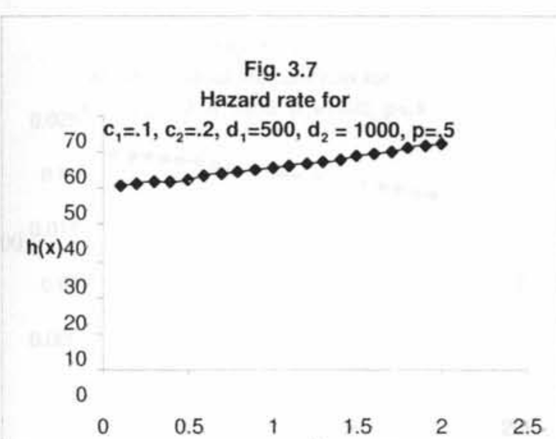



Fig. 3.7
Hazard rate for
$c_1$=.1, $c_2$=.2, $d_1$=500, $d_2$ = 1000, p=.5

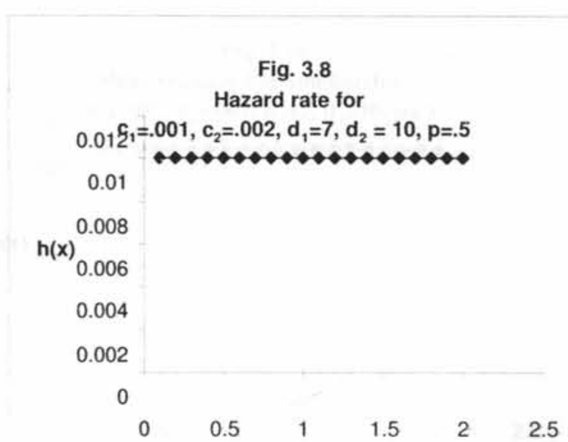



Fig. 3.8
Hazard rate for
$c_1$=.001, $c_2$=.002, $d_1$=7, $d_2$ = 10, p=.5

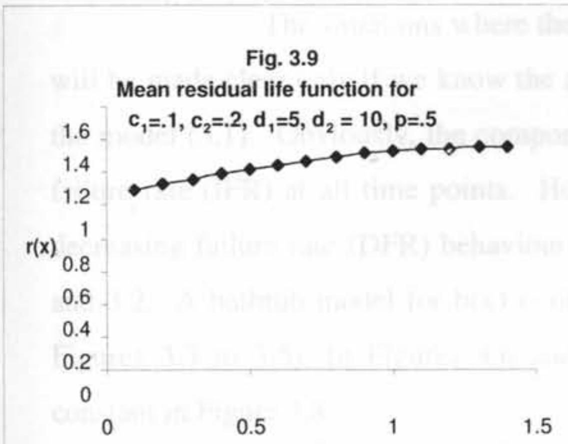


Fig. 3.9 Mean residual life function for $c_1$=.1, $c_2$=.2, $d_1$=5, $d_2$ = 10, p=.5

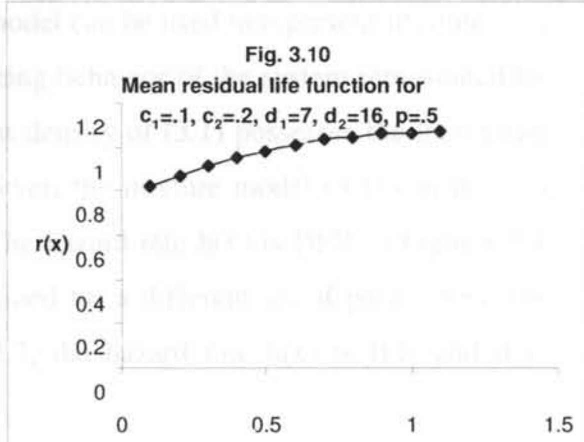


Fig. 3.10 Mean residual life function for $c_1$=.1, $c_2$=.2, $d_1$=7, $d_2$=16, p=.5

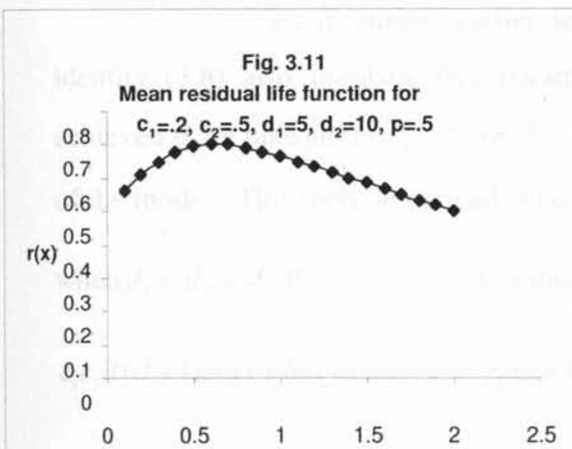


Fig. 3.11 Mean residual life function for $c_1$=.2, $c_2$=.5, $d_1$=5, $d_2$=10, p=.5

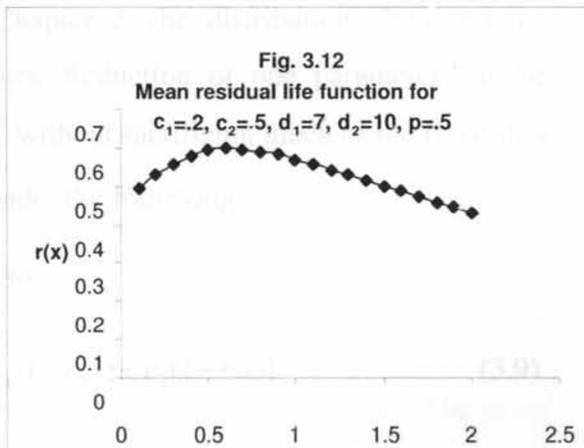


Fig. 3.12 Mean residual life function for $c_1$=.2, $c_2$=.5, $d_1$=7, $d_2$=10, p=.5

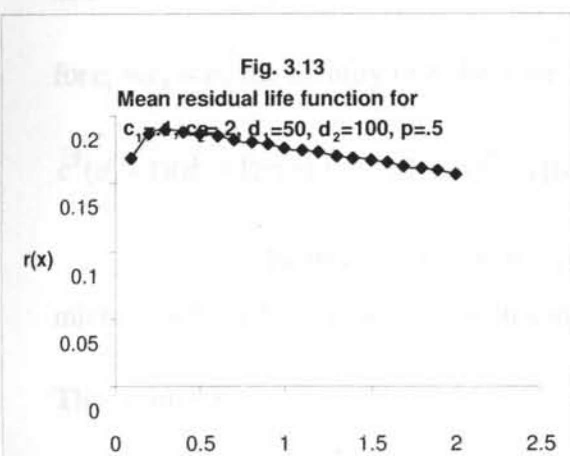


Fig. 3.13 Mean residual life function for $c_1$=.1, $c_2$=.2, $d_1$=50, $d_2$=100, p=.5

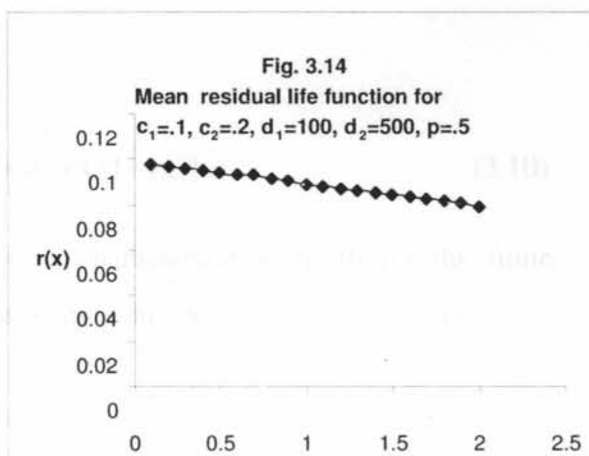


Fig. 3.14 Mean residual life function for $c_1$=.1, $c_2$=.2, $d_1$=100, $d_2$=500, p=.5

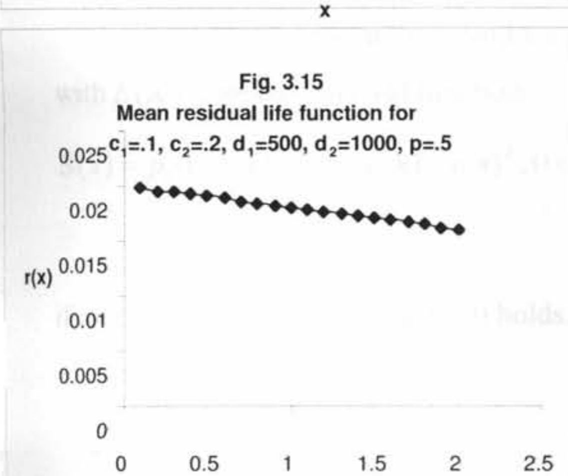


Fig. 3.15 Mean residual life function for $c_1$=.1, $c_2$=.2, $d_1$=500, $d_2$=1000, p=.5

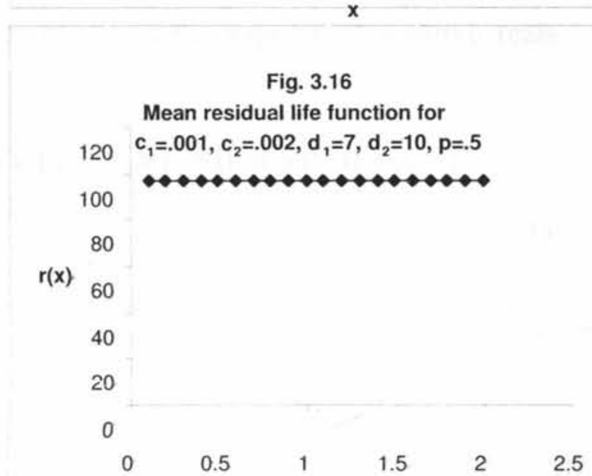


Fig. 3.16 Mean residual life function for $c_1$=.001, $c_2$=.002, $d_1$=7, $d_2$=10, p=.5

The situations where the model can be used to represent lifetime data will be made clear only if we know the ageing behavior of the system represented by the model (3.1). Obviously, the component density of (3.1) possesses the increasing failure rate (IFR) at all time points. However, the mixture model (3.1) can possess decreasing failure rate (DFR) behaviour. The hazard rate h(x) is DFR in Figures 3.1 and 3.2. A bathtub model for h(x) is obtained for a different set of parameters (see Figures 3.3 to 3.5). In Figures 3.6 and 3.7, the hazard rate h(x) is IFR and it is constant in Figure 3.8.

As discussed earlier in Chapter 2, the distribution (3.1) and the identity (3.8) also involves five parameters. Reduction of one parameter can be achieved if we take either $c_1 = c_2$ or $d_1 = d_2$ without sacrificing much of the flexibility of the model. This motivation leads to consider the following,

when $d_1 = d_2 = d$, the identity (3.8) reduces to

$$c_1 c_2 d(d+1) r(x) + h(x)(1-c_1 x)(1-c_2 x) = d[c_1(1-c_2 x) + c_2(1-c_1 x)] \tag{3.9}$$

and

for $c_1 = c_2 = c$, the identity (3.8) becomes

$$c^2(d_1+1)(d_2+1) r(x) + h(x)(1-cx)^2 = c[(d_1+d_2+1)(1-cx)] \; . \tag{3.10}$$

In the following, we prove a characterization result for the finite mixture of beta distributions through simple relationship between $h(x)$ and $r(x)$.

**Theorem 3.3.1**

A continuous random variable $X$ in the support of positive reals with $E(X) < \infty$, has survival function

$$S(x) = p_1(1-c_1 x)^d + (1-p_1)(1-c_2 x)^d \,, 0 < x < 1/c_1 \,,\; c_2 \ge c_1 > 0 \,, d > 0 \,,\; 0 < p_1 < 1 . \tag{3.11}$$

if and only if the relationship (3.9) holds.

**Proof.**

Suppose that the identity (3.9) holds, and then we have,

$$c_1c_2d(d+1)\int_x^{\infty}S(t)dt + f(x)(1-c_1x)(1-c_2x) = d[c_1+c_2-2c_1c_2x]S(x). \quad (3.12)$$

Differentiating (3.12) twice with respect to $x$, we get,

$$c_1c_2(d-1)(d-2)f(x)+(d-2)f'(x)[c_1(1-c_2x)+c_2(1-c_1x)]+f''(x)(1-c_1x)(1-c_2x)=0 \quad (3.13)$$

where $f'(x)$ and $f''(x)$ respectively denotes the first and the second derivatives of $f(x)$ with respect to $x$. The solution of the differential equation (3.13) is obtained as

$$S(x) = e_1(1-c_1x)^d + e_2(1-c_2x)^d. \quad (3.14)$$

Since $S(0)=1$, we have $e_2 = 1-e_1$, which gives (3.14) as required in (3.11).The proof of the converse part is direct.

**Theorem 3.3.2**

The relationship (3.10) characterizes a mixture model with survival function

$$S(x) = p_1(1-cx)^{d_1} + (1-p_1)(1-cx)^{-d_2}.$$

For the proof, see Abraham and Nair (2001).

## 3.4 Identifiability

In the following, we show that a finite mixture of two beta densities is identifiable.

**Theorem 3.4.1**

Finite mixture of beta densities is identifiable.

**Proof:**

From Teicher (1961), it follows that a finite mixture of $k$ exponential component densities is identifiable. If $Z$ follows exponential with parameter $\lambda$ and $Y = \frac{1-e^{-z}}{c}$, then $Y$ follows beta distribution with density

$$g(y) = c\lambda(1-cy)^{\lambda-1},\ 0 < y < 1/c.$$

Since the transformation $Y = \frac{1-e^{-z}}{c}$ is one to one and onto, a finite mixture of beta with component densities $g_i(y) = c_i\lambda_i(1-c_i y)^{\lambda_i-1}$, $i = 1,2$ is identifiable. The rest of the proof is analogues to that of Al-Hussaini et.al. (2000).

### 3.5 Estimation of parameters

In this section, we discuss the estimation of parameters $c_1, c_2, d_1, d_2$ and $p_1$ of the model (3.1), using different methods under the complete as well as censored sample set up.

### 3.6 Complete sample set up

#### 3.6.1 Estimation of parameters when the observations belonging to each subpopulation are known

We consider the situation, when there are only two subpopulations with mixing proportions $p_1$ and $(1-p_1)$ and $f_1(x)$ and $f_2(x)$ are beta densities with parameters $(c_1, d_1)$ and $(c_2, d_2)$. Assuming that items that fail can be classified and can be attributed to the appropriate subpopulations so that the data would consist of the $n$ lifetimes grouped according to the subpopulations $\{(x_{11}, x_{12}, \ldots, x_{1n_1}), (x_{21}, x_{22}, \ldots, x_{2n_2})\}$, where it is assumed that $n_1$ and $n_2$ are the observed frequencies in the sample of the units belonging to respective subpopulations .

Then the likelihood of the complete sample is given by

$L(c_1, c_2, d_1, d_2, p_1 \mid \underline{x}) = .$

$$\frac{n!}{n_1!, n_2!} p_1^{n_1}(1-p_1)^{n_2} c_1^{n_1} d_1^{n_1} c_2^{n_2} d_2^{n_2} \prod_{j=1}^{n_1}(1-c_1x_{1j})^{d_1-1} \prod_{j=1}^{n_2}(1-c_2x_{2j})^{d_2-1}. \tag{3.15}$$

which can be written as

$L(c_1, c_2, d_1, d_2, p_1 \mid \underline{x})$

$$= \frac{n!}{n_1!, n_2!} p_1^{n_1}(1-p_1)^{n_2} c_1^{n_1} d_1^{n_1} c_2^{n_2} d_2^{n_2} \exp[(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})$$

$$\exp[(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j}). \tag{3.16}$$

**(a) Maximum likelihood estimation (M.L.E).**

To estimate parameters, we consider the logarithm of (3.16), which provides

$$\log L = \log C + n_1 \log p_1 + n_2 \log(1-p_1) + n_1 \log c_1 + n_1 \log d_1 + n_2 \log c_2 + n_2 \log d_2$$

$$+(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j}) + (d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j}), \tag{3.17}$$

where $C = \dfrac{n!}{n_1!n_2!}$.

Maximization of (3.17) with respect to the model parameters yields the following likelihood equations,

$$\frac{n_1}{c_1} + \frac{(d_1-1)}{c_1}\sum_{j=1}^{n_1}(1-\frac{1}{c_1x_{1j}})^{-1} = 0, \tag{3.18}$$

$$\frac{n_2}{c_2} + \frac{(d_2-1)}{c_2}\sum_{j=1}^{n_2}(1-\frac{1}{c_2x_{2j}})^{-1} = 0, \tag{3.19}$$

$$\frac{n_1}{d_1} + \sum_{j=1}^{n_1}\log(1-c_1x_{1j}) = 0, \tag{3.20}$$

$$\frac{n_2}{d_2} + \sum_{j=1}^{n_2}\log(1-c_2x_{2j}) = 0, \tag{3.21}$$

and

$$\frac{n_1}{p_1} - \frac{n_2}{(1-p_1)} = 0. \tag{3.22}$$

The equation (3.22) provides,

$$\hat{p}_1 = \frac{n_1}{n_1 + n_2}.$$

The solution of the equations (3.18), (3.19), (3.20) and (3.21) by Newton-Raphson iterative procedure yields the M.L.E.'s of $c_1, c_2, d_1$ and $d_2$.

For obtaining variance of the estimates, we need to compute the observed information matrix $I(c_1, c_2, d_1, d_2, p_1)$.

The second derivative of $\log L(c_1, c_2, d_1, d_2, p_1 \mid \underline{x})$ leads to the following

$$\frac{\partial^2 \log L}{\partial c_1^2} = -\left[\frac{n_1}{c_1^2} - \frac{(d_1-1)}{c_1}\sum_{j=1}^{n_1} -\frac{1}{x_{1j}c_1^2}(1-\frac{1}{c_1x_{1j}})^{-2} + \frac{(d_1-1)}{c_1^2}\sum_{j=1}^{n_1}(1-\frac{1}{c_1x_{1j}})^{-1}\right],$$

$$\frac{\partial^2 \log L}{\partial c_1 \partial d_1} = \frac{\partial^2 \log L}{\partial d_1 \partial c_1} = \frac{1}{c_1}\sum_{j=1}^{n_1}(1-\frac{1}{c_1x_{1j}})^{-1},$$

$$\frac{\partial^2 \log L}{\partial c_2^2} = -\left[\frac{n_2}{c_2^2} - \frac{(d_2-1)}{c_2}\sum_{j=1}^{n_2} -\frac{1}{x_{2j}c_2^2}(1-\frac{1}{c_2x_{2j}})^{-2} + \frac{(d_2-1)}{c_2^2}\sum_{j=1}^{n_2}(1-\frac{1}{c_2x_{2j}})^{-1}\right],$$

$$\frac{\partial^2 \log L}{\partial c_2 \partial d_2} = \frac{\partial^2 \log L}{\partial d_2 \partial c_2} = \frac{1}{c_2}\sum_{j=1}^{n_2}(1-\frac{1}{c_2x_{2j}})^{-1},$$

$$\frac{\partial^2 \log L}{\partial d_1^2} = -\frac{n_1}{d_1^2},$$

$$\frac{\partial^2 \log L}{\partial d_2^2} = -\frac{n_2}{d_2^2},$$

and

$$\frac{\partial^2 \log L}{\partial p_1^2} = -\frac{(n_1+n_2)}{p_1^2}.$$

and all other second derivatives are zero.

The observed information matrix is thus

$$I(c_1,c_2,d_1,d_2,p_1)=\begin{pmatrix} -\frac{\partial^2 \log L}{\partial c_1^2} & -\frac{\partial^2 \log L}{\partial c_1 \partial c_2} & -\frac{\partial^2 \log L}{\partial c_1 \partial d_1} & -\frac{\partial^2 \log L}{\partial c_1 \partial d_2} & -\frac{\partial^2 \log L}{\partial c_1 \partial p_1} \\ & & & & \\ -\frac{\partial^2 \log L}{\partial p_1 \partial c_1} & -\frac{\partial^2 \log L}{\partial p_1 \partial c_2} & -\frac{\partial^2 \log L}{\partial p_1 \partial d_1} & -\frac{\partial^2 \log L}{\partial p_1 \partial d_2} & -\frac{\partial^2 \log L}{\partial p_1^2} \end{pmatrix}$$

For large $n$, the joint distribution of the estimates $\hat{c}_1,\hat{c}_2,\hat{d}_1,\hat{d}_2$ and $\hat{p}_1$ is approximately multivariate normal with mean $(c_1,c_2,d_1,d_2,p_1)$ and covariance matrix $I^{-1}(\hat{c}_1,\hat{c}_2,\hat{d}_1,\hat{d}_2,\hat{p}_1)$, which is the inverse of $I(\hat{c}_1,\hat{c}_2,\hat{d}_1,\hat{d}_2,\hat{p}_1)$.

**(b) Bayes estimation.**

Using Jeffrey's invariant prior for $c_1,c_2,d_1$ and $d_2$ and uniform prior over $(0,1)$ for $p_1$, the joint prior distribution of $c_1,c_2,d_1,d_2$ and $p_1$ is given by

$$g(c_1,c_2,d_1,d_2,p_1) \;\alpha\; \frac{1}{c_1c_2d_1d_2}. \tag{3.23}$$

where we assume that $c_1,c_2,d_1,d_2$ and $p_1$ are independent.

From (3.16) and (3.23), the joint posterior distribution of $c_1,c_2,d_1,d_2$ and $p_1$ is given by

$$\Pi(c_1,c_2,d_1,d_2,p_1 \mid \underline{x}) \;\alpha\; p_1^{n_1}(1-p_1)^{n_2} c_1^{n_1-1} c_2^{n_2-1} d_1^{n_1-1} d_2^{n_2-1}$$

$$\exp[(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]\exp[(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]. \tag{3.24}$$

From equation (3.24), we obtain the marginal posterior distributions of $c_1,c_2,d_1,d_2$ and $p_1$ as

$$\Pi_1(c_1 \mid \underline{x}) = K_1 \frac{c_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]}{[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]^{n_1}} \int_{c_1}^{\infty} \frac{c_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]}{[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]^{n_2}} dc_2, \tag{3.25}$$

$$\Pi_2(c_2 \mid \underline{x}) = K_2 \frac{c_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]}{[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]^{n_2}} \int_0^{c_2} \frac{c_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]}{[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]^{n_1}} dc_1 , \quad (3.26)$$

$$\Pi_3(d_1 \mid \underline{x}) = K_3 d_1^{n_1-1}\left\{\int_0^\infty d_2^{n_2-1}\left[\int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j})}\left(\int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})} dc_1\right) dc_2\right] dd_2\right\},$$

(3.27)

$$\Pi_4(d_2 \mid \underline{x}) = K_4 d_2^{n_2-1}\left\{\int_0^\infty d_1^{n_1-1}\left[\int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j})}\left(\int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})} dc_1\right) dc_2\right] dd_1\right\},$$

(3.28)

and

$$\Pi_5(p_1 \mid \underline{x}) = K_5 p_1^{n_1}(1-p_1)^{n_2} , \quad (3.29)$$

where

$$K_1^{-1} = \left\{\int_0^\infty \frac{c_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]}{[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]^{n_1}}\left[\int_{c_1}^\infty \frac{c_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]}{[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]^{n_2}} dc_2\right] dc_1\right\}, \quad (3.30)$$

$$K_2^{-1} = \left\{\int_0^\infty \frac{c_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]}{[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]^{n_2}}\left[\int_0^{c_2} \frac{c_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]}{[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]^{n_1}} dc_1\right] dc_2\right\}, \quad (3.31)$$

$$K_3^{-1} = \int_0^\infty d_1^{n_1-1}\left\{\int_0^\infty d_2^{n_2-1}\left[\int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j})}\left(\int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})} dc_1\right) dc_2\right] dd_2\right\} dd_1 ,$$

(3.32)

$$K_4^{-1} = \int_0^\infty d_2^{n_2-1}\left\{\int_0^\infty d_1^{n_1-1}\left[\int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j})}\left(\int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})} dc_1\right)dc_2\right]dd_1\right\}dd_2\,, \tag{3.33}$$

and

$$K_5^{-1} = B(n_1+1, n_2+1)\,. \tag{3.34}$$

Under squared error loss function, we obtain the Bayes estimators of $c_1, c_2, d_1, d_2$ and $p_1$ as

$$c_1^* = K_1\left\{\int_0^\infty \frac{c_1^{n_1}\exp[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]}{[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]^{n_1}}\left[\int_{c_1}^\infty \frac{c_2^{n_2-1}\exp[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]}{[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]^{n_2}}dc_2\right]dc_1\right\}, \tag{3.35}$$

$$c_2^* = K_2\left\{\int_0^\infty \frac{c_2^{n_2}\exp[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]}{[-\sum_{j=1}^{n_2}\log(1-c_2x_{2j})]^{n_2}}\left[\int_0^{c_2} \frac{c_1^{n_1-1}\exp[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]}{[-\sum_{j=1}^{n_1}\log(1-c_1x_{1j})]^{n_1}}dc_1\right]dc_2\right\}, \tag{3.36}$$

$$d_1^* = K_3\int_0^\infty d_1^{n_1}\left\{\int_0^\infty d_2^{n_2-1}\left[\int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j})}\left(\int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})} dc_1\right)dc_2\right]dd_2\right\}dd_1\,, \tag{3.37}$$

$$d_2^* = K_4\int_0^\infty d_2^{n_2}\left\{\int_0^\infty d_1^{n_1-1}\left[\int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2}\log(1-c_2x_{2j})}\left(\int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1}\log(1-c_1x_{1j})} dc_1\right)dc_2\right]dd_1\right\}dd_2\,, \tag{3.38}$$

and

$$p_1^* = K_5 B(n_1+2, n_2+1)\,, \tag{3.39}$$

where $B(p,q)$ is the beta function of the first kind.

Bayes estimators of $c_1, c_2, d_1$ and $d_2$ will obtained by numerical integration procedure.

The posterior variance of $c_1$ is obtained as,

$$V(c_1 \mid \underline{x}) = E(c_1^2 \mid \underline{x}) - E(c_1 \mid \underline{x}), \tag{3.40}$$

where

$$E(c_1^2 \mid \underline{x}) = K_1 \left\{ \int_0^\infty \frac{c_1^{n_1+1} \exp[-\sum_{j=1}^{n_1} \log(1-c_1 x_{1j})]}{[-\sum_{j=1}^{n_1} \log(1-c_1 x_{1j})]^{n_1}} \left[ \int_{c_1}^\infty \frac{c_2^{n_2-1} \exp[-\sum_{j=1}^{n_2} \log(1-c_2 x_{2j})]}{[-\sum_{j=1}^{n_2} \log(1-c_2 x_{2j})]^{n_2}} dc_2 \right] dc_1 \right\}.$$

and $E(c_1 \mid \underline{x})$ is given by (3.35).

Similarly, we obtain the posterior variances of $c_2, d_1, d_2$ and $p_1$ using the following identities,

$$E(c_2^2 \mid \underline{x}) = K_2 \left\{ \int_0^\infty \frac{c_2^{n_2+1} \exp[-\sum_{j=1}^{n_2} \log(1-c_2 x_{2j})]}{[-\sum_{j=1}^{n_2} \log(1-c_2 x_{2j})]^{n_2}} \left[ \int_0^{c_2} \frac{c_1^{n_1-1} \exp[-\sum_{j=1}^{n_1} \log(1-c_1 x_{1j})]}{[-\sum_{j=1}^{n_1} \log(1-c_1 x_{1j})]^{n_1}} dc_1 \right] dc_2 \right\},$$

$$E(d_1^2 \mid \underline{x}) = K_3 \int_0^\infty d_1^{n_1+1} \left\{ \int_0^\infty d_2^{n_2-1} \left[ \int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2} \log(1-c_2 x_{2j})} \left( \int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1} \log(1-c_1 x_{1j})} dc_1 \right) dc_2 \right] dd_2 \right\} dd_1,$$

$$E(d_2^2 \mid \underline{x}) = K_4 \int_0^\infty d_2^{n_2+1} \left\{ \int_0^\infty d_1^{n_1-1} \left[ \int_0^\infty c_2^{n_2-1} e^{(d_2-1)\sum_{j=1}^{n_2} \log(1-c_2 x_{2j})} \left( \int_0^{c_2} c_1^{n_1-1} e^{(d_1-1)\sum_{j=1}^{n_1} \log(1-c_1 x_{1j})} dc_1 \right) dc_2 \right] dd_1 \right\} dd_2,$$

and

$$E(p_1^2 \mid \underline{x}) = K_5 B(n_1+3, n_2+1).$$

**Remark 3.1**

One can use various types of priors depending on the situation for the Bayesian analysis. However, when we employ conjugate priors for $c_1, c_2, d_1, d_2$ and $p_1$ the estimation procedure will become complex.

### 3.6.2 Estimation of parameters when the observations belonging to each subpopulation are unknown

We now consider the case when there are only two subpopulations, with mixing proportions $p_1$ and $(1-p_1)$ and $f_1(x)$ and $f_2(x)$ are beta densities with parameters $(c_1, d_1)$ and $(c_2, d_2)$ respectively. The likelihood based on sample $(x_1, x_2, ..., x_n)$ is given by

$$L(c_1, c_2, d_1, d_2, p_1 \mid \underline{x}) = \prod_{j=1}^{n} [p_1 c_1 d_1 (1-c_1 x_j)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_j)^{(d_2-1)}]. \qquad (3.41)$$

#### (a) Maximum likelihood estimation (M.L.E).

Maximization of log-likelihood function (3.41) with respect to the model parameters yields the following likelihood equations,

$$\sum_{j=1}^{n} \frac{p_1 d_1 (1-c_1 d_1 x_j)(1-c_1 x_j)^{(d_1-2)}}{p_1 c_1 d_1 (1-c_1 x_j)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_j)^{(d_2-1)}} = 0, \qquad (3.42)$$

$$\sum_{j=1}^{n} \frac{(1-p_1) d_2 (1-c_2 d_2 x_j)(1-c_2 x_j)^{(d_2-2)}}{p_1 c_1 d_1 (1-c_1 x_j)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_j)^{(d_2-1)}} = 0, \qquad (3.43)$$

$$\sum_{j=1}^{n} \frac{p_1 c_1 (1+d_1 \log(1-c_1 x_j)(1-c_1 x_j)^{(d_1-1)}}{p_1 c_1 d_1 (1-c_1 x_j)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_j)^{(d_2-1)}} = 0, \qquad (3.44)$$

$$\sum_{j=1}^{n} \frac{(1-p_1) c_2 (1+d_2 \log(1-c_2 x_j)(1-c_2 x_j)^{(d_2-1)}}{p_1 c_1 d_1 (1-c_1 x_j)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_j)^{(d_2-1)}} = 0, \qquad (3.45)$$

and

$$\sum_{j=1}^{n} \frac{c_1 d_1 (1-c_1 x_j)^{(d_1-1)} - c_2 d_2 (1-c_2 x_j)^{(d_2-1)}}{p_1 c_1 d_1 (1-c_1 x_j)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_j)^{(d_2-1)}} = 0. \qquad (3.46)$$

The solution of the above equations provides the M.L.E.'s of $c_1, c_2, d_1, d_2$ and $p_1$.

**(b) Bayes estimation.**

From (3.23) and (3.41), we have the joint posterior distribution of $c_1, c_2, d_1, d_2$ and $p_1$ is obtained as

$$\prod(c_1, c_2, d_1, d_2, p_1 \mid \underline{x})\ \alpha\ [\prod_{i=1}^{n}(p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]\ \frac{1}{c_1 c_2 d_1 d_2} \tag{3.47}$$

From (3.47), we obtain the marginal posterior distributions as,

$$\Pi_1(c_1 \mid \underline{x}) = M_1 \int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_1 d_2}\, dp_1 dd_2 dd_1 dc_2, \tag{3.48}$$

$$\Pi_2(c_2 \mid \underline{x}) = M_2 \int_{0}^{c_2}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_1 d_2}\, dp_1 dd_2 dd_1 dc_1, \tag{3.49}$$

$$\Pi_3(d_1 \mid \underline{x}) = M_3 \int_{0}^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_1 d_2}\, dp_1 dc_2 dc_1 dd_2, \tag{3.50}$$

$$\Pi_4(d_2 \mid \underline{x}) = M_4 \int_{0}^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_1 d_2}\, dp_1 dc_2 dc_1 dd_1\ , \tag{3.51}$$

and

$$\Pi_5(p_1 \mid \underline{x}) = M_5 \int_{0}^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}[\prod_{i=1}^{n}(p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_1 d_2}\, dd_2 dd_1 dc_2 dc_1, \tag{3.52}$$

where

$$M_1^{-1} = \int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1c_1d_1(1-c_1x_i)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1c_2d_1d_2}dp_1dd_2dd_1dc_2dc_2 . \qquad (3.53)$$

$$M_2^{-1} = \int_0^{\infty}\int_0^{c_2}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1c_1d_1(1-c_1x_i)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1c_2d_1d_2}dp_1dd_2dd_1dc_1dc_2 . \qquad (3.54)$$

$$M_3^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1c_1d_1(1-c_1x_i)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1c_2d_1d_2}dp_1dc_2dc_1dd_2dd_1 . \qquad (3.55)$$

$$M_4^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1c_1d_1(1-c_1x_i)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1c_2d_1d_2}dp_1dc_2dc_1dd_1dd_2 . \qquad (3.56)$$

$$M_5^{-1} = \int_0^{1}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}[\prod_{i=1}^{n}(p_1c_1d_1(1-c_1x_i)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1c_2d_1d_2}dd_2dd_1dc_2dc_1dp_1 . \qquad (3.57)$$

Then Bayes estimates of the parameters, under squared error loss function, are given as,

$$c_1^* = M_1\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1c_1d_1(1-c_1x_i)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)})]$$

$$\frac{1}{c_2d_1d_2}dp_1dd_2dd_1dc_2dc_2 , \qquad (3.58)$$

$$c_2^* = M_2\int_0^{\infty}\int_0^{c_2}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{n}(p_1c_1d_1(1-c_1x_i)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1d_1d_2}dp_1dd_2dd_1dc_1dc_2 , \qquad (3.59)$$

$$d_1^* = M_3 \int_0^\infty \int_0^\infty \int_0^\infty \int_{c_1}^\infty \int_0^1 [\prod_{i=1}^{n} (p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_2} dp_1 dc_2 dc_1 dd_2 dd_1 , \qquad (3.60)$$

$$d_2^* = M_4 \int_0^\infty \int_0^\infty \int_0^\infty \int_{c_1}^\infty \int_0^1 [\prod_{i=1}^{n} (p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_1} dp_1 dc_2 dc_1 dd_1 dd_2 , \qquad (3.61)$$

and

$$p_1^* = M_5 \int_0^1 \int_0^\infty \int_{c_1}^\infty \int_0^\infty \int_0^\infty [\prod_{i=1}^{n} (p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)})]$$

$$\frac{1}{c_1 c_2 d_1 d_2} dd_2 dd_1 dc_2 dc_1 dp_1 . \qquad (3.62)$$

Bayes estimates of $c_1, c_2, d_1, d_2$ and $p_1$ will obtain by numerical integration procedure.

