FLUID MECHANICS

A Study On Vortex Knots

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Certificate

This is to certify that the work reported in the thesis entitled '**A Study on Vortex Knots**' that is being submitted by Smt.M.B.Rajeswari Devi for the award of Doctor of Philosophy, to Cochin University of Science and Technology is based on bonafide research work carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Declaration

I hereby declare that the thesis entitled '**A Study On Vortex Knots**' is based on the original work done by me under the supervision of Dr. M. Jathavedan in the Department of Mathematics Cochin University of Science and Technology,Cochin-22, Kerala. This thesis contains no material which had been accepted for any other Degree or Diploma in any University to the best of my knowledge and belief. It contains no material previously published by any person, except where due reference is made in the text of the thesis.

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Dedicated to my loving parents

Prof(Late)R.KrishnaWarrier and (Late) M.S.BhavaniAmma.

Contents

Chapter 1

Introduction

1.1 Topological fluid mechanics

The use of topological ideas in fluid mechanics and physics in general dates back to the very origin of topology as an independent science in the days of Karl Gauss [48], Johan Benedit Listing et.al[91]. But later developments using differential and integral calculus dominated leaving behind the topological approach.

Ordered or disordered vortical motion in nature, such as a tornado or a whirlpool, has long attracted attention in fluid dynamics. The physical quantity associated with vortical motion in a flow domain is the vorticity field which has provided a powerful qualitative description for many of the important phenomena of fluid mechanics. The formation and separation of boundary layers have been described in terms of production, convection and diffusion of vorticity. In turbulent flows the dissipation of energy at a rate independent of viscosity is explained by the amplification of vorticity by the stretching of vortex lines. The lift on an air wing is explained by the bound vorticity and trailing vortex structure. The concept of coherent structures in turbulent shear flows has led to the picture of such flows as a superposition of organized deterministic vortices whose evolution and interaction is the turbulence.

The non-linearity of the equation of vortex motion has made quantitative use of the concept difficult for the great scientist who founded and developed the subject. In the words of Truesdell [172], "vorticity generates those beautiful, intricate and perplexing phenomena which make the challenge of the theory of the motion of fluids, perfect or viscous, of more complicated in their dynamical response - a challenge for the most part declined by classical hydrodynamics - and that analysis of the basic kinematical properties of vorticity initiates a frontal attack upon the citadel of the non-linear convective acceleration". Trusedell's work, though published 50 years back, points out to many hitherto unsolved problems of the kinematics of vorticity.

The earlier concept of vorticity can be traced back at least to Leonardo da Vinchi and Descartes, though the first treatment of vorticity occurs in the work of d'Alembert and Euler; Langrange and Cauchy were the first to introduce single letters to stand for vorticity components. The kinematical significance of vorticity was recognized only when Mac-Cullagh and Cauchy proved that the components of the curl of the velocity vector satisfy the vectorial law of transformation.

The mathematical understanding of vortex motions begins with the renowned work of Helmholtz [59]. The name 'vorticity' was introduced by Lamb [81]. The original physical problem that motivated Helmholtz's great study was the non-linear study of motion of an ideal incompressible fluid governed by Euler's equation. He found interesting invariant properties of a dual analytic and topological nature for vorticity vector. The purely mathematical aspects of Helmholtz's ideas have been developed into the modern Hodge-Kodaira decomposition theorem for differential form on Riemannian manifolds [12].

The credit for the creation and unification of the discipline of vorticity transport goes to Truesdell [172]. The great significance of vorticity is aptly and beautifully recorded in [172], in the following words "before our eyes opens forth now the splendid prospect of three-dimensional kinematics, the mother tongue for man's perception of the changing world about him. It's peculiar and characteristic glory is the vorticity vector $\overrightarrow{\omega}$, for whose existence it is both requisite and sufficient that the number of dimensions be three."

It is not easy to measure the vorticity directly nor to analyze the three dimensional data of a flow field. In most experiments a passive scalar such as a dye or smoke ring has been used as a tracer of vortex tubes, but special care should be taken in interpreting these results because the evolution of a passive scalar Θ is quite different from that of vorticity field.

$$\frac{D\overrightarrow{\omega}}{Dt} = (\overrightarrow{\omega}.\nabla)\vec{u} + v\nabla^2\overrightarrow{\omega}. \tag{1.1}$$

There is only diffusion term in the evolution equation of Θ where as in the vorticity equation there is a term of stretching along the velocity field $\vec{u}$. The vorticity is stretched or weakened in the direction of eigen vector of the operator $(.,\nabla)\vec{u}$. The velocity field only advects the passive scalar field, on the otherhand. There have been many numerical simulations published so far for vortex interaction. But for understanding the fundamental mechanism of vortex interaction it is better to investigate the simplest situation that still retain some essential mechanism. Among others the interaction of two vortex tubes, either straight or curved (Pumir and Kerr [134], Melander and Zabusky [100], Melander and Hussain [99, 101, 102] ,Zabusky and Melander [177], Meiron et.al [98], Zabusky et.al [178], Boratev et.al [14], Shelly et.al [152]) and two vortex rings (Ashurst and Meiron [5],Kida et.al [75]) have been extensively studied. With relation to helicity dynamics, the time evolution of the trefoiled vortex tubes

(Kida and Takaoka [74, 76]) and the interaction of two elliptical vortex rings (Aref and Zawadzki [2]) have also been simulated. Computer simulation is of great advantage that the visualization of a complicated three-dimensional flow field can be relatively easily obtained. Two iso-surfaces of vorticity of magnitude $|\vec{\omega}|$ have often been used for visualization of vortical structure and for discussion of reconnection. Low-pressure region (Douady et.al [35]) and high strain region (Tanaka and Kida [161]) also characterize vortical structure.

Understanding the dynamics and mutual interaction among various types of vortical motion including vortex reconnection is a key ingredient in classifying and controlling fluid motion. The first analytical work on vortex reconnection goes back to the study of sinusoidal instability in a pair of counter rotating vortex tubes shed from the wings of an airplane (Crow [29]). The systematic study of vortex reconnection was initiated by observations and laboratory experiments of highly ordered vortical motions. Special and constant attentions have been paid to the topological structure of high vorticity concentrated region. One of the simplest and most fundamental experiments on vortex reconnection is the interaction of two colliding circular rings (Fohl and Turner [40], Oshima and Asaka [124], Oshima and Izutsu [125]). The motion of an elliptic vortex ring was studied by Hussain and Hussain [60]. They found a bending motion, a self-collision and a reconnection in the case of larger ellipticity.

Viscous effect is indispensable for vorticity reconnection to occur. Vortex reconnection in laboratory experiments is usually understood as a change in topology of a passive scalar, which behaves quite differently from vorticity field. In numerical simulation the reconnection is usually discussed for the iso-surfaces of vorticity magnitude, which is composed of vortex lines. It is therefore necessary to distinguish three types of reconnections - scalar, vortex and vorticity - scalar and vortex reconnection will respectively be referred to when the topology of iso-surfaces of passive scalar and of vorticity magnitude has changed, whereas vorticity reconnection apply when the topology of vorticity lines has changed [76].

According to Saffman [150] a vortex is a compact region of vorticity surrounded by irrotational fluid. Vortex dynamics then refers to the motion of such vortices under the influence of other vortices and (or) their own self induced velocity and possible external irrotational strain. Because the flow is entirely determined by the distribution of vorticity the evolution of a turbulent velocity distribution is a problem of vortex dynamics.

The general theme of vortex dynamics in turbulence may be expressed in the hypothesis that certain ranges of the broad spectrum of turbulence scales can be described as comprising ensembles of more or less coherent laminar vortex structures that either evolve internally or interact dynamically. The discovery by Batchelor and Townsend [8] and prediction by Landau, of intermittency in high Reynolds number, turbulent flow has led to extensive research in to the existence of "organized eddy structure" or "coherent structure" (Townsend[169]). One important issue is the shape or geometry of coherent eddies. Two principal candidates are tubes and sheets of vorticity. The work of Kuo and Corrsin [80] presented convincing evidence that the fine scale structure was more likely to consist of tubes than other possibilities. In addition to the fine scale tubular structures, the experiments of Brown and Roshko [17] demonstrated existence of a large-scale tube like structures in turbulent mixing layers. Ideas of this kind suggested simple and heuristic physical models of both the fine scale and the large-scale characteristics of the flow.

Saffman[148] developed a heuristic vortex sheet model and made prediction about the way in which flatness and skewness factors of arbitrary order depends on the Reynolds number. In a pioneering paper Synge and Lin [154] examined the consequences of assuming that isotropic turbulence could be modeled as a random superposition of Hill's spherical vortices. Hill's spherical vortex is a sphere containing vorticity in which the vortex lines are circles about the direction of propagation and the magnitude of vorticity is proportional to the radius of the circle. Tube like structures have been reported in many numerical simulations (Kerr[69, 70], Ashurst et.al [6], Vincent and Meneguzzi[175], Jimenez et.al[62] and Porter [131]). In these papers the term tube appears to mean a locally compact nearly axi-symmetric distribution of vorticity. The most recognizable tubes seem to constitute a region of the most intense vorticity. But it is not known what proportion of the total vorticity lies either within tube like structures or within the dominant velocity field of identifiable tubes. Townsend [167] used random ensembles of axi-symmetric and plane Burger's vortices to calculate velocity spectra in the dissipation range for isotropic turbulence.

There have been attempts to construct vortex based kinematical models based on more extended vorticity structures. Chorin [21, 22, 23] has developed a vortex lattice model of the inertial range scales. The vortex structure is represented by lines composed of contiguous elements, each positioned on the orthogonal generators of a three dimensional rectangular lattice. There may be several such vortices on the lattice, each with constant circulation, and these are not allowed to intersect. Energy functional is defined as that models in a discrete way the energy of continuous vorticity distribution. In this sense the model has kinematics content. Vortex folding is observed in numerical simulation. The energy spectrum is obtained by first estimating the Hausdorff dimension of the lattice vortex structure, followed by the use of scaling and dimensional arguments. Another simple kinematic model of turbulence is provided by the fields induced by a distribution of large number of discrete vortex elements. Min et.al [103] discuss the probability density function for the velocity and the velocity difference for singular and blob-like vortex elements in two and three dimensions.

A vortex-based description of energy containing scales, in real turbulent flow should introduce inhomogenity and account for the boundary condition that creates vorticity. Townsend's 'attached eddy' hypothesis [169] for the turbulent boundary layer has been developed in a quantitative way by Perry and Chong [127] and Perry and Marasic [128]. They suppose that the region of a turbulent boundary layer is composed of slender vortex rods with a Λ -like shape. The significance of the work lies in the attempt to construct quantitative vortex morphology of a classical turbulent flow namely the highly non-homogenous, constant pressure gradient turbulent boundary layer.

The possibility that the dynamics of vortex filament could help to understand the large-scale properties of nonhomogenous turbulent flows was examined by Robbinson and Saffman [147]. They examined the stability of three classical steady vortex configuration of filaments to three -dimensional disturbances in an incompressible inviscid fluid in the limit of small vortex cross sectional area and long axial disturbance wavelength. The motion of the filament is then given by the Moore and Saffman cut off approximation to the Biot-Savart law [119]. Leonard [86] also describes calculation of fine interacting vortex filaments, each of finite core radius. The evolution of filaments was calculated by a vortex numerical method. Follow up calculations using a similar method and various initial geometries of vortex rings were done by Kiya and Ishii [77].

Approximate solution to the unsteady Navier-Stokes equations have been known since the 1930's and have found application to the structure of the starting vortex by Saffman [150] and the laminar trailing vortex by Moore and Saffman [119]. Lundgren

[94] adapted this structure to model the fine scale turbulence, replacing the steady Burger's vortices in the Townsend's ensemble by unsteady stretched spiral vortices. The dual properties of axial straining combined with a non axi-symmetric vorticity structure give the model a rich predictive capability. The presence of axi-symmtric strain that exponentially stretches vortex lines is a crucial model ingredient.

A challenging and still open question is the problem of the finite time singularity for unsteady solution of the Euler equation. This is the problem of vortex dynamics for the case in which the initial vorticity distribution consists of a number of vortices or, in particular, a vortex filament. A paper of Battacharjee et.al [13] demonstrates the existence of a singularity. A second open question in the significance for turbulence theory is that of existence of a finite time singularity. In [13] he also claim to have established a connection by showing that the presence of spiral structure in the initial conditions leads naturally to the Lundgren spiral vortex model[94]. An alternative approach has been presented by Constantin et.al in [28]. This work appears to consider constraints imposed on a turbulent velocity field by the requirement that a singularity does not occur.

Apart from these studies using differential and integral calculus, recent years have witnessed renewed interest in topological studies in fluid flows. This has resulted in the birth of a new branch of research called Topological Fluid Mechanics.

In fact Herman Helmholtz's paper [59] on vortex motion made it possible to apply topological ideas to Fluid Mechanics. Impressed by Helmholtz's work on vorticity Lord Kelvin [67, 68] believed in the eternal existence of vortex atoms as fundamental constituents of nature. In his theory atoms are thought to be tiny vortex filaments in the fluid ether. The different chemical compounds are given birth by topological combinations of linked and knotted vortices. His work was seminal in the development of topological approach to fluid flow analysis. The work of J.J.Thomson [164] on vortex links and studies of fluid flows are notable. Lichtenstein [90] dedicated two of the eleven chapters in his book on hydrodynamics to topological ideas. But the difficulty of an immediate application limited, for many years, the use of these concepts. In recent years the application of new research from topology and knot theory and greater access to the direct numerical simulation of fluid flows have led to new developments in the qualitative study of fluid mechanics. According to H.K. Moffatt [109, 111] "topological, rather than analytical techniques and language provide the natural frame work for many aspects of fluid mechanical research that are now attracting intensive study."

In an ideal fluid there are no dissipating effects. Therefore fluid structures will not diffuse or die out. It is known from the studies of Helmholtz and Kelvin that vortex line topology is frozen in the case of ideal fluids. But structures of these lines in continuous motion can be distorted by the background flow. Thus if vortex lines or tubes are initially knotted or linked, they may evolve or deform but preserve the type of knot or link. It follows that topological properties of ideal fluids are flow invariant and physical information expressed in pure topological terms is bound to be conserved during flows. Based on topological interpretation of a new fluid invariant known as 'helicity' Moffatt established new fundamental connection between ideal fluid mechanics and topology. Here helicity is a measure of the knottedness of vortex lines or tubes [104].

The study of knots and links is a part of topology. Thus it has developed, independent of physics or fluid mechanics, as a branch of geometry that deals with flexible and deformable spaces. Thus knot theory is the study of embedding of circles in spaces, independent of materials forming the circles. In the context of fluid mechanics this circle is made of vortex lines.

It was Listing, a student of C.F.Gauss, who first undertook research on knots. In honour of his accomplishment the knot, at present known as 'Figure 8 knot' was originally called 'Listing knot'. But it was the American mathematician J.W.Alexander [1] who pointed out the importance of knot theory in the study of three dimensional topology. The polynomial invariant known as 'Alexander polynomial' bears testimony to this. H.Seifert further developed the work.

After Second World War, research on knot theory progressed at a pace in the United States and Japan. In the 1970's it was shown that knot theory is connected to Algebraic number theory by way of solution of Smith's Conjecture concerning periodic mappings. After the epoch making discovery of the invariant associated with Jone's polynomial at the beginning of 1980's knot theory moved from the pure mathematics' realm of topology to mathematical physics. It is seen that knot theory is closely related to solvable models of statistical mechanics. It has found applications in divorce fields like Mathematical biology and Chemistry in Murasugi [79]. For example, in Biology, certain types of DNA molecules are seen to take the form of some wellknown knots [63].

Relationship between topology and dynamics of fluid structures have been investigated from the late 1980's. In ideal conditions, ie, in the absence of dissipative effects, all topological properties and physical quantities are conserved. These form a set of scalar and vector invariants that guide the evolution of the system towards homotopy solution whose existence is guaranteed by the diffeomorphisms associated with the flow maps. Changes in the topology of the system occur only if the singularities, bifurcation and dissipative effects present. In a model of a real flow these dissipative effects are not completely negligible. Thus results obtained by the techniques of ideal topological fluid mechanics are in fact preliminary to the real flows.

1.2 Knots

A mathematical knot is a closed curve in 3-dimensional space that does not self intersect and has no thickness.

Here we are not going to the details of knot theory. What are relevant to our study only are considered.

Definition 1. *A knot K in R^3 is a subset of points homeomorphic to a circle.*

The simplest example is a circle itself, which is called a trivial knot. We know that the above definition says nothing about how the points should be arranged in space. Two possible ways are a polygon and a smooth curve.

Like knots in a string we want mathematical knots to be flexible. i.e., a mechanism by which a thread can be deformed into different positions. A mere homotopy is of no use for deforming knots because it allows the curve to pass through itself, thus making all knots homotopic to the trivial knot. Though isotopy, i.e., a one-one homotopy, avoids this problem of curve passing through itself, all knots are again isotopic to the trivial knot. Because a mathematical knot has no thickness, we can reduce it until it became a point and disappear. Such a continuous transformation is called Batchelor's unknotting. This problem can be overcomed by a minor modification so that the space containing the knot moves continuously along with the knot. One-way to visualize this is to imagine a knot to be embedded in a viscous syrup. Stirring syrup causes the knot to be moved. Mathematically it is called an ambient isotopy.

An ambient isotopy is an equivalence relation on knots. Two knots are equivalent if one can be deformed into the other by an ambient isotopy. Each equivalence class is called a knot type.

We will concentrate on knots that lie on the surface of a standard torus. These are called torus knots and are the simplest knots to describe parametrically.
Taking a circle in the YZ-plane of radius r, centered on the Y-axis at distance R+r from the origin, then rotating it about the Z-axis, can generate a torus. If we parameterize the circle by angle $\Theta \in [0, 2\pi]$, and the rotation by an angle $\phi \in [0, 2\pi]$, we express the torus as

$$\underbrace{\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\textbf{Rotation}} \underbrace{\begin{pmatrix} 0 \\ R + r\cos\Theta \\ r\sin\Theta \end{pmatrix}}_{\textbf{Circle}} = \begin{pmatrix} -\sin\phi(R + r\cos\Theta) \\ \cos\phi(R + r\cos\Theta) \\ r\sin\Theta \end{pmatrix}$$

The parameters r and R control the geometry of the torus; r is the radius of the hole. The angles form a co-ordinate system; any point on the torus can be labelled by a pair of the form (Θ, ϕ).

The above torus can be described as the cross product of two circles of radii R and r, $R > r$. A parameterized curve on the torus is defined as $\phi = \phi(t), \Theta = \Theta(t)$.

Definition 2. *Winding number of a curve on the surface of a torus* $\phi = \phi(t), \Theta = \Theta(t)$ *is the number* $W = \lim_{t \to \infty} \frac{\Theta(t)}{\phi(t)}$

This is the average number of wraps of the curve over small radius per unit wrap around the larger radius. The winding number exists and is unique if the curve is not self intersecting. This number is an invariant for the curve under any deformation without self-intersection.

Following [65] we define a torus knot as follows:

Definition 3. *A non-trivial torus knot T (p,q) is a closed non-self intersecting curve on the surface of a torus with winding number* p/q, $(p > q > 1)$, *where p and q are relatively prime.*

There are two distinguished curves on the torus: the meridian T(1,0) and the longitude T (0,1). Clearly these are trivial knots. An example of non-trivial knot is a 'trefoil knot' whose winding number is $\frac{3}{2}$. This is a curve that wraps three times around the small radius of the torus for two wraps around the large radius of the torus.

In general any knot constructed as a wrapping of a torus is called a torus knot.

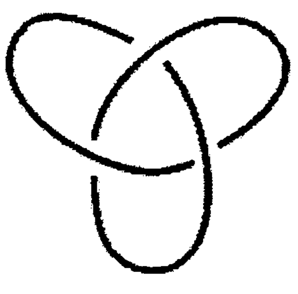

Figure 1.1: Trefoil Knot

1.3 Dynamics of line vortices

In the past four decades the study of vortices and vortex motion has received continuing attention due to problems arising in physics, engineering and mathematics. It is said that vortices are the 'sinews and muscles' of fluid motion. The discovery of coherent structures in turbulence has kindled the hope that the study of vortices will lead to models and an understanding of turbulence - the greatest unsolved problem of classical physics.

The governing equations of fluid dynamics in Eulerian approach resulted in an infinite dynamical system. But when we concentrate on vorticity, by the theorems of Kelvin and Helmholtz, vortices in the case of inviscid liquids are confined to well-defined domains that results in coherent structures mentioned above. Given a flow with a given vorticity distribution, there is a well defined relation between velocity distribution and vorticity. The advances in computation techniques and computer technology has made possible the solutions of problems of vortex motion which are not amenable to analytical methods. An added advantage in the recent years is remarkable advances in the experimental techniques.

A special case is a flow field characterized by values of the magnitude of vorticity in the neighborhood of a certain line in the fluid much larger than that elsewhere. Tornados and whirlpool are examples from nature. A useful mathematical idealization is derived from such cases by supposing the line to be a vortex tube whose cross sectional area tends to zero, the strength remaining constant, say Γ. This gives a line singularity of the vorticity distribution, specified by the value of ω and the position of the line. Such an idealized line is called a 'line vortex of strength Γ'. The generators of a vortex tube are vortex lines and hence by our definition the vorticity is tangential to the line vortex everywhere.

Now to the ideal geometry of line vortex we add the dynamics of vortex tubes. Thus, as a limiting case of a vortex tube, line vortex moves through the fluid without change of strength. It can be seen that velocity distribution has singularities at points on the line vortex. There is a circulatory motion around any portion of the line vortex. But this circulatory motion can only rotate the infinitesimal cross section about its center and can not translate it. In addition to this, there is another weaker singularity of the velocity distribution associated with the local curvature of the line vortex. Thus the fluid in the neighborhoods of the line vortex has large velocity in the direction of the binormal. It is seen that an ideal curved line vortex in the mathematical sense move with infinite speed and also changes its shape with infinite speed [9].

The above facts point to the difficulty of considering line vortex as a geometrical line in the limiting case of vanishingly small cross sectional area. Thus what we consider in our study is in fact vortex filaments of infinitesimal thickness so that the deformations and speed are finite.

1.4 Localised induction concept

The invention of high-speed computers has replaced the earlier laboratory experiments by mathematical modeling and simulation. This is applicable to the study of the onset of turbulence in fluid flows. It was found that a curved vortex loop is formed in one way or another before the final breakdown to turbulence takes place. Thus the last stage of transition process hinges upon further developments of vortex loop. In the numerical study by Hama and Nutant [52] it was found that the self-induced velocity of a curved vortex filament is maximum in regions where the curvature of the vortex filament is maximum.

Another problem is the generation of large-scale vortex structures in turbulent media. By large-scale structures we mean large-scale velocity fields, averaged over an ensemble of realizations of random flow, for which the spatial and temporal scales exceed considerably the respective scales of the back ground turbulence (Moiseev and Sagdeev et.al [118]). It is known that turbulent convection in rotating volumes in atmosphere is spiral and not mirror - symmetrical. Under the action of the Coriolis force, convection cells swirls around the vertical. The directions of the swirl of convection cells are opposite in the northern and southern hemispheres.

The velocity induced by a line vortex is given by Biot- Savart Law. But direct application of these in a numerical study presents problems of instability. To overcome these, the localized induction approximation (LIA) was first applied by Hama [53]. The method is to neglect the long distance induction, hence the name localized induction approximation. In a later paper by Hama [55] has briefly discussed the origin of the LIA. Arms and Hama [4] have studied the motion of an elliptic vortex ring using this method. Another later study is that by Betchov [137] on the curvature and torsion of an isolated vortex filament. He found that an isolated vortex filament starting from some random initial state may find statistical equilibrium between production and dispersion of regions of concentrated torsion. This points to the possibility that a mass of turbulent fluid can perhaps be considered as a system of entangled vortex filaments.

The LIA is developed to study the behavior of a curved vortex filament in an inviscid incompressible fluid. The essential feature of this method is the approximation of the local motion of the filament by that of a thin circular vortex with the same curvature. The self -induced velocity at a point evaluated through the Biot-Savart integral is dominated by only neighboring segments in addition to a short cut -off introduced to avoid the logarithmic infinity present in the expression. The essence of the method is to retain the minimum of the details for the self -induced motion of a vortex filament. Thus a point on the filament is considered to be influenced only by the neighboring segments of length 2L and evolve according to

$$\frac{\partial \vec{X}}{\partial t} = \frac{\Gamma}{4\pi} \log\left(\frac{L}{\sigma}\right) \kappa \vec{B}, \tag{1.2}$$

where $\vec{X} = \vec{X}(s,t)$ denotes a point on the filament as a function of arc-length 's' and time 't'. Further, κ is the curvature and $\vec{B}$ is the binormal vector. Γ is the circulation and σ the short-range cut off introduced to avoid the logarithmic singularity in the Biot -Savart Integral. Da-Rios [31] transformed this equation into a coupled system of intrinsic equation for the curvature κ and torsion τ. They are now known as Betchov Da-Rios equation as they were independently derived by Betchov also. Travelling wave solutions of these equations were sought by Da-Rios [31] and Levi-Civita [88]. Later Hasimoto [57] discovered that this coupled equations lead to a non-linear Schrödinger equation with the introduction of a complex variable with curvature as amplitude and torsion angle as phase. It is seen that this equation admits a solution describing a solitary wave propagating along a line vortex filament which induces various types of motion of the filament according to the value of the torsion.

