SOME PROPERTIES OF WEIGHTED DISTRIBUTIONS FOR TRUNCATED RANDOM VARIABLES

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CERTIFICATE

Certified that the thesis entitled 'SOME PROPERTIES OF WEIGHTED DISTRIBUTIONS FOR TRUNCATED RANDOM VARIABLES' is a bonafide record of work done by Smt. Maya S. S. under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

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26th October 2007

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

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CONTENTS

CHAPTER ONE

WEIGHTED DISTRIBUTIONS

1.1 Introduction

The Concept of weighted distributions can be traced to the work of Fisher (1934) in connection with his studies on how methods of ascertainment can influence the form of distribution of recorded observations. Later it was introduced and formulated in general terms by Rao (1965) in connection with modeling statistical data where the usual practice of using standard distributions for the purpose was not found to be appropriate. In Rao's paper, he identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distributions. This may occur due to non-observability of some events or damage caused to the original observation resulting in a reduced value, or adoption of a sampling procedure which gives unequal chances to the units in the original. Rao's paper gave much attention on damage models, characterization of discrete distributions and sampling mechanism generating a wide variety of weighted distributions.

A mathematical definition of the weighted distribution is as follows. Let $(\Omega, \Im, P)$ be a probability space, $X : \Omega \rightarrow H$ be a random variable (rv) where $H = (a,b)$ be an interval on real line with $a > 0$ and $b(> a)$ can be finite or infinite. When the distribution function (df) $F(t)$ of X is absolutely continuous with probability density function (pdf) $f(t)$ and $w(t)$ be a non-negative weight function satisfying $\mu_w = E(w(X)) < \infty$, then the rv X_w having pdf

$$f^w(t) = \frac{w(t)f(t)}{\mu_w}, \qquad a < t < b \tag{1.1}$$

is said to have weighted distribution, corresponding to the distribution of X. The definition in the discrete case is analogous.

One of the basic problems when one use weighted distributions as a tool in the selection of suitable models for observed data is the choice of the weight function that fits the data. Depending upon the choice of weight function $w(t)$, we have different weighted models. For example, when the weight function depends on the lengths of units of interest ($i.e.\ w(t) = t$), the resulting distribution is called length-biased. In this case, the pdf of a length-biased rv X_L is defined as

$$f^L(t) = \frac{tf(t)}{\mu}; \ a < t < b \tag{1.2}$$

where $\mu = E(X) < \infty$. The statistical interpretation of length-biased distributions was originally identified by Cox (1962) in the context of renewal theory. Length-biased sampling situations may occur in clinical trials, reliability theory, survival analysis and population studies, where a proper sampling frame is absent. In such situations, items are sampled at a rate proportional to their length, so that larger values of the quantity being measured are sampled with higher probabilities. Numerous works on various aspects of length-biased sampling are available in literature which include family size and sex ratio (Rao (1965), Neel and Schull (1966)), wild life population and line transect sampling (Eberhardt (1968, 1978)), analysis of family data (Fisher (1934), Haldane (1938)), cell cycle analysis (Zelen (1974)), efficacy of early screening for disease (Zelen (1971, 1974)), aerial survey and visibility bias (Cook and Martin (1974), Patil and Rao (1977, 1978)). For some recent publications related to length-biased sampling, we refer to Sankaran and Nair (1993), Sen and Khattree (1996), Oluyede (1999, 2000), Van Es. et al. (2000), Sunoj (2004) and Bar-Lev and Shouten (2004).

More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, *i.e.*, when $w(t) = t^{\alpha}; \alpha > 0$, then the resulting distribution is called size-biased. This type of sampling is a generalization of length-biased sampling and majority of the literature is centered on this weight function. Denoting $\mu_{\alpha} = E\left(X^{\alpha}\right) < \infty$, distribution of the size-biased rv X_S of order α is specified by the pdf

$$f^{S}(t) = \frac{t^{\alpha} f(t)}{\mu_{\alpha}}; \; a < t < b. \tag{1.3}$$

Clearly, when $\alpha = 1$, (1.3) reduces to the pdf of a length-biased rv. Size-biased distributions arise in life lenth studies (Bluementhal (1967), Scheaffer (1972), Gupta (1984)), etiological studies (Simon (1980)) and in the studies of wildlife population and human families (Patil and Rao (1977, 1978), Rao (1965, 1977, 1985) and Patil and Ord (1976)). The effect of size-biased sampling in cell kinetics problems and the distribution associated with cell populations have been studied by several authors including Takahasi (1968), Bartlett (1969) and Zelen (1974).

However, there are many other weight functions being studied by different authors such as Haldane (1938), Rao (1965), Neel and Schull (1966), Cook and Martin (1974), Patil and Ord (1976), Gupta (1975), Kemp (1973) and Patil and Rao (1977). The important weight functions which are used in discrete and continuous set up are listed below.

$$w(t) = t$$

$$w(t) = t^{\alpha}; \alpha > 0$$

$$w(t) = \left(1 - (1-\beta)^{t}\right); 0 < \beta < 1$$

$$w(t) = (t+1)$$

$$w(t) = t(t-1)\ldots(t-r+1); r > 0$$

$$w(t) = \phi^{t}; 0 < \phi < 1$$

$$w(t) = \exp(\phi t).$$

These weight functions are also useful for modeling through the identities connecting the original and weighted random variables. Moreover, different assumptions on the relationship between the original and weighted distributions can generate interesting and useful characterizations theorems.

1.1.1 Some general results

Patil and Rao (1978) have given some useful comparison between the mean values of X and X_w based on the monotonic behavior of the weight functions. Mahfoud and Patil (1982) studied some properties of weighted distributions in comparison with those of original distribution. They also examined the relation connecting the parameters of weighted distributions to that of original distributions and characterized log normal, gamma and Poisson distributions based on it. Further, they investigated the effect of size-biased sampling on the mixtures of specific distribution. Kochar and Gupta (1987) studied some properties of weighted distributions in comparison with that of the original distributions for the positive valued random variables. Later, Jones (1990) discussed the relationships between moments of weighted and original distributions and examined some structural properties of weighted distributions and studied the weighted mixture distributions as mixtures of weighted distributions and vice versa.

Apart from the properties related to the moments of original and weighted distributions, another well-studied property of weighted distributions is that of form invariance. Let a rv X follows a pdf $f(t;\theta)$, where θ is the parameter of the distribution, then $f(t,\theta)$ is form invariant under the weight function $w(t)=t^{\alpha}$, if X_S follows the pdf $f(t;\eta)$, where η is a new parameter depending on θ and α. More specifically, under size-biased sampling of order α, the observed rv retains the same functional form as that of the original rv except the parameters. A major contribution of this property is due to Patil and Ord (1976) and they proved that under some mild regularity conditions, log exponential family is form-invariant under size-biased sampling. This is stated in the following theorem.

Theorem 1.1 (Patil and Ord (1976)): Let the rv X have pdf $f(t;\theta)$ and have size bias with weight function t^{α}. Then a necessary and sufficient condition for $f^{S}(t;\theta,\alpha) = f(t;\eta)$, where $\eta = \eta(\theta,\alpha)$, is that

$$f(t;\theta)=\frac{t^{\theta}a(t)}{m(\theta)}=exp\left(\theta\log t+A(t)-B(\theta)\right),$$

where $a(t)=\exp\left(A(t)\right)$, $m(\theta)=\exp\left(B(\theta)\right)$ and $E(\log X)=\frac{m^{'}(\theta)}{m(\theta)}=B(\theta)$. In this case, $f^{S}(t;\theta,\alpha)=f(t;\theta+\alpha)$. This result holds under certain regularity conditions.

The form invariance of certain discrete distributions under size-bias of factorial order is also studied by Patil and Ord. They also examined form invariance property for the distributions belonging to Power series family under length-biased sampling. Further, Patil and Ratnaparki (1986) studied this concept for mixtures of distributions. However, Sankaran and Nair (1993) focused attention to the study of form invariance for Pearson family and its discrete version, namely Ord's family.

1.1.2 Characterizations

Characterization problem usually identifies some unique property possessed by a distribution and it helps to obtain an exact model followed by the observations through the consideration of the physical characteristics that governs the pattern of the data. A plethora of work is being done in connection with the characterization of original and weighted distributions under different considerations such as the properties of moments, form invariance and reliability characteristics using different weight functions. The present section explains a brief review of characterizations associated with weighted distributions.

The characterization of log normal distribution came out by Krumbein and Pettijohn (1938) with the fact that this distribution provides a good fit to the observed particle sizes. Gupta (1975) provided certain characterizations to some discrete distributions using the properties of length-biased distributions. Gupta (1976) characterized exponential distribution using the property that the mean of length-biased rv is twice as that of original rv. Mahfoud and Patil (1982) characterized log normal distribution using the equality of variances of the logarithms of original and weighted distributions under size-biased sampling. They also characterized log normal, Poisson and gamma distributions using the relationships between parameters of the original distribution and those of the weighted distributions. Kirmani and Ahsanullah (1987) considered the relationship between characteristic functions of the length-biased form and the original version to develop a characterization for the Inverse Gaussian distribution. Pakes and Khattree (1992) and Pakes et al. (1996, 2003) characterized various distributions under length-biased sampling and using the property of infinite divisibility. Characterization results on invariant length-biased distribution of the Pearson family can be found in Sankaran and Nair (1993). Some other characterizations obtained by using rather different approaches appeared in Sen and Khattree (1996) and Lingappaiah (1988). Some characterizations of length-biased Inverse Gaussian distribution are studied by Gupta and Akman (1995). Recently, Bar-Lev and Schouten (2004) obtained a unified approach for characterizing exponential dispersion models which are invariant up to translations under various types of length-biased sampling.

1.1.3 Characterizations based on reliability concepts

Length-biased models are widely used in the context of reliability and survival analysis. Cox (1962) described the importance of length-biased distributions in the context of renewal theory. Suppose that a component operating in a system is replaced upon failure by another component possessing the same life distributions and the process is repeated. Then the sequence of component life lengths forms a renewal process. Let $L(t)=U(t)+V(t)$ denotes the total life of the component, where $U(t)$ and $V(t)$ respectively denotes the age and remaining life of the component at any time t. Then the limiting pdf of $L(t)$ follows a length-biased distribution. The equilibrium distribution of the backward and forward recurrence times in the limiting case is described by Cox (1962), Bluementhal (1967), Scheaffer (1972), Gupta (1984), Rao (1985) and Sen and Khattree (1995). Equilibrium distribution is a special case of weighted distributions and it is useful in reliability analysis.

Gupta and Keating (1986) initially examined some structural relationships between original and length-biased rv in the context of reliability, they are useful for life length studies. Later, Kochar and Gupta (1987) and Jain et al. (1989) extended these relationships for general weight function $w(t)$. Motivated by these results, Gupta and Kirmani (1990) studied extensively the various relationships between original and weighted distributions in the context of reliability and life testing and surveyed the results available in literature useful in distribution theory and its applied problems. Gupta and Akman (1995) studied a mixture of Inverse Gaussian distribution and its length-biased version from a reliability point of view. As a continuation of results proved in Sankaran and Nair (1993), Asadi (1998) characterized Pearson system of distributions using the relation between X and X_S and also studied it for the Ord's family. Navarro et al. (2001) also obtained some general characterizations of probability distributions and obtain some useful relationships between reliability concepts of original distribution and the associated weighted distribution. However, Oluyede and George (2002) and Oluyede (2002) derived some reliability inequalities for weighted distributions. Sunoj (2004) characterized certain probability distributions based on the concept of partial moments in the context of length-biased and equilibrium models. Recently, Bartoszewicz and Skolimowska (2004, 2006) represented weighted distributions by Lorenz curve and obtained some results concerning stochastic ordering for weighted distributions using reliability concepts.

1.1.4 Bivariate weighted distributions

The bivariate extension of weighted distribution is discussed in Mahfoud and Patil (1982). For a pair of non-negative random variables (X_1, X_2) with joint density function $f(t_1, t_2)$ and a non-negative weight function $w(t_1, t_2)$ such that $E\left(w(X_1, X_2)\right) < \infty$, the random vector (X_{1_w}, X_{2_w}) with density function

$$f^w(t_1, t_2) = \frac{w(t_1, t_2)}{E\left(w(X_1, X_2)\right)} f(t_1, t_2)\ ;\ a_i < t_i < b_i\ ;\ i = 1, 2 \qquad (1.4)$$

is said to have bivariate weighted distribution corresponding to (X_1, X_2). The extension to p- variate case is straightforward. Mahfoud and Patil studied the nature of some weight functions and characterized some probability distributions based on it. Patil et al. (1987) studied different weight functions in the bivariate set up and their applications in different models. Jain and Nanda (1995) extended the idea of weighted distributions in multivariate case and presented some partial ordering in connection with the original and weighted distributions. Some properties of bivariate weighted distributions under different weight functions are also studied by Arnold and Nagaraja (1991). However, Sunoj and Nair (1998, 1999) characterized some bivariate life distributions using weighted conditionals and weighted marginal respectively. In the view of the usefulness of form invariance property, Sunoj and Nair (2000) and Nair and Sunoj (2003) further examined this concept in the bivariate case and proved certain theorems which characterize family of distributions viz. Pearson and log exponential and studied some applications of it. Sunoj and Sankaran (2005) characterized certain bivariate distributions in the context of reliability modeling using different weight functions. However, the bivariate weighted distributions for discrete random variables are studied by Kocherlakota (1995) and Gupta and Tripathi (1996).

1.1.5 Applications

In addition to various applications mentioned in Section 1.1, the concept of weighted distributions has also been applied in variety of fields such as analysis of data relating to human populations and ecology (Patil and Rao (1977, 1978)), biomedicine (Zelen (1974), Simon (1975)), demography (Sheps and Menken (1972)), Economics (Ord (1975)), forestry (Warren (1975)), reliability (Cox (1962), Scheaffer (1972)), small particle Physics and sedimentology (Herdan (1960)). For more applications and characterizations results related to weighted distributions one can see the references given in Sunoj (2000), Navarro et al. (2001) and Di Crescenzo and Longobardi (2006).

Apart from these, some of the known and important distributions in statistics and applied probability may be expressed as weighted distributions. Truncated distributions, equilibrium-renewal distributions, distribution of order statistics, distribution in proportional hazards and proportional reversed hazard models (see Bartosezewicz and Skolimowska (2004) are some of the examples. Thus the theory of weighted distributions is appropriate whenever these distributions are applied.

1.2 Some basic concepts useful in the present study

In this section, we discuss some basic concepts and definitions which are useful in the present investigation.

1.2.1 Truncation

The problem of truncation occurs mainly due to the sampling methods that we follow. Due to truncation, some events may be missed in the record, even though they occur in reality. Statistical problem of truncation arise when a standard statistical model is appropriate for analysis except the values of the rv falling below or above some values are not measured at all. For example, in a study of particle size, particles below the resolving power of the observational equipment will not be seen at all. Truncation is sometimes usefully regarded as a special case of selection. Depending up on the nature of the data that we have observed, there are mainly 3 types of truncation namely left, right and double (interval) truncations.

More particularly, if the values below a certain lower limit L are not observed at all, the distribution is said to be truncated on the left. If the values are larger than an upper limit U are not observed, the distribution is said to be truncated on the right. If only values lying between L and U are observed, the distribution is said to be doubly (interval) truncated. For examples related to various types of above discussed truncations, one can refer Lawless (2003), International encyclopedia of Statistics (1968, page no.1060-1065), Efron and Pertosian (1999), Betensky and Martin (2003).

1.2.2 Basic concepts in reliability theory

The term reliability of a system/component is the probability that it will perform its intended functions for a specific period of time when operating under normal environmental conditions. In studies related to reliability, there are certain basic concepts which are extensively studied by different authors. They are explained below.

1.2.2.1 Reliability function

Consider a rv X represents the lifetime of a device/component then the reliability function (survival function) of X is denoted as $R(t)$ and it is defined by

$$\begin{aligned} R(t) &= P(X > t)\ ;\ t > 0 \\ &= 1 - F(t) \end{aligned} \tag{1.5}$$

where $F(t)$ is the df of the rv X. It gives the probability of failure free operation of the device at time t (≥ 0).

1.2.2.2 Failure rate (Hazard rate)

Defining the right extremity of $F(t)$ by

$$b = \inf\{t : F(t) = 1\}, \text{ for } t < b,$$

the failure (hazard) rate of X is defined as

$$h(t) = \frac{f(t)}{R(t)} = -\frac{d}{dt}\log R(t). \tag{1.6}$$

This function uniquely determines the df $F(t)$ through a relation

$$F(t) = 1 - \exp\left(-\int_0^t h(x)dx\right). \tag{1.7}$$

In general case, for a rv X with support $-\infty < X < \infty$, Kotz and Shanbhag (1980) defined the failure rate as a Radom Nikodym derivative with respect to Lebesgue measure on $\{t : F(t) = 1\}$, of hazard measure

$$H(B) = \int_B \frac{dF(x)}{1 - F(x)},$$

for every Borel set B of $(-\infty, b)$. Further the distribution of X is uniquely determined by the relationship

$$R(t) = \prod_{u<t}\left[1 - H(u)\right]\exp\left[-H_C(-\infty, b)\right]$$

where H_C is the continuous part of H.

1.2.2.3 Mean Residual Function

For a continuous rv with $E(X) < \infty$, the mean residual function (MRLF) is defined as the Borel measurable function

$$r(t) = E\left(X - t \mid X > t\right) \tag{1.8}$$

for all t such that $R(t) > 0$. It measures the average residual life of a component when it has completed t units of time. If X is absolutely continuous, (1.8) can be expressed as

$$r(t) = \frac{1}{R(t)} \int_t^{\infty} R(x)dx. \tag{1.9}$$

Further, MRLF is related to failure rate and reliability function by the relations

$$h(t) = \frac{1 + r'(t)}{r(t)} \tag{1.10}$$

and

$$R(t) = \frac{r(0)}{r(t)} \exp\left(-\int_0^t \frac{dx}{r(x)}\right) \tag{1.11}$$

for every t in $(0, b)$, where $r'(t)$ denotes the derivative of $r(t)$ with respect to t and $r(0) = E(X)$.

The concepts of failure rate and MRLF are extensively applied in modeling equipment behavior and in defining various criteria for aging. When the specification of the functional form of the failure rate is possible based on the physical characteristics of the process governing the failure of a system/device, the result that failure rate uniquely determines a distribution helps the identification the failure time model.

Further, Gupta and Keating (1986) established some structural relationships between original and length-biased rv using reliability concepts. The major relationships are

$$R^L(t) = \left(\frac{t + r(t)}{\mu}\right) R(t) \tag{1.12}$$

$$h^L(t) = \left(\frac{t}{t + r(t)}\right) h(t) \tag{1.13}$$

$$r^L(t) = \left(\frac{r(t)}{t + r(t)}\right) \int_t^{\infty} \left(\frac{x + r(x)}{r(x)}\right) \exp\left(\int_t^x \frac{du}{r(u)}\right) dx \tag{1.14}$$

where $R^L(t)$, $h^L(t)$ and $r^L(t)$ respectively denotes the reliability function, failure rate and MRLF corresponding to the length-biased models. Using the measures (1.12), (1.13) and (1.14), Gupta and Keating characterized Pareto II, exponential and beta distributions based on the ratio of the reliability functions and failure rates of original and length-biased random variables.

1.2.2.4 Vitality function

The concept of vitality function is introduced by Kupka and Loo (1989) as a Borel measurable function on the real line as

$$m(t) = E\left(X \middle| X > t\right) = \frac{1}{R(t)}\int_t^{\infty} xf(x)dx. \tag{1.15}$$

Clearly, vitality function (1.15) measures the expected life of a component, when it has survived t units of time. The vitality function is closely related to MRLF and it is clear from the definition (1.15) that

$$m(t) = t + r(t) \tag{1.16}$$

and

$$m'(t) = r(t)h(t),$$

where $m'(t)$ is the derivative of $m(t)$.

1.2.2.5 Reversed hazard rate

For a non-negative rv X, the reversed hazard rate (RHR) is defined by

$$\lambda(t) = \frac{f(t)}{F(t)} \tag{1.17}$$

where $\lambda(t)dt$ can be interpreted as an approximate probability of failure in $(t-dt, t]$ given that the failure had occurred in $[0,t]$. Keilson and Sumita (1982) were among the first to define RHR and called it the dual failure function. The RHR uniquely determines $F(t)$ through a relation

$$F(t) = \exp\left(-\int_{t}^{b} \lambda(x)dx \right);\ t \in (a,b). \tag{1.18}$$

Later, Block, Savits and Singh (1998) studied RHR, its properties and characterized a class of distributions having constant RHR in their interval of support. Shaked and Shanthikumar (1994) also proved several results related to RHR ordering. Now we restate the following definitions and theorems for the consideration of later chapters.

Definition 1.1 (Shaked and Shantikumar (1994)): Let X and Y be two random variables with absolutely continuous distribution functions F_X and F_Y and reversed hazard rates λ_X and λ_Y respectively such that

$$\lambda_X(t) \le \lambda_Y(t);\ t \ge 0 \tag{1.19}$$

then X is said to be smaller than Y in RHR order and it is denoted as $X \overset{rh}{\le} Y$.

From the definition 1.1, it is easy to verify that (1.19) holds if and only if $\frac{F_X(t)}{F_Y(t)}$ decreases in t. Based on the RHR ordering, Shaked and Shantikumar (1994) obtained the following result.

Theorem 1.2 (Shaked and Shantikumar (1994)): If X and Y be two random variables such that $X \overset{rh}{\le} Y$, then $X \overset{st}{\le} Y$ ($X \overset{st}{\le} Y$ if and only if $P(X \ge u) \le P(Y \ge u)$ for all $u \in (-\infty, \infty)$).

Nanda and Shaked (2001) proved some useful results in stochastic ordering in the context of RHR. However, Nair et al. (2005) characterized certain models using the relation between RHR and conditional expectation. Nair and Asha (2004) gave a review of literature on RHR and developed certain identities connecting df, pdf and reliability function in terms of failure rate and RHR and characterized some well known families of distributions using these identities. They explained these concepts in discrete set up also. Recently, Bartoszewicz and Skolimowska (2006) proved certain theorems based on the monotonic properties of RHR. They are given below.