**(c) Method of moments.**

As discussed earlier in Chapter 2, let $m_1^{'}, m_2^{'}$, $m_3^{'}$ $m_4^{'}$ and $m_5^{'}$ denotes the sample raw moments. Now we equate the sample moments to the population moments, which provide the following equations,

$$\frac{p_1}{c_1(d_1+1)} + \frac{(1-p_1)}{c_2(d_2+1)} = m_1^{'}, \qquad (3.63)$$

$$\frac{2p_1}{c_1^2(d_1+1)(d_1+2)} + \frac{2(1-p_1)}{c_2^2(d_2+1)(d_2+2)} = m_2^{'}, \qquad (3.64)$$

$$\frac{6p_1}{c_1^3(d_1+1)(d_1+2)(d_1+3)} + \frac{6(1-p_1)}{c_2^3(d_2+1)(d_2+2)(d_2+3)} = m_3^{'}, \qquad (3.65)$$

$$\frac{24p_1}{c_1^4(d_1+1)(d_1+2)(d_1+3)(d_1+4)} + \frac{24(1-p_1)}{c_2^4(d_2+1)(d_2+2)(d_2+3)(d_2+4)} = m_4^{'}, \qquad (3.66)$$

$$\frac{120p_1}{c_1^5(d_1+1)(d_1+2)(d_1+3)(d_1+4)(d_1+5)}+\frac{120(1-p_1)}{c_2^5(d_2+1)(d_2+2)(d_2+3)(d_2+4)(d_2+5)}=m_5'.$$

(3.67)

The moment estimates of $c_1, c_2, d_1, d_2$ and $p_1$ can be obtained by solving the equations (3.63), (3.64), (3.65), (3.66) and (3.67).

**(d) Maximum product of spacing.**

In this section, we obtain the estimate of parameters $c_1, c_2, d_1, d_2$ and $p_1$ using the maximum product of spacing method introduced by Cheng and Amin (1983). The estimates by this method are obtained by maximizing the geometric mean of the spacing based on a sample $(x_1, x_2, ..., x_n)$.

$$G=\frac{1}{n}\sum_{i=1}^{n}\log\{p_1[(1-c_1x_{i-1})^{d_1}-(1-c_1x_i)^{d_1}]+(1-p_1)[(1-c_2x_{i-1})^{d_2}-(1-c_2x_i)^{d_2}]\}. \quad (3.68)$$

Differentiating $G$ with respect to $c_1, c_2, d_1, d_2$ and $p_1$ and equating to zero we have,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p_1d_1[(1-c_1x_i)^{(d_1-1)}x_i-(1-c_1x_{i-1})^{(d_1-1)}x_{i-1}]}{p_1[(1-c_1x_{i-1})^{d_1}-(1-c_1x_i)^{d_1}]+(1-p_1)[(1-c_2x_{i-1})^{d_2}-(1-c_2x_i)^{d_2}]}=0, \quad (3.69)$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(1-p_1)c_2[(1-c_2x_i)^{(d_2-1)}x_i-(1-c_2x_{i-1})^{(d_2-1)}x_{i-1}]}{p_1[(1-c_1x_{i-1})^{d_1}-(1-c_1x_i)^{d_1}]+(1-p_1)[(1-c_2x_{i-1})^{d_2}-(1-c_2x_i)^{d_2}]}=0, \quad (3.70)$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p_1[(1-c_1x_{i-1})^{d_1}\log(1-c_1x_{i-1})-(1-c_1x_i)^{d_1}\log(1-c_1x_i)]}{p_1[(1-c_1x_{i-1})^{d_1}-(1-c_1x_i)^{d_1}]+(1-p_1)[(1-c_2x_{i-1})^{d_2}-(1-c_2x_i)^{d_2}]}=0, \quad (3.71)$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(1-p_1)[(1-c_2x_{i-1})^{d_2}\log(1-c_2x_{i-1})-(1-c_2x_i)^{d_2}\log(1-c_2x_i)]}{p_1[(1-c_1x_{i-1})^{d_1}-(1-c_1x_i)^{d_1}]+(1-p_1)[(1-c_2x_{i-1})^{d_2}-(1-c_2x_i)^{d_2}]}=0, \quad (3.72)$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\frac{[(1-c_1x_{i-1})^{d_1}-(1-c_1x_i)^{d_1}+(1-c_2x_i)^{d_2}-(1-c_2x_{i-1})^{d_2}]}{p_1[(1-c_1x_{i-1})^{d_1}-(1-c_1x_i)^{d_1}]+(1-p_1)[(1-c_2x_{i-1})^{d_2}-(1-c_2x_i)^{d_2}]}=0. \quad (3.73)$$

Solving the above equations by Newton-Raphson iterative procedure, we obtain the estimates of $c_1, c_2, d_1, d_2$ and $p_1$

## 3.7 Censored set up

### 3.7.1 Estimation based on type I censored samples when the observations belonging to each subpopulation are known

Consider the situation, when there are only two subpopulations with mixing proportions $p_1$ and $(1-p_1)$ and $f_1(x)$ and $f_2(x)$ are respective beta densities with parameters $(c_1, d_1)$ and $(c_2, d_2)$. Let $t_{ij}$ denote the failure time of the $j^{th}$ unit belonging to the $i^{th}$ subpopulation, $t_{ij} \leq T$, $j = 1, 2, ..., r_i$; $i = 1, 2$ and let $\gamma_i = c_i T$, $i = 1, 2$ and $x_{ij} = \frac{t_{ij}}{T}$ and observed $\underline{x} = \{x_{11}, x_{12}, ..., x_{1r_1}; x_{21}, x_{22}, ..., x_{2r_2}\}$. Now we discuss the estimation of parameters.

#### (a) Maximum likelihood Estimation (M.L.E).

The likelihood function based on a type I censored sample is given by (see Mendenhall and Hader (1958) and Sinha (1986))

$$L(\gamma_1, \gamma_2, d_1, d_2, p_1 \mid \underline{x}) = \frac{n!}{r_1! r_2! (n-r)!} \frac{p_1^{r_1}(1-p_1)^{r_2}}{T^r} \prod_{j=1}^{r_1} \{d_1 \gamma_1 (1-\gamma_1 x_{1j})^{(d_1-1)}\}$$

$$\prod_{j=1}^{r_2} \{d_2 \gamma_2 (1-\gamma_2 x_{2j})^{(d_2-1)}\} \{p_1(1-\gamma_1)^{d_1} + (1-p_1)(1-\gamma_2)^{d_2}\}^{(n-r)}. \tag{3.74}$$

Maximization of log-likelihood function with respect to the model parameters yields the following likelihood equations,

$$\frac{r_1}{\gamma_1} + \frac{(d_1-1)}{\gamma_1} \sum_{j=1}^{r_1} (1 - \frac{1}{\gamma_1 x_{1j}})^{-1} - \frac{(n-r) p_1 d_1 (1-\gamma_1)^{(d_1-1)}}{[p_1(1-\gamma_1)^{d_1} + (1-p_1)(1-\gamma_2)^{d_2}]} = 0, \tag{3.75}$$

$$\frac{r_2}{\gamma_2} + \frac{(d_2-1)}{\gamma_2} \sum_{j=1}^{r_2} (1 - \frac{1}{\gamma_2 x_{2j}})^{-1} - \frac{(n-r)(1-p_1) d_2 (1-\gamma_2)^{(d_2-1)}}{[p_1(1-\gamma_1)^{d_1} + (1-p_1)(1-\gamma_2)^{d_2}]} = 0, \tag{3.76}$$

$$\frac{r_1}{d_1} + \sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j}) + \frac{(n-r) p_1 (1-\gamma_1)^{d_1} \log(1-\gamma_1)}{[p_1(1-\gamma_1)^{d_1} + (1-p_1)(1-\gamma_2)^{d_2}]} = 0, \tag{3.77}$$

$$\frac{r_2}{d_2} + \sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j}) + \frac{(n-r)(1-p_1)(1-\gamma_2)^{d_2} \log(1-\gamma_2)}{[p_1(1-\gamma_1)^{d_1} + (1-p_1)(1-\gamma_2)^{d_2}]} = 0, \tag{3.78}$$

and

$$\frac{r_1}{p_1}-\frac{r_2}{(1-p_1)}+\frac{(n-r)[(1-\gamma_1)^{d_1}-(1-\gamma_2)^{d_2}]}{[p_1(1-\lambda_1)^{d_1}+(1-p_1)(1-\gamma_2)^{d_2}]}=0. \qquad (3.79)$$

The solution of the above five equations, using numerical iteration method, yields the maximum likelihood estimate (M.L.E) of $\gamma_1, \gamma_2, d_1, d_2$ and $p_1$ and thus we obtain the maximum likelihood estimate of $c_1, c_2, d_1, d_2$ and $p_1$.

**(b) Bayes estimation.**

We assume that $\gamma_1, \gamma_2, d_1, d_2$ and $p_1$ are independent. Using Jeffrey's invariant prior for $\gamma_1, \gamma_2, d_1$ and $d_2$ and uniform prior over $(0,1)$ for $p_1$, then joint prior distribution of $\gamma_1, \gamma_2, d_1, d_2$ and $p_1$ is given by

$$g(\gamma_1,\gamma_2,d_1,d_2,p_1) \;\alpha\; \frac{1}{\gamma_1\gamma_2 d_1 d_2} \qquad (3.80)$$

From (3.74) and (3.80), the joint posterior distribution of $\gamma_1, \gamma_2, d_1, d_2$ and $p_1$ is given by

$$\Pi(\gamma_1,\gamma_2,d_1,d_2,p_1 \mid \underline{x}) \;\alpha\; \sum_{k=0}^{n-r}\binom{n-r}{k} p_1^{\,n-r_2-k}(1-p_1)^{r_2+k}\gamma_1^{r_1-1}\gamma_2^{r_2-1}d_1^{r_1-1}d_2^{r_2-1}$$

$$\exp[(d_1-1)\sum_{j=1}^{r_1}\log(1-\gamma_1 x_{1j})]\exp[(d_2-1)\sum_{j=1}^{r_2}\log(1-\gamma_2 x_{2j})](1-\gamma_1)^{d_1(n-r-k)}(1-\gamma_2)^{d_2 k}. \qquad (3.81)$$

From equation (3.81), we obtain the marginal posterior distributions of $\gamma_1, \gamma_2, d_1, d_2$ and $p_1$ as,

$$\Pi_1(\gamma_1 \mid \underline{x}) = A_1\sum_{k=0}^{n-r}\binom{n-r}{k}B(n-r_2-k+1, r_2+k+1)\frac{\gamma_1^{r_1-1}\exp[-\sum_{j=1}^{r_1}\log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1}\log(1-\gamma_1 x_{1j})-(n-r-k)\log(1-\gamma_1)]^{r_1}}$$

$$\int_{\gamma_1}^{\infty}\frac{\gamma_2^{r_2-1}\exp[-\sum_{j=1}^{r_2}\log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2}\log(1-\gamma_2 x_{2j})-k\log(1-\gamma_2)]^{r_2}}\,d\gamma_2, \qquad (3.82)$$

$$\prod_2(\gamma_2 \mid \underline{x}) = A_2 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1) \frac{\gamma_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}}$$

$$\int_0^{\gamma_2} \frac{\gamma_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} d\gamma_1 , \tag{3.83}$$

$$\prod_3(d_1 \mid \underline{x}) = A_3 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1) d_1^{r_1-1}$$

$$\left\{ \int_0^{\infty} \gamma_1^{r_1-1} e^{-d_1(-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j}) - (n-r-k)\log(1-\gamma_1))} e^{-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})} \left[ \int_{\gamma_1}^{\infty} \frac{\gamma_2^{r_2-1} e^{-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})}}{\left[ (-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j}) - k\log(1-\gamma_2) \right]^{r_2}} d\gamma_2 \right] d\gamma_1 \right\}, \tag{3.84}$$

$$\prod_4(d_2 \mid \underline{x}) = A_4 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1) d_2^{r_2-1}$$

$$\left\{ \int_0^{\infty} \gamma_2^{r_2-1} e^{-d_2(-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j}) - k\log(1-\gamma_2))} e^{-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})} \left[ \int_0^{\gamma_2} \frac{\gamma_1^{r_1-1} e^{-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})}}{\left[ (-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j}) - (n-r-k)\log(1-\gamma_1) \right]^{r_1}} d\gamma_2 \right] d\gamma_1 \right\}, \tag{3.85}$$

and

$$\prod_5(p_1 \mid \underline{x}) = A_5 \sum_{k=0}^{n-r} \binom{n-r}{k} p_1^{n-r_2-k} (1-p_1)^{r_2+k}$$

$$\left\{ \int_0^{\infty} \frac{\gamma_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} \left[ \int_{\gamma_1}^{\infty} \frac{\gamma_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}} d\gamma_2 \right] d\gamma_1 \right\}. \tag{3.86}$$

where

$$A_1^{-1} = \sum_{k=0}^{n-r} \binom{n-r}{k} B(n - r_2 - k + 1, r_2 + k + 1)$$

$$\left\{ \int_0^\infty \frac{\gamma_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} \left[ \int_{\gamma_1}^\infty \frac{\gamma_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}} d\gamma_2 \right] d\gamma_1 \right\}, \qquad (3.87)$$

$$A_2^{-1} = \sum_{k=0}^{n-r} \binom{n-r}{k} B(n - r_2 - k + 1, r_2 + k + 1)$$

$$\left\{ \int_0^\infty \frac{\gamma_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}} \left[ \int_0^{\gamma_2} \frac{\gamma_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} d\gamma_1 \right] d\gamma_2 \right\},$$

(3.88)

$$A_3^{-1} = \sum_{k=0}^{n-r} \binom{n-r}{k} B(n - r_2 - k + 1, r_2 + k + 1)$$

$$\left\{ \int_0^\infty d_1^{r_1-1} \left[ \int_0^\infty \gamma_1^{r_1-1} e^{-d_1(-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j}) - (n-r-k)\log(1-\gamma_1))} e^{-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})} \left( \int_{\gamma_1}^\infty \frac{\gamma_2^{r_2-1} e^{-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})}}{(-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j}) - k\log(1-\gamma_2))} d\gamma_2 \right) d\gamma_1 \right] dd_1 \right\},$$

(3.89)

$$A_4^{-1} = \sum_{k=0}^{n-r} \binom{n-r}{k} B(n - r_2 - k + 1, r_2 + k + 1)$$

$$\left\{ \int_0^\infty d_2^{r_2-1} \left[ \int_0^\infty \gamma_2^{r_2-1} e^{-d_2(-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j}) - k\log(1-\gamma_2))} e^{-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})} \left( \int_0^{\gamma_2} \frac{\gamma_1^{r_1-1} e^{-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})}}{\left[(-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j}) - (n-r-k)\log(1-\gamma_1)\right]^{r_1}} d\gamma_2 \right) d\gamma_1 \right] dd_2 \right\},$$

(3.90)

and

$$A_5^{-1} = \sum_{k=0}^{n-r} \binom{n-r}{k} B(n - r_2 - k + 1, r_2 + k + 1)$$

$$\left\{ \int_0^\infty \frac{\gamma_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} \left[ \int_{\gamma_1}^\infty \frac{\gamma_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}} d\gamma_2 \right] d\gamma_1 \right\}.$$

(3.91)

with

$B(p,q)$ is the beta function of the first kind.

Under squared error loss function, we obtain the Bayes estimators of $\gamma_1, \gamma_2, d_1, d_2$ and $p_1$ as,

$$\gamma_1^* = A_1 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1)$$

$$\left\{ \int_0^\infty \frac{\gamma_1^{r_1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} \left[ \int_{\gamma_1}^\infty \frac{\gamma_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}} d\gamma_2 \right] d\gamma_1 \right\}, \tag{3.92}$$

$$\gamma_2^* = A_2 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1)$$

$$\left\{ \int_0^\infty \frac{\gamma_2^{r_2} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}} \left[ \int_0^{\gamma_2} \frac{\gamma_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} d\gamma_1 \right] d\gamma_2 \right\}, \tag{3.93}$$

$$d_1^* = A_3 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1)$$

$$\left\{ \int_0^\infty d_1^{r_1} \left[ \int_0^\infty \gamma_1^{r_1-1} e^{-d_1(-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j}) - (n-r-k)\log(1-\gamma_1))} e^{-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})} \left( \int_{\gamma_1}^\infty \frac{\gamma_2^{r_2-1} e^{-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})}}{(-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j}) - k\log(1-\gamma_2))} d\gamma_2 \right) d\gamma_1 \right] dd_1 \right\}, \tag{3.94}$$

$$d_2^* = A_4 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+1, r_2+k+1)$$

$$\left\{ \int_0^\infty d_2^{r_2} \left[ \int_0^\infty \gamma_2^{r_2-1} e^{-d_2(-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j}) - k\log(1-\gamma_2))} e^{-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})} \left( \int_0^{\gamma_2} \frac{\gamma_1^{r_1-1} e^{-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})}}{\left[(-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j}) - (n-r-k)\log(1-\gamma_1)\right]^{r_1}} d\gamma_2 \right) d\gamma_1 \right] dd_2 \right\}, \tag{3.95}$$

and

$$p_1^* = A_5 \sum_{k=0}^{n-r} \binom{n-r}{k} B(n-r_2-k+2, r_2+k+1)$$

$$\left\{ \int_0^\infty \frac{\gamma_1^{r_1-1} \exp[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})]}{[-\sum_{j=1}^{r_1} \log(1-\gamma_1 x_{1j})] - (n-r-k)\log(1-\gamma_1)]^{r_1}} \left[ \int_{\gamma_1}^\infty \frac{\gamma_2^{r_2-1} \exp[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})]}{[-\sum_{j=1}^{r_2} \log(1-\gamma_2 x_{2j})] - k\log(1-\gamma_2)]^{r_2}} d\gamma_2 \right] d\gamma_1 \right\}. \quad (3.96)$$

Bayes estimators of $\gamma_1, \gamma_2, d_1, d_2$ and $p_1$ and hence that of $c_1, c_2, d_1, d_2$ and $p_1$ will obtained by numerical integration procedure.

**Remark 3.2**

One can use different types of priors for the analysis. However, when we use conjugate priors for $c_1, c_2, d_1, d_2$ and $p_1$, the estimation procedure will become complex.

### 3.7.2 Estimation based on type I censored samples when the observations belonging to each subpopulation are unknown

#### (a) Maximum likelihood Estimation (M.L.E)

In this Section, we develop maximum likelihood estimators of parameters under type I censoring. The likelihood function based on a sample $(x_1, x_2, ..., x_n)$ from the model (3.1) is given by

$$L(c_1, c_2, d_1, d_2, p_1 \mid \underline{x}) = \prod_{i=1}^{n} \{ p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)} \}^{\delta_i}$$

$$\{ p_1 (1-c_1 T)^{d_1} + (1-p_1)(1-c_2 T)^{d_2} \}^{1-\delta_i}. \quad (3.97)$$

where $\delta_i = 1$ if $x_i \le T$ and $\delta_i = 0$ if $x_i > T$.

Maximization of log-likelihood function with respect to the model parameters yields the following likelihood equations,

$$\sum_{i=1}^{n}\Big[\frac{\delta_i[p_1d_1(1-c_1x_i)^{(d_1-1)} - p_1d_1c_1(d_1-1)(1-c_1x_i)^{(d_1-2)}x_i]}{[p_1c_1d_1(1-c_1x_i)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)}]}$$

$$-\frac{(1-\delta_i)p_1d_1(1-c_1T)^{(d_1-1)}T}{[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]}\Big] = 0, \qquad (3.98)$$

$$\sum_{i=1}^{n}\Big[\frac{\delta_i[(1-p_1)d_2(1-c_2x_i)^{(d_2-1)} - (1-p_1)d_2c_2(d_2-1)(1-c_2x_i)^{(d_2-2)}x_i]}{[p_1c_1d_1(1-c_1x_i)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)}]}$$

$$-\frac{(1-\delta_i)(1-p_1)d_2(1-c_2T)^{(d_2-1)}T}{[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]}\Big] = 0\ , \qquad (3.99)$$

$$\sum_{i=1}^{n}\Big[\Big\{\frac{\delta_i\{p_1c_1(1-c_1x_i)^{-1}[(1-c_1x_i)^{d_1} + d_1(1-c_1x_i)^{d_1}\log(1-c_1x_i)]}{[p_1c_1d_1(1-c_1x_i)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)}]}\Big\}$$

$$+\frac{(1-\delta_i)[p_1(1-c_1T)^{d_1}\log(1-c_1T)}{[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]}\Big] = 0, \qquad (3.100)$$

$$\sum_{i=1}^{n}\Big[\Big\{\frac{\delta_i\{(1-p_1)c_2(1-c_2x_i)^{-1}[(1-c_2x_i)^{d_2} + d_2(1-c_2x_i)^{d_2}\log(1-c_2x_i)]}{[p_1c_1d_1(1-c_1x_i)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)}]}\Big\}$$

$$+\frac{(1-\delta_i)[(1-p_1)(1-c_2T)^{d_2}\log(1-c_2T)}{[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]}\Big] = 0, \qquad (3.101)$$

and

$$\sum_{i=1}^{n}\Big[\frac{\delta_i[c_1d_1(1-c_1x_i)^{(d_1-1)} - c_2d_2(1-c_2x_i)^{(d_2-1)}]}{[p_1c_1d_1(1-c_1x_i)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x_i)^{(d_2-1)}]}$$

$$+\frac{(1-\delta_i)[(1-c_1T)^{d_1} - (1-c_2T)^{d_2}]}{[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]}\Big] = 0. \qquad (3.102)$$

The solution of the above equations using numerical iteration scheme, yields the maximum likelihood estimate (M.L.E) of $c_1, c_2, d_1, d_2$ and $p_1$ .

**(b) Bayes estimation.**

From (3.23) and (3.97), we obtain the joint posterior distribution of $c_1, c_2, d_1, d_2$ and $p_1$ as

$$\prod(c_1, c_2, d_1, d_2, p_1 \mid \underline{x}) \;\alpha\; [\prod_{i=1}^{n}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1 d_2}. \qquad (3.103)$$

Then, we obtain the marginal posterior distributions as,

$$\Pi_1(c_1 \mid \underline{x}) = B_1 \int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\prod_{i=1}^{n}[p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_2, \qquad (3.104)$$

$$\Pi_2(c_2 \mid \underline{x}) = B_2 \int_{0}^{c_2}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\prod_{i=1}^{n}[p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_1, \qquad (3.105)$$

$$\Pi_3(d_1 \mid \underline{x}) = B_3 \int_{0}^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1}\prod_{i=1}^{n}[p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dc_2 dc_1 dd_2, \qquad (3.106)$$

$$\Pi_4(d_2 \mid \underline{x}) = B_4 \int_{0}^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1}\prod_{i=1}^{n}[p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dc_2 dc_1 dd_1, \qquad (3.107)$$

and

$$\Pi_5(p_1 \mid \underline{x}) = B_5 \int_{0}^{\infty}\int_{c_2}^{\infty}\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{n}[p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1 d_2} dd_2 dd_1 dc_2 dc_1. \qquad (3.108)$$

where

$$B_1^{-1} = \int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1} \prod_{i=1}^{n} [p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_2 dc_1 , \qquad (3.109)$$

$$B_2^{-1} = \int_0^{\infty}\int_0^{c_2}\int_0^{\infty}\int_0^{\infty}\int_0^{1} \prod_{i=1}^{n} [p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 d_1 d_2} dp_1 dd_1 dd_2 dc_1 dc_2 , \qquad (3.110)$$

$$B_3^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1} \prod_{i=1}^{n} [p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_2} dp_1 dc_2 dc_1 dd_2 dd_1 , \qquad (3.111)$$

$$B_4^{-1} = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{1} \prod_{i=1}^{n} [p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1} dp_1 dc_2 dc_1 dd_1 dd_2 , \qquad (3.112)$$

and

$$B_5^{-1} = \int_0^{1}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty} p_1 \prod_{i=1}^{n} [p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_1 c_2 d_1 d_2} dd_2 dd_1 dc_2 dc_1 dp_1 . \qquad (3.113)$$

Then Bayes estimates of the parameters, under squared error loss function, are given as,

$$c_1^* = B_1 \int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1} \prod_{i=1}^{n} [p_1 c_1 d_1 (1-c_1 x_i)^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_i)^{(d_2-1)}]^{\delta_i}$$

$$[p_1(1-c_1T)^{d_1} + (1-p_1)(1-c_2T)^{d_2}]^{(1-\delta_i)} \frac{1}{c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_2 dc_1 . \qquad (3.114)$$

$$\sum_{i=1}^{r}\frac{p_1d_1(1-c_1d_1x_{(i)})(1-c_1x_{(i)})^{(d_1-2)}}{p_1c_1d_1(1-c_1x_{(i)})^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_{(i)})^{(d_2-1)}}-\frac{p_1d_1(n-r)x_{(r)}(1-c_1x_{(r)})^{(d_1-1)}}{p_1(1-c_1x_{(r)})^{d_1}+(1-p_1)(1-c_2x_{(r)})^{d_2}}=0,$$

(3.120)

$$\sum_{i=1}^{r}\frac{(1-p_1)d_2(1-c_2d_2x_{(i)})(1-c_2x_{(i)})^{(d_2-2)}}{p_1c_1d_1(1-c_1x_{(i)})^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_{(i)})^{(d_2 1)}}-\frac{(1-p_1)d_2(n-r)x_{(r)}(1-c_2x_{(r)})^{(d_2-1)}}{p_1(1-c_1x_{(r)})^{d_1}+(1-p_1)(1-c_2x_{(r)})^{d_2}}=0,$$

(3.121)

$$\sum_{i=1}^{r}\frac{p_1c_1(1+d_1\log(1-c_1x_{(i)})(1-c_1x_{(i)})^{(d_1-1)}}{p_1c_1d_1(1-c_1x_{(i)})^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_{(i)})^{(d_2-1)}}+\frac{(n-r)p_1(1-c_1x_{(r)})^{d_1}\log(1-c_1x_{(r)})}{p_1(1-c_1x_{(r)})^{d_1}+(1-p_1)(1-c_2x_{(r)})^{d_2}}=0,$$

(3.122)

$$\sum_{i=1}^{r}\frac{(1-p_1)c_2(1+d_2\log(1-c_2x_{(i)})(1-c_2x_{(i)})^{(d_2-1)}}{p_1c_1d_1(1-c_1x_{(i)})^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_{(i)})^{(d_2-1)}}+\frac{(n-r)(1-p_1)(1-c_2x_{(r)})^{d_2}\log(1-c_2x_{(r)})}{p_1(1-c_1x_{(r)})^{d_1}+(1-p_1)(1-c_2x_{(r)})^{d_2}}=0,$$

(3.123)

and

$$\sum_{i=1}^{r}\frac{c_1d_1(1-c_1x_{(i)})^{(d_1-1)}-c_2d_2(1-c_2x_{(i)})^{(d_2-1)}}{p_1c_1d_1(1-c_1x_{(i)})^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x_{(i)})^{(d_2-1)}}+\frac{(n-r)[(1-c_1x_{(r)})^{d_1}-(1-c_2x_{(r)})^{d_2}]}{p_1(1-c_1x_{(r)})^{d_1}+(1-p_1)(1-c_2x_{(r)})^{d_2}}=0.$$

(3.124)

Solving the above equations we obtain the M.L.E s of $c_1, c_2, d_1, d_2$ and $p_1$.

**(b) Bayes estimation.**

From (3.23) and (3.119), we obtain the joint posterior distribution of $c_1, c_2, d_1, d_2$ and $p_1$ as,

$$\prod(c_1, c_2, d_1, d_2, p_1 \mid \underline{x}) \;\alpha\; [\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} \;. \tag{3.125}$$

From (3.125), we obtain the marginal posterior distributions,

$$\Pi_1(c_1 \mid \underline{x}) = H_1 \int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_2 \,, \tag{3.126}$$

$$\Pi_2(c_2 \mid \underline{x}) = H_2 \int_{0}^{c_2}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_1 \,, \tag{3.127}$$

$$\Pi_3(d_1 \mid \underline{x}) = H_3 \int_{0}^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dd_2 dc_2 dc_1 \,, \tag{3.128}$$

$$\Pi_4(d_2 \mid \underline{x}) = H_4 \int_{0}^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} dp_1 dd_1 dc_2 dc_1 \,, \tag{3.129}$$

and

$$\Pi_5(p_1 \mid \underline{x}) = H_5 \int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1)c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} dd_1 dd_2 dc_2 dc_1 . \qquad (3.130)$$

where

$$H_1^{-1} = \int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1)c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_2 dc_1 , \qquad (3.131)$$

$$H_2^{-1} = \int_0^{\infty}\int_0^{c_2}\int_0^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1)c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 d_1 d_2} dp_1 dd_1 dd_2 dc_1 dc_2 , \qquad (3.132)$$

$$H_3^{-1} = \int_0^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1)c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_2} dp_1 dd_2 dc_2 dc_1 dd_1 , \qquad (3.133)$$

$$H_4^{-1} = \int_0^{\infty}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{1}[\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1)c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1} dp_1 dd_1 dc_2 dc_1 dd_2 , \qquad (3.134)$$

and

$$H_5^{-1} = \int_0^{1}\int_0^{\infty}\int_{c_1}^{\infty}\int_0^{\infty}\int_0^{\infty} p_1 [\prod_{i=1}^{r}(p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1)c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1(1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} dd_1 dd_2 dc_2 dc_1 dp_1 . \qquad (3.135)$$

Then Bayes estimates of the parameters, under squared error loss function, are given as,

$$c_1^* = H_1 \int_0^\infty \int_{c_1}^\infty \int_0^\infty \int_0^\infty \int_0^1 [\prod_{i=1}^{r} (p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1 (1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_2 d_1 d_2} dp_1 dd_1 dd_2 dc_2 dc_1 , \qquad (3.136)$$

$$c_2^* = H_2 \int_0^\infty \int_0^{c_2} \int_0^\infty \int_0^\infty \int_0^1 [\prod_{i=1}^{r} (p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1 (1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 d_1 d_2} dp_1 dd_1 dd_2 dc_1 dc_2 , \qquad (3.137)$$

$$d_1^* = H_3 \int_0^\infty \int_0^\infty \int_{c_1}^\infty \int_0^\infty \int_0^1 [\prod_{i=1}^{r} (p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1 (1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_2} dp_1 dd_2 dc_2 dc_1 dd_1 , \qquad (3.138)$$

$$d_2^* = H_4 \int_0^\infty \int_0^\infty \int_{c_1}^\infty \int_0^\infty \int_0^1 [\prod_{i=1}^{r} (p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1 (1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1} dp_1 dd_1 dc_2 dc_1 dd_2 , \qquad (3.139)$$

and

$$p_1^* = H_5 \int_0^1 \int_0^\infty \int_{c_1}^\infty \int_0^\infty \int_0^\infty p_1 [\prod_{i=1}^{r} (p_1 c_1 d_1 (1-c_1 x_{(i)})^{(d_1-1)} + (1-p_1) c_2 d_2 (1-c_2 x_{(i)})^{(d_2-1)}]$$

$$[p_1 (1-c_1 x_{(r)})^{d_1} + (1-p_1)(1-c_2 x_{(r)})^{d_2}]^{(n-r)} \frac{1}{c_1 c_2 d_1 d_2} dd_1 dd_2 dc_2 dc_1 dp_1 . \qquad (3.140)$$

**Remark 3.3**

When $r$=$n$, the results obtained under type II censoring becomes complete sample case.