The reduction of LIA equations to non-linear Schrödinger (NLS) equation has lead to the possibility of using the rich theory of solitons to the study of vortex filament motion. But as pointed out by Kida [72] it has limitations also. For example, it is possible only in principle and rather difficult in practice to determine the shape of the vortex filament from the solutions of NLS. Also the physical concepts such as the translational and rotational velocities of the vortex filament appear explicitly in the solutions of LIA equations. Thus it is interesting that Kida was able to obtain, by solving the LIA equation, a family of vortex filaments which move without change of form like solitons. These solutions were expressed in terms of elliptic integrals of the first, second and third kinds. In general they represent the vortex filaments of infinite length and not closed ones. In some cases they take the form of torus knots also.

In essence LIA is the study of geometry of line vortices. A line vortex has a geometric structure of its own and its curvature places a decisive role in its dynamics. Since the motion is related to local geometry it is described in terms of intrinsic properties of a curve that do not depend on the choice of the co-ordinate system.

We know that in the case of space curve there are three orthogonal unit vectors tangent $\vec{T}$, normal $\vec{N}$ and binormal $\vec{B}$. These define Frenet trihedron at a point. Curvature κ and torsion τ are the two intrinsic parameters that define the rotation of the Frenet tri-hedron. Therefore this determines the shape of a smooth curve. If the curvature and torsion are known functions of the arc-length parameter s the curve $\vec{X} = \vec{X}(s,t)$ can be reconstructed at least in principle by integrating the Frenet-Serret equation

$$\begin{aligned} \vec{X}_s &= \vec{T}, \\ \vec{T}_s &= \kappa\vec{N}, \\ \vec{N}_s &= -\kappa\vec{T} + \tau\vec{B} \text{ and} \\ \vec{B}_s &= -\tau\vec{N}. \end{aligned} \tag{1.3}$$

Since the arc length changes as the curve moves, the derivative with respect to t and s do not commute. In [65] another parameter x is used so the arc length $s = Ax$. As the curve moves in space its motion can be described by the equation:

$$\vec{X}_t = \gamma\vec{T} + \alpha\vec{N} + \beta\vec{B}, \tag{1.4}$$

Here the functions α and β are determined by the physics of the problem. Thus, in a purely geometrical study we have to consider the evolution equation for κ, τ, A and γ. From the Frenet-Serret equation we can obtain the evolution equations for κ, τ and A. Thus we get three evolution equations in four unknowns. Then the system of equations is closed by imposing a gauge condition. Most commonly used gauge conditions are the commoving gauge γ=0 and the isometric gauge A=1. In the latter case we have $\gamma_s = \kappa\alpha$[130].

Motion along binormal preserves the local metric of the curve. Thus its motion is particularly simple and the comoving gauge is also isometric. Historically it was such motion that was first studied by directly solving the Frenet -Serret equation. The basic equation is given by

$$\vec{X}_t = \vec{X}_s \times \vec{X}_{ss}. \tag{1.5}$$

In fact, this is the equation derived by Da-Rios in [31].

We are not going to the details of the derivation of LIA. We shall point out some of the limitations inherent in the LIA. It is well known that the velocity of slender vortex filament depends on its core structure and is not defined in the limit of zero core size, in Lamb[82]. Because of the filament stretching, the core structure variation is coupled with the velocity of the filament. But by Helmholtz theorem, the circulation along any closed curve around the filament is a constant. In fact, the circulation is equal to the surface integral of the vorticity on the cross section of the tube with surface having the closed curve as its boundary. In the ideal situation of a filament this section is shrinked to the corresponding point of the tube. Then the vorticity is identified by a δ-vector measure in the direction of the tangent vector at a given point with modulus Γ, the strength of the tube. From this point of view it is natural to assume that the flow preserves the arc-length parameterization. This implies that if we start with a closed filament the total length will be preserved. This is highly unexpected when a vortex tube of positive cross section is considered.

In the isometric case the partial derivatives with respect to t and s commute so that the partial differential equation (1.5) can be solved numerically in a straight forward way. But the assumption of isometry that makes complete integrability of the equation possible is destroyed by any big advection in the normal direction. Moreover the induction is usually non-local, making the local velocity dependent on the instantaneous shape of the curve. An accumulated action of even weak, normal and/or nonlocal advection causes the filament to shrink or stretch at a different location and typically leads to highly convoluted or even singular shapes. Thus the advantages of computational simplicity are all wiped out when non-local induction is important. Another criticism of LIA is due to the structural instability of the locally induced binormal motion, Pismen [129, 130]. Inspite of all these criticisms, LIA has been extensively used due to its mathematical elegance.

Writhing number of knots is analogous to the helicity for vector fields [10, 61]. In fact it is the linking number of two disjoint closed space curves defined by Gauss [48]. The concept of helicity was introduced by Woltjer [176] and was so named by Moffatt [104] in the context of Fluid mechanics. Writhing number was introduced by Călugăreanu [18, 19] though the name was introduced by Fuller [47]. It has applications in molecular biology in the study of knotted duplex DNA and of the enzymes that affect it, Jason Cantarella et.al [61]. There are several unavoidable problems in the attempt to reconcile knot theory and fluid dynamics. The basic problem is that little is known about the rigorous behaviour of fluid flows as evident from the fact that the global existence of solutions to the Euler and Navier-Stoke's equations in 3-dimensional case are unknown and perhaps not true.

1.5 About the thesis

In this thesis an attempt is made to study vortex knots based on the work of Keener [65]. It is seen that certain mistakes have been crept in to the details of this paper. We have chosen this study for an investigation as it is the first attempt to study vortex knots. Other works had given attention to this. In chapter 2 we have considered these corrections in detail. In chapter 3 we have tried a simple extension by introducing vorticity in the evolution of vortex knots. In chapter 4 we have introduced a stress tensor related to vorticity. Chapter 5 is the general conclusion.

Knot theory is a branch of topology and has been developed as an independent branch of study. It has wide applications and vortex knot is one of them. As pointed out earlier, most of the studies in fluid dynamics exploits the analogy between vorticity and magnetic induction in the case of MHD. But vorticity is more general than magnetic induction and so it is essential to discuss the special properties of vortex knots, independent of MHD flows. This is what is being done in this thesis.

Chapter 2

Keener's Analysis on Vortex Knots

The dynamical behavior of closed vortex filaments is a complicated problem. The perturbation method introduced by Keener [65] gives new opening for dealing with this. Here a knotted curve is realized as bifurcating from a circular curve. The basic principle is that a circle is an unknot and other knots are obtained starting from a circle in the reverse order. For example a trefoil knot can be viewed as a curve on a torus with winding number $3/2$. As the small radius of the torus is allowed to go to zero, the trefoil knot becomes a twisted multiple cover of a circle. Since a circle has constant curvature and zero torsion, bifurcation of a trefoil knot from a circle occurs if the deviation of the curvature from a constant and the torsion oscillate sinusoidally three times as the circle is traversed twice.

In spite of interesting results, we find that there are some corrections needed in the equations and solutions of [65]. We are presenting the required corrections.

2.1 Intrinsic Equations

Following [65], we consider a curve $\vec{R} = \vec{R}(x,t)$ in 3- dimensional space described in terms of its tangent, normal and bi-normal unit vectors $\vec{T}, \vec{N}$ and$\vec{B}$ where x is a parameter and t time. The quantities $\vec{R}, \vec{T}, \vec{N}$ and $\vec{B}$ are related by the following Frenet-Serret equations (F-S equations).

$$\begin{aligned} \vec{R}_x &= A\vec{T}, \ |\vec{R}_x| = A, \\ \vec{T}_x &= A\vec{N}, \\ \vec{N}_x &= A(-\kappa\vec{T} + \tau\vec{B}), \\ \vec{B}_x &= -\tau A\vec{N}. \end{aligned} \tag{2.1}$$

Here the parameter x is not the arc length. A is taken as independent of the parameter x and the length of the curve is A times the variation of the independent variable x so that the arc length co-ordinate for the curve is Ax. A is used to adjust the total length of the curve so that it remains closed.
The motion of the curve in space is described by

$$\vec{R}_t = \gamma\vec{T} + \alpha\vec{N} + \beta\vec{B}. \tag{2.2}$$

where α and β are the components of velocity $\overrightarrow{R_t}$ in the normal and bi-normal direction respectively. The velocity component γ in the direction of the tangent does not contribute to the change in shape.

Keener studied the behavior of closed curves according to equation (2.2). A simple example is a circle that moves without change of shape in the direction normal to its plane with constant velocity β evaluated at $\kappa = \kappa_0$ and $\tau = 0$. In order to find other solutions, the equation (2.2) is reformulated so that the shape of solution curve is obtained without reference to a co-ordinate system. For this, the equation of motion of curvature, torsion and the tangential velocity are derived which do not make any reference to the position vector $\vec{R}$.

Thus he obtained the equations

$$\begin{aligned}
(\kappa A)_t &= v_x - \tau u A, \\
(\tau A)_t &= z_x + \kappa u A \text{ and} \\
A_t &= \gamma_x - \alpha \kappa A.
\end{aligned} \tag{2.3}$$

Here u, v, and z are defined by

$$\begin{aligned}
u &= \beta_x + \alpha \tau, \\
v &= \tfrac{\alpha_x}{A} + \gamma \kappa - \tau \beta \text{ and} \\
\kappa z &= \tau v + \tfrac{u_x}{A}.
\end{aligned} \tag{2.4}$$

Since the curve considered is closed, the position vector must be a periodic function of arc-length co-ordinates. This in turn implies that the curvature and torsion should be periodic and

$$\int_0^p \vec{T} dx = 0, \tag{2.5}$$

where P is the period of the solution and $\vec{T}$ is the tangent to the curve $\vec{R}$.

2.2 Simplified form of Frenet-Serret equations

To show that equation (2.5) is satisfied, the evolution equation must be solved along with the F-S equations. Since $\vec{T}$, $\vec{N}$ and $\vec{B}$ are unit vectors, Keener [65] expressed $\vec{T}$ and $\vec{N}$ in spherical co-ordinates as

$$\vec{T} = \begin{pmatrix} \cos\phi\cos\Theta \\ \cos\phi\sin\Theta \\ \sin\phi \end{pmatrix} \text{ and } \vec{N} = \begin{pmatrix} \cos\psi\cos\eta \\ \cos\psi\sin\eta \\ \sin\psi \end{pmatrix}. \tag{2.6}$$

Here Θ, η, ϕ and ψ are such that angles Θ and η correspond to longitude and angles Θ and ψ to latitude in a spherical co-ordinates system. Taking $\eta - \Theta = \frac{\pi}{2} + q$ the condition $\vec{T}\cdot\vec{N} = 0$ gives the relation

$$\sin\phi\sin\psi - \cos\phi\cos\psi\sin q = 0. \tag{2.7}$$

This expresses the angle η in terms of independent angular variables ϕ, ψ and Θ. Then the Frenet-Serret equations can be expressed as three equations in terms of the three angular variables ϕ, ψ and Θ as

$$\begin{aligned} (\sin\phi)_x &= \kappa\sin\psi, \\ (\sin\psi)_x &= A(\tau\cos\phi\cos\psi\cos q - \kappa\sin\phi) \text{ and} \\ \Theta_x &= \kappa A\frac{\cos\psi\cos q}{\cos\phi}. \end{aligned} \tag{2.8}$$

With the substitution $\sin\phi = \Phi$ and $\sin\psi = \Psi$ the F-S equations can be written as

$$\begin{aligned} \Phi_x &= A\kappa\Psi, \\ \Psi_x &= A(\tau X - \kappa\Phi) \text{ and} \\ \Theta_x &= \frac{A\kappa X}{1-\Phi^2}, \end{aligned} \tag{2.9}$$

so that

$$X^2 + \Phi^2 + \psi^2 = 1 \text{ where} X = \cos\phi\cos\psi\cos q. \tag{2.10}$$

Then the vector $\vec{T}$ has components

$$\vec{T} = \begin{bmatrix} \sqrt{1-\Phi^2}\cos\Theta \\ \sqrt{1-\Phi^2}\sin\Theta \\ \Phi \end{bmatrix}. \tag{2.11}$$

To find the closed curve solution, he sought spatially periodic solution of the equations (2.3) and (2.4). F-S equation (2.9) was solved to determine the tangent vector $\vec{T}$ through (2.11). To get closed curve solution $\vec{T}$ should satisfy the condition (2.5).

2.3 Torus knots

In [65] it is showed that torus knot can be constructed as a solution of F-S equations. Suppose the curvature and torsion of nearly circular closed curves are known. Let the curvature and torsion be small-amplitude oscillatory functions that oscillate m times as the circle is traversed n times. Clearly this is a closed curve solution of F-S equation and is a torus knot with winding number m/n. The existence of such solution is given by the following theorem, ,referred as theorem 3.3 in [65]).

Theorem 1. *Suppose that $\kappa_1(x)$ and $\tau_1(x)$ are periodic functions of period P, and that there are integers m and n so that $mP\kappa_0 = 2n\pi$. Suppose further that*

$$\begin{aligned} &\int_0^{mP} \kappa_1(x)\sin\kappa_0(x)dx = 0, \int_0^{mP} \kappa_1(x)\cos\kappa_0(x)dx = 0, \\ &\int_0^{mP} \tau_1(x)\sin\kappa_0(x)dx = 0, \int_0^{mP} \tau_1(x)\cos\kappa_0(x)dx = 0 \ \text{and} \int_0^{mP} \tau_1(x)dx = 0. \end{aligned} \tag{2.12}$$

Then for all ε sufficiently small there are functions $A = A(\varepsilon)$ and $\tau_0 = \varepsilon^2\tau_2(\varepsilon)$ and a closed curve whose curvature and torsion are given by $\kappa(x) = \kappa_0 + \varepsilon\kappa_1(x)$, $\tau(x) = \varepsilon\tau_1(x) + \varepsilon^2\tau_2(\varepsilon)$. The arc length variable for the curve is $s = A(\varepsilon)x$.

The proof of the theorem needs correction and is given below. Subsequent changes are needed in the remaining part.

The variables of equation (2.9) have power series expansions in ε of the form

$$\begin{aligned}\Phi(x) &= \varepsilon\Phi_1(x) + \varepsilon^2\Phi_2(x) + \cdots\cdots,\\ \Psi(x) &= \varepsilon\Psi_1(x) + \varepsilon^2\Psi_2(x) + \cdots\cdots,\\ \Theta(x) &= \kappa_0 x + \varepsilon\Theta_1(x) + \varepsilon^2\Theta_2(x) + \cdots\cdots \\ \text{and } X &= 1 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots\cdots \\ \text{where } A &= 1 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots\cdots\end{aligned} \tag{2.13}$$

Keener does not consider an expansion for X. In fact equation (2.10) shows that X also is to be expanded in powers of ε. The equation also shows that an expression for X should be of the above form.

The tangent vector is given approximately by

$$\vec{T} = \begin{pmatrix} \cos\kappa_0 x \\ \sin\kappa_0 x \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -\Theta_1(x)\sin\kappa_0 x \\ \Theta_1(x)\cos\kappa_0 x \\ \Phi_1(x) \end{pmatrix} + \cdots \tag{2.14}$$

Substituting in equations (2.9) we get, to the order of ε

$$\begin{aligned}\frac{d\Phi_1}{dx} &= \kappa_0\Psi_1(x),\\ \frac{d\Psi_1}{dx} &= (\tau_1(x) - \kappa_0\Phi_1) \text{ and}\\ \frac{d\Theta_1}{dx} &= A_1\kappa_0 + \kappa_1(x) + \kappa_0 X_1.\end{aligned} \tag{2.15}$$

But Keener [65] did not consider the power series expansion for X, as in equation (2.15). X is also periodic so has to satisfy

$$\frac{1}{mP}\int_0^{mP} X dx = 0. \tag{2.16}$$

The periodic solution of equation (2.15) exists only if the solvability conditions are satisfied; i.e, if and only if $\tau_1(x)$ is orthogonal to both $\sin\kappa_0 x$ and $\cos\kappa_0 x$ in the interval $0 \leq x \leq mP$ and

$$A_1\kappa_0 = -\frac{1}{mP}\left[\int_0^{mP}\kappa_1(x)dx + \kappa_0\int_0^{mP} X_1 dx\right], \tag{2.17}$$

instead of what Keener [65] obtains in his equation (3.7).

Θ_1 is periodic. So if the average values of X_1 and that of $\kappa_1(x)$ is zero, we get $A_1 = 0$. Further $\int_0^{mP}\vec{T}dx = 0$ since the curve is closed. To the first order in ε, this implies that $\kappa_1(x)$ is orthogonal to both $\sin\kappa_0 x$ and $\cos\kappa_0 x$ and that the average value of $\tau_1(x)$ is zero. i.e, $\frac{1}{mP}\int_0^{mP}\tau_1(x)dx = 0$.

Here the scale factor $A = A(\varepsilon)$ and the average value of τ is used to ensure that average value of Φ is zero. These observations enable us to write equations(2.5), (2.9) and (2.11) in a form to which the theorem can be applied so that there is a solution to the problem for all ε sufficiently small.

Following [65], we consider

$$\begin{aligned} \kappa(x) &= \kappa_0 + \varepsilon a \sin \mu x + \varepsilon b \cos \mu x \text{ and} \\ \tau(x) &= \tau_0 + \varepsilon c \cos \mu x. \end{aligned} \tag{2.18}$$

Taking,

$$\kappa_1(x) = a \sin \mu x + b \cos \mu x \text{ and}$$

$$\tau_1(x) = c \cos \mu x$$

and applying the theorem, the solution of the equation (2.15) to leading order is

$$\begin{aligned} \Phi_1(x) &= \frac{c\kappa_0 \cos \mu x}{\kappa_0^2 - \mu^2}, \\ \Psi_1(x) &= \frac{c\mu \sin \mu x}{\mu^2 - \kappa_0^2} \text{ and} \\ \Theta_1(x) &= -\frac{a}{\mu} \cos \mu x + \frac{b \sin \mu x}{\mu} + \kappa_0 \int X_1 dx. \end{aligned} \tag{2.19}$$

At each order in the perturbation calculation, the coefficients A_i is used to guarantee that $\Theta(x) - \kappa_0 x$ is periodic and average value of torsion is adjusted so that the curve is closed. The second and third expressions in solution (2.19) are corrections to the referred paper of Keener [65].

We have

$$\frac{d\Theta_2}{dx} - \kappa_0\Phi_1^2 = A_0(\kappa_0 X_2 + \kappa_1 X_1) + A_1(\kappa_1 + \kappa_0 X_1) + A_2\kappa_0$$

corrected to the second order of perturbation. Since Θ_2 is periodic, taking the average values of X_1 and that of X_2 as zero, A_2 can be calculated as follows.

We have

$$A_2\kappa_0 \int_0^{mP} dx = -\int_0^{mP} \kappa_0 X_2 dx - \int_0^{mP} \kappa_1(x) X_1 dx - \kappa_0 \int_0^{mP} \Phi_1^2 dx$$

This gives

$$A_2 = -\frac{1}{mP}\int_0^{mP} \frac{c^2\kappa_0^2(1+\cos 2\mu x)}{(\kappa_0^2-\mu^2)^2} = -\frac{c^2\kappa_0^2}{2(\kappa_0^2-\mu^2)^2} < 0. \tag{2.20}$$

Therefore

$$\begin{aligned} A(\varepsilon) &= A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots\cdots \\ &= 1 + 0 - \frac{c^2\kappa_0^2\varepsilon^2}{2(\kappa_0^2-\mu^2)^2} + \cdots\cdots \end{aligned} \tag{2.21}$$

This shows that the length of the curve changes as a function of ε and $A(\varepsilon) < 1$. So for non-zero ε the arc length of the curve is $2n\pi\frac{A(\varepsilon)}{\kappa_0} < \frac{2n\pi}{\kappa_0}$ in contrast to what is obtained in [65].

If the curve $\vec{R}$ is closed satisfying the conditions laid down in the theorem, then it will form a torus knot having winding number m/n where m and n are co-prime integers. In [65] this result is verified by showing that the curve $\vec{R}$ lies on a torus and by calculating its winding number. For this, following [65], we write the curve $\vec{R}(x)$ in toroidal co-ordinates as

$$\vec{R}(x) = \vec{R}_0(t) + \alpha(t)\vec{N}(t) + \beta(t)\vec{B}(t)$$

where $\vec{R}_0(t)$ is the centreline of the torus, and $\vec{N}$ and $\vec{B}$ are the normal and binormal vectors for the centreline. The centreline $\vec{R}_0(t)$ is taken to have components

$$r_{01}(t) = \frac{1}{\kappa_0}\sin\kappa_0 t,\ r_{02}(t) = -\frac{1}{\kappa_0}\cos\kappa_0 t \text{ and } \ r_{03}(t) = 0,$$

which is a circle.

Then

$$\alpha(t) = \frac{1-\kappa_0^2\vec{R}(x).\vec{R}_0(t)}{\kappa_0}$$
$$= \frac{1}{\kappa_0}\left[\begin{array}{l} 1-\cos\kappa_0(x-t)+\frac{\varepsilon\kappa_0}{\mu(\mu^2-\kappa_0^2)}\{a\left[\begin{array}{l}\kappa_0\cos\mu x\sin\kappa_0(x-t)- \\ \mu\sin\mu x\cos\kappa_0(x-t)\end{array}\right]- \\ b\left[\kappa_0\sin\mu x\sin\kappa_0(x-t)+\mu\cos\mu x\cos\kappa_0(x-t)\right]\} \end{array}\right], \tag{2.22}$$

instead of equation (3.13) in [65], taking t as some function of x. And to the order ε i.e, $x=t$

we get

$\alpha(t) = \frac{\varepsilon}{\kappa_0^2-\mu^2}(a\sin\mu x + b\cos\mu x)$ as in equation (3.14) of [65].

Also we have

$$\beta(t) = \frac{-\varepsilon\kappa_0 c\sin\mu x}{\mu(\mu^2-\kappa^2)}. \tag{2.23}$$

The curve $\vec{R}(x)$ given by $\vec{R}(x) = \vec{R}_0(t) + \alpha(t)\vec{N}(t) + \beta(t)\vec{B}(t)$ where $\vec{R_0}(t)$ is the centreline of the torus, is in a plane orthogonal to this line. The trajectory of curves $\alpha(t)$ and $\beta(t)$ determines the behavior of the curve $\vec{R}(x)$ and satisfies the equation

$$(c\kappa_0\alpha - a\mu\beta)^2 + (b\mu\beta)^2 = \left(\frac{cb\kappa_0\varepsilon}{\mu^2-\kappa_0^2}\right)^2. \tag{2.24}$$

If b and c are non-zero, the curve $\vec{R}$ lies on a torus with elliptical cross-section. As shown in [65], the curve traverses the ellipse in a single direction m times for every n times that the circular centreline $\vec{R}_0(t)$ is traversed so that the winding number is

m/n. Theorem 3.4 of [65] also needs correction in the expression for $A(\varepsilon)$. This gives the following theorem.

Theorem 2. *Suppose that the functions $\kappa_1(x,\varepsilon)$ and $\tau_1(x,\varepsilon)$ are periodic in x with period $P = \frac{2\pi n}{m\kappa_0}$, with m/n an irreducible rational number greater than one. Suppose further that $\kappa_1(x,\varepsilon) = a\sin\mu x + b\cos\mu x + o(\varepsilon), \tau_1(x,\varepsilon) = c\cos\mu x + o(\varepsilon)$ with $\int_0^P \kappa_1(x,\varepsilon)dx = \varepsilon\kappa_2(\varepsilon)$ and $\int_0^P \tau_1(x,\varepsilon)dx = 0$.*
Then for all ε sufficiently small there are functions $A(\varepsilon)$ and $\tau_2(\varepsilon)$ so that the curve with curvature and torsion, given by $\kappa(x) = \kappa_0 + \varepsilon\kappa_1(x,\varepsilon)$, $\tau(x) = \varepsilon\tau_1(x,\varepsilon) + \varepsilon^2\tau_2(\varepsilon)$ is a closed curve with arc length variable $s = A(\varepsilon)x$. Furthermore

$$\begin{aligned} A(\varepsilon) &= 1 - \varepsilon^2\left(\frac{c^2\kappa_0^2}{2(\kappa_0^2-\mu^2)^2} - \frac{\kappa_2(0)}{\kappa_0}\right) + \cdots\cdots \text{ and} \\ \tau_2(\varepsilon) &= -bc\frac{\kappa_0}{2(\mu^2-\kappa_0^2)} + \cdots\cdots \end{aligned} \tag{2.25}$$

If $bc \neq 0$, the curve $\vec{R}$ is a closed torus knot with winding number m/n for all non-zero ε sufficiently small. The knot is right handed if $bc > 0$, and left handed if $bc < 0$.