Theorem 1.3 (Bartoszewicz and Skolimowska (2006)): Let $w(.)$ be a monotone left continuous function.

a) If $w(t)$ is increasing and $w(t)\lambda(t)$ is decreasing, then X_w is decreasing reversed hazard rate (DRHR). (A distribution F is said to be DRHR if $\log F$ is concave on (a,b)).

b) If $w(t)$ is decreasing and $w(t)\lambda(t)$ is increasing, then X_w is increasing reversed hazard rate (IRHR).

c) If $w(t)h(t)$ is decreasing, then X_w is DRHR.

1.2.2.6 Expected inactivity time (Mean waiting time)

Another important rv closely associated with RHR is the expected inactivity time (EIT) or mean waiting time (MWT). Similar to MRLF in left truncated situation, the EIT in right truncated situation is defined as

$$\bar{r}(t) = E\left(t - X \middle| X \le t\right). \tag{1.20}$$

Similar to equation (1.9), the EIT for an item failed in an interval $[0,t]$ as

$$\bar{r}(t) = \frac{1}{F(t)} \int_0^t F(x)dx. \tag{1.21}$$

Assuming $\bar{r}(t)$ is differentiable, as in (1.10), EIT is related to RHR through the relation

$$\lambda(t) = \frac{1 - \bar{r}'(t)}{\bar{r}(t)} \tag{1.22}$$

or

$$F(t) = \exp\left(-\int_t^{\infty} \frac{1 - \bar{r}'(x)}{\bar{r}(x)} dx \right). \tag{1.23}$$

Chandra and Roy (2001) studied some properties of waiting time with respect to RHR. Finkelstein (2002) focused the importance of EIT (MWT) in defining RHR and studied its properties. Li and Lu (2003) established some stochastic comparisons on inactivity time and the residual life of a series and parallel system respectively and presented some applications based on these comparisons. Several preservation properties of stochastic comparisons based on the MIT order under the reliability operations of convolution and mixture is studied by Kayid and Ahmad (2004). Further, Nanda et al. (2003) studied some reliability properties of EIT and Nanda et al. (2006) proposed a stochastic order based on this function for a rv with support (l_x,∞) where l_x may be $-\infty$ and studied its properties.

1.2.2.7 Generalized failure rate

For a continuous rv, X^* which is doubly truncated at the points t_1 and t_2, then the generalized failure rate (GFR) defined by Navarro and Ruiz (1996) is defined as

$$h_1(t_1,t_2)=\frac{f(t_1)}{F(t_2)-F(t_1)} \tag{1.24}$$

$$h_2(t_1,t_2)=\frac{f(t_2)}{F(t_2)-F(t_1)} \tag{1.25}$$

where $X^*=\left(X:t_1\le X\le t_2\right)$ defined for $(t_1,t_2)\in D=\left\{(u,v)\in\Re^{+^2};F(u)<F(v)\right\}$.

Based on the definitions (1.24) and (1.25), Navarro and Ruiz proved that the GFR satisfies the following properties.

1) D is not an open, not empty such that if $(t_1,t_2)\in D$ then $t_1<t_2$ and $(t_1,t)\in D$ or $(t,t_2)\in D$ for all $t\in\Re^+$.
2) $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ are positive functions, if $f(t)$ is continuous then h_i; $i=1,2$ are continuous functions.
3) Let $(c,d)\notin D$; $h_1(t_1,t_2)=0$ in almost every point $t_1\in[c,d]$ and $h_2(t_1,t_2)=0$ for almost every point $t_2\in[c,d]$.
4) The integrals

$$I_1(t_1,t_2)=\int_a^{t_1}h_1(t,t_2)dt \tag{1.26}$$

$$I_2(t_1,t_2)=\int_{t_2}^{b}h_2(t_1,t)dt \tag{1.27}$$

are finite for all $(t_1,t_2)\in D$.

5) For all $(t_1,t_2)\in D$, $G_{t_1}(t_2)=\dfrac{\exp[I_1(t_1,t_2)]}{Denom_C(t_1,t_2)}$ depends only on t_2 and $H_{t_2}(t_1)=\dfrac{\exp[I_1(t_1,t_2)]-1}{Denom_C(t_1,t_2)}$ depends only on t_1, where $Denom_C(t_1,t_2)=\exp[I_1(t_1,t_2)]+\exp[I_2(t_1,t_2)]-1$.

Further, they characterized class of these functions using the following theorem.

Theorem 1.4 (Navarro and Ruiz (1996))**:** Let h_i; $i=1,2$ be real functions with domain $D\in\Re^{+^2}$. Then, there exists a cumulative distribution function F for which h_1 and h_2 are its GFR if and only if h_1 and h_2 satisfy the properties 1 to 5.

Ruiz and Navarro (1996) obtained a necessary and sufficient condition that any function $m:\Re^2\to\Re$ is the conditional expectation $E\left(X\middle|t_1<X<t_2\right)$ of a rv X with continuous df and they related this function to order statistics. Navarro and Ruiz (2004) studied some characterizations using the relationships between GFR functions and the conditional moment function $m_u(t_1,t_2)=E\left(u(X)\middle|t_1<X<t_2\right)$. Subsequently, they also obtained some characterizations based on RHR and EIT. Sankaran and Sunoj (2004) further studied some relationships between failure rate and the MRLF for doubly truncated random variables and proved certain characterizations for exponential, Pareto II, beta distributions and some families of distributions. Further they explored some applications of these measures in the context of length-biased models.

1.2.3 Families of distributions

A standard practice in modeling statistical data is either to derive an appropriate model based on the physical properties of the system or to choose a flexible family of distributions, and then find a member of the family that fits to the data. In latter case, it is necessary to identify the data generating process to explain why the empirical distribution appropriates the data. When using families of distribution as the starting point, often some general properties of the family will be of considerable use in identifying a suitable member. It also helps to unify the results obtained in the case of individual distributions that are obtained in separate studies. Apart from these, there should be some simple criterion that distinguishes the members of the family so that it is easy to choose a member that fits the data. This shows the importance of families of distributions in modeling problems. So this section explains some important families of distributions that are used in the present study.

1.2.3.1 Exponential family

The exponential family of distributions includes all density functions and its pdf can be written in the form

$$f(t)=\exp\left[\theta t+C(t)+D(\theta)\right] \tag{1.28}$$

where $C(.)$ and $D(.)$ are arbitrary functions. This class was recognized nearly simultaneously by Darmois and Koopman. So it is often called Darmois-Koopman class.

1.2.3.2 Log exponential family

The distribution of a rv X belong to log exponential family if the pdf of X is of the form

$$f(t)=\frac{t^{\theta}C(t)}{A(\theta)};\ t>0,\ \theta>0 \tag{1.29}$$

where $C(t)$ is a non-negative function of t, which is differentiable and $A(\theta)$ is a non-negative function of θ satisfying $A(\theta)=\int_0^{\infty}x^{\theta}C(x)dx$ (see Patil and Ord (1976)).

1.2.3.3 Pearson family of distributions

Pearson family of probability distributions was introduced by Karl Pearson in 1895. The df $F(t)$ of a rv X belong to the Pearson family of distributions if the pdf $f(t)$ satisfies a differential equation of the form

$$\frac{d}{dt}\log f(t) = -\frac{(t+d)}{\left(b_0 + b_1 t + b_2 t^2\right)} \tag{1.30}$$

where $f(t)$ is differentiable, b_0, b_1, b_2 and d are real constants. The shape of the distribution depends on the values of the parameters. The form of this distribution depends on nature of the roots of the equation $b_0 + b_1 t + b_2 t^2 = 0$ and various types correspond to the roots of the quadratic equation in the denominator of (1.30). For various properties and applications of this family we refer to Nair and Sankaran (1991), Johnson et al. (1994), Ruiz and Navarro (1994), Asadi (1998) and Sankaran and Nair (2000).

1.2.3.4 General class of distributions

Consider a class of distributions whose pdf defined by Ruiz and Navarro (1994) is given by

$$\frac{f'(t)}{f(t)} = \frac{\mu - t - g'(t)}{g(t)} \tag{1.31}$$

where μ is a constant, $g(t)$ is a real function in (a,b) and it satisfies the first order differential equation

$$g'(t) + \frac{f'(t)}{f(t)} g(t) = \mu - t.$$

Ruiz and Navarro proved certain theorems using the equivalence relation connecting (1.31) and the vitality function. Alternatively, we may view $g(t)$ as given, and then $f(t)$ is uniquely determined by (1.31). By appropriately choosing $g(t)$, one can obtain many of the important cases that have appeared in the literature and which includes Pearson family, beta, gamma and Maxwell distributions etc (see Gupta and Bradely (2003)).

1.2.3.5 Generalized Pearson family of distributions

The Pearson family of distributions is widely used in reliability models as it contains many other important probability models such as exponential, gamma, Pareto II, beta etc. But there are certain other distributions such as Inverse Gaussian, random walk distribution etc. that do not belong to this family. Ord (1972) proposed another family called Generalized Pearson family of distributions and for this family, the pdf of X satisfies a differential equation

$$\frac{d}{dt}\log f(t) = -\frac{\left(a_0 + a_1 t + a_2 t^2\right)}{\left(b_0 + b_1 t + b_2 t^2\right)} \tag{1.32}$$

where a_i, b_i; $i = 0,1,2$ are real constants. When $a_2 = 0$, this family reduces to Pearson family (1.30). Like Pearson family of distributions, the generalized version can be classified into various types according to the nature of the roots of the quadratic function given in the denominator of (1.32) (see Sindu (2002)).

1.2.3.6 Burr system of distributions

The Burr system of distributions was constructed in 1941 by Irving W. Burr. Since the corresponding density functions have a wide variety of shapes, this system is useful for approximating histograms, particularly when a simple mathematical structure for the fitted cumulative distribution function is required. A number of standard theoretical distributions are limiting forms of Burr distributions. A rv X with df $F(t)$ satisfying a differential equation

$$\frac{dy}{dt} = y(1-y)k(t) \tag{1.33}$$

where $y = F(t)$ and $k(t)$ must be positive for $0 \le y \le 1$ (see Burr (1942)). Different choices of $k(t)$ generate various solutions for $F(t)$. In the next section, we explain some basic concepts relating measures of uncertainty.

1.2.4 Measures of uncertainty - preliminaries

In this section, we examine some important measures of uncertainty which are useful in the present study.

1.2.4.1 Shannon's entropy

The concept of entropy was extensively used in literature as a quantitative measure of uncertainty associated with random phenomena. In the context of equilibrium thermodynamics, physicists originally developed the notion of entropy which was later extended through the development of statistical mechanics and information theory. Shannon (1948) was the one who formally introduced entropy, known as Shannon's entropy or Shannon's information measure into information theory. For a rv X, having pdf $f(t)$, the Shannon's entropy is defined as

$$H = -\int_0^{\infty} f(x)\log f(x)dx. \tag{1.34}$$

In the pioneering work of Shannon, the properties and virtue of H have been thoroughly investigated in the literature. This entropy finds applications in several areas such as communication theory, flow of electricity and visual communications from artist to viewers etc.

1.2.4.2 Residual entropy

For a continuous non-negative rv X representing lifetime of a component, Ebrahimi (1996) defines the residual entropy function as the Shannon's entropy associated with the rv $(X-t)$ truncated at $t>0$, namely

$$H(t) = -\frac{1}{R(t)}\int_t^{\infty} f(x)\log\left(\frac{f(x)}{R(t)}\right)dx\,;\; R(t)>0. \tag{1.35}$$

The residual entropy (1.35) measures the expected uncertainty contained in the conditional density of $(X-t)$ given $X>t$ about the predictability of remaining lifetime of the component. It is noticed that when $t=0$, (1.35) reduces to Shannon's entropy defined over $(0,\infty)$. For some properties and useful characterizations, on can refer Ebrahimi and Pellerey (1995), Nair and Rajesh (1998), Rajesh and Nair (1998), Sankaran and Gupta (1999) and Asadi and Ebrahimi(2000) and Belzunce et al. (2004).

1.2.4.3 Conditional measure of uncertainty

For a non-negative rv X with pdf $f(t)$ and reliability function $R(t)$, Sankaran and Gupta (1999) defined a conditional measure of uncertainty as

$$M(t) = -E\left(\log f(X) \middle| X > t\right)$$

$$= -\frac{1}{R(t)}\int_t^{\infty} f(x)\log f(x)dx. \tag{1.36}$$

$M(t)$ measures the uncertainty contained in $f(t)$ about the predictability of the total lifetime of a unit which has survived to age t. This measure can be represented as the sum of residual entropy and total failure rate as

$$M(t) = H(t) - \log R(t) \tag{1.37}$$

where $-\log R(t) = \int_0^t h(x)dx$ is the total failure rate. For some properties and characterizations based on this measure, one could refer Sankaran and Gupta (1999), Rajesh (2001).

1.2.4.4 Renyi's measure of entropy

Entropies of higher order are defined by several authors and their properties are being examined. The works of Renyi (1961) and Kapur (1968) proceed in this direction. For a continuous rv X admitting an absolutely continuous distribution, Renyi defined the entropy of order β as

$$I_R = \frac{1}{(1-\beta)}\log \int_0^{\infty} f(x)dx. \tag{1.38}$$

When $\beta \to 1$, (1.38) reduces to Shannon's entropy. For the rv $(X-t)$ truncated at $t > 0$, the Renyi's entropy measure becomes

$$I_R(t) = \frac{1}{(1-\beta)}\log \int_t^{\infty}\left(\frac{f(x)}{R(t)}\right)^{\beta} dx. \tag{1.39}$$

Further $I_R(t)$ reduces to the residual entropy (1.35) when $\beta \to 1$. For some properties and characterization of this measure one can refer Song (2001) and Abraham and Sankaran (2005).

1.3 Present study

The present work is organized into six chapters. In reliability and survival studies, the identification of probability models of lifetimes is of prime concern and it is often achieved through studying the characteristics of various measures such as failure rate, mean residual life function, vitality function, coefficient of variation etc. However, in recent years considerable attention has been paid to the problem of characterizing probability distributions of a rv based on conditional expectations of left and right truncated data. For details regarding conditional expectations one can refer Zoroa et al. (1990), Navarro and Ruiz (2004) and the references therein. Similarly, characterization problems using weighted distributions have been studied by different authors. Even if several research works were carried out on weighted distributions in the context of left truncated case, but a very little has been explored for the right (past lifetime) and interval (doubly) truncated random variables. Motivated by this, in the present study, we focus attention on studying the mathematical relationships between weighted and original random variables using various measures such as maintainability function, reversed repair rate, log odds rate, measures of uncertainty and discrimination and lower partial moments for the right and doubly truncated random variables and prove certain characterization theorems arising out of it. Most of the results that we have obtained in the present thesis based on the right truncation, are unique in nature compared to the existing results based on left truncation. However, some results show certain similarity in their functional forms.

In continuation of the present chapter, in Chapter 2, we discuss the weighted distributions in the context of repairable systems. Accordingly, characterization theorems are established in respect of exponential, Pareto II, beta distributions and some important families of distributions based on the ratio of maintainability functions of original and length-biased distributions and conditional expectations.

In Chapter 3, we explain the significance of log odds ratio in reliability modeling and characterize certain families of distributions based on this concept. Further we examine the usefulness of this concept in the context of weighted models and extend the log odds rate to the bivariate case.

Chapter 4 is devoted to the study of measures of uncertainty based on past life. All measures of uncertainty have much relevance in characterizing and classifying distributions based on the properties showed by them. Accordingly, Chapter 4 explores the concept of past entropy introduced by Di Crescenzo and Longobardi (2002) and studies some of its properties. Further we obtained some characterizations of distributions based on the relationship between past entropy and other reliability measures and study this concept for weighted models. We also examine the properties and usefulness of some measures of discrimination in the context of weighted models.

We extend the concept of past entropy and other uncertainty measures to doubly truncated random variables in Chapter 5. In this chapter, we introduce some new conditional measures uncertainty and obtained various useful results for some well known families/distributions in the context of interval truncated data. We also compare these measures for weighted models and proved certain characterization theorems arising out of it.

Chapter 6 considers the right truncated rv into an economic point of view. In this chapter we discuss the concept of Lower Partial Moments (LPMs). LPMs are useful for the studies related to risks and poverty in Economics. Here also we present some characterizations of certain important distributions and families of distributions based on the r^{th} order LPM. The concept is considered in the context of weighted models and studied some of its particular cases. Finally, we explain the application of LPM for the studies related to poverty.

CHAPTER TWO

WEIGHTED DISTRIBUTIONS USEFUL IN REPAIRABLE SYSTEMS[1]

2.1. Introduction

In reliability theory, the term reliability of a system is the probability that it will perform its intended functions for a specified period of time when operating under normal environmental conditions. This definition is explained with reference to the failure behavior of the system and it plays an important role in modeling lifetime distributions. In a similar way, maintainability also plays a vital role in studying the effectiveness of a system. This is more useful to those situations where a repair is possible. Thus for the modeling problems related to repairable systems, maintainability is more practical than reliability. In this view, in the present chapter, we discuss some properties of weighted distributions in the context of repairable systems and provide some interesting results based on it.

2.2. Maintainability function and Reversed repair rate

The purpose of maintenance is to restore a deteriorating or failed system to its normal operating state. The maintainable and repairable system can be restored into service at

[1] Some of the results in this Chapter have been published as entitled "Some properties of weighted distributions in the context of repairable systems", *Communication in Statistics – Theory and Methods*, 35(2), 223-228 (*see* Sunoj and Maya (2006)). Another paper has also been communicated.

regular intervals of time or after each of its failure. Generally, maintainability of a system provides a measure of the reparability of a system when it fails or it is defined as the probability of repairing a failed component or system in a specified period of time. Mathematically, it is defined as follows. Let X be an absolutely continuous rv representing the repair time of a component/system, then the maintainability function (df) of X can be defined as

$$F(t) = P(X \leq t) \tag{2.1}$$

and it gives the probability that required maintenance will be successfully completed in a given time period. The maintainability functions are used to predict the probability that a repair beginning at time $X = 0$ will be accomplished in time $X = t$ (see Rao (1992)). Various probability distributions may be used to present an item's repair time data. Of these, the most frequently used repair time models include exponential, log normal, gamma, Poisson and uniform distributions. For example, if the components of a system that fail frequently have a relatively short repair times compared to those components that fail infrequently, then the repair times can be assumed to follow exponential distribution. On the other hand, if every component of the system has the same failure rate and the same repair time, the repair times can be assumed to follow uniform distributions. Once the repair time distributions are identified, the corresponding maintainability function can be easily obtained. An important measure closely associated with the maintainability function is the reversed repair rate, which is defined as the ratio of the pdf of repair time and the maintainability function and is given by

$$\lambda(t) = \frac{f(t)}{F(t)}. \tag{2.2}$$

When X represents the repair time of a component, then the probability that it is repaired during the time $(t-\varepsilon, t)$ (where ε is a small positive number) is approximately equal to $\varepsilon\lambda(t)$. When X represents the lifetime, $\lambda(t)$ then termed as reversed hazard rate (RHR). In this context $\varepsilon\lambda(t)$ is the approximate probability that a device survived time $(t-\varepsilon)$ given that it failed at t.

From the definition (2.2), it is clear that

$$\lambda(t) = \frac{d \log F(t)}{dt}. \tag{2.3}$$

Integrating (2.3) over the limits t to b and assume that $\lim_{t \to b} \log F(t) = 0$, then we obtain

$$F(t) = \exp\left(-\int_t^b \lambda(x) dx \right). \tag{2.4}$$

From (2.4) it is clear that the reversed repair rate uniquely determines the maintainability function. Let

$$\bar{m}_w(t) = E\left(w(X) \middle| X \le t \right) \tag{2.5}$$

be a conditional moment function. Differentiating (2.5) with respect to t and assume that $\lim_{t \to a} w(t) f(t) = 0$ and simplifying, we have

$$\lambda(t) = \frac{\frac{d}{dt} \bar{m}_w(t)}{w(t) - \bar{m}_w(t)}. \tag{2.6}$$

Substituting (2.6) in (2.4), we get

$$F(t) = \exp\left(-\int_t^b \frac{d\bar{m}_w(x)}{w(x) - \bar{m}_w(x)} \right). \tag{2.7}$$

Thus (2.7) implies that any conditional moment of repair time also uniquely determines the maintainability function.

2.3 Mathematical relationships between original and weighted random variables

Let X be a non-negative rv which denote the repair time of a component/system possessing an absolutely continuous maintainability function $F(t)$ and survival function $R(t) = P(X > t)$ and assume that $w(t)$ is positive and differentiable with $\lim_{t \to a} w(t) F(t) = 0$, then the maintainability function of the rv X_w using (1.1) is given by

$$F^{w}(t) = P(X_w \le t) = \int_a^t f^{w}(x)dx = \frac{\bar{m}_w(t)}{\mu_w} F(t) \tag{2.8}$$

and the corresponding reversed repair rate $\lambda^{w}(t)$ becomes

$$\lambda^{w}(t) = \frac{f^{w}(t)}{F^{w}(t)} = \frac{w(t)}{\bar{m}_w(t)} \lambda(t). \tag{2.9}$$

From (2.8) and (2.9), the following theorem is immediate.

Theorem 2.1: a.) If $w(t)$ is monotone increasing (decreasing), then $\lambda^{w}(t) \ge (\le) \lambda(t)$, and $F^{w}(t) \le (\ge) F(t) \left(X \overset{st}{\le} \overset{st}{(\ge)} X_w \right)$ for all t.

b) $\lambda^{w}(t) \ge \lambda(t) \Leftrightarrow \frac{F^{w}(t)}{F(t)}$ is non increasing $\Leftrightarrow \frac{\bar{m}_w(t)}{\mu_w}$ is non increasing.

c) $X \overset{st}{\le} \overset{st}{(\ge)} X_w \Leftrightarrow \mu_w \ge (\le) \bar{m}_w(t)$

Proof: Part (a) of the Theorem is clearly obtained from the definition of weighted distribution and from (2.8) and (2.9). The first part of (b) is obtained from Shaked and Shanthikumar (1994) (see Chapter 1) and the second part is obtained from (2.8). To prove (c), consider $X \overset{st}{\le} X_w$. This implies $F(t) \ge F^{w}(t) \Leftrightarrow \frac{F^{w}(t)}{F(t)} \le 1$. From (2.8), we get $\bar{m}_w(t) \le \mu_w$.