## 3.8 Simulation study.

We have described four methods for estimation of parameters of the model (3.1). Now, we illustrate these procedures using simulated data. We generate observations from two beta densities with various combinations of parameters $c_1, c_2, d_1, d_2$ and $p_1$. The samples from the finite mixture of beta distributions are generated using Bernoulli probability $p_1 (0 < p_1 < 1)$. Then, the estimates of parameters for complete as well as censored cases are computed. The estimates for one set of parameters are given in

Tables 3.1 to 3.4. The variances of the estimates are presented in brackets. For Bayes estimates, the posterior risks are given in brackets. We compared the estimates using risk improvement factor as mentioned in Chapter 2. We use the software 'Mathematica' for numerical calculations. The bias and variances (posterior risks) of the estimates decreases as sample size increases. Table 3.5 gives the risk improvement of the estimator by Bayesian over the different estimators in the complete sample case.

**Table 3.1 Estimates of parameters under complete sample for** $c_1 = .1, c_2 = .6, d_1 = 2, d_2 = 3$ **and** $p_1 = 0.5$

| Estimate | M.L.E. | Bayes | Method of moments | Maximum product of spacing |
|---|---|---|---|---|
| n=30 | $\hat{c}_1$=.098544<br>(.0079187) | $c_1^*$=.102<br>(.000214) | $\tilde{c}_1$=.150339<br>(.00017) | $\breve{c}_1$=.1012<br>(2.34E-05) |
| | $\hat{c}_2$=.606886<br>(.0012954) | $c_2^*$=.600364<br>(.00214) | $\tilde{c}_2$=.6190161<br>(.465196) | $\breve{c}_2$=.602212<br>(1.7E-05) |
| | $\hat{d}_1$=1.93653<br>(.0053) | $d_1^*$=1.96168<br>(.005195) | $\tilde{d}_1$=2.34851<br>(.075758) | $\breve{d}_1$=1.829962<br>(.008508) |
| | $\hat{d}_2$=2.987<br>(.00059) | $d_2^*$=3.01<br>(.00054) | $\tilde{d}_2$=3.0911<br>(.03348) | $\breve{d}_2$=3.1201<br>(4.98E-05) |
| | $\hat{p}_1$=.500693<br>(.000409) | $p_1^*$=.500816<br>(.00022) | $\tilde{p}_1$=.5071<br>(.005623) | $\breve{p}_1$=.5028<br>(.000762) |
| n=50 | $\hat{c}_1$=.089<br>(.000153) | $c_1^*$=.09949<br>(.000237) | $\tilde{c}_1$=.1764<br>(.0005368) | $\breve{c}_1$=.091<br>(1.887E-05) |
| | $\hat{c}_2$=.600474<br>(8.05E-05) | $c_2^*$=.6033154<br>(.00125) | $\tilde{c}_2$=.554747<br>(.04244) | $\breve{c}_2$=.60214<br>(3.8E-06) |
| | $\hat{d}_1$=2.01248<br>(.000237) | $d_1^*$=2.039314<br>(.00508) | $\tilde{d}_1$=1.83284<br>(.04841) | $\breve{d}_1$=1.918252<br>(.00252) |
| | $\hat{d}_2$=3.0042<br>(.0013) | $d_2^*$=3.001<br>(.0023) | $\tilde{d}_2$=2.60939<br>(.00184) | $\breve{d}_2$=3.020872<br>(5.72E-06) |
| | $\hat{p}_1$=.49894<br>(5.17E-06) | $p_1^*$=.499592<br>(8.65E-05) | $\tilde{p}_1$=.50089<br>(.000171) | $\breve{p}_1$=.50181<br>(.000518) |
| n=100 | $\hat{c}_1$=.1034<br>(.0009316) | $c_1^*$=.102<br>(.0002136) | $\tilde{c}_1$=.1142<br>(.00027) | $\breve{c}_1$=.0915<br>(1.38E-05) |
| | $\hat{c}_2$=.600758<br>(.000197) | $c_2^*$=.601<br>(.00002) | $\tilde{c}_2$=.5807978<br>(.06039) | $\breve{c}_2$=.60022<br>(1.4E-06) |
| | $\hat{d}_1$=1.989<br>(.0023) | $d_1^*$=2.0009<br>(.00041) | $\tilde{d}_1$=1.9011<br>(.03013) | $\breve{d}_1$=1.967<br>(.001636) |
| | $\hat{d}_2$=3.0032<br>(.000098) | $d_2^*$=3.00362<br>(2.7E-05) | $\tilde{d}_2$=2.89221<br>(.03396) | $\breve{d}_2$=3.022694<br>(2.6E-06) |
| | $\hat{p}_1$=.49961<br>(.00011) | $p_1^*$=.50224<br>(.00016) | $\tilde{p}_1$=.49979<br>(.000231) | $\breve{p}_1$=.50032<br>(.000227) |

**Table 3.2 Estimates of parameters under type I censoring for $c_1 = .1, c_2 = .6, d_1 = 2, d_2 = 3$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are known.**

| Censoring time | Estimate | n=30 | n=50 | n=100 |
|---|---|---|---|---|
| T=15 | M.L.E | $\hat{c}_1$=.10377 (.000101) | $\hat{c}_1$=.10289 (.0002714) | $\hat{c}_1$=.1025556 (1.254E-05) |
| | | $\hat{c}_2$=.5765176 (.002683) | $\hat{c}_2$=.5815027 (.00069) | $\hat{c}_2$=.60581409 (.00001859) |
| | | $\hat{d}_1$=2.089226 (.085652) | $\hat{d}_1$=2.083178 (.011878) | $\hat{d}_1$=1.988662 (.00814) |
| | | $\hat{d}_2$=2.954112 (.005939) | $\hat{d}_2$=3.030418 (.002761) | $\hat{d}_2$=3.00084 (8.28E-05) |
| | | $\hat{p}_1$=.511808 (.000628) | $\hat{p}_1$=.49159 (.000182) | $\hat{p}_1$=.501296 (1.13E-05) |
| | Bayes estimate | $c_1^*$=.103922 (.000121) | $c_1^*$=.103746 (.0001) | $c_1^*$=.1036792 (.0000134) |
| | | $c_2^*$=.58208 (.000445) | $c_2^*$=.59124005 (.0002638) | $c_2^*$=.595432 (.0000938) |
| | | $d_1^*$=1.973064 (.051677) | $d_1^*$=2.026086 (.00608) | $d_1^*$=2.009246 (.005297) |
| | | $d_2^*$=3.104115 (.051996) | $d_2^*$=3.019295 (.048365) | $d_2^*$=3.003463 (.004104) |
| | | $p_1^*$=.519985 (.000618) | $p_1^*$=.507185 (.000546) | $p_1^*$=.5019385 (.000442) |
| T=19 | M.L.E | $\hat{c}_1$=.11822 (.0032826) | $\hat{c}_1$=.1106404 (.0001595) | $\hat{c}_1$=.107364 (1.89E-05) |
| | | $\hat{c}_2$=.5840347 (.002485) | $\hat{c}_2$=.5890734 (.000215) | $\hat{c}_2$=.595951 (7.63E-05) |
| | | $\hat{d}_1$=1.908292 (.032296) | $\hat{d}_1$=1.947618 (.02298) | $\hat{d}_1$=1.981732 (.000372) |
| | | $\hat{d}_2$=2.838636 (.026488) | $\hat{d}_2$=3.05251 (.005917) | $\hat{d}_2$=2.971 (.004022) |
| | | $\hat{p}_1$=.520284 (.000415) | $\hat{p}_1$=.518547 (.000477) | $\hat{p}_1$=.50048 (.000032) |
| | Bayes estimate | $c_1^*$=.106812 (.0001538) | $c_1^*$=.10529 (.000128) | $c_1^*$=.10464 (.0001232) |
| | | $c_2^*$=.5829069 (.000648) | $c_2^*$=.5919147 (.000818) | $c_2^*$=.595122 (.000485) |
| | | $d_1^*$=1.970258 (.003023) | $d_1^*$=1.989864 (.001013) | $d_1^*$=2.0104686 (.00042186) |
| | | $d_2^*$=3.1078 (.05431) | $d_2^*$=3.06085 (.037744) | $d_2^*$=3.01443 (.0042331) |
| | | $p_1^*$=.500481 (.00028) | $p_1^*$=.499283 (1.97E-05) | $p_1^*$=.499421 (3.97E-06) |

**Table 3.3 Estimates of parameters under type I censoring for $c_1 = .1, c_2 = .6, d_1 = 2, d_2 = 3$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are unknown.**

| Censoring time | Estimate | n=30 | n=50 | n=100 |
|---|---|---|---|---|
| T=15 | M.L.E | $\hat{c}_1$=.120847 (8.84167E-6) | $\hat{c}_1$=.11535 (2.093E-7) | $\hat{c}_1$=.105021 (1.61564E-7) |
| | | $\hat{c}_2$=.623705 (3.7157E-7) | $\hat{c}_2$=.58986 (4.524E-8) | $\hat{c}_2$=.5899636 (8.37E-9) |
| | | $\hat{d}_1$=1.6995 (.0389693) | $\hat{d}_1$=2.05192 (.0216674) | $\hat{d}_1$=2.04821 (.0158107) |
| | | $\hat{d}_2$=2.7917 (.387985) | $\hat{d}_2$=3.07168 (.25694) | $\hat{d}_2$=3.03136 (.14817) |
| | | $\hat{p}_1$=.50592 (5.55E-5) | $\hat{p}_1$=.4956 (1.0356E-6) | $\hat{p}_1$=.5009347 (1.01E-7) |
| | Bayes estimate | $c_1^*$=.098265 (.00002027) | $c_1^*$=.10593 (5.485E-5) | $c_1^*$=.09989 (4.387E-6) |
| | | $c_2^*$=.660765 (3.103E-4) | $c_2^*$=.610666 (1.5481E-5) | $c_2^*$=.606123 (3.7448E-8) |
| | | $d_1^*$=1.944494 (.462066) | $d_1^*$=1.98061 (.240225) | $d_1^*$=1.998 (.10416) |
| | | $d_2^*$=2.88663 (.244156) | $d_2^*$=3.011622 (.1255) | $d_2^*$=3.05533 (.11254) |
| | | $p_1^*$=.502641 (8.34E-5) | $p_1^*$=.505561 (2.087E-6) | $p_1^*$=.50569 (4.50117E-7) |
| T=19 | M.L.E | $\hat{c}_1$=.0958657 (1.877E-6) | $\hat{c}_1$=.0990615 (8.7407E-7) | $\hat{c}_1$=.1094232 (9.8030E-8) |
| | | $\hat{c}_2$=.65287 (7.37E-6) | $\hat{c}_2$=.5979633 (1.574E-7) | $\hat{c}_2$=.60624 (2.8456E-9) |
| | | $\hat{d}_1$=1.8689 (.297627) | $\hat{d}_1$=2.11241 (.008808) | $\hat{d}_1$=2.02673 (.000259) |
| | | $\hat{d}_2$=2.9735 (.911787) | $\hat{d}_2$=2.89674 (.232003) | $\hat{d}_2$=2.90433 (.0013144) |
| | | $\hat{p}_1$=.502694 (1.38E-6) | $\hat{p}_1$=.501733 (3.19135E-7) | $\hat{p}_1$=.500493 (4.98E-10) |
| | Bayes estimate | $c_1^*$=.123137 (.00003422) | $c_1^*$=.1209 (1.53073E-6) | $c_1^*$=.110868 (2.67083E-7) |
| | | $c_2^*$=.6170304 (.00001157) | $c_2^*$=.6103 (3.225E-6) | $c_2^*$=.60163 (5.76747E-8) |
| | | $d_1^*$=2.10241 (.374128) | $d_1^*$=2.05645 (.0127) | $d_1^*$=1.96753 (.00564882) |
| | | $d_2^*$=2.82344 (.168282) | $d_2^*$=2.93161 (.034698) | $d_2^*$=2.989 (.001055) |
| | | $p_1^*$=.501724 (1.01612E-6) | $p_1^*$=.503564 (2.5009E-7) | $p_1^*$=.501262 (6.57077E-8) |

**Table 3.4 Estimates of parameters under type II censoring for $c_1 = .1, c_2 = .6, d_1 = 2, d_2 = 3$ and $p_1 = 0.5$**

| Estimate | n=30 | | n=50 | | n=100 | |
|---|---|---|---|---|---|---|
| | r=10 | r=30 | r=30 | r=50 | r=60 | r=100 |
| M.L.E | $\hat{c}_1$=.09764 (.008179) | $\hat{c}_1$=.098544 (.0079187) | $\hat{c}_1$=.108 (.002353) | $\hat{c}_1$=.089 (.000153) | $\hat{c}_1$=.09865 (.003364) | $\hat{c}_1$=.1034 (.0009316) |
| | $\hat{c}_2$=.60734 (.013239) | $\hat{c}_2$=.60688 (.0012954) | $\hat{c}_2$=.600654 (.000243) | $\hat{c}_2$=.600474 (8.05E-05) | $\hat{c}_2$=.59328 (.019827) | $\hat{c}_2$=.60075 (.000197) |
| | $\hat{d}_1$=1.91236 (.014286) | $\hat{d}_1$=1.93653 (.0053) | $\hat{d}_1$=2.9167 (.028538) | $\hat{d}_1$=2.01248 (.000237) | $\hat{d}_1$=2.03508 (.009212) | $\hat{d}_1$=1.989 (.0023) |
| | $\hat{d}_2$=2.94572 (.004187) | $\hat{d}_2$=2.987 (.00059) | $\hat{d}_2$=3.08145 (.009609) | $\hat{d}_2$=3.0042 (.0013) | $\hat{d}_2$=2.98562 (.000534) | $\hat{d}_2$=3.0032 (.000098) |
| | $\hat{p}_1$=.5092 (.000757) | $\hat{p}_1$=.500693 (.000409) | $\hat{p}_1$=.4991 (.000293) | $\hat{p}_1$=.49894 (5.17E-06) | $\hat{p}_1$=.48852 (.000323) | $\hat{p}_1$=.49961 (.00011) |
| Bayes estimate | $c_1^*$=.098246 (.0006511) | $c_1^*$=.102 (.000214) | $c_1^*$=.1066 (.00247) | $c_1^*$=.9949 (.000237) | $c_1^*$=.097432 (.001504) | $c_1^*$=.102 (.0002136) |
| | $c_2^*$=.59829 (.005643) | $c_2^*$=.60036 (.00214) | $c_2^*$=.606782 (.009528) | $c_2^*$=.603315 (.00125) | $c_2^*$=.59925 (.000249) | $c_2^*$=.601 (.00002) |
| | $d_1^*$=1.93876 (.006492) | $d_1^*$=1.96168 (.005195) | $d_1^*$=2.04128 (.049616) | $d_1^*$=2.039314 (.00508) | $d_1^*$=2.0601 (.0006274) | $d_1^*$=2.0009 (.00041) |
| | $d_2^*$=2.9607 (.002012) | $d_2^*$=3.01 (.00054) | $d_2^*$=3.05456 (.014672) | $d_2^*$=3.001 (.0023) | $d_2^*$=3.02 (1.7E-04) | $d_2^*$=3.00362 (2.7E-05) |
| | $p_1^*$=.50324 (.000372) | $p_1^*$=.50081 (.00022) | $p_1^*$=.49978 (.000103) | $p_1^*$=.499592 (8.65E-05) | $p_1^*$=.49965 (.000451) | $p_1^*$=.50224 (.00016) |

**Table 3.5 Risk improvement of the estimator by Bayesian over the different estimators for the set of parameters $c_1 = .1, c_2 = .6, d_1 = 2, d_2 = 3$ and $p_1 = 0.5$ in the complete sample set up.**

| Estimate | M.L.E. (%) | Method of moments (%) | Maximum product of spacing (%) |
|---|---|---|---|
| n=30 | 91 | 98 | 96 |
| | 87 | 95 | 99 |
| | 90 | 97 | 95 |
| | 91 | 94 | 94 |
| | 92 | 95 | 93 |
| n=50 | 90 | 97 | 94 |
| | 91 | 96 | 95 |
| | 92 | 95 | 96 |
| | 89 | 98 | 93 |
| | 87 | 96 | 92 |
| n=100 | 91 | 99 | 95 |
| | 89 | 98 | 94 |
| | 86 | 97 | 97 |
| | 87 | 94 | 95 |
| | 89 | 97 | 93 |

## 3.9 Data analysis

For the illustration of the role of finite mixture of beta distributions in practical situations, we consider a data on survival times of cancer patients with advanced cancer of the bronchus or colon were treated with ascorbate (see Cameron and Pauling, 1978). The data is given in Table 3.6. We estimate the parameters using the method of M.L.E and Bayes technique. Table 3.7 provides the values of the estimates by M.L.E and Bayes. We then used the Kolmogorov-Smirnov statistic to test the goodness of fit. The values of the test statistic $D_n$ are given in Table 3.8. The table value at 5 % significance level is 0.224. From the analysis, it concludes that the model (3.1) is a plausible model for the data with parameters given in Table 3.7. Table 3.9 gives the maximum likelihood estimate of the survival function at various time points.

**Table 3.6 Survival times of cancer patients**

| Bronchus | Colon |
|---|---|
| 81 | 248 |
| 461 | 377 |
| 20 | 189 |
| 450 | 1843 |
| 246 | 180 |
| 166 | 537 |
| 63 | 519 |
| 64 | 455 |
| 155 | 406 |
| 859 | 365 |
| 151 | 942 |
| 166 | 776 |
| 37 | 372 |
| 223 | 163 |
| 138 | 101 |
| 72 | 20 |
| 245 | 283 |

**Table 3.7 Estimates of parameters of survival times of cancer patients.**

| M.L.E | Bayes method |
|---|---|
| $\hat{c}_1 = 0.0001758$ | $c_1^* = 0.000231$ |
| $\hat{c}_2 = 0.00164$ | $c_2^* = 0.00172$ |
| $\hat{d}_1 = 25.8821$ | $d_1^* = 28.93$ |
| $\hat{d}_2 = 12.2915$ | $d_2^* = 14.537$ |
| $\hat{p}_1 = 0.5$ | $p_1^* = 0.5$ |

**Table 3.8 Kolmogorov-Smirnov test statistic**

| Max $D_n$ (M.L.E) | Max $D_n$ (Bayes) |
|---|---|
| 0.063211 | 0.1067 |

**Table 3.9 Maximum likelihood estimate of survival probability at various time points**

| $x$ | 20 | 100 | 500 | 1000 |
|---|---|---|---|---|
| $\hat{S}(x)$ | .847882 | .457781 | .05028 | .00402 |

### 3.10 Conclusion

The role of finite mixture of beta distributions in reliability analysis was extensively studied. We derived some important reliability characteristics of the model. Obviously, the component density of finite mixture of two beta distributions possesses increasing failure rate pattern at all time points. However, the finite mixture of beta distributions can possess decreasing failure rate behaviour, which will be useful in many practical situations. We then proved that finite mixture of two beta distributions is identifiable. The estimation of the parameters by different techniques for the complete as well as censored samples was discussed. Simulation studies were carried out to assess the performance of the estimators. Both M.L.E and Bayes estimates provide estimates with small bias and the variance of the estimates decreases as n increases. The result developed under type II censoring can be specialized to the complete sample situation by taking $r = n$. We illustrated the use of finite mixture of beta distributions based on a real data on survival times of cancer patients. The part of the work in this chapter has appeared in Sankaran and Maya (2004).

# Chapter 4

# BAYESIAN PREDICTIVE DISTRIBUTION

## 4.1 Introduction

Statistical prediction is the problem of inferring the values of unknown future variables or functions of such variables from current available informative observations. Whitmore (1986), Nelson (1982) and Dudewicz (1976) have illustrated the application of prediction limits in government industry and quality assurance. For example, the government may wish to predict the total revenue income for the next fiscal year from a historical record of revenue accumulation to formulate the taxation policy or allocation of resources for various projects. A wide range of potential applications of statistical prediction includes density function estimation, calibration, classification, regulation, model comparison and model criticism. For more applications and references, see, Bernardo and Smith (1994), Johnson et.al. (1995).

As in estimation, a predictor can be either a point or an interval predictor. Parametric and nonparametric prediction has been considered in literature. Frequentist and Bayesian approaches have been used to obtain predictors and study their properties. Reviews on parametric point and interval predictors may be found in Patel (1989), Kaminsky and Nelson (1998) and Al-Hussaini (2001).The problem of prediction can be solved fully within the Bayes framework (Geisser, 1993). Several researchers have studied Bayesian prediction based on homogeneous populations. Among others are Dunsmore (1983), Lingappaiah (1989), Howlader and Hossain (1995).

Recently, many authors have investigated the Bayesian prediction based on heterogeneous populations. Al-Hussaini (1999) obtained Bayesian prediction for a mixture of two exponential distributions based on type I censored sample. Bayesian prediction bounds based on type I censoring for a finite mixture of Lomax components were obtained by Al-Hussaini et.al. (2001). Al-Hussaini (2001) provided Bayesian predictive density of finite mixture models based on order statistics. Recently, Jaheen (2003) obtained Bayesian prediction for a mixture of two-component Gompertz lifetime model.

In this chapter, we develop Bayesian predictive distribution for the future observations based on type I censored samples from a finite mixture of Pareto II as well as beta distributions. The rest of the chapter is organized as follows. We give the definition of the Bayesian predictive distribution in Section 4.2. Section 4.3 discusses the reliability function of the Bayesian predictive distribution. The Bayesian two sample predictive distribution when the underlying distribution is finite mixture of Pareto II is given in Section 4.4. Section 4.5 illustrates the procedure using the real data given in Chapter 2. In Section 4.6, we discuss the Bayesian predictive distribution when the underlying distribution is finite mixture of beta. In Section 4.7, we give the conclusion of the chapter.

## 4.2 Definition

Suppose $n$ units have been subjected to a life testing experiment. Let the failure times $\underline{t}=(t_1,t_2,\ldots,t_n)$ be a random sample from the distribution with probability density function $f(t\mid\theta)$. Suppose $m$ units of the same kind are to be put into future use and let the future failure times $\underline{y}=(y_1,y_2,\ldots,y_m)$ be the second independent sample from the same distribution. In life testing and quality control problems, it is often important to make prediction about some function of $\underline{y}=(y_1,y_2,\ldots,y_m)$ on the basis of the available data $\underline{t}=(t_1,t_2,\ldots,t_n)$. Then, the Bayesian predictive distribution of a future observation $Y$ is defined as the posterior expectation of $f(y\mid\theta)$, and is given by

$$h(y\mid\underline{t})=C\int_{\Theta}f(y\mid\theta)\Pi(\theta\mid\underline{t})d\theta, \tag{4.1}$$

where $C$ is the normalizing constant and $\Pi(\theta\mid\underline{t})$ is the posterior distribution of $\theta$ with $\Theta$ is the range space of $\theta$.

Then $100\tau\%$ prediction interval $(L,U)$ of a future observation $Y$ is defined by

$$\tau = P\left[L < Y < U\right]$$

$$= \int_{L}^{U} h(y \mid \underline{t})dy .$$

One can usually take

$$\int_{L}^{\infty} h(y \mid \underline{t})dy = \frac{1+\tau}{2}$$

and

$$\int_{U}^{\infty} h(y \mid \underline{t})dy = \frac{1-\tau}{2} . \tag{4.2}$$

Let $\{t_{(1)}, t_{(2)}, \ldots, t_{(n)}\}$ be the ordered failure times of $n$ components of a system and let $\{y_{(1)}, y_{(2)}, \ldots, y_{(m)}\}$ be the ordered failure times of a future sample of $m$ similar components. Given $\{t_{(1)}, t_{(2)}, \ldots, t_{(n)}\}$, we are interested in the predictive distribution and the prediction interval of the $k^{th}$ failure time $Y_{(k)}$. Let $\underline{t}^{*} = (t_{(1)}, t_{(2)}, \ldots, t_{(n)})$ and $\underline{y}^{*} = (y_{(1)}, y_{(2)}, \ldots, y_{(m)})$. Then, we have, the Bayesian predictive distribution of $Y_{(k)}$ as

$$h_{Y_{(k)}}(y \mid \underline{t}^{*}) = C \int_{\Theta} f_{Y_{(k)}}(y \mid \theta)\Pi(\theta \mid \underline{t}^{*})d\theta . \tag{4.3}$$

Then, the $100\tau\%$ prediction limits $(L,U)$ of $Y_{(k)}$ are solutions of

$$\int_{L}^{\infty} h_{Y_{(k)}}(y \mid \underline{t}^{*})dy = \frac{1+\tau}{2}$$

and

$$\int_{U}^{\infty} h_{Y_{(k)}}(y \mid \underline{t}^{*})dy = \frac{1-\tau}{2} . \tag{4.4}$$

## 4.3 Predictive distribution and reliability estimation

Let $X$ be a random variable representing the life of an item or a component with probability density function $f(x \mid \theta)$. The reliability function at any time $t$ is defined by

$$S(t) = P(X \geq t)$$

$$= \int_t^{\infty} f(x \mid \theta) dx.$$

Under squared-error loss function Bayes estimate of $S(t)$ is given by

$$S^*(t) = E_\theta(S(t) \mid \underline{x})$$

$$= E_\theta[P(Y \geq t \mid \underline{x}]$$

$$= E_\theta\left[\int_t^{\infty} f(y \mid \theta) dy \mid \underline{x}\right]$$

$$= \int_\Omega \Pi(\theta \mid \underline{x})\left[\int_t^{\infty} f(y \mid \theta) dy\right] d\theta$$

$$= \int_t^{\infty}\left[\int_\Omega f(y \mid \theta) \Pi(\theta \mid \underline{x}) d\theta\right] dy.$$

From (4.1), we get

$$S^*(t) = \int_t^{\infty} h(y \mid \underline{x}) dy, \tag{4.5}$$

which is the reliability function of the predictive distribution.

Thus, under squared-error loss function, Bayes estimator of the reliability function of a distribution is the reliability function of the predictive distribution.

## 4.4 Bayesian two sample prediction for finite mixture of Pareto II distributions

In this section, Bayesian prediction bounds for the $k$-th future observation from a heterogeneous population represented by a finite mixture of two component Pareto II life time distribution under type I censoring are obtained.

Suppose that $n$ units from a population with probability density function (2.2) are subjected to a life testing experiment and the test is terminated after a predetermined time $T$. It is assumed that an item can be attributed to the appropriate subpopulation after it had failed. Suppose that $r$ units have failed during the interval $(0,T)$ with $r_1$ from the first and $r_2$ from the second subpopulation, such that $r = r_1 + r_2$. The remaining $n-r$ units cannot be identified, which are still functioning. Naturally, neither $r_1$ nor $r_2$ (and consequently $r$) is fixed, but they are rather determined by knowing $T$. Let $t_{ij}$ denote the failure time of the $j^{th}$ unit that belongs to the $i^{th}$ subpopulation and that $t_{ij} \le T$, $j = 1,2,...,r_i$ and $i = 1,2$. Such scheme of sampling was suggested by Mendenhall and Hader (1958).

### 4.4.1 The scale parameters $a_1$ and $a_2$ are known.

In this section, we compute Bayesian predictive density function of finite mixture of Pareto II components based on order statistics when the prior distribution is Jeffrey's invariant prior when $a_1$ and $a_2$ are known. Al-Hussaini et.al. (2001) obtained the Bayesian predictive density of finite mixture of Lomax components based on order statistics using natural conjugate prior when $a_1$ and $a_2$ are known.

Rewriting (2.70), the likelihood function is given by

$$L(b_1,b_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\,n-r_2-j_1} (1-p_1)^{r_2+j_1} b_1^{r_1} b_2^{r_2}$$

$$\exp[-b_1 \sum_{j=1}^{r_1} \log(1+a_1 t_{1j})] \exp[-b_2 \sum_{j=1}^{r_2} \log(1+a_2 t_{2j})] (1+a_1 T)^{-b_1(n-r-j_1)} (1+a_2 T)^{-b_2 j_1}. \qquad (4.6)$$

where $\underline{t}^* = \left(t_{11}, t_{12}, ..., t_{1r_1}, t_{21}, t_{22}, ..., t_{2r_2}\right)$, $t_{ij}$ denotes the failure time of the $j^{th}$ unit belonging to the $i^{th}$ subpopulation, $t_{ij} \le T$, $j = 1,2,...,r_i$, $i = 1,2$ and $r_1 + r_2 = r$.

Rearranging (4.6), we have

$$L(b_1,b_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\,n-r_2-j_1} (1-p_1)^{r_2+j_1} b_1^{r_1} b_2^{r_2}$$

$$\exp[-b_1 \left\{ \sum_{j=1}^{r_1} \log(1+a_1 t_{1j}) + (n-r-j_1)\log(1+a_1 T) \right\} - b_2 \left\{ \sum_{i=1}^{r_2} \log(1+a_2 t_{2j}) + j_1 \log(1+a_2 T) \right\}.$$

which can be written as

$$L(b_1,b_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\,n-r_2-j_1}(1-p_1)^{r_2+j_1} b_1^{r_1} b_2^{r_2} \exp\left[-A_{1j_1}b_1 - A_{2j_1}b_2\right]. \tag{4.7}$$

with

$$A_{ij_1} = \sum_{j=1}^{r_i} \log(1+a_i t_{ij}) + \left[(n-r)(2-i)-(-1)^{(i-1)} j_1\right]\log(1+a_i T)\,,\ i=1,2.$$

Now, suppose that $p_1$, $b_1$ and $b_2$ are independent such that $p_1$ follows $U(0,1)$ and for $i=1,2$, $b_i$ follows Jeffrey's invariant prior. Then the joint prior density is given by

$$g(b_1,b_2,p_1) \propto \frac{1}{b_1 b_2}\,, 0<p_1<1\,,\ b_i>0. \tag{4.8}$$

From (4.7) and (4.8), the joint posterior density function of $p_1$, $b_1$ and $b_2$ is given by

$$\prod(b_1,b_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} b_1^{r_1-1} b_2^{r_2-1} \exp\left[-A_{1j_1}b_1 - A_{2j_1}b_2\right]. \tag{4.9}$$

where

$$\delta_1 = n - r_2 - j_1$$

and

$$\delta_2 = r_2 + j_1\ .$$

A future sample of size $m$ is assumed to be independent of the past (informative) sample of size $n$ and is obtained from the same population with probability density function (2.2). Let $Y_{(k)}$ be the ordered lifetime of the $k$-th components to fail in a future sample of size $m$, $1 \le k \le m$. The $k$-th order statistic in a sample of size $m$ represents the life length of an $(m-k+1)$ out of $m$ system which is an important technical structure in reliability theory. The density function of $Y_{(k)}$ is then given by

$$f_{Y_{(k)}}(y \mid b_1,b_2,p_1) = \left[1-S(y)\right]^{k-1}\left[S(y)\right]^{m-k} f(y)\,.$$

$$= \sum_{j_2=0}^{k-1} (-1)^{j_2}\binom{k-1}{j_2}\left[S(y)\right]^{m-k+j_2} f(y)\,, \tag{4.10}$$

where $S(y)$ and $f(y)$ are the reliability function and probability density function of the finite mixture model (2.2). For the model (2.2), (4.10) becomes,

$$f_{Y_{(k)}}(y \mid b_1, b_2, p_1) = \sum_{j_2=0}^{k-1} (-1)^{j_2} \binom{k-1}{j_2} \left[ p_1(1+a_1 y)^{-b_1} + (1-p_1)(1+a_2 y)^{-b_2} \right]^{m-k+j_2} f(y)$$

$$= \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{m-k+j_2} (-1)^{j_2} \binom{k-1}{j_2} \binom{m-k+j_2}{j_3} p_1^{\,m-k+j_2-j_3} (1-p_1)^{j_3}$$

$$\exp\left[-b_1(m-k+j_2-j_3)\log(1+a_1 y) - b_2 j_3 \log(1+a_2 y))\right] f(y)$$

$$= \sum C_1 p_1^{\delta_3 - 1} (1-p_1)^{j_3} \exp\left[-b_1(\delta_3 - 1)\log(1+a_1 y) - b_2 j_3 \log(1+a_2 y))\right] f(y). \qquad (4.11)$$

where

$$\sum = \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{m-k+j_2}, \quad C_1 = (-1)^{j_2} \binom{k-1}{j_2} \binom{m-k+j_2}{j_3} \text{ and } \delta_3 = m-k+j_2-j_3+1.$$

The product of (4.9) and (4.11), then yields

$$f_{Y_{(k)}}(y \mid b_1, b_2, p_1) \Pi(b_1, b_2, p_1 \mid \underline{t}^*) \propto$$

$$\sum C_1 p_1^{\delta_3 - 1} (1-p_1)^{j_3} \exp\left[-b_1(\delta_3 - 1)\log(1+a_1 y) - b_2 j_3 \log(1+a_2 y))\right]$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1} (1-p_1)^{\delta_2} b_1^{r_1 - 1} b_2^{r_2 - 1} \exp\left[-A_{1_{j_1}} b_1 - A_{2_{j_1}} b_2\right]$$

$$[p_1 a_1 b_1 (1+a_1 y)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 y)^{-(b_2+1)}].$$

$$\propto \sum{}^* \binom{n-r}{j_1} C_1 \{\varepsilon_1(y) p_1^{\delta_1+\delta_3} (1-p_1)^{\delta_2+j_3} b_1^{r_1} b_2^{r_2-1} \exp\left[-\eta_1 b_1 - \eta_2 b_2\right]$$

$$+ \varepsilon_2(y) p_1^{\delta_1+\delta_3-1} (1-p_1)^{\delta_2+j_3+1} b_1^{r_1-1} b_2^{r_2} \exp\left[-\eta_{11} b_1 - \eta_{22} b_2\right]\}. \qquad (4.12)$$

where

$$\sum{}^* = \sum_{j_1=0}^{n-r} \sum, \quad \varepsilon_i(y) = (a_i (1+a_i y)^{-1}), \quad i = 1, 2,$$

$$\eta_1 = A_{1_{j_1}} + \delta_3 \log(1+a_1 y),$$

$$\eta_2 = A_{2_{j_1}} + j_3 \log(1+a_2 y),$$

$$\eta_{11} = A_{1_{j_1}} + (\delta_3 - 1)\log(1+a_1 y),$$

and

$\eta_{22} = A_{2j_1} + (j_3+1)\log(1+a_2 y)$.