2.4 Invariant knotted vortex filament

The self induced motion of a vortex filament is given by the equation

$$\vec{R}_t = \kappa\vec{B}, \tag{2.26}$$

by taking $\alpha = 0$and $\beta = \kappa$ in equation (2.2). The corresponding evolution equations (2.3) and (2.4) of curvature and torsion are

$$\begin{aligned} (\kappa^2)_t &= (\gamma\kappa^2 - 2\tau\kappa^2)_s, \\ (\tau)_t &= (\frac{1}{2}\kappa^2 + z)_s \text{ and} \\ z\kappa &= \kappa_{ss} + \tau\kappa(\gamma - \tau), \end{aligned} \tag{2.27}$$

where γ is arbitrary and 's' is the arc length co-ordinate. Keener tried to determine the structure of vortex filaments when the above equations have oscillatory curvature and torsion of the correct type. He analyzed equation (2.27) by looking for invariant solutions. These solutions are the travelling wave solutions where the speed of translation is the unknown constant γ. In order to work on a fixed spatial interval, 's' is transformed to x by $s = Ax$, where A is determined by the condition that the curve is closed as given in theorem 2. Thus we get

$$\begin{aligned} \gamma\kappa^2 - 2\tau\kappa^2 &= C_1, \\ \kappa^2 + 2z &= C_2 \text{ and} \\ \kappa_{xx} &= A^2\left[z\kappa - \tau\kappa(\gamma - \tau)\right] \end{aligned} \tag{2.28}$$

where the constants C_1 and C_2 are undetermined constants of integration.

In [65] Keener expressed the solution as perturbations of a constant solution. Note that $\kappa = \kappa_0$, $\tau = 0$ and $z = 0$ is a solution for any constant $\gamma = \gamma_0$. To find the spatially periodic solutions in a neighborhood of this solution, he solved the equations

$$\begin{aligned} \kappa_{1xx} + (\kappa_0^2 + \gamma_0^2)\kappa_1 &= c_1\gamma_0 + c_2\kappa_0, \\ \gamma_0\kappa_1 - \kappa_0\tau_1 &= c_1 \text{ and} \\ z_1 + \kappa_0\kappa_1 &= c_2, \end{aligned} \tag{2.29}$$

where κ_1, τ_1 and z_1 are the small deviations of κ, τ and z respectively from the constant solution.

The solutions are of the form

$$\begin{aligned} \kappa_1(x) &= a\kappa_0 \cos \mu x \text{ and} \\ \tau_1(x) &= a\gamma_0 \cos \mu x \end{aligned} \tag{2.30}$$

where $\gamma_0^2 = \mu^2 - \kappa_0^2$. By proper choice of μ and γ_0, the above solution gives an $\frac{m}{n}$ torus knot.

Knotted solution of equation (2.28) is obtained by seeking a power series solution in powers of small parameter ε using standard perturbation calculations [26, 66] in the form

$$\begin{aligned} \kappa(x) &= \kappa_0 + \varepsilon\kappa_0 \cos \mu x + \varepsilon^2 \kappa_2(x) + \cdots \text{ and} \\ \tau(x) &= \varepsilon\gamma_0 \cos \mu x + \varepsilon^2 \tau_2(x) + \cdots\cdots\cdots. \end{aligned} \tag{2.31}$$

From theorem 2, it follows that the scale factor A and the average torsion should satisfy

$$\begin{aligned} A(\varepsilon) &= 1 - \frac{\kappa_0 \varepsilon^2}{2\gamma_0^2} + \cdots\cdots \text{ and} \\ \frac{1}{P}\int_0^P \tau_2(x)dx &= -\frac{\varepsilon^2 \kappa_0^2}{2\gamma_0} + \cdots\cdots \end{aligned} \tag{2.32}$$

We note the difference from equation (4.10) of [65]. The perturbation solution is obtained in [65] using "REDUCE". Since the solutions are not used further, we are not looking for the corrected solutions corresponding to (4.11) of [65].

As in [65] it follows from the theorem 2 that for γ in a one-sided neighbourhood of γ_0 determined by γ_2, there are closed knotted invariant solutions of the equation of motion. γ determines the rate and direction of rotation of the knot about its axis of symmetry. Thus a right handed knot rotates in the direction of its tangent vector and a left-handed knot rotates in the direction opposite to its tangent vector. The knot moves as a rigid body in the direction of the binormal to the circle which underlies the torus knot.

2.5 Dynamics of torus knots

In Keener [65] it is proved that there are torus knots with any winding number that move as rigid bodies under the dynamics $\vec{R}_t = \kappa\vec{B}$ as γ has no influence in the shape of the solution. The invariant solutions are of solitary structure as the equations of motion for curvature and torsion (2.27) can be transformed into the nonlinear Schrödinger equation [57, 84]. Thus the invariant knots correspond to spatially periodic invariant travelling wave solutions of the Non-linear Schrödinger (NLS) equation. This equation is completely integrable and has solution that can be found using the inverse scattering transform. The invariant solution act as solitons and can be superposed to find temporarily quasi- periodic, spatially periodic solution[171]. By analogy he tried to determine how the invariant knotted solution curves can be viewed as 'solitons' which can be superposed to give closed curve which are not invariant and may have a different topology from the invariant torus knots. The evolution equation $\vec{R}_t = \kappa\vec{B}$ with an arbitrary torus knots will not guarantee that the motion is invariant since the equation of motion only take into account local effects.

Following [65] we consider the solution of the evolution equation (2.26). The system will be periodic in space and has small amplitude in their deviation from a circle. The two-soliton solution of equation (2.26) can be written in the following form by method of harmonic balance [156]:

$$\begin{aligned}\kappa(x,t,\varepsilon) &= \kappa_0(t) + \sum_{j=1}^{\infty}\varepsilon^j[\kappa_j(t)e^{ij\mu x} + \kappa_j^*(t)e^{-ij\mu x}],\\ \tau(x,t,\varepsilon) &= \tau_0 + \sum_{j=1}^{\infty}\varepsilon^j[\tau_j(t)e^{ij\mu x} + \tau_j^*(t)e^{-ij\mu x}] \text{ and}\\ z(x,t,\varepsilon) &= z_0(t) + \sum_{j=1}^{\infty}\varepsilon^j[z_j(t)e^{ij\mu x} + z_j^*(t)e^{-ij\mu x}].\end{aligned} \tag{2.33}$$

where '*' denotes complex conjugate. Taking $\gamma = 0$ and

$$\begin{aligned}A(\varepsilon) &= 1 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots\cdots\cdot\\ &= 1 - \frac{c^2\varepsilon^2\kappa_0^2}{2(\kappa_0^2 - \mu^2)^2} + \cdots\cdots\end{aligned}$$

the equation to be solved can be written as

$$\begin{aligned}\kappa_t A &= -(\tau_x\kappa + 2\tau\kappa_x),\\ \tau_t A &= \kappa\kappa_x + z_x \text{ and}\\ \kappa z &= \frac{\kappa_{xx}}{A^2} - \tau^2\kappa.\end{aligned} \tag{2.34}$$

Substituting from (2.33) in equation (2.34) (taking $A = 1$) and collecting like exponential terms, Keener got system of linear equations of which some of them need correction. The following equations (2.35) and (2.36) are the corrected linear equations referred as (5.3) and (5.4) in [65].

$$\begin{aligned}\frac{d\kappa_0}{dt} &= i\mu\varepsilon^2(\tau_1\kappa_1^* - \kappa_1\tau_1^*),\\ \kappa_0 z_0 + \tau_0^2\kappa_0 &= -\varepsilon^2(\kappa_1 z_1^* + \kappa_1^* z_1 + 2\kappa_0\tau_1\tau_1^*).\end{aligned} \tag{2.35}$$

$$\begin{aligned}&\frac{d\kappa_1}{dt} + i\mu(\kappa_0\tau_1 + 2\tau_0\kappa_1) = -3i\mu\varepsilon^2\kappa_2\tau_1^*,\\ &\frac{d\tau_1}{dt} - i\mu(\kappa_0\kappa_1 + z_1) = i\mu\varepsilon^2\kappa_2\kappa_1^*,\\ &z_1\kappa_0 + \mu^2\kappa_1 + 2\kappa_0\tau_0\tau_1 + z_0\kappa_1 + \varepsilon^2(2\kappa_0\tau_2\tau_1^* + 2\kappa_1\tau_1\tau_1^* + \kappa_1^*\tau_1^2 + \kappa_1^* z_2 + \kappa_2 z_1^*) = 0.\end{aligned} \tag{2.36}$$

$$
\begin{aligned}
&\frac{d\kappa_2}{dt} + 2i\mu(\tau_2\kappa_0 + 2\tau_0\kappa_2) = -3i\mu\tau_1\kappa_1, \\
&\frac{d\tau_2}{dt} - 2i\mu(\kappa_0\kappa_2 + z_2) = i\mu\kappa_1^2, \\
&z_1\kappa_0 + z_0\kappa_2 + 4\mu^2\kappa_2 + z_1\kappa_1 + \kappa_0\tau_1^2 = 0.
\end{aligned}
\tag{2.37}
$$

In the above equations we are retaining only the terms up to $o(\varepsilon^2)$. The solutions of these equations is approximated using multiscale techniques given in Cole[26] and Keener [66].

Taking $\varepsilon = 0$ in equation (2.35) and in (2.36) the solution to the leading order is obtained as follows.

$$
\begin{aligned}
\frac{d\kappa_0}{dt} &= 0 \Rightarrow \kappa_0 = constant \\
\kappa_0 z_0 &= 0 \Rightarrow z_0 = 0. \\
\frac{d\kappa_1}{dt} + i\mu\kappa_0\tau_1 &= 0, \\
\frac{d\tau_1}{dt} - i\mu(\kappa_0\kappa_1 + z_1) &= 0, \\
z_1\kappa_0 + \mu^2\kappa_1 &= 0.
\end{aligned}
\tag{2.38}
$$

These equations give

$$
\begin{aligned}
\frac{d\kappa_1}{dt} &= -i\mu\kappa_0\tau_1 \text{ and} \\
\frac{d\tau_1}{dt} &= i\mu(\kappa_0^2 - \mu^2)\frac{\kappa_1}{\kappa_0}.
\end{aligned}
\tag{2.39}
$$

Solving we get

$$
\begin{aligned}
\kappa_1(t) &= \kappa_+ e^{i\varpi t} + \kappa_- e^{-i\varpi t} \text{ and} \\
\tau_1(t) &= \frac{\varpi}{\mu\kappa_0}(\kappa_- e^{-i\varpi t} - \kappa_+ e^{i\varpi t}),
\end{aligned}
\tag{2.40}
$$

where κ_+ and κ_- are arbitrary constants with $\varpi^2 = \mu^2(\mu^2 - \kappa_0^2)$. This corresponds to the superposition of travelling waves moving in opposite directions with speed $\frac{\varpi}{\mu} = \gamma$.

2.6 Approximation by multiscale techniques

The two timing assumption is made to determine the effect of the order- ε^2 correction terms on the solution (2.40). i.e. the behaviour of the solution can be described in terms of two timescales - a fast time T_0, and a slow time $T_1 = \varepsilon^2 T_0$.

Taking

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon^2 \frac{\partial}{\partial T_1} \text{ and } (\varepsilon^2 = \varepsilon_1)$$

the equations (2.35), (2.36) and (2.37) can be rewritten as

$$\left(\frac{\partial}{\partial T_0} + \varepsilon_1 \frac{\partial}{\partial T_1}\right)(\kappa_{00}+\varepsilon_1\kappa_{01}) = i\mu\varepsilon_1\left[(\tau_{10}+\varepsilon_1\tau_{11})(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*) - (\tau_{10}^*+\varepsilon_1\tau_{11}^*)(\kappa_{10}+\varepsilon_1\kappa_{11})\right]. \tag{2.41}$$

$$\left[(z_{00}+\varepsilon_1 z_{01}) + (\tau_{00}+\varepsilon_1\tau_{01})^2\right](\kappa_{00}+\varepsilon_1\kappa_{01}) = -\varepsilon_1 \begin{bmatrix} (\kappa_{10}+\varepsilon_1\kappa_{11})(z_{10}^*+\varepsilon_1 z_{11}^*) \\ +(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*)(z_{10}+\varepsilon_1 z_{11}) \\ +2(\kappa_{00}+\varepsilon_1\kappa_{01})(\tau_{10}^*+\varepsilon_1\tau_{11}^*)(\tau_{10}+\varepsilon_1\tau_{11}) \end{bmatrix}. \tag{2.42}$$

$$\left(\frac{\partial}{\partial T_0} + \varepsilon_1 \frac{\partial}{\partial T_1}\right)(\kappa_{10}+\varepsilon_1\kappa_{11})$$
$$+ i\mu\left[(\kappa_{00}+\varepsilon_1\kappa_{01})(\tau_{10}+\varepsilon_1\tau_{11}) + 2(\tau_{00}+\varepsilon_1\tau_{01})(\kappa_{10}+\varepsilon_1\kappa_{11})\right] = -3i\mu\varepsilon_1(\kappa_{20}+\varepsilon_1\kappa_{21})(\tau_{10}^*+\varepsilon_1\tau_{11}^*). \tag{2.43}$$

$$\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\tau_{10}+\varepsilon_1\tau_{11})$$
$$-i\mu\left[(\kappa_{00}+\varepsilon_1\kappa_{01})(\kappa_{10}+\varepsilon_1\kappa_{11})+(z_{10}+\varepsilon_1 z_{11})\right]=i\mu\varepsilon_1(\kappa_{20}+\varepsilon_1\kappa_{21})(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*). \tag{2.44}$$

$$(z_{10}+\varepsilon_1 z_{11})(\kappa_{00}+\varepsilon_1\kappa_{01})+\mu^2(\kappa_{10}+\varepsilon_1\kappa_{11})+2(\kappa_{00}+\varepsilon_1\kappa_{01})(\tau_{00}+\varepsilon_1\tau_{01})(\tau_{10}+\varepsilon_1\tau_{11})$$
$$+(z_{00}+\varepsilon_1 z_{01})(\kappa_{10}+\varepsilon_1\kappa_{11})+\varepsilon_1\left[\begin{array}{l}2(\kappa_{00}+\varepsilon_1\kappa_{01})(\tau_{20}+\varepsilon_1\tau_{21})(\tau_{10}^*+\varepsilon_1\tau_{11}^*)\\ +2(\kappa_{10}+\varepsilon_1\kappa_{11})(\tau_{10}+\varepsilon_1\tau_{11})(\tau_{10}^*+\varepsilon_1\tau_{11}^*)+\\ (\kappa_{10}^*+\varepsilon_1\kappa_{11}^*)(\tau_{10}+\varepsilon_1\tau_{11})^2+(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*)(z_{20}+\varepsilon_1 z_{21})\\ +(\kappa_{20}+\varepsilon_1\kappa_{21})(z_{10}^*+\varepsilon_1 z_{11}^*)\end{array}\right]=0. \tag{2.45}$$

$$\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{20}+\varepsilon_1\kappa_{21})$$
$$+2i\mu\left[(\tau_{20}+\varepsilon_1\tau_{21})(\kappa_{00}+\varepsilon_1\kappa_{01})+2(\tau_{00}+\varepsilon_1\tau_{01})(\kappa_{20}+\varepsilon_1\kappa_{21})\right]=-3i\mu(\tau_{10}+\varepsilon_1\tau_{11})(\kappa_{10}+\varepsilon_1\kappa_{11}). \tag{2.46}$$

$$\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\tau_{20}+\varepsilon_1\tau_{21})-2i\mu\left[(\kappa_{00}+\varepsilon_1\kappa_{01})(\kappa_{20}+\varepsilon_1\kappa_{21})+(z_{20}+\varepsilon_1 z_{21})\right]=i\mu(\kappa_{10}+\varepsilon_1\kappa_{11})^2. \tag{2.47}$$

$$(z_{20}+\varepsilon_1 z_{21})(\kappa_{00}+\varepsilon_1\kappa_{01})+(z_{00}+\varepsilon_1 z_{01})(\kappa_{20}+\varepsilon_1\kappa_{21})$$
$$+4\mu^2(\kappa_{20}+\varepsilon_1\kappa_{21})+(z_{10}+\varepsilon_1 z_{11})(\kappa_{10}+\varepsilon_1\kappa_{11})+(\kappa_{00}+\varepsilon_1\kappa_{01})(\tau_{10}+\varepsilon_1\tau_{11})^2=0. \tag{2.48}$$

The system of equations from (2.41) to (2.48) gives rise to 2 sets of linear equations. The Set I contains the system of $o(1)$ linear equations obtained by collecting terms independent of ε_1 and the Set II contains the system of $o(\varepsilon_1)$ linear equations obtained by collecting coefficients of ε_1 in the above system.

SetI

$$
\begin{aligned}
&\frac{\partial \kappa_{00}}{\partial T_0} = 0, \\
&z_{00}\kappa_{00} + \tau_{00}^2\kappa_{00} = 0, \\
&\frac{\partial \kappa_{10}}{\partial T_0} + i\mu(\kappa_{00}\tau_{10} + 2\tau_{00}\kappa_{10}) = 0, \\
&\frac{\partial \tau_{10}}{\partial T_0} - i\mu(\kappa_{00}\kappa_{10} + z_{10}) = 0, \\
&z_{00}\kappa_{00} + \mu^2\kappa_{10} + 2(\tau_{00}\kappa_{00}\tau_{10} + z_{00}\kappa_{10}) = 0, \\
&\frac{\partial \kappa_{20}}{\partial T_0} + 2i\mu(\tau_{20}\kappa_{00} + 2\tau_{00}\kappa_{20}) = -3i\mu\tau_{10}\kappa_{10}, \\
&\frac{\partial \tau_{20}}{\partial T_0} - 2i\mu(\kappa_{00}\kappa_{20} + z_{20}) = i\mu\kappa_{10}^2 \text{ and} \\
&z_{20}\kappa_{00} + z_{00}\kappa_{20} + z_{10}\kappa_{10} + \kappa_{00}\tau_{10}^2 + 4\mu^2\kappa_{20} = 0.
\end{aligned}
\tag{2.49}
$$

Set II

$$\frac{\partial\kappa_{01}}{\partial T_0}+\frac{\partial\kappa_{00}}{\partial T_1}=i\mu(\tau_{10}\kappa_{10}^*-\tau_{10}^*\kappa_{10}),$$

$$z_{00}\kappa_{01}+z_{01}\kappa_{00}+2\tau_{00}\tau_{01}\kappa_{00}+\tau_{00}^2\kappa_{01}+(\kappa_{10}z_{10}^*+\kappa_{10}^*z_{10}+2\kappa_{00}\tau_{10}\tau_{10}^*)=0,$$

$$\frac{\partial\kappa_{11}}{\partial T_0}+\frac{\partial\kappa_{10}}{\partial T_1}+i\mu(\kappa_{01}\tau_{10}+\kappa_{00}\tau_{11}+2\kappa_{10}\tau_{01}+2\kappa_{11}\tau_{00})=-3i\mu\kappa_{20}\tau_{10}^*,$$

$$\frac{\partial\tau_{11}}{\partial T_0}+\frac{\partial\tau_{10}}{\partial T_1}-i\mu(\kappa_{00}\kappa_{11}+\kappa_{01}\kappa_{10}+z_{11})=i\mu\kappa_{20}\kappa_{10}^*,$$

$$z_{11}\kappa_{00}+z_{10}\kappa_{01}+\mu^2\kappa_{11}+z_{01}\kappa_{10}+2(\kappa_{01}\tau_{00}\tau_{10}+\kappa_{00}\tau_{01}\tau_{10}+$$

$$\tau_{11}\tau_{00}\kappa_{00}+z_{00}\kappa_{11}+z_{01}\kappa_{10})+(2\kappa_{00}\tau_{20}\tau_{10}^*+2\kappa_{10}\tau_{10}\tau_{10}^*+\kappa_{10}^*\tau_{10}^2+\kappa_{10}^*z_{20}+\kappa_{20}z_{10}^*)=0,$$

$$\frac{\partial\kappa_{21}}{\partial T_0}+\frac{\partial\kappa_{20}}{\partial T_1}+2i\mu(\tau_{21}\kappa_{00}+\tau_{20}\kappa_{01}+2\tau_{00}\kappa_{21}+2\tau_{01}\kappa_{20})=-3i\mu(\tau_{11}\kappa_{10}+\tau_{10}\kappa_{11}),$$

$$\frac{\partial\tau_{21}}{\partial T_0}+\frac{\partial\tau_{20}}{\partial T_1}-2i\mu(\kappa_{00}\kappa_{21}+\kappa_{20}\kappa_{01}+z_{21})=2i\mu\kappa_{10}\kappa_{11}\quad\text{and}$$

$$z_{21}\kappa_{00}+z_{20}\kappa_{01}+z_{10}\kappa_{11}+z_{11}\kappa_{10}+z_{00}\kappa_{21}+z_{01}\kappa_{20}+2\kappa_{00}\tau_{10}\tau_{11}+\kappa_{01}\tau_{10}^2+4\mu^2\kappa_{21}=0. \tag{2.50}$$

We use the method of variation of parameters to solve the above system of equations. Also we see that the solution to the linear order is (2.40), with a modification that κ_- and κ_+ both depends on T_1; that is, they are slowly varying functions of time.

From equation (2.49) we get

$$\begin{aligned}\kappa_{00}&=\text{constant},\\ \tau_{00}&=0, z_{00}=0\text{ and}\\ z_{10}&=-\frac{\mu^2\kappa_{10}^2}{\kappa_{00}}.\end{aligned}\tag{2.51}$$

Then equations third and fourth of (2.49) can be rewritten as

$$\begin{aligned}\frac{\partial\kappa_{10}}{\partial T_0}&=-i\mu\kappa_{00}\tau_{10}\text{ and}\\ \frac{\partial\tau_{10}}{\partial T_0}&=i\mu(\kappa_{00}^2-\mu^2)\frac{\kappa_{10}}{\kappa_{00}}.\end{aligned}\tag{2.52}$$

Solving, by the method of variation of parameters we get the solution as

$$\begin{aligned}\kappa_{10} &= \kappa_{-}\mathrm{e}^{-\mathrm{i}\varpi \mathrm{T}_0} + \kappa_{+}\mathrm{e}^{\mathrm{i}\varpi \mathrm{T}_0} \text{ and} \\ \tau_{10} &= \frac{\varpi}{\mu\kappa_{00}}(\kappa_{-}\mathrm{e}^{-\mathrm{i}\varpi \mathrm{T}_0} - \kappa_{+}\mathrm{e}^{\mathrm{i}\varpi \mathrm{T}_0}).\end{aligned} \tag{2.53}$$

Again from equation eight of (2.49) and (2.51)

$$z_{20} = \frac{1}{\kappa_{00}}(\frac{\mu^2\kappa_{10}^2}{\kappa_{00}} - \kappa_{00}\tau_{10}^2 - 4\mu^2\kappa_{20}) \tag{2.54}$$

Then the equations sixth and seventh of (2.49) can be written as

$$\begin{aligned}\frac{\partial\kappa_{20}}{\partial T_0} + 2i\mu\tau_{20}\kappa_{00} &= -3i\mu\tau_{10}\kappa_{10} \text{ and} \\ \frac{\partial\tau_{20}}{\partial T_0} + 2i\mu\beta^2\frac{\kappa_{20}}{\kappa_{00}} &= i\mu\kappa_{10}^2 + 2i\mu^3\frac{\kappa_{10}^2}{\kappa_{00}^2} - 2i\mu\tau_{10}^2\end{aligned} \tag{2.55}$$

where $\beta^2 = 4\mu^2 - \kappa_{00}^2$.

The equations (2.55) are solved by the method of variation of parameters as they are ordinary differential equations in the independent variable T_0.

The solution of the homogeneous part of the differential equation is

$$\begin{aligned}\kappa_{20} &= A_1 e^{2i\mu\beta T_0} + A_2 e^{2i\mu\beta T_0} \text{and} \\ \tau_{20} &= -\frac{1}{2i\mu\kappa_{00}}\frac{\partial\kappa_{20}}{\partial T_0} = \frac{\beta}{\kappa_{00}}(A_2 e^{-2i\mu\beta T_0} - A_1 e^{2i\mu\beta T_0})\end{aligned} \tag{2.56}$$

where A_1 and A_2 are functions of T_0 and T_1.