The following theorems prove that the ratio of the maintainability functions or reversed repair rates of weighted and original rv determines the distribution.

Theorem 2.2: If $\bar{\alpha}(t) = \frac{F^{w}(t)}{F(t)}$, then

$$F(t) = \exp\left(-\int_t^b \frac{\bar{\alpha}'(x)}{\frac{w(x)}{\mu_w} - \bar{\alpha}(x)} dx \right). \tag{2.10}$$

Proof: Given that $\bar{\alpha}(t)=\dfrac{F^{w}(t)}{F(t)}$, then from (2.8), we have $\bar{\alpha}(t)=\dfrac{\bar{m}_{w}(t)}{\mu_{w}}$. Now using the relation (2.7) we get (2.10).

Theorem 2.3: If $w(t)$ is differentiable and $\bar{\beta}(t)=\dfrac{\lambda(t)}{\lambda^{w}(t)}$, then

$$F(t)=\exp\left(-\int_{t}^{b}\frac{\bar{\beta}'(x)w(x)+\bar{\beta}(x)w'(x)}{\left(1-\bar{\beta}(x)\right)w(x)}dx\right). \tag{2.11}$$

Proof: We have $\bar{\beta}(t)=\dfrac{\lambda(t)}{\lambda^{w}(t)}$, then from (2.9) $m_{w}(t)=w(t)\bar{\beta}(t)$. Now using (2.7) we obtain the form (2.11).

The major relationship between the original rv X and the size-biased rv X_{S} are obtained by using (1.3) in (2.8) and (2.9) and they are

$$F^{S}(t)=\frac{\bar{m}^{\alpha}(t)}{\mu_{\alpha}}F(t) \tag{2.12}$$

and

$$\lambda^{S}(t)=\frac{t^{\alpha}}{\bar{m}^{\alpha}(t)}\lambda(t), \tag{2.13}$$

where $\bar{m}^{\alpha}(t)=E\left(X^{\alpha}\middle|X\le t\right)$, $\mu_{\alpha}=E\left(X^{\alpha}\right)$ and $F^{S}(t)$ and $\lambda^{S}(t)$ respectively denote the maintainability function and the reversed repair rate of the size-biased rv X_{S}. The corresponding maintainability function $F^{L}(t)$ and the reversed repair rate $\lambda^{L}(t)$ for the length-biased model (1.2) is given by

$$F^{L}(t)=\frac{\bar{m}(t)}{\mu}F(t) \tag{2.14}$$

and

$$\lambda^{L}(t)=\frac{t}{\bar{m}(t)}\lambda(t) \tag{2.15}$$

where $\mu=E(X)<\infty$ and $\bar{m}(t)=E(X|X\le t)$, the right truncated mean of X.

Theorem 2.4: Let $\bar{r}(t) = E(t - X | X \le t)$. If X is increasing reversed repair rate (IRRR) and $\frac{\bar{r}(t)}{t}$ is non-decreasing, then the length-biased rv X_L is also increasing reversed repair rate.

Proof: X is IRRR implies $\lambda(t_1) \le \lambda(t_2)$ for all $t_1 \le t_2$.

i.e.,
$$\left(\frac{t_1 - \bar{r}(t_1)}{t_1}\right)\lambda^L(t_1) \le \left(\frac{t_2 - \bar{r}(t_2)}{t_2}\right)\lambda^L(t_2) \text{ for all } t_1 \le t_2.$$

$$\Rightarrow \lambda^L(t_1) \le \frac{\left(1 - \frac{\bar{r}(t_2)}{t_2}\right)}{\left(1 - \frac{\bar{r}(t_1)}{t_1}\right)}\lambda^L(t_2)$$

$$\Rightarrow \lambda^L(t_1) \le \lambda^L(t_2) \quad \text{for all } t_1 \le t_2.$$

Next we prove a theorem characterizing Pareto II, exponential and beta distributions using the functional relationship connecting the ratio of the maintainability functions of original and length-biased rv.

Theorem 2.5: Let $\lim_{t \to a} F(t) = 0$, then the ratio

$$\frac{F^L(t)}{F(t)} = 1 - t(1 + Ct)\lambda(t) \tag{2.16}$$

holds for all $t > 0$, if and only if X has Pareto II distribution with

$$F(t) = 1 - (1 + pt)^{-q}; \quad t > 0, p, q > 0 \tag{2.17}$$

for $C > 0$, exponential distribution with

$$F(t) = 1 - e^{-\lambda t}; \quad t > 0, \lambda > 0 \tag{2.18}$$

for $C = 0$, or beta distribution with

$$F(t) = 1 - (1 - Rt)^d; \quad 0 < t < \tfrac{1}{R}, \; d > 0, R > 0 \tag{2.19}$$

for $C < 0$.

Proof: When X is specified by Pareto II distribution (2.17), then from (2.14), we have

$$\frac{F^L(t)}{F(t)}=\frac{1}{\mu}\left(t-\frac{1}{F(t)}\int_a^t F(x)dx\right).$$

For the Pareto II distribution we obtain

$$\frac{1}{F(t)}\int_a^t F(x)dx=\frac{t}{F(t)}-\frac{1}{p(q-1)}+\frac{t(1+pt)\lambda(t)}{pq(q-1)}$$

and therefore

$$\frac{1}{\mu}\left(t-\frac{1}{F(t)}\int_a^t F(x)dx\right)=1-t(1+pt)\lambda(t). \tag{2.20}$$

Now (2.20) is of the form (2.16).

Conversely assume that (2.16) holds, then using (2.14), we get

$$t-\frac{1}{F(t)}\int_a^t F(x)dx=\mu-\mu t(1+Ct)\frac{f(t)}{F(t)}$$

or

$$(t-\mu)F(t)=\int_a^t F(x)dx-\mu t(1+Ct)f(t). \tag{2.21}$$

Differentiating (2.21) with respect to t, we get

$$\frac{f'(t)}{f(t)}=-\frac{(1+2\mu C)}{\mu(1+Ct)}. \tag{2.22}$$

Integrating (2.22) with respect to t, we obtain (2.17) for $C>0$, (2.18) for $C=0$ and (2.19) for $C<0$ respectively.

Next we prove some characterization theorems that provide the relationships between the ratios of the maintainability function of certain important families of distributions under length-biased sampling.

Theorem 2.6: Assume that $\lim_{t\to a}\left(g(t)f(t)\right)=0$, then the ratio of the relationship

$$\frac{F^L(t)}{F(t)}=1-k(t)\lambda(t) \tag{2.23}$$

where $k(t)=\dfrac{g(t)}{\mu}$ holds for all $t\geq 0$ if and only if the pdf $f(t)$ belongs to general family of distributions (1.31).

Proof: From (1.31), we have

$$\frac{d}{dt}\left(f(t)g(t)\right)=(\mu-t)f(t). \tag{2.24}$$

Integrating (2.24) over the limits a to t and dividing each term by $F(t)$ gives

$$g(t)\lambda(t)=\mu-\bar{m}(t). \tag{2.25}$$

Now using (2.14), we obtain (2.23).

Conversely when (2.23) holds, then using (2.14), we get

$$\frac{\bar{m}(t)}{\mu}=1-k(t)\lambda(t)$$

or

$$\frac{1}{F(t)}\int_a^t xf(x)dx=\mu-\mu k(t)\frac{f(t)}{F(t)}. \tag{2.26}$$

Assuming $\lim_{t\to a} tf(t)=0$, and multiplying both sides of (2.26) by $F(t)$ and on differentiation we get (1.31).

Corollary 2.1: When $g(t)=b_0+b_1t+b_2t^2$, a quadratic form, then (1.31) becomes Pearson system of distributions and the relation (2.23) reduces to

$$\frac{F^L(t)}{F(t)}=1-\left(k_0+k_1t+k_2t^2\right)\lambda(t) \tag{2.27}$$

where $k_i=\dfrac{b_i}{\mu}$ for $i=0,1,2$.

Theorem 2.7: Let $\lim_{t\to a}\left(b_0+b_1t+b_2t^2\right)f(t)=0$ then the pdf $f(t)$ of a rv X belongs to generalized Pearson family of distribution (1.32) if and only if it satisfies the relationship

$$\frac{F^L(t)}{F(t)}=\left(c_0+c_1t+c_2t^2\right)\lambda(t)-d_0-d_2\bar{m}^2(t) \tag{2.28}$$

where $\bar{m}^2(t)=E\left(X^2\,|X\le t\right)$, c_0,c_1,c_2,d_0 and d_2 are real constants and $\left(a_1+2b_2\right)\neq 0$.

Proof: From (1.32), we have

$$\left(b_0+b_1t+b_2t^2\right)\frac{d}{dt}f(t)=\left(a_0+a_1t+a_2t^2\right)f(t). \tag{2.29}$$

Integrating (2.29) over the limits a to t and assuming the boundary condition, we get (2.28), with $c_i=\frac{b_i}{\mu(a_1+2b_2)}$; $i=0,1,2$, $d_0=\frac{(a_0+b_1)}{\mu(a_1+2b_2)}$, $d_2=\frac{a_2}{\mu(a_1+2b_2)}$ provided $\left(a_1+2b_2\right)\neq 0$.

Conversely assuming (2.28) for all t, from (2.14) we get

$$\bar{m}(t)=\mu\left[(c_0+c_1t+c_2t^2)\lambda(t)-d_0-d_2E\left(X^2\,|X\le t\right)\right]$$

or

$$\frac{1}{F(t)}\int_a^t xf(x)dx=\mu\left[(c_0+c_1t+c_2t^2)\frac{f(t)}{F(t)}-d_0-d_2\frac{1}{F(t)}\int_a^t x^2f(x)dx\right]. \tag{2.30}$$

Multiplying both sides of (2.30) by $F(t)$ and on differentiation and applying $\lim_{t\to a}t^if(t)=0$ for $i=1,2$, we get

$$\left(\frac{t}{\mu}-2c_2t+e_0+d_2t^2\right)f(t)=(c_0+c_1t+c_2t^2)\frac{d}{dt}f(t),$$

where $e_0=d_0-c_1$. Then

$$\frac{d}{dt}\ln f(t)=\frac{\left[e_0+\left(\frac{1}{\mu}-2c_2\right)t+d_2t^2\right]}{(c_0+c_1t+c_2t^2)},$$

which reduces to the generalized Pearson family (1.32).

2.4 Conditional moments for some families of distributions

In this section we prove some characterizations of the Pearson and Generalized Pearson families of distributions based on the conditional moments and also examined its relationships in the context of weighted models.

Let X be a rv with df $F(t)$. Assume that the regularity condition $\lim_{t\to a}(l_o + l_1 t + l_2 t^2)t^r f(t) = 0$ for $r = 0,1,2,...$ where l_i is any a_i or b_i for $i = 0,1,2$, in (1.32) holds, and let $\bar{m}^r(t) = E(X^r | X \le t)$ denote the r^{th} order conditional moment of X, then we have the following theorems.

Theorem 2.8: Under the regularity condition, the pdf of a rv X belongs to generalized Pearson family (1.32) if and only if its r^{th} order conditional moments satisfies a recurrence relation of the form

$$\bar{m}^r(t) = (b_{0r} + b_{1r}t + b_{2r}t^2)t^{r-2}\lambda(t) - (rb_{2r} + a_{1r})\bar{m}^{r-1}(t)$$
$$-\left((r-1)b_{1r} + a_{0r}\right)\bar{m}^{r-2}(t) - (r-2)b_{0r}\bar{m}^{r-3}(t), \qquad (2.31)$$

where b_{ir} ; $i = 0,1,2$ and a_{jr} ; $j = 0,1$, are real constants and $a_2 \ne 0$.

Proof: Multiplying both sides of (1.32) by t^{r-2} and on integrating over the limits a to t, we obtain

$$(b_0 + b_1 t + b_2 t^2)t^{r-2} f(t) - \left((r-1)b_1 + a_0\right)\int_a^t x^{r-2} f(x)dx - (rb_2 + a_1)\int_a^t x^{r-1} f(x)dx$$
$$-b_0(r-2)\int_a^t x^{r-3} f(x)dx - a_2\int_a^t x^r f(x)dx = 0. \qquad (2.32)$$

Dividing equation (2.32) by $F(t)$, we get (2.31) with $a_{jr} = \frac{a_j}{a_2}$; $j = 0,1$ and $b_{ir} = \frac{b_i}{a_2}$; $i = 0,1,2$.

Conversely assume that (2.31) holds for all t, then we have

$$\frac{1}{F(t)}\int_a^t x^r f(x)dx = (b_{0r} + b_{1r}t + b_{2r}t^2)t^{r-2}\frac{f(t)}{F(t)} - \frac{(rb_{2r} + a_{1r})}{F(t)}\int_a^t x^{r-1} f(x)dx$$

$$-\frac{((r-1)b_{1r} + a_{0r})}{F(t)}\int_a^t x^{r-2} f(x)dx - \frac{(r-2)b_{0r}}{F(t)}\int_a^t x^{r-3} f(x)dx. \quad (2.33)$$

Multiply both sides of (2.33) by $F(t)$ and on differentiation using the regularity condition, yields,

$$t^r f(t) = (b_{0r} + b_{1r}t + b_{2r}t^2)t^{r-2}\frac{d}{dt}f(t) + \left(a_{0r}t^{r-2} + a_{1r}t^{r-1}\right)f(t) \quad (2.34)$$

Dividing each term of (3.14) by t^{r-2}, we get

$$(a_{0r} + a_{1r}t + t^2)f(t) = (b_{0r} + b_{1r}t + b_{2r}t^2)\frac{d}{dt}f(t),$$

which is the generalized Pearson family (1.32). Table 2.1 provides some of the important members of the family (1.32) and the parameters involved in Theorem 2.8.

Table 2.1: Values of $b_{0r}, b_{1r}, b_{2r}, a_{0r}$ and a_{1r} based on Theorem 2.8

Members and Distribution	b_{0r}	b_{1r}	b_{2r}	a_{0r}	a_{1r}
Inverse Gaussian $\left(\frac{\lambda}{2\pi t^3}\right)^{1/2}\exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right)$; $t, \lambda, \mu > 0$	0	0	$\frac{-2\mu^2}{\lambda}$	$-\mu^2$	$\frac{3\mu^2}{\lambda}$
Maxwell $4\left(\frac{\lambda^3}{\pi}\right)^{1/2} t^2\exp\left(-\lambda t^2\right)$; $t, \lambda > 0$	0	$\frac{-1}{2\lambda}$	0	$\frac{-1}{\lambda}$	0
Rayleigh $2\lambda t\exp\left(-\lambda t^2\right)$; $t, \lambda > 0$	0	$\frac{-1}{2\lambda}$	0	$\frac{-1}{2\lambda}$	0

We now prove a characterization theorem for Pearson family of distributions.

Theorem 2.9: If $\lim_{t\to a}\left(b_0+b_1t+b_2t^2\right)t^r f(t)=0$, then the rv X with df $F(t)$ in (a,b) belongs to Pearson family of distributions (1.30) if and only if its r^{th} order conditional moments satisfies a recurrence relation

$$\bar{m}^r(t)=(c_{0r}+c_{1r}t+c_{2r}t^2)t^{r-1}\lambda(t)-(rc_{1r}-d_{0r})\bar{m}^{r-1}(t)-(r-1)c_{0r}\bar{m}^{r-2}(t), \tag{2.35}$$

provided $\left((r+1)b_2-1\right)\neq 0$, where c_{ir} ; $i=0,1,2$ and d_{0r} are real constants.

Proof: Suppose the df $F(t)$ of X belongs to Pearson family (1.30). Multiplying both sides of (1.30) by t^{r-1} and on integrating over the limits a to t and on using the boundary condition, we get

$$(b_0+b_1t+b_2t^2)t^{r-1}f(t)-(r-1)b_0\int_a^t x^{r-2}f(x)dx-(rb_1-d)\int_a^t x^{r-1}f(x)dx$$

$$-\left((r+1)b_2-1\right)\int_a^t x^r f(x)dx. \tag{2.36}$$

Dividing each term of (2.36) by $F(t)$ and using the definition of conditional moments, we obtain (2.35) with $c_{ir}=\dfrac{b_i}{\left((r+1)b_2-1\right)};i=0,1,2$ and $d_{0r}=\dfrac{d}{\left((r+1)b_2-1\right)}$ provided $\left((r+1)b_2-1\right)\neq 0$.

Conversely assume (2.35), then

$$\frac{1}{F(t)}\int_a^t x^r f(x)dx=(c_{0r}+c_{1r}t+c_{2r}t^2)t^{r-1}\frac{f(t)}{F(t)}-\frac{(rc_{1r}+d_{0r})}{F(t)}\int_a^t x^{r-1}f(x)dx$$

$$-\frac{(r-1)c_{0r}}{F(t)}\int_a^t x^{r-2}f(x)dx. \tag{2.37}$$

Multiply both sides of (2.37) by $F(t)$ and on differentiation, we get

$$t^r f(t)=(c_{0r}+c_{1r}t+c_{2r}t^2)t^{r-1}\frac{d}{dt}f(t)+(r+1)c_{2r}t^r f(t)+d_{0r}t^{r-1}f(t). \tag{2.38}$$

Simplifying (2.38), we obtain the Pearson family (1.30) with $d = \frac{d_{0r}}{((r+1)c_{2r}-1)}$ and $b_i = \frac{c_{ir}}{((r+1)c_{2r}-1)}$; $i = 0,1,2$.

Corollary 2.2: When $a_2 = 0$, the differential equation (1.32) becomes that of Pearson family of distributions. In this case the Theorem 2.8 reduces to Theorem 2.9.

Table 2.2 provides some members of the Pearson family of distributions and the values of the constants involved in Theorem 2.9.

Table 2.2: Values of c_{0r}, c_{1r}, c_{2r} and d_{0r} based on Theorem 2.9

Members and Distribution	c_{0r}	c_{1r}	c_{2r}	d_{0r}
Gamma $\frac{m^p}{\Gamma p}\exp(-mt)t^{p-1}$; $t > 0, m, p > 0$	0	$\frac{-1}{m}$	0	$\frac{(p-1)}{m}$
Pareto I $ck^c t^{-(c+1)}$; $t \geq k$, $k, c > 0$	0	0	$\frac{1}{(r-c)}$	0
Normal $\frac{1}{\sqrt{2\pi}\sigma}\exp-1/2\left(\frac{t-\mu}{\sigma}\right)^2$; $-\infty < t, \mu < \infty, \sigma > 0$	$-\sigma^2$	0	0	μ
Beta $\frac{d}{R}\left(1-\frac{t}{R}\right)^{d-1}$; $0 < t < R, d > 1$	0	$\frac{-R}{(d+r)}$	$\frac{1}{(d+r)}$	0
Exponential $\lambda\exp(-\lambda t)$; $t, \lambda > 0$	0	$\frac{-1}{\lambda}$	0	0

Motivated by the relevance of form-invariance in characterizing families of distributions and usefulness of the same in modeling various families of distributions, Sankaran and

Nair (1993) derived the conditions under which the Pearson and Ord families are form-invariant with respect to length-biased sampling. Later, Asadi (1998) further extended it to the size-biased sampling of order α. Further, Sindu (2002) proved that a generalized Pearson family (1.32) satisfies the form-invariant property under size-biased sampling if and only if $b_0 = 0$. In the size biased case, the differential equation for X_s becomes

$$\frac{d}{dt}\log f^S(t) = \frac{p_0 + p_1 t + p_2 t^2}{q_1 t + q_2 t^2} \tag{2.39}$$

where $p_0 = (\alpha b_1 + a_0)$, $p_1 = (\alpha b_2 + a_1)$, $p_2 = a_2$ and $q_i = b_i$ for $i = 1,2$.

Therefore an analogous statement for Theorem 2.8 in the context of size-biased models which are form-invariant is immediate, which is stated as follows.

Theorem 2.10: Assume $\lim_{t\to a}\left(p_0 + p_1 t + p_2 t^2\right)t^r f(t) = 0$, then the pdf of a rv X belongs to generalized Pearson family (2.39) under $w(t) = t^\alpha$ if and only if it satisfies the recurrence relation

$$\bar{m}^r(t) = (q_{1r\alpha} + q_{2r\alpha}t)t^{r-1}\lambda(t) - p_{0r\alpha}\bar{m}^{r-2}(t) - p_{1r\alpha}\bar{m}^{r-1}(t), \text{ for } p_2 \neq 0 \tag{2.40}$$

where $q_{ir\alpha}$; $i = 1,2$, $p_{jr\alpha}$; $j = 0,1$ are constants, and

$$\bar{m}^r(t) = (q_{1r\alpha} + q_{2r\alpha}t)t^r\lambda(t) - p_{0r\alpha}\bar{m}^{r-1}(t), \text{ for } p_2 = 0 \tag{2.41}$$

where $q_{ir\alpha}$; $i = 1,2$ and $p_{0r\alpha}$ are real constants provided $\left((r+1-\alpha)q_2 + p_1\right) \neq 0$.

Proof: Case I: when $p_2 \neq 0$

Multiplying both sides of (2.39) by x^{u-2} and on integrating over the limits a to t, we obtain

$$\left(q_1 t + q_2 t^2\right)t^{u-2} f^S(t) - \left((u-1)q_1 + p_0\right)\int_a^t x^{u-2} f^S(x)dx$$
$$-(uq_2 + p_1)\int_a^t x^{u-1} f^S(x)dx - p_2\int_a^t x^u f^S(x)dx = 0. \tag{2.42}$$

Dividing equation (2.42) by $F(t)$ and using the definition of $f^{S}(t)$, we get

$$p_2\bar{m}^{u+\alpha}(t)=\left(q_1+q_2t^2\right)t^{u+\alpha-1}\lambda(t)$$
$$-\left((u-1)q_1+p_0\right)\bar{m}^{u+\alpha-2}(t)-(uq_2+p_1)\bar{m}^{u+\alpha-1}(t) \qquad (2.43)$$

Changing u to $(r-\alpha)$ in (2.43) and on simplification we obtain the required result with $p_{0r\alpha}=\dfrac{\left(p_0+(r-\alpha-1)q_1\right)}{p_2}$, $p_{1r\alpha}=\dfrac{\left(p_1+(r-\alpha)q_2\right)}{p_2}$ and $q_{jr\alpha}=\dfrac{q_j}{p_2}$; $j=1,2$ provided $p_2\neq 0$.