Substituting (4.12) in (4.3), we have

$$h_{Y_{(k)}}(y \mid \underline{t}^*) = C\int_0^\infty\int_0^\infty\int_0^1 f_{Y_{(k)}}(y \mid b_1, b_2, p_1)\Pi(b_1, b_2, p_1 \mid \underline{t}^*)dp_1 db_1 db_2 . \qquad (4.13)$$

On simplifying (4.13), we get the predictive distribution as

$$h_{Y_{(k)}}(y \mid \underline{t}^*) = C\sum{}^*\binom{n-r}{j_1}C_1\int_0^\infty\int_0^\infty\int_0^1\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}b_1^{r_1}b_2^{r_2-1}\exp[-\eta_1 b_1 - \eta_2 b_2]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}b_1^{r_1-1}b_2^{r_2}\exp[-\eta_{11}b_1 - \eta_{22}b_2]\}dp_1 db_1 db_2$$

$$= C\sum{}^*\binom{n-r}{j_1}C_1\{\varepsilon_1(y)B(\delta_1+\delta_3+1, \delta_2+j_3+1)\frac{\Gamma(r_1+1)}{(\eta_1)^{(r_1+1)}}\frac{\Gamma(r_2)}{(\eta_2)^{(r_2)}}$$

$$+\varepsilon_2(y)B(\delta_1+\delta_3, \delta_2+j_3+2)\frac{\Gamma(r_1)}{(\eta_{11})^{(r_1)}}\frac{\Gamma(r_2+1)}{(\eta_{22})^{(r_2+1)}}\}$$

$$= C\sum{}^* C_1^*\left[\phi_1(y)+\phi_2(y)\right], \qquad (4.14)$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{(k)}}(y \mid \underline{t}^*)dy = 1$,

$$C_1^* = C_1\binom{n-r}{j_1}\left[\frac{\Gamma(r_1)\Gamma(r_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y) = \varepsilon_1(y)\frac{r_1(\delta_1+\delta_3)}{(\eta_1)^{(r_1+1)}(\eta_2)^{(r_2)}},$$

and

$$\phi_2(y) = \varepsilon_2(y)\frac{r_2(\delta_2+j_3+1)}{(\eta_{11})^{(r_1)}(\eta_{22})^{(r_2+1)}}.$$

**Remark 4.1**

Al-Hussaini et.al. (2001) obtained the Bayesian predictive density of finite mixture of Lomax components based on order statistics using natural conjugate prior when $a_1$ and $a_2$ are known. They have considered the joint prior density as

$$g(b_1,b_2,p_1) \propto p_1^{\,b-1}(1-p_1)^{d-1} b_1^{\beta_1-1} b_2^{\beta_2-1} \exp[-\gamma_1 b_1 - \gamma_2 b_2], \qquad 0<p_1<1,\ b_i>0. \qquad (4.15)$$

From (4.7) and (4.15), the joint posterior density function of $p_1$, $b_1$ and $b_2$ is given by

$$\prod(b_1,b_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} b_1^{\omega_1-1} b_2^{\omega_2-1} \exp\left[-\psi_1 b_1 - \psi_2 b_2\right]. \qquad (4.16)$$

where

$\delta_1 = n - r_2 - j_1 + b - 1$ , $\delta_2 = r_2 + j_1 + d - 1$ and for $i = 1,2$, $\omega_i = r_i + \beta_i$, $\psi_i = A_{ij_1} + \gamma_i$.

From (4.11) and (4.16), we have,

$$f_{Y_{(k)}}(y \mid b_1,b_2,p_1)\Pi(b_1,b_2,p_1 \mid \underline{t}^*) \propto$$

$$\sum D_1 p_1^{\delta_3-1}(1-p_1)^{j_3} \exp\left[-b_1(\delta_3-1)\log(1+a_1 y) - b_2 j_3 \log(1+a_2 y)\right]$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} b_1^{\omega_1-1} b_2^{\omega_2-1} \exp\left[-\psi_1 b_1 - \psi_2 b_2\right]$$

$$\left[p_1 a_1 b_1 (1+a_1 y)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 y)^{-(b_2+1)}\right].$$

where

$$\sum = \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{m-k+j_2}, \quad D_1 = (-1)^{j_2}\binom{k-1}{j_2}\binom{m-k+j_2}{j_3} \text{ and } \delta_3 = m-k+j_2-j_3+1.$$

After simplification, the above product becomes,

$$\propto \sum{}^{*} \binom{n-r}{j_1} D_1 \{\varepsilon_1(y) p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3} b_1^{\omega_1} b_2^{\omega_2-1} \exp\left[-\eta_1 b_1 - \eta_2 b_2\right]$$

$$+\varepsilon_2(y) p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1} b_1^{\omega_1-1} b_2^{\omega_2} \exp\left[-\eta_{11} b_1 - \eta_{22} b_2\right]\}. \qquad (4.17)$$

where

$$\sum{}^{*} = \sum_{j_1=0}^{n-r} \sum, \quad \text{for } i=1,2, \ \varepsilon_i(y) = (a_i(1+a_i y)^{-1}),$$

$$\eta_1 = \psi_1 + \delta_3 \log(1+a_1 y),$$

$$\eta_2 = \psi_2 + j_3 \log(1+a_2 y),$$

$$\eta_{11} = \psi_1 + (\delta_3 - 1)\log(1+a_1 y),$$

and

$\eta_{22} = \psi_2 + (j_3 + 1)\log(1 + a_2 y)$.

Then, the predictive density is obtained as

$$h_{Y_{k1}}(y \mid \underline{t}^*) = C\sum{}^*\binom{n-r}{j_1} D_1 \int_0^\infty\int_0^\infty\int_0^1 \{\varepsilon_1(y) p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3} b_1^{\omega_1} b_2^{\omega_2-1} \exp[-\eta_1 b_1 - \eta_2 b_2]$$

$$+\varepsilon_2(y) p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1} b_1^{\omega_1-1} b_2^{\omega_2} \exp[-\eta_{11} b_1 - \eta_{22} b_2]\} dp_1 db_1 db_2$$

$$= C\sum{}^*\binom{n-r}{j_1} D_1 \{\varepsilon_1(y) B(\delta_1+\delta_3+1, \delta_2+j_3+1) \frac{\Gamma(\omega_1+1)}{(\eta_1)^{(\omega_1+1)}} \frac{\Gamma(\omega_2)}{(\eta_2)^{(\omega_2)}}$$

$$+\varepsilon_2(y) B(\delta_1+\delta_3, \delta_2+j_3+2) \frac{\Gamma(\omega_1)}{(\eta_{11})^{(\omega_1)}} \frac{\Gamma(\omega_2+1)}{(\eta_{22})^{(\omega_2+1)}}\}$$

$$= C\sum{}^* D_1^* [\phi_1(y) + \phi_2(y)], \tag{4.18}$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{k1}}(y \mid \underline{t}^*) dy = 1$,

$$D_1^* = D_1\binom{n-r}{j_1}\left[\frac{\Gamma(\omega_1)\Gamma(\omega_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y) = \varepsilon_1(y)\frac{\omega_1(\delta_1+\delta_3)}{(\eta_1)^{(\omega_1+1)}(\eta_2)^{(\omega_2)}},$$

and

$$\phi_2(y) = \varepsilon_2(y)\frac{\omega_2(\delta_2+j_3+1)}{(\eta_{11})^{(\omega_1)}(\eta_{22})^{(\omega_2+1)}}.$$

### 4.4.2 Parameters $a_1, a_2, b_1, b_2$ and $p_1$ are unknown

In this section, we compute Bayesian predictive density function of finite mixture of Pareto II distributions based on order statistics when prior distribution is Jeffrey's invariant as well as natural conjugate prior when $a_1, a_2, b_1, b_2$ and $p_1$ are unknown.

From (2.70), the likelihood function is given by,

$$L(a_1,a_2,b_1,b_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\,n-r_2-j_1}(1-p_1)^{r_2+j_1} a_1^{r_1} a_2^{r_2} b_1^{r_1} b_2^{r_2}$$

$$\exp[-b_1 \sum_{j=1}^{r_1} \log(1+a_1 t_{1j})] \exp[-b_2 \sum_{j=1}^{r_2} \log(1+a_2 t_{2j})] (1+a_1 T)^{-b_1(n-r-j_1)} (1+a_2 T)^{-b_2 j_1}$$

$$\exp[-\left\{\sum_{j=1}^{r_1} \log(1+a_1 t_{1j}) + \sum_{j=1}^{r_2} \log(1+a_2 t_{2j})]\right\}. \tag{4.19}$$

which leads to

$$L(a_1,a_2,b_1,b_2,p_1 \mid \underline{t}^*) \propto$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\,n-r_2-j_1}(1-p_1)^{r_2+j_1} a_1^{r_1} a_2^{r_2} b_1^{r_1} b_2^{r_2} \exp\left[-A_{1j_1} b_1 - A_{2j_1} b_2\right] \exp\left[-B_1 - B_2\right], \tag{4.20}$$

where, for $i = 1,2$, $A_{ij_1}$ is given in (4.7) and $B_i = \sum_{j=1}^{r_i} \log(1+a_i t_{ij})]$.

Now, assume that $p_1$, $a_1, a_2, b_1$ and $b_2$ are independent such that $p_1$ follows $U(0,1)$ and for $i=1,2$, $a_i$ and $b_i$ follows jeffrey's invariant prior. Then, the joint prior density is given by

$$g(a_1,a_2,b_1,b_2,p_1) \propto \frac{1}{a_1 a_2 b_1 b_2}, 0 < p_1 < 1,\ a_i > 0,\ b_i > 0. \tag{4.21}$$

The posterior density function of $p_1, a_1, a_2, b_1$ and $b_2$ is given by

$$\Pi(a_1,a_2,b_1,b_2,p_1 \mid \underline{t}^*) \propto$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} a_1^{r_1-1} a_2^{r_2-1} b_1^{r_1-1} b_2^{r_2-1} \exp\left[-A_{1j_1} b_1 - A_{2j_1} b_2\right] \exp[-B_1 - B_2]. \tag{4.22}$$

where $\delta_1 = n - r_2 - j_1$ and $\delta_2 = r_2 + j_1$ .

The density function of $k$-th order statistic $Y_{(k)}$ is given in (4.11). The product of (4.11) and (4.22) then yields

$f_{Y_{(k)}}(y \mid a_1, a_2, b_1, b_2, p_1)\Pi(a_1, a_2, b_1, b_2, p_1 \mid \underline{t}^*) \propto$

$$\sum M_1 p_1^{\delta_3 - 1}(1-p_1)^{j_3} \exp[-b_1(\delta_3 - 1)\log(1+a_1 y) - b_2 j_3 \log(1+a_2 y))]$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} a_1^{r_1-1} a_2^{r_2-1} b_1^{r_1-1} b_2^{r_2-1} \exp\left[-A_{1j_1} b_1 - A_{2j_1} b_2\right]\exp[-B_1 - B_2]$$

$$[p_1 a_1 b_1 (1+a_1 y)^{-(b_1+1)} + (1-p_1) a_2 b_2 (1+a_2 y)^{-(b_2+1)}]$$

where

$$\sum = \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{m-k+j_2}, \quad M_1 = (-1)^{j_2} \binom{k-1}{j_2}\binom{m-k+j_2}{j_3} \text{ and } \delta_3 = m-k+j_2-j_3+1 .$$

$$\propto \sum{}^{*} \binom{n-r}{j_1} M_1 \exp[-B_1 - B_2]$$

$$\{\varepsilon_1(y) p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3} a_1^{r_1} a_2^{r_2-1} b_1^{r_1} b_2^{r_2-1} \exp[-\eta_1 b_1 - \eta_2 b_2]$$

$$+\varepsilon_2(y) p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1} a_1^{r_1-1} a_2^{r_2} b_1^{r_1-1} b_2^{r_2} \exp[-\eta_{11} b_1 - \eta_{22} b_2]\} . \qquad (4.23)$$

where

$$\sum{}^{*} = \sum_{j_1=0}^{n-r} \sum, \quad \varepsilon_i(y) = (1+a_i y)^{-1}), \ i = 1,2,$$

$$\eta_1 = A_{1j_1} + \delta_3 \log(1+a_1 y),$$

$$\eta_2 = A_{2j_1} + j_3 \log(1+a_2 y),$$

$$\eta_{11} = A_{1j_1} + (\delta_3 - 1)\log(1+a_1 y),$$

and

$$\eta_{22} = A_{2j_1} + (j_3+1)\log(1+a_2 y).$$

Substituting (4.23) in (4.3), we have

$$h_{Y_{(k)}}(y \mid \underline{t}^*) = C \sum{}^{*} \binom{n-r}{j_1} M_1 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 \exp[-B_1 - B_2]$$

$$\{\varepsilon_1(y) p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3} a_1^{r_1} a_2^{r_2-1} b_1^{r_1} b_2^{r_2-1} \exp[-\eta_1 b_1 - \eta_2 b_2]$$

$$+\varepsilon_2(y) p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1} a_1^{r_1-1} a_2^{r_2} b_1^{r_1-1} b_2^{r_2} \exp[-\eta_{11} b_1 - \eta_{22} b_2]\} dp_1 db_1 db_2 da_1 da_2 .$$

$$= C\sum{}^{*}\binom{n-r}{j_1}M_1[\{B(\delta_1+\delta_3+1,\delta_2+j_3+1)\Gamma(r_1+1)\Gamma(r_2)$$

$$\int_0^\infty \frac{a_1^{r_1}\varepsilon_1(y)\exp[-B_1]}{(\eta_1)^{(r_1+1)}}da_1\int_0^\infty \frac{a_2^{r_2-1}\exp[-B_2]}{(\eta_2)^{(r_2)}}da_2 + B(\delta_1+\delta_3,\delta_2+j_3+2)\Gamma(r_1)\Gamma(r_2+1)$$

$$\int_0^\infty \frac{a_2^{r_2}\varepsilon_2(y)\exp[-B_2]}{(\eta_{22})^{(r_2+1)}}da_2\int_0^\infty \frac{a_1^{r_1-1}\varepsilon_2(y)\exp[-B_1]}{(\eta_{11})^{(r_1)}}da_1]$$

$$= C\sum{}^{*}M_1^{*}\left[\phi_1(y)+\phi_2(y)\right], \tag{4.24}$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{(1)}}(y\mid \underline{t}^*)dy = 1$,

$$M_1^* = M_1\binom{n-r}{j_1}\left[\frac{\Gamma(r_1)\Gamma(r_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y) = r_1(\delta_1+\delta_3)\int_0^\infty \frac{a_1^{r_1}\varepsilon_1(y)\exp[-B_1]}{(\eta_1)^{(r_1+1)}}da_1\int_0^\infty \frac{a_2^{r_2-1}\exp[-B_2]}{(\eta_2)^{(r_2)}}da_2,$$

and

$$\phi_2(y) = r_2(\delta_2+j_3+1)\int_0^\infty \frac{a_2^{r_2}\varepsilon_2(y)\exp[-B_2]}{(\eta_{22})^{(r_2+1)}}da_2\int_0^\infty \frac{a_1^{r_1-1}\varepsilon_2(y)\exp[-B_1]}{(\eta_{11})^{(r_1)}}da_1.$$

Now, we consider that $p_1, a_1, a_2, b_1$ and $b_2$ are independent random variables such that $p_1$ follows beta$(b,d)$, for $i=1,2$, $a_i$ follows gamma$(\mu_i,\nu_i)$ and for $i=1,2$, $b_i$ follows gamma$(\beta_i,\gamma_i)$. Then the joint prior density is given by

$$g(a_1,a_2,b_1,b_2,p_1) \propto$$

$$p_1^{b-1}(1-p_1)^{d-1}a_1^{\mu_1-1}a_2^{\mu_2-1}b_1^{\beta_1-1}b_2^{\beta_2-1}\exp[-\nu_1 a_1-\nu_2 a_2]\exp[-\gamma_1 b_1-\gamma_2 b_2],$$

$$0<p_1<1,\ b_i>0,\ a_i>0. \tag{4.25}$$

From (4.20) and (4.25), the joint posterior density function of $a_1, a_2, b_1, b_2$ and $p_1$ given $\underline{t}^*$ is given by

$$\Pi(a_1,a_2,b_1,b_2,p_1 \mid \underline{t}^*) \propto$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\,n-r_2-j_1}(1-p_1)^{r_2+j_1} a_1^{r_1} a_2^{r_2} b_1^{r_1} b_2^{r_2} \exp[-A_{1j_1}b_1 - A_{2j_2}b_2]\exp[-B_1 - B_2]$$

$$p_1^{\,b-1}(1-p_1)^{d-1} a_1^{\mu_1-1} a_2^{\mu_2-1} b_1^{\beta_1-1} b_2^{\beta_2-1} \exp[-\nu_1 a_1 - \nu_2 a_2]\exp[-\gamma_1 b_1 - \gamma_2 b_2]$$

$$\propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} a_1^{\sigma_1-1} a_2^{\sigma_2-1} b_1^{\omega_1-1} b_2^{\omega_2-1} \exp[-\psi_1 b_1 - \psi_2 b_2]\exp[-\chi_1 - \chi_2]. \qquad (4.26)$$

where

$\delta_1 = n - r_2 - j_1 + b - 1$, $\delta_2 = r_2 + j_1 + d - 1$

and for $i = 1,2$, $\omega_i = r_i + \beta_i$, $\sigma_i = r_i + \mu_i$, $\psi_i = A_{ij_1} + \gamma_i$ and $\chi_i = B_i + \nu_i a_i$.

The product of (4.11) and (4.26) is given by

$$f_{Y_{(k)}}(y \mid a_1,a_2,b_1,b_2,p_1)\Pi(a_1,a_2,b_1,b_2,p_1 \mid \underline{t}^*) \propto$$

$$\sum Q_1 p_1^{\delta_3-1}(1-p_1)^{j_3} \exp\left[-b_1(\delta_3-1)\log(1+a_1 y) - b_2 j_3 \log(1+a_2 y))\right] f(y)$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} a_1^{\sigma_1-1} a_2^{\sigma_2-1} b_1^{\omega_1-1} b_2^{\omega_2-1} \exp[-\psi_1 b_1 - \psi_2 b_2]\exp[-\chi_1 - \chi_2]$$

where

$$\sum = \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{m-k+j_2}, \quad Q_1 = (-1)^{j_2}\binom{k-1}{j_2}\binom{m-k+j_2}{j_3} \text{ and } \delta_3 = m-k+j_2-j_3+1.$$

$$\propto \sum{}^* \binom{n-r}{j_1} Q_1 \exp[-\chi_1 - \chi_2]\{\varepsilon_1(y) p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3} a_1^{\sigma_1} a_2^{\sigma_2-1} b_1^{\omega_1} b_2^{\omega_2-1} \exp\left[-\eta_1 b_1 - \eta_2 b_2\right]$$

$$+ \varepsilon_2(y) p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1} a_1^{\sigma_1-1} a_2^{\sigma_2} b_1^{\omega_1-1} b_2^{\omega_2} \exp\left[-\eta_{11} b_1 - \eta_{22} b_2\right]\}. \qquad (4.27)$$

where

$$\sum{}^* = \sum_{j_1=0}^{n-r} \sum, \quad \varepsilon_i(y) = (1+a_i y)^{-1}, \ i = 1,2,$$

$$\eta_1 = \psi_1 + \delta_3 \log(1+a_1 y),$$

$$\eta_2 = \psi_2 + j_3 \log(1+a_2 y),$$

$\eta_{11} = \psi_1 + (\delta_3 - 1)\log(1 + a_1 y)$,

and

$\eta_{22} = \psi_2 + (j_3 + 1)\log(1 + a_2 y)$.

By substituting (4.27) in (4.3), the predictive density function is given by

$$h_{Y_{(k)}}(y \mid \underline{t}^*) = C\sum{}^*\binom{n-r}{j_1}Q_1$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^1 \exp[-\chi_1 - \chi_2]\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}a_1^{\sigma_1}a_2^{\sigma_2-1}b_1^{\omega_1}b_2^{\omega_2-1}\exp[-\eta_1 b_1 - \eta_2 b_2]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}a_1^{\sigma_1-1}a_2^{\sigma_2}b_1^{\omega_1-1}b_2^{\omega_2}\exp[-\eta_{11}b_1 - \eta_{22}b_2]\}dp_1 db_1 db_2 da_1 da_2$$

$$= C\sum{}^*\binom{n-r}{j_1}Q_1\{\varepsilon_1(y)B(\delta_1+\delta_3+1,\delta_2+j_3+1)\Gamma(\omega_1+1)\Gamma(\omega_2)$$

$$\int_0^\infty\frac{a_1^{\sigma_1}\varepsilon_1(y)\exp[-\chi_1]}{(\eta_1)^{(\omega_1+1)}}da_1\int_0^\infty\frac{a_2^{\sigma_2-1}\exp[-\chi_2]}{(\eta_2)^{(\omega_2)}}da_2 + B(\delta_1+\delta_3,\delta_2+j_3+2)\Gamma(\omega_1)\Gamma(\omega_2+1)$$

$$\int_0^\infty\frac{a_2^{\sigma_2}\varepsilon_2(y)\exp[-\chi_2]}{(\eta_{22})^{(\omega_2+1)}}da_2\int_0^\infty\frac{a_1^{\sigma_1-1}\varepsilon_2(y)\exp[-\chi_1]}{(\eta_{11})^{(\omega_1)}}da_1.$$

$$= C\sum{}^*Q_1^*\left[\phi_1(y)+\phi_2(y)\right], \tag{4.28}$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{(k)}}(y \mid \underline{t}^*)dy = 1$,

$$Q_1^* = Q_1\binom{n-r}{j_1}\left[\frac{\Gamma(\omega_1)\Gamma(\omega_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y) = \omega_1(\delta_1+\delta_3)\int_0^\infty\frac{a_1^{\sigma_1}\varepsilon_1(y)\exp[-\chi_1]}{(\eta_1)^{(\omega_1+1)}}da_1\int_0^\infty\frac{a_2^{\sigma_2-1}\exp[-\chi_2]}{(\eta_2)^{(\omega_2)}}da_2,$$

and

$$\phi_2(y) = \omega_2(\delta_2+j_3+1)\int_0^\infty\frac{a_2^{\sigma_2}\varepsilon_2(y)\exp[-\chi_2]}{(\eta_{22})^{(\omega_2+1)}}da_2\int_0^\infty\frac{a_1^{\sigma_1-1}\varepsilon_2(y)\exp[-\chi_1]}{(\eta_{11})^{(\omega_1)}}da_1.$$

### 4.5 Data analysis

In this section, we illustrate the procedure given in Section 4.4.2 using the data given in Chapter 2. We, here, estimate the reliability function using predictive density function. We obtained the lower and upper prediction bounds for $Y_{(1)}$ and $Y_{(m)}$, the first and last failure times in a future sample of size $m = 10$. The failure times are assumed to follow finite mixture of Pareto II distribution with density function given in (2.2). Table 4.1 provides the Bayes estimate of parameters when the data is censored at the point $T$ and $r_1$ and $r_2$ are the number of observations belonging to the respective subpopulations. Using our results in (4.4) and (4.24) with $\tau = 0.95$, the lower and upper 95% prediction bounds for $Y_{(1)}$, the first failure time, are 7 and 12 respectively. Whereas the 95% prediction bounds for $Y_{(10)}$, the last failure times are given by 89 and 97 respectively. The reliability $S^*(t)$ at $t = 25$ is obtained as 0.4231 by putting $k = 1$ in (4.24) and using (4.5).

**Table 4.1 Bayes estimates of parameters of survival times of leukaemia patients.**

| $T$=80 $(r_1 = 13, r_2 = 16)$ | $a_1^* = .0336$ | $a_2^* = .0662$ | $b_1^* = .9743$ | $b_2^* = 1.138$ | $p_1^* = .4519$ |
|---|---|---|---|---|---|

### 4.6 Bayesian two sample prediction for finite mixture of beta distributions

In this section, we consider finite mixture of two-component beta model and two-sample prediction using Bayesian technique is employed.

Suppose that $n$ units from a population with probability density function (3.1) are subjected to a life testing experiment and the test is terminated after a predetermined time $T$. It is assumed that an item can be attributed to the appropriate subpopulation after it had failed. Suppose that $r$ units have failed during the interval $(0, T)$ with $r_1$ from the first and $r_2$ from the second subpopulation, such that $r = r_1 + r_2$. The remaining $n - r$ units which cannot be identified as to subpopulation are still functioning. Naturally, neither $r_1$ nor $r_2$ (and consequently $r$) is fixed, but they are rather determined by knowing $T$. Let $t_{ij}$ denote the failure time of the $j^{th}$ unit that belongs to the $i^{th}$ subpopulation and that $t_{ij} \le T$, $j = 1,2,...,r_i$, $i = 1,2$.

### 4.6.1 The scale parameters $c_1$ and $c_2$ are known

In this section, we compute Bayesian predictive distribution when the prior distribution is Jeffrey's invariant as well as natural conjugate prior when $c_1$ and $c_2$ are known.

Rewriting (3.74), the likelihood function is given by,

$$L(d_1,d_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{n-r_2-j_1}(1-p_1)^{r_2+j_1} d_1^{r_1} d_2^{r_2}$$

$$\exp[d_1 \sum_{j=1}^{r_1} \log(1-c_1 t_{1j})]\exp[d_2 \sum_{j=1}^{r_2} \log(1-c_2 t_{2j})](1-c_1T)^{d_1(n-r-j_1)}(1-c_2T)^{d_2 j_1}. \tag{4.29}$$

Rearranging (4.29), we have

$$L(d_1,d_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{n-r_2-j_1}(1-p_1)^{r_2+j_1} d_1^{r_1} d_2^{r_2}$$

$$\exp[-d_1\left\{-\sum_{j=1}^{r_1}\log(1-c_1t_{1j})-(n-r-j_1)\log(1-c_1T)\right\}-d_2\left\{-\sum_{j=1}^{r_2}\log(1-c_2t_{2j})-j_1\log(1-c_2T)\right\}.$$

$$L(d_1,d_2,p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{n-r_2-j_1}(1-p_1)^{r_2+j_1} d_1^{r_1} d_2^{r_2} \exp[-A_{1j_1}d_1 - A_{2j_1}d_2] \tag{4.30}$$

with

$$A_{ij_1} = -\sum_{j=1}^{r_i}\log(1-c_it_{ij}) - \left[(n-r)(2-i)-(-1)^{(i-1)}j_1\right]\log(1-c_iT),\ i=1,2.$$

Now, assume that $p_1$, $d_1$ and $d_2$ are independent such that $p_1$ follows $U(0,1)$ and for $i=1,2$, $b_i$ follows Jeffrey's invariant prior. Accordingly, the joint prior density is given by

$$g(d_1,d_2,p_1) \propto \frac{1}{d_1d_2}, 0<p_1<1,\ d_i>0. \tag{4.31}$$

The joint posterior density function of $p_1$, $d_1$ and $d_2$ is given by

$$\Pi(d_1,d_2,p_1\mid \underline{t}^*)\propto\sum_{j_1=0}^{n-r}\binom{n-r}{j_1}p_1^{\delta_1}(1-p_1)^{\delta_2}d_1^{r_1-1}d_2^{r_2-1}\exp[-A_{1j_1}d_1-A_{2j_1}d_2] \tag{4.32}$$

where $\delta_1=n-r_2-j_1$ and $\delta_2=r_2+j_1$.

The density function of $k$-th order statistic $Y_{(k)}$ is given in (4.10), where $S(y)$ and $f(y)$ are the reliability function and probability density function of the finite mixture model (3.1). For the finite mixture of beta distributions (3.1), (4.10) becomes,

$$f_{Y_{(k)}}(y\mid d_1,d_2,p_1)\propto\sum_{j_2=0}^{k-1}(-1)^{j_2}\binom{k-1}{j_2}\left[p_1(1-c_1y)^{d_1}+(1-p_1)(1-c_2y)^{d_2}\right]^{m-k+j_2}f(y)$$

$$\propto\sum_{j_2=0}^{k-1}\sum_{j_3=0}^{m-k+j_2}(-1)^{j_2}\binom{k-1}{j_2}\binom{m-k+j_2}{j_3}p_1^{m-k+j_2-j_3}(1-p_1)^{j_3}$$

$$\exp\left[d_1(m-k+j_2-j_3)\log(1-c_1y)+d_2j_3\log(1-c_2y))\right]f(y)$$

$$=\sum N_1p_1^{\delta_3-1}(1-p_1)^{j_3}\exp[d_1(\delta_3-1)\log(1-c_1y)+d_2j_3\log(1-c_2y)]f(y). \tag{4.33}$$

where

$$\sum=\sum_{j_2=0}^{k-1}\sum_{j_3=0}^{m-k+j_2},\quad N_1=(-1)^{j_2}\binom{k-1}{j_2}\binom{m-k+j_2}{j_3}\text{ and }\delta_3=m-k+j_2-j_3+1.$$

The product of (4.32) and (4.33), then yields

$$f_{Y_{(k)}}(y\mid d_1,d_2,p_1)\Pi(d_1,d_2,p_1\mid\underline{t}^*)\propto$$

$$\sum N_1p_1^{\delta_3-1}(1-p_1)^{j_3}\exp[d_1(\delta_3-1)\log(1-c_1y)+d_2j_3\log(1-c_2y)]$$

$$\sum_{j_1=0}^{n-r}\binom{n-r}{j_1}p_1^{\delta_1}(1-p_1)^{\delta_2}d_1^{r_1-1}d_2^{r_2-1}\exp[-A_{1j_1}d_1-A_{2j_1}d_2]$$

$$[p_1c_1d_1(1-c_1y)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2y)^{(d_2-1)}].$$

$$\propto\sum{}^*\binom{n-r}{j_1}N_1\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}d_1^{r_1}d_2^{r_2-1}\exp\left[-\eta_1d_1-\eta_2d_2\right]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}d_1^{r_1-1}d_2^{r_2}\exp\left[-\eta_{11}d_1-\eta_{22}d_2\right]\}. \tag{4.34}$$

where

$$\sum{}^{*}=\sum_{j_1=0}^{n-r}\sum,\quad \varepsilon_i(y)=(c_i(1-c_i y)^{-1}),\quad i=1,2,$$

$$\eta_1=A_{1j_1}-\delta_3\log(1-c_1y),$$

$$\eta_2=A_{2j_1}-j_3\log(1-c_2y),$$

$$\eta_{11}=A_{1j_1}-(\delta_3-1)\log(1-c_1y),$$

and

$$\eta_{22}=A_{2j_1}-(j_3+1)\log(1-c_2y).$$

Substituting (4.34) in (4.3), we have

$$h_{Y_{(k)}}(y\mid \underline{t}^*)=C\int_0^\infty\int_0^\infty\int_0^1 f_{Y_{(k)}}(y\mid d_1,d_2,p_1)\Pi(d_1,d_2,p_1\mid \underline{t}^*)dp_1dd_1dd_2. \tag{4.35}$$

On simplifying (4.35), we get the predictive distribution as

$$h_{Y_{(k)}}(y\mid \underline{t}^*)=C\sum{}^{*}\binom{n-r}{j_1}N_1\int_0^\infty\int_0^\infty\int_0^1\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}d_1^{r_1}d_2^{r_2-1}\exp[-\eta_1d_1-\eta_2d_2]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}d_1^{r_1-1}d_2^{r_2}\exp[-\eta_{11}d_1-\eta_{22}d_2]\}dp_1dd_1dd_2$$

$$=C\sum{}^{*}\binom{n-r}{j_1}N_1\{\varepsilon_1(y)B(\delta_1+\delta_3+1,\delta_2+j_3+1)\frac{\Gamma(r_1+1)}{(\eta_1)^{(r_1+1)}}\frac{\Gamma(r_2)}{(\eta_2)^{(r_2)}}$$

$$+\varepsilon_2(y)B(\delta_1+\delta_3,\delta_2+j_3+2)\frac{\Gamma(r_1)}{(\eta_{11})^{(r_1)}}\frac{\Gamma(r_2+1)}{(\eta_{22})^{(r_2+1)}}\}$$

$$=C\sum{}^{*}N_1^*\left[\phi_1(y)+\phi_2(y)\right], \tag{4.36}$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{(k)}}(y\mid \underline{t}^*)dy=1$,

$$N_1^*=N_1\binom{n-r}{j_1}\left[\frac{\Gamma(r_1)\Gamma(r_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y)=\varepsilon_1(y)\frac{r_1(\delta_1+\delta_3)}{(\eta_1)^{(r_1+1)}(\eta_2)^{(r_2)}},$$

and

$$\phi_2(y)=\varepsilon_2(y)\frac{r_2(\delta_2+j_3+1)}{(\eta_{11})^{(r_1)}(\eta_{22})^{(r_2+1)}}.$$

Now, consider the Bayesian analysis when the prior distribution is conjugate prior. Suppose that $p_1, d_1$ and $d_2$ are independent such that $p_1$ follows $\text{beta}(b,d)$ and for $i=1,2,$ $d_i$ follows gamma $(\beta_i,\gamma_i)$.