Thus

$$\begin{aligned}\kappa_{20} &= v_1(T_0,T_1)\kappa_{20}^{(1)} + v_2(T_0,T_1)\kappa_{20}^{(2)} \text{and} \\ \tau_{20} &= \frac{\beta}{\kappa_{00}}\left[v_1(T_0,T_1)\tau_{20}^{(1)} + v_2(T_0,T_1)\tau_{20}^{(2)}\right]\end{aligned} \tag{2.57}$$

will be a particular solution of the equation (2.54) if v_1 and v_2 satisfy the following system of equations

$$\begin{aligned}v_1'\kappa_{20}^{(1)} + v_2'\kappa_{20}^{(2)} &= -3i\mu\kappa_{10}\tau_{10} \text{ and} \\ v_1'\tau_{20}^{(1)} + v_2'\tau_{20}^{(2)} &= i\mu\kappa_{10}^2 + 2i\mu^3\frac{\kappa_{10}^2}{\kappa_{00}^2} - 2i\mu\tau_{10}^2.\end{aligned} \tag{2.58}$$

Therefore,

$$\begin{aligned}v_1'(T_0,T_1) &= \frac{\begin{vmatrix} -3i\mu\kappa_{10}\tau_{10} & e^{-2i\mu\beta T_0} \\ i\mu\kappa_{10}^2 - 2i\mu\tau_{10}^2 + 2i\mu^3\frac{\kappa_{10}^2}{\kappa_{00}^2} & \frac{\beta}{\kappa_{00}}e^{-2i\mu\beta T_0}\end{vmatrix}}{\left[\kappa_{20}^{(1)}(T_0)\tau_{20}^{(2)}(T_0) - \kappa_{20}^{(2)}(T_0)\tau_{20}^{(1)}(T_0)\right]} \\ &= -\frac{i\mu e^{-2i\mu\beta T_0}}{\beta\kappa_{00}}\left[\begin{array}{l}\frac{3\beta\varpi}{\mu}\left(\kappa_-^2 e^{-2i\varpi T_0} - \kappa_+^2 e^{2i\varpi T_0}\right) + \\ \left(2\mu^2 + \kappa_{00}^2\right)\left(\kappa_-^2 e^{-2i\varpi T_0} + 2\kappa_-\kappa_+ + \kappa_+^2 e^{2i\varpi T_0}\right) - \\ \frac{2\varpi^2}{\mu^2}\left(\kappa_-^2 e^{-2i\varpi T_0} - 2\kappa_-\kappa_+ + \kappa_+^2 e^{2i\varpi T_0}\right)\end{array}\right]\end{aligned} \tag{2.59}$$

On integration we get,

$$v_1(T_0,T_1) = \frac{3\varpi}{4\kappa_{00}(\varpi^2-\mu^2\beta^2)}\left[\kappa_-^2(\varpi-\mu\beta)e^{-2i(\varpi+\mu\beta)T_0}+\kappa_+^2(\varpi+\mu\beta)e^{2i(\varpi-\mu\beta)T_0}\right]$$
$$+\frac{2\mu^2+\kappa_{00}^2}{4\beta^2\kappa_{00}(\varpi^2-\mu^2\beta^2)}\begin{bmatrix}\kappa_-^2(\varpi-\mu\beta)\mu\beta e^{-2i(\varpi+\mu\beta)T_0}\\ +2\kappa_-\kappa_+(\varpi^2-\mu^2\beta^2)e^{-2i\mu\beta T_0}\\ -\kappa_+^2\mu\beta(\varpi+\mu\beta)e^{2i(\varpi-\mu\beta)T_0}\end{bmatrix}$$
$$-\frac{\varpi^2}{2\beta^2\kappa_{00}\mu^2(\varpi^2-\mu^2\beta^2)}\begin{bmatrix}\kappa_-^2(\varpi-\mu\beta)\mu\beta e^{-2i(\varpi+\mu\beta)T_0}\\ -2\kappa_-\kappa_+(\varpi^2-\mu^2\beta^2)e^{-2i\mu\beta T_0}\\ -\kappa_+^2\mu\beta(\varpi+\mu\beta)e^{2i(\varpi-\mu\beta)T_0}\end{bmatrix} \tag{2.60}$$

Similarly we get

$$v_2(T_0,T_1) = \frac{2\mu^2+\kappa_{00}^2}{4\beta^2\kappa_{00}^2(\varpi^2-\mu^2\beta^2)}\begin{bmatrix}\kappa_-^2(\varpi+\mu\beta)\mu\beta e^{-2i(\varpi-\mu\beta)T_0}+\\ 2\kappa_-\kappa_+(\varpi^2-\mu^2\beta^2)e^{2i\mu\beta T_0}+\\ \kappa_+^2\mu\beta(\varpi-\mu\beta)e^{2i(\varpi+\mu\beta)T_0}\end{bmatrix}$$
$$+\frac{\varpi^2}{2\beta^2\kappa_{00}\mu^2(\varpi^2-\mu^2\beta^2)}\begin{bmatrix}\kappa_-^2(\varpi+\mu\beta)\mu\beta e^{-2i(\varpi-\mu\beta)T_0}+\\ 2\kappa_-\kappa_+(\varpi^2-\mu^2\beta^2)e^{-2i\mu\beta T_0}-\\ \kappa_+^2\mu\beta(\varpi-\mu\beta)e^{2i(\varpi+\mu\beta)T_0}\end{bmatrix}$$
$$+\frac{3\varpi\begin{pmatrix}\kappa_-^2(\varpi+\mu\beta)e^{-2i(\varpi-\mu\beta)T_0}+\\ \kappa_+^2(\varpi-\mu\beta)e^{2i(\varpi+\mu\beta)T_0}\end{pmatrix}}{4\kappa_{00}(\varpi^2-\mu^2\beta^2)} \tag{2.61}$$

Substituting in equation (2.56) and simplifying, we get

$$\begin{aligned}\kappa_{20} &= v_1(T_0,T_1)e^{2i\mu\beta T_0} + v_2(T_0,T_1)e^{-2i\mu\beta T_0} \\ &= \frac{(\kappa_-^2 e^{-2i\varpi T_0} + \kappa_+^2 e^{2i\varpi T_0})(2\kappa_{00}^2 - \mu^2)}{2\kappa_{00}\mu^2} + \frac{\kappa_+\kappa_-}{\kappa_{00}} \text{ and}\end{aligned} \tag{2.62}$$

$$\tau_{20} = -\frac{1}{2i\mu\kappa_{00}}\frac{\partial\kappa_{20}}{\partial T_0} = -\frac{(2\kappa_{00}^2 - \mu^2)(-2i\varpi)}{2i\mu^3\kappa_{00}^2}(\kappa_-^2 e^{-2i\varpi T_0} - \kappa_+^2 e^{2i\varpi T_0}) \tag{2.63}$$

Then we calculate κ_{01}, z_{01} and z_{11} from equation (2.50).

Thus

$$\begin{aligned}\kappa_{01} &= -\frac{1}{\kappa_{00}}\left[\kappa_+(T_1)\kappa_-^*(T_1)e^{2i\varpi T_1} + \kappa_-(T_1)\kappa_+^*(T_1)e^{-2i\varpi T_1}\right] \\ z_{01}\kappa_{00} &= -(\kappa_{10}z_{10}^* + \kappa_{10}^* z_{10} + 2\kappa_{00}\tau_{10}\tau_{10}^*) \\ &= \frac{2\mu^2\kappa_{10}^*\kappa_{10}}{\kappa_{00}} - 2\kappa_{00}\frac{\varpi^2}{\mu^2\kappa_{00}^2}\left[\kappa_- e^{-i\varpi T_0} - \kappa_+ e^{i\varpi T_0}\right]\left[\kappa_-^* e^{i\varpi T_0} - \kappa_+^* e^{-i\varpi T_0}\right] \\ &= 2\frac{\mu^2}{\kappa_{00}}\left[|\kappa_-|^2 + |\kappa_+|^2 + \kappa_-\kappa_+^* e^{-2i\varpi T_0} + \kappa_+\kappa_-^* e^{2i\varpi T_0}\right] \\ &\quad - 2\left(\frac{\mu^2 - \kappa_{00}^2}{\kappa_{00}}\right)\left[|\kappa_-|^2 + |\kappa_+|^2 - \kappa_-\kappa_+^* e^{-2i\varpi T_0} - \kappa_+\kappa_-^* e^{2i\varpi T_0}\right] \\ &= 2\kappa_{00}\left(|\kappa_-|^2 + |\kappa_+|^2\right) + 2(2\mu^2 - \kappa_{00}^2)\frac{(\kappa_-\kappa_+^* e^{-2i\varpi T_0} + \kappa_+\kappa_-^* e^{2i\varpi T_0})}{\kappa_{00}}\end{aligned} \tag{2.64}$$

This gives

$$z_{01} = 2\left(|\kappa_+|^2 + |\kappa_-|^2\right) - 2(2\mu^2 - \kappa_{00}^2)\frac{\kappa_{01}}{\kappa_{00}} \tag{2.65}$$

where κ_{01} is given by equation (2.64). Again from (2.50) we get

$$z_{11}\kappa_{00} = \frac{1}{2\kappa_{00}^2\mu^2}\left[P_1\kappa_- e^{-i\varpi T_0} + P_2\kappa_+ e^{i\varpi T_0}\right] + \frac{1}{\kappa_0^2\mu^2}\left[8\kappa_{00}^2\mu^2 - 2\kappa_{00}^4 - \frac{15\mu^4}{2}\right] - \mu^2\kappa_{11} \tag{2.66}$$

where

$P_1 = -4\mu^4\left|\kappa_+\right|^2 + (4\kappa_{00}^4 - 9\mu^4 + 4\kappa_{00}^2\mu^2)\left|\kappa_-\right|^2$ and

$P_2 = -4\mu^4\left|\kappa_-\right|^2 + (4\kappa_{00}^4 - 9\mu^4 + 4\kappa_{00}^2\mu^2)\left|\kappa_+\right|^2$.

Thus we have $\kappa_{00}, \kappa_{01}, \kappa_{10}, \kappa_{20}, \tau_{00}, \tau_{01}, \tau_{10}, \tau_{20}, z_{00}, z_{01}, z_{11}$ and z_{20}.

Since the solutions required are to be periodic in T_0, we get the differential equations for the slow evolution of the functions κ_- and κ_+. We can rewrite the equations third and fourth of (2.50) as

$$\frac{\partial\kappa_{11}}{\partial T_0} + i\mu\kappa_{00}\tau_{11} = -\frac{\partial\kappa_{10}}{\partial T_1} - i\mu\kappa_{01}\tau_{10} - 3i\mu\kappa_{20}\tau_{10}^* \text{ and} \tag{2.67}$$

$$\frac{\partial\tau_{11}}{\partial T_0} - i\mu(\kappa_{00}^2 - \mu^2)\frac{\kappa_{11}}{\kappa_{00}} = -\frac{\partial\tau_{10}}{\partial T_1} + i\mu\left[\begin{array}{l}\kappa_{20}\kappa_{10}^* + \kappa_{01}\kappa_{10} + \frac{1}{2\kappa_{00}^3\mu^2}(P_1\kappa_- e^{-i\varpi T_0} + P_2\kappa_+ e^{i\varpi T_0}) + \\ \frac{1}{\kappa_{00}^3\mu^2}(8\kappa_{00}^2\mu^2 - 2\kappa_{00}^4 - \frac{15\mu^4}{2})(\kappa_+^2\kappa_-^* e^{3i\varpi T_0} + \kappa_-^2\kappa_+^* e^{-3i\varpi T_0})\end{array}\right]. \tag{2.68}$$

From equation (2.67) we get

$$\begin{aligned}\frac{\partial\kappa_{11}}{\partial T_0} + i\mu\kappa_{00}\tau_{11} &= e^{i\varpi T_0}\left[\begin{array}{l}-\frac{\partial\kappa_+}{\partial T_1} + \frac{i\varpi}{2\kappa_{00}^2\mu^2}\kappa_+[-4\mu^2\left|\kappa_-\right|^2 \\ +3(2\kappa_{00}^2 - \mu^2)\left|\kappa_+\right|^2]\end{array}\right] \\ &+ e^{-i\varpi T_0}\left[\begin{array}{l}-\frac{\partial\kappa_-}{\partial T_1} - \frac{i\varpi}{2\kappa_{00}^2\mu^2}\kappa_-[-4\mu^2\left|\kappa_+\right|^2 \\ +3(2\kappa_{00}^2 - \mu^2)\left|\kappa_-\right|^2]\end{array}\right] \\ &+ e^{3i\varpi T_0}\frac{\varpi i}{2\kappa_{00}^2\mu^2}A(\mu^2) + c.c,\end{aligned} \tag{2.69}$$

where $A(\mu^2) = (\mu^2 - 6\kappa_{00}^2)\kappa_+^2\kappa_-^*$ and c.c denotes complex conjugate.

The conditions for non secularity are

$$
\begin{aligned}
-\frac{\partial \kappa_+}{\partial T_1} + \frac{i\varpi}{2\kappa_{00}^2\mu^2}\kappa_+\left[-4\mu^2\left|\kappa_-\right|^2 + 3(2\kappa_{00}^2-\mu^2)\left|\kappa_+\right|^2\right] = 0,\\
-\frac{\partial \kappa_-}{\partial T_1} - \frac{i\varpi}{2\kappa_{00}^2\mu^2}\kappa_-\left[-4\mu^2\left|\kappa_+\right|^2 + 3(2\kappa_{00}^2-\mu^2)\left|\kappa_-\right|^2\right] = 0 \text{ and}\\
\mu^2 - 6\kappa_{00}^2 = 0.
\end{aligned}
\tag{2.70}
$$

Also equation (2.68) gives

$$
\begin{aligned}
&\frac{\partial \tau_{11}}{\partial T_0} - i\mu(\kappa_{00}^2-\mu^2)\frac{\kappa_{11}}{\kappa_{00}}\\
&= e^{i\varpi T_0}\left\{\frac{\varpi}{\mu\kappa_{00}}\frac{\partial\kappa_+}{\partial T_1} + i\frac{1}{2\kappa_{00}^3\mu}\left[(6\kappa_{00}^4+3\kappa_{00}^2\mu^2-9\mu^4)\left|\kappa_+\right|^2 - 4\mu^4\left|\kappa_-\right|^2\right]\kappa_+\right\}\\
&- e^{-i\varpi T_0}\left\{\frac{\varpi}{\mu\kappa_{00}}\frac{\partial\kappa_-}{\partial T_1} - i\frac{1}{2\kappa_{00}^3\mu}\left[(6\kappa_{00}^4+3\kappa_{00}^2\mu^2-9\mu^4)\left|\kappa_-\right|^2 - 4\mu^4\left|\kappa_-\right|^2\right]\kappa_-\right\}\\
&+ i\frac{1}{2\kappa_{00}^3\mu}\left[(13\kappa_{00}^2\mu^2 - 15\mu^4 - 2\kappa_{00}^4)(\kappa_+^2\kappa_-^* e^{3i\varpi T_0} + \kappa_-^2\kappa_+^* e^{-3i\varpi T_0})\right].
\end{aligned}
\tag{2.71}
$$

Here the conditions for non - secularity are

$$
\begin{aligned}
\frac{\varpi}{\mu\kappa_{00}}\frac{\partial\kappa_+}{\partial T_1} + i\frac{1}{2\kappa_{00}^3\mu}\left[3(2\kappa_{00}^4+\kappa_{00}^2\mu^2-3\mu^4)\left|\kappa_+\right|^2 - 4\mu^4\left|\kappa_-\right|^2\right]\kappa_+ = 0,\\
\frac{\varpi}{\mu\kappa_{00}}\frac{\partial\kappa_-}{\partial T_1} - i\frac{1}{2\kappa_{00}^3\mu}\left[3(2\kappa_{00}^4+\kappa_{00}^2\mu^2-3\mu^4)\left|\kappa_-\right|^2 - 4\mu^4\left|\kappa_+\right|^2\right]\kappa_- = 0 \text{ and}\\
13\kappa_{00}^2\mu^2 - 15\mu^4 - 2\kappa_{00}^4 = 0.
\end{aligned}
\tag{2.72}
$$

Then consider equation (2.70) and (2.72). We get

$$
\begin{aligned}
&-\varpi\frac{\partial\kappa_+}{\partial T_1} + i\frac{(\mu^2-\kappa_{00}^2)}{2\kappa_{00}^2}\left[3(2\kappa_{00}^2-\mu^2)\left|\kappa_+\right|^2 - 4\mu^2\left|\kappa_+\right|^2\right]\kappa_+\\
&= \varpi\frac{\partial\kappa_+}{\partial T_1} + \frac{i\kappa_+}{2\kappa_{00}^2}\begin{bmatrix} 3(2\kappa_{00}^4+\kappa_{00}^2\mu^2-3\mu^4)\left|\kappa_+\right|^2 \\ -4\mu^4\left|\kappa_-\right|^2 \end{bmatrix}
\end{aligned}
\tag{2.73}
$$

Rearranging the terms, we get

$$2\varpi\kappa_{00}^2\frac{\partial\kappa_+}{\partial T_1} = i\left[3(\mu^4 - 2\kappa_{00}^4 + \kappa_{00}^2\mu^2)\left|\kappa_+\right|^2 + 2\mu^2\kappa_{00}^2\left|\kappa_-\right|^2\right]\kappa_+. \tag{2.74}$$

Considering again the equations (2.70) and (2.72) and proceeding as before we get

$$2\varpi\kappa_{00}^2\frac{\partial\kappa_-}{\partial T_1} = i\left[3(2\kappa_{00}^4 - \mu^4 - \kappa_{00}^2\mu^2)\left|\kappa_-\right|^2 - 2\mu^2\kappa_{00}^2\left|\kappa_+\right|^2\right]\kappa_-. \tag{2.75}$$

Thus we get equations (2.74) and (2.75) as the evolution equations for κ_+ and κ_- in terms of the slow time variableT_1. The equations above can be rewritten as follows:

$$\begin{aligned} 2\kappa_{00}^2\varpi\frac{\partial\kappa_+}{\partial T_1} &= i(p_2(\mu)\left|\kappa_-\right|^2 - p_1(\mu)\left|\kappa_+\right|^2)\kappa_+ \text{ and} \\ 2\kappa_{00}^2\varpi\frac{\partial\kappa_-}{\partial T_1} &= i(p_1(\mu)\left|\kappa_-\right|^2 - p_2(\mu)\left|\kappa_+\right|^2)\kappa_-, \end{aligned} \tag{2.76}$$

where

$$p_1(\mu) = 3(2\kappa_{00}^4 - \mu^4 - \kappa_{00}^2\mu^2)\text{and } p_2(\mu) = 2\kappa_{00}^2\mu^2.$$

These equations correspond to equation (5.9) of [65].

Solving we get

$$\begin{aligned} \kappa_+(T_1) &= \kappa_+(0)e^{i\varpi_2(|\kappa_+|,|\kappa_-|)T_1} \text{ and} \\ \kappa_-(T_1) &= \kappa_-(0)e^{-i\varpi_2(|\kappa_-|,|\kappa_+|)T_1} \end{aligned} \tag{2.77}$$

where

$$\begin{aligned} \varpi_2(x,y) &= \frac{1}{2\kappa_{00}^2\varpi}(p_2(\mu)y^2 - p_1(\mu)x^2), \\ p_2(\mu) &= 2\kappa_{00}^2\mu^2 \text{ and} \\ p_1(\mu) &= 3(2\kappa_{00}^4 - \mu^4 - \kappa_{00}^2\mu^2). \end{aligned}$$

Here the amplitudes of κ_+ and κ_- are constant.

Then we have, to the linear order in ε

$$\begin{aligned}\tau_0(\varepsilon) &= \frac{-\varepsilon^2\kappa_{00}}{\mu^2-\kappa_{00}^2}(\kappa_1\tau_1^* + \kappa_1^*\tau_1) \\ &= -2\frac{\varepsilon^2\mu}{\varpi}(|\kappa_-|^2 - |\kappa_+|^2)\end{aligned} \tag{2.78}$$

and

$$\begin{aligned}A_2(\varepsilon) &= \frac{\tau_1\tau_1^*}{(\mu^2-\kappa_{00}^2)} - \frac{\kappa_{02}}{\kappa_{00}} \\ &= \frac{1}{\kappa_{00}^2}(|\kappa_+|^2 + |\kappa_-|^2)\end{aligned} \tag{2.79}$$

and that both τ_0 and A_2 are also constants.

From equations (2.33) and (2.40), in [65] he observed that the evolution of vortex knots was very much like the super-position of two travelling waves. i.e

$$\begin{aligned}\kappa_1(T_0,T_1) &= \kappa_- e^{-i\varpi T_0} + \kappa_+ e^{i\varpi T_0} \text{and} \\ \tau_1(T_0,T_1) &= \frac{\varpi}{\mu\kappa_{00}}(\kappa_- e^{-i\varpi T_0} - \kappa_+ e^{i\varpi T_0})\end{aligned}$$

where κ_+ and κ_- are arbitrary constant functions of slow time variable T_1.

The speed of propagation of the κ_+ wave component is obtained as follows: Consider equations (2.33), (2.40) and (2.64). We have

$$\kappa(x,t,\varepsilon) = \begin{bmatrix} \kappa_{00} - \frac{\varepsilon^2}{\kappa_{00}}[\kappa_+(T_1)\kappa_-^*(T_1)e^{2i\varpi T_0} + \\ \kappa_-(T_1)\kappa_+^*(T_1)e^{-2i\varpi T_0}] \end{bmatrix} + \varepsilon \begin{bmatrix} \kappa_+(0)e^{i\varpi_2(|\kappa_+|,|\kappa_-|)T_1}e^{i\varpi T_0} + \\ \kappa_-(0)e^{-i\varpi_2(|\kappa_-|,|\kappa_+|)T_1}e^{-i\varpi T_0} \end{bmatrix} e^{i\mu x} + \cdots\cdots$$

In the order of ε, we have

$\mu x + \varpi_2(|\kappa_+|, |\kappa_-|)T_1 + \varpi T_0 = \text{constant}$

Therefore

$\mu\frac{\mathrm{dx}}{\mathrm{dT_0}} + \varpi_2(|\kappa_+|, |\kappa_-|)\varepsilon^2 + \varpi = 0$ and

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}T_0} &= -\frac{\varpi}{\mu} - \varpi_2\frac{(|\kappa_+|, |\kappa_-|)\varepsilon^2}{\mu} \\ &= -\{\gamma_0 + \frac{\varepsilon^2}{2\mu^2\kappa_{00}^2\gamma_0}\left[2\mu^2\kappa_{00}^2|\kappa_+|^2 - 3(\kappa_{00}^2\mu^2 + \mu^4 - 2\kappa_{00}^4)|\kappa_-|^2\right]\} \end{aligned}$$

Thus the speed of κ_+ component in the order of ε, $C+$ is

$$-\{\gamma_0 + \frac{\varepsilon^2}{2\mu^2\kappa_{00}^2\gamma_0}\left[2\mu^2\kappa_{00}^2|\kappa_+|^2 - 3(\kappa_{00}^2\mu^2 + \mu^4 - 2\kappa_{00}^4)|\kappa_-|^2\right]\} \tag{2.80}$$

instead of equation (5.12) in [65]. A similar expression can be obtained for the speed of the κ_- component. The shape and topology of the knots is understood by the following observations:

- From the equation $\kappa_0(T) = \kappa_{00} - \frac{\varepsilon^2}{\kappa_{00}}\left[\kappa_+(T_1)\kappa_-^*(T_1)e^{2i\varpi T_0} + \kappa_+^*(T_1)\kappa_-(T_1)e^{-2i\varpi T_0}\right]$, we see that the solution curve lies on a torus whose large radius is oscillatory so that the whole torus is breathing.

- Comparing the equations (5.1), (5.6) with (3.1) in [65], the topology of the solution curve can be determined. First find two constants c and Θ (phase shift) such that $2\tau_1 = ce^{i\Theta}$. Then find a and b such that $2\kappa_1 = (b - ia)e^{i\Theta}$.

Taking $\kappa_1 = |\kappa_+| e^{i\phi_+} + |\kappa_-| e^{i\phi_-}$, $\tau_1 = \frac{\varpi}{\mu\kappa_{00}}(|\kappa_-| e^{i\phi_-} - |\kappa_+| e^{i\phi_+})$ we get

$$\begin{aligned}
c^2 &= 4\tau_1\tau_1^* \\
&= 4\left[\frac{\varpi}{\mu\kappa_{00}}\right]^2 \left[|\kappa_-|^2 + |\kappa_+|^2 - 2|\kappa_+||\kappa_-|\cos(\phi_+ - \phi_-)\right] \\
ac &= \frac{8\varpi}{\mu\kappa_{00}}\sin(\phi_- - \phi_+) \text{ and } bc = \frac{-4\varpi}{\mu\kappa_{00}^2}\left[|\kappa_+|^2 - |\kappa_-|^2\right]
\end{aligned}$$

Since $|\kappa_+|$ and $|\kappa_-|$ are constants in time, the quantity bc is also a constant independent of time. So by theorem 3.4 in [65] if $bc \neq 0$, the topology of the knot is invariant. Also if $|\kappa_+|$ is larger than $|\kappa_-|$ then a right-handed torus knot and if $|\kappa_+| < |\kappa_-|$, a left-handed torus knot is obtained. But the topology is unchanged during the course of the motion of the curve.

2.7 Discussion

As pointed out earlier this chapter deals with certain corrections needed in the work by Keener in [65]. The changes incorporated by us are the following:

- Equations (3.6) in [65] are obtained from (2.22) using (3.4). It is to be noted that X is defined by equation (2.10).

  So when Keener considered power series expansion for Φ and Ψ, he has not considered such expansion for X. This is also included in our work. This leads to corrections in the expression for $\Theta_1(x)$.

- In equation (3.6) of [65], the expression for $\Psi_1(x)$ should be $-\frac{c\mu\sin\mu x}{\kappa_0^2-\mu^2}$ instead of $\frac{c\kappa_0\sin\mu x}{\mu(\kappa_0^2-\mu^2)}$ as given in [65]. This follows directly from (2.15) and (2.19).