Case II: when $p_2=0$

In this case equation (2.43) becomes

$$\left(q_1+q_2t^2\right)t^{u+\alpha-1}\lambda(t)-\left((u-1)q_1+p_0\right)\bar{m}^{u+\alpha-2}(t)-(uq_2+p_1)\bar{m}^{u+\alpha-1}(t)=0 \qquad (2.44)$$

Changing u to $(r+1-\alpha)$ in (2.44) and on simplification yields (2.41).

Conversely assume the relation (2.40) holds true, then by using the definition of conditional expectation, we get

$$\int_a^t x^r f(x)dx=(q_{1r\alpha}+q_{2r\alpha}t)t^{r-1}\lambda(t)-p_{0r\alpha}\int_a^t x^{r-2}f(x)dx-p_{0r\alpha}\int_a^t x^{r-1}f(x)dx. \qquad (2.45)$$

Assume $\lim_{x\to a}x^r f(x)=0$ and on differentiation (2.45) with respect to t implies

$$(q_{1r\alpha}+q_{2r\alpha}t)tf'(t)=\left[t^2-(r-1)q_{1r\alpha}-\left((r-1)+t\right)q_{2r\alpha}+p_{0r\alpha}+tp_{1r\alpha}\right]f(t). \qquad (2.46)$$

Multiplying both sides of (2.46) by $\left((q_{1r\alpha}+q_{2r\alpha}t)\alpha f(t)\right)\dfrac{t^{\alpha}}{\mu_{\alpha}}$ and on simplification, we obtain

$$\frac{f^{S'}(t)}{f^{S}(t)}=\frac{p_0+p_1t+p_2t^2}{q_1t+q_2t^2}, \qquad (2.47)$$

where $p_0 = p_{0r\alpha} - (r-\alpha-1)q_{1r\alpha}$, $p_1 = p_{1r\alpha} - (r-\alpha)q_{2r\alpha}$, $p_2 = 1$ and $q_i = q_{ir\alpha}$; $i = 1,2$. From (2.47) we get the pdf of X_S belongs to generalized Pearson family (2.39). The proof Case II is similar.

Corollary 2.3: When $p_2 = 0$, (2.39) reduces to the form-invariant Pearson family of distribution and in this case the differential equation becomes

$$\frac{f^{S'}(t)}{f^S(t)} = -\frac{(t+d_1)}{\left(k_1 t + k_2 t^2\right)} \tag{2.48}$$

where $d_1 = \frac{p_0}{p_1}$ and $k_i = \frac{-q_i}{p_1}$; $i = 1,2$, and the corresponding recurrence relation of the rth order conditional moments for the model given in (2.48) becomes

$$\bar{m}^r(t) = \left(k_{1r\alpha} + k_{2r\alpha} t\right) t^r \lambda(t) + d_{1r\alpha} \bar{m}^{r-1}(t), \tag{2.49}$$

with $k_{ir\alpha} = \frac{k_i}{\left((r-\alpha+1)k_2 - 1\right)}$; $i = 1,2$ and $d_{1r\alpha} = \frac{\left(d_1 - (r-\alpha)k_1\right)}{\left((r-\alpha+1)k_2 - 1\right)}$, $\left((r-\alpha+1)k_2 - 1\right) \neq 0$.

The Table 2.3 and Table 2.4 provide some members of the generalized Pearson family and the parameters involved in Theorem 2.10.

Table 2.3: Values of $p_{0r\alpha}$, $p_{1r\alpha}$, $q_{1r\alpha}$ and $q_{2r\alpha}$ based on Theorem 2.10 when $p_2 \neq 0$

Members and Distribution	$p_{0r\alpha}$	$p_{1r\alpha}$	$q_{1r\alpha}$	$q_{2r\alpha}$
Inverse Gaussian $\left(\frac{\lambda}{2\pi t^3}\right)^{1/2} \exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right)$; $t, \lambda, \mu > 0$	$-\mu^2$	$\frac{-(2r-3)\mu^2}{\lambda}$	0	$\frac{-2\mu^2}{\lambda}$
Maxwell $4\left(\frac{\lambda^3}{\pi}\right)^{1/2} t^2 \exp\left(-\lambda t^2\right)$; $t, \lambda > 0$	$\frac{-(r+1)}{2\lambda}$	0	$\frac{-1}{2\lambda}$	0
Rayleigh $2\lambda t \exp\left(-\lambda t^2\right)$; $t, \lambda > 0$	$\frac{-r}{2\lambda}$	0	$\frac{-1}{2\lambda}$	0

Table 2.4: Values of $p_{0r\alpha}$, $p_{1r\alpha}$, $q_{1r\alpha}$ and $q_{2r\alpha}$ based on Theorem 2.10 when $p_2 = 0$

Members and Distribution	$p_{0r\alpha}$	$q_{1r\alpha}$	$q_{2r\alpha}$
Gamma $\frac{m^p}{\Gamma p}\exp(-mt)t^{p-1}$; $t>0, m, p>0$	$\frac{(1-r-p)}{m}$	$\frac{-1}{m}$	0
Pareto I $ck^c t^{-(c+1)}$; $t \ge k$, $k, c>0$	0	0	$\frac{1}{(r-c)}$
Beta $\frac{d}{R}\left(1-\frac{t}{R}\right)^{d-1}$; $0<t<R, d>1$	$\frac{-rR}{(d+r)}$	$\frac{-R}{(d+r)}$	$\frac{1}{(d+r)}$
Exponential $\lambda\exp(-\lambda t)$; $t, \lambda>0$	$\frac{r}{\lambda}$	$\frac{1}{\lambda}$	0

We now prove a characterization theorem that provide the relationships between reversed repair rate and right truncated moments of the original and weighted rv for the generalized Pearson system of distributions.

Theorem 2.11: Let X be a non-negative, non-degenerate rv with density function $f(t)$ and suppose that $w(t)=t^\alpha$. Then the pdf of X is a member of generalized Pearson system of distributions of the form (1.32) with $b_0=0$ and $\lim_{t\to a}(b_1 t+b_2 t^2)f(t)=0$ if and only if

$$\frac{\lambda^S(t)}{\lambda(t)} = K\left(\frac{\bar{m}_S(t)+\mu_\alpha+k_1\bar{m}_S^2(t)}{\bar{m}(t)+\mu+k_2\bar{m}^2(t)}\right) \tag{2.50}$$

where $\bar{m}_s(t)=E(X_s|X_s\le t), \bar{m}_s^2(t)=E(X_s^2|X_s\le t)$, K, k_1 and k_2 are real constants such that $Kk_1=k_2$ provided $(a_1+(\alpha+2)b_2)\neq 0$ and $(a_1+2b_2)\neq 0$.

Proof: Let the pdf of X be a member of generalized Pearson system (1.32) with $b_0=0$. Then by using (2.39) and integrating between the limits a to t, we have

$$(b_1 t+b_2 t^2)\lambda^s(t)=(a_0+(\alpha+1)b_1)+(a_1+(\alpha+2)b_2)\bar{m}_s(t)+a_2\bar{m}_s^2(t). \tag{2.51}$$

Again from (1.32) and integrating between the limits a to t, we get

$$(b_1t + b_2t^2)\lambda(t) = (a_0 + b_1) + (a_1 + 2b_2)\bar{m}(t) + a_2\bar{m}^2(t). \qquad (2.52)$$

From (2.51) and (2.52), we obtain (2.50) with $K = \frac{(a_1 + (\alpha+2)b_2)}{(a_1 + 2b_2)}$, $\mu_\alpha = \frac{(a_0 + (\alpha+1)b_1)}{(a_1 + (\alpha+2)b_2)}$, $k_1 = \frac{a_2}{(a_1 + (\alpha+2)b_2)}$, $\mu = \frac{(a_0 + b_1)}{(a_1 + 2b_2)}$ and $k_2 = \frac{a_2}{(a_1 + 2b_2)}$ such that $Kk_1 = k_2$, $(a_1 + (\alpha+2)b_2) \neq 0$ and $(a_1 + 2b_2) \neq 0$.

Conversely assume (2.50) holds, then using (2.13) we get

$$t^\alpha\left(\int_a^t xf(x)dx + \mu F(t) + k_2\int_a^t x^2 f(x)dx\right)$$

$$= K\left(\int_a^t x^{\alpha+1} f(x)dx + \mu_\alpha \bar{m}_\alpha(t)F(t) + k_1\int_a^t x^{\alpha+2} f(x)dx\right). \qquad (2.53)$$

By twice differentiating both sides of (2.53) with respect to t and simplifying and using the condition $Kk_1 = k_2$ we obtain the form (1.32).

Theorem 2.12: The pdf of a rv X is a member of Pearson family (1.30) with $b_0 = 0$ and $\lim_{t\to a}(b_1t + b_2t^2)f(t) = 0$ if and only if

$$\frac{\lambda^S(t)}{\lambda(t)} = \bar{K}\left(\frac{\bar{m}_S(t) + \mu_\alpha}{\bar{m}(t) + \mu}\right) \qquad (2.54)$$

where $\bar{K}$ is a real constant provided $(1 - (\alpha+2)b_2) \neq 0$ and $(1 - 2b_2) \neq 0$.

Proof: The proof is similar to that of the Theorem 2.11.

2.5 Finite mixture models

Finite mixtures of distributions have provided a mathematical based approach to the statistical modeling of a wide variety of random phenomena. Because of their usefulness as an extremely flexible method of modeling, finite mixture models have continued to receive interesting attention over the years. There is variety of fields such as biology, genetics, medicine, astronomy, psychiatry, economics, engineering, and marketing etc. in which finite mixture models have been successfully applied. A simplest form of the finite mixture model is defined as

$$f(t) = pf_1(t) + (1-p)f_2(t) \tag{2.55}$$

where $f_i(x)$; $i = 1,2$ are component densities of the mixture and p (where $0 \le p \le 1$) is called the mixing proportion or weight. For more details on finite mixture models, we refer to Mc Lachlan and Peel (2001). Then the corresponding maintainability and reversed hazard rate functions for the model (2.55) is given by

$$F(t) = pF_1(t) + (1-p)F_2(t) \tag{2.56}$$

$$\lambda(t) = p\lambda_1(t) + (1-p)\lambda_2(t). \tag{2.57}$$

Under length-biased sampling, equations (2.55), (2.56) and (2.57) becomes

$$f^L(t) = pf_1^L(t) + (1-p)f_2^L(t) \tag{2.58}$$

$$F^L(t) = pF_1^L(t) + (1-p)F_2^L(t) \tag{2.59}$$

$$\lambda^L(t) = p\lambda_1^L(t) + (1-p)\lambda_2^L(t) \tag{2.60}$$

where $f_i^L(t) = \dfrac{tf_i(t)}{\mu_i}$, $F_i^L(t) = \dfrac{\bar{m}_i(t)F_i(t)}{\mu_i}$, $\lambda_i^L(t) = \dfrac{t\lambda_i(t)}{\bar{m}_i(t)}$; $i = 1,2$ respectively.

Now we prove a characterization theorem for exponential, Lomax or beta densities.

Theorem 2.13: The following relationships

a) $$\frac{F^L(t)}{F(t)} = 1 - t(1+Ct)\lambda(t) \tag{2.61}$$

b) $$\lambda^L(t) = \frac{t(1+Ct)\left(\lambda'(t) + \lambda^2(t)\right) + 2Ct\lambda(t)}{\left(t(1+Ct)\lambda(t) - 1\right)} \tag{2.62}$$

provided $t(1+Ct)\lambda(t)-1\neq 0$ is satisfied for all $t\in(a,b)$ if and only if the component densities are

$$f_i(t)=q_i p(1+pt)^{-(q_i+1)},\ t>0,q_i,p>0\ ;\ i=1,2\ ; \tag{2.63}$$

$$f_i(t)=\lambda_i\exp(-\lambda_i t),\ a_i>0,t>0\ ;\ i=1,2\ ; \tag{2.64}$$

$$f_i(t)=d_i R(1-Rt)^{d_i-1},\ 0<t<\frac{1}{R},d_i,R>0,\ i=1,2\ ; \tag{2.65}$$

according as $C>0$, $C=0$ or $C<0$.

Proof: To prove (a), from the model (2.65), we have

$$\bar{m}_i(t)=\frac{1}{R(d_i+1)}\left[1-t(1+Rt)\lambda_i(t)\right],\ i=1,2\,. \tag{2.66}$$

Using (2.66), the relation (2.59) becomes

$$\frac{F^L(t)}{F(t)}=p\left(1-t(1+Rt)\lambda_1(t)\right)+(1-p)\left(1-t(1+Rt)\lambda_2(t)\right),$$

which is the form (2.61). The proofs of other two models are similar.

To prove the converse part, assume that (2.61) holds. Using (2.57) and the definition of right truncated moment, we have

$$\frac{p}{\mu_1 F_1(t)}\int_a^t xf_1(x)dx+\frac{(1-p)}{\mu_2 F_2(t)}\int_a^t xf_2(x)dx=1-t(1+Ct)\left(p\lambda_1(t)+(1-p)\lambda_2(t)\right). \tag{2.67}$$

Equating coefficients of p and constants in (2.67), we get

$$\frac{1}{\mu_1 F_1(t)}\int_a^t xf_1(x)dx-\frac{1}{\mu_2 F_2(t)}\int_a^t xf_2(x)dx=-t(1+Ct)\left(\lambda_1(t)-\lambda_2(t)\right) \tag{2.68}$$

and

$$\frac{1}{\mu_2 F_2(t)}\int_a^t xf_2(x)dx=1-t(1+Ct)\lambda_2(t)\,. \tag{2.69}$$

Substituting (2.69) in (2.68), and on simplification we obtain

$$\frac{1}{\mu_1 F_1(t)} \int_0^t x f_1(x)dx = 1 - t(1+Ct)\frac{f_1(t)}{F_1(t)}. \quad (2.70)$$

Multiply both sides of (2.70) by $F_1(t)$ and on differentiation using the assumption $\lim_{t \to a} tf(t) = 0$, we get

$$\frac{f_1'(t)}{f_1(t)} = \frac{-1(1+2C\mu_1)}{\mu_1(1+Ct)}. \quad (2.71)$$

Proceeding similar lines, from (2.69), we obtain

$$\frac{f_2'(t)}{f_2(t)} = \frac{-1(1+2C\mu_2)}{\mu_2(1+Ct)}. \quad (2.72)$$

Integrating (2.71) and (2.72) with respect to t and on simplification, we get (2.63), (2.64) and (2.65) respectively according as $C > 0$, $C = 0$ and $C < 0$.

To prove (b), taking logarithm on both sides of (2.61) and differentiating with respect to t obtain the required form (2.62).

Conversely assume (2.62) holds. Now using the definition of $\lambda^L(t)$ and (2.61), we have

$$\lambda^L(t) = \frac{f^L(t)}{F^L(t)} = \frac{f^L(t)}{F(t)}\frac{F(t)}{F^L(t)} = \frac{t\lambda(t)}{\mu\left(1 - t(1+Ct)\lambda(t)\right)}. \quad (2.73)$$

Also from the definition of $\lambda^L(t)$, and using (2.73), we have

$$\left(1 - t(1+Ct)\lambda(t)\right) = -\frac{\bar{m}(t)}{\mu}. \quad (2.74)$$

Differentiating (2.2) with respect to t, we get

$$f'(t) = F(t)\left(\lambda'(t) + \lambda^2(t)\right). \quad (2.75)$$

Substituting (2.74) and (2.75) in (2.62), we obtain

$$(2C\mu + 1)f(t) = -\mu(1+Ct)f'(t). \quad (2.76)$$

Simplifying (2.76), we get (2.63), (2.64) and (2.65) according to the values of C. This completes the proof of the theorem.

CHAPTER THREE

LOG ODDS RATE[2]

3.1 Introduction

The failure rate/hazard rate is one of the fundamental elements of reliability theory and therefore in many practical situations it has been considered as a useful measure in modeling statistical data to derive appropriate model. Based on the physical properties of the component, the monotone behavior of the failure pattern is also an effective method to identify the underlying model.

Recently, with the need of high reliability of the components, non-monotone hazard or failure rates has also been played an important role in the study of engineering reliability and biological survival analysis. The important distributions such as lognormal, Burr, Inverse Gaussian and truncated normal are appropriate in such situations. The use of odds ratio and proportional odds is becoming more common in the field of reliability or survival analysis when the data exhibits non-proportional hazards (see Kirmani and Gupta (2001)). However, there are certain other situations in which the survival data indicate a

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[2] Some of the results in this Chapter have been published entitled "Characterizations of distributions using log odds rate", *Statistics*, 41(5), 443-451, *see* Sunoj, Sankaran and Maya (2007).

non-monotone failure rate, and then the modeling by either proportional hazard or proportional odds may be inappropriate for the description of the situation of failure.

Accordingly, it has been identified recently that log odds rate (LOR) is a useful measure to model statistical data that shows a non-monotone failure rate (see Wang et al. (2003)). A formal definition of LOR is as follows. Let X be a random variable representing the lifetime of a component/system, $F(t)$ is the cumulative distribution function (cdf) and $R(t) = 1 - F(t)$ is the reliability function, then the log odds function is

$$LO(t) = \ln \frac{F(t)}{R(t)} = \ln F(t) - \ln R(t). \tag{3.1}$$

Wang et al. (2003) have shown that the distributions that are non-monotone in terms of failure rate are monotone in terms of LOR in $\log t$ or $\log(\log t)$. They established some bounds on reliability based on increasing LOR and characterized logistic distribution in terms of constant LOR.

In view of the usefulness of LOR for modeling statistical data that exhibits non-monotone failure rate, the present chapter focuses attention to examine the relationships between LOR and various reliability measures such as hazard rate and reversed hazard rate in the context of repairable systems. Some families of distributions are characterized and discuss the properties and applications of log odds ratio in weighted models. Further we extend this concept to the bivariate set up and study its properties.

3.2 Properties and characterizations

In this section, we discuss some properties of LOR and characterize some families of distributions viz. general family of distributions, Burr, Pearson and log exponential models.

From the definition of log odds function (3.1),

$$\frac{F(t)}{R(t)} = \exp\left(LO(t)\right)$$

or equivalently,

$$F(t) = \frac{\exp\left(LO(t)\right)}{1+\exp\left(LO(t)\right)}. \qquad (3.2)$$

Thus the log odds function determines the distribution uniquely through the relation (3.2). Then log odds rate

$$\psi(t) = LO'(t) = \frac{f(t)}{F(t)R(t)}. \qquad (3.3)$$

As mentioned in the previous chapter, reliability and maintainability are important measures to study the effectiveness of systems/components. The major difference between these two measures is that reliability is the probability that a component has survived (or does not failure) in a particular time, whereas maintainability is the probability that required maintenance will be successfully completed in a given time period. Let Y denotes the repair time of a component and $\lambda_Y(t)$ be the corresponding reversed repair rate. When X and Y are independent and identically distributed (i.i.d.) random variables, using the definitions of hazard and reversed repair rate, the LOR (3.3) becomes

$$\psi(t) = \lambda(t) + h(t). \qquad (3.4)$$

Therefore $\psi(t)$ reduces to the sum of reversed repair rate and failure rate. One important property (3.4) posses is that even if the survival data shows a non-monotone failure rate, the log odds rate might be monotone. For various properties of $\psi(t)$, one could refer to Wang et al. (2003).

Consider a random variable X with the support of (a,b) with an absolutely continuous cdf $F(t)$, the system of distributions, introduced by Burr (1942), is given by

$$f(t) = F(t)\left(1-F(t)\right)k(t) \qquad (3.5)$$

where $k(t)$ is some convenient function, which must be non-negative in $0 \le F(t) \le 1$ and the range of X. The solution to this differential equation, for given $k(t)$ is obtained as

$$F(t) = \left(1+\exp\left(-K(t)\right)\right)^{-1}$$

where $K(t)=\int_a^t k(u)du$ with $\lim_{t\to a} K(t)=-\infty$ and $\lim_{t\to b} K(t)=\infty$. Therefore $k(t)$ uniquely determine the df. From (3.5), we have

$$dF(t)\left(\frac{1}{F(t)}+\frac{1}{R(t)}\right)=k(t)dt$$

i.e.
$$\lambda(t)+h(t)=k(t). \tag{3.6}$$

Equations (3.4) and (3.6) together implies that

$$\psi(t)=k(t).$$

Hence for Burr family of distributions, $k(t)$ directly gives the log odds rate and vice versa.

We now prove a characterization theorem for Pearson family of distributions using the relation connecting LOR and the conditional expectations.

Theorem 3.1: Let X be a rv having an absolutely continuous df $F(t)$ with the support of (a,b), a subset of the real line. Assume that $E(X)<\infty$, $m(t)=E(X|X>t)$ and $\bar{m}(t)=E(X|X\le t)$ denotes the conditional expectations of X. Then the relationship

$$m(t)=\bar{m}(t)+(c_0+c_1t+c_2t^2)\psi(t) \tag{3.7}$$

holds for all $t\in(a,b)$ if and only if the pdf of X satisfies the equation (1.30).

Proof: The family of distributions (1.30) is characterized by the identity

$$m(t)=\mu+(c_0+c_1t+c_2t^2)h(t) \tag{3.8}$$

where $\mu=E(X)$ (see Nair and Sankaran (1991)). One can also establish that for the family (1.30),

$$\bar{m}(t)=\mu-(c_0+c_1t+c_2t^2)\lambda(t) \tag{3.9}$$

(see Navarro and Ruiz (2004) and Nair et al. (2005)).

From (3.8) and (3.9), we get

$$m(t) = \bar{m}(t) + (c_0 + c_1 t + c_2 t^2)\left(h(t) + \lambda(t)\right)$$

which yields (3.7).

Conversely, assume that (3.7) holds, multiplying (3.7) by $F(t)R(t)$ and on simplification we get,

$$F(t)\int_t^b x f(x)dx = R(t)\int_a^t x f(x)dx + (c_0 + c_1 t + c_2 t^2) f(t) . \qquad (3.10)$$

Differentiating (3.10) with respect to t, and simplifying we obtain the result (1.30). This completes the proof.

Examples: Here we consider some of the important members of the Pearson family and their respective forms (3.7).

1. Normal: $f(t) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right\}$; $-\infty < t < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$.