A joint prior density is then given by

$$g(d_1,d_2,p_1)\propto p_1^{b-1}(1-p_1)^{d-1}d_1^{\beta_1-1}d_2^{\beta_2-1}\exp[-\gamma_1 d_1-\gamma_2 d_2],\ 0<p_1<1,\ d_i>0. \quad (4.37)$$

From (4.30) and (4.37), the joint posterior density of $d_1, d_2$ and $p_1$ given $\underline{t}^*$ is given by

$$\prod(d_1,d_2,p_1\mid\underline{t}^*)\propto\sum_{j_1=0}^{n-r}\binom{n-r}{j_1}p_1^{\delta_1}(1-p_1)^{\delta_2}d_1^{\omega_1-1}d_2^{\omega_2-1}\exp[-\psi_1 d_1-\psi_2 d_2]. \quad (4.38)$$

where

$\delta_1=n-r_2-j_1+b-1$ , $\delta_2=r_2+j_1+d-1$ and for $i=1,2,$ $\omega_i=r_i+\beta_i$, $\psi_i=A_{ij_1}+\gamma_i$

The product of (4.33) and (4.38), we obtain,

$$f_{Y_{(k)}}(y\mid d_1,d_2,p_1)\Pi(d_1,d_2,p_1\mid\underline{t}^*)\propto$$

$$\sum R_1 p_1^{\delta_3-1}(1-p_1)^{j_3}\exp[d_1(\delta_3-1)\log(1-c_1y)+d_2 j_3\log(1-c_2y)]f(y)$$

$$\sum_{j_1=0}^{n-r}\binom{n-r}{j_1}p_1^{\delta_1}(1-p_1)^{\delta_2}d_1^{\omega_1-1}d_2^{\omega_2-1}\exp[-\psi_1 d_1-\psi_2 d_2].$$

where

$$\sum=\sum_{j_2=0}^{k-1}\sum_{j_3=0}^{m-k+j_2},\quad R_1=(-1)^{j_2}\binom{k-1}{j_2}\binom{m-k+j_2}{j_3}\ \text{and}\ \ \delta_3=m-k+j_2-j_3+1.$$

which can be reduced to

$$\propto\sum{}^{*}\binom{n-r}{j_1}R_1\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}d_1^{\omega_1}d_2^{\omega_2-1}\exp[-\eta_1 d_1-\eta_2 d_2]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}d_1^{\omega_1-1}d_2^{\omega_2}\exp[-\eta_{11}d_1-\eta_{22}d_2]\}. \quad (4.39)$$

where

$\sum^{*} = \sum_{j_3=0}^{n-r}\sum$, for $i = 1,2$, $\varepsilon_i(y) = (c_i(1-c_i y)^{-1})$,

$\eta_1 = \psi_1 - \delta_3 \log(1-c_1 y)$,

$\eta_2 = \psi_2 - j_3 \log(1-c_2 y)$,

$\eta_{11} = \psi_1 - (\delta_3 - 1)\log(1-c_1 y)$,

$\eta_{22} = \psi_2 - (j_3 + 1)\log(1-c_2 y)$.

Then, the predictive density is obtained as

$$h_{Y_{(k)}}(y \mid \underline{t}^*) = C\sum{}^*\binom{n-r}{j_3}N_1\int_0^\infty\int_0^\infty\int_0^1\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}d_1^{\omega_1}d_2^{\omega_2-1}\exp[-\eta_1 d_1 - \eta_2 d_2]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}d_1^{\omega_1-1}d_2^{\omega_2}\exp[-\eta_{11}d_1 - \eta_{22}d_2]\}dp_1 dd_1 dd_2$$

$$= C\sum{}^*\binom{n-r}{j_3}N_1\{\varepsilon_1(y)B(\delta_1+\delta_3+1,\delta_2+j_3+1)\frac{\Gamma(\omega_1+1)}{(\eta_1)^{(\omega_1+1)}}\frac{\Gamma(\omega_2)}{(\eta_2)^{(\omega_2)}}$$

$$+\varepsilon_2(y)B(\delta_1+\delta_3,\delta_2+j_3+2)\frac{\Gamma(\omega_1)}{(\eta_{11})^{(\omega_1)}}\frac{\Gamma(\omega_2+1)}{(\eta_{22})^{(\omega_2+1)}}\}$$

$$= C\sum{}^*N_1^*\left[\phi_1(y)+\phi_2(y)\right], \tag{4.40}$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{(k)}}(y \mid \underline{t}^*)dy = 1$,

$$N_1^* = N_1\binom{n-r}{j_3}\left[\frac{\Gamma(\omega_1)\Gamma(\omega_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y) = \varepsilon_1(y)\frac{\omega_1(\delta_1+\delta_3)}{(\eta_1)^{(\omega_1+1)}(\eta_2)^{(\omega_2)}},$$

and

$$\phi_2(y) = \varepsilon_2(y)\frac{\omega_2(\delta_2+j_3+1)}{(\eta_{11})^{(\omega_1)}(\eta_{22})^{(\omega_2+1)}}.$$

### 4.6.2 The parameters $c_1, c_2, d_1, d_2$ and $p_1$ are unknown

In this section, we obtain Bayesian predictive distribution function when the prior distribution is Jeffrey's invariant as well as natural conjugate prior when $c_1, c_2, d_1, d_2$ and $p_1$ are unknown.

From (3.74), the likelihood function is given by,

$$L(c_1, c_2, d_1, d_2, p_1 \mid \underline{t}^*) \propto \sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{n-r_2-j_1} (1-p_1)^{r_2+j_1} c_1^{r_1} c_2^{r_2} d_1^{r_1} d_2^{r_2}$$

$$\exp[d_1 \sum_{j=1}^{r_1} \log(1-c_1 t_{1j})] \exp[d_2 \sum_{j=1}^{r_2} \log(1-c_2 t_{2j})] (1-c_1 T)^{d_1(n-r-j_1)} (1-c_2 T)^{d_2 j_1} .$$

$$\exp[-\left\{ \sum_{j=1}^{r_1} \log(1-c_1 t_{1j}) + \sum_{j=1}^{r_2} \log(1-c_2 t_{2j}) \right] \right\}. \qquad (4.41)$$

Rearranging (4.41), we have

$$L(c_1, c_2, d_1, d_2, p_1 \mid \underline{t}^*) \propto$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{n-r_2-j_1} (1-p_1)^{r_2+j_1} c_1^{r_1} c_2^{r_2} d_1^{r_1} d_2^{r_2} \exp\left[-A_{1j_1} d_1 - A_{2j_1} d_2\right] \exp\left[-B_1 - B_2\right], \qquad (4.42)$$

where, for $i = 1,2,$ $A_{ij_1}$ is given in (4.30) and $B_i = \sum_{j=1}^{r_i} \log(1-c_i t_{ij})\}$.

Now, consider that $p_1$, $c_1, c_2, d_1$ and $d_2$ are independent such that $p_1$ follows $U(0,1)$ and for $i = 1,2$, $c_i$ and $d_i$ follows jeffrey's invariant prior.

A joint prior density is then given by

$$g(c_1, c_2, d_1, d_2, p_1) \propto \frac{1}{c_1 c_2 d_1 d_2}, 0 < p_1 < 1,\ d_i > 0. \qquad (4.43)$$

The joint posterior density function of $p_1, c_1, c_2, d_1$ and $d_2$ is given by

$$\Pi(c_1, c_2, d_1, d_2, p_1 \mid \underline{t}^*) \propto$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1} (1-p_1)^{\delta_2} c_1^{r_1} c_2^{r_2} d_1^{r_1} d_2^{r_2} \exp\left[-A_{1j_1} d_1 - A_{2j_1} d_2\right] \exp\left[-B_1 - B_2\right]. \qquad (4.44)$$

where $\delta_1 = n - r_2 - j_1$ and $\delta_2 = r_2 + j_1$.

The density function of $k$-th order statistic $Y_{(k)}$ is given in (4.33).

The product of (4.33) and (4.44) gives

$$f_{Y_{i,k}}(y \mid c_1, c_2, d_1, d_2, p_1)\Pi(c_1, c_2, d_1, d_2, p_1 \mid \underline{t}^*) \propto$$

$$\sum K_1 p_1^{\delta_3 - 1}(1-p_1)^{j_3} \exp[d_1(\delta_3 - 1)\log(1-c_1 y) + d_2 j_3 \log(1-c_2 y)] f(y)$$

$$\sum_{j_1=0}^{n-r} \binom{n-r}{j_1} p_1^{\delta_1}(1-p_1)^{\delta_2} c_1^{r_1} c_2^{r_2} d_1^{r_1} d_2^{r_2} \exp\left[-A_{1j_1} d_1 - A_{2j_1} d_2\right] \exp\left[-B_1 - B_2\right]$$

where

$$\sum = \sum_{j_2=0}^{k-1} \sum_{j_3=0}^{m-k+j_2}, \quad K_1 = (-1)^{j_2} \binom{k-1}{j_2} \binom{m-k+j_2}{j_3} \text{ and } \delta_3 = m - k + j_2 - j_3 + 1.$$

$$\propto \sum{}^{*} \binom{n-r}{j_1} K_1 \exp[-B_1 - B_2]$$

$$\{\varepsilon_1(y) p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3} c_1^{r_1} c_2^{r_2-1} d_1^{r_1} d_2^{r_2-1} \exp\left[-\eta_1 d_1 - \eta_2 d_2\right]$$

$$+\varepsilon_2(y) p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1} c_1^{r_1-1} c_2^{r_2} d_1^{r_1-1} d_2^{r_2} \exp\left[-\eta_{11} d_1 - \eta_{22} d_2\right]\}. \quad (4.45)$$

where

$$\sum{}^{*} = \sum_{j_1=0}^{n-r} \sum, \quad \varepsilon_i(y) = (1 - c_i y)^{-1}), \; i = 1, 2,$$

$$\eta_1 = A_{1j_1} - \delta_3 \log(1 - c_1 y),$$

$$\eta_2 = A_{2j_1} - j_3 \log(1 - c_2 y),$$

$$\eta_{11} = A_{1j_1} - (\delta_3 - 1)\log(1 - c_1 y),$$

and

$$\eta_{22} = A_{2j_1} - (j_3 + 1)\log(1 - c_2 y).$$

Substituting (4.45) in (4.3), we have

$$h_{Y_{i,k}}(y \mid \underline{t}^*) = C \sum{}^{*} \binom{n-r}{j_1} K_1 \int_0^\infty \int_0^{C_2} \int_0^\infty \int_0^\infty \int_0^1 \exp[-B_1 - B_2]$$

$$\{\varepsilon_1(y) p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3} c_1^{r_1} c_2^{r_2-1} d_1^{r_1} d_2^{r_2-1} \exp\left[-\eta_1 d_1 - \eta_2 d_2\right]$$

$$+\varepsilon_2(y) p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1} c_1^{r_1-1} c_2^{r_2} d_1^{r_1-1} d_2^{r_2} \exp\left[-\eta_{11} d_1 - \eta_{22} d_2\right]\} dp_1 dd_1 dd_2 dc_1 dc_2.$$

$$= C\sum{}^{*}\binom{n-r}{j_1}K_1[\{B(\delta_1+\delta_3+1,\delta_2+j_3+1)\Gamma(r_1+1)\Gamma(r_2)$$

$$\int_0^\infty \frac{c_2^{r_2-1}\exp[-B_2]}{(\eta_2)^{(r_2)}}\int_0^{c_2}\frac{c_1^{r_1}\varepsilon_1(y)\exp[-B_1]}{(\eta_1)^{(r_1+1)}}dc_1dc_2 + B(\delta_1+\delta_3,\delta_2+j_3+2)\Gamma(r_1)\Gamma(r_2+1)$$

$$\int_0^\infty \frac{c_2^{r_2}\varepsilon_2(y)\exp[-B_2]}{(\eta_{22})^{(r_2+1)}}\int_0^{c_2}\frac{c_1^{r_1-1}\varepsilon_2(y)\exp[-B_1]}{(\eta_{11})^{(r_1)}}dc_1dc_2\,]$$

$$= C\sum{}^{*}K_1^{*}\left[\phi_1(y)+\phi_2(y)\right], \qquad (4.46)$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{(k)}}(y\mid \underline{t}^{*})dy = 1$,

$$K_1^{*} = K_1\binom{n-r}{j_1}\left[\frac{\Gamma(r_1)\Gamma(r_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y) = r_1(\delta_1+\delta_3)\int_0^\infty \frac{c_2^{r_2-1}\exp[-B_2]}{(\eta_2)^{(r_2)}}\int_0^{c_2}\frac{c_1^{r_1}\varepsilon_1(y)\exp[-B_1]}{(\eta_1)^{(r_1+1)}}dc_1dc_2,$$

and

$$\phi_2(y) = r_2(\delta_2+j_3+1)\int_0^\infty \frac{c_2^{r_2}\varepsilon_2(y)\exp[-B_2]}{(\eta_{22})^{(r_2+1)}}\int_0^{c_2}\frac{c_1^{r_1-1}\varepsilon_2(y)\exp[-B_1]}{(\eta_{11})^{(r_1)}}dc_1dc_2.$$

Consider the situation of conjugate prior. Now, assume that $p_1, c_1, c_2, d_1$ and $d_2$ are independent such that $p_1$ follows beta$(b,d)$, for $i=1,2$, $c_i$ follows gamma$(\mu_i,\nu_i)$ and for $i=1,2$, $d_i$ follows gamma$(\beta_i,\gamma_i)$.

A joint prior density is then given by

$$g(c_1,c_2,d_1,d_2,p_1) \propto p_1^{b-1}(1-p_1)^{d-1}c_1^{\mu_1-1}c_2^{\mu_2-1}d_1^{\beta_1-1}d_2^{\beta_2-1}\exp[-\nu_1c_1-\nu_2c_2]\exp[-\gamma_1d_1-\gamma_2d_2]$$

$$0<p_1<1,\ d_i>0,\ c_i>0 \qquad (4.47)$$

From (4.42) and (4.47), the joint posterior density function of $c_1, c_2, d_1, d_2$ and $p_1$ given $\underline{t}^{*}$ is given by

$$\Pi(c_1,c_2,d_1,d_2,p_1\mid \underline{t}^*)\propto$$

$$\sum_{j_1=0}^{n-r}\binom{n-r}{j_1}p_1^{\,n-r_2-j_1}(1-p_1)^{r_2+j_1}c_1^{r_1}c_2^{r_2}d_1^{r_1}d_2^{r_2}\exp\left[-A_{1j_1}d_1-A_{2j_1}d_2\right]\exp\left[-B_1-B_2\right]$$

$$p_1^{\,b-1}(1-p_1)^{d-1}c_1^{\mu_1-1}c_2^{\mu_2-1}d_1^{\beta_1-1}d_2^{\beta_2-1}\exp[-\nu_1c_1-\nu_2c_2]\exp[-\gamma_1d_1-\gamma_2d_2]$$

$$\propto\sum_{j_1=0}^{n-r}\binom{n-r}{j_1}p_1^{\delta_1}(1-p_1)^{\delta_2}c_1^{\sigma_1-1}c_2^{\sigma_2-1}d_1^{\omega_1-1}d_2^{\omega_2-1}\exp[-\psi_1d_1-\psi_2d_2]\exp[-\chi_1-\chi_2]. \qquad (4.48)$$

where

$\delta_1=n-r_2-j_1+b-1$, $\delta_2=r_2+j_1+d-1$

and for $i=1,2$, $\omega_i=r_i+\beta_i$, $\sigma_i=r_i+\mu_i$, $\psi_i=A_{ij_1}+\gamma_i$ and $\chi_i=B_i+\nu_ia_i$.

From (4.33) and (4.48), we get,

$$f_{Y_{(k)}}(y\mid c_1,c_2,d_1,d_2,p_1)\Pi(c_1,c_2,d_1,d_2,p_1\mid \underline{t}^*)\propto$$

$$\sum L_1p_1^{\delta_3-1}(1-p_1)^{j_3}\exp\left[d_1(\delta_3-1)\log(1-c_1y)+d_2j_3\log(1-c_2y)\right]f(y)$$

$$\sum_{j_1=0}^{n-r}\binom{n-r}{j_1}p_1^{\delta_1}(1-p_1)^{\delta_2}c_1^{\sigma_1-1}c_2^{\sigma_2-1}d_1^{\omega_1-1}d_2^{\omega_2-1}\exp[-\psi_1d_1-\psi_2d_2]\exp[-\chi_1-\chi_2]$$

where

$$\sum=\sum_{j_2=0}^{k-1}\sum_{j_3=0}^{m-k+j_2},\quad L_1=(-1)^{j_2}\binom{k-1}{j_2}\binom{m-k+j_2}{j_3}\ \text{and}\ \ \delta_3=m-k+j_2-j_3+1.$$

$$\propto\sum{}^*\binom{n-r}{j_1}L_1\exp[-\chi_1-\chi_2]\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}c_1^{\sigma_1}c_2^{\sigma_2-1}d_1^{\omega_1}d_2^{\omega_2-1}\exp\left[-\eta_1d_1-\eta_2d_2\right]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}c_1^{\sigma_1-1}c_2^{\sigma_2}d_1^{\omega_1-1}d_2^{\omega_2}\exp\left[-\eta_{11}d_1-\eta_{22}d_2\right]\}. \qquad (4.49)$$

where

$$\sum{}^*=\sum_{j_1=0}^{n-r}\sum,\quad \varepsilon_i(y)=(1-c_iy)^{-1},\quad i=1,2,$$

$\eta_1=\psi_1-\delta_3\log(1-c_1y)$,

$\eta_2=\psi_2-j_3\log(1-c_2y)$,

$\eta_{11}=\psi_1-(\delta_3-1)\log(1-c_1y)$,

and

$$\eta_{22} = \psi_2 - (j_3 + 1)\log(1 - c_2 y).$$

Substituting (4.49) in (4.3) , we have the predictive distribution as

$$h_{Y_{(k)}}(y \mid \underline{t}^*) = C\sum{}^*\binom{n-r}{j_1}L_1$$

$$\int_0^\infty\int_0^{c_2}\int_0^\infty\int_0^\infty\int_0^1 \exp[-\chi_1 - \chi_2]\{\varepsilon_1(y)p_1^{\delta_1+\delta_3}(1-p_1)^{\delta_2+j_3}c_1^{\sigma_1}c_2^{\sigma_2-1}d_1^{\omega_1}d_2^{\omega_2-1}\exp[-\eta_1 d_1 - \eta_2 d_2]$$

$$+\varepsilon_2(y)p_1^{\delta_1+\delta_3-1}(1-p_1)^{\delta_2+j_3+1}c_1^{\sigma_1-1}c_2^{\sigma_2}d_1^{\omega_1-1}d_2^{\omega_2}\exp[-\eta_{11}d_1 - \eta_{22}d_2]\}dp_1\,dd_1\,dd_2\,dc_1\,dc_2$$

$$= C\sum{}^*\binom{n-r}{j_1}L_1\{\varepsilon_1(y)B(\delta_1+\delta_3+1, \delta_2+j_3+1)\Gamma(\omega_1+1)\Gamma(\omega_2)$$

$$\int_0^\infty \frac{c_2^{\sigma_2-1}\exp[-\chi_2]}{(\eta_2)^{(\omega_2)}}\int_0^{c_2}\frac{c_1^{\sigma_1}\varepsilon_1(y)\exp[-\chi_1]}{(\eta_1)^{(\omega_1+1)}}dc_1\,dc_2 + B(\delta_1+\delta_3, \delta_2+j_3+2)\Gamma(\omega_1)\Gamma(\omega_2+1)$$

$$\int_0^\infty \frac{c_2^{\sigma_2}\varepsilon_2(y)\exp[-\chi_2]}{(\eta_{22})^{(\omega_2+1)}}\int_0^{c_2}\frac{c_1^{\sigma_1-1}\varepsilon_2(y)\exp[-\chi_1]}{(\eta_{11})^{(\omega_1)}}dc_1\,dc_2.$$

$$= C\sum{}^*L_1^*[\phi_1(y)+\phi_2(y)], \qquad (4.50)$$

where $C$ is the normalizing constant satisfying $\int_0^\infty h_{Y_{(k)}}(y \mid \underline{t}^*)dy = 1$,

$$L_1^* = L_1\binom{n-r}{j_1}\left[\frac{\Gamma(\omega_1)\Gamma(\omega_2)\Gamma(\delta_1+\delta_3)\Gamma(\delta_2+j_3+1)}{\Gamma(\delta_1+\delta_2+\delta_3+j_3+2)}\right],$$

$$\phi_1(y) = \omega_1(\delta_1+\delta_3)\int_0^\infty \frac{c_2^{\sigma_2-1}\exp[-\chi_2]}{(\eta_2)^{(\omega_2)}}\int_0^{c_2}\frac{c_1^{\sigma_1}\varepsilon_1(y)\exp[-\chi_1]}{(\eta_1)^{(\omega_1+1)}}dc_1\,dc_2,$$

and

$$\phi_2(y) = \omega_2(\delta_2+j_3+1)\int_0^\infty \frac{c_2^{\sigma_2}\varepsilon_2(y)\exp[-\chi_2]}{(\eta_{22})^{(\omega_2+1)}}\int_0^{c_2}\frac{c_1^{\sigma_1-1}\varepsilon_2(y)\exp[-\chi_1]}{(\eta_{11})^{(\omega_1)}}dc_1\,dc_2.$$

## 4.7 Data analysis

For the illustration of the role of finite mixture of beta distributions for predicting future observations, we consider a data given in Chapter 3, on survival times of cancer patients with advanced cancer of the bronchus or colon were treated with ascorbate. We, here, estimate the reliability function using predictive density function. We obtained the lower and upper prediction bounds for $Y_{(1)}$ and $Y_{(m)}$, the first and last failure times in a future sample of size $m = 15$. The failure times are assumed to follow finite mixture of beta distribution with density function given in (3.1). Table 4.2 provides the Bayes estimate of parameters when the data is censored at the point $T$ and $r_1$ and $r_2$ are the number of observations belonging to the respective subpopulations. Using our results in (4.4) and (4.46) with $\tau = 0.95$, the lower and upper 95% prediction bounds for $Y_{(1)}$, the first failure time, are 21 and 26 respectively. Whereas the 95% prediction bounds for $Y_{(15)}$, the last failure times are given by 542 and 551 respectively. The reliability $S^*(t)$ at $t = 250$ is obtained as .00147 by putting $k = 1$ in (4.46) and using (4.5).

**Table 4.2 Bayes estimate of parameters of survival times of cancer patients.**

| | | | | | |
|---|---|---|---|---|---|
| $T = 800$ $(r_1 = 16, r_2 = 15)$ | $c_1^* = .00017$ | $c_2^* = .0012$ | $d_1^* = 15.77$ | $d_2^* = 1.45$ | $p_1^* = .5516$ |

## 4.8 Conclusion.

In this chapter, the role of finite mixture of Pareto II and finite mixture of beta distributions on prediction of future observations are explored using Bayesian approach. The role of finite mixture of Pareto II distributions is illustrated using a real life data on time to death of two groups of leukaemia patients. We, also here, illustrated the use of finite mixture of beta distributions based on a real data on survival times of cancer patients

# Chapter 5

# ESTIMATION OF RELIABILITY UNDER STRESS-STRENGTH MODEL

## 5.1 Introduction

In reliability theory, stress- strength models describe the life of a component, which is having a random strength $X$ that is subjected to a random stress $Y$. The component fails at the instant the stress applied to it exceeds the strength. Thus $R = P(Y < X)$ measures the component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the ageing of concrete pressure vessels. Accordingly, in practical applications, the estimation of $R = P(Y < X)$ is important. The estimation of reliability $R$ when the distribution of $X$ and $Y$ have exponential, Weibull, lognormal etc. has already been done in the literature.

However, the estimation of $R$ when the distributions of $X$ and $Y$ have mixture form is not seen widely discussed in literature. The estimation of reliability based on finite mixture of inverse Gaussian distributions was studied by Akman et.al. (1999). Al-Hussaini et.al. (1997) considered finite mixture of lognormal components for the estimation of $R = P(Y < X)$. As mentioned in Chapter I, censored data are commonly encountered in reliability analysis. So it is worthwhile to estimate the reliability $R$ for type I and type II censored data. Accordingly in the present chapter, we consider the estimation problem of $R$ when the distribution of $X$ is finite mixture of Pareto II. We consider different distributions for $Y$ such as exponential, Pareto II and a finite mixture of Pareto II. We develop the maximum likelihood estimation of the system reliability for complete as well as censored samples.

The rest of the chapter is organized as follows. The definition of stress-strength reliability is given in Section 5.2. In Section 5.3, we derive the reliability, when the strength $X$ follows finite mixture of Pareto II, and the stress $Y$ takes different distributions such as exponential, Pareto II or finite mixture of Pareto II. The maximum likelihood estimation of parameters for the different models under type I and type II censoring is discussed in Section 5.4. In Section 5.5, we develop the maximum likelihood estimate of reliability in various situations. A simulation study is conducted to assess the performance of the estimator in Section 5.6. We apply the method to a real data in Section 5.7. In Sections 5.8, 5.9 and 5.10, we discuss the estimation of reliability, when the strength $X$ follows finite mixture of beta, and the stress $Y$ takes different distributions such as exponential, beta and finite mixture of beta. Section 5.11 provides the simulation studies to assess the performance of the estimates. Finally, Section 5.12 gives the conclusion of the chapter.

## 5.2 Definition

Let $X$ and $Y$ be two non-negative random variables having absolutely continuous distribution functions $F(x)$ and $G(y)$. Let $f(x)$ and $g(y)$ respectively denote the probability density functions of $X$ and $Y$. If $X$ and $Y$ are independent, then the stress-strength reliability $R$ is given by

$$R = P(Y < X)$$

$$= \int_0^{\infty}\int_0^{x} g(y) f(x)\,dy\,dx,$$

which is already mentioned in Chapter I.

## 5.3 Finite mixture of Pareto II distributions

In this section, we assume that the strength $X$ follows finite mixture of Pareto II distribution with p.d.f (2.2). Lindley and Singpurwalla (1986) pointed out that if the distribution of life times measured in a laboratory environment is exponential, then the distribution of life lengths in a different environment is Pareto II. Accordingly in this Section, we consider various distributions such as exponential and Pareto II for the stresses. We also consider the situations, where the stress has finite mixture of Pareto II distribution.

### 5.3.1 The stress $Y$ follows exponential distribution

When the stress $Y$ follows exponential with p.d.f

$$g(y)=\lambda e^{-\lambda y}; \qquad y>0, \lambda>0. \tag{5.1}$$

and if $X$ and $Y$ are independent we obtain the reliability $R$ as

$$R=\int_0^\infty\int_0^x(\lambda e^{-\lambda y})[p_1a_1b_1(1+a_1x)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dydx$$

$$=\int_0^\infty(1-e^{-\lambda x})[p_1a_1b_1(1+a_1x)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dx$$

$$=1-p_1a_1b_1\int_0^\infty e^{-\lambda x}(1+a_1x)^{-(b_1+1)}dx-(1-p_1)a_2b_2\int_0^\infty e^{-\lambda x}(1+a_2x)^{-(b_2+1)}dx$$

$$=1-p_1a_1b_1[(\lambda)^{b_1}e^{(\lambda/a_1)}\Gamma(-b_1,\frac{\lambda}{a_1})a_1^{-1-b_1}]-(1-p_1)a_2b_2[(\lambda)^{b_2}e^{(\lambda/a_2)}\Gamma(-b_2,\frac{\lambda}{a_2})a_2^{-1-b_2}]$$

$$=1-p_1b_1(\lambda/a_1)^{b_1}e^{(\lambda/a_1)}\Gamma(-b_1,\frac{\lambda}{a_1})-(1-p_1)b_2(\lambda/a_2)^{b_2}e^{(\lambda/a_2)}\Gamma(-b_2,\frac{\lambda}{a_2}), \tag{5.2}$$

where $\Gamma(-b_i,\frac{\lambda}{a_i})$, $i=1,2$ is the incomplete gamma function.

### 5.3.2 The stress $Y$ follows Pareto II distribution

Suppose that the stress $Y$ follows Pareto II with p.d.f.

$$g(y)=ab(1+ay)^{-(b+1)}; \qquad a>0, b>0. \tag{5.3}$$

When $X$ and $Y$ are independent, the reliability $R$ is obtained as

$$R=\int_0^\infty\int_0^x(ab(1+ay)^{-(b+1)})[p_1a_1b_1(1+a_1x)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dydx$$

$$= \int_0^\infty [1-(1+ax)^{-b}][p_1a_1b_1(1+a_1x)^{-(b_1+1)}+(1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dx$$

$$= 1 - p_1a_1b_1\int_0^\infty (1+ax)^{-b}(1+a_1x)^{-(b_1+1)}dx - (1-p_1)a_2b_2\int_0^\infty (1+ax)^{-b}(1+a_2x)^{-(b_2+1)}dx$$

$$= 1 - p_1b_1\left\{-\pi\csc[\pi b_1]\frac{\Gamma(b_1+b)}{\Gamma b\Gamma(1+b_1)}a_1^{-b_1}[1-\frac{a}{a_1}]^{-(b_1+b)}a^{b_1} + \frac{Hy2F_1[1,b,1-b_1,\frac{a}{a_1}]}{b_1}\right\}$$

$$-(1-p_1)b_2\left\{-\pi\csc[\pi b_2]\frac{\Gamma(b_2+b)}{\Gamma b\Gamma(1+b_2)}a_2^{-b_2}[1-\frac{a}{a_2}]^{-(b_2+b)}a^{b_2} + \frac{Hy2F_1[1,b,1-b_2,\frac{a}{a_2}]}{b_2}\right\},$$

(5.4)

where $Hy2F_1\left[1,b,1-b_i,\frac{a}{a_i}\right]$, $(i=1,2)$ is the hypergeometric function and $\csc[\pi b]$ is the cosecant of $\pi b$

### 5.3.3 The stress $Y$ follows finite mixture of Pareto II distributions

When the stress $Y$ follows finite mixture of Pareto II with p.d.f

$$g(y) = p_3a_3b_3(1+a_3y)^{-(b_3+1)} + (1-p_3)(1+a_4y)^{-(b_4+1)}; \quad a_3, a_4, b_3 \text{ and } b_4 > 0 \qquad (5.5)$$

and if $X$ and $Y$ are independent, we obtain the reliability $R$ as

$$R = \int_0^\infty\int_0^x [p_3a_3b_3(1+a_3y)^{-(b_3+1)} + (1-p_3)(1+a_4y)^{-(b_4+1)}]$$

$$[p_1a_1b_1(1+a_1x)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dydx$$

$$= \int_0^\infty [1-p_3(1+a_3x)^{-b_3} - (1-p_3)(1+a_4x)^{-b_4}][p_1a_1b_1(1+a_1x)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dx$$

$$= 1 - \int_0^{\infty} p_3(1+a_3x)^{-b_3}[p_1a_1b_1(1+a_1x)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dx$$

$$-\int_0^{\infty}(1-p_3)(1+a_4x)^{-b_4}[p_1a_1b_1(1+a_1x)^{-(b_1+1)} + (1-p_1)a_2b_2(1+a_2x)^{-(b_2+1)}]dx$$

$$= 1 - p_3p_1a_1b_1\int_0^{\infty}(1+a_3x)^{-b_3}(1+a_1x)^{-(b_1+1)}dx - p_3(1-p_1)a_2b_2\int_0^{\infty}(1+a_3x)^{-b_3}(1+a_2x)^{-(b_2+1)}dx$$

$$-(1-p_3)p_1a_1b_1\int_0^{\infty}(1+a_4x)^{-b_4}(1+a_1x)^{-(b_1+1)}dx - (1-p_3)(1-p_1)a_2b_2\int_0^{\infty}(1+a_4x)^{-b_4}(1+a_2x)^{-(b_2+1)}dx$$

$$= 1 - p_3p_1a_1b_1\left\{\pi\csc[\pi b_3]\frac{\Gamma(b_1+b_3)}{\Gamma b_3\Gamma(1+b_1)}a_1^{-1+b_3}[1-\frac{a_1}{a_3}]^{-(b_1+b_3)}a_3^{-b_3} + \frac{Hy2F_1[1,1+b_1,2-b_3,\frac{a_1}{a_3}]}{a_3(-1+b_3)}\right\}$$

$$-p_3(1-p_1)a_2b_2\left\{\pi\csc[\pi b_3]\frac{\Gamma(b_2+b_3)}{\Gamma b_3\Gamma(1+b_2)}a_2^{-1+b_3}[1-\frac{a_2}{a_3}]^{-(b_2+b_3)}a_3^{-b_3} + \frac{Hy2F_1[1,1+b_2,2-b_3,\frac{a_2}{a_3}]}{a_3(-1+b_3)}\right\}$$

$$-(1-p_3)p_1a_1b_1\left\{\pi\csc[\pi b_4]\frac{\Gamma(b_1+b_4)}{\Gamma b_4\Gamma(1+b_1)}a_1^{-1+b_4}[1-\frac{a_1}{a_4}]^{-(b_1+b_4)}a_4^{-b_4} + \frac{Hy2F_1[1,1+b_1,2-b_4,\frac{a_1}{a_4}]}{a_4(-1+b_4)}\right\}$$

$$-(1-p_3)(1-p_1)a_2b_2\left\{\pi\csc[\pi b_4]\frac{\Gamma(b_2+b_4)}{\Gamma b_4\Gamma(1+b_2)}a_2^{-1+b_4}[1-\frac{a_2}{a_4}]^{-(b_2+b_4)}a_4^{-b_4} + \frac{Hy2F_1[1,1+b_2,2-b_4,\frac{a_2}{a_4}]}{a_4(-1+b_4)}\right\}, \tag{5.6}$$

## 5.4 Estimation of parameters

In this section, we estimate the parameters of the models by the method of maximum likelihood, for the complete as well as censored samples.