- The expression for $A(\varepsilon)$ given in [65] needs correction and is given by (2.21). The corresponding changes are needed in inferences that follow.

- The expression for $\alpha(t)$ given by (3.13) of [65] needs correction and is given by equation(2.22)

- Next corrections needed are in the set of linear equations (5.3) and (5.4) of [65] and are given by equations (2.36) and (2.37). Consequently equation (5.9) of [65] are replaced by equation (2.76). Corresponding changes are needed in equation (5.12)of [65] and is given by equation (2.80).

- In equation (5.1) of [65], on right hand side, the coefficients $\kappa_j, \kappa_j^*, \tau_j, \tau_j^*, z_j$ and z_j^* are functions of t only.

Keener uses a variable x defined by $s = Ax$ instead of the arc-length parameter x. Here A is independent of x. Later he considers the case $A = 1$ only, which implies $s = x$. Instead we are considering the variable x defined by $\frac{ds}{dx} = \omega$ and with $\omega \neq 1$, is the vorticity and x is not the arc-length. We consider these corrections because a similar method is used in chapter 3 of this thesis.

Chapter 3

The Contribution of 'ω' in the Evolution of Vortex Knots

The dynamical behavior of closed vortex filament is a complicated problem. The perturbation method introduced in [65] gives new opening for dealing with this. Here a knotted curve is realized as bifurcating from a circular curve. The basic principle is that a circle is an unknot and other knots are obtained starting from a circle in the reverse order.

Though Keener calls his knots as knotted vortex filament, he deals only with the geometry of a closed curve. Hence this method has the drawback that it does not take into account the stretching of vortex lines or filaments, which is the key-hydrodynamic contribution.

Here an attempt is made to overcome this drawback. The difference in our approach is that vorticity is assumed to adjust corresponding to stretching so that the curve remains closed.

We consider a moving vortex filament. Each point on it traces a curve. Then the velocity of a point on it is the local fluid velocity of the point (neglecting external flow), the induced velocity due to the vortex line being given by Biot-Savart-law[9]. Treating the vorticity $\overrightarrow{\omega}$ as a vector, associated with the moving point we have $\overrightarrow{\omega_s} = \overrightarrow{d} \times \overrightarrow{\omega}$ with s as arc-length, where $\overrightarrow{d}$ is the Darbaux vector[37].

Here we make use of the following assumptions:

- For consistency, $\overrightarrow{T}$ the unit tangent vector should be in the direction of vorticity $\overrightarrow{\omega}$ i.e.$\overrightarrow{\omega} = |\overrightarrow{\omega}|\, \overrightarrow{T}$.
- Corresponding to the perturbations in κ and τ, we must have perturbations in the vorticity also; since $\overrightarrow{\omega}$ is associated with the curve, like $\kappa$and τ, it should also be periodic.

3.1 Equations of motion of vortex filaments

A vortex filament in a fluid can be described in terms of its tangent, normal and binormal vectors $\overrightarrow{T}$, $\overrightarrow{N}$ and $\overrightarrow{B}$ respectively through Frenet -Serret equations in the form

$$\begin{aligned}
\overrightarrow{R}_x &= \omega \overrightarrow{T}, \\
\overrightarrow{T}_x &= \kappa\omega \overrightarrow{N}, \\
\overrightarrow{N}_x &= \omega\left(-\kappa\overrightarrow{T} + \tau\overrightarrow{B}\right) \text{ and} \\
\overrightarrow{B}_x &= -\tau\omega\overrightarrow{N},
\end{aligned} \tag{3.1}$$

where κ and τ are the local curvature and torsion respectively of the vortex filament, 'ω' denotes the local vorticity such that $\omega = |\overrightarrow{\omega}| = \left|\overrightarrow{R}_x\right|$ and$\overrightarrow{\omega} = |\overrightarrow{\omega}|\, \overrightarrow{T}$.

Keener has considered purely the geometry of a vortex filament (or a curve) in space. The motion of the vortex filament can be described by the equation

$$\overrightarrow{R}_t = \gamma\overrightarrow{T} + \alpha\overrightarrow{N} + \beta\overrightarrow{B}, \tag{3.2}$$

which can be used to obtain the evolution equation for curvature κ and torsion τ for a given vorticity $\overrightarrow{\omega}$. To find the shape of solution curves without reference to a specific co-ordinate system, we seek equations of motion for curvature and torsion that do not make any reference to the position vector $\overrightarrow{R}$. For this, differentiate $\overrightarrow{R}_t$ in equation (3.2) with respect to x and $\overrightarrow{R}_x$ in equation (3.1) with respect to t, and set

$$\overrightarrow{R}_{tx} = \overrightarrow{R}_{xt}.$$

Then we find that

$$\left(\omega\overrightarrow{T}\right)_t = \left(\gamma\overrightarrow{T} + \alpha\overrightarrow{N} + \beta\overrightarrow{B}\right)_x, \tag{3.3}$$

which gives

$$\omega_t = \gamma_x - \alpha\kappa\omega \text{ and} \tag{3.4}$$

$$\overrightarrow{T}_t = \nu\overrightarrow{N} + u\overrightarrow{B}, \tag{3.5}$$

where

$$\begin{aligned} \nu &= \tfrac{\alpha_x}{\omega} + \gamma\kappa - \tau\beta \text{ and} \\ u &= \tfrac{\beta_x}{\omega} + \alpha\tau. \end{aligned} \tag{3.6}$$

Differentiating $\overrightarrow{T}_x$ with respect to t and $\overrightarrow{T}_t$ with respect to x, and setting

$$\overrightarrow{T}_{xt} = \overrightarrow{T}_{tx}, \tag{3.7}$$

we get

$$
\begin{aligned}
(\kappa\omega)_t &= \nu_x - \tau u\omega \text{ and} \\
\overrightarrow{N}_t &= -\nu\overrightarrow{T} + z\overrightarrow{B},
\end{aligned}
\tag{3.8}
$$

where

$$
z = \frac{u_x + \tau\omega\nu}{\kappa\omega}. \tag{3.9}
$$

Also $\overrightarrow{B}_{tx} = \overrightarrow{B}_{xt}$ where $\overrightarrow{B}_t = -u\overrightarrow{T} - z\overrightarrow{N}$ gives $(\tau\omega)_t = u\kappa\omega + z_x$.

Thus if α and β are the normal and binormal velocities respectively of a moving vortex filament, then the curvature, torsion, vorticity and the tangential velocity γ evolve according to

$$
\begin{aligned}
(\kappa\omega)_t &= \nu_x - \tau u\omega, \\
(\tau\omega)_t &= u\kappa\omega + z_x, \\
\omega_t &= \gamma_x - \alpha\kappa\omega, \\
\kappa z &= \tau\nu + \frac{u_x}{\omega}, \\
\nu &= \frac{\alpha_x}{\omega} + \gamma\kappa - \tau\beta \text{ and} \\
u &= \frac{\beta_x}{\omega} + \alpha\tau.
\end{aligned}
\tag{3.10}
$$

When $\alpha = 0$ and $\beta = \kappa$ in equation (3.2), we get the self-induced motion of a vortex filament. Then the equation (3.10) reduces to

$$
\begin{aligned}
& u = \tfrac{\kappa_x}{\omega}, \nu = \kappa(\gamma - \tau), \omega_t = \gamma_x, \\
& \kappa z = \tfrac{\kappa_{xx}}{\omega^2} + \tau\kappa(\gamma - \tau), \\
& (\kappa^2\omega)_t = (\gamma\kappa^2 - 2\tau\kappa^2)_x \text{ and} \\
& (\tau\omega)_t = \left(\tfrac{\kappa^2}{2} + z\right)_x.
\end{aligned}
\tag{3.11}
$$

For a given value of ω, we can write

$$\begin{aligned} \kappa_t &= \frac{(\gamma-\tau)\,\kappa_x}{\omega} - \frac{(\tau\kappa)_x}{\omega} \text{ and} \\ \tau_t &= \frac{1}{\omega}\left[\kappa\kappa_x - \tau\gamma_x + \tau(\gamma-\tau) + \frac{1}{\omega\kappa}\left(\kappa_{xx} - \frac{\kappa_x\omega_x}{\omega}\right)\right]. \end{aligned} \tag{3.12}$$

These are the evolution equations for κ and τ in the self induced motion. As in Chapter 2, κ and τ are periodic and the tangent vector $\overrightarrow{T}$ must satisfy the condition

$$\int_0^P \overrightarrow{T}\,dx = 0, \tag{3.13}$$

where P is the period.

3.2 F-S Equations in the simplified form

In order to show that condition (3.13) is satisfied, first we have to solve the evolution equation (3.10) along with F-S equations that can be expressed as

$$\begin{aligned} \Phi_x &= \omega\kappa\Psi, \\ \Psi_x &= \omega\,(\tau X - \kappa\Phi) \text{ and} \\ \Theta_x &= \frac{\omega\kappa X}{1-\Phi^2}, \text{ where } X^2+\Phi^2+\Psi^2 = 1. \end{aligned} \tag{3.14}$$

In terms of these variables, the vectors $\overrightarrow{T}$ and $\overrightarrow{N}$ have components

$$\overrightarrow{T} = \begin{bmatrix} \sqrt{(1-\Phi^2)}\cos\Theta \\ \sqrt{(1-\Phi^2)}\sin\Theta \\ \Phi \end{bmatrix} \text{ and } \overrightarrow{N} = \begin{bmatrix} -\sqrt{(1-\Psi^2)}\sin(q+\Theta) \\ \sqrt{(1-\Psi^2)}\cos(q+\Theta) \\ \Psi \end{bmatrix} \tag{3.15}$$

Thus to find the closed curve solutions, we seek spatially periodic solutions of the equations of curvature, torsion and vorticity. F-S equation (3.14) must also be solved to determine the tangent vector $\overrightarrow{T}$ through equation (3.15). Finally we require that the tangent vector must satisfy equation (3.13) so that the solution curve is closed. For the self-induced motion of a vortex filament,ie the equation of motion for it's curvature and torsion, equations (3.11) are considered.

3.3 Torus knots - the solution to F-S equations

Following [65], we look for the solution of the F-S equations that are closed knotted curves. For this, Keener uses the characteristics of the curvature and torsion of the curve that guarantee it closed and knotted. No fluid mechanical characteristics are involved in his method.

For a circular closed curve, curvature is constant and torsion is zero. In the case of a nearly circular closed curve, perturbations of curvature and torsion are considered. The fact that curve is closed makes the curvature and torsion periodic. Since vorticity is associated with the closed curve, we consider 'ω' also periodic.

Any curve drawn on the surface of a torus has a winding number defined as in section (1.2). i.e. the ratio of average number of wraps of the curve around the small radius of the torus to the number of wraps around the large radius. The winding number exits and is unique if the curve is not self intersecting and is an invariant for the curve. Any knot constructed as a wrapping of a torus is called a torus knot but in the class of knots, torus knots forms a small subclass.
We state the following theorem without proof (Massey [119]).

Theorem 3. *A closed non-self intersecting curve T(m,n) on the surface of a torus with winding number* $\frac{m}{n}$; $m > n > 1$, *m and n are relatively prime, is a non trivial knot.*

Remarks:

- Vortex knots we consider here is a thin vortex filament in the shape of a torus knot $T(m,n)$, $m > n > 1$, m and n relatively prime.

- If the small radius of a torus is allowed to approach zero the torus approaches a circle and the knot becomes an n-cover of the circle with the curve twisted around itself m times.

Next we show how to find a closed curve (or vortex filament) solution of the F-S equations and also that the resulting vortex filament is a torus knot with winding number $\frac{m}{n}$. Suppose that the curvature, torsion, and vorticity of the vortex filament are given by

$$\begin{aligned}\kappa(x) &= \kappa_0 + \varepsilon a \sin \mu x + \varepsilon b \cos \mu x \\ \tau(x) &= \tau_0 + \varepsilon c \cos \mu x \text{ and} \\ \omega(x) &= \omega_0 + \varepsilon d \sin \mu x + \varepsilon e \cos \mu x.\end{aligned} \tag{3.16}$$

to the leading order in $O(\varepsilon)$, where $\mu = \frac{\kappa_0 m}{n}$. Here we take the independent variable x to vary over the fixed interval $0 \leq x \leq \frac{2\pi n}{\kappa_0}$. We seek solutions of the F-S equations (3.14) that are closed vortex filaments for small ε and use the fact that ω is also periodic like κ and τ. The function $\tau_0 = \tau_0(\varepsilon)$ is to be adjusted so that the vortex filament is closed.

Instead of 'A' in [65] we are taking vorticity ω, which should be periodic. Thus in the theorem 3.3 of [65] we have to make the following modifications:

Theorem 4. *Suppose that the curvature* $\kappa_1(x)$, *torsion* $\tau_1(x)$ *and vorticity* $\omega_1(x)$ *of a simple closed vortex filament are periodic functions of period* P *and that there are integers* m *and* n *so that* $mP\kappa_0 = 2n\pi$. *Suppose further that*

$$\begin{aligned}
&\int_0^{mP} \kappa_1(x)\sin\kappa_0 x dx = 0,\ \int_0^{mP} \kappa_1(x)\cos\kappa_0 x dx = 0,\ \int_0^{mP} \kappa_1(x)dx = 0,\\
&\int_0^{mP} \tau_1(x)\sin\kappa_0 x dx = 0,\ \int_0^{mP} \tau_1(x)\cos\kappa_0 x dx = 0,\ \int_0^{mP} \tau_1(x)dx = 0,\\
&\int_0^{mP} \omega_1(x)\sin\kappa_0 x dx = 0,\ \int_0^{mP} \omega_1(x)\cos\kappa_0 x dx = 0 \text{ and } \int_0^{mP} \omega_1(x)dx = 0.
\end{aligned} \tag{3.17}$$

Then for all ε *sufficiently small, there exists a function* $\tau_0 = \varepsilon^2\tau_2(\varepsilon)$ *and a closed vortex filament whose curvature, torsion and vorticity are given by*

$$\kappa(x) = \kappa_0 + \varepsilon\kappa_1(x), \tau(x) = \tau_0 + \varepsilon\tau_1(x), \omega(x) = \omega_0 + \varepsilon\omega_1(x) \tag{3.18}$$

The arc-length variable s *for the vortex filament is such that* $\frac{ds}{dx} = \omega$.

Following Keener [65], we assume that variables Θ, Φ, Ψ and ω in (3.14) have power series expansions

$$\begin{aligned}
\Phi(x) &= \varepsilon\Phi_1(x) + \varepsilon^2\Phi_2(x) + \dots\dots\dots\dots,\\
\Psi(x) &= \varepsilon\Psi_1(x) + \varepsilon^2\Psi_2(x) + \dots\dots\dots\dots,\\
\Theta(x) &= \kappa_0 x + \varepsilon\Theta_1(x) + \varepsilon^2\Theta_2(x) + \dots\dots,\\
X &= 1 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots\dots\dots\dots \text{ and}\\
\omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots\dots\dots\dots
\end{aligned} \tag{3.19}$$

For $\varepsilon = 0$, the vortex filament corresponds to a closed curve with constant curvature κ_0, torsion zero and constant vorticity ω_0. For small ε the tangent vector is given approximately by

$$\overrightarrow{T} = \begin{bmatrix} \cos\kappa_0 x \\ \sin\kappa_0 x \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -\Theta_1(x)\sin\kappa_0 x \\ \Theta_1(x)\cos\kappa_0 x \\ \Phi_1(x) \end{bmatrix} + O\left(\varepsilon^2\right) \tag{3.20}$$

Substituting from equation(3.19) in equation(3.14) and collecting like powers of ε, we get the leading order equations as

$$\begin{aligned} \frac{d\Phi_1}{dx} &= \kappa_0 \Psi_1(x)\omega_0, \\ \frac{d\Psi_1}{dx} &= (\tau_1 - \kappa_0\Phi_1)\,\omega_0 \text{ and} \\ \frac{d\Theta_1}{dx} &= \omega_0\kappa_1(x) + \omega_1(x)\kappa_0 + \kappa_0 X_1. \end{aligned} \tag{3.21}$$

If we take into account the stretching of vortex lines, ω is not a constant along the curve so that $\overrightarrow{\omega} = \overrightarrow{\omega}(x)$. When we start from a uniform circular vortex ring as in the case of Keener, we may assume $\omega = \omega_0$ constant initially.

The periodic solutions of equation(3.21) with period $mP = \frac{2\pi n}{\kappa_0}$ exist if and only if the solvability conditions are satisfied. Take $\omega_0 = 1$ for normalization. Consider

$$\begin{aligned} \frac{d\Phi_1}{dx} - \kappa_0\Psi_1 &= 0 \text{ and} \\ \frac{d\Psi_1}{dx} + \kappa_0\Phi_1 &= \tau_1(x). \end{aligned}$$

By the method of variation of parameters we can solve this first order linear system of equations (3.21).

Take

$$\begin{aligned} \Phi_1(x) &= \nu_1\left(\sin\kappa_0 x + \cos\kappa_0 x\right) + \nu_2\left(\sin\kappa_0 x - \cos\kappa_0 x\right) \text{ and} \\ \Psi_1(x) &= \nu_1\left(\cos\kappa_0 x - \sin\kappa_0 x\right) + \nu_2\left(\sin\kappa_0 x + \cos\kappa_0 x\right), \end{aligned} \tag{3.22}$$

such that

$$\Phi_1' = \nu_1'\Phi_1^{(1)} + \nu_1\Phi_1^{(1)'} + \nu_2'\Phi_1^{(2)} + \nu_2\Phi_1^{(2)'} \text{ where } \nu_1'\Phi_1^{(1)} + \nu_2'\Phi_1^{(2)} = 0 \text{ and}$$
$$\Psi_1' = \nu_1'\Psi_1^{(1)} + \nu_1\Psi_1^{(1)'} + \nu_2'\Psi_1^{(2)} + \nu_2\Psi_1^{(2)'} \text{ where } \nu_1'\Psi_1^{(1)} + \nu_2'\Psi_1^{(2)} = \tau_1(x).$$

Solving for ν_1' and ν_2', we get

$$\nu_1' = \frac{\tau_1(x)\Phi_1^{(2)}(x)}{\Psi_1^{(1)}\Phi_1^{(2)} - \Phi_1^{(1)}\Psi_1^{(2)}} \text{ and}$$
$$\nu_2' = \frac{-\tau_1(x)\Phi_1^{(1)}(x)}{\Psi_1^{(1)}\Phi_1^{(2)} - \Phi_1^{(1)}\Psi_1^{(2)}} \text{ where}$$
$$\Psi_1^{(1)}\Phi_1^{(2)} - \Phi_1^{(1)}\Psi_1^{(2)} = (\sin\kappa_0 x - \cos\kappa_0 x)(\cos\kappa_0 x - \sin\kappa_0 x) - (\sin\kappa_0 x + \cos\kappa_0 x)^2 = -2.$$

Then

$$\nu_1' = \frac{-\tau_1(x)\Phi_1^{(2)}(x)}{2} \text{ and}$$
$$\nu_2' = \frac{\tau_1(x)\Phi_1^{(1)}(x)}{2}.$$

On integration, we get

$$\begin{aligned} \nu_1 &= \frac{1}{2}\int_0^{mP} \tau_1(x)(\sin\kappa_0 x - \cos\kappa_0 x)\,dx \text{ and} \\ \nu_2 &= \frac{1}{2}\int_0^{mP} \tau_1(x)(\sin\kappa_0 x + \cos\kappa_0 x)\,dx, \end{aligned} \tag{3.23}$$

so that equation(3.22) become

$$\Phi_1 = \nu_1\Phi_1^{(1)} + \nu_2\Phi_1^{(2)} \text{ and}$$
$$\Psi_1 = \nu_1\Psi_1^{(1)} + \nu_2\Psi_1^{(2)}$$

To get periodic solutions, the integral of the tangent vector $\overrightarrow{T}$ is to be zero. To the first order in ε , this implies that

$$\begin{aligned} &\int_0^{mP} \Theta_1(x)\sin\kappa_0 x dx = 0, \\ &\int_0^{mP} \Theta_1(x)\cos\kappa_0 x dx = 0 \text{ and} \\ &\int_0^{mP} \Phi_1(x) dx = 0. \end{aligned} \tag{3.24}$$

Then using (3.22) and (3.23) in (3.24) we get

$$\begin{aligned} \int_0^{mP} \Phi_1(x)dx &= \int_0^{mP} \nu_1(\sin\kappa_0 x + \cos\kappa_0 x)dx + \int_0^{mP} \nu_2(\sin\kappa_0 x - \cos\kappa_0 x)dx \\ &= \int_0^{mP}\left[\int_0^{mP} \tau_1(x)\left(\sin\kappa_0 x - \cos\kappa_0 x\right)\right](\sin\kappa_0 x + \cos\kappa_0 x)dx \\ &+ \int_0^{mP}\left[\int_0^{mP} \tau_1(x)\left(\sin\kappa_0 x + \cos\kappa_0 x\right)\right](\sin\kappa_0 x - \cos\kappa_0 x)dx = 0 \end{aligned}$$

This implies that

$$\begin{aligned} &\int_0^{mP} \tau_1(x)\cos\kappa_0 x dx = 0 \text{ and} \\ &\int_0^{mP} \tau_1(x)\sin\kappa_0 x dx = 0; \end{aligned} \tag{3.25}$$

i.e $\tau_1(x)$ is orthogonal to $\sin\kappa_0 x$ and $\cos\kappa_0 x$ in $0 \le x \le mP$. Since $\tau_0 = 0$ we get $\tau(x)$is orthogonal to $\sin\kappa_0 x$ and $\cos\kappa_0 x$ in $0 \le x \le mP$.

Also $\int_0^{mP} \Theta_1(x)\sin\kappa_0 x dx = 0$ gives

$$\begin{aligned} &\int_0^{mP} \Theta_1'(x)\cos\kappa_0 x dx = 0 \text{ and} \\ \text{similarly } &\int_0^{mP} \Theta_1'(x)\sin\kappa_0 x dx = 0. \end{aligned} \tag{3.26}$$

But

$$\frac{d\Theta_1}{dx} = \kappa_1(x) + \omega_1\kappa_0 + \kappa_0 X_1, \tag{3.27}$$

so that $\int_0^{mP} [\kappa_1(x) + \omega_1\kappa_0 + \kappa_0 X_1] \cos\kappa_0 x dx = 0$ where$\Theta_1(x)$ and X_1 are both periodic. X_1 is so chosen that the average of X_1 is zero. i.e.

$$\frac{1}{mP}\int_0^{mP} X_1 dx = 0. \tag{3.28}$$

Since $\kappa_0 \neq 0$ (a constant), we get

$$\begin{gathered} \int_0^{mP} \kappa_1(x)\cos\kappa_0 x dx = 0 \text{ and } \int_0^{mP} \omega_1(x)\cos\kappa_0 x dx = 0, \\ \text{Similarly } \int_0^{mP} \kappa_1(x)\sin\kappa_0 x dx = 0 \text{ and } \int_0^{mP} \omega_1(x)\sin\kappa_0 x dx = 0. \end{gathered} \tag{3.29}$$

These results show that $\kappa_1(x)$, $\omega_1(x)$ and $\tau_1(x)$ must be orthogonal to both $sin\kappa_0 x$ and $cos\kappa_0 x$ in $0 \leq x \leq mP$ and the average value of $\tau_1(x)$ must be zero in the interval $0 \leq x \leq mP$.

Consider equation (3.27). Since $\kappa_1(x)$ and $\omega_1(x)$ are periodic functions of x with period mP and X_1 is so chosen as in equation (3.28), $\Theta_1(x)$ is also periodic in $0 \leq x \leq mP$. Thus we get $\int_0^{mP} \kappa_1(x)dx + \kappa_0 \int_0^{mP} \omega_1(x)dx = 0$ so that the average value of both $\kappa_1(x)$ and $\omega_1(x)$ are zero. Also we have

$$\Theta_1(x) = \Theta_1(0) + \int_0^x \kappa_0\omega_1(\alpha)d\alpha + \int_0^x \kappa_1(\alpha)d\alpha \tag{3.30}$$

These are the conditions for the existence of the periodic solutions of equation (3.21) in the order $o(\varepsilon)$.

This is true for each order of perturbation. i.e. periodic requirements are satisfied at each order of perturbation scheme. The factor $\omega = \omega(x,t)$ is periodic, is used to ensure that $\Theta(x)$ is periodic and the average value of $\tau = 0$ is used to ensure that value of Φ is zero. These are sufficient for applying implicit function theorem to guarantee that there is a solution of the full problem for all ε sufficiently small.

For the example given by (3.16) where the curvature, torsion and vorticity are simple trigonometric functions, the solution of the leading order equation is given by

$$\begin{aligned} \Phi_1(x) &= \frac{c\kappa_0 \cos\mu x}{\kappa_0^2 - \mu^2}, \\ \Psi_1(x) &= \frac{\mu c \sin\mu x}{\mu^2 - \kappa_0^2} \end{aligned} \tag{3.31}$$

and $\Theta_1(x) = \Theta_1(0) + \int_0^x \kappa_0 \omega_1(\alpha) d\alpha + \int_0^x \kappa_1(\alpha) d\alpha$.