Comparing with equation (1.30), we have $c_0 = \sigma^2, c_1 = 0$ and $c_2 = 0$. Then equation (3.7) becomes

$$m(t) = \bar{m}(t) + \sigma^2 \psi(t) .$$

2. Beta: $f(t) = \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1}$; $0 < t < 1$, $a,b > 0$.

Here $c_0 = 0$, $c_1 = \frac{1}{(a+b)}$ and $c_2 = \frac{-1}{(a+b)}$, equation (3.7) yields

$$m(t) = \bar{m}(t) + \frac{t(1-t)}{(a+b)}\psi(t) .$$

3. Gamma: $f(t) = \frac{m^p}{\Gamma(p)} t^{p-1}\exp(-mt)$; $0 < t < \infty$, $m, p > 0$.

In this case, $c_0 = 0, c_1 = \frac{1}{m}$ and $c_2 = 0$. Then (3.7) becomes

$$m(t) = \bar{m}(t) + \frac{t}{m}\psi(t).$$

Theorem 3.2: The df $F(t)$ of a rv X belong to the general family of distributions (1.31) if and only if it satisfies the relationship

$$m(t) = \bar{m}(t) + g(t)\psi(t). \tag{3.11}$$

Proof: For the general family of distributions (1.31), we have

$$m(t) = \mu + g(t)h(t) \tag{3.12}$$

(see Ruiz and Navarro (1994)).

Similarly from (2.25), the right truncated moment function of the family (1.31),

$$\bar{m}(t) = \mu - g(t)\lambda(t). \tag{3.13}$$

Now (3.12) and (3.13) together implies (3.11). The proof of the converse part is similar to that of the Theorem 3.1.

Next we prove a characterization theorem using $\psi(t)$ for the one parameter log exponential family. Let $m_C(t) = E\left(\frac{XC'(X)}{C(X)} \middle| X > t\right)$ and $\bar{m}_C(t) = E\left(\frac{XC'(X)}{C(X)} \middle| X < t\right)$ and $E\left(\frac{XC'(X)}{C(X)}\right) < \infty$.

Theorem 3.3: Assume that $\lim_{\substack{t \to a \\ t \to b}} C(t)t^{\theta+1} = 0$. Then the distribution of X belongs to one parameter log exponential family (1.29) if and only if

$$m_C(t) = \bar{m}_C(t) - t\psi(t). \tag{3.14}$$

Proof: For the family (1.29), we have

$$R(t)=\frac{-C(t)t^{\theta+1}}{A(\theta)(\theta+1)}-\frac{1}{A(\theta)(\theta+1)}\int_t^b C'(x)x^{\theta+1}dx$$

or

$$m_C(t)=-th(t)-(\theta+1)\,. \qquad (3.15)$$

Similarly, one can obtain the df of the log exponential family (1.29) as

$$F(t)=\frac{C(t)t^{\theta+1}}{A(\theta)(\theta+1)}-\frac{1}{A(\theta)(\theta+1)}\int_t^b C'(x)x^{\theta+1}dx$$

or

$$\bar{m}_C(t)=t\lambda(t)-(\theta+1)\;. \qquad (3.16)$$

Combining (3.15) and (3.16), we obtain the required form (3.14). The converse part is straightforward.

3.3 Weighted models

In this section we examine the application of LOR in the context of weighted models. Denoting $R^w(t)=P(X_w>t)$, the survival function of the weighted rv X_w, then the log odds function denoted by $LO^w(t)$ is given by

$$LO^w(t)=\ln\left(\frac{F^w(t)}{R^w(t)}\right)=\ln F^w(t)-\ln R^w(t)\,. \qquad (3.17)$$

But it can be obtained directly from the relations (1.1) and (2.8), as

$$R^w(t)=\frac{m_w(t)}{\mu_w}R(t) \qquad (3.18)$$

where $m_w(t)=E\left(w(X)\middle|X>t\right)$, is the conditional mean of $w(X)$. From (2.8), (3.1), (3.17) and (3.18), the log odds function becomes

$$LO^{w}(t) = LO(t) + \ln\left(\frac{\bar{m}_w(t)}{m_w(t)}\right).$$

The corresponding weighted log odds rate is given by

$$\psi^{w}(t) = LO^{w'}(t) = \frac{d}{dt}LO^{w}(t) = \lambda^{w}(t) + h^{w}(t)$$

where $\lambda^{w}(t)$ and $h^{w}(t)$ are the reversed hazard rate and hazard rate of the rv X_w respectively. Using (1.1), (2.8), (3.3) and (3.18), we obtain

$$\psi^{w}(t) = \frac{f^{w}(t)}{F^{w}(t)R^{w}(t)} = \frac{w(t)\mu_w}{\bar{m}_w(t)m_w(t)}\psi(t). \tag{3.19}$$

In view of the form-invariance property for families (1.29) and (1.30), the analogous statements for Theorems 3.1 and 3.3 in the context of weighted models are immediate, which are stated as follows.

Theorem 3.4: Let X_S be a size-biased rv associated to X with $w(t) = t^{\alpha}, \alpha > 0$. Then the pdf of X is a member of the Pearson system of distributions (2.48) with $c_0 = 0$ and $\lim_{t \to a}(c_1 t + c_2 t^2) f(t) = 0$ if and only if

$$m^{S}(t) = \bar{m}^{S}(t) + (v_1 t + v_2 t^2)\psi^{S}(t) \tag{3.20}$$

where $m^{S}(x) = E(X_S | X_S > t)$, $\bar{m}^{S}(t) = E(X_S | X_S \le t)$, and $\psi^{S}(t) = \frac{t^{\alpha}\mu_{\alpha}}{\bar{m}^{\alpha}(t)m^{\alpha}(t)}$.

Proof: Under the weight function $w(t) = t^{\alpha}$ and $c_0 = 0$, the Pearson system of distributions is characterized by the relationship,

$$m^{S}(t) = \mu_{\alpha} + \frac{(k_1 t + k_2 t^2)}{(1 - 2k_2)} h^{S}(t) \tag{3.21}$$

and

$$\bar{m}^{S}(t) = \mu_{\alpha} - \frac{(k_1 t + k_2 t^2)}{(1 - 2k_2)} \lambda^{S}(t) \tag{3.22}$$

where $\mu_\alpha = \dfrac{k_1 - d_1}{1-2k_2}$, $1-2k_2 \neq 0$. Using (3.21) and (3.22), we obtain the relationship (3.20).

Conversely, assume that (3.20) holds. Then multiplying (3.20) by $R^S(t)F^S(t)$, we get

$$F^S(t)\int_t^b xf^S(x)dx = R^S(t)\int_a^t xf^S(x)dx - \int_a^t xf^S(x)dx + (v_1 t + v_2 t^2)f^S(t) \qquad (3.23)$$

Differentiating (3.23) with respect to t and on simplification, we obtain

$$\left[(\mu_\alpha - v_1) - (1+2v_2)t\right]f^S(t) = (v_1 t + v_2 t^2)f^{S'}(t)$$

which on further simplification, yields (2.48) with $d_1 = \left(\dfrac{v_1 - \mu_\alpha}{1+2v_2}\right)$ and $k_i = \dfrac{v_i}{(1+2v_2)}$; $i = 1,2$ provided $(1+2v_2) \neq 0$.

Theorem 3.5: Assume that $\lim_{t\to b} c(t)t^{\theta+1} = 0$, with $w(t) = t^\alpha, \alpha > 0$, the relationship

$$m_g^S(t) = \bar{m}_g^S(t) - t\psi^S(t)$$

if and only if the pdf of X_S belongs to the one parameter log exponential family (1.29), where $m_g^S(t) = E\left(\dfrac{X_S C'(X_S)}{C(X_S)} \middle| X_S > t\right)$, $\bar{m}_g^S(t) = E\left(\dfrac{X_S C'(X_S)}{C(X_S)} \middle| X_S < t\right)$ and $E\left(\dfrac{X_S C'(X_S)}{C(X_S)}\right) < \infty$.

Proof: When $w(t) = t^\alpha$, (1.29) becomes $f^S(t) = \dfrac{t^{\theta+\alpha}C(t)}{A(\theta+\alpha)}$. Since $\mu_\alpha = \dfrac{A(\theta+\alpha)}{A(\theta)}$, the rest of the proof is similar to the proof of the Theorem 3.3.

3.4 Bivariate case

In this section, we extend the concept of log odds function and log odds rate to higher dimensions. We confine our study to the bivariate setup. The extensions to higher dimensions are direct. Let $X=(X_1,X_2)$ be a bivariate random vector in the support of $R_2^+=\{(t_1,t_2)|0<t_i<\infty\}$; $i=1,2$ with an absolutely continuous distribution function $F(t_1,t_2)$ and survival function $R(t_1,t_2)$ and pdf $f(t_1,t_2)$. Let $F_i(t_i)$ and $R_i(t_i)$; $i=1,2$ denote the marginal distribution function and survival function of X_i. Let $f_i(t_i)$ be the density function of X_i. Then we propose the bivariate log-odds function by

$$L=LO(t_1,t_2)=\ln\frac{F(t_1,t_2)}{R(t_1,t_2)}=\ln F(t_1,t_2)-\ln R(t_1,t_2) \tag{3.24}$$

which gives

$$\frac{F(t_1,t_2)}{R(t_1,t_2)}=\exp\left(LO(t_1,t_2)\right). \tag{3.25}$$

The corresponding LOR is defined as a vector

$$\psi(t_1,t_2)=\left(\psi_1(t_1,t_2),\psi_2(t_1,t_2)\right) \tag{3.26}$$

where

$$\psi_i(t_1,t_2)=\frac{\partial LO(t_1,t_2)}{\partial t_i};\ i=1,2. \tag{3.27}$$

Using the bivariate vector failure rate due to Johnson and Kotz (1975) and bivariate reversed hazard rate due to Roy(2002), (3.27) becomes

$$\psi_i(t_1,t_2)=\lambda_i(t_1,t_2)+h_i(t_1,t_2) \tag{3.28}$$

where $\lambda_i(t_1,t_2)=\frac{\partial}{\partial t_i}\ln F(t_1,t_2)$ and $h_i(t_1,t_2)=-\frac{\partial}{\partial t_i}\ln R(t_1,t_2)$; $i=1,2$, are the i^{th} components of the reversed hazard rates and failure rates respectively.

Examples: Here we consider some bivariate densities having simple vector valued log odds rate.

1. Bivariate normal:

$$f(t_1,t_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho2)}\left[\frac{t_1^2}{\sigma_1^2}-\frac{2\rho t_1t_2}{\sigma_1\sigma_2}+\frac{t_2^2}{\sigma_2^2}\right]\right\}; \tag{3.29}$$

$$-\infty<t_1,t_2<\infty,\sigma_1,\sigma_2>0,|\rho|<1.$$

Taking logarithm on both sides of (3.29) and differentiating with respect to t_j, we obtain

$$(1-\rho^2)\sigma_i^2\sigma_j \frac{\partial f}{\partial t_i} = (\rho\sigma_i t_j - \sigma_j t_i) f, \; i \neq j = 1,2. \tag{3.30}$$

Now integrating (3.30) twice between the limits t_i to b_i and t_j to b_j, $i \neq j = 1,2$, we get

$$(1-\rho^2)\sigma_i^2\sigma_j h_i(t_1,t_2) = \sigma_j m_i(t_1,t_2) - \rho\sigma_i m_j(t_1,t_2), \; i \neq j = 1,2 \tag{3.31}$$

where $m_i(t_1,t_2) = E\left(X_i \middle| X_i > t_i, X_j > t_j\right)$, $i \neq j = 1,2$. Similarly integrating (3.30) twice between the limits a_i to t_i and a_j to t_j, $i \neq j = 1,2$, we obtain

$$(1-\rho^2)\sigma_i^2\sigma_j \lambda_i(t_1,t_2) = \rho\sigma_i n_j(t_1,t_2) - \sigma_j \bar{m}_i(t_1,t_2), \; i \neq j = 1,2 \tag{3.32}$$

where $\bar{m}_i(t_1,t_2) = E\left(X_i \middle| X_i < t_i, X_j < t_j\right), i \neq j = 1,2$. Now adding (3.31) and (3.32), we get

$$(1-\rho^2)\sigma_i^2\sigma_j \psi_i(t_1,t_2) = \sigma_j\left(m_i(t_1,t_2) - \bar{m}_i(t_1,t_2)\right) - \rho\sigma_i\left(\bar{m}_j(t_1,t_2) - m_j(t_1,t_2)\right), \; i \neq j = 1,2.$$

Example 2: Bivariate exponential

The joint density function of the exponential conditional due to Arnold and Strauss (1988) is

$$f(t_1,t_2) = C\exp\left(-\alpha_1 t_1 - \alpha_2 t_2 - \beta t_1 t_2\right), \; t_1, t_2, \alpha_1, \alpha_2 > 0, \beta \geq 0 \tag{3.33}$$

where $C = -\beta \exp\left[\dfrac{-\alpha_1\alpha_2}{\beta}\right]\left(E_i\left(\dfrac{-\alpha_1\alpha_2}{\beta}\right)\right)^{-1}$.

Now proceeding in the similar manner as above, the identity connecting the vector valued log odds rate and the conditional moments for (3.33) becomes

$$\psi_i(t_1,t_2) = \beta\left(m_j(t_1,t_2) - \bar{m}_j(t_1,t_2)\right), \; i = 1,2 \text{ and } j = 3-i.$$

Theorem 3.6: The relationship

$$LO_{X_1,X_2}(t_1,t_2) = LO_{X_1}(t_1) + LO_{X_2}(t_2) \tag{3.34}$$

holds for all t_1, t_2, if and only if X_1 and X_2 are independent.

Proof: Suppose (3.34) holds, then

$$\frac{F(t_1,t_2)}{R(t_1,t_2)}=\frac{F_1(t_1)}{R_1(t_1)}\frac{F_2(t_2)}{R_2(t_2)}$$

which is equivalent to

$$F(t_1,t_2)(1-F_1(t_1))(1-F_2(t_2))=F_1(t_1)F_2(t_2)(1-F_1(t_1)-F_2(t_2)+F(t_1,t_2)).$$

On simplification, we obtain

$$F(t_1,t_2)(1-F_2(t_2)-F_1(t_1))=F_1(t_1)F_2(t_2)(1-F_1(t_1)-F_2(t_2))$$

or

$$F(t_1,t_2)=F_1(t_1)F_2(t_2).$$

which proves the result. The converse part is straightforward.

Remark: Theorem 3.6 can be useful to test the independence among the variables. This might be helpful in reliability analysis to study the dependence structure between the components of a system.

CHAPTER FOUR

SOME MEASURES OF UNCERTAINTY IN PAST LIFETIME*

4.1 Introduction

It is well known that the knowledge and use of various methods for information coding and transmission play a vital role in understanding and modeling many aspects of biological system features. As explained in the first Chapter, Shannon's entropy plays an important role in the context of information theory. Since Shannon's entropy is not sufficient enough for the study of the remaining life of system that have survived for some units of time, Ebrahimi and Pellerey (1995) proposed a new measure of uncertainty called residual entropy, a measure that plays a vital role in left truncated data sets. However, Di Crescenzo and Longobardi (2002) showed that in many realistic situations, uncertainty is not necessarily related to the future but can also be refer to the past. For instance, if at time t, a system that is observed only at certain pre assigned inspection times is found to be down, then the uncertainty of the system life relies on the past, *i.e.*, on which instant in $(0,t)$ it has failed. Based on this idea, Di Crescenzo and Longobardi (2002) introduced the past entropy over $(0,t)$. They showed the necessity of past entropy using an example and discussed its relationship with residual entropy and studied the monotonic behaviors of it. Let the rv X denote the lifetime of a component/system or of living organism, then past entropy of X at time t is defined as

* Some of the results in this Chapter have been communicated to two International Journals.

$$\bar{H}(t) = -\int_0^t \frac{f(x)}{F(t)}\left(\log\frac{f(x)}{F(t)}\right)dx. \tag{4.1}$$

Note that (4.1) can be rewritten as

$$\bar{H}(t) = 1 - \frac{1}{F(t)}\int_0^t (\log\lambda(x))\, f(x)dx. \tag{4.2}$$

Recently, Nanda and Paul (2006) proved some ordering properties based on past entropy and some sufficient conditions for these orders to hold. They also introduced a non parametric class based on past entropy and studied its properties and examined it under the discrete setup. However, Di Crescenzo and Longobardi (2006) introduced the notion of weighted residual and past entropies and studied its properties and monotone behavior of it. In view of the usefulness of measure of uncertainty (4.1) in past time, in the present chapter, we further explore the same and also define a new conditional measure and study its properties. In Section 4.4, we study Renyi's entropy for the past lifetime and proved some theorems arising out of it. Further, in Section 4.5, we extend these concepts in the context of weighted models and also study some ordering and aging properties based on these measures. In the final two sections, we discuss some measures of discrimination proposed by Di Crescenzo and Longobardi (2004) and Asadi et al. (2005) and study its applications in the context of weighted models.

4.2 Properties

Differentiating (4.1) with respect to t, we get

$$\bar{H}'(t) = \lambda(t)\left[1 - \bar{H}(t) - \log\lambda(t)\right]. \tag{4.3}$$

Nanda and Paul (2006) proved that if X has absolutely continuous distribution function $F(t)$ and an increasing past entropy $\bar{H}(t)$, then $\bar{H}(t)$ uniquely determines $F(t)$.

Next we prove a characterization theorem for the power distribution using the functional relationships between reversed hazard rate and past entropy.

Theorem 4.1: Let X be a non-negative rv admitting an absolutely continuous df such that $E(X) < \infty$ and let $\bar{H}(t)$ be defined as in (4.1). The relationship

$$\bar{H}(t) = k - \log \lambda(t)\,; \quad 0 < k < 1 \tag{4.4}$$

where k is a constant, holds for all $t > 0$ if and only if X follows power distribution with df

$$F(t) = \left(\frac{t}{b}\right)^c,\ 0 < t < b,\ b, c > 0. \tag{4.5}$$

Proof: Suppose the relation (4.4) holds. Differentiating (4.4) with respect to t implies that

$$\bar{H}'(t) = -\frac{\lambda'(t)}{\lambda(t)}. \tag{4.6}$$

From (4.6) and (4.3) we get

$$\lambda'(t) + \lambda^2(t)(1-k) = 0. \tag{4.7}$$

Divide each term of (4.7) by $\lambda^2(t)$, we get

$$\frac{\lambda'(t)}{\lambda^2(t)} = k - 1. \tag{4.8}$$

Putting $u(t) = \frac{1}{\lambda(t)}$, (4.8) becomes

$$-\frac{du}{dt} = (k-1). \tag{4.9}$$

Solving the differential equation (4.9), we obtain $u(t) = At + B$, where $A = (1-k)$. This implies $\lambda(t) = \frac{1}{(At+B)}$, now from the uniqueness property of reversed hazard rate, we obtain the required result (4.5).

Conversely when (4.5) holds, substituting (4.5) in (4.2) and on direct calculation, we obtain (4.4) with $k = \frac{(c-1)}{c}$.

The following Theorems characterize the exponential distribution and exponential family of distributions using the possible relationships between RHR, EIT and the past entropy.

Theorem 4.2: For the rv X considered in Theorem 4.1, with $\lim_{t\to 0} F(t) = 0$, the relation

$$\bar{H}(t) + \log \lambda(t) = -C\bar{r}(t), \tag{4.10}$$

where C(>0) is a constant holds for all $t \geq 0$ if and only if X follows exponential distribution with distribution function (2.18).

Proof: Suppose the relation (4.10) holds, then differentiating (4.10) with respect to t

$$\bar{H}'(t) + \frac{\lambda'(t)}{\lambda(t)} = -C\bar{r}'(t). \tag{4.11}$$

Using (4.3), (1.22) and (4.10), we obtain

$$\lambda'(t) + C\lambda(t) + \lambda^2(t) = 0. \tag{4.12}$$

Now solving (4.12) following the similar steps as that of the Theorem 4.1, we obtain $\lambda(t) = \frac{C\exp(-Ct)}{1-\exp(-Ct)}$. Using the uniqueness property of $\lambda(t)$, we get the required model (2.18). Substitution of (2.18) in (4.2) and by direct calculation we obtain the converse part of the theorem with $C = \lambda$.

Theorem 4.3: Let $\lim_{t\to 0} \log a(t) f(t) = 0$, $\lim_{t\to 0} F(t) = 0$, $\bar{m}_P(t) = E\left(P(X)\middle|X \leq t\right)$ and $E\left(P(X)\right) < \infty$, then the past entropy of a non-negative rv satisfies a relation of the form

$$\bar{H}(t) + \log \lambda(t) = P(t) - \bar{m}_P(t) + \theta\bar{r}(t), \tag{4.13}$$

where $P(t)$ is any function of t holds for all $t \geq 0$ if and only if the pdf of X belongs to exponential family (1.28).

Proof: Assume that (4.13) holds. On differentiating (4.13) with respect to t, we obtain

$$\bar{H}'(t) + \frac{\lambda'(t)}{\lambda(t)} = P'(t) - \bar{m}_P'(t) + \theta\bar{r}'(t). \tag{4.14}$$

From the definition of $\bar{m}_P(t)$, we have

$$\bar{m}_P'(t) = \lambda(t)\left(P(t) - \bar{m}_P(t)\right). \quad (4.15)$$

Using (4.3), (1.22), (4.15) and (4.13) we get (4.14) as

$$\lambda'(t) - \left(P'(t) + \theta\right)\lambda(t) + \lambda^2(t) = 0. \quad (4.16)$$

Dividing each term by $\lambda^2(t)$ and putting $u(t) = \frac{-1}{\lambda(t)}$, (4.16) becomes

$$\frac{du}{dt} + \left(\theta + P'(t)\right)u(t) + 1 = 0. \quad (4.17)$$

Solving the differential equation (4.17), we get $\lambda(t) = \frac{\exp\left(P(t) + \theta t\right)}{\int_0^t \exp\left(P(x) + \theta x\right)dx}$. Now from the uniqueness property of $\lambda(t)$ we obtain (1.28).

Conversely, substitution of (1.28) in (4.1) and on simplification, we get

$$\bar{H}(t) = -\theta\bar{m}(t) - E\left(C(X)\middle|X \le t\right) - D(\theta) - \log F(t). \quad (4.18)$$

Add and subtract $\log f(t)$ in (4.18) yields (4.13) with $P(t) = C(t)$.