### 5.4.1 Exponential distribution

Suppose the stress $Y$ is a non-negative random variable which follows an exponential distribution with probability density function (5.1).

**(a) Estimation of parameter based on type I censored samples.**

In this section, we develop maximum likelihood estimators of parameters under type I censoring. Suppose there is a random sample of $n$ units with lifetimes $Y_1, Y_2, ..., Y_n$ having distribution (5.1). We observe $Y_i$ only if $Y_i \le T$, where $T$ is a fixed censoring time. The observed data therefore consists of pairs $(\tilde{y}_i, \delta_i)$, $i = 1, 2, ..., n$ where $\tilde{y}_i = \min(Y_i, T)$ and

$\delta_i = 1$ if $Y_i \le T$

$= 0$ if $Y_i > T$.

The general form of the likelihood function for type I censored sample is given by (2.89) and for the exponential model (5.1), (2.89) becomes

$$L(\lambda \mid \underline{y}) = \prod_{i=1}^{n} (\lambda e^{-\lambda y_i})^{\delta_i} e^{-\lambda T(1-\delta_i)} \tag{5.7}$$

where $\underline{y} = (\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)$.

Maximization of log-likelihood function with respect to the parameter yields the estimate of $\lambda$ as

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{n} [\tilde{y}_i \delta_i + T(1-\delta_i)]}, \tag{5.8}$$

where $r = \sum_{i=1}^{n} \delta_i$ is the observed number of lifetimes.

**(b) Estimation of parameter based on type II censored samples.**

Suppose that only the first $r$ smallest observations $y_{(1)} \le y_{(2)} \le ... \le y_{(r)}$ are available in a total sample of size $n$. From (2.108), the likelihood function for type II censored sample is given by

$$L(\lambda \mid \underline{y}) = \frac{n!}{(n-r)!} [\prod_{i=1}^{r} (\lambda e^{-\lambda y_{(i)}} ] e^{-\lambda y_{(r)}(n-r)} \tag{5.9}$$

By maximizing the log-likelihood function with respect to the parameter, the maximum likelihood estimate of $\lambda$ can be obtained as

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} y_{(i)} + (n-r) y_{(r)}} . \tag{5.10}$$

**Remark 5.1**

When $r=n$, the results obtained under type II censoring becomes complete sample case.

### 5.4.2 Pareto II distribution

The stress $Y$ follows Pareto II distribution with probability density function given in (5.3).

**(a) Estimation of parameters based on type I censored samples.**

For the model (5.3), the likelihood function (2.89) based on a sample $(\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)$ becomes

$$L(a,b \mid \underline{y}) = \prod_{i=1}^{n} \left\{ (ab(1+ay_i)^{-(b+1)})^{\delta_i} (1+aT)^{-b(1-\delta_i)} \right\} \tag{5.11}$$

Maximization of log-likelihood function with respect to the model parameters yields the following likelihood equations,

$$\frac{r}{a} = (b+1) \sum_{i=1}^{n} \frac{\delta_i \tilde{y}_i}{(1+a\tilde{y}_i)} + b \frac{\sum_{i=1}^{n} (1-\delta_i) T}{(1+aT)} . \tag{5.12}$$

$$\frac{r}{b} = \sum_{i=1}^{n} \delta_i \log(1+a\tilde{y}_i) + \sum_{i=1}^{n} (1-\delta_i) \log(1+aT) . \tag{5.13}$$

where $r = \sum_{i=1}^{n} \delta_i$ is the observed number of lifetimes.

Solving the equations (5.12) and (5.13) numerically, we obtain M.L.E. s of $a$ and $b$

**(b) Estimation of parameters based on type II censored samples.**

Based on a sample of size $n$, we observe only $r$ smallest lifetimes $y_{(1)} \leq y_{(2)} \leq ... \leq y_{(r)}$, then, for the model (5.3), the likelihood function (2.108) becomes

$$L(a,b \mid \underline{y}) = \frac{n!}{(n-r)!}[\prod_{i=1}^{r} ab(1+ay_{(i)})^{-(b+1)}](1+ay_{(r)})^{-b(n-r)} \tag{5.14}$$

Maximization of log-likelihood function with respect to the model parameters yields the following likelihood equations,

$$\frac{r}{a} = (b+1)\sum_{i=1}^{r}\frac{y_{(i)}}{(1+ay_{(i)})} + \frac{b(n-r)y_{(r)}}{1+ay_{(r)}}, \tag{5.15}$$

$$\frac{r}{b} = \sum_{i=1}^{r}\log(1+ay_{(i)}) + (n-r)\log(1+ay_{(r)}). \tag{5.16}$$

The estimates of $a$ and $b$ can be obtained by solving the equations (5.15) and (5.16) by Newton- Raphson iterative method.

**Remark 5.2**

When $r=n$, the results obtained under type II censoring becomes complete sample case.

### 5.4.3 Finite mixture of Pareto II distribution

Estimation of parameters of finite mixture of Pareto II is already discussed in Chapter 2.

## 5.5 Estimation of stress-strength reliability

When the strength $X$ follows finite mixture of Pareto II with parameters $a_1, a_2, b_1, b_2$ and $p_1$ and the stress $Y$ follows exponential distribution with parameter $\lambda$, then the maximum likelihood estimate of $R$ is given as

$$\hat{R} = 1 - \hat{p}_1\hat{b}_1(\hat{\lambda}/\hat{a}_1)^{\hat{b}_1} e^{(\hat{\lambda}/\hat{a}_1)}\Gamma(-\hat{b}_1, \frac{\hat{\lambda}}{\hat{a}_1}) - (1-\hat{p}_1)\hat{b}_2(\hat{\lambda}/\hat{a}_2)^{\hat{b}_2} e^{(\hat{\lambda}/\hat{a}_2)}\Gamma(-\hat{b}_2, \frac{\hat{\lambda}}{\hat{a}_2}).$$

Then from appendix B of Lawless (2003), the asymptotic variance of $\hat{R}$ is given by

$$Var\left(\hat{R}\right)=\left(\frac{\partial R}{\partial a_1}\right)^2 Var(\hat{a}_1)+\left(\frac{\partial R}{\partial a_2}\right)^2 Var(\hat{a}_2)+...+\left(\frac{\partial R}{\partial p_1}\right)^2 Var(\hat{p}_1)$$
$$+\frac{\partial R}{\partial a_1}\frac{\partial R}{\partial a_2}Cov(\hat{a}_1,\hat{a}_2)+\frac{\partial R}{\partial a_1}\frac{\partial R}{\partial b_1}Cov(\hat{a}_1,\hat{b}_1)+...+\frac{\partial R}{\partial b_2}\frac{\partial R}{\partial p_1}Cov(\hat{b}_2,\hat{p}_1).$$

where variances and covariances are directly obtained from the inverse of the matrix $I(a_1,a_2,b_1,b_2,p_1)$ with

$$I(a_1,a_2,b_1,b_2,p_1)=\begin{pmatrix} -\frac{\partial^2\log R}{\partial a_1^2} & -\frac{\partial^2\log R}{\partial a_1\partial a_2} & -\frac{\partial^2\log R}{\partial a_1\partial b_1} & -\frac{\partial^2\log R}{\partial a_1\partial b_2} & -\frac{\partial^2\log R}{\partial a_1\partial p_1} \\ & & & & \\ \cdot & & & & \\ -\frac{\partial^2\log R}{\partial p_1\partial a_1} & -\frac{\partial^2\log R}{\partial p_1\partial a_2} & -\frac{\partial^2\log R}{\partial p_1\partial b_1} & -\frac{\partial^2\log R}{\partial p_1\partial b_2} & -\frac{\partial^2\log R}{\partial p_1^2} \end{pmatrix}$$

When the strength $X$ follows finite mixture of Pareto II with parameters $a_1,a_2,b_1,b_2$ and $p_1$ and the stress $Y$ follows Pareto II distribution with parameters $a$ and $b$, then the maximum likelihood estimate of $R$ is obtained as

$$\hat{R}=1-\hat{p}_1\hat{b}_1\left\{-\pi\csc[\pi\hat{b}_1]\frac{\Gamma(\hat{b}_1+\hat{b})}{\Gamma\hat{b}\Gamma(1+\hat{b}_1)}\hat{a}_1^{-\hat{b}_1}[1-\frac{\hat{a}}{\hat{a}_1}]^{-(\hat{b}_1+\hat{b})}\hat{a}^{\hat{b}_1}+\frac{Hy2F_1[1,\hat{b},1-\hat{b}_1,\frac{\hat{a}}{\hat{a}_1}]}{\hat{b}_1}\right\}$$

$$-(1-\hat{p}_1)\hat{b}_2\left\{-\pi\csc[\pi\hat{b}_2]\frac{\Gamma(\hat{b}_2+\hat{b})}{\Gamma\hat{b}\Gamma(1+\hat{b}_2)}\hat{a}_2^{-\hat{b}_2}[1-\frac{\hat{a}}{\hat{a}_2}]^{-(\hat{b}_2+\hat{b})}\hat{a}^{\hat{b}_2}+\frac{Hy2F_1[1,\hat{b},1-\hat{b}_2,\frac{\hat{a}}{\hat{a}_2}]}{\hat{b}_2}\right\}.$$

When the strength $X$ follows finite mixture of Pareto II with parameters $a_1,a_2,b_1,b_2$ and $p_1$ and the stress $Y$ follows finite mixture of Pareto II distribution with parameters $a_3,a_4,b_3,b_4$ and $p_3$, then the maximum likelihood estimate of $R$ is given as

$$\hat{R} = 1 - \hat{p}_3\hat{p}_1\hat{a}_1\hat{b}_1 \left\{ \pi \csc[\pi \hat{b}_3] \frac{\Gamma(\hat{b}_1 + \hat{b}_3)}{\Gamma \hat{b}_3 \Gamma(1+\hat{b}_1)} \hat{a}_1^{-1+\hat{b}_3} [1 - \frac{\hat{a}_1}{\hat{a}_3}]^{-(\hat{b}_1+\hat{b}_3)} \hat{a}_3^{-\hat{b}_3} + \frac{Hy2F_1[1, 1+\hat{b}_1, 2-\hat{b}_3, \frac{\hat{a}_1}{\hat{a}_3}]}{\hat{a}_3(-1+\hat{b}_3)} \right\}$$

$$-\hat{p}_3(1-\hat{p}_1)\hat{a}_2\hat{b}_2 \left\{ \pi \csc[\pi \hat{b}_3] \frac{\Gamma(\hat{b}_2 + \hat{b}_3)}{\Gamma \hat{b}_3 \Gamma(1+\hat{b}_2)} \hat{a}_2^{-1+\hat{b}_3} [1 - \frac{\hat{a}_2}{\hat{a}_3}]^{-(\hat{b}_2+\hat{b}_3)} \hat{a}_3^{-\hat{b}_3} + \frac{Hy2F_1[1, 1+\hat{b}_2, 2-\hat{b}_3, \frac{\hat{a}_2}{\hat{a}_3}]}{\hat{a}_3(-1+\hat{b}_3)} \right\}$$

$$-(1-\hat{p}_3)\hat{p}_1\hat{a}_1\hat{b}_1 \left\{ \pi \csc[\pi \hat{b}_4] \frac{\Gamma(\hat{b}_1 + \hat{b}_4)}{\Gamma \hat{b}_4 \Gamma(1+\hat{b}_1)} \hat{a}_1^{-1+\hat{b}_4} [1 - \frac{\hat{a}_1}{\hat{a}_4}]^{-(\hat{b}_1+\hat{b}_4)} \hat{a}_4^{\hat{b}_4} + \frac{Hy2F_1[1, 1+\hat{b}_1, 2-\hat{b}_4, \frac{\hat{a}_1}{\hat{a}_4}]}{\hat{a}_4(-1+\hat{b}_4)} \right\}$$

$$-(1-\hat{p}_3)(1-\hat{p}_1)\hat{a}_2\hat{b}_2 \left\{ \pi \csc[\pi \hat{b}_4] \frac{\Gamma(\hat{b}_2 + \hat{b}_4)}{\Gamma \hat{b}_4 \Gamma(1+\hat{b}_2)} \hat{a}_2^{-1+\hat{b}_4} [1 - \frac{\hat{a}_2}{\hat{a}_4}]^{-(\hat{b}_2+\hat{b}_4)} \hat{a}_4^{-\hat{b}_4} + \frac{Hy2F_1[1, 1+\hat{b}_2, 2-\hat{b}_4, \frac{\hat{a}_2}{\hat{a}_4}]}{\hat{a}_4(-1+\hat{b}_4)} \right\}.$$

## 5.6 Simulation Study

In this section, we carry out simulation studies to observe the finite sample properties of the estimates and their robustness.

### (a) Exponential distribution

We generate observations from exponential distribution with parameter $\lambda = 0.7$. Table 5.1 and 5.2 provides maximum likelihood estimates of parameters under complete as well as censored situations with various combinations of $n$ (sample size), $T$ (censoring time) and $r$ (number of failures). The variances of the estimates are given in brackets.

### (b) Pareto II distribution

We generate samples from Pareto II distribution with parameters $a = 1, b = 4$. The estimates under complete as well as type I and type II censoring set up are given in Tables 5.3 and 5.4 for various combinations of $n$, $T$ and $r$

**(c) Finite mixture of Pareto II distribution**

For the type I censoring, in which observations belonging to each subpopulation are known, we generate two sets of samples, one from a Pareto II distribution with parameters $a_1$ and $b_1$ and the second from a Pareto II distribution with parameters $a_2$ and $b_2$.

For the type I censoring (in which observations belonging to each subpopulation are unknown) and type II censoring, first we generate two sets of samples, one from a Pareto II distribution with parameters $a_1$ and $b_1$ and the second from a Pareto II distribution with parameters $a_2$ and $b_2$ and then, we obtained finite mixture of Pareto II distributions using Bernoulli probability $p_1 (0 < p_1 < 1)$.

The estimates of parameters by the method of maximum likelihood under type I censoring and type II censoring for the set of parameters $a_1 = .03, a_2 = 3, b_1 = 1, b_2 = 2$ and $p_1 = 0.5$ with various combinations of $n$ (sample size) and $T$ (censoring time) are given in Chapter 2. The result obtained under type II censoring is extended to the complete sample by taking $r=n$.

Tables 5.5 to 5.7 give the maximum likelihood estimates for another set of parameters $a_3 = .005, a_4 = 2, b_3 = 3, b_4 = 5$ and $p_3 = 0.6$ under complete as well as censoring for same combinations of $n, T$ and $r$, where $p_3$ is the mixing Bernoulli probability $(0 < p_3 < 1)$.

Stress strength reliability when strength is finite mixture of Pareto II and stress is exponential is given in Table 5.8. Table 5.9 provides the Stress strength reliability when strength is finite mixture of Pareto II and stress is Pareto II. When stress and strength are finite mixture of Pareto II, the reliability is given in Table 5.10. The variances of the estimates are given in brackets.

We use the software 'Mathematica' to evaluate the integrals numerically.

**Table 5.1 Maximum likelihood estimates of parameters under type I censoring for $\lambda = 0.7$ (exponential)**

| Censoring time | n=30 | n=50 | n=100 |
|---|---|---|---|
| T=30 | $\hat{\lambda}$=0.670027 (0.001320) | $\hat{\lambda}$=0.6827 (0.000271) | $\hat{\lambda}$=0.70022 (0.000050) |
| T=100 | $\hat{\lambda}$=0.71027 (0.000871) | $\hat{\lambda}$=0.6987 (0.000154) | $\hat{\lambda}$=0.70001 (0.000011) |

**Table 5.2 Maximum likelihood estimates of parameters under type II censoring for $\lambda = 0.7$ (exponential)**

| n=30 | | n=50 | | n=100 | |
|---|---|---|---|---|---|
| r=10 | r=30 | r=30 | r=50 | r=60 | r=100 |
| $\hat{\lambda}$=0.68175 (0.002764) | $\hat{\lambda}$=0.68886 (0.001159) | $\hat{\lambda}$=0.68892 (0.001042) | $\hat{\lambda}$=0.7019 (0.000987) | $\hat{\lambda}$=0.68899 (0.000125) | $\hat{\lambda}$=.7012 (.0000654) |

**Table 5.3 Maximum likelihood estimates of parameters under type I censoring for $a = 1, b = 4$ (Pareto II)**

| Censoring time | n=30 | n=50 | n=100 |
|---|---|---|---|
| T=30 | $\hat{a}$=1.13372 (2.7E-2) | $\hat{a}$=1.05943 (1.08E-4) | $\hat{a}$=1.00058 (1.01E-6) |
| | $\hat{b}$=3.78796 (.00821) | $\hat{b}$=3.87436 (.000275) | $\hat{b}$=4.01902 (2.7E-7) |
| T=100 | $\hat{a}$=1.1164 (1.2E-2) | $\hat{a}$=1.02179 (1.01E-4) | $\hat{a}$=1.00012 (1.01E-7) |
| | $\hat{b}$=3.864 (3.4E-3) | $\hat{b}$=3.9019 (2.9E-5) | $\hat{b}$=4.0017 (1.3E-7) |

**Table 5.4 Maximum likelihood estimates of parameters under type II censoring for $a = 1, b = 4$ (Pareto II)**

| n=30 | | n=50 | | n=100 | |
|---|---|---|---|---|---|
| r=10 | r=30 | r=30 | r=50 | r=60 | r=100 |
| $\hat{a}$=0.9764 (.000453) | $\hat{a}$=0.9889 (0.000124) | $\hat{a}$=0.9899 (2.6E-4) | $\hat{a}$=1.015 (2.56E-5) | $\hat{a}$=1.011 (1.2E-5) | $\hat{a}$=1.007 (1.1E-6) |
| $\hat{b}$=3.8643 (.0001975) | $\hat{b}$=3.9117 (.000124) | $\hat{b}$=3.9547 (1.8E-4) | $\hat{b}$=4.0743 (1.01E-5) | $\hat{b}$=3.998 (1.1E-5) | $\hat{b}$=4.0012 (1.02E-6) |

**Table 5.5 Maximum likelihood estimates of parameters under type I censoring for $a_3 = .005, a_4 = 2, b_3 = 3, b_4 = 5$ and $p_3 = 0.6$ in which observations belonging to each subpopulation are known.**

| Censoring time | n=30 | n=50 | n=100 |
|---|---|---|---|
| T=30 | $\hat{a}_3$=.0051004<br>(9.08E-05) | $\hat{a}_3$=.00491342<br>(5.53E-05) | $\hat{a}_3$=.00497522<br>(1.13E-05) |
| | $\hat{a}_4$=2.1098<br>(9.44E-05) | $\hat{a}_4$=2.02653<br>(6.81E-05) | $\hat{a}_4$=2.02132<br>(1.49E-05) |
| | $\hat{b}_3$=2.829962<br>(.000386) | $\hat{b}_3$=2.918252<br>(.000124) | $\hat{b}_3$=3.029122<br>(1.1E-4) |
| | $\hat{b}_4$=5.0976<br>(.0437) | $\hat{b}_4$=4.9973<br>(.00631) | $\hat{b}_4$=4.99843<br>(.000289) |
| | $\hat{p}_3$=.608161<br>(5.9E-11) | $\hat{p}_3$=.6008275<br>(9.7E-12) | $\hat{p}_3$=.6001367<br>(4.97E-12) |
| T=100 | $\hat{a}_3$=.0051106<br>(2.37E-05) | $\hat{a}_3$=.0051058<br>(1.892E-05) | $\hat{a}_3$=.0050117<br>(1.882E-05) |
| | $\hat{a}_4$=1.9124<br>(8.83E-06) | $\hat{a}_4$=1.975<br>(5.72E-06) | $\hat{a}_4$=2.0232<br>(4.98E-06) |
| | $\hat{b}_3$=3.1096<br>(.040889) | $\hat{b}_3$=3.0864<br>(.040573) | $\hat{b}_3$=3.00031<br>(.007085) |
| | $\hat{b}_4$=5.1087<br>(.00963) | $\hat{b}_4$=5.0965<br>(.00149) | $\hat{b}_4$=5.0132<br>(.000397) |
| | $\hat{p}_3$=.59998<br>(5.18E-11) | $\hat{p}_3$=.59999<br>(8.57E-12) | $\hat{p}_3$=.6002<br>(3.18E-12) |

**Table 5.6 Maximum likelihood estimates of parameters under type I censoring for $a_3 = .005, a_4 = 2, b_3 = 3, b_4 = 5$ and $p_3 = 0.6$ in which observations belonging to each subpopulation are unknown.**

| Censoring time | n=30 | n=50 | n=100 |
|---|---|---|---|
| T=30 | $\hat{a}_3$=.00512 (3.7E-03) | $\hat{a}_3$=.005098 (2.4E-03) | $\hat{a}_3$=.005087 (1.5E-03) |
| | $\hat{a}_4$=2.0912 (2.4E-03) | $\hat{a}_4$=1.92786 (1.9E-03) | $\hat{a}_4$=2.01764 (1.8E-03) |
| | $\hat{b}_3$=3.101 (2.7E-03) | $\hat{b}_3$=3.107 (2.1E-03) | $\hat{b}_3$=3.028 (1.5E-03) |
| | $\hat{b}_4$=5.0875 (5.4E-03) | $\hat{b}_4$=5.07121 (3.7E-03) | $\hat{b}_4$=5.019 (2.2E-03) |
| | $\hat{p}_3$=.6009 (4.2E-12) | $\hat{p}_3$=.59989 (3.7E-12) | $\hat{p}_3$=.6002 (1.8E-12) |
| T=100 | $\hat{a}_3$=.00509 (3.2E-03) | $\hat{a}_3$=.005054 (1.98E-03) | $\hat{a}_3$=.00508 (1.32E-03) |
| | $\hat{a}_4$=1.9469 (2.6E-03) | $\hat{a}_4$=1.98 (1.7E-03) | $\hat{a}_4$=2.0031 (1.3E-03) |
| | $\hat{b}_3$=3.0942 (2.1E-03) | $\hat{b}_3$=3.023 (1.1E-03) | $\hat{b}_3$=2.9976 (2.7E-04) |
| | $\hat{b}_4$=5.1087 (4.1E-03) | $\hat{b}_4$=5.0965 (2.1E-03) | $\hat{b}_4$=4.9832 (1.08E-03) |
| | $\hat{p}_3$=.59917 (7.6E-14) | $\hat{p}_3$=.60034 (3.9E-14) | $\hat{p}_3$=.60001 (2.2E-14) |

**Table 5.7 Maximum likelihood estimates of parameters under type II censoring for $a_3 = .005, a_4 = 2, b_3 = 3, b_4 = 5$ and $p_3 = 0.6$**

| n=30 | | n=50 | | n=100 | |
|---|---|---|---|---|---|
| r=10 | r=30 | r=30 | r=50 | r=60 | r=100 |
| $\hat{a}_3$=.00511 (2.9E-4) | $\hat{a}_3$=.00504 (1.22E-5) | $\hat{a}_3$=.00494 (1.7E-5) | $\hat{a}_3$=.00498 (1.01E-5) | $\hat{a}_3$=.00508 (2.4E-6) | $\hat{a}_3$=.00502 (1.98E-6) |
| $\hat{a}_4$=1.9537 (.0711) | $\hat{a}_4$=2.087 (.0532) | $\hat{a}_4$=1.987 (.0532) | $\hat{a}_4$=1.995 (.02187) | $\hat{a}_4$=2.05 (.00721) | $\hat{a}_4$=1.999 (.00210) |
| $\hat{b}_3$=3.10 (.0057) | $\hat{b}_3$=3.0987 (.0032) | $\hat{b}_3$=2.9897 (.00654) | $\hat{b}_3$=3.0043 (.00191) | $\hat{b}_3$=3.014 (.0007) | $\hat{b}_3$=3.001 (.00012) |
| $\hat{b}_4$=5.075 (.1001) | $\hat{b}_4$=5.045 (.0978) | $\hat{b}_4$=4.9855 (.0325) | $\hat{b}_4$=4.997 (.00951) | $\hat{b}_4$=4.998 (.007821) | $\hat{b}_4$=5.001 (.00019) |
| $\hat{p}_3$=.6103 (6.7E-12) | $\hat{p}_3$=.5987 (1.09E-13) | $\hat{p}_3$=.5988 (3.1E-15) | $\hat{p}_3$=.5999 (1.1E-15) | $\hat{p}_3$=.6002 (7.2E-18) | $\hat{p}_3$=.6001 (2.0E-19) |

**Table 5.8 Stress-strength reliability when strength is finite mixture of Pareto II and stress is exponential**

| Sample size | n=30 | n=50 | n=100 |
|---|---|---|---|
| Stress-strength reliability $R$ | 0.553244 | 0.555229 | 0.559292 |

**Table 5.9 Stress-strength reliability when strength is finite mixture of Pareto II and stress is Pareto II**

| Sample size | n=30 | n=50 | n=100 |
|---|---|---|---|
| Stress-strength reliability $R$ | 0.79128 | 0.7991 | 0.80127 |

**Table 5.10 Stress-strength reliability when strength is finite mixture of Pareto II and stress is finite mixture of Pareto II**

| Sample size | n=30 | n=50 | n=100 |
|---|---|---|---|
| Stress-strength reliability $R$ | 0.6759 | 0.69131 | 0.71297 |

## 5.7 Data analysis.

For the illustration of the method, we consider a stress-strength data which is given in Table 5.11. (see Rao (1992)). The stresses induced in power transmitting shafts and the experimental values of the strength of the shaft material are given as stress and strength values. The values are given in k(lb/in2). We then estimate the parameters using the method of maximum likelihood. We used the Kolmogorov Smirnov statistic to test the goodness of fit. From the analysis, it concludes that model (2.2) with $a_1 = 0.59, a_2 = 3.35\times10^{12}, b_1 = b_2 = b = .05$ and $p_1 = 0.5$ is a good model for strength data and the model (5.1) with $\lambda = .87$ is a plausible model for stress data. Then we estimate the reliability of the shafts as 0.213

**Table 5.11 Stress-strength data**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| strength | 31.8 | 25.2 | 28.3 | 29.6 | 30.1 | 32.7 | 26.5 | 28.9 | 30.8 | 33.5 |
| stress | 23.4 | 28.6 | 23.9 | 25.2 | 25.5 | 26.7 | 20.3 | 27.7 | 19.2 | 22.8 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| strength | 27.4 | 29.1 | 30.7 | 32.2 | 29.9 | 30.6 | 31.7 | 31.3 | 33.9 | 34.3 |
| stress | 23.1 | 24.6 | 25.9 | 26.8 | 27.5 | 21.7 | 24.1 | 24.9 | 24.8 | 22.6 |

## 5.8 Finite mixture of beta distribution

In this section, we consider the strength $X$ as finite mixture of beta distribution with p.d.f (3.1). As in the previous case, we consider three different distributions for the stress $Y$.

### 5.8.1 The stress $Y$ follows exponential distribution

The stress $Y$ follows exponential with probability density function given in (5.1) and assumes that $X$ and $Y$ are independent. Then the reliability $R$ is obtained as,

$$R = \int_0^{1/c_1}\int_0^{x}(\lambda e^{-\lambda y})[p_1c_1d_1(1-c_1x)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dydx$$

$$= \int_0^{1/c_1}(1-e^{-\lambda x})[p_1c_1d_1(1-c_1x)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dx$$

$$= 1 - p_1c_1d_1\int_0^{1/c_1} e^{-\lambda x}(1-c_1x)^{(d_1-1)} + (1-p_1)c_2d_2\int_0^{1/c_2} e^{-\lambda x}(1-c_2x)^{(d_2-1)}dx$$

$$= 1 - p_1c_1d_1\left[(-1)^{-d_1} e^{-\lambda/c_1}\lambda^{-d_1}\left(\Gamma d_1 - \Gamma\left(d_1, -\lambda/c_1\right)\right)c_1^{-1+d_1}\right]$$

$$-(1-p_1)c_2d_2\left[(-1)^{-d_2} e^{-\lambda/c_2}\lambda^{-d_2}\left(\Gamma d_2 - \Gamma\left(d_2, -\lambda/c_2\right)\right)c_2^{-1+d_2}\right]$$

$$= 1 - p_1d_1\left[(-1)^{-d_1} e^{-\lambda/c_1}\left(c_1/\lambda\right)^{d_1}\left(\Gamma d_1 - \Gamma\left(d_1, -\lambda/c_1\right)\right)\right]$$

$$-(1-p_1)d_2\left[(-1)^{-d_2} e^{-\lambda/c_2}\left(c_2/\lambda\right)^{d_2}\left(\Gamma d_2 - \Gamma\left(d_2, -\lambda/c_2\right)\right)\right]. \qquad (5.17)$$

### 5.8.2 The stress $Y$ follows beta distribution

The stress $Y$ follows beta with probability density function

$$f(y) = cd(1-cy)^{(d-1)} \quad ; c > 0, d > 0, 0 < y < \frac{1}{c} \qquad (5.18)$$

If $X$ and $Y$ are independent, the reliability $R$ is given by

$$R = \int_0^{1/c_1}\int_0^{x}(cd(1-cy)^{(d-1)})[p_1c_1d_1(1-c_1x)^{(d_1-1)} + (1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dydx$$

$$= \int_0^{1/c_1} [1-(1-cx)^d][p_1c_1d_1(1-c_1x)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dx$$

$$=1-p_1c_1d_1\int_0^{1/c_1}(1-cx)^d(1-c_1x)^{(d_1-1)}dx-(1-p_1)c_2d_2\int_0^{1/c_2}(1-cx)^d(1-c_2x)^{(d_2-1)}dx$$

$$=1-p_1\left[Hy2F1\left[1,-d,1+d_1,\frac{c}{c_1}\right]\right]-(1-p_1)\left[Hy2F1\left[1,-d,1+d_2,\frac{c}{c_2}\right]\right] \quad (5.19)$$

### 5.8.3 The stress $Y$ follows finite mixture of beta distribution

The stress $Y$ follows finite mixture of beta with probability density function

$$f(y)=p_3c_3d_3(1-c_3y)^{(d_3-1)}+(1-p_3)c_4d_4(1-c_4y)^{(d_4-1)};$$

$$c_3,c_4,d_3,d_4>0,0<p_3<1,0<y<\frac{1}{c_3}. \quad (5.20)$$

When $X$ and $Y$ are independent, the reliability is given by

$$R=\int_0^{1/c_1}\int_0^{x}[p_3c_3d_3(1-c_3y)^{(d_3-1)}+(1-p_3)c_4d_4(1-c_4y)^{(d_4-1)}]$$

$$[p_1c_1d_1(1-c_1x)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dydx$$

$$=\int_0^{1/c_1}[1-p_3(1-c_3x)^{d_3}-(1-p_3)(1-c_4x)^{d_4}][p_1c_1d_1(1-c_1x)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dx$$

$$=1-\int_0^{1/c_1}p_3(1-c_3x)^{d_3}[p_1c_1d_1(1-c_1x)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dx$$

$$-\int_0^{1/c_1}(1-p_3)(1-c_4x)^{d_4}[p_1c_1d_1(1-c_1x)^{(d_1-1)}+(1-p_1)c_2d_2(1-c_2x)^{(d_2-1)}]dx$$

$$=1-p_3p_1c_1d_1\int_0^{1/c_1}(1-c_3x)^{d_3}(1-c_1x)^{(d_1-1)}dx-p_3(1-p_1)c_2d_2\int_0^{1/c_2}(1-c_3x)^{d_3}(1-c_2x)^{(d_2-1)}dx$$

$$-(1-p_3)p_1c_1d_1\int_0^{1/c_1}(1-c_4x)^{d_4}(1-c_1x)^{(d_1-1)}dx-(1-p_3)(1-p_1)c_2d_2\int_0^{1/c_2}(1-c_4x)^{d_4}(1-c_2x)^{(d_2-1)}dx$$

$$= 1 - p_3 p_1 d_1 \left[ \Gamma d_1 Hy2F1R\left[1, -d_3, 1+d_1, \frac{c_3}{c_1}\right]\right] - p_3(1-p_1)d_2 \left[\Gamma d_2 Hy2F1R\left[1, -d_3, 1+d_2, \frac{c_3}{c_2}\right]\right]$$

$$-(1-p_3)p_1 d_1 \left[\Gamma d_1 Hy2F1R\left[1, -d_4, 1+d_1, \frac{c_4}{c_1}\right]\right] - (1-p_3)(1-p_1)d_2 \left[\Gamma d_2 Hy2F1R\left[1, -d_4, 1+d_2, \frac{c_4}{c_2}\right]\right].$$

(5.21)

where $Hy2F1\text{R}\left[1, -d_j, 1+d_i, \frac{c_j}{c_i}\right]$, $i = 1,2$ and $j = 3,4$ is the regularized hypergeometric function.