Substituting for κ_1 and ω_1 we get the corresponding solution for $\Theta_1(x)$ as

$$\begin{aligned} \Theta_1(x) &= \Theta_1(0) - \frac{p}{\mu}\cos\mu x + \frac{q}{\mu}\sin\mu x - \frac{p}{\mu} \text{ where} \\ p &= a + \kappa_0 d \text{ and } q = b + \kappa_0 e. \end{aligned} \tag{3.32}$$

We have for small ε, the tangent vector is given approximately by

$$\vec{T} = \begin{bmatrix} \cos\kappa_0 x \\ \sin\kappa_0 x \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -\Theta_1(x)\sin\kappa_0 x \\ \Theta_1(x)\cos\kappa_0 x \\ \Theta_1(x) \end{bmatrix} + \dots\dots\dots\dots$$

Integrating the tangent vector, we find the position vector $\vec{R}$ with components

r_1, r_2 and r_3 given by

$$
\begin{aligned}
r_1(x) &= \int \cos\kappa_0 x dx + \varepsilon \int -\Theta_1(x)\sin\kappa_0 x dx + ... \\
&= \frac{\sin\kappa_0 x}{\kappa_0} + \frac{\varepsilon}{\kappa_0^2-\mu^2}\left[\begin{array}{l} q\left(\frac{\kappa_0}{\mu}\cos\kappa_0 x\sin\mu x - \sin\kappa_0 x\cos\mu x\right) \\ -p\left(\frac{\kappa_0}{\mu}\cos\kappa_0 x\cos\mu x + \sin\kappa_0 x\sin\mu x\right)\end{array}\right] \\
&+ \varepsilon\left(\Theta_1(0) - \frac{p}{\mu}\right)\frac{\cos\kappa_0 x}{\kappa_0}
\end{aligned} \tag{3.33}
$$

$$
\begin{aligned}
r_2(x) &= -\frac{\cos\kappa_0 x}{\kappa_0} + \frac{\varepsilon}{\kappa_0^2-\mu^2}\left[\begin{array}{l} q\left(\cos\kappa_0 x\cos\mu x + \frac{\kappa_0}{\mu}\sin\kappa_0 x\sin\mu x\right) \\ +p\left(\cos\kappa_0 x\sin\mu x - \frac{\kappa_0}{\mu}\sin\kappa_0 x\cos\mu x\right)\end{array}\right] \\
&+ \varepsilon\left(\Theta_1(0) - \frac{p}{\mu}\right)\frac{\sin\kappa_0 x}{\kappa_0} + ...
\end{aligned} \tag{3.34}
$$

and

$$
r_3(x) = \frac{\varepsilon\kappa_0 c\sin\mu x}{\mu\left(\kappa_0^2-\mu^2\right)} + ... \tag{3.35}
$$

to leading order in ε.

At each order in the perturbation calculations, the periodicity of ω is used to guarantee that $\Theta(x) - \kappa_0 x$ is a periodic function and the average value of torsion must be adjusted so that the curve is closed.

Collecting $o(\varepsilon^2)$ terms in equation (3.14) we get

$$
\begin{aligned}
(\Phi_2)_x &= \omega_0\kappa_0\Psi_2 + (\omega_0\kappa_1 + \kappa_0\omega_1)\Psi_1 \\
(\Psi_2)_x &= \omega_0(\tau_2 + \tau_1 X_1) - \omega_0\kappa_0\Phi_2 + \tau_1\omega_1 - (\kappa_1\omega_0 + \omega_1\kappa_0)\Phi_1 \\
(\Theta_2)_x &= \kappa_0\Phi_1^2 + \omega_0(\kappa_0 X_2 + \kappa_1 X_1) + \omega_1(\kappa_0 X_1 + \kappa_1) + \omega_2\kappa_0
\end{aligned} \tag{3.36}
$$

From the first two of these equations we get (taking ω_0=1)

$$
\frac{d^2\Psi_2}{dx^2} + \kappa_0^2\Psi_2 = -\kappa_0(\kappa_1+\kappa_0)\omega_1\Psi_1 + \tau_1 X_1 + \tau_2 - (\kappa_1 + \omega_1\kappa_0)(\Phi_1)_x + \tau_1\omega_1 \tag{3.37}
$$

From the above equations we see that both Φ_2 and Ψ_2 are periodic with period mP.

We have

$$\int_0^{mP} \tau(x)dx = \int_0^{mP} (\tau_0 + \varepsilon\tau_1(x) + \varepsilon^2\tau_2(x) + ...)dx$$

$$= \varepsilon^2 \int_0^{mP} \tau_2(x)dx \text{ since } \int_0^{mP} \tau_1 dx = 0 \text{ and} \tau_0 = 0$$

$$= \varepsilon^2 \left[\int_0^{mP} (\Psi_2)_x dx - \int_0^{mP} \tau_1 X_1 dx + \int_0^{mP} (\kappa_1\Phi_1 + \kappa_0\Phi_1\omega_1 + \kappa_0\Phi_2 - \tau_1\omega_1)dx \right]$$

$$= \varepsilon^2 \left(\begin{array}{l} c\kappa_0 \displaystyle\int_0^{mP} \frac{(a\sin\mu x + b\cos\mu x)\cos\mu x}{\kappa_0^2 - \mu^2} dx + \kappa_0^2 c \int_0^{mP} \frac{(d\sin\mu x + e\cos\mu x)\cos\mu x}{\kappa_0^2 - \mu^2} dx - \\ \displaystyle\int_0^{mP} c(d\sin\mu x + e\cos\mu x)\cos\mu x dx \end{array} \right)$$

$$= \varepsilon^2 \left[\frac{bc\kappa_0}{2\left(\kappa_0^2 - \mu^2\right)} \int_0^{mP} (1 + \cos 2\mu x)dx + \frac{ce\kappa_0^2}{2\left(\kappa_0^2 - \mu^2\right)} \int_0^{mP} (1 + \cos 2\mu x)dx \right]$$

$$- \frac{\varepsilon^2 ce}{2} \int_0^{mP} (1 + \cos 2\mu x)dx$$

$$= -\varepsilon^2 c \frac{(b\kappa_0 + e\mu^2)}{2\left(\mu^2 - \kappa_0^2\right)} mP$$

Therefore

$$\frac{1}{mP} \int_0^{mP} \tau(x)dx = -\,\varepsilon^2 c \frac{(b\kappa_0 + e\mu^2)}{2\left(\mu^2 - \kappa_0^2\right)}. \tag{3.38}$$

If $\overline{R}$ is a closed curve satisfying the conditions laid down in the theorem 4, then this curve will be a closed vortex filament in the form of a torus knot having winding number m/n where m and n are co -prime integers such that $m > n > 1$.

Suppose the curve $\overrightarrow{R}(x)$ can be written as

$$\overrightarrow{R}(x) = \overrightarrow{R}(t) + \alpha(t)\overrightarrow{N}(t) + \beta(t)\overrightarrow{B}(t) \tag{3.39}$$

where $\overrightarrow{R_0}(t)$ is the centreline of the torus, $\overrightarrow{N}$ and $\overrightarrow{B}$ are the normal and bi-normal vectors for the centreline $\overrightarrow{R_0}(t)$ in toroidal co ordinates and t is some function of x (but not time). The components of the centreline can be taken as

$$r_{01}(t) = \frac{1}{\kappa_0}\sin\kappa_0 t, r_{02}(t) = -\frac{1}{\kappa_0}\cos\kappa_0 t, r_{03} = 0 \tag{3.40}$$

i.e. a circle in the XY -plane. $'t'$ is chosen as a function of x so that the tangent vector of $\overrightarrow{R_0}(t)$ is orthogonal to the vector $\overrightarrow{R}(x)$. i.e.

$$r_1(x)\cos\kappa_0 t + r_2(x)\sin\kappa_0 t = 0 \tag{3.41}$$

It follows that

$$\begin{aligned}\sin\kappa_0(x-t) + \frac{\varepsilon\kappa_0}{\mu\left(\kappa_0^2-\mu^2\right)}&\left[\begin{array}{c} q\left[\kappa_0\cos\kappa_0(x-t)\sin\mu x - \mu\sin\kappa_0(x-t)\cos\mu x\right] \\ -p\left[\kappa_0\cos\kappa_0(x-t)\cos\mu x + \mu\sin\kappa_0(x-t)\sin\mu x\right]\end{array}\right] \\ +\varepsilon\left(\Theta_1(0)-\frac{p}{\mu}\right)&\cos\kappa_0(x-t) = 0\end{aligned} \tag{3.42}$$

or to the leading order in ε i.e.$x = t$ and $\Theta_1(0) = \frac{p}{\mu}$.

Also we have

$$\overrightarrow{R}(x)\cdot\overrightarrow{R_0}(t) = \frac{1}{\kappa_0^2} - \frac{\alpha(t)}{\kappa_0}$$

so that

$$
\begin{aligned}
\alpha(t) &= \frac{1-\kappa_0^2\overrightarrow{R}(x)\cdot\overrightarrow{R_0}(t)}{\kappa_0} \\
&= \frac{1}{\kappa_0}\left[1-\kappa_0^2\left\{\begin{array}{l}
\frac{1}{\kappa_0^2}\cos\kappa_0(x-t) \\
-\frac{\varepsilon\kappa_0^2}{\mu\kappa_0\left(\mu^2-\kappa_0^2\right)}\left[\begin{array}{l}
q\left(\kappa_0\sin\kappa_0(x-t)\sin\mu x+\mu\cos\kappa_0(x-t)\cos\mu x\right) \\
+p\left(\kappa_0\sin\kappa_0(x-t)\cos\mu x-\mu\cos\kappa_0(x-t)\sin\mu x\right)
\end{array}\right] \\
-\left(\Theta_1(0)-\frac{p}{\mu}\right)\sin\kappa_0(x-t)
\end{array}\right\}\right]
\end{aligned}
\tag{3.43}
$$

Then to the leading order in ε, i.e.$x = t$ and $\Theta_1(0) = \frac{p}{\mu}$, we have

$$
\begin{aligned}
&\alpha(t) = \frac{-\varepsilon}{\left(\mu^2-\kappa_0^2\right)}\left(q\cos\mu x+p\sin\mu x\right) \text{ where} \\
&p = a+\kappa_0 d \text{ and } q = b+\kappa_0 e
\end{aligned}
\tag{3.44}
$$

and the z-component of $\overrightarrow{R}$ is given by

$$
\beta(t) = -\frac{\varepsilon\kappa_0 c\sin\mu x}{\mu\left(\mu^2-\kappa_0^2\right)}
\tag{3.45}
$$

The trajectory of the curves $\alpha(t)$ and $\beta(t)$ determines the behavior of the closed vortex filament in the plane orthogonal to the centreline. This trajectory satisfy the equation

$$
\left(c\kappa_0\alpha-p\mu\beta\right)^2+\left(q\mu\beta\right)^2 = \left(\frac{cq\varepsilon\kappa_0}{\mu^2-\kappa_0^2}\right)^2
\tag{3.46}
$$

It follows that if q and c are nonzero, the curve $\overrightarrow{R}$ lies on a torus with elliptical cross section as in [65]. We modify the theorem 3.4 in [65] as follows.

Theorem 5. *Suppose that the curvature, torsion and vorticity of a simple closed vortex filament are $\kappa_1(x,\varepsilon)$, $\tau_1(x,\varepsilon)$ and $\omega_1(x,\varepsilon)$ which are periodic functions with period $P = \frac{2\pi n}{m\kappa_0}$ where m and n are relatively prime integers. Suppose further that*

$\kappa_1(x,\varepsilon) = a\ sin\mu x + b\ cos\mu x + o(\varepsilon)$

$\tau_1(x,\varepsilon) = c\ cos\mu x + o(\varepsilon)$ *and*

$\omega_1(x,\varepsilon) = d\ sin\mu x + e\ cos\mu x + o(\varepsilon)$ *with*

$$\begin{aligned}
&\int_0^P \kappa_1(x,\varepsilon)\,dx = \varepsilon^2\kappa_2(\varepsilon),\\
&\int_0^P \tau_1(x,\varepsilon)\,dx = 0 \text{ and}\\
&\int_0^P \omega_1(x,\varepsilon)\,dx = 0.
\end{aligned} \tag{3.47}$$

Then at each order of perturbation, there exists a closed vortex filament with curvature, torsion and vorticity given by

$$\begin{aligned}
\kappa(x) &= \kappa_0 + \varepsilon\kappa_1(x,\varepsilon),\\
\tau(x) &= \tau_1 + \varepsilon^2\tau_2(x,\varepsilon) \quad \textit{and}\\
\omega(x) &= \omega_0 + \varepsilon\omega_1(x,\varepsilon).
\end{aligned} \tag{3.48}$$

Also

$$\tau_2(\varepsilon) = -\frac{c\left(e\mu^2 + \kappa_0 b\right)}{2\left(\mu^2 - \kappa_0^2\right)}.$$

If $c \neq 0$, the vortex filament is in the form of a closed torus knot with winding number $\frac{m}{n}$ for every $o(\varepsilon)$ for $\varepsilon \neq 0$.

3.4 Invariant knotted vortex filament

Following [65], the evolution equations of the vortex filament can be written as

$$\begin{aligned}
\left(\kappa^2\omega\right)_t &= \left(\gamma\kappa^2 - 2\tau\kappa^2\right)_s, \\
\left(\tau\omega\right)_t &= z_s + \kappa\kappa_s, \\
\kappa z &= \kappa_{ss} + \tau\kappa\left(\gamma - \tau\right) \text{ and} \\
\omega_t &= \gamma_s,
\end{aligned} \tag{3.49}$$

where γ is arbitrary and 's' is the arc-length co ordinate. We try to determine the structure of vortex filament when the above equations have oscillatory curvature and torsion of the correct type.

As in [65] we analyze (3.49) for invariant solution. These are travelling wave solutions where the speed of translation is γ. In order to work on a fixed spatial interval, 's' is transformed to x by $\frac{ds}{dx} = \omega$ where ω is periodic function of x so that the curve is closed as given by theorem 5. Like in [65], we express the solutions as perturbations of a constant solution. Consider the equations

$$\begin{aligned}
\gamma\kappa^2 - 2\tau\kappa^2 &= C_1, \\
\kappa^2 + 2z &= C_2, \\
\kappa_{xx} &= \omega^2\left[z\kappa - \tau\kappa\left(\gamma - \tau\right)\right] \text{ and} \\
\gamma &= C_3,
\end{aligned} \tag{3.50}$$

where the constants C_1, C_2 and C_3 are arbitrary constants of integration. Here the fluid parameter ω will be periodic by the condition that the curve be closed as given in theorem 4. Also $\kappa = \kappa_0, \tau = 0, z = 0$ is a solution for any constant $\gamma = \gamma_0$. To find the spatially periodic solutions in a neighborhood of this constant solution, linearize the equations (3.50) about the known constant solution. i.e put $\kappa = \kappa_0 + \varepsilon\kappa_1, \tau = \varepsilon\tau_1, z = \varepsilon z_1$

and $\gamma = \gamma_0$ in equation (3.50).

Collecting coefficients of ε, we get

$$\begin{aligned} &\kappa_{1xx} + \left(\kappa_0^2 + \gamma_0^2\right)\kappa_1 = c_1\gamma_0 + c_2\kappa_0, \\ &\gamma_0\kappa_1 - \gamma_1\kappa_0 = c_1, \\ &z_1 + \kappa_0\kappa_1 = c_2 \text{ and} \\ &\omega_{1x} = c_3, \end{aligned} \tag{3.51}$$

where κ_1, τ_1 and z_1 are the small deviations of κ, τ and z respectively from the constant solution. The solutions are of the form

$$\begin{aligned} &\kappa_1\,(x) = a\ \kappa_0\ cos\mu x, \\ &\tau_1\,(x) = a\ \gamma_0\ cos\mu x \text{ and} \\ &\omega_1 = c_3 x \text{ where } \gamma_0^2 = \mu^2 - \kappa_0^2. \end{aligned} \tag{3.52}$$

For periodicity, $c_3 = 0$ so that $\omega_1 = 0$. This implies that $\omega = \omega_0$, a constant. By proper choice of μ and γ_0, the above solution give a torus knot. Thus the exact structure is a closed vortex filament in the form of a m/n torus knot.

Knotted solution of equation (3.49) is obtained by seeking a power series solution of the form

$$\begin{aligned} &\kappa\,(x) = \kappa_0 + \varepsilon\ \kappa_0\ cos\mu x + \varepsilon^2\ \kappa_2(x) + \ldots\ldots\ldots\ldots, \\ &\tau\,(x) = \varepsilon\ \gamma_0\ cos\mu x + \varepsilon^2\tau_2(x) + \ldots\ldots\ldots\ldots\ldots\ldots \text{ and} \\ &\quad\omega = \omega_0 + \varepsilon\omega_1(x) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3.53}$$

Then the average torsion should satisfy the equation

$$\frac{1}{P}\int_0^P \tau_2(x)dx = -\frac{\varepsilon^2\kappa_0^2}{2\gamma_0}. \tag{3.54}$$

It follows from theorem 5 that for γ in a one sided neighbourhood of γ_0, there are closed knotted invariant solutions of the equations of motion (3.49). The knot can be right handed or left handed according as γ is positive or negative. Also γ determines the speed of translation of the knot along its tangential direction. A right-handed knot rotates in the direction of its tangent vector and a left-handed knot rotates in the direction opposite to its tangent vector. The knot moves as a rigid body in the direction of the binormal to the circle that underlies the torus knot.

3.5 Dynamics of knotted vortex filaments

As in chapter 2 section (2.5) we consider the simplest two soliton solutions of the non-linear Schrödinger equation for the curvature, torsion and vorticity. In section (3.4) we note that the simple solitary wave solution corresponds to an invariant torus knot and that for each rational number m/n, there are left-handed and right-handed torus knots that rotate about their own axis of symmetry in opposite directions. The evolution equation $\vec{R}_t = \kappa\vec{B}$, with an arbitrary torus knot, will not guarantee that the motion is invariant since the equations of motion only take into account local effects.
The evolution equation (3.11) can be written as (taking $\gamma = 0$)

$$\begin{aligned} \kappa_t\omega &= -\tau_x\kappa - 2\tau\kappa_x, \\ \tau_t\omega &= \kappa\kappa_x + z_x, \\ \kappa_{xx} &= \omega^2\kappa\left(z+\tau^2\right) \text{ and} \\ \omega_t &= 0. \end{aligned} \tag{3.55}$$

The solution will be periodic in space and has small amplitude in their deviations from a circle.

Suppose that the solution of (3.55) can be written in the form as in Section (2.5)

$$\begin{aligned}
\kappa(x,t,\varepsilon) &= \kappa_0(t) + \sum_{j=1}^{\infty} \varepsilon^j \left[\kappa_j(t)e^{ij\mu x} + \kappa_j^*(t)e^{-ij\mu x}\right], \\
\tau(x,t,\varepsilon) &= \tau_0 + \sum_{j=1}^{\infty} \varepsilon^j \left[\tau_j(t)e^{ij\mu x} + \tau_j^*(t)e^{-ij\mu x}\right], \\
z(x,t,\varepsilon) &= z_0(t) + \sum_{j=1}^{\infty} \varepsilon^j \left[z_j(t)e^{ij\mu x} + z_j^*(t)e^{-ij\mu x}\right] \text{and} \\
\omega(x,\varepsilon) &= \omega_0 + \sum_{j=1}^{\infty} \varepsilon^j \left[\omega_j e^{ij\mu x} + \omega_j^* e^{-ij\mu x}\right].
\end{aligned} \tag{3.56}$$

where $*$ denotes the complex conjugate. Note that 'ω' is a periodic function of x (i.e. a space variable).

Following [65], we substitute from equation(3.56) in equation(3.55). Then, collecting the coefficients of like exponential terms and taking $\omega_0 = 1$ (for normalization), we get the following system of linear equations:

$$\begin{aligned}
&\frac{d\kappa_0}{dt} + \varepsilon^2\left(\frac{d\kappa_1}{dt}\omega_1^* + \frac{d\kappa_1^*}{dt}\omega_1\right) = -i\mu\varepsilon^2\left(\tau_1^*\kappa_1 - \tau_1\kappa_1^*\right), \\
&\frac{d\tau_0}{dt} = 0, \\
&z_0\kappa_0 + \tau_0^2\kappa_0 = -\varepsilon^2\left(\kappa_1 z_1^* + \kappa_1^* z_1 + 2\kappa_0\tau_1\tau_1^*\right), \\
&\frac{d\omega_0}{dt} = 0; \frac{d\omega_1}{dt} = 0; \frac{d\omega_2}{dt} = 0,
\end{aligned} \tag{3.57}$$

$$
\begin{aligned}
&\frac{d\kappa_1}{dt} + \omega_1 \frac{d\kappa_0}{dt} + i\mu\left(\kappa_0\tau_1 + 2\kappa_1\tau_0\right) = -3\varepsilon^2 i\mu\kappa_2\tau_1^*,\\
&\frac{d\tau_1}{dt} - i\mu\left(\kappa_0\kappa_1 + z_1\right) = i\mu\varepsilon^2\kappa_2\kappa_1^*,\\
&\kappa_0\left(z_1 + 2\tau_0\tau_1^*\right) + z_0\kappa_1 + \varepsilon^2\left(2\tau_2\tau_1^*\kappa_0 + 2\tau_1\tau_1^*\kappa_1 + \kappa_1^*\tau_1^2 + \kappa_1^* z_2 + \kappa_2 z_1^*\right)\\
&+ 2\omega_1\kappa_0\left(z_0 + \tau_0^2\right) = -\mu^2\kappa_1.
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
&\frac{d\kappa_2}{dt} + \frac{d\kappa_1}{dt}\omega_1 + \frac{d\kappa_0}{dt}\omega_2 + 2i\mu\left(\tau_2\kappa_0 + 2\tau_0\kappa_2\right) = -3i\mu\tau_1\kappa_1 - \kappa_0\omega_2^{'},\\
&\frac{d\tau_2}{dt} + \frac{d\tau_1}{dt}\omega_1 - 2i\mu\left(\kappa_0\kappa_2 + z_2\right) = i\mu\kappa_1^2 - \tau_0\omega_2^{'},\\
&\kappa_0\left(z_2 + \tau_1^2\right) + \kappa_1\left(z_1 + 2\tau_1\tau_0\right) + \kappa_2 z_0 + \kappa_2\tau_0^2 + \omega_1^2\kappa_0\left(z_0 + \tau_0^2\right) = -4\mu^2\kappa_2.
\end{aligned}
\tag{3.59}
$$

retaining only the terms up to $o(\varepsilon^2)$.

Following [65] the solutions of these equations can be approximated using multiscale techniques as in chapter 2[26, 66]. Taking $\varepsilon = 0$ in equations (3.57) and (3.58), the behaviour of the solution to leading order is obtained as follows.

$$
\begin{aligned}
&\frac{d\kappa_0}{dt} = 0 \Rightarrow \kappa_0 = \text{ a constant.}\\
&\frac{d\tau_0}{dt} = 0 \Rightarrow \tau_0 = 0.\\
&\kappa_0\left(z_0 + \tau_0^2\right) = 0 \Rightarrow z_0 = 0.\\
&\frac{d\kappa_1}{dt} + i\mu\kappa_0\tau_1 = 0,\\
&\frac{d\tau_1}{dt} - i\mu\left(\kappa_0\kappa_1 + z_1\right) = 0 \text{ and}\\
&z_1\kappa_0 = -\mu^2\kappa_1.
\end{aligned}
\tag{3.60}
$$

The last three equations above give

$$\begin{aligned}\frac{d\kappa_1}{dt} + i\mu\kappa_0\tau_1 &= 0 \text{ and} \\ \frac{d\tau_1}{dt} &= i\mu\left(\kappa_0^2 - \mu^2\right)\frac{\kappa_1}{\kappa_0}.\end{aligned} \tag{3.61}$$

We substitute $\mu^2 - \kappa_0^2 = \gamma^2$

Solving we get the solution of the system in the form

$$\begin{aligned}\kappa_1(t) &= \kappa_+ e^{i\mu\gamma t} + \kappa_- e^{-i\mu\gamma t} \text{and} \\ \tau_1(t) &= \tfrac{\gamma}{\kappa_0}\left(\kappa_- e^{-i\mu\gamma t} - \kappa_+ e^{i\mu\gamma t}\right),\end{aligned} \tag{3.62}$$

where κ_+ and κ_- are arbitrary.

This corresponds to the superposition of travelling waves, moving in opposite directions with speed γ.

3.6 Approximation by multiscale techniques

The two timing assumption is made to determine the effect of the order- ε^2 correction terms on the solution of equation(3.61) i.e. the behavior of the solution can be described in terms of two time scales, - a fast time T_0 and a slow time $T_1 = \varepsilon^2 T_0$.