Our next results provide characterization theorems for the Pareto I distribution and log exponential family of distributions using a functional relationship between the past entropy and geometric vitality function in past time denoted by $\log \bar{G}(t) = E(\log X \mid X \le t)$.

Theorem 4.4: Let X be a non-negative rv in the support $[k, \infty)$, $k > 0$, admitting an absolutely continuous df such that $E(\log X) < \infty$ and $\log \bar{G}(t) = E\left(\log X \mid X \le t\right)$. Then the relationship

$$\bar{H}(t) + \log \lambda(t) = C \log\left(\frac{\bar{G}(t)}{t}\right), \quad (4.19)$$

where $C > 1$ is a constant holds for all $t \ge 0$ if and only if X follows Pareto I distribution with cdf

$$F(t) = 1 - \left(\frac{k}{t}\right)^c \; ; \; t > k \, , \; k, c > 0 \, . \tag{4.20}$$

Proof: Assuming (4.19) and differentiating with respect to t, we have

$$\bar{H}'(t) + \frac{\lambda'(t)}{\lambda(t)} = C\frac{d}{dt}\left(\log \bar{G}(t)\right) - \frac{C}{t} \, . \tag{4.21}$$

Substituting (4.3), (4.19) and $\frac{d}{dt}\left(\log \bar{G}(t)\right) = \lambda(t)\left(\log t - \log \bar{G}(t)\right)$, and on simplification, (4.21) implies

$$\lambda(t) + \frac{\lambda'(t)}{\lambda(t)} + \frac{C}{t} = 0 \, . \tag{4.22}$$

Now following the similar steps as that of the Theorem 4.2, the solution of the differential equation (4.22) is $\lambda(t) = \frac{(C-1)}{t^k\left(k^{-C+1} - t^{-C+1}\right)}$. From the uniqueness property of $\lambda(t)$, we obtain (4.20).

To prove the converse part, assume (4.20). From a direct calculation we obtain

$$\bar{H}(t) = \log F(t) - \log ck^c + (c+1)\log \bar{G}(t) \, . \tag{4.23}$$

Add and subtract $\log f(t)$ in (4.23) and on simplification, we get (4.19).

Theorem 4.5: For the rv considered in Theorem 4.1, let $\lim_{t \to 0} \log Q(t) f(t) = 0$, $\log \bar{m}_Q(t) = E\left(\log Q(X) \middle| X \le t\right)$ and $E\left(\log Q(X)\right) < \infty$, a relation of the form

$$\bar{H}(t) + \log \lambda(t) = \log Q(t) - \log \bar{m}_Q(t) - \theta \log\left(\frac{\bar{G}(t)}{t}\right) \tag{4.24}$$

where $Q(t)$ is any function of t, holds for all $t \ge 0$ if and only if the pdf of X belongs to log exponential family with probability density function (1.29).

Proof: Assuming (4.24) for all $t \ge 0$, using the similar steps as that of Theorem (4.4), we have the proof.

4.3 Conditional measure of uncertainty for past lifetime

In continuation of the measure of uncertainty of residual lifetime (1.35) proposed by Ebrahimi and Pellerey (1995), Sankaran and Nair (1999) introduced a conditional measure of uncertainty (1.36) which is defined in Chapter 1. Analogous to $M(t)$ defined in (1.36), for a non-negative rv X, we define a conditional measure of uncertainty for the past life as

$$\bar{M}(t) = E\left(-\log f(X) \middle| X \le t\right)$$

$$= -\frac{1}{F(t)} \int_0^t f(x) \log f(x) dx. \tag{4.25}$$

Clearly $\bar{M}(t)$ gives the measure of uncertainty of the past lifetime of a unit. Using (4.25) and (4.1), $\bar{M}(t)$ can be directly related to $\bar{H}(t)$ and $\lambda(t)$ through the following relationships

$$\bar{M}(t) = \bar{H}(t) - \log F(t) \tag{4.26}$$

and

$$\lambda(t) = \bar{H}'(t) - \bar{M}'(t). \tag{4.27}$$

We now give a characterization theorem for the exponential distribution using the conditional measure of uncertainty for the past life defined by (4.25) and the right truncated conditional moment $\bar{m}(t)$.

Theorem 4.6: For a rv X considered in Theorem 4.1 with $\lim_{t \to 0} tf(t) = 0$ and $\bar{M}(t)$ as defined in (4.25). A relation of the form

$$\bar{M}(t) - \frac{1}{\mu} \bar{m}(t) = k \tag{4.28}$$

where k is a constant, is satisfied for all $t \ge 0$ if and only if X have an exponential distribution with distribution function (2.18).

Proof: Assume that the relation (4.28) holds, then by substituting (4.26) and the definition of $\bar{m}(t)$, we get

$$-\frac{1}{F(t)}\int_0^t f(x)\log f(x)dx-\frac{1}{\mu F(t)}\int_0^t xf(x)dx=k. \qquad (4.29)$$

Multiply both sides of (4.29) by $F(t)$ and on differentiation with respect to t using the condition, we obtain $f(t)=c\exp\left(-\frac{t}{\mu}\right)$. Applying the boundary conditions, we have $c=\mu$.

Conversely, when X is specified by exponential distribution (2.18), from direct calculation of $\bar{M}(t)$ using (4.25), we obtain (4.28) with $k=-\log\lambda$.

In the following theorems we prove certain characterizations to some well-known distributions viz power and Pareto I and families of distributions such as exponential and log exponential using the functional form of $\bar{M}(t)$ and $\log\bar{G}(t)$.

Theorem 4.7: Let X be a non-negative rv having an absolutely continuous df with $E(\log X)<\infty$ and $\log\bar{G}(t)$ is defined as in Theorem 4.4. Then a relationship

$$\bar{M}(t)+(c-1)\log\bar{G}(t)=k, \qquad (4.30)$$

where k is a constant, holds for all $t\geq 0$ if and only if X follows power distribution (4.5).

Proof: suppose that the relation (4.30) holds. Using (4.26) and the definition of $\log\bar{G}(t)$ we get

$$\frac{-1}{F(t)}\int_0^t \log f(x)f(x)dx+\frac{(c-1)}{F(t)}\int_0^t \log xf(x)dx=k. \qquad (4.31)$$

Now proceeding the similar steps as that of the Theorem 4.6, the remaining part of the theorem can be proved. A direct substitution of (4.5) in (4.26) gives the converse part of the theorem.

Theorem 4.8: For a rv X defined in Theorem 4.7 with a support $[k,\infty)$, $k>0$, a relation of the form

$$\bar{M}(t)-(c-1)\log\bar{G}(t)=K\,, \tag{4.32}$$

where K is a constant and $c>1$, is satisfied for all $t>0$ if and only if X follows a Pareto I distribution (4.20).

4.4 Renyi's entropy for past lifetime

As pointed out in Chapter I, Renyi's entropy measure for the residual life also being a measure of uncertainty of component. Based on the past life of a system, Asadi et.al (2005) defined the Renyi entropy for the past lifetime $X|X\le t$ as

$$\bar{I}_R(t)=\frac{1}{(1-\beta)}\log\int_0^t\frac{f^{\beta}(x)}{F^{\beta}(t)}dx\,. \tag{4.33}$$

As a measure of uncertainty, $\bar{I}_R(t)$ can be used to describe the physical characteristics of the failure mechanism and so characterization theorems using this concept helps one to determine the lifetime distribution through the knowledge of the form of the Renyi entropy for the past life $\bar{I}_R(t)$. Now (4.33) can be rewritten as

$$(1-\beta)\bar{I}_R(t)=\log\left(\int_0^t f^{\beta}(x)dx\right)-\beta\log F(t)\,. \tag{4.34}$$

The following Theorem characterizes power distribution using the functional relationship between Renyi's past entropy and the reversed hazard rate.

Theorem 4.9: For a rv X defined in Theorem 4.1 with Renyi entropy for past life $\bar{I}_R(t)$ is defined in (4.34), then a relationship

$$\bar{I}_R(t)=K-\log\lambda(t)\,, \tag{4.35}$$

where K is a constant holds if and only if X follows a power distribution with cdf (4.5).

Proof: Assume that (4.35) holds. Using (4.33) and on simplification, (4.35) implies

$$\log\int_0^t\frac{f^{\beta}(x)}{F^{\beta}(t)}dx=K_1-\log\left(\lambda(t)\right)^{(1-\beta)},\ \text{where } K_1=K(1-\beta)\,.$$

i.e.

$$\int_0^t \frac{f^{\beta}(x)}{F^{\beta}(t)}dx = K^* - (\lambda(t))^{(\beta-1)}$$

or

$$\int_0^t f^{\beta}(x)dx = K^* f^{\beta-1}(t)F(t) \text{, where } K^* = \exp(K_1). \tag{4.36}$$

Differentiating (4.36) using the assumption $\lim_{t\to 0} f^{\beta}(t) = 0$ with respect to t, we get

$$f^{\beta}(t) = K^*\left(f^{\beta}(t) + (\beta-1)f^{\beta-2}(t)f'(t)F(t)\right). \tag{4.37}$$

Divide each term of (4.37) by $f^{\beta-2}(t)F^2(t)$ and on simplification using $\lambda'(t) = \frac{f'(t)}{F(t)} - \lambda^2(t)$, we obtain

$$(1-K^*\beta)\lambda^2(t) - K^*(\beta-1)\frac{d}{dt}\lambda(t) = 0. \tag{4.38}$$

Solving the differential equation (4.38), we obtain the required result. The converse part is obtained by direct calculation.

4.5 Weighted models

In this section, we study the usefulness of these uncertainty measures viz past entropy (4.1), conditional measure of uncertainty for the past life (4.25) and the Renyi's entropy for the past life (4.33) in the context of weighted distributions. The mathematical relationships between the weighted and original variables for (4.1), (4.25) and (4.33) are given by

$$\bar{H}^w(t) = 1 - \left[F(t)\bar{m}_w(t)\right]^{-1}\int_0^t w(x)f(x)\log\left(\frac{w(x)\lambda(x)}{\bar{m}_w(x)}\right)dx, \tag{4.39}$$

$$\bar{M}^w(t) = -\left[F(t)\bar{m}_w(t)\right]^{-1}\int_0^t w(x)f(x)\log\left(\frac{w(x)f(x)}{E(w(X))}\right)dx \tag{4.40}$$

and

$$\bar{I}_R^w(t)=\frac{1}{(1-\beta)[\bar{m}_w(t)]^\beta F^{(\beta-1)}(t)}\log\left[E\left(w\beta(X)f^{(\beta-1)}(X)\middle|X\le t\right)\right], \tag{4.41}$$

where $\bar{H}^w(t), \bar{M}^w(t)$ and $I_R^w(t)$ respectively denote the past entropy, conditional measure of uncertainty for the past life and Renyi's entropy for the past life for the weighted rv X_w.

Remark 4.1: When the weight function $w(t)=t^\alpha$, the model reduces to size-biased model.

The following theorem characterizes the exponential and log exponential families of distributions using the weighted conditional measure for past life and the weighted form of $\log \bar{G}(t)$.

Theorem 4.10: Let X_w be a weighted rv with weight function $w(t)$ and $\log \bar{G}^w(t)=E(\log X_w|X_w\le t)$. Assume $\lim_{t\to 0}\log V(t)f(t)=0$, then a relationship

$$\bar{M}^w(t)=\log U(\theta)-\theta\log\bar{G}^w(t)-E\left(\log V(X_w)\middle|X_w\le t\right), \tag{4.42}$$

where U and V are any functions of θ and x_w respectively, satisfies if and only if the pdf of X belongs to one parameter log exponential family (1.29).

Proof: For the one-parameter log exponential family (1.29), we have

$$f^w(t)=\frac{w(t)t^\theta C(t)}{A(\theta)E\left(w(X)\right)}=\frac{t^\theta C^*(t)}{A^*(\theta)},\text{ where } C^*(t)=w(t)C(t) \text{ and } A^*(\theta)=A(\theta)E\left(w(X)\right).$$

Now proceeding the similar steps as that of the Theorem 4.3, we obtain the result.

Theorem 4.11: For a rv considered in Theorem 4.10 and assume that $\lim_{t\to 0}B(t)f(t)=0$. A relation of the form

$$\bar{M}^w(t)+\theta\bar{m}^w(t)=-A(\theta)-E\left(B(X_w)\middle|X_w\le t\right), \tag{4.43}$$

where A and B are any functions of θ and x_w, is satisfied for all $t \geq 0$ if and only if the pdf of X belongs to one parameter exponential family (1.28).

Proof: For the weighted rv $X_w, f^w(t) = \exp\left(\theta t + C^*(t) + D^*(\theta)\right)$, belongs to the exponential family with $C^*(t) = \left(\log w(t)\right) + C(t)$ and $D^*(\theta) = D(\theta) - \log E\left(w(X)\right)$. Rest of the proof is similar to that of Theorem 4.3.

4.6 Some new classes of distributions

Recently Di Crescenzo and Longobardi (2002) observed that even if the Shannon's entropy of two components with lifetimes X and Y are same, the expected uncertainty contained in the conditional density of X given $X \leq t$ (*i.e.*, past entropy of X) is different from that contained in the conditional density of Y given $Y \leq t$ (*i.e.*, past entropy of Y). Motivated from this, Nanda and Paul (2006) defined the following ordering based on past entropy.

Definition 4.1 (Nanda and Paul (2006)): Let X and Y be two random variables denoting the lifetimes of two components. Then X is said to be greater than Y in past entropy order (written as $X \overset{PE}{\geq} Y$) if $\bar{H}_X(t) \leq \bar{H}_Y(t)$ for all $t > 0$.

Definition 4.2 (Nanda and Paul (2006)): A rv X is said to have increasing (decreasing) uncertainty of life (or increasing (decreasing) past entropy) if $\bar{H}(t)$ is increasing (decreasing) in $t \geq 0$.

Theorem 4.12: Let X_w be a weighted rv with the weight function $w(t)$ if (a) $\dfrac{w(t)}{\bar{m}_w(t)}$ is decreasing and (b) X is DRHR, then X_w has increasing past entropy (IPE).

Proof: Using (2.9), and from the conditions (a) and (b) implies $\lambda^w(t)$ is DRHR. Now using Theorem 3.1 in Nanda and Paul (2006) (*i.e.*, If X is DRHR then $X \in$ IPE .), we get X_w is IPE. Nanda and Paul (2006) have shown with an example that IPE property does not imply DRHR property. Using this argument the converse of the Theorem 3.1 does not hold.

Theorem 4.12 can be illustrated by the following example.

Example 4.1: Let X be a non-negative rv following a power distribution with df (4.5) and weight function $w(t)=t$. Then this rv satisfies the conditions given in Theorem 4.14 and hence the past entropy of its weighted version is increasing. Figure 4.1 shows that the increasing nature of past entropy of weighted version of power distribution for $c=2$ and $t\in(0,100)$.

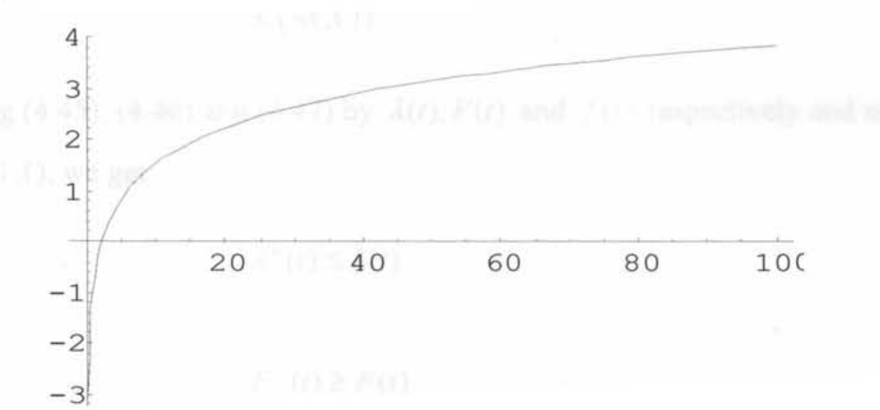

Figure 4.1: Plot of $\bar{H}^{w}(t)$ against $t\in(0,100)$ when $w(t)=t$ and $c=2$

Theorem 4.13: (i) If $w(t)$ is increasing and $w(t)\lambda(t)$ is decreasing then X_w is IPE.

(ii) If $w(t)h(t)$ is decreasing then X_w is IPE.

Proof: Using the theorems given in Bartoszewicz and Skolimowska (2006) (see Chapter 1) and Nanda and Paul (2006, Theorem 3.1), we can prove (i) and (ii).

Example 4.2: Consider a rv having exponential distribution with weight function $w(t)=t$. X satisfies the conditions of the Theorem 4.15 (i) and hence the past entropy of its length-biased rv is increasing.

Theorem 4.14: If $w(t)\le(\ge)E\left(w(X)\right)\le(\ge)E\left(w(X)|X\le t\right)$, then $X\overset{PE}{\ge}(\overset{PE}{\le})X_w$.

Proof: Assume that

$$w(t)\le E\left(w(X)\right)\le E\left(w(X)|X\le t\right), \tag{4.44}$$

holds. Then we have

$$\frac{w(t)}{E\left(w(X)\middle|X \le t\right)} \le 1 \tag{4.45}$$

$$\frac{E\left(w(X)\middle|X \le t\right)}{E\left(w(X)\right)} \ge 1 \tag{4.46}$$

and

$$\frac{w(t)}{E\left(w(X)\right)} \le 1. \tag{4.47}$$

Multiplying (4.45), (4.46) and (4.47) by $\lambda(t), F(t)$ and $f(t)$ respectively and using (2.9), (2.8) and (1.1), we get

$$\lambda^{w}(t) \le \lambda(t) \tag{4.48}$$

$$F^{w}(t) \ge F(t) \tag{4.49}$$

and

$$f^{w}(t) \le f(t). \tag{4.50}$$

Substituting (4.48), (4.49) and (4.50) in (4.1) we get $\bar{H}^{w}(t) \ge \bar{H}(t)$.

The following result is direct from the definitions 4.1 and 4.2.

Result 4.1: If $X \overset{RHR}{\ge} X_w$ and $X \overset{PE}{\ge} X_w$ then

(i) X is IPE implies X_w is IPE.

(ii) X_w is DPE implies X is DPE.

In connection with the ordering based on past entropy, we define the following order based on the conditional measure of uncertainty.

Definition 4.3: Let X and Y be two random variables denoting the lifetimes of two components, then X is said to be greater than Y in conditional measure of uncertainty life order (written as $X \overset{CMUL}{\ge} Y$) if $\bar{M}_X(t) \le \bar{M}_Y(t)$ for all $t > 0$.

Definition 4.4: A df $F(t)$ is said to have increasing (decreasing) conditional measure of uncertainty life if $\bar{M}(t)$ is increasing (decreasing) in $t \geq 0$.

Theorem 4.15: (1) If $F(t)$ is ICMUL, then X has IPE. The converse is not true always.

Proof: The first part is direct from (4.26) and to prove the converse consider the following example

Example 4.3: Let X be a non-negative rv having df

$$F(t) = \begin{cases} \frac{t^2}{2}; 0 \leq t < 1 \\ \frac{t^2+2}{6}; 1 \leq t < 2. \\ 1; t \geq 2 \end{cases}$$

For this distribution, the past entropy $\bar{H}(t)$ and $\bar{M}(t)$ are given by

$$\bar{H}(t) = \begin{cases} \log\left(\frac{t}{2}\right) + \frac{1}{2}; 0 \leq t \leq 1 \\ \log\left(\frac{t^2+2}{6}\right) + \left(\frac{t^2-1}{t^2+2}\right)\log 3 - \left(\frac{t^2}{t^2+2}\right)\log t + \frac{1}{2}; 1 \leq t \leq 2 \\ \frac{1}{2}\log 3 - \frac{2}{3}\log 2 + \frac{1}{2}; t \geq 2 \end{cases}$$

and

$$\bar{M}(t) = \begin{cases} \frac{1}{2} - \log t; 0 \leq t \leq 1 \\ \left(\frac{t^2-1}{t^2+2}\right)\log 3 - \left(\frac{t^2}{t^2+2}\right)\log t + \frac{1}{2}; 1 \leq t \leq 2. \\ \frac{1}{2}\log 3 - \frac{2}{3}\log 2 + \frac{1}{2}; t \geq 2 \end{cases}$$

For this distribution, $\bar{H}(t)$ is increasing in t (see Nanda and Paul (2006)), but $\bar{M}(t)$ is not increasing in $t \in [0,1]$ as shown in figure 4.2.

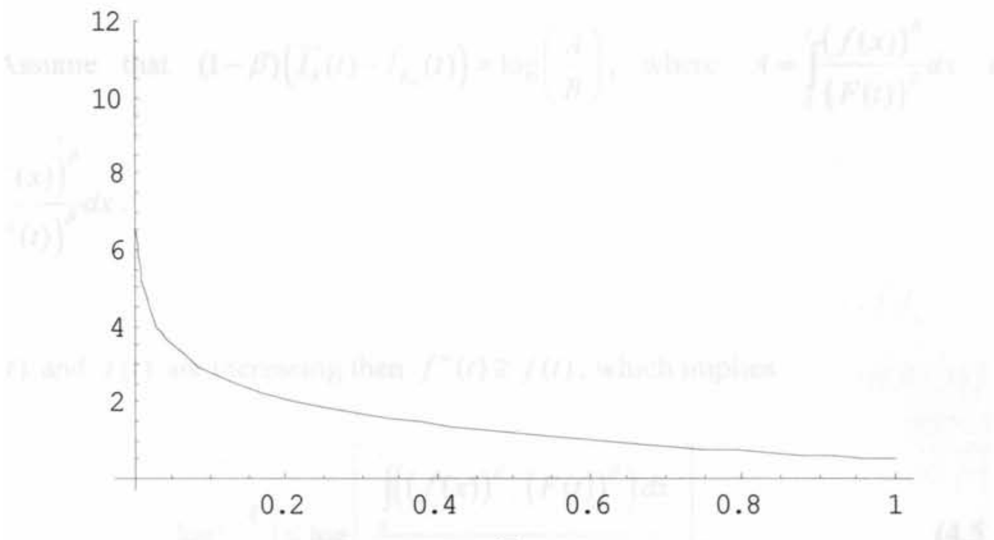

Figure 4.2: Plot of $\bar{M}(t)$ against $t \in [0,1]$

Theorem 4.16: If $w(t) \leq (\geq) E\left(w(X)\right) \leq (\geq) E\left(w(X) \middle| X \leq t\right)$, then $X \overset{CMUL}{\geq} \left(\overset{CMUL}{\leq} \right) X_w$.