## 5.9 Estimation of parameters

In this section, we discuss the estimation of parameters of the different models by the method of maximum likelihood for the complete as well as censored situations.

### 5.9.1 Exponential distribution

Estimation of parameters in this case is mentioned in Section 5.4.1.

### 5.9.2 Beta distribution

The stress $Y$ follows beta distribution with probability density function given in (5.18).

#### (a) Estimation of parameters based on type I censored samples.

In this section, we develop maximum likelihood estimators of parameters under type I censoring. Suppose there is a random sample of n units with lifetimes $Y_1, Y_2, ..., Y_n$ from the model (5.18) and associated with each unit, a fixed censoring time $T$. We observe $Y_i$ only if $Y_i \le T$ and the data therefore consists of pairs $(\tilde{y}_i, \delta_i)$, $i = 1,2,...,n$ and $\tilde{y}_i = \min(Y_i, T)$

where $\delta_i = 1$ if $Y_i \le T$ and $\delta_i = 0$ if $Y_i > T$.

The general form of the likelihood function for type I censored sample is given by (2.89) and for the beta model (5.18), (2.89) becomes

$$L(c,d \mid \underline{y}) = \prod_{i=1}^{n}\left\{(cd(1-c\tilde{y}_i)^{d-1})^{\delta_i}(1-cT)^{d(1-\delta_i)}\right\}. \tag{5.22}$$

where $\underline{y} = (y_1, y_2, \ldots, y_n)$.

Maximization of log likelihood function with respect to the model parameters yields the following differential equations,

$$\frac{r}{c} = d\frac{\sum_{i=1}^{n}(1-\delta_i)T}{(1-cT)} + (d-1)\sum_{i=1}^{n}\frac{\delta_i\tilde{y}_i}{(1-c\tilde{y}_i)}, \tag{5.23}$$

$$\frac{r}{d} = \sum_{i=1}^{n}\delta_i\log(1-c\tilde{y}_i) + \sum_{i=1}^{n}(1-\delta_i)\log(1-cT) \ . \tag{5.24}$$

where $r = \sum_{i=1}^{n}\delta_i$ is the observed number of lifetimes.

Solving the equations (5.23) and (5.24) using Newton-Raphson method, we obtain M.L.E.s of $c$ and $d$.

**(b) Estimation of parameters based on type II censored samples.**

In type II censoring, the number of observations $r$ is decided before the data are collected. The data consists of the $r$ smallest lifetimes $y_{(1)} \le y_{(2)} \le \ldots \le y_{(r)}$ out of a random sample of $n$ lifetimes. From (2.108), the likelihood function is given by

$$L(c,d \mid \underline{y}) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r}cd(1-cy_{(i)})^{(d-1)}\right](1-cy_{(r)})^{d(n-r)}. \tag{5.25}$$

Maximization of log likelihood function with respect to the parameters yields the following equations,

$$\frac{r}{c} = \frac{d(n-r)y_{(r)}}{1-cy_{(r)}} + (d-1)\sum_{i=1}^{r}\frac{y_{(i)}}{(1-cy_{(i)})} \tag{5.26}$$

$$\frac{r}{d} = (n-r)\log(1-cy_{(r)}) + \sum_{i=1}^{r}\log(1-cy_{(i)}) \tag{5.27}$$

Solving the equations (5.26) and (5.27) using Newton-Raphson method, we obtain maximum likelihood estimates of $c$ and $d$.

### 5.9.3 Finite mixture of beta distribution

Estimation of parameters of finite mixture of beta is discussed in Chapter 3.

## 5.10 Estimation of stress-strength reliability

When the strength $X$ follows finite mixture of Beta with parameters $c_1, c_2, d_1, d_2$ and $p_1$ and the stress $Y$ follows exponential distribution with parameter $\lambda$, then the maximum likelihood estimate of $R$ is given as

$$\hat{R} = 1 - \hat{p}_1\hat{d}_1\left[(-1)^{-\hat{d}_1} e^{-\hat{\lambda}/\hat{c}_1}\left(\hat{c}_1/\hat{\lambda}\right)^{\hat{d}_1}\left(\Gamma\hat{d}_1 - \Gamma\left(\hat{d}_1, -\hat{\lambda}/\hat{c}_1\right)\right)\right]$$

$$-(1-\hat{p}_1)\hat{d}_2\left[(-1)^{-\hat{d}_2} e^{-\hat{\lambda}/\hat{c}_2}\left(\hat{c}_2/\hat{\lambda}\right)^{\hat{d}_2}\left(\Gamma\hat{d}_2 - \Gamma\left(\hat{d}_2, -\hat{\lambda}/\hat{c}_2\right)\right)\right].$$

When the strength $X$ follows finite mixture of Beta with parameters $c_1, c_2, d_1, d_2$ and $p_1$ and the stress $Y$ follows Beta distribution with parameter $c$ and $d$, then the maximum likelihood estimate of $R$ is obtained as

$$\hat{R} = 1 - \hat{p}_1\left[Hy2F1\left[1, -\hat{d}, 1+\hat{d}_1, \frac{\hat{c}}{\hat{c}_1}\right]\right] - (1-\hat{p}_1)\left[Hy2F1\left[1, -\hat{d}, 1+\hat{d}_2, \frac{\hat{c}}{\hat{c}_2}\right]\right]$$

When the strength $X$ follows finite mixture of Beta with parameters $c_1, c_2, d_1, d_2$ and $p_1$ and the stress $Y$ follows finite mixture of Beta distribution with parameters $c_3, c_4, d_3, d_4$ and $p_3$ then the maximum likelihood estimate of $R$ is provided as

$$\hat{R} = 1 - \hat{p}_3\hat{p}_1\hat{d}_1\left[\Gamma\hat{d}_1 Hy2F1R\left[1, -\hat{d}_3, 1+\hat{d}_1, \frac{\hat{c}_3}{\hat{c}_1}\right]\right] - \hat{p}_3(1-\hat{p}_1)\hat{d}_2\left[\Gamma\hat{d}_2 Hy2F1R\left[1, -\hat{d}_3, 1+\hat{d}_2, \frac{\hat{c}_3}{\hat{c}_2}\right]\right]$$

$$-(1-\hat{p}_3)\hat{p}_1\hat{d}_1\left[\Gamma\hat{d}_1 Hy2F1R\left[1, -\hat{d}_4, 1+\hat{d}_1, \frac{\hat{c}_4}{\hat{c}_1}\right]\right] - (1-\hat{p}_3)(1-\hat{p}_1)\hat{d}_2\left[\Gamma\hat{d}_2 Hy2F1R\left[1, -\hat{d}_4, 1+\hat{d}_2, \frac{\hat{c}_4}{\hat{c}_2}\right]\right].$$

## 5.11 Simulation Study

In this section, we carry out simulation studies to assess the performance of the estimators and we obtained stress strength reliability for simulated datum.

### (a) Exponential distribution

We generate observations from exponential distribution with parameter $\lambda = .8$. Table 5.12 and 5.13 provide maximum likelihood estimate of parameter under complete as well as censored situations with various combinations of $n$ (sample size), $T$ (censoring time) and $r$ (number of failures).

### (b) Beta distribution

We generate samples from beta distribution with parameters $c = 1, d = 5$. The estimates under complete as well as type I and type II censoring set up are given in Tables 5.14 and 5.15 for various combinations of $n$, $T$ and $r$

### (c) Finite mixture of beta distributions

For the type I censoring, in which observations belonging to each subpopulation are known, we generate two sets of samples, one from a beta distribution with parameters $c_3$ and $d_3$ and the second from a beta distribution with parameters $c_4$ and $d_4$.

For the type I censoring (in which observations belonging to each subpopulation are unknown) and type II censoring, first we generate two sets of samples, one from a beta distribution with parameters $c_3$ and $d_3$ and the second from a beta distribution with parameters $c_4$ and $d_4$ and then, we obtained finite mixture of beta distributions using Bernoulli probability $p_3 (0 < p_3 < 1)$.

The estimates of parameters by the method of maximum likelihood under type I censoring, in which observations belonging to each subpopulation are known, for $c_3 = .1, c_4 = .9, d_3 = 3, d_4 = 1$ and $p_3 = 0.7$ with various combinations of $n$ (sample size) and $T$ (censoring time) are given in Table 5.16. The variances of the estimates are given in brackets.

Stress strength reliability when strength is finite mixture of beta and stress is exponential is given in Table 5.17. Table 5.18 provides the stress strength reliability when strength is finite mixture of beta and stress is beta. When stress and strength are finite mixture of beta, the reliability is given in Table 5.19. We use the software 'Mathematica' to evaluate the integrals numerically.

**Table 5.12 Maximum likelihood estimates of parameters under type I censoring for $\lambda = 0.8$ (exponential)**

| Censoring time | n=30 | n=50 | n=100 |
|---|---|---|---|
| T=30 | $\hat{\lambda}$=0.794166 (0.0297) | $\hat{\lambda}$=0..794354 (0.00581) | $\hat{\lambda}$=0.805804 (0.00185) |
| T=100 | $\hat{\lambda}$=0.819436 (0.0476) | $\hat{\lambda}$=0.8171227 (0.00328) | $\hat{\lambda}$=0.805074 (0.00012) |

**Table 5.13 Maximum likelihood estimates of parameters under type II censoring for $\lambda = 0.8$ (exponential)**

| n=30 | | n=50 | | n=100 | |
|---|---|---|---|---|---|
| r=10 | r=30 | r=30 | r=50 | r=60 | r=100 |
| $\hat{\lambda}$=0.821529 (0.0115) | $\hat{\lambda}$=0.794501 (0.00276) | $\hat{\lambda}$=0.789441 (0.00173) | $\hat{\lambda}$=0.80875 (0.000111) | $\hat{\lambda}$=0.79951 (0.0001) | $\hat{\lambda}$=.80035 (.00001) |

**Table 5.14 Maximum likelihood estimates of parameters for beta $c=1, d=5$ under type I censoring**

| Censoring time | n=30 | n=50 | n=100 |
|---|---|---|---|
| T=30 | $\hat{c}$=.98976 (2.5E-2) | $\hat{c}$=1.1023 (2.1E-4) | $\hat{c}$=1.003 (1.1E-5) |
| | $\hat{d}$=4.965 (1.7E-3) | $\hat{d}$=5.118 (1.4E-3) | $\hat{d}$=4.991 (2.7E-5) |
| T=100 | $\hat{c}$=1.1098 (3.3E-3) | $\hat{c}$=1.1003 (2.01E-4) | $\hat{c}$=0.9989 (2.1E-5) |
| | $\hat{d}$=4.985 (3.8E-4) | $\hat{d}$=5.076 (1.9E-5) | $\hat{d}$=5.0021 (2.4E-6) |

**Table 5.15 Maximum likelihood estimates of parameters for beta $c=1, d=5$ under type II censoring**

| n=30 | | n=50 | | n=100 | |
|---|---|---|---|---|---|
| r=10 | r=30 | r=30 | r=50 | r=60 | r=100 |
| $\hat{c}$=1.1002 (0.00467) | $\hat{c}$=1.0085 (0.00013) | $\hat{c}$=0.9965 (4.1E-4) | $\hat{c}$=1.0032 (3.8E-5) | $\hat{c}$=0.9998 (2.54E-5) | $\hat{c}$=1.00012 (2.1E-6) |
| $\hat{d}$=4.973 (0.00041) | $\hat{d}$=5.0064 (0.000032) | $\hat{d}$=5.0012 (2.1E-5) | $\hat{d}$=5.0027 (1.9E-5) | $\hat{d}$=4.9987 (1.6E-5) | $\hat{d}$=5.001 (1.01E-6) |

**Table 5.16 Maximum likelihood estimates of parameters under type I censoring for $c_3 = 0.1, c_4 = 0.9, d_3 = 3, d_4 = 1$ and $p_3 = 0.7$ in which observations belonging to each subpopulation are known.**

| Censoring time | n=30 | n=50 | n=100 |
|---|---|---|---|
| T=19 | $\hat{c}_3$=.0893 (.0000221) | $\hat{c}_3$=.0953976 (5.25E-7) | $\hat{c}_3$=.107286 (3.062E-7) |
| | $\hat{c}_4$=.86014 (.000017) | $\hat{c}_4$=.91216 (9.85E-8) | $\hat{c}_4$=.90179 (1.22E-9) |
| | $\hat{d}_3$=3.1105 (.842805) | $\hat{d}_3$=2.8069 (.100405) | $\hat{d}_3$=2.98003 (.0833) |
| | $\hat{d}_4$=1.0268 (.265423) | $\hat{d}_4$=1.0197 (.19532) | $\hat{d}_4$=1.00527 (.000135) |
| | $\hat{p}_3$=.7211 (5.56E-6) | $\hat{p}_3$=.70653 (1.15E-6) | $\hat{p}_3$=.70423 (2.06E-7) |

**Table 5.17 Stress-strength reliability when strength is finite mixture of beta and stress is exponential**

| Sample size | n=30 | n=50 | n=100 |
|---|---|---|---|
| Stress-strength reliability $R$ | 0.6591 | 0.6927 | 0.69816 |

**Table 5.18 Stress-strength reliability when strength is finite mixture of beta and stress is beta**

| Sample size | n=30 | n=50 | n=100 |
|---|---|---|---|
| Stress-strength reliability $R$ | 0.8127 | 0.84316 | 0.84417 |

**Table 5.19 Stress-strength reliability when strength is finite mixture of beta and stress is finite mixture of beta**

| Sample size | n=30 | n=50 | n=100 |
|---|---|---|---|
| Stress-strength reliability $R$ | 0.7128 | 0.7517 | 0.7556 |

## 5.12 Conclusion

In this chapter, the role of finite mixture of Pareto II distributions and finite mixture of beta distributions as stress-strength models is examined. To study the finite sample properties of the estimates we carried out a simulation work. The use of finite mixture of Pareto II distributions is illustrated using a real life data. As $n$ increases, the bias and variance of the estimate decreases.

# Chapter 6

# ESTIMATION OF THE LORENZ CURVE

## 6.1 Introduction

As mentioned earlier in Chapter I, the Lorenz curve is an important tool for the measurement of income inequality. Lorenz curve can be interpreted as the cumulative proportion of income to the cumulative proportion of population, after ordering the population according to the increasing level of income. Most of the popular parametric models have been employed in different situations of income analysis. For more details on this, one could refer to Kleiber and Kotz (2003). When population of income of individuals can be divided into population of income from different sources or any set of income data with earners in different categories, it is usual to model the data using finite mixture of distributions. However, the finite mixture of Pareto II and finite mixture of beta have not been explored much in literature in the context of income analysis.

Censored data are commonly encountered in practical applications to income and wealth distributions, for several reasons. Some high-income observations may have been removed from the sample because of concern for confidentiality-if it were not so, a skilful data-detective would have no difficulty, in identifying the individuals or households corresponding to individual observations in the sparse upper tail. In certain other instances, low- income observations may have been removed or modified for reasons of convenience. As pointed out in Fichtenbaum and Shahidi (1988), type I censoring is commonly used in such situations. So it is worthwhile to estimate Lorenz curve for type I censored data.

Motivated by these facts, in this chapter, we study the properties and applications of both finite mixture of Pareto II and finite mixture of beta distributions in the context of income analysis. The definition of the Lorenz curve is given in Section 6.2. Sections 6.3 and 6.4 give the maximum likelihood estimation of Lorenz curve for the finite mixture of Pareto II distributions. Simulation studies are carried out in Section 6.5. We, then, illustrate the method using a real data on household expenditure of men and women in Section 6.6. We obtain the estimate of the Lorenz curve for the finite mixture of beta distributions in Sections 6.7 and 6.8. Simulation studies are carried out to asses the performance of the estimators in Section 6.9. Finally in Section 6.10, we give the conclusion of the chapter.

## 6.2 Definition

The Lorenz curve, one of the important measures of income inequality, plays a significant role in analyzing income data. For a non-negative random variable $X$ with distribution function $F(x)$ and finite mean $\mu$, the Lorenz curve $L(F(x))$ is defined by

$$L(p) = \frac{1}{\mu}\int_0^x tf(t)dt, \tag{6.1}$$

where $f(t)$ is the probability density function of $X$ and $p = F(x)$ which is already defined in equations (1.24) and (1.25).

## 6.3 Finite mixture of Pareto II distribution

Any set of income data with earners in different categories will be well described by a finite mixture of distributions. For example, if we have income data for families in Kerala, we would expect that families of different sizes and /or different makeup in terms of the gender distribution of the adults in the families would have incomes that would follow different Pareto or other skewed distributions. Then the data will be considered as a mixture of such distributions.

Assume that the density function of the income distribution in the whole population is a finite mixture of Pareto densities, defined in (2.2). Then, the Lorenz curve $L(F(x))$ is obtained as

$$L(F(x)) = \frac{p_1 a_2 (b_2 - 1)[1-(1+a_1 x)^{-b_1}(1+a_1 b_1 x)] + (1-p_1) a_1 (b_1 - 1)[1-(1+a_2 x)^{-b_2}(1+a_2 b_2 x)]}{[p_1 a_2 (b_2 - 1) + (1-p_1) a_1 (b_1 - 1)]}. \tag{6.2}$$

### 6.4 Estimation of Lorenz curve

In Chapter 2, we have discussed different estimation procedures and estimated the parameters of finite mixture of Pareto II for complete as well as censored samples. Then the M.L.E of $L(F(x))$ is given by

$$\hat{L}(F(x)) = \frac{\hat{p}_1\hat{a}_2(\hat{b}_2-1)[1-(1+\hat{a}_1x)^{-\hat{b}_1}(1+\hat{a}_1\hat{b}_1x)]+(1-\hat{p}_1)\hat{a}_1(\hat{b}_1-1)[1-(1+\hat{a}_2x)^{-\hat{b}_2}(1+\hat{a}_2\hat{b}_2x)]}{[\hat{p}_1\hat{a}_2(\hat{b}_2-1)+(1-\hat{p}_1)\hat{a}_1(\hat{b}_1-1)]}. \tag{6.3}$$

Then from appendix B of Lawless (2003), the asymptotic variance of $\hat{L}$ is given by

$$Var(\hat{L}) = \left(\frac{\partial L}{\partial a_1}\right)^2 Var(\hat{a}_1) + \left(\frac{\partial L}{\partial a_2}\right)^2 Var(\hat{a}_2) + \ldots + \left(\frac{\partial L}{\partial p_1}\right)^2 Var(\hat{p}_1)$$

$$+\frac{\partial L}{\partial a_1}\frac{\partial L}{\partial a_2}Cov(\hat{a}_1,\hat{a}_2) + \frac{\partial L}{\partial a_1}\frac{\partial L}{\partial b_1}Cov(\hat{a}_1,\hat{b}_1) + \ldots + \frac{\partial L}{\partial b_2}\frac{\partial L}{\partial p_1}Cov(\hat{b}_2,\hat{p}_1). \tag{6.4}$$

where variances and covariances are directly obtained from the inverse of the matrix $I(a_1,a_2,b_1,b_2,p_1)$ with

$$I(a_1,a_2,b_1,b_2,p_1) = \begin{pmatrix} -\frac{\partial^2 \log L}{\partial a_1^2} & -\frac{\partial^2 \log L}{\partial a_1 \partial a_2} & -\frac{\partial^2 \log L}{\partial a_1 \partial b_1} & -\frac{\partial^2 \log L}{\partial a_1 \partial b_2} & -\frac{\partial^2 \log L}{\partial a_1 \partial p_1} \\ & & & & \\ \cdot & & & & \\ -\frac{\partial^2 \log L}{\partial p_1 \partial a_1} & -\frac{\partial^2 \log L}{\partial p_1 \partial a_2} & -\frac{\partial^2 \log L}{\partial p_1 \partial b_1} & -\frac{\partial^2 \log L}{\partial p_1 \partial b_2} & -\frac{\partial^2 \log L}{\partial p_1^2} \end{pmatrix}$$

### 6.5 Simulation study

In this section, we carry out simulation studies to find out the maximum likelihood estimates of the Lorenz curve under complete as well as censored samples.

For type I censoring, in which observations belonging to each subpopulation are known, we generate two sets of samples, one from a Pareto distribution with parameters $a_1$ and $b_1$ and the second from a Pareto distribution with parameters $a_2$ and $b_2$.

For type I censoring, in which observations belonging to each subpopulation are unknown, we obtained finite mixture of Pareto distributions using a Bernoulli distribution with probability of success $p_1 (0 < p_1 < 1)$.

The estimates of parameters by the method of maximum likelihood under complete as well as type I censoring, for the set of parameters $a_1 = .004, a_2 = .09, b_1 = 7, b_2 = 3$ and $p_1 = 0.5$ with various combinations of $n$ (sample size) and $N$ (censoring income) are given in Tables 6.1 to 6.3. Tables 6.4 to 6.6 provide the estimates for another set of parameters $a_1 = 2, a_2 = 3, b_1 = 5, b_2 = 7$ and $p_1 = 0.5$ under complete as well as type I censoring with various combinations of $n$ and $N$. The values in brackets provide the variance of the estimates. Simulation studies reported here shows that M.L.E provide estimates with small bias and less variance. The variance of the estimates decreases as n increases. Graphs of the actual and estimated Lorenz curve are given in Figures 6.1 to 6.3.

**Table 6.1 Maximum likelihood estimates of parameters under complete sample for** $a_1 = .004, a_2 = .09, b_1 = 7, b_2 = 3$ **and** $p_1 = 0.5$

| n=30 | n=50 | n=100 |
|---|---|---|
| $\hat{a}_1$=.00411472<br>(3.38321E-8) | $\hat{a}_1$=.00423549<br>(2.46863E-8) | $\hat{a}_1$=.00410912<br>(2.4321E-8) |
| $\hat{a}_2$=.0983347<br>(.0000321937) | $\hat{a}_2$=.0986351<br>(.00001894) | $\hat{a}_2$=.100476<br>(3.85598E-6) |
| $\hat{b}_1$=6.89626<br>(.0739452) | $\hat{b}_1$=6.93539<br>(.0661982) | $\hat{b}_1$=6.87722<br>(.00993038) |
| $\hat{b}_2$=2.86146<br>(.0139852) | $\hat{b}_2$=2.8976<br>(.0124295) | $\hat{b}_2$=2.91442<br>(.00361627) |
| $\hat{p}_1$=.500057<br>(3.2202E-9) | $\hat{p}_1$=.500052<br>(2.70565E-9) | $\hat{p}_1$=.500046<br>(2.09203E-9) |

**Table 6.2 Maximum likelihood estimates of parameters under type I censoring for $a_1=.004, a_2=.09, b_1=7, b_2=3$ and $p_1=0.5$ in which observations belonging to each subpopulation are known.**

| Censoring income | n=30 | n=50 | n=100 |
|---|---|---|---|
| N=30 | $\hat{a}_1$=.0044872 (8.421E-7) | $\hat{a}_1$=.00438394 (4.669E-7) | $\hat{a}_1$=.004984 (1.19918E-8) |
| | $\hat{a}_2$=.0963709 (.0008286) | $\hat{a}_2$=.0980649 (1.8715E-6) | $\hat{a}_2$=.094886 (1.8042E-6) |
| | $\hat{b}_1$=6.85789 (.151983) | $\hat{b}_1$=6.88304 (.0848708) | $\hat{b}_1$=6.9353 (.001556) |
| | $\hat{b}_2$=2.92781 (.0500188) | $\hat{b}_2$=2.94408 (.000855) | $\hat{b}_2$=2.96472 (.0006696) |
| | $\hat{p}_1$=.500443 (7.826E-7) | $\hat{p}_1$=.500446 (2.007E-7) | $\hat{p}_1$=.501203 (1.98099E-7) |
| N=50 | $\hat{a}_1$=.00380753 (1.183E-6) | $\hat{a}_1$=.00375961 (1.492E-7) | $\hat{a}_1$=.0044017 (2.5361E-8) |
| | $\hat{a}_2$=.0988792 (.0000328) | $\hat{a}_2$=.100693 (6.25E-7) | $\hat{a}_2$=.0981161 (4.62E-7) |
| | $\hat{b}_1$=7.00632 (.0308752) | $\hat{b}_1$=7.10751 (.006559) | $\hat{b}_1$=7.04669 (.0006628) |
| | $\hat{b}_2$=2.94612 (.0149049) | $\hat{b}_2$=2.97302 (.0002166) | $\hat{b}_2$=2.97256 (.00021) |
| | $\hat{p}_1$=.499885 (3.643E-8) | $\hat{p}_1$=.49968 (2.265E-8) | $\hat{p}_1$=.500298 (2.1404E-8) |
| N=100 | $\hat{a}_1$=.00420753 (1.0903E-7) | $\hat{a}_1$=.00407865 (1.024E-7) | $\hat{a}_1$=.004077 (2.1194E-8) |
| | $\hat{a}_2$=.098026 (6.6085E-6) | $\hat{a}_2$=.09778 (1.78E-6) | $\hat{a}_2$=.0984021 (1.6642E-6) |
| | $\hat{b}_1$=6.94212 (.006307) | $\hat{b}_1$=7.0007 (.000314) | $\hat{b}_1$=6.984 (.00004) |
| | $\hat{b}_2$=2.96212 (.001755) | $\hat{b}_2$=2.9459 (.00148) | $\hat{b}_2$=2.96334 (.0002764) |
| | $\hat{p}_1$=.50006 (8.6623E-8) | $\hat{p}_1$=.500086 (3.726E-9) | $\hat{p}_1$=.500153 (3.35E-9) |

**Table 6.3 Maximum likelihood estimates of parameters under type I censoring for $a_1 = .004, a_2 = .09, b_1 = 7, b_2 = 3$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are unknown.**

| Censoring income | n=30 | n=50 | n=100 |
|---|---|---|---|
| N=30 | $\hat{a}_1$=.0043864 (7.53E-8) | $\hat{a}_1$=.004298 (1.6192E-8) | $\hat{a}_1$=.00418 (2.022E-9) |
| | $\hat{a}_2$=.0973671 (.00001113) | $\hat{a}_2$=.09767 (1.799E-6) | $\hat{a}_2$=.098766 (4.506E-7) |
| | $\hat{b}_1$=6.80949 (.141223) | $\hat{b}_1$=6.83318 (.0527144) | $\hat{b}_1$=6.94737 (.034792) |
| | $\hat{b}_2$=2.87988 (.00386) | $\hat{b}_2$=2.94875 (.001495) | $\hat{b}_2$=2.95472 (.0004162) |
| | $\hat{p}_1$=.500074 (4.8236E-8) | $\hat{p}_1$=.500101 (3.8091E-8) | $\hat{p}_1$=.500116 (4.945E-11) |
| N=50 | $\hat{a}_1$=.00417876 (6.18E-9) | $\hat{a}_1$=.0042373 (1.60344E-9) | $\hat{a}_1$=.004172 (1.548E-9) |
| | $\hat{a}_2$=.097445 (.0000237) | $\hat{a}_2$=.0986273 (.00001751) | $\hat{a}_2$=.09959 (.0000111) |
| | $\hat{b}_1$=6.88576 (.0126) | $\hat{b}_1$=6.91828 (.002838) | $\hat{b}_1$=6.92217 (.0003794) |
| | $\hat{b}_2$=2.88065 (.0101) | $\hat{b}_2$=2.92881 (.0011) | $\hat{b}_2$=2.93605 (.000867) |
| | $\hat{p}_1$=.50062 (8.5511E-9) | $\hat{p}_1$=.500009 (2.1141E-9) | $\hat{p}_1$=.50005 (1.07E-9) |
| N=100 | $\hat{a}_1$=.00412299 (3.414E-9) | $\hat{a}_1$=.004205 (1.57E-9) | $\hat{a}_1$=.0041796 (1.044E-9) |
| | $\hat{a}_2$=.096988 (.0000231) | $\hat{a}_2$=.098146 (.0000169) | $\hat{a}_2$=.09915 (.0000108) |
| | $\hat{b}_1$=6.86767 (.00716) | $\hat{b}_1$=6.91419 (.00287) | $\hat{b}_1$=6.9157 (.0004095) |
| | $\hat{b}_2$=2.89201 (.010018) | $\hat{b}_2$=2.94047 (.001141) | $\hat{b}_2$=2.94684 (.0008588) |
| | $\hat{p}_1$=.500097 (9.6565E-9) | $\hat{p}_1$=.500047 (1.4751E-9) | $\hat{p}_1$=.500084 (6.7675E-10) |

**Table 6.4 Maximum likelihood estimates of parameters under complete sample for $a_1 = 2, a_2 = 3, b_1 = 5, b_2 = 7$ and $p_1 = 0.5$**

| n=30 | n=50 | n=100 |
|---|---|---|
| $\hat{a}_1$=2.1796 (.015002) | $\hat{a}_1$=1.90602 (.0071) | $\hat{a}_1$=2.02431 (.001621) |
| $\hat{a}_2$=2.77746 (.090189) | $\hat{a}_2$=3.13001 (.063607) | $\hat{a}_2$=2.99554 (.014771) |
| $\hat{b}_1$=4.61384 (.044781) | $\hat{b}_1$=5.24161 (.033323) | $\hat{b}_1$=4.94152 (.01437) |
| $\hat{b}_2$=7.56975 (.46587) | $\hat{b}_2$=6.80461 (.325679) | $\hat{b}_2$=7.16267 (.1713) |
| $\hat{p}_1$=.5 (2.7E-11) | $\hat{p}_1$=.5 (2.1E-11) | $\hat{p}_1$=.5 (1.7E-12) |

**Table 6.5 Maximum likelihood estimates of parameters under type I censoring for $a_1 = 2, a_2 = 3, b_1 = 5, b_2 = 7$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are known.**

| Censoring income | n=30 | n=50 | n=100 |
|---|---|---|---|
| N=0.05 | $\hat{a}_1$=2.18731 (.014204) | $\hat{a}_1$=1.945499 (.0097261) | $\hat{a}_1$=2.04146 (.005322) |
| | $\hat{a}_2$=2.81789 (.092232) | $\hat{a}_2$=3.08213 (.0610755) | $\hat{a}_2$=3.03947 (.020721) |
| | $\hat{b}_1$=4.60094 (.042512) | $\hat{b}_1$=5.15087 (.032336) | $\hat{b}_1$=4.91116 (.0299144) |
| | $\hat{b}_2$=7.46988 (.45217) | $\hat{b}_2$=6.89746 (.294549) | $\hat{b}_2$=7.07043 (.174879) |
| | $\hat{p}_1$=.499975 (3.5198E-11) | $\hat{p}_1$=.500033 (1.8122E-11) | $\hat{p}_1$=.499973 (1.2789E-11) |
| N=0.1 | $\hat{a}_1$=2.1819 (.0113455) | $\hat{a}_1$=1.90403 (.006345) | $\hat{a}_1$=2.02279 (.001711) |
| | $\hat{a}_2$=2.77861 (.090253) | $\hat{a}_2$=3.13203 (.0637089) | $\hat{a}_2$=2.99652 (.015009) |
| | $\hat{b}_1$=4.60695 (.046137) | $\hat{b}_1$=5.24602 (.044466) | $\hat{b}_1$=4.94477 (.0141144) |
| | $\hat{b}_2$=7.56688 (.465398) | $\hat{b}_2$=6.8007 (.325851) | $\hat{b}_2$=7.1607 (.27219) |
| | $\hat{p}_1$=.465398 (7.14726E-13) | $\hat{p}_1$=.4999 (2.4857E-14) | $\hat{p}_1$=.5 (1.3237E-14) |

**Table 6.6 Maximum likelihood estimates of parameters under type I censoring for $a_1 = 2, a_2 = 3, b_1 = 5, b_2 = 7$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are unknown.**

| Censoring income | n=30 | n=50 | n=100 |
|---|---|---|---|
| N=0.05 | $\hat{a}_1$=1.96906 (.04259) | $\hat{a}_1$=2.05246 (.010876) | $\hat{a}_1$=2.14809 (.004623) |
| | $\hat{a}_2$=2.94507 (.178502) | $\hat{a}_2$=3.23991 (.11828) | $\hat{a}_2$=3.0688 (.0296502) |
| | $\hat{b}_1$=5.15519 (.217949) | $\hat{b}_1$=4.8908 (.09505) | $\hat{b}_1$=4.75426 (.026343) |
| | $\hat{b}_2$=7.17026 (.84553) | $\hat{b}_2$=6.67554 (.807092) | $\hat{b}_2$=6.97098 (.79998) |
| | $\hat{p}_1$=.500052 (6.2789E-9) | $\hat{p}_1$=.4999 (3.295E-9) | $\hat{p}_1$=.500011 (3.0716E-9) |
| N=0.1 | $\hat{a}_1$=1.98499 (.029985) | $\hat{a}_1$=2.02314 (.016288) | $\hat{a}_1$=2.1451 (.0092313) |
| | $\hat{a}_2$=3.02692 (.072712) | $\hat{a}_2$=3.23613 (.01157) | $\hat{a}_2$=3.07318 (.002619) |
| | $\hat{b}_1$=5.10479 (.134448) | $\hat{b}_1$=4.92734 (.0945907) | $\hat{b}_1$=4.76164 (.0441035) |
| | $\hat{b}_2$=6.96226 (.252676) | $\hat{b}_2$=6.69581 (.090537) | $\hat{b}_2$=6.94939 (.0768141) |
| | $\hat{p}_1$=.49999 (1.3833E-12) | $\hat{p}_1$=.4999 (4.778E-13) | $\hat{p}_1$=.4999 (1.0539E-14) |