Taking $\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon^2 \frac{\partial}{\partial T_1}$ (and $\varepsilon^2 = \varepsilon_1$) the equations (3.57), (3.58) and (3.59) can be rewritten as

$$\omega_0\left(\frac{\partial}{\partial T_0} + \varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{00} + \varepsilon_1\kappa_{01}) + \varepsilon_1\begin{bmatrix}\omega_1^*\left(\frac{\partial}{\partial T_0} + \varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{10} + \varepsilon_1\kappa_{11}) + \\ \omega_1\left(\frac{\partial}{\partial T_0} + \varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{10}^* + \varepsilon_1\kappa_{11}^*)\end{bmatrix} \tag{3.63}$$

$$= i\mu\varepsilon_1\left[(\tau_{10} + \varepsilon_1\tau_{11})(\kappa_{10}^* + \varepsilon_1\kappa_{11}^*) - (\tau_{10}^* + \varepsilon_1\tau_{11}^*)(\kappa_{10} + \varepsilon_1\kappa_{11})\right].$$

$$\omega_0\left(\frac{\partial}{\partial T_0} + \varepsilon_1\frac{\partial}{\partial T_1}\right)(\tau_{00} + \varepsilon_1\tau_{01}) = 0. \tag{3.64}$$

$$(z_{00}+\varepsilon_1 z_{01})(\kappa_{00}+\varepsilon_1\kappa_{01})+(\tau_{00}+\varepsilon_1\tau_{01})^2(\kappa_{00}+\varepsilon_1\kappa_{01})$$
$$=-\varepsilon_1\left[\begin{array}{l}(\kappa_{10}+\varepsilon_1\kappa_{11})(z_{10}^*+\varepsilon_1 z_{11}^*)\\+(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*)(z_{10}+\varepsilon_1 z_{11})\\+2\left\{\begin{array}{l}(\kappa_{00}+\varepsilon_1\kappa_{01})(\tau_{10}+\varepsilon_1\tau_{11})\times\\(\tau_{10}^*+\varepsilon_1\tau_{11}^*)\end{array}\right\}\end{array}\right]. \quad (3.65)$$

$$\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{10}+\varepsilon_1\kappa_{11})+\omega_1\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{00}+\varepsilon_1\kappa_{01})$$
$$=-i\mu\left[\begin{array}{l}(\kappa_{00}+\varepsilon_1\kappa_{01})(\tau_{10}+\varepsilon_1\tau_{11})+2(\kappa_{10}+\varepsilon_1\kappa_{11})(\tau_{00}+\varepsilon_1\tau_{01})+\\3\varepsilon_1(\kappa_{20}+\varepsilon_1\kappa_{21})(\tau_{10}^*+\varepsilon_1\tau_{11}^*)\end{array}\right]. \quad (3.66)$$

$$\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\tau_{10}+\varepsilon_1\tau_{11})=i\mu\left[\begin{array}{l}(\kappa_{00}+\varepsilon_1\kappa_{01})(\kappa_{10}+\varepsilon_1\kappa_{11})+\\(z_{10}+\varepsilon_1 z_{11})+\varepsilon_1\left\{\begin{array}{l}(\kappa_{20}+\varepsilon_1\kappa_{21})\times\\(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*)\end{array}\right\}\end{array}\right]. \quad (3.67)$$

$$(\kappa_{00}+\varepsilon_1\kappa_{01})(z_{10}+\varepsilon_1 z_{11})+(z_{00}+\varepsilon_1 z_{01})(\kappa_{10}+\varepsilon_1\kappa_{11})$$
$$+2\omega_1(\kappa_{00}+\varepsilon_1\kappa_{01})(z_{00}+\varepsilon_1 z_{01})$$
$$+\varepsilon_1\left[\begin{array}{l}2(\tau_{20}+\varepsilon_1\tau_{21})(\tau_{10}^*+\varepsilon_1\tau_{11}^*)(\kappa_{01}+\varepsilon_1\kappa_{11})+\\2(\tau_{10}+\varepsilon_1\tau_{11})(\tau_{10}^*+\varepsilon_1\tau_{11}^*)(\kappa_{10}+\varepsilon_1\kappa_{11})+\\(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*)(\tau_{10}+\varepsilon_1\tau_{11})^2+(\kappa_{10}^*+\varepsilon_1\kappa_{11}^*)(z_{20}+\varepsilon_1 z_{21})+\\(\kappa_{20}+\varepsilon_1\kappa_{21})(z_{10}^*+\varepsilon_1 z_{11}^*)\end{array}\right]=-\mu^2(\kappa_{10}+\varepsilon_1\kappa_{11}). \quad (3.68)$$

$$\omega_2\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{00}+\varepsilon_1\kappa_{01})+\omega_1\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{10}+\varepsilon_1\kappa_{11})$$
$$+\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\kappa_{20}+\varepsilon_1\kappa_{21})+2i\mu\left[\begin{array}{l}(\tau_{20}+\varepsilon_1\tau_{21})(\kappa_{00}+\varepsilon_1\kappa_{01})\\+2(\tau_{00}+\varepsilon_1\tau_{01})(\kappa_{20}+\varepsilon_1\kappa_{21})\end{array}\right] \quad (3.69)$$
$$=-3i\mu(\tau_{10}+\varepsilon_1\tau_{11})(\kappa_{10}+\varepsilon_1\kappa_{11}).$$

$$\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\tau_{20}+\varepsilon_1\tau_{21})+\omega_1\left(\frac{\partial}{\partial T_0}+\varepsilon_1\frac{\partial}{\partial T_1}\right)(\tau_{10}+\varepsilon_1\tau_{11})$$
$$-2i\mu\left[(\kappa_{00}+\varepsilon_1\kappa_{01})(\kappa_{20}+\varepsilon_1\kappa_{21})+(z_{20}+\varepsilon_1 z_{21})\right] \tag{3.70}$$
$$=i\mu(\kappa_{10}+\varepsilon_1\kappa_{11})^2.$$

$$(\kappa_{00}+\varepsilon_1\kappa_{01})\left[(z_{20}+\varepsilon_1 z_{21})+(\tau_{10}+\varepsilon_1\tau_{11})^2\right]$$
$$+(\kappa_{10}+\varepsilon_1\kappa_{11})\left[\begin{array}{l}(z_{10}+\varepsilon_1 z_{11})+2(\tau_{10}+\varepsilon_1\tau_{11})(\tau_{00}+\varepsilon_1\tau_{01})+\\(\kappa_{20}+\varepsilon_1\kappa_{21})(z_{00}+\varepsilon_1 z_{01})+(\kappa_{20}+\varepsilon_1\kappa_{21})(\tau_{00}+\varepsilon_1\tau_{01})^2\end{array}\right] \tag{3.71}$$
$$+\omega^2(\kappa_{00}+\varepsilon_1\kappa_{01})\left[(z_{00}+\varepsilon_1 z_{01})+(\tau_{00}+\varepsilon_1\tau_{01})^2\right]+4\mu^2(\kappa_{20}+\varepsilon_1\kappa_{21})=0.$$

The above system of equations from equation(3.63) to equation(3.71) give rise to 2 set of linear equations. The set I contains the system of o(1) linear equations, obtained by collecting terms independent of ε_1 and the Set II contains the system of $o(\varepsilon_1)$ linear equations, obtained by collecting coefficients of ε_1 in the above system.

SetI

$$\frac{\partial\kappa_{00}}{\partial T_0}=0,$$
$$\frac{\partial\tau_{00}}{\partial T_0}=0,$$
$$z_{00}\kappa_{00}+\tau_{00}^2\kappa_{00}=0,$$
$$\frac{\partial\kappa_{10}}{\partial T_0}+\omega_1\frac{\partial\kappa_{00}}{\partial T_0}=-i\mu\kappa_{00}\tau_{10},$$
$$\frac{\partial\tau_{10}}{\partial T_0}=i\mu(\kappa_{00}\kappa_{10}+z_{10}),$$
$$z_{10}\kappa_{00}+\mu^2\kappa_{10}+z_{00}\kappa_{10}+2\kappa_{00}\left[\tau_{00}\tau_{10}+\omega_1\left(\tau_{00}^2+z_{00}\right)\right]=0,$$
$$\frac{\partial\kappa_{20}}{\partial T_0}+\omega_1\frac{\partial\kappa_{10}}{\partial T_0}+\omega_2\frac{\partial\kappa_{00}}{\partial T_0}=-2i\mu(\tau_{20}\kappa_{00}+2\tau_{00}\kappa_{20})-3i\mu\tau_{10}\kappa_{10},$$
$$\frac{\partial\tau_{20}}{\partial T_0}+\omega_1\frac{\partial\tau_{10}}{\partial T_0}-2i\mu(\kappa_{00}\kappa_{20}+z_{20})=i\mu\kappa_{10}^2 \text{ and}$$
$$z_{20}\kappa_{00}+\kappa_{00}\tau_{10}^2+z_{10}\kappa_{10}+2\kappa_{00}\tau_{00}\tau_{10}+z_{00}\kappa_{20}+\tau_{00}^2\kappa_{20}+\omega_1^2\kappa_{00}\left(\tau_{00}^2+z_{00}\right)=-4\mu^2\kappa_{20}. \tag{3.72}$$

Set II

$$
\begin{aligned}
&\frac{\partial\kappa_{01}}{\partial T_0}+\omega_1^*\frac{\partial\kappa_{10}}{\partial T_0}+\omega_1\frac{\partial\kappa_{10}^*}{\partial T_0}=i\mu\left(\kappa_{10}^*\tau_{10}-\kappa_{10}\tau_{10}^*\right),\\
&\frac{\partial\tau_{01}}{\partial T_0}+\frac{\partial\tau_{00}}{\partial T_1}=0,\\
&z_{00}\kappa_{01}+z_{01}\kappa_{00}+2\tau_{00}\left[\kappa_{00}\tau_{01}+\kappa_{01}\tau_{00}\right]=-\left(\kappa_{10}z_{10}^*+\kappa_{10}^*z_{10}+2\kappa_{00}\tau_{10}\tau_{10}^*\right),\\
&\frac{\partial\kappa_{11}}{\partial T_0}+\frac{\partial\kappa_{10}}{\partial T_1}+\omega_1\frac{\partial\kappa_{01}}{\partial T_0}=-3i\mu\kappa_{20}\tau_{10}^*-i\mu\left(\kappa_{01}\tau_{01}+\kappa_{00}\tau_{11}+2\kappa_{10}\tau_{01}\right),\\
&\frac{\partial\tau_{11}}{\partial T_0}+\frac{\partial\tau_{10}}{\partial T_1}=i\mu\left(\kappa_{01}\kappa_{10}+\kappa_{00}\kappa_{11}+z_{11}+\kappa_{20}\kappa_{10}^*\right),\\
&z_{11}\kappa_{00}+z_{10}\kappa_{01}+z_{01}\kappa_{10}+2\left(\tau_{01}\tau_{10}\kappa_{00}+\tau_{20}\tau_{10}^*\kappa_{00}+\tau_{10}\tau_{10}^*\kappa_{10}\right),\\
&+\kappa_{10}^*\tau_{10}^2+\kappa_{10}^*z_{20}+z_{10}^*\kappa_{20}+2\omega_1\kappa_{00}z_{01}=-\mu^2\kappa_{11},\\
&\frac{\partial\kappa_{21}}{\partial T_0}+\frac{\partial\kappa_{20}}{\partial T_1}+\omega_1\left(\frac{\partial\kappa_{11}}{\partial T_0}+\frac{\partial\kappa_{10}}{\partial T_1}\right)+\omega_2\frac{\partial\kappa_{01}}{\partial T_0},\\
&=-2i\mu\left(\tau_{20}\kappa_{01}+\tau_{21}\kappa_{00}+2\tau_{01}\kappa_{20}\right)-3i\mu\left(\tau_{10}\kappa_{11}+\tau_{11}\kappa_{10}\right),\\
&\frac{\partial\tau_{21}}{\partial T_0}+\frac{\partial\tau_{20}}{\partial T_1}+\omega_1\left(\frac{\partial\tau_{11}}{\partial T_0}+\frac{\partial\tau_{10}}{\partial T_1}\right)=2i\mu\left[\kappa_{01}\kappa_{20}+\kappa_{00}\kappa_{21}+z_{21}+\kappa_{10}\kappa_{11}\right],\\
&z_{21}\kappa_{00}+z_{20}\kappa_{01}+\kappa_{00}\tau_{10}^2+z_{11}\kappa_{10}+z_{10}\kappa_{11}+\kappa_{20}z_{01}\\
&+2\tau_{10}\left(\kappa_{00}\tau_{11}+\tau_{01}\kappa_{10}\right)+\omega_1^2\kappa_{00}z_{01}=-4\mu^2\kappa_{21}.
\end{aligned}
\tag{3.73}
$$

We use the method of variation of parameters to solve the above linear system of equations. Also we see that the solution to the linear order given by (3.61) is (3.62), with a modification that κ_+ and κ_- depend on both T_0 and T_1.

We are looking for the evolution of curvature and torsion of vortex knots for a given function $\omega(x,t)$ of 't'. Here $\omega(x,t)$ is taken as a periodic function of x, a space variable. Without loss of generality, we take $\omega_0 = 1$ for normalization. So the derivatives of ω_0 occurring in the above set of equations will be zero. From equation (3.72) we get the following.

$$\begin{aligned}\kappa_{00} &= \text{ constant,} \\ \tau_{00} &= 0 \text{ and} \\ z_{00} &= 0.\end{aligned} \tag{3.74}$$

Again from equations 6^{th} and 9^{th} of (3.72), we get

$$\begin{aligned} z_{10} &= -\frac{\mu^2\kappa_{10}}{\kappa_{00}} \text{ and} \\ z_{20} &= \frac{1}{\kappa_{00}^2}\left(\mu^2\kappa_{10}^2 - 4\mu^2\kappa_{20} - \kappa_{00}\tau_{10}^2\right). \end{aligned} \tag{3.75}$$

Therefore we can write the equations 4 and 5 of equation (3.72) as

$$\begin{aligned} \frac{\partial\kappa_{10}}{\partial T_0} + i\mu\kappa_{00}\tau_{10} = 0 \text{ and} \\ \frac{\partial\tau_{10}}{\partial T_0} + i\mu\left(\mu^2 - \kappa_{00}^2\right)\frac{\kappa_{10}}{\kappa_{00}} = 0. \end{aligned} \tag{3.76}$$

Solving, by the method of variation of parameters we get the solution as

$$\begin{aligned} \kappa_{10} &= \kappa_+ e^{i\mu\gamma T_0} + \kappa_- e^{-i\mu\gamma T_0} \text{ and} \\ \tau_{10} &= \frac{\gamma}{\kappa_{00}}\left[\kappa_- e^{-i\mu\gamma T_0} - \kappa_+ e^{i\mu\gamma T_0}\right]. \end{aligned} \tag{3.77}$$

These are the same as the solutions (3.62) obtained earlier. As in chapter 2, then consider the equations 7 and 8 of (3.72). They can be rewritten as

$$\begin{aligned} &\frac{\partial\kappa_{20}}{\partial T_0} + 2i\mu\kappa_{00}\tau_{20} = -\omega_1\frac{\partial\kappa_{10}}{\partial T_0} - 3i\mu\kappa_{10}\tau_{10} \text{ and} \\ &\frac{\partial\tau_{20}}{\partial T_0} + 2i\mu\left(4\mu^2 - \kappa_{00}^2\right)\frac{\kappa_{20}}{\kappa_{00}} = 2i\frac{\mu}{\kappa_{00}^2}\left(\mu^2\kappa_{10}^2 - \kappa_{00}^2\tau_{10}^2\right) + i\mu\kappa_{10}^2 - \omega_1\frac{\partial\tau_{10}}{\partial T_0}. \end{aligned} \tag{3.78}$$

Considering the homogenous part of the system (3.78) we get the solution as

$$\kappa_{20} = A_1 e^{2i\mu\beta T_0} + A_2 e^{-2i\mu\beta T_0} \text{and}$$
$$\tau_{20} = -\frac{\beta}{\kappa_{00}}\left[A_1 e^{2i\mu\beta T_0} - A_2 e^{-2i\mu\beta T_0}\right] \text{ where } \beta^2 = 4\mu^2 - \kappa_{00}^2 \qquad (3.79)$$

Here A_1 and A_2 are functions of T_0 and T_1 .

The evaluation of A_1 and A_2 is done as follows:

We use the method of variation of parameters to find the solution of the non-homogeneous system (3.78).

Let the solution be

$$\kappa_{20} = A_1(T_0, T_1)\,\kappa_{20}^{(1)}(T_0) + A_2(T_0, T_1)\,\kappa_{20}^{(2)}(T_0) \quad \text{and}$$
$$\tau_{20} = A_1(T_0, T_1)\,\tau_{20}^{(1)}(T_0) + A_2(T_0, T_1)\,\tau_{20}^{(2)}(T_0) \qquad (3.80)$$

where

$$A_1'(T_0, T_1)\,\kappa_{20}^{(1)} + A_2'(T_0, T_1)\,\kappa_{20}^{(2)} = -\omega_1 \kappa_{10}' - 3i\mu\tau_{10}\kappa_{10} \quad \text{and}$$
$$A_1'(T_0, T_1)\,\tau_{20}^{(1)} + A_2'(T_0, T_1)\,\tau_{20}^{(2)} = i\mu\kappa_{10}^2 - \omega_1\tau_{10}' + 2\frac{i\mu}{\kappa_{00}^2}\left(\mu^2\kappa_{10}^2 - \kappa_{00}^2\tau_{10}^2\right). \qquad (3.81)$$

Solving equation (3.81), we get

$$A_1'(T_0, T_1) = \frac{\begin{vmatrix} -\omega_1\kappa_{10}' - 3i\mu\tau_{10}\kappa_{10} & e^{-2i\mu\beta T_0} \\ i\mu\kappa_{10}^2 - \omega_1\tau_{10}' + 2\frac{i\mu}{\kappa_{00}^2}\left(\mu^2\kappa_{10}^2 - \kappa_{00}^2\tau_{10}^2\right) & \frac{\beta}{\kappa_{00}} e^{-2i\mu\beta T_0} \end{vmatrix}}{\begin{vmatrix} e^{2i\mu\beta T_0} & e^{-2i\mu\beta T_0} \\ -\frac{\beta}{\kappa_{00}} e^{2i\mu\beta T_0} & \frac{\beta}{\kappa_{00}} e^{-2i\mu\beta T_0} \end{vmatrix}}$$
$$= -\frac{e^{-2i\mu\beta T_0}}{2}\left[\omega_1\kappa_{10}' + 3i\mu\tau_{10}\kappa_{10} + 2\frac{i\mu}{\beta\kappa_{00}}\left(\mu^2\kappa_{10}^2 - \kappa_{00}^2\tau_{10}^2\right) + \frac{\kappa_{00}}{\beta}\left(i\mu\kappa_{10}^2 - \omega_1\tau_{10}'\right)\right]$$

Integrating partially with respect to T_0 and then multiplying with $e^{2i\mu\beta T_0}$, we get

$$\begin{aligned}
A_1(T_0,T_1)\,e^{2i\mu\beta T_0} &= \frac{\omega_1\gamma}{2(\gamma^2-4\beta^2)}\left[\kappa_+(\gamma+2\beta)\,e^{i\mu\gamma T_0}+\kappa_-(\gamma-2\beta)\,e^{-i\mu\gamma T_0}\right]\\
&+\frac{3\gamma}{4(\gamma^2-\beta^2)\,\kappa_{00}}\left[\kappa_+^2(\gamma+\beta)\,e^{2i\mu\gamma T_0}+\kappa_-^2(\gamma-\beta)\,e^{-2i\mu\gamma T_0}\right]\\
&-\frac{2\mu^2+\kappa_{00}^2}{4(\gamma^2-\beta^2)\,\beta^2\kappa_{00}}\left[\begin{array}{l}\kappa_+^2\beta(\gamma+\beta)\,e^{2i\mu\gamma T_0}-2\kappa_+\kappa_-\left(\gamma^2-\beta^2\right)\\ -\kappa_-^2\beta(\gamma-\beta)\,e^{-2i\mu\gamma T_0}\end{array}\right]\\
&+\frac{\gamma^2}{2\kappa_{00}\beta^2(\gamma^2-\beta^2)}\left[\begin{array}{l}\kappa_+^2\beta(\gamma+\beta)\,e^{2i\mu\gamma T_0}+2\kappa_+\kappa_-\left(\gamma^2-\beta^2\right)\\ -\kappa_-^2\beta(\gamma-\beta)\,e^{-2i\mu\gamma T_0}\end{array}\right]\\
&-\frac{\omega_1\gamma^2}{2\beta(\gamma^2-4\beta^2)}\left[\kappa_+(\gamma+2\beta)\,e^{i\mu\gamma T_0}-\kappa_-(\gamma-2\beta)\,e^{-i\mu\gamma T_0}\right]
\end{aligned} \tag{3.82}$$

Proceeding as above, starting from equation (3.81), we get

$$\begin{aligned}
A_2(T_0,T_1)\,e^{-2i\mu\beta T_0} &= -\frac{\omega_1\gamma}{2(\gamma^2-4\beta^2)}\left[\kappa_+(\gamma-2\beta)\,e^{i\mu\gamma T_0}+\kappa_-(\gamma+2\beta)\,e^{-i\mu\gamma T_0}\right]\\
&+\frac{3\gamma}{4(\gamma^2-\beta^2)\,\kappa_{00}}\left[\kappa_+^2(\gamma-\beta)\,e^{2i\mu\gamma T_0}+\kappa_-^2(\gamma+\beta)\,e^{-2i\mu\gamma T_0}\right]\\
&-\frac{2\mu^2+\kappa_{00}^2}{4(\gamma^2-\beta^2)\,\beta^2\kappa_{00}}\left[\begin{array}{l}-\kappa_+^2\beta(\gamma-\beta)\,e^{2i\mu\gamma T_0}-2\kappa_+\kappa_-\left(\gamma^2-\beta^2\right)\\ +\kappa_-^2\beta(\gamma+\beta)\,e^{-2i\mu\gamma T_0}\end{array}\right]\\
&+\frac{\gamma^2}{2\kappa_{00}\beta^2(\gamma^2-\beta^2)}\left[\begin{array}{l}-\kappa_+^2\beta(\gamma-\beta)\,e^{2i\mu\gamma T_0}+2\kappa_+\kappa_-\left(\gamma^2-\beta^2\right)\\ +\kappa_-^2\beta(\gamma+\beta)\,e^{-2i\mu\gamma T_0}\end{array}\right]\\
&-\frac{\omega_1\gamma^2}{2\beta(\gamma^2-4\beta^2)}\left[-\kappa_+(\gamma-2\beta)\,e^{i\mu\gamma T_0}+\kappa_-(\gamma+2\beta)\,e^{-i\mu\gamma T_0}\right]
\end{aligned} \tag{3.83}$$

Substituting in equation(3.80) and on simplification, we get

$$\kappa_{20}=\frac{2\kappa_{00}^2-\mu^2}{2\kappa_{00}\mu^2}\left[\kappa_+^2e^{2i\mu\gamma T_0}+\kappa_-^2e^{-2i\mu\gamma T_0}\right]+\frac{\omega_1\gamma^2}{(5\mu^2-\kappa_{00}^2)}\left[\kappa_+e^{i\mu\gamma T_0}+\kappa_-e^{-i\mu\gamma T_0}\right]+\frac{\kappa_+\kappa_-}{\kappa_{00}} \tag{3.84}$$

Then

$$\begin{aligned}
\tau_{20} &= -\frac{1}{2i\mu\kappa_{00}}\frac{\partial\kappa_{20}}{\partial T_0} \\
&= \frac{\left(\mu^2-2\kappa_{00}^2\right)\gamma}{2\kappa_{00}^2\mu^2}\left\{\kappa_+^2 e^{2i\mu\gamma T_0} - \kappa_-^2 e^{-2i\mu\gamma T_0}\right\} - \omega_1\frac{\gamma^3}{2\kappa_{00}\left(5\mu^2-\kappa_{00}^2\right)}\left(\kappa_+ e^{i\mu\gamma T_0} - \kappa_- e^{-i\mu\gamma T_0}\right) \\
&= \frac{2\kappa_{00}^2-\mu^2}{2\kappa_{00}\mu^2}\kappa_{10}\tau_{10} + \frac{\omega_1\gamma^2}{2\left(5\mu^2-\kappa_{00}^2\right)}\tau_{10}
\end{aligned} \tag{3.85}$$

Then substituting for $\kappa_{10}, \kappa_{10}^*, \tau_{10}, \tau_{10}^*$ in the first equation of (3.73)and integrating w.r.t T_0, we get,

$$\kappa_{01} = -\frac{1}{\kappa_0}\left(\kappa_-^*\kappa_+ e^{2i\mu\gamma T_0} + \kappa_-\kappa_+^* e^{-2i\mu\gamma T_0}\right) - \omega_1^*\kappa_{10} - \omega_1\kappa_{10}^*. \tag{3.86}$$