Proof: Proof is similar to that of Theorem 4.16.

Similarly we can define the following order based on Renyi's entropy.

Definition 4.5: A df $F(.)$ is said to be increasing (decreasing) Renyi's past entropy if $\bar{I}_R(t)$ is increasing (decreasing) for all $t \geq 0$.

Theorem 4.17: If $f(t)$ is increasing (decreasing) in t, then

$$\bar{I}_R(t) \leq (\geq) - \log \lambda(t) \text{ for all } \beta. \tag{4.51}$$

Proof: When $f(t)$ is increasing, then $f(x) \leq f(t)$ for all $x \leq t$. Using (4.34), we have

$$\bar{I}_R(t) \leq (1-\beta)^{-1} \log \int_0^t \frac{f^{(\beta-1)}(t)}{F^{(\beta-1)}(t)} \frac{f(x)}{F(t)} dx \text{, for all } x \leq t. \tag{4.52}$$

On simplification, (4.52) implies (4.51). In a similar manner when $f(t)$ is decreasing, then the inequality is reversed.

Theorem 4.18: When $f(t)$ and $w(t)$ are increasing (decreasing), then

$$\bar{I}_R(t) \leq (\geq) \bar{I}_{R^w}(t)$$

Proof: Assume that $(1-\beta)\left(\bar{I}_R(t)-\bar{I}_{R_w}(t)\right)=\log\left(\frac{A}{B}\right)$, where $A=\int_0^t \frac{(f(x))^\beta}{(F(t))^\beta}dx$ and $B=\int_0^t \frac{(f^w(x))^\beta}{(F^w(t))^\beta}dx$.

When $w(t)$ and $f(t)$ are increasing then $f^w(t)\geq f(t)$, which implies

$$\log\left(\frac{A}{B}\right)\leq\log\left[\frac{\int_0^t\left((f(x))^\beta/(F(t))^\beta\right)dx}{\int_0^t\left((f(x))^\beta/(F^w(t))^\beta\right)dx}\right]. \tag{4.53}$$

From (4.53) we get

$$\log\left(\frac{A}{B}\right)\leq\log\left(\frac{F^w(t)}{F(t)}\right)^\beta=\beta\log\left(\frac{F^w(t)}{F(t)}\right). \tag{4.54}$$

But when $w(t)$ is increasing (decreasing), then $F^w(t)\leq(\geq)F(t)$ (Sunoj and Maya (2006)). Therefore from (4.54), we get $\beta\log\left(\frac{F^w(t)}{F(t)}\right)\leq\beta\log 1=0$. This implies that $(1-\alpha)\left(\bar{I}_R(t)-\bar{I}_{R_w}(t)\right)\leq 0$, therefore $\bar{I}_R(t)\leq\bar{I}_{R_w}(t)$.

The following tables give the measures of uncertainty for various distributions.

Table: 4.1 $\bar{H}(t)$ for various distributions

Distribution	pdf	$\bar{H}(t)$
Exponential	$\lambda\exp(-\lambda t)$; $t,\lambda>0$	$1-\log\left(\frac{\lambda}{1-\exp(-\lambda t)}\right)-\left(\frac{\lambda t\exp(-\lambda t)}{1-\exp(-\lambda t)}\right)$
Pareto I	$ck^ct^{-(c+1)}$; $t\geq k,k,c>0$	$\left(\frac{c+1}{c}\right)-\log\left(\frac{ck^c}{1-(\frac{k}{t})^c}\right)+\left(\frac{(c+1)(\frac{k}{t})^c}{1-(\frac{k}{t})^c}\right)$

Pareto II	$pq^p(t+q)^{-(p+1)}$; $t>0,q,p>0$	$\left(\frac{p+1}{p}\right)-\log\left(\frac{pq^p}{1-(\frac{q}{t+q})^p}\right)-\left(\frac{(a+1)}{1-(\frac{c}{t+c})^a}\right)\left(\log q-(\frac{q}{t+q})^a\log(t+q)\right)$
Beta	$dR^{-d}(R-t)^{(d-1)}$; $0<t<R,d>0$	$\left(\frac{1-d}{d}\right)-\log\left(\frac{dR^{-d}}{1-(\frac{R}{R-t})^{-d}}\right)-\left(\frac{c-1}{1-(\frac{R}{R-t})^{-c}}\right)\left(\log R-(\frac{R}{R-t})^{-d}\log(R-t)\right)$
Power	$\frac{ct^{(c-1)}}{b^c}$; $0\le t\le b,b,c>0$	$\left(\frac{c-1}{c}\right)-\log\left(\frac{c}{t}\right)$

Table: 4.2 $\bar{M}(t)$ and $(1-\beta)\bar{I}_R(\beta,t)$ for various distributions

Pdf	$\bar{M}(t)$	$(1-\beta)\bar{I}_R(\beta,t)$
$\lambda\exp(-\lambda t)$; $t,\lambda>0$	$1-\log\lambda-\left(\frac{\lambda t\exp(-\lambda t)}{1-\exp(-\lambda t)}\right)$	$(\beta-1)\log\lambda-\log\beta+\left(\frac{1-\exp(-\lambda\beta t)}{(1-\exp(-\lambda t))^\beta}\right)$
$ck^ct^{-(c+1)}$; $t\ge k,k,c>0$	$\left(\frac{c+1}{c}\right)-\log\left(ck^c\right)+\left(\frac{(c+1)(\frac{k}{t})^c}{1-(\frac{k}{t})^c}\right)\log\left(\frac{k}{t}\right)$	$\log\left(c^\beta k^{(1-\beta)}\right)-\log\left((c+1)\beta-1\right)+\log\left(\frac{1-(\frac{k}{t})^{\beta(c+1)-1}}{\left(1-(\frac{k}{t})^c\right)^\beta}\right)$
$pq^p(t+q)^{-(p+1)}$; $t>0,q,p>0$	$\left(\frac{p+1}{p}\right)-\log\left(pq^p\right)-\left(\frac{(a+1)}{1-(\frac{c}{t+c})^a}\right)\left(\log q-(\frac{q}{t+q})^p\log(t+q)\right)$	$\log\left(p^\beta q^{(1-\beta)}\right)-\log\left(1-(p+1)\beta\right)+\log\left(\frac{1-(\frac{q}{t+q})^{\beta(a+1)-1}}{\left(1-(\frac{q}{t+q})^a\right)^\beta}\right)$
$dR^{-d}(R-t)^{(d-1)}$; $0<t<R,d>0$	$\left(\frac{1-d}{d}\right)-\log\left(dR^{-d}\right)-\left(\frac{d-1}{1-(\frac{R}{R-t})^{-d}}\right)\left(\log R-(\frac{R}{R-t})^{-d}\log(R-t)\right)$	$\log\left(d^\beta R^{(1-\beta)}\right)+\log\left((d-1)\beta+1\right)+\log\left(\frac{1-(\frac{R}{R-t})^{-[\beta(d-1)+1]}}{\left(1-(\frac{R}{R-t})^{-d}\right)^\beta}\right)$
$\frac{ct^{(c-1)}}{b^c}$; $0\le t\le b,b,c>0$	$\left(\frac{c-1}{c}\right)-\log c+c\log b-(c-1)\log t$	$\beta\log c-\log[(c-1)\beta+1]+(1-\beta)\log t$

4.7 Measures of discrimination

In this section, we discuss some measures of discrimination proposed by Di Crescenzo and Longobardi (2004) and Asadi et al. (2005). Further we derive the bounds and inequalities for the comparison of weighted distributions and their unweighted counterparts using these measures.

Let X and Y be two non-negative random variables admitting absolutely continuous distribution functions $F(t)$ and $G(t)$ respectively, then Kullback and Leibler (1951) extensively studied the concept of directed divergence which gives the discriminations between two populations and it is defined as

$$I(X,Y) = I(F,G) = \int_0^\infty f(x)\log\frac{f(x)}{g(x)}dx \tag{4.55}$$

where $f(t)$ and $g(t)$ are the corresponding pdistribution functions of X and Y respectively. Motivated by this, Ebrahimi and Kirmani (1996) modified (4.55) useful to measure the discrimination between two residual lifetime distributions and is given by

$$I_{X,Y}(t) = \int_t^\infty \frac{f(x)}{R(t)}\log\left(\frac{f(x)/R(t)}{g(x)/S(t)}\right)dx; \quad t>0, \tag{4.56}$$

where $R(t)=1-F(t)$ and $S(t)=1-G(t)$. $I_{X,Y}(t)$ measures the relative entropy of $(X-t|X>t)$ and $(Y-t|Y>t)$ and it is useful for comparing the residual lifetimes of two items, which have both survived up to time t. Along the similar lines of the measure (4.56), Di Crescenzo and Longobardi (2004) defined the information distance between the past lives $(X|X\le t)$ and $(Y|Y\le t)$ as

$$\bar{I}_{X,Y}(t) = \int_0^t \frac{f(x)}{F(t)}\log\left(\frac{f(x)/F(t)}{g(x)/G(t)}\right)dx; \; t>0. \tag{4.57}$$

Given that at time t, two items have been found to be failing, $\bar{I}_{X,Y}(t)$ measures the discrepancy between their past lives. Similarly, Renyi divergence between the residual distributions proposed by Asadi et al. (2005) is given by

$$I_{X,Y}(\beta,t)=\frac{1}{(\beta-1)}\log\int_{t}^{\infty}\frac{f^{\beta}(x)g^{(1-\beta)}(x)}{R^{\beta}(t)S^{(1-\beta)}(t)}dx. \tag{4.58}$$

In a similar way, Asadi et al. proposed the Renyi discrimination for the past lives implied by F and G as

$$\bar{I}_{X,Y}(\beta,t)=\frac{1}{(\beta-1)}\log\int_{0}^{t}\frac{\left(f(x)\right)^{\beta}\left(g(x)\right)^{1-\beta}}{\left(F(t)\right)^{\beta}\left(G(t)\right)^{1-\beta}}dx. \tag{4.59}$$

In view of the wide applicability of the discrimination measures (4.57) and (4.59) in past lifetime, in the present section, we investigate its relationships between original and weighted random variables and prove certain results.

Now for the random variables X and X_w, the measure (4.57) is defined as

$$\bar{I}_{X,X_w}(t)=\int_{0}^{t}\frac{f(x)}{F(t)}\log\left(\frac{f(x)/F(t)}{f^{w}(x)/F^{w}(t)}\right)dx. \tag{4.60}$$

The measure (4.60) gives the measure of discrepancy between original and weighted rv and it directly related to past entropy $\bar{H}(t)$ through a relation

$$\bar{I}_{X,X_w}(t)=-\int_{0}^{t}\frac{f(x)}{F(t)}\log\left(\frac{f^{w}(x)}{F^{w}(t)}\right)dx-\bar{H}(t). \tag{4.61}$$

Using (1.1) and (2.8) in (4.60), we obtain

$$\bar{I}_{X,X_w}(t)=\log\left[E\left(w(X)\middle|X\le t\right)\right]-E\left(\log w(X)\middle|X\le t\right). \tag{4.62}$$

For a size-biased model, (4.62) reduces to the form

$$\bar{I}_{X,X_s}(t)=\log\bar{m}^{\alpha}(t)-\alpha\left(\log\bar{G}(t)\right) \tag{4.63}$$

where $\bar{m}^{\alpha}(t)=E\left(X^{\alpha}\middle|X\le t\right)$ and $\log\bar{G}(t)=E(\log X|X\le t)$. When $\alpha=1$, (4.63) reduces to the discrimination measure between the original and length-biased model.

Theorem 4.19: If $\overline{I}_{X,X_w}(t)$ is independent of t if and only if the weight function takes the form $w(t)=\left(F(t)\right)^{\theta-1};\theta>0$.

Proof: Suppose that $\overline{I}_{X,X_w}(t)$ is independent of t.

i.e.
$$\overline{I}_{X,X_w}(t)=K, \tag{4.64}$$

where K is independent of t. Using (4.60) and differentiating (4.64) with respect to t, we get

$$\lambda(t)\left[\log\left(\frac{\lambda(t)}{\lambda^{w}(t)}\right)-k\right]+\lambda^{w}(t)-\lambda(t)=0. \tag{4.65}$$

Divide each term of (4.65) by $\lambda(t)$ yields

$$\left[\log\frac{\lambda(t)}{\lambda^{w}(t)}-k\right]+\frac{\lambda^{w}(t)}{\lambda(t)}-1=0. \tag{4.66}$$

Substitute $u(t)=\dfrac{\lambda(t)}{\lambda^{w}(t)}$ and on differentiating (4.66), we get

$$\frac{u'(t)}{u(t)}\left(1-\frac{1}{u(t)}\right)=0 \tag{4.67}$$

which implies that either $u'(t)=0$ or $u(t)=1$. But as X and X_w are not equal $u(t)\neq 1$. So $u'(t)=0$. Hence we have $u'(t)=0$, which implies that there exists a non-negative constant θ such that $\lambda^{w}(t)=\theta\lambda(t)$. Now using (2.9) we get $w(t)=\left(F(t)\right)^{\theta-1};\theta>0$.

Conversely assuming $w(t)=\left(F(t)\right)^{\theta-1}$ and using (2.8), we obtain

$$F^{w}(t)=\left(F(t)\right)^{\theta}. \tag{4.68}$$

From (4.68) and (4.60) we get that for $t>0$ $\overline{I}_{X,X^{w}}(t)=\theta-1-\log\theta$, which is independent of t.

Corollary 4.1: When $F(t) = t$, then Theorem 4.19 characterizes power distribution with df (4.5).

The discrimination measure (4.59) proposed by Asadi et al. (2005) for the random variables X and X_w is defined as

$$\bar{I}_{X,X_w}(\beta,t) = \frac{1}{(\beta-1)} \log \int_0^t \frac{(f(x))^\beta (f^w(x))^{1-\beta}}{(F(t))^\beta (F^w(t))^{1-\beta}} dx \tag{4.69}$$

Using (1.1) and (2.8), (4.69) becomes

$$\bar{I}_{X,X_w}(\beta,t) = \log\left[E\left(w(X)|X \le t\right)\right] + \frac{1}{(\beta-1)} \log\left[E\left(w^{1-\beta}(X)|X \le t\right)\right]. \tag{4.70}$$

For the size-biased model, (4.70) becomes

$$\bar{I}_{X,X_s}(\beta,t) = \log \bar{m}^\alpha(t) + \frac{1}{(\beta-1)} \log \bar{m}^{\alpha(1-\beta)}(t). \tag{4.71}$$

Remark 4.2: When $\beta = 0$, then (4.71) reduces to the measure (4.60).

Theorem 4.20: The Renyi divergence measure for the past life $\bar{I}_{X,X_w}(\beta,t)$ is independent of t if and only if the weight function is $w(t) = (F(t))^{\theta-1}$; $\theta > 0$.

Proof: The proof of this theorem is similar to that of Theorem 4.19.

Corollary 4.2: When $F(t) = t$, then Theorem 4.20 characterizes power distribution with df (4.5).

4.8 Inequalities for measures of discrimination

In this section, we present some results including inequalities and comparisons of discrimination measures for weighted and unweighted or parent distributions. Under some mild constraints, bounds for these measures are also presented here.

Theorem 4.21: If the weight function $w(t)$ is increasing (decreasing) in $t > 0$, then

(a) $\bar{I}_{X,X_w}(t) \geq (\leq) \log\left(\frac{\lambda(t)}{\lambda^w(t)}\right)$

(b) $\bar{I}_{X,X_w}(\beta,t) \geq (\leq) \frac{\beta}{(\beta-1)} \log\left(\frac{\lambda(t)}{\lambda^w(t)}\right), \quad \beta \neq 1.$

Proof: Suppose $w(t)$ is increasing, then from (1.1) we get $\frac{f(t)}{f^w(t)}$ is decreasing which implies

$$\frac{f(t)}{f^w(t)} \leq \frac{f(x)}{f^w(x)} \text{ for all } x \leq t. \tag{4.72}$$

Now from (4.60) we have

$$\int_0^t \frac{f(x)}{F(t)} \log\left(\frac{f(x) F(t)}{f^w(x) F^w(t)}\right) dx \geq \int_0^t \frac{f(x)}{F(t)} \log\left(\frac{f(t) F(t)}{f^w(t) F^w(t)}\right) dx$$

which implies that

$$\bar{I}_{X,X_w}(t) \geq \log\left(\frac{\lambda(t)}{\lambda^w(t)}\right) \text{ for all } x \leq t.$$

When $w(t)$ is decreasing then the inequality is reversed.

Proof of (b) is similar to that of (a).

Theorem 4.22: When (i) $w(t)$ is decreasing (increasing) and (ii) $X \overset{RHR}{\leq} \left(\overset{RHR}{\geq}\right) X_w$, then $\bar{I}_{X,X_w}(t)$ is increasing (decreasing) for all $t > 0$.

Proof: From the definition (4.60)

$$\bar{I}_{X,X_w}(t) = \log\left(\frac{F^w(t)}{F(t)}\right) + \int_0^t \frac{f(x)}{F(t)} \log\left(\frac{f(x)}{f^w(x)}\right) dx. \tag{4.73}$$

The first term of (4.73) is increasing using Theorem 2 (see Sunoj and Maya (2006)) and

$$\int_0^t \frac{f(x)}{F(t)} \log\left(\frac{f(x)}{f^{w}(x)}\right) dx = \log \mu_w - \frac{1}{F(t)} \int_0^t f(x) \log w(x) dx. \qquad (4.74)$$

Now the second term of (4.74) is given by

$$-\frac{1}{F(t)} \int_0^t f(x) \log w(x) dx = -\log(w(t)) + \frac{1}{F(t)} \int_0^t \frac{w'(x)}{w(x)} F(x) dx. \qquad (4.75)$$

Differentiating (4.75) with respect to t and on simplification we get

$$\frac{d}{dt}\left(-\int_0^t \frac{f(x)\log w(x)}{F(t)} dx\right) = -\frac{\lambda(t)}{F(t)} \int_0^t \frac{w'(x)}{w(x)} F(x) dx \geq 0. \qquad (4.76)$$

Thus (4.73) is the sum of two increasing functions. It implies that $\bar{I}_{X,X_w}(t)$ also increasing. Using the similar steps as above, the inequality in the reverse direction can be proved.

Theorem 4.23: When $w(t)$ is increasing (decreasing) and $\dfrac{E\left(w(X)\middle|X \leq t\right)}{w(t)}$ is increasing (decreasing), then $\bar{I}_{X,X_w}(t)$ is increasing (decreasing) for all $t > 0$.

Proof: When $w(t)$ is increasing, from Theorem 4.21,

$$\bar{I}_{X,X_w}(t) \geq \log\left(\frac{\lambda(t)}{\lambda^{w}(t)}\right).$$

Now using (2.9) and the condition given in theorem, we get $\log\left(\dfrac{\lambda(t)}{\lambda^{w}(t)}\right)$ increases, which imply the required result. Similarly one can prove the inequality in the reverse direction.

CHAPTER FIVE

MEASURES OF UNCERTAINTY FOR DOUBLY TRUNCATED RANDOM VARIABLES*

5.1 Introduction

In the previous chapter, we have discussed some measures of uncertainty for the right truncated random variables and characterized certain models arising out of them. But in reliability/survival analysis there may be situations in which the data is doubly truncated. As pointed out in chapter 1, a doubly truncated failure time arises if an individual is potentially observed only if its failure time falls within a certain interval, unique to that individual. In such type of truncation, the individual whose event time lies within a certain time interval are only observed. Thus an individual whose event time is not in this interval is not observed and therefore the information on this subject outside the interval is not available to the investigator (see Ruiz and Navarro (1996), Efron and Petrosian (1999), Betensky and Martin (2003), Navarro and Ruiz (1996, 2004), Sankaran and Sunoj (2004) and Bairamov and Gebizlioglu (2005)). Such types of truncation happen in lifetime studies also. Therefore the properties related to these type of datasets are important both in reliability and survival analysis. In addition, the measures of uncertainty and reliability are closely related. All measures of uncertainty have much relevance in characterizing and classifying life distributions according to the behavior of them.

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* Some of the results in this Chapter have been communicated to an International Journal.

Accordingly in the present chapter, we focus on characterizing some probability models based on different measures of uncertainty and conditional expectations of doubly truncated random variables. Because of the wide applicability of conditional expectations for interval truncated data in survival studies and reliability life testing, in the present chapter, we study the different uncertainty measures considered in the previous chapter to the doubly truncated case and examine its relationships. We also extend these studies in weighted models. Many of the results that we have obtained in the present chapter are generalizations of some of the existing results.

5.2 Definitions and properties

5.2.1 Geometric vitality function

Kupka and Loo (1989) studied the vitality function extensively in connection with their studies on ageing process. It provides a useful tool in modeling lifetime data. Recently, Nair and Rajesh (2000) defined a conditional geometric vitality function and it has been found a useful tool in modeling and analysis of lifetime data. For a non-negative rv X representing the lifetime of a component with an absolutely continuous df $F(t)$ and $E(\log X) < \infty$, then the geometric vitality function of a left truncated rv is

$$\log G(t) = E\left(\log X \middle| X > t\right). \tag{5.1}$$

In reliability theory, (5.1) gives the geometric mean of the lifetimes of components, which has survived t units of time. Nair and Rajesh studied this measure in detail and characterize some probability distributions based on it. Using (5.1), a straightforward generalization of geometric vitality function for a doubly truncated rv $\left(X \middle| t_1 \le X \le t_2\right)$, where $(t_1, t_2) \in D = \left\{(u,v) \in \Re^{+^2}; F(u) < F(v)\right\}$ is given by

$$\log G(t_1, t_2) = E\left(\log X \middle| t_1 < X < t_2\right)$$

$$= \frac{1}{\left(F(t_2) - F(t_1)\right)} \int_{t_1}^{t_2} (\log x) f(x) dx \tag{5.2}$$

$\log G(t_1,t_2)$ gives the geometric mean life of a rv truncated at two points t_1 and t_2. It is clear that when $t_2 \to \infty$ (5.2) reduces to (5.1). The following properties are immediate from the definition (5.2),

$$(1) \lim_{\substack{t_1 \to 0 \\ t_2 \to \infty}} \log G(t_1,t_2) = E(\log X), \text{ and} \tag{5.3}$$

$$(2)\ m(t_1,t_2) \geq \log G(t_1,t_2) \text{ for all } t_1 < t_2, \tag{5.4}$$

where $m(t_1,t_2) = E(X \mid t_1 < X < t_2)$.