**Fig. 6.1 Graph of actual ($a_1 = .004, a_2 = .09, b_1 = 7, b_2 = 3$ and $p_1 = .5$) and estimated Lorenz curve ($\hat{a}_1 = .0041, \hat{a}_2 = .1, \hat{b}_1 = 6.88, \hat{b}_2 = 2.91$ and $\hat{p}_1 = .5$) for the complete sample case.**

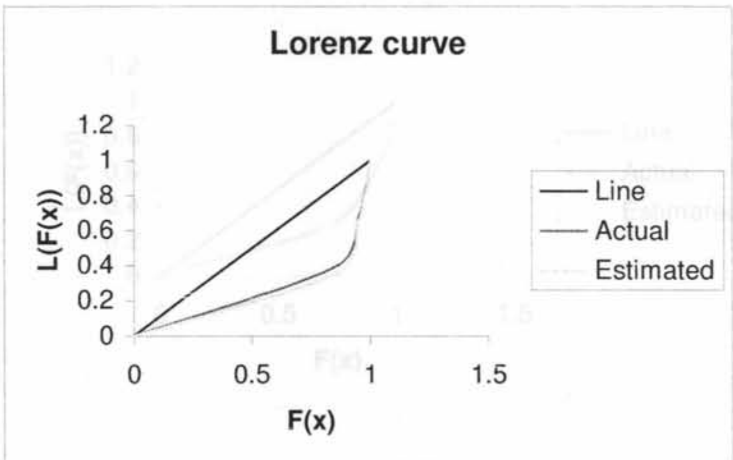



**Fig. 6.2 Graph of actual ($a_1 = .004, a_2 = .09, b_1 = 7, b_2 = 3$ and $p_1 = .5$) and estimated Lorenz curve ($\hat{a}_1 = .0044, \hat{a}_2 = .0963, \hat{b}_1 = 6.86, \hat{b}_2 = 2.93$ and $\hat{p}_1 = .5$) for the censored sample case.**

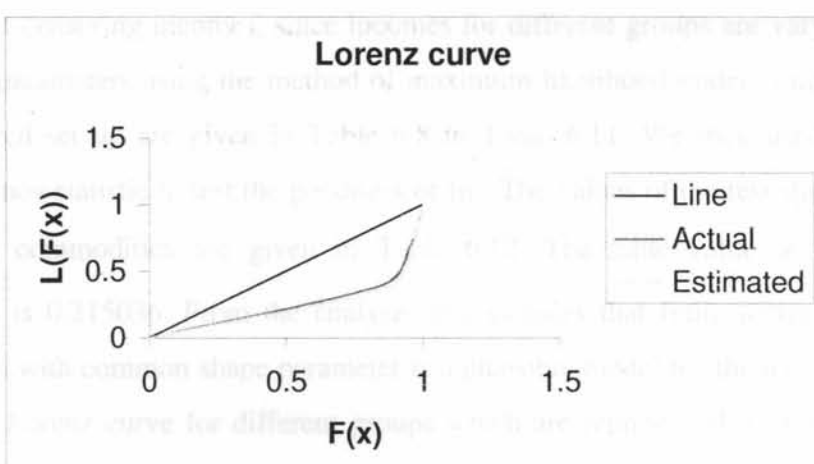
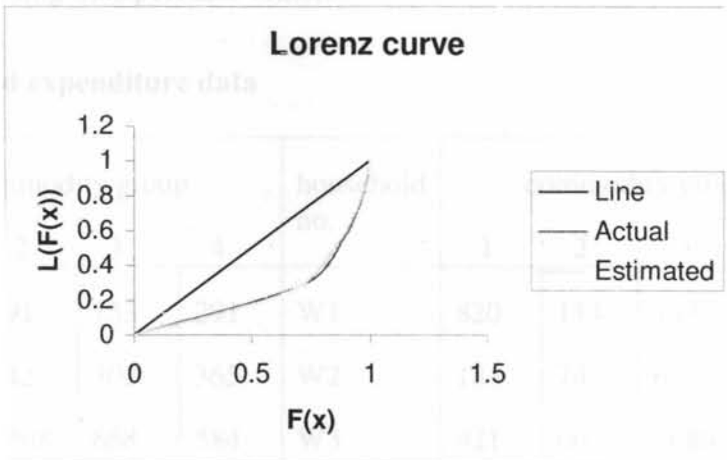


**Fig. 6.3 Graph of actual ($a_1 = 2, a_2 = 3, b_1 = 5, b_2 = 7$ and $p_1 = .5$) and estimated Lorenz curve ($\hat{a}_1 = 1.96, \hat{a}_2 = 2.94, \hat{b}_1 = 5.2, \hat{b}_2 = 7.2$ and $\hat{p}_1 = .5$) for the censored sample case.**




## 6.6 Data analysis

For the illustration of the method, we consider a data on household expenditures for four commodity groups of 20 men and 20 women (see Aitchison, 1986). The expenditures are given in Hong Kong dollars, and the commodity groups are Housing, Foodstuffs, Other items and Services. The data is given in Table 6.7. To understand the behaviour of Lorenz curve in the censored situation, we censored the data with different censoring incomes, since incomes for different groups are varying. The estimates of parameters using the method of maximum likelihood under complete as well as censored set up are given in Table 6.8 to Table 6.11. We then used the Kolmogorov smirnov statistic to test the goodness of fit. The values of the test statistic $D_n$ for different commodities are given in Table 6.12. The table value at 5 % significance level is 0.215036. From the analysis, it concludes that finite mixture of Pareto distribution with common shape parameter is a plausible model for the data. We then estimate the Lorenz curve for different groups which are represented by Figures 6.4 to 6.8.

The data come from a survey of household expenditure and give the expenditure of 20 single men (M) and 20 single women (W) on four commodity groups. The units of expenditure are Hong Kong dollars, and the commodity groups are as follows.

1 Housing, including fuel and light

2 Foodstuffs, including alcohol and tobacco

3 Other goods, including clothing, footwear and durable goods

4 Services, including transport and vehicles

**Table 6.7 Household expenditure data**

| household no. | commodity group | | | | household no. | commodity group | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | | 1 | 2 | 3 | 4 |
| M1 | 497 | 591 | 153 | 291 | W1 | 820 | 114 | 183 | 154 |
| M2 | 839 | 942 | 302 | 365 | W2 | 184 | 74 | 6 | 20 |
| M3 | 798 | 1308 | 668 | 584 | W3 | 921 | 66 | 1686 | 455 |
| M4 | 892 | 842 | 287 | 395 | W4 | 488 | 80 | 103 | 115 |
| M5 | 1585 | 781 | 2476 | 1740 | W5 | 721 | 83 | 176 | 104 |
| M6 | 755 | 764 | 428 | 438 | W6 | 614 | 55 | 441 | 193 |
| M7 | 388 | 655 | 153 | 233 | W7 | 801 | 56 | 357 | 214 |
| M8 | 617 | 879 | 757 | 719 | W8 | 396 | 59 | 61 | 80 |
| M9 | 248 | 438 | 22 | 65 | W9 | 864 | 65 | 1618 | 352 |
| M10 | 1641 | 440 | 6471 | 2063 | W10 | 845 | 64 | 1935 | 414 |
| M11 | 1180 | 1243 | 768 | 813 | W11 | 404 | 97 | 33 | 47 |
| M12 | 619 | 684 | 99 | 204 | W12 | 781 | 47 | 1906 | 452 |
| M13 | 253 | 422 | 15 | 48 | W13 | 457 | 103 | 136 | 108 |
| M14 | 661 | 739 | 71 | 188 | W14 | 1029 | 71 | 244 | 189 |
| M15 | 1981 | 869 | 1489 | 1032 | W15 | 1047 | 90 | 653 | 298 |
| M16 | 1746 | 746 | 2662 | 1594 | W16 | 552 | 91 | 185 | 158 |
| M17 | 1865 | 915 | 5184 | 1767 | W17 | 718 | 104 | 583 | 304 |
| M18 | 238 | 522 | 29 | 75 | W18 | 495 | 114 | 65 | 74 |
| M19 | 1199 | 1095 | 261 | 344 | W19 | 382 | 77 | 230 | 147 |
| M20 | 1524 | 964 | 1739 | 1410 | W20 | 1090 | 59 | 313 | 177 |

**Table 6.8 Maximum likelihood estimates of parameters of housing expenses of men and women under complete as well as censoring.**

| Item | Censoring income(in dollars) N=2000 | N=1500 | N=1000 |
|---|---|---|---|
| Housing expenses | $\hat{a}_1$=0.00007 | $\hat{a}_1$=0.00009 | $\hat{a}_1$=0.000125 |
| | $\hat{a}_2$=0.0001 | $\hat{a}_2$=0.000085 | $\hat{a}_2$=0.000112 |
| | $b_1 = b_2 = b$<br>$\hat{b}$=14.17 | $b_1 = b_2 = b$<br>$\hat{b}$=11.45 | $b_1 = b_2 = b$<br>$\hat{b}$=8.86776 |
| | $\hat{p}_1$=0.5 | $\hat{p}_1$=0.408 | $\hat{p}_1$=0.4047 |

**Table 6.9 Maximum likelihood estimates of parameters of food expenses of men and women under complete as well as censoring.**

| Item | Censoring income(in dollars) N=1400 | N=1000 | N=800 |
|---|---|---|---|
| Food Expenses | $\hat{a}_1$=0.00007 | $\hat{a}_1$=0.0000719 | $\hat{a}_1$=0.000082 |
| | $\hat{a}_2$=0.0007 | $\hat{a}_2$=0.00081369 | $\hat{a}_2$=0.001328 |
| | $b_1 = b_2 = b$<br>$\hat{b}$=17.42 | $b_1 = b_2 = b$<br>$\hat{b}$=15.6138 | $b_1 = b_2 = b$<br>$\hat{b}$=9.56 |
| | $\hat{p}_1$=0.5 | $\hat{p}_1$=0.5002 | $\hat{p}_1$=0.499616 |

**Table 6.10 Maximum likelihood estimates of parameters of other expenses of men and women under complete as well as censoring.**

| Item | Censoring income(in dollars) N=6500 | N=5000 | N=1000 |
|---|---|---|---|
| Other Expenses | $\hat{a}_1$=0.0011 | $\hat{a}_1$=0.00232 | $\hat{a}_1$=0.002918 |
| | $\hat{a}_2$=0.0019 | $\hat{a}_2$=0.002815 | $\hat{a}_2$=0.00343 |
| | $b_1 = b_2 = b$ $\hat{b}$=1.625 | $b_1 = b_2 = b$ $\hat{b}$=1.4675 | $b_1 = b_2 = b$ $\hat{b}$=1.409215 |
| | $\hat{p}_1$=0.5 | $\hat{p}_1$=0.4763 | $\hat{p}_1$=0.476815 |

**Table 6.11 Maximum likelihood estimates of parameters of service expenses of men and women under complete as well as censoring.**

| Item | Censoring income(in dollars) N=2100 | N=1500 | N=1000 |
|---|---|---|---|
| Service Expenses | $\hat{a}_1$=0.00011 | $\hat{a}_1$=0.0001158 | $\hat{a}_1$=0.0001598 |
| | $\hat{a}_2$=0.0004 | $\hat{a}_2$=0.00044 | $\hat{a}_2$=0.000514 |
| | $b_1 = b_2 = b$ $\hat{b}$=12.45 | $b_1 = b_2 = b$ $\hat{b}$=11.2505 | $b_1 = b_2 = b$ $\hat{b}$=9.06372 |
| | $\hat{p}_1$=0.5 | $\hat{p}_1$=0.498103 | $\hat{p}_1$=0.487185 |

**Table 6.12 Calculated values of the test statistic for different expenses.**

| Different commodities | Max $D_n$ (M.L.E) |
|---|---|
| Housing Expenses | 0.200094 |
| Food Expenses | 0.133804 |
| Other Expenses | 0.0430059 |
| Service Expenses | 0.089086 |

**Fig. 6.4 Graph of maximum likelihood estimate of Lorenz curve of housing expenses of men and women**

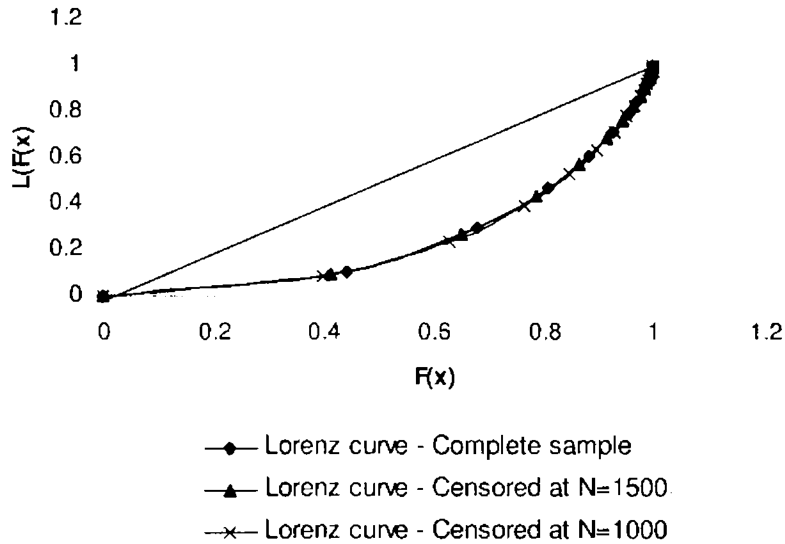
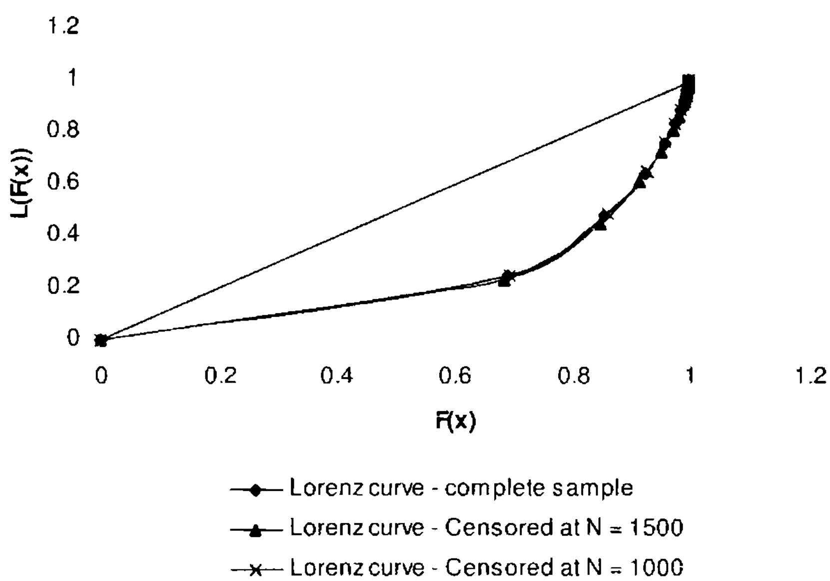



**Fig. 6.5 Graph of maximum likelihood estimate of Lorenz curve of service expenses of men and women**




**Remark 6.1**

Lorenz curves for complete as well as censored samples are almost identical for housing expenses and service expenses of men and women.

**Fig. 6.6 Graph of maximum likelihood estimate of Lorenz curve of food expenses of men and women**

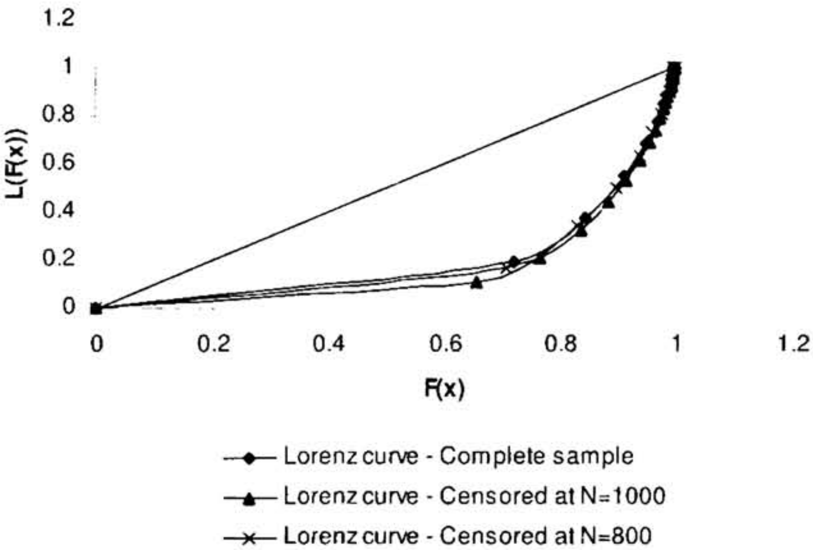
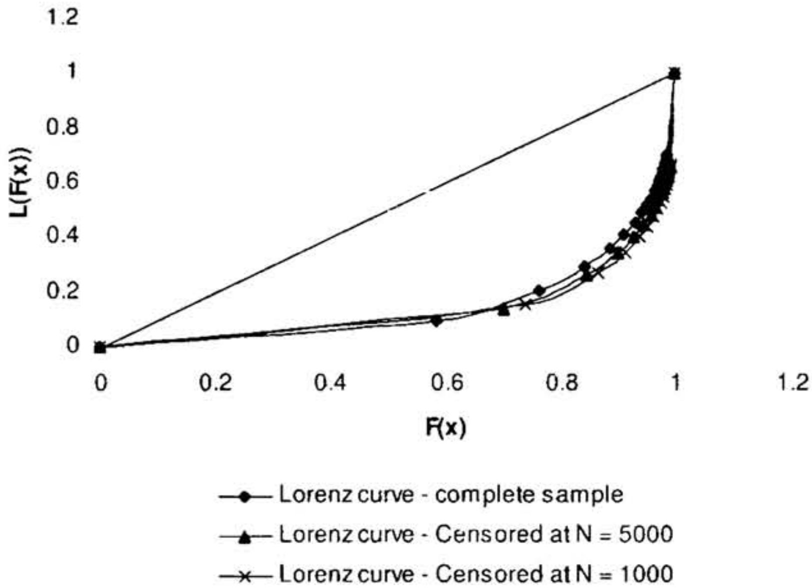


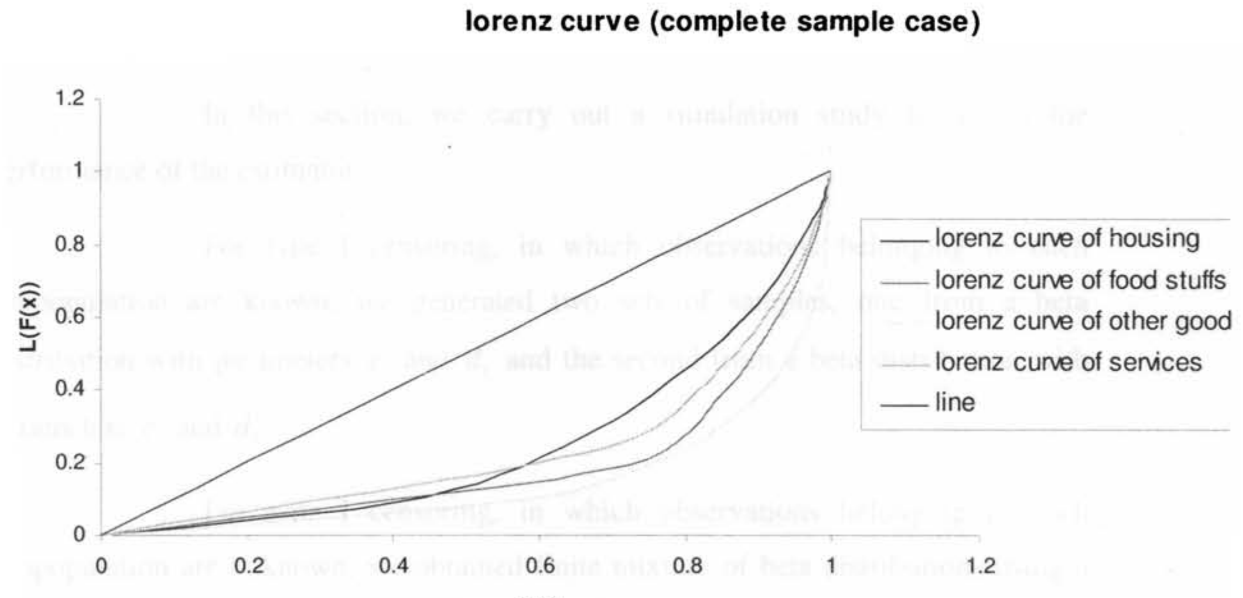

**Fig. 6.7 Graph of maximum likelihood estimate of Lorenz curve of other expenses of men and women**




**Remark 6.2**

However, Lorenz curve in the censored set up provide more disparity for food expenses and other expenses of men and women.

**Fig. 6.8 Graph of maximum likelihood estimate of Lorenz curve of different expenses of men and women**




**Remark 6.3**

In the complete sample set up, Lorenz curve gives more disparity for other expenses than housing, foodstuffs and service expenses of men and women.

## 6.7 Finite mixture of beta distribution

In this section, we study the role of finite mixture of beta distribution (3.1) in the context of income analysis. For the model (3.1), the Lorenz curve $L(F(x))$ is obtained as

$L(F(x)) =$

$$\frac{p_1 c_2(d_2+1)[1-(1-c_1 x)^{d_1}(1+c_1 d_1 x)]+(1-p_1)c_1(d_1+1)[1-(1-c_2 x)^{d_2}(1+c_2 d_2 x)]}{[p_1 c_2(d_2+1)+(1-p_1)c_1(d_1+1)]}. \quad (6.5)$$

## 6.8 Estimation of Lorenz curve

For estimating the Lorenz curve, we need the estimate of parameters of the model (3.1). The estimation of parameters of the model is discussed in Chapter 3.

Thus M.L.E of $L(F(x))$ as

$$\hat{L}(F(x)) = \frac{\hat{p}_1\hat{c}_2(\hat{d}_2+1)[1-(1-\hat{c}_1x)^{\hat{d}_1}(1+\hat{c}_1\hat{d}_1x)]+(1-\hat{p}_1)\hat{c}_1(\hat{d}_1+1)[1-(1-\hat{c}_2x)^{\hat{d}_2}(1+\hat{c}_2\hat{d}_2x)]}{[\hat{p}_1\hat{c}_2(\hat{d}_2+1)+(1-\hat{p}_1)\hat{c}_1(\hat{d}_1+1)]}. \tag{6.6}$$

## 6.9 Simulation study

In this section, we carry out a simulation study to assess the performance of the estimator.

For type I censoring, in which observations belonging to each subpopulation are known, we generated two sets of samples, one from a beta distribution with parameters $c_1$ and $d_1$ and the second from a beta distribution with parameters $c_2$ and $d_2$.

For type I censoring, in which observations belonging to each subpopulation are unknown, we obtained finite mixture of beta distributions using a Bernoulli distribution with probability of success $p_1 (0 < p_1 < 1)$.

The estimates of parameters by the method of maximum likelihood under complete and type I censoring, for the set of parameters $c_1 = .1, c_2 = .2, d_1 = 5, d_2 = 10$ and $p_1 = 0.5$ with various combinations of $n$ and $N$ are given in Table 6.13 to 6.15. The values in brackets provide the variance of the estimates. Simulation study reported here shows that M.L.E provides estimates with small bias. The variance of the estimates decreases as n increases. Graphs of the actual and estimated Lorenz curve are given in Fig 6.9 and 6.10.

**Table 6.13 Maximum likelihood estimates of parameters under complete sample for $c_1=.1, c_2=.2, d_1=5, d_2=10$ and $p_1=0.5$**

| n=30 | n=50 | n=100 |
|---|---|---|
| $\hat{c}_1$=.1098236 (8.322E-6) | $\hat{c}_1$=.109352 (2.74217E-8) | $\hat{c}_1$=.0916796 (1.31172E-9) |
| $\hat{c}_2$=.188062 (1.23116E-7) | $\hat{c}_2$=.1789255 (3.2446E-6) | $\hat{c}_2$=.179949 (1.26327E-6) |
| $\hat{d}_1$=4.92574 (.04452) | $\hat{d}_1$=5.07017 (.0151182) | $\hat{d}_1$=5.02106 (.00848767) |
| $\hat{d}_2$=10.3343 (.08585) | $\hat{d}_2$=10.4858 (.01558) | $\hat{d}_2$=10.4986 (.00303049) |
| $\hat{p}_1$=.5 (3.9E-8) | $\hat{p}_1$=.5 (2.7E-9) | $\hat{p}_1$=.5 (1.2E-11) |

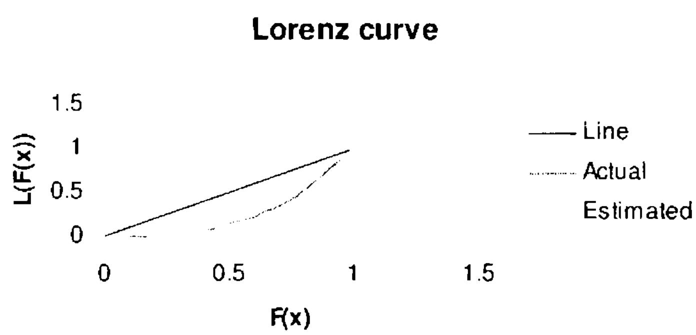
**Table 6.14 Maximum likelihood estimates of parameters under type I censoring for $c_1=.1, c_2=.2, d_1=5, d_2=10$ and $p_1=0.5$ in which observations belonging to each subpopulation are known.**

| Censoring income | n=30 | n=50 | n=100 |
|---|---|---|---|
| N=15 | $\hat{c}_1$=.1220745 (1.40616E-9) | $\hat{c}_1$=.0904950 (1.03046E-9) | $\hat{c}_1$=.0903679 (2.76269E-11) |
| | $\hat{c}_2$=.20328749 (1.32434E-8) | $\hat{c}_2$=.280551 (1.06227E-8) | $\hat{c}_2$=.24784 (1.80183E-9) |
| | $\hat{d}_1$=4.83494 (.52921) | $\hat{d}_1$=5.0323 (.266372) | $\hat{d}_1$=5.06633 (.00135048) |
| | $\hat{d}_2$=10.1772 (.17411) | $\hat{d}_2$=10.078881 (.038415) | $\hat{d}_2$=10.0850 (.0007159) |
| | $\hat{p}_1$=.586043 (.0158459) | $\hat{p}_1$=.50153 (.008962) | $\hat{p}_1$=.50697 (.00099) |
| N=19 | $\hat{c}_1$=.128464 (7.33582E-7) | $\hat{c}_1$=.170163 (4.38079E-7) | $\hat{c}_1$=.132481 (1.194E-8) |
| | $\hat{c}_2$=.1742113 (2.597E-8) | $\hat{c}_2$=.20692 (9.55987E-9) | $\hat{c}_2$=.199065 (2.61837E-9) |
| | $\hat{d}_1$=5.2529 (.0036046) | $\hat{d}_1$=4.96169 (.0199362) | $\hat{d}_1$=4.76143 (.002968) |
| | $\hat{d}_2$=9.75696 (.000887) | $\hat{d}_2$=10.12323 (.0005169) | $\hat{d}_2$=10.40191 (.000348) |
| | $\hat{p}_1$=.482237 (.000853) | $\hat{p}_1$=.439791 (.0001795) | $\hat{p}_1$=.532104 (.0000211) |

**Table 6.15 Maximum likelihood estimates of parameters under type I censoring for $c_1 = .1, c_2 = .2, d_1 = 5, d_2 = 10$ and $p_1 = 0.5$ in which observations belonging to each subpopulation are unknown.**

| Censoring income | Estimate | n=30 | n=50 | n=100 |
|---|---|---|---|---|
| N=15 | M.L.E | $\hat{c}_1$=.103728 (.000666) | $\hat{c}_1$=.103320 (.000466) | $\hat{c}_1$=.10266 (.00001924) |
| | | $\hat{c}_2$=.25195 (.000535) | $\hat{c}_2$=.20506 (2.36588E-6) | $\hat{c}_2$=.205053 (1.6508E-6) |
| | | $\hat{d}_1$=5.9867 (.02679) | $\hat{d}_1$=5.2926 (.00149) | $\hat{d}_1$=5.03388 (.000904) |
| | | $\hat{d}_2$=10.32588 (.4479) | $\hat{d}_2$=10.0336 (.00312) | $\hat{d}_2$=10.0319 (.002064) |
| | | $\hat{p}_1$=.51162 (.09115) | $\hat{p}_1$=.50804 (.000272) | $\hat{p}_1$=.507314 (.0000748) |
| N=19 | M.L.E | $\hat{c}_1$=.1129162 (.0000126) | $\hat{c}_1$=.1168359 (2.9314E-6) | $\hat{c}_1$=.102233 (8.36481E-7) |
| | | $\hat{c}_2$=.199692 (1.4724E-6) | $\hat{c}_2$=.20114 (3.6192E-7) | $\hat{c}_2$=.188321 (6.433E-8) |
| | | $\hat{d}_1$=5.90832 (.704024) | $\hat{d}_1$=5.0816 (.004299) | $\hat{d}_1$=5.05114 (.0010644) |
| | | $\hat{d}_2$=10.94874 (.611743) | $\hat{d}_2$=10.96933 (.013504) | $\hat{d}_2$=10.08138 (.0013461) |
| | | $\hat{p}_1$=.43484 (.0000388) | $\hat{p}_1$=.447885 (.0000204) | $\hat{p}_1$=.514562 (2.1671E-7) |

**Fig. 6.9 Graph of actual ($c_1 = .1, c_2 = .2, d_1 = 5, d_2 = 10$ and $p_1 = .5$) and estimated Lorenz curve ($\hat{c}_1 = .112, \hat{c}_2 = .199, \hat{d}_1 = 5.91, \hat{d}_2 = 10.95$ and $\hat{p}_1 = .5$) for the censored sample case.**



# Chapter 7

# CONCLUSION

## 7.1 Introduction

The remarkable fact about mixture distributions is that there are lots of real life situations where the concept of mixture distributions can be applied. For example, in life testing experiments, the systems will be failed due to different causes and the times to failure due to different reasons are likely to follow different distributions. A knowledge of these distributions is essential to eliminate cause of failures and thereby to improve the reliability.

In the foregone chapters, we have examined the role of finite mixture of Pareto and finite mixture of beta distributions in the context of reliability and income analysis.We have proved that finite mixture of Pareto and finite mixture of beta distributions are identifiable. The estimation of parameters has been done using various techniques for complete as well as censored samples. Maximum likelihood and Bayes methods provide estimates with small bias and less variance (in the case of Bayes, we calculated posterior risk) and the variance of the estimates decreases as $n$ increases. As expected, the method of moments was shown to be inferior to maximum likelihood estimation, Bayes method and maximum product of spacing. We have employed finite mixture models for prediction of future observations and for this purpose Bayesian approach has been used. It should be remarked that the results obtained in Bayesian prediction seems to be quite good. Maximum likelihood estimation of stress-strength reliability and Lorenz curve using finite mixture models has also been carried out.

Finite mixture of Pareto always possesses decreasing hazard rate function. However, finite mixture of beta possesses decreasing hazard rate, increasing hazard rate, constant hazard rate and bathtub pattern depending upon the values of the parameters. We have employed the finite mixture of Pareto and finite mixture of beta distributions in modelling four different decreasing hazard rate situations. Since decreasing hazard rate is most commonly encountered in many real life situations, finite mixture of Pareto and finite mixture of beta can be an ideal choice for modelling a wide range of such situations when parametric methods are resorted to.

## 7.2 Future works

Estimation of the number of components is a special kind of model choice problem in the analysis of mixture model, for which there is a number of possible solutions. There is no universal method, and the implementation of the previous methods is often very difficult. Hence there exists a wide scope for attempting to find tractable solutions for the number of components in a mixture model, either by refining the existing ones or by trying some new approach.

Progressive censoring and hybrid censoring occur frequently in many real life situations and therefore analysis using mixtures in this context is an area for future study.

Bayesian estimation of stress-strength reliability using finite mixture models under complete as well as censored cases may also be attempted.

So far we have considered the maximum likelihood estimation of the Lorenz curve for measuring income inequality; another important area which would require attention is the Bayesian estimation of Lorenz curve and other income inequality measures using finite mixture models.

The analysis of finite mixture of discrete distributions is an area of research work that remains to be explored.

In this study, we have attempted parametric approach using finite mixture of Pareto and finite mixture of beta models in the univariate set up. However, a systematic study on finite mixture of multivariate distributions is not yet carried out. Accordingly, analysis of multivariate mixture models is also an area worth exploring.

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