Also from Set II we get

$$z_{11} = -\frac{1}{\kappa_{00}}\left[\mu^2\kappa_{11} + \kappa_{01}z_{10} + \kappa_{20}z_{10}^* + \kappa_{10}^*\tau_{10}^2 + 2\tau_{10}\tau_{10}^*\kappa_{10} + 2\tau_{20}\tau_{10}^*\kappa_{10} + \kappa_{10}^*z_{20}\right] \tag{3.87}$$

$$\frac{\partial\kappa_{11}}{\partial T_0} + i\mu\kappa_{00}\tau_{11} = -\omega_1\frac{\partial\kappa_{01}}{\partial T_0} - \frac{\partial\kappa_{10}}{\partial T_1} - i\mu\left(\kappa_{01}\tau_{10} + 3\kappa_{20}\tau_{10}^*\right) \tag{3.88}$$

$$\begin{aligned}
\frac{\partial\tau_{11}}{\partial T_0} + i\mu\left(\mu^2-\kappa_{00}^2\right)\frac{\kappa_{11}}{\kappa_{00}} &= -\frac{\partial\tau_{10}}{\partial T_0} + i\mu\left[\kappa_{20}\kappa_{10}^* + \kappa_{01}\kappa_{10} - 2\tau_{20}\tau_{10}^*\right] \\
&\quad + i\mu\left[\begin{array}{l} \frac{\mu^2\kappa_{10}\kappa_{01}}{\kappa_{00}^2} - \frac{2|\tau_{10}|^2\kappa_{10}}{\kappa_{00}} - \frac{\kappa_{10}^*\tau_{10}^2}{\kappa_{00}} \\ -\frac{\kappa_{10}^*}{\kappa_{00}^3}\left(\mu^2\kappa_{10}^2 - \kappa_{00}^2\tau_{10}^2 - 4\mu^2\kappa_{20}\kappa_{00}\right) + \frac{\mu^2\kappa_{20}\kappa_{10}^*}{\kappa_{00}^2} \end{array}\right]
\end{aligned} \tag{3.89}$$

Now equation (3.88) gives

$$\begin{aligned}
&\frac{\partial \kappa_{11}}{\partial T_0} + i\mu\kappa_{00}\tau_{11} \\
&= e^{i\mu\gamma T_0}\left[-\frac{\partial \kappa_+}{\partial T_1} + \frac{i\gamma}{2\mu\kappa_{00}^2}\left\{\begin{array}{l}\left(2\mu^2\kappa_{00}^2\left|\omega_1\right|^2 + 3\left(2\kappa_{00}^2-\mu^2\right)\left|\kappa_+\right|^2 - 4\mu^2\left|\kappa_-\right|^2\right)\kappa_+ \\ +2\omega_1^2\mu^2\kappa_{00}^2\kappa_-^*\end{array}\right\}\right] \\
&+ e^{-i\mu\gamma T_0}\left[-\frac{\partial \kappa_-}{\partial T_1} - \frac{i\gamma}{2\mu\kappa_{00}^2}\left\{\begin{array}{l}\left(2\mu^2\kappa_{00}^2\left|\omega_1\right|^2 + 3\left(2\kappa_{00}^2-\mu^2\right)\left|\kappa_-\right|^2 - 4\mu^2\left|\kappa_+\right|^2\right)\kappa_- \\ +2\omega_1^2\mu^2\kappa_{00}^2\kappa_+^*\end{array}\right\}\right] \\
&+ 4i\mu\gamma\omega_1\frac{\left(\kappa_{00}^2+\mu^2\right)}{\kappa_{00}\left(5\mu^2-\kappa_{00}^2\right)}\left[\kappa_+\kappa_-^* e^{2i\mu\gamma T_0} + \kappa_-\kappa_+^* e^{-2i\mu\gamma T_0}\right] \\
&- \frac{4i\mu\gamma}{\kappa_{00}^2}\left[\kappa_+^2\kappa_-^* e^{3i\mu\gamma T_0} - \kappa_-^2\kappa_+^* e^{-3i\mu\gamma T_0}\right]
\end{aligned} \tag{3.90}$$

The conditions for non secularity are

$$\begin{aligned}
-\frac{\partial \kappa_+}{\partial T_0} + \frac{i\gamma}{2\mu\kappa_{00}^2}\left[\left(2\mu^2\kappa_{00}^2\left|\omega_1\right|^2 + 3\left(2\kappa_{00}^2-\mu^2\right)\left|\kappa_+\right|^2 - 4\mu^2\left|\kappa_-\right|^2\right)\kappa_+ + 2\omega_1^2\mu^2\kappa_{00}^2\kappa_-^*\right] &= 0, \\
-\frac{\partial \kappa_-}{\partial T_0} - \frac{i\gamma}{2\mu\kappa_{00}^2}\left[\left(2\mu^2\kappa_{00}^2\left|\omega_1\right|^2 + 3\left(2\kappa_{00}^2-\mu^2\right)\left|\kappa_-\right|^2 - 4\mu^2\left|\kappa_+\right|^2\right)\kappa_- + 2\omega_1^2\mu^2\kappa_{00}^2\kappa_+^*\right] &= 0,
\end{aligned}$$

along with $\omega_1 = 0$ and $\kappa_+^2\kappa_-^* e^{6i\mu\gamma T_0} = \kappa_-^2\kappa_+^*$

(3.91)

Proceeding as above equation(3.89) gives

$$
\begin{aligned}
&\frac{\partial \tau_{11}}{\partial T_0} + i\mu\left(\mu^2 - \kappa_{00}^2\right)\frac{\kappa_{11}}{\kappa_{00}} \\
&= e^{i\mu\gamma T_0}\left\{\frac{\gamma}{\kappa_{00}}\frac{\partial \kappa_+}{\partial T_1} + \frac{i}{2\kappa_{00}^3\mu}\left[4\mu^4\left|\kappa_-\right|^2 + \left(6\kappa_{00}^4 + 7\mu^2\kappa_{00}^2 - 9\mu^4\right)\left|\kappa_+\right|^2\right]\right\}\kappa_+ \\
&- e^{-i\mu\gamma T_0}\left\{\frac{\gamma}{\kappa_{00}}\frac{\partial \kappa_-}{\partial T_1} - \frac{i}{2\kappa_{00}^3\mu}\left[4\mu^4\left|\kappa_+\right|^2 + \left(6\kappa_{00}^4 + 7\mu^2\kappa_{00}^2 - 9\mu^4\right)\left|\kappa_-\right|^2\right]\right\}\kappa_- \\
&+ \frac{ie^{2i\mu\gamma T_0}\mu}{\kappa_{00}^2}\left[\left(\kappa_{00}^2 + \mu^2\right)\left(\omega_1^*\kappa_+^2 + \omega_1\kappa_+\kappa_-^*\right) + \frac{\left(5\mu^2 + \kappa_{00}^2 + \gamma^2\right)}{5\mu^2 - \kappa_{00}^2}\omega_1\gamma^2\kappa_+\kappa_-^*\right] \\
&- \frac{ie^{-2i\mu\gamma T_0}\mu}{\kappa_{00}^2}\left[\left(\kappa_{00}^2 + \mu^2\right)\left(\omega_1^*\kappa_-^2 - \omega_1\kappa_+^*\kappa_-\right) + \frac{\left(5\mu^2 + \kappa_{00}^2 + \gamma^2\right)}{5\mu^2 - \kappa_{00}^2}\omega_1\gamma^2\kappa_-\kappa_+^*\right] \\
&- \frac{\left[\mu^4 - 2\kappa_{00}^4 + \mu^2\kappa_{00}^2\right]}{2\kappa_{00}^3\mu^2}\left\{3i\mu e^{3i\mu\gamma T_0}\kappa_+^2\kappa_-^* - 3i\mu e^{-3i\mu\gamma T_0}\kappa_-^2\kappa_+^*\right\}
\end{aligned}
\tag{3.92}
$$

The conditions for non secularity are

$$
\begin{aligned}
&\frac{\partial \kappa_+}{\partial T_1} + \frac{i}{2\kappa_{00}^2\mu\gamma}\left[4\mu^4\left|\kappa_-\right|^2 + \left(6\kappa_{00}^4 + 7\mu^2\kappa_{00}^2 - 9\mu^4\right)\left|\kappa_+\right|^2\right]\kappa_+ = 0 \\
&\frac{\partial \kappa_-}{\partial T_1} - \frac{i}{2\kappa_{00}^2\mu\gamma}\left[4\mu^4\left|\kappa_+\right|^2 + \left(6\kappa_{00}^4 + 7\mu^2\kappa_{00}^2 - 9\mu^4\right)\left|\kappa_-\right|^2\right]\kappa_- = 0 \\
&\omega_1 = 0 \text{ and } \kappa_+^2\kappa_-^* e^{6i\mu\gamma T_0} = \kappa_-^2\kappa_+^*.
\end{aligned}
\tag{3.93}
$$

From the set of equations (3.91) and (3.93) we see that both κ_+ and κ_- satisfy the following equations in slow time variable on the assumption that they are periodic functions in T_0.

$$
\begin{aligned}
&2\kappa_{00}^2\mu\gamma\frac{\partial \kappa_+}{\partial T_1} = i\left[2\mu^2\left(\kappa_{00}^2 - 2\mu^2\right)\left|\kappa_-\right|^2 - \left(6\kappa_{00}^4 - 2\kappa_{00}^2\mu^2 - 3\mu^4\right)\left|\kappa_+\right|^2\right]\kappa_+ \quad \text{and} \\
&2\kappa_{00}^2\mu\gamma\frac{\partial \kappa_-}{\partial T_1} = -i\left[\left(6\kappa_{00}^4 - 2\kappa_{00}^2\mu^2 - 3\mu^4\right)\left|\kappa_-\right|^2 - 2\mu^2\left(\kappa_{00}^2 - 2\mu^2\right)\left|\kappa_+\right|^2\right]\kappa_-
\end{aligned}
\tag{3.94}
$$

These are the equations we obtained, similar to (5.9) of Keener[65].

Keener obtains other solutions assuming $|\kappa_+|$ and $|\kappa_-|$ as constants. Otherwise it is not easy to solve the non-linear equations. In our case this assumption is also not sufficient to solve it.

3.7 Discussion

In this chapter we have considered a vortex filament in an ideal fluid. We have followed Keener throughout, with the correction discussed in chapter 2.

As mentioned in the introduction what Keener considers is only a geometrical knot evolving under Frenet-Serret formulae. In fact he is not considering even the constant circulation that is taken into account in a LIA approximation. In our study we have tried a simple extension where instead of constant circulation we have considered the effect of vorticity, which is aligned along the vortex line. Along with the periodicity of curvature and torsion we have used the periodicity of 'ω' also in our study. It is assumed that 'ω' as a function of t i.e. evolution of 'ω', is known from the hydro dynamical equations.

Chapter 4

Vorticity Stress Tensor

The recent studies on vortex knots concentrate more on the MHD and plasma, mentioning analogy between vortex dynamics and electromagnetic theory. This analogy needs deeper studies to analyze the physics of fluid flows.

Thus treating line vortices as physical structures, it is possible to study the forces acting on it and its deformations. We can talk of the force between vortex lines, the pressure and the stress that deform vortex tubes. This stress can be reduced to principal form which involves the enstrophy. It is to be noted that enstrophy is associated with the work done in stretching of vortex lines, Doering and Gibbon [34] and Chorin [24].

Inspite of the analogies it may be noted that there are differences in the properties of MHD and incompressible inviscid fluid flows. For example, Moffatt [108, 107] points out the question of stablity of Euler flow.

Suppose we have a knotted closed curve C, which runs along the center-line of a tube of small uniform radius r so that the curve becomes a string[115]. If for a given configuration of C, r is gradually increased, the process corresponds to knot tightening and r cannot increase above a maximum value. If the initial configuration of C is varied, with length held constant, the maximum value of r, say $r_{\max}$ is reached,Moffatt[115]. Katritch et.al [63] have developed a numerical technique by which $r_{\max}$ may be increased and the configuration of C is called ideal. Alternatively, the process can be explained in terms of energy. Here the knot is identified with an imaginary magnetic flux tube, which is allowed to contract due to Maxwell's tension. This is followed by corresponding increase in cross-section to maintain total tube volume. The process is stopped when the tube comes into contact with itself and a minimum energy equilibrium state is reached.

The analogy between Maxwell's equation of electrodynamics and vorticity equation is well known. But recent studies concentrates more on electromagnetic equation and magnetic induction in the case of plasma flows. Though the analogy is mentioned everywhere, it is seldom studied in detail. In this chapter we attempt to exploit this analogy and to interpret the quantities associated with vorticity.

4.1 Vorticity stress tensor

In the case of incompressible flows, the vorticity field is frozen-in and thus satisfies the equation

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}) \tag{4.1}$$

This equation admits solution given by

$$\omega_i(X,t) = \omega_j(x,0)\frac{\partial X_i}{\partial x_j} \tag{4.2}$$

This is the well-known Cauchy equation, which relates the current vorticity to the initial vorticity and thus establishes a topological equivalence between them. Corresponding to magnetic energy, the energy associated with vorticity field is defined by enstrophy $\frac{1}{2}\omega^2_i$ Moffatt [105, 107, 116, 112]; Moffatt and Ricca [112]

The rate of change of this energy density is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\omega^2}{2}\right) &= \vec{\omega}\cdot\frac{\partial\vec{\omega}}{\partial t} \\ &= \vec{\omega}\cdot[\nabla\times(\vec{v}\times\vec{\omega})] \\ &= -\vec{v}\cdot[(\nabla\times\vec{\omega})\times\vec{\omega}] - \nabla\cdot[(\vec{v}\times\vec{\omega})\times\vec{\omega}] \\ &= -\vec{v}\cdot\left[\vec{f}\times\vec{\omega}\right] - \nabla\cdot[(\vec{v}\times\vec{\omega})\times\vec{\omega}]\end{aligned} \tag{4.3}$$

Here $\vec{f} = \nabla\times\vec{\omega}$ is the flexion field.

As pointed out by Truesdell [172], the importance of flexion lies in its occurrence in the Navier-Stokes equation. If vorticity is confined to a sub-domain of the fluid, the divergence term vanishes on integrating over the entire volume.

Thus we get

$$\frac{dM}{dt} = -\int \vec{v}\cdot\vec{F}dV. \tag{4.4}$$

Here $M = \int \frac{\omega^2}{2}dV$ is the total enstrophy and $\vec{F} = \vec{f}\times\vec{\omega}$ is a force analogous to

Lorentz force in electrodynamics, Ferraro et.al. [39] and Moffatt [116].

The Lorentz force is associated with the Maxwell's stress tensor. Analogously we associate a stress tensor

$$T_{ij} = \omega_i \omega_j - \frac{\omega^2}{2}\delta_{ij} \tag{4.5}$$

with this force $\overrightarrow{F}$. The normal components of this stress represents the tension in the line vortices and the terms $\omega_i \omega_j$ are the shearing forces between adjacent vortex lines of the filament whose limiting case is the line vortex.

$[T_{ij}]$ is a symmetric matrix and can be diagonalised. Choosing the principal axis OX_1 in the direction of vorticity $(\omega, 0, 0)$, we get

$$[T_{ij}] = \begin{bmatrix} \frac{w^2}{2} & 0 & 0 \\ 0 & -\frac{w^2}{2} & 0 \\ 0 & 0 & -\frac{w^2}{2} \end{bmatrix} \tag{4.6}$$

Thus the principal stress components constitute a tension $\frac{1}{2}\omega^2$ along the line vortex and an equal pressure normal to it.

We can rewrite this tensor as

$$\begin{bmatrix} w^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{w^2}{2} & 0 & 0 \\ 0 & -\frac{w^2}{2} & 0 \\ 0 & 0 & -\frac{w^2}{2} \end{bmatrix}$$

In this form ω^2 gives the effect of the force $\overrightarrow{F}$ and the second term represents pressure in the direction of vortex line.

4.2 Discussion

Any property that depends only on the frozen-in-character of the B-lines of magnetic induction apply equally to the vorticity $\vec{\omega}$. Unlike MHD flows, the vorticity is defined in terms of velocity. The geometrical object corresponding to B-lines is a line vortex.

The question naturally arises in what sense T_{ij} can be called a vorticity stress tensor. While a magnetic flux ring shrinks in radius towards the axis, a vortex ring propagates in a direction parallel to it's axis. In spite of these differences, the concept of line vortex can be made use of in mathematical studies as an approximate representation of real vorticity distribution.

Recent references to enstrophy are in the Kolmogrov theory of turbulence associated with viscous dissipation of energy, [24, 34]. The reference to flexion can be seen in Truesdell [172] where he mentions its importance as appearing in Navier-Stokes equation.

In a private communication, Moffatt had pointed out the difference between the deformation of a closed magnetic flux tube (under Maxwell stress) and that of a vortex ring (which moves along its axis) and expressed the doubt whether the tensor can be called a vortex stress tensor. The question naturally arises whether it is meaningful to identify a geometric knot with a magnetic flux tube and in this case to what extend the kinematics of vortex knots is included in that of knots of magnetic lines. We also note that the vortex stress tensor is related to the enstrophy in the same way as the Reynolds stress tensor is related to kinetic energy.

Suppose we identify a knot with an imaginary vortex flux tube instead of magnetic flux tube. It is to be noted that motion of the tube depends on its cross-section. The velocity of the tube is infinite if the size of the cross-section is zero and is indeterminate unless a finite size of the cross-section is assigned, Batchelor [9]. Thus the classical inviscid theory of vortex ring is unsatisfactory and incomplete in describing its motion. This is applicable to more complicated structures like vortex knots.

Chapter 5

Conclusion

As pointed out in the introduction, though vortex knots have attracted attention of the scientists of late nineteenth century like Kelvin and Tait, the interest was renewed in 1960's due to the works of Moreau [122]and Moffatt et.al [104, 110, 109, 107]. While in ideal fluid flows the topology of the knots and links are preserved in a flow, in real fluid flows these are changed due to some sort of dissipation. Thus both topological invariants and topological changes have been the subjects of study for the last forty years.

The research works on vorticity has led to the emergence of a new field called Topological Fluid Mechanics . A major milestone was the annual conference of IUTAM, the International Union for Theoretical and Applied Mechanics, held at Cambridge in the year 1989. This symposium addressed the problems of fluid mechanics and magneto-hydrodynamics that could be described as topological rather than exclusively analytical in character. These include purely kinematical problems such as classification of possible streamline structures in three-dimensions, the deformation of convected lines and surfaces in a prescribed flow field and the relation between Lagrangian and Eulerian properties for laminar and turbulent flows, to mention a few. On the other hand there are dynamical problems such as treatment of changes in flow topology associated with symmetry breaking instability, topological invariants associated with Euler equations and the manner in which the topological invariants may be broken in high Reynolds number flows. Underlying all such problems is a desire to identify structures if any that are characteristic of the fully developed turbulent flows with a view to constructing an improved statistical theory of turbulence.

A vortex filament in an ideal barotropic or incompressible flow is an individual entity preserving identity for a long period of time. It exhibits vigorous bending and twisting motion owing to self-induction as well as the influence of neighboring vortices and background flow, by Fukumoto [45]. Investigation of the competitive and co-operative action of these ingredients will give some insight into the complicated structure of high Reynolds number flows. Apart from these, turbulence in super fluid helium II manifest itself as a tangle of quantized vortices [33]. The study of the tangle was initially motivated by engineering applications of heat transfer and to understand a peculiar form of disorder in the vicinity of absolute zero. The more recent interest comes from the study of isothermal helium turbulence and its possible connection with classical turbulence. Also in cosmology, experiments are conducted in which a tangle of vortex knots is interpreted as a model of cosmic strings in the early universe In all these cases since the core structure is very small and dissipative effects are negligible at sufficiently low temperatures these vortices can be well approximated by vortex line singularity which move in an ideal Euler fluid.[33].

The study of vortex knots belongs to the topological fluid mechanics as it is primarily concerned with structures within a flow field that retain some coherence over a significant period of time. In ideal situation this coherence may remain indefinitely, but in real flow we may have to consider the manner in which the structural or topological properties of flow change with time. The related problems cannot be solved in general and can be solved only by considering particular problems as in the case of

Navier-Stoke's equation of viscous fluids.

In this context Moffatt [117] has identified eight problems which he calls the twenty first century problems. Of these, problems four to seven are related to knots. As coherent structures in a fluid are related to vorticity, the above remark shows the significance of knots in general.

Related to these problems are many problems of magneto hydrodynamics of highly conducting fluids like relaxation to magneto-static equilibrium under the constraint of conserved topology and the resulting formation of discontinuity, dynamo action due to fluid motion either with or without helicity and problems of solar and stellar magnetism. There are other fields like molecular biology in which entanglement of DNA chains are studied [79, 25].

Coming back to Fluid Mechanics, vortex knots and knots of magnetic lines of force in MHD flows are considered together in most of the studies. The wellknown parallel between vortex lines and magnetic lines of force is made use of in giving attention to the problems of the latter, rather than vortex knots. Thus most of the studies in fluid mechanics are concerned with magnetic knots rather than vortex knots. In this context we recall that in a paper "Evolution of Vortex Knots" Ricca et.al [145] claim, "for the first time since Lord Kelvin's original conjectures of 1875, we address and study the time evolution of vortex knots in the context of the Euler equations". The authors did not refer to the paper by Keener [65] to whom the credit should really go. Though Kida [72] has studied the motion of a vortex filament and found the existence of torus knot solutions, the work of Keener can be singled out as the one which addresses the evolution of a vortex knot for the first (It may be recalled that Ricca refers to Keener, in his earlier papers [112, 143]). This has led us to choose Keener's work as starting point for our study.

Though much study has been done on evolution of vortex knots, only a few authors have addressed the question as to the influence of external flow on the motion of a knotted vortex filament. This situation is of interest not only in its own right but also in the practical view point as possible idealization of the distant vortices and boundary. In this connection we mention the works of Hama [54], Takaki [160], Aref and Flinchem [3] and Fukumoto [45]. All these are based on LIA approximation to Biot Savart law for vortex line [9].

LIA is a crude approximation for the study of vortex motion based on Biot-Savart law governing vortex motion which can be derived from the Euler equation of incompressible flow as well as Ginzburg-Landau model for super fluids. The fundamental flaw of the LIA lies in the absence of vortex stretching. But this aspect is crucial as even small non-local correction causing changes of the length of the vortex line are apt to cause strong deviation from LIA results [129]. The inadequacy of LIA should be felt, particularly strongly, when the stability of the line vortex configuration is studied. Due to the self focusing nature of NLS explained by solution corresponds to spiral configuration, must be unstable [123]. The exceptional case is a circle which turns out to be neutrally stable. But a neutrally stable situation might be resolved when weaker effects are taken into account. And even in this case abandoning the LIA should be the first step.

LIA model preserves three fundamental properties of Euler equation.

- Vortex tube traveling with the fluid.
- The binormal flow reversible in time.
- Circulation around any closed curve encircling the vortex line.

In the ideal situation of a vortex filament the vorticity is identified by a δ-vector measure in the direction of the tangent with circulation as magnitude. From this point of view, the flow preserves the arc length parameterization. This prevents the possibility of vortex stretching.

It is to be noted that vortex line stretching for vorticity production belongs to truly three-dimensional property that distinguish turbulence from other random and quasi-random flows. This vortex stretching mechanism is represented in the vorticity equation by the term

$$(\overrightarrow{\omega} \cdot \nabla) \overrightarrow{u} = \omega_j \frac{\partial u_i}{\partial x_j}.$$

This shows the relevance of the quantity $\gamma_{ij} = u_i \omega_j$ which is called the helicity tensor related to knottedness of vortex lines as in Moffatt[104]. This tensor can be interpreted as the lagrangian tensor of vorticity density flux. It is interesting to note that the helicity invariant is simply an integral of the helicity tensor trace (like the kinetic energy is the integral of the trace of the Reynolds stress tensor) [173]. Local rate of change of helicity is the skew-symmetric part of helicity tensor [51]. The result concerning the conservation of helicity was first proved by Moreau [122] who has pointed out its topological significance also.

In Fluid Mechanics the fundamental assumption is that the fluid is a continuum. In mathematical language, this says that, at least locally, the fluid looks like usual two or three-dimensional space. A fluid particle is identified with mathematical point in this space. The basic model of fluid mechanics is a set consisting of the fluid body and the various additional structures that allow one to discuss such properties as continuity, volume, velocity and deformation. The flow itself is a transformation of the fluid body to itself and the basic object of study is the transport of structures under the flow [16]. The more deeply one penetrates into the general character of fluid motion, the more apparent it becomes that the dynamical properties of fluids in the main are but names, in the interpretations and methods of measuring purely kinematical quantities, and that in general, the flow of fluid whether perfect or viscous may be defined by purely kinematical conditions. Since fluid flows occur in three-dimensional space, the above consideration leads to the conclusion that the fluid flow is related to the geometry of three-dimensional space. This has been investigated by Tur and Yanosky [174] using differential forms (0-form, 1-form, 2-form and 3-form) in a three-dimensional manifold and they have obtained 4 types of invariants. Knots and Links are also discussed as topological invariants for frozen-in fields associated with closed 2-forms. The underlying physical idea is rather simple. In the case of such fields, the field lines are either closed or extends to infinity. When such field lines are linked, their frozenness prevents the flow from unlinking them provided the flow is continuous.

But following Drobot and Rybarsky [36], Mathew and Vedan [96, 97], Geetha, Thomas and Vedan [49], Subin and Vedan[157] has used a four dimensional space-time manifold to study inviscid flows. It is to be noted that in such a space-time manifold there are five kinds of forms including 4-forms also so that we expect additional geometrical properties associated with it. The use of such a four-dimensional manifold has great significance in the study of knots and links. This gives a new direction for future work.

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