Denoting $h_1(t_1,t_2) = \dfrac{f(t_1)}{(R(t_1)-R(t_2))}$ and $h_2(t_1,t_2) = \dfrac{f(t_2)}{(R(t_1)-R(t_2))}$ as the GFR functions of Navarro and Ruiz (1996), (2.1) is related to $h_i(t_1,t_2)$; $i = 1,2$ as

$$h_1(t_1,t_2) = \frac{\frac{\partial}{\partial t_1}\left(\log G(t_1,t_2)\right)}{\log\left(\frac{G(t_1,t_2)}{t_1}\right)} \tag{5.5}$$

and

$$h_2(t_1,t_2) = \frac{\frac{\partial}{\partial t_2}\left(\log G(t_1,t_2)\right)}{\log\left(\frac{t_2}{G(t_1,t_2)}\right)}. \tag{5.6}$$

for all $(t_1,t_2) \in D$.

Table 5.1 provides the relationships between geometric vitality function $\log G(t_1,t_2)$ and GFR functions $h_i = h_i(t_1,t_2)$; $i = 1,2$ of certain probability distributions.

Table 5.1: Relationships between geometric vitality function and GFR functions

Distribution	$R(t)$	$\log G(t_1,t_2)$
Exponential	$\exp(-\lambda t)\,;t>0,$ $\lambda>0$	$\frac{1}{\lambda}\left[h_1\log t_1-h_2\log t_2+A(t_1,t_2)\right]$
Beta	$(1-Rt)^d\,;0<t<\frac{1}{R},$ $d>0,R>0$	$\frac{1}{Rd}\left[(1-Rt_1)h_1\log t_1-(1-Rt_2)h_2\log t_2+A(t_1,t_2)-R\right]$
Pareto II	$(1+pt)^{-q}\,;\ t>0,$ $p>0,q>0$	$\frac{1}{pq}\left[(1+pt_1)h_1\log t_1-(1+pt_2)h_2\log t_2+A(t_1,t_2)+p\right]$
Power	$1-(t/b)^c\,;\ 0\le t\le b,$ $b>0,c>0$	$\frac{1}{c}\left[t_2h_2\log t_2-t_1h_1\log t_1-1\right]$
Pareto I	$(k/t)^c\,;\ t>k,$ $k>0,c>0$	$\frac{1}{c}\left[t_1h_1\log t_1-t_2h_2\log t_2-1\right]$

Here $h_i=h_i(t_1,t_2)=\dfrac{f(t_i)}{\left(R(t_1)-R(t_2)\right)}$; $i=1,2$ and $A(t_1,t_2)=E\left(\dfrac{1}{X}\middle|t_1<X<t_2\right)$.

Theorem 5.1: If $h_i(t_1,t_2)$; $i=1,2$ satisfy the properties given in Navarro and Ruiz (1996), then the geometric vitality function (5.2) determine distribution uniquely.

Proof: The proof follows from (5.5), (5.6) and Theorem 4.1 of Navarro and Ruiz (1996) (see Chapter 1).

5.2.2 Measure of uncertainty

Combining the residual entropy (1.35) defined by Ebrahimi and Pellerey (1995) and the past entropy (4.1) defined by Di Crescenzo and Longobardi (2002), we introduce a new measure of uncertainty which generalize (1.35) and (4.1) to the doubly truncated random variables. Defining a rv $\left(X\,|\,t_1\le X\le t_2\right)$ which represents the life of a unit between t_1 and t_2, a measure of uncertainty for the doubly truncated rv is given by

$$H(t_1,t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{\left(R(t_1)-R(t_2)\right)} \log\left(\frac{f(x)}{\left(R(t_1)-R(t_2)\right)}\right) dx. \tag{5.7}$$

Clearly, $\lim_{t_1\to 0} H(t_1,t_2) = \bar{H}(t_2)$ and $\lim_{t_2\to\infty} H(t_1,t_2) = H(t_1)$.

Equation (5.7) can be written as

$$H(t_1,t_2) = 1 - \frac{1}{\left(R(t_1)-R(t_2)\right)} \int_{t_1}^{t_2} f(x)\left(\log h(x)\right) dx$$
$$+\frac{1}{\left(R(t_1)-R(t_2)\right)}\left[R(t_2)\log R(t_2) - R(t_1)\log R(t_1)\right] + \log\left(R(t_1)-R(t_2)\right). \tag{5.8}$$

$$H(t_1,t_2) = 1 - \frac{1}{\left(F(t_2)-F(t_1)\right)} \int_{t_1}^{t_2} f(x)\left(\log \lambda(x)\right) dx$$
$$-\frac{1}{\left(F(t_2)-F(t_1)\right)}\left[F(t_2)\log F(t_2) - F(t_1)\log F(t_1)\right] + \log\left(F(t_2)-F(t_1)\right), \tag{5.9}$$

where (5.8) and (5.9) are the expressions of $H(t_1,t_2)$ in terms of hazard rate $h(t)$ and reversed hazard rate $\lambda(t)$, respectively.

Now using (4.1), (1.35) and (5.7), the Shannon entropy (1.34) can be decomposed as

$$H = F(t_1)\bar{H}(t_1) + \left(R(t_1)-R(t_2)\right)H(t_1,t_2) + R(t_2)H(t_2) - \left[F(t_1)\log F(t_1)\right.$$
$$\left. + \left(R(t_1)-R(t_2)\right)\log\left(R(t_1)-R(t_2)\right) + R(t_2)\log R(t_2)\right]. \tag{5.10}$$

The identity (5.10) can be interpreted in the following way. The uncertainty about the failure of an item can be decomposed into 4 parts: (i) the uncertainty about the failure time in $(0,t_1)$ given that the item has failed before t_1, (ii) the uncertainty about the failure time in the interval (t_1,t_2) given that the item has failed after t_1 but before t_2, (iii) the uncertainty about the failure time in $(t_2,+\infty)$ given that it has failed after t_2 and (iv) the uncertainty of the item that has failed before t_1 or in between t_1 and t_2 or after t_2.

On partially differentiating $H(t_1,t_2)$ with respect to t_1 and t_2, we get

$$\frac{\partial}{\partial t_1} H(t_1,t_2) = h_1(t_1,t_2)\left[\log h_1(t_1,t_2) + H(t_1,t_2) - 1\right] \tag{5.11}$$

and

$$\frac{\partial}{\partial t_2} H(t_1,t_2) = h_2(t_1,t_2)\left[1 - \log h_2(t_1,t_2) + H(t_1,t_2)\right]. \tag{5.12}$$

When $H(t_1,t_2)$ is increasing in t_1 and in t_2, then, (5.11) and (5.12) together implies

$$1 - \log h_1(t_1,t_2) \le H(t_1,t_2) \le 1 - \log h_2(t_1,t_2). \tag{5.13}$$

Thus when the uncertainty measure is increasing, then it lies between $\left(1 - \log h_1(t_1,t_2)\right)$ and $\left(1 - \log h_2(t_1,t_2)\right)$. We can also write the bounds (5.13) as

$$h_2(t_1,t_2) \le \exp\left(1 - H(t_1,t_2)\right) \le h_1(t_1,t_2).$$

Table 5.2 provides the relationships between the measure of uncertainty $H(t_1,t_2)$, the conditional expectation $m(t_1,t_2) = E\left(X \mid t_1 < X < t_2\right)$ and GFR functions $h_i = h_i(t_1,t_2)$; $i = 1,2$ for various distributions.

Table 5.2: Relationships between $H(t_1,t_2)$, the conditional expectation and GFR functions for various distributions

Distribution	$R(t)$	$H(t_1,t_2)$
Exponential	$\exp(-\lambda t)\,;t>0,$ $\lambda>0$	$\lambda m(t_1,t_2) - \lambda t_2 - \log h_2(t_1,t_2)$ or $\lambda m(t_1,t_2) - \lambda t_1 - \log h_1(t_1,t_2)$
Beta	$(1-Rt)^d\,;0<t<\frac{1}{R},$ $d>0, R>0$	$-(d-1)E\left[\log(1-RX)\mid t_1 < X < t_2\right] - \log\left[\frac{Rd}{(1-Rt_1)^d - (1-Rt_2)^d}\right]$
Pareto II	$(1+pt)^{-q}\,;\ t>0,$ $p>0, q>0$	$(q+1)E\left[\log(1+pX)\mid t_1 < X < t_2\right] - \log\left[\frac{pq}{(1+pt_1)^{-q} - (1+pt_2)^{-q}}\right]$
Power	$1-(t/b)^c\,;\ 0\le t\le b,$ $b>0, c>0$	$1 + \log G(t_1,t_2) + t_1 h_1 \log(t_1/b) - t_2 h_2 \log(t_2/b) - \log\left[\frac{c}{(t_2/b)^c - (t_1/b)^c}\right]$
Pareto I	$(k/t)^c\,;\ t>k,$ $k>0, c>0$	$1 + \log G(t_1,t_2) + t_2 h_2 \log(k/t_2) - t_1 h_1 \log(k/t_1) - \log\left[\frac{c}{(k/t_1)^c - (k/t_2)^c}\right]$

5.2.3 Conditional measure of uncertainty

As an extension of the definition (4.25) given in the previous chapter, we define the conditional measure of uncertainty for the doubly truncated rv as

$$M(t_1,t_2) = -E\left[\log f(X) \middle| t_1 < X < t_2\right]$$

$$= \frac{-1}{\left(R(t_1)-R(t_2)\right)} \int_{t_1}^{t_2} f(x)\left(\log f(x)\right)dx \tag{5.14}$$

where $(t_1,t_2) \in D$. Using (5.14), $M(t_1,t_2)$ can be easily related to $H(t_1,t_2)$ through the relation

$$M(t_1,t_2) = H(t_1,t_2) - \log\left(R(t_1)-R(t_2)\right). \tag{5.15}$$

Differentiating (5.15) with respect to t_1 and t_2 respectively provide the relationships with GFR functions, which are given by

$$\frac{\partial M(t_1,t_2)}{\partial t_1} = \frac{\partial H(t_1,t_2)}{\partial t_1} + h_1(t_1,t_2)$$

and

$$\frac{\partial M(t_1,t_2)}{\partial t_2} = \frac{\partial H(t_1,t_2)}{\partial t_2} - h_2(t_1,t_2).$$

The various relationships between the conditional measure of uncertainty $M(t_1,t_2)$ for doubly truncated random variables and GFR functions $h_i = h_i(t_1,t_2)$; $i = 1,2$ for some probability models are given in Table 5.3.

Table5.3: Relationships between $M(t_1,t_2)$ and GFR functions

Distribution	$R(t)$	$M(t_1,t_2)$
Exponential	$\exp(-\lambda t)\,; t > 0,$ $\lambda > 0$	$\lambda m(t_1,t_2) - \log\lambda$
Beta	$(1-Rt)^d\,; 0 < t < \frac{1}{R},$ $d > 0, R > 0$	$-\log Rd - (d+1)E\left[\log(1-RX) \middle| t_1 < X < t_2\right]$

Pareto II	$(1+pt)^{-q}$; $t>0$, $p>0, q>0$	$(q+1)E\left[\log(1+pX)\middle\|t_1<X<t_2\right]-\log pq$
Power	$1-(t/b)^c$; $0\le t\le b$, $b>0, c>0$	$\left(\frac{c-1}{c}\right)[1+t_1h_1\log t_1-t_2h_2\log t_2+\log c-c\log b]$
Pareto I	$(k/t)^c$; $t>k$, $k>0, c>0$	$\left(\frac{c+1}{c}\right)[t_1h_1\log t_1-t_2h_2\log t_2+1-\log c-c\log k]$
Weibull	$\exp(-t^p);t>0$, $p>0$	$-\log p-(p-1)\log G(t_1,t_2)+E\left[X^p\middle\|t_1<X<t_2\right]$
Rayleigh	$\exp(-t^2);t>0$	$-\log 2-\log G(t_1,t_2)+E\left[X^2\middle\|t_1<X<t_2\right]$

5.3 Characterizations

In this section we prove certain characterization theorems for some life distributions and certain family of distributions using GFR functions, geometric vitality function (5.2) and conditional Shannon's measure of uncertainties (5.7) and (5.14).

The following theorem gives a characterization to a family of distributions such as exponential, Pareto II and beta using a possible relation connecting the geometric vitality function and the GFR functions.

Theorem 5.2: Let X be a rv with support $(0,\infty)$ admitting an absolutely continuous distribution function $F(t)$ with respect to Lebesgue measure. Then a relation of the form

$$\log G(t_1,t_2)=\frac{1}{k}\left[(1+Ct_1)h_1(t_1,t_2)\log t_1-(1+Ct_2)h_2(t_1,t_2)\log t_2+A(t_1,t_2)+C\right] \tag{5.16}$$

where $k>0$ and C are constants holds for all $(t_1,t_2)\in D$ if and only if X follows exponential distribution for $C=0$, Pareto II distribution for $C>0$ and Beta distribution for $C<0$ with distribution functions (2.18), (2.17) and (2.19) respectively.

Proof: Assume that the relation (5.16) holds. From the definitions of $A(t_1,t_2)$, $h_i(t_1,t_2)$; $i=1,2$ and $\log G(t_1,t_2)$, (5.16) becomes

$$\int_{t_1}^{t_2} f(x)\log x dx = \frac{1}{k}\Big[(1+Ct_1)f(t_1)\log t_1 - (1+Ct_2)f(t_2)\log t_2 + \int_{t_1}^{t_2}\frac{1}{x}f(x)dx + C\big(R(t_1)-R(t_2)\big)\Big]. \tag{5.17}$$

Differentiating (5.17) with respect to t_i; $i=1,2$ and on simplification, we obtain

$$\frac{f'(t_i)}{f(t_i)} = -\frac{(k+C)}{(1+Ct_2)}, \text{ for } (t_1,t_2)\in D$$

or

$$\frac{d}{dt}\log f(t) = -\frac{(k+C)}{(1+Ct)}. \tag{5.18}$$

From (5.18.), it follows that X follows exponential, Pareto II and Beta distributions according as $C=0, C>0$ and $C<0$.

The converse part is obtained from Table 5.1.

Theorem 5.3 gives a characterization to the exponential distribution using the functional relation connecting the conditional measure of uncertainty and the conditional moment function $m(t_1,t_2)$.

Theorem 5.3: For a non-negative rv X, a relation of the form

$$M(t_1,t_2) - \mu^{-1}m(t_1,t_2) = k, \tag{5.19}$$

where $k>0$ is a constant, holds for all $(t_1,t_2)\in D$ if and only if X follows exponential distribution with distribution function (2.18).

Proof: Assume (5.19) holds. From (5.14), we can write

$$-\int_{t_1}^{t_2} f(x)\big(\log f(x)\big)dx - \mu^{-1}\int_{t_1}^{t_2} xf(x)dx = k\big(R(t_1)-R(t_2)\big). \tag{5.20}$$

Differentiating (5. 20) with respect to $t_i; i = 1,2$ gives

$$\log f(t_i) = -k - \mu^{-1} t_i \tag{5.21}$$

or $f(t) = K\exp(-\mu^{-1}t)$, which provides the result.

The converse part is obtained from Table 5.3.

The following two theorems identify Pareto I and Power distributions using the functional relation connecting conditional measure of uncertainty and geometric vitality function.

Theorem 5.4: For a non-negative random variable X in the support $[k,\infty)$, $k > 0$, admitting an absolutely continuous df, then a relation of the form

$$M(t_1,t_2) - (c+1)\log G(t_1,t_2) = K, \tag{5.22}$$

where $K > 0$ and c are constants, $c > 0$, holds for $k < t_1 < t_2$ with $F(t_1) < F(t_2)$ if and only if X follows a Pareto I with df (4.20).

Proof: Assuming (5.22), then by using (5.14), and (5.2), (5.22) we have

$$\frac{-1}{\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2} f(x)\left(\log f(x)\right)dx - \frac{(c+1)}{\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2} f(x)\log x dx = K \tag{5.23}$$

or

$$-\int_{t_1}^{t_2} f(x)\left(\log f(x)\right)dx - (c+1)\int_{t_1}^{t_2} f(x)\log x dx = K\left(R(t_1)-R(t_2)\right). \tag{5.24}$$

Differentiating (5.24) with respect to t_i; $i = 1,2$ and simplifying we get $f(t) = Kt^{-(c+1)}$, which corresponds to Pareto type I with $K = ck^c$. The converse part can be easily verified by direct calculation and it is obtained from Tables 5.1 and 5.3

Theorem 5.5: For a non-negative random variable X having an absolutely continuous df $F(t)$ then a relation

$$M(t_1,t_2) + (c-1)\log G(t_1,t_2) = k, \text{ a constant} \tag{5.25}$$

is satisfied for $0 < t_1 < t_2 < b$ with $F(t_1) < F(t_2)$ and $c > 1$ if and only if X follows Power distribution with distribution function (4.5).

Proof: The proof is similar to that of Theorem 5.4.

The following theorem characterizes log exponential family using the functional relation connecting the conditional measure of uncertainty and geometric vitality function.

Theorem 5.6: The distribution of X belongs to one-parameter log exponential family (1.29) if and only if

$$M(t_1,t_2) = \log A(\theta) - \theta \log G(t_1,t_2) - \log m_C(t_1,t_2), \quad (5.26)$$

where $\log m_C(t_1,t_2) = E\left[\log C(X) \middle| t_1 < X < t_2\right]$ with $E\left(\log C(X)\right) < \infty$, for all $(t_1,t_2) \in D$.

Proof: Assume (5.26) holds. From the definition (5.14), we get

$$\frac{-1}{\left(R(t_1)-R(t_2)\right)} \int_{t_1}^{t_2} f(x)\left(\log f(x)\right)dx = \log A(\theta) - \frac{\theta}{\left(R(t_1)-R(t_2)\right)} \int_{t_1}^{t_2} f(x)\log x dx$$
$$+ \frac{1}{\left(R(t_1)-R(t_2)\right)} \int_{t_1}^{t_2} f(x)\left(\log C(x)\right)dx. \quad (5.27)$$

Multiplying both sides of (5.27) by $\left(R(t_1)-R(t_2)\right)$, we obtain

$$-\int_{t_1}^{t_2} f(x)\left(\log f(x)\right)dx = \log A(\theta)\left(R(t_1)-R(t_2)\right) - \theta \int_{t_1}^{t_2} f(x)\log x dx + \int_{t_1}^{t_2} f(x)\left(\log C(x)\right)dx.$$
$$(5.28)$$

Differentiating (5.28) with respect to t_i; $i = 1,2$ and simplifying, we get (1.29).

Conversely assume (1.29), by direct calculation and using the definition of $\log m_C(t_1,t_2)$, we obtain (5.26).

The next theorem characterizes exponential family using the possible relation connecting $M(t_1,t_2)$ and $m(t_1,t_2)$.

Theorem 5.7: The df of a non-negative rv X belongs to one parameter exponential family (1.28) if and only if the relation

$$M(t_1,t_2)+\theta m(t_1,t_2)+m_C(t_1,t_2)+D(\theta)=0 \tag{5.29}$$

where $m_C(t_1,t_2)=E\left[C(X)\middle|t_1<X<t_2\right]$ with $E\left(C(X)\right)<\infty$ is satisfied for all $(t_1,t_2)\in D$.

Proof: The proof is similar to that of the Theorem 5.6.

The next result characterizes generalized Pearson family of distributions using the relation connecting the r^{th} order conditional moment functions and the GFR functions. For a doubly truncated rv $\left(X\middle|t_1<X<t_2\right)$, the conditional moment function of order r is given by

$$m^r(t_1,t_2)=E\left(X^r\middle|t_1<X<t_2\right)=\frac{1}{\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2}x^r f(x)dx,\ (t_1,t_2)\in D. \tag{5.30}$$

Theorem 5.8: The df of a rv X belongs to generalized Pearson family of distributions (1.32) if and only if its r^{th} order conditional moments satisfies a recurrence relation of the form

$$m^r(t_1,t_2)=(b_{0r}+b_{1r}t_2+b_{2r}t_2^2)t_2^{r-2}h_2(t_1,t_2)-(b_{0r}+b_{1r}t_1+b_{2r}t_1^2)t_1^{r-2}h_1(t_1,t_2)$$
$$-a_{0r}m^{r-2}(t_1,t_2)-a_{1r}m^{r-1}(t_1,t_2)-b_{0r}(r-2)\bar{m}^{r-3}(t_1,t_2)$$

where $b_{ir}=\dfrac{b_i}{a_2}$; $i=0,1,2$, $a_{0r}=\dfrac{\left(a_0+(r-1)b_1\right)}{a_2}$, $a_{1r}=\dfrac{\left(a_1+rb_2\right)}{a_2}$ provided $a_2\neq 0$

and

$$m^r(t_1,t_2)=(b_{0r}+b_{1r}t_2+b_{2r}t_2^2)t_2^{r-1}h_2(t_1,t_2)-(b_{0r}+b_{1r}t_1+b_{2r}t_1^2)t_1^{r-1}h_1(t_1,t_2)$$
$$-a_{0r}m^{r-1}(t_1,t_2)-b_{0r}(r-1)m^{r-2}(t_1,t_2) \tag{5.31}$$

where $b_{ir} = \frac{b_i}{((r+1)b_2 + a_1)}$; $i = 0,1,2$, $a_{0r} = \frac{(a_0 + rb_1)}{((r+1)b_2 + a_1)}$, provided $((r+1)b_2 + a_1) \neq 0$ and $a_2 = 0$.

Proof: Case I: When $a_2 \neq 0$, using (5.30), (1.24) and (1.25), (5.31) becomes

$$\int_{t_1}^{t_2} x^r f(x)dx = (b_{0r} + b_{1r}t_2 + b_{2r}t_2^2)t_2^{r-2} f(t_2) - (b_{0r} + b_{1r}t_1 + b_{2r}t_1^2)t_1^{r-2} f(t_1)$$

$$-a_{0r}\int_{t_1}^{t_2} x^{r-2} f(x)dx - a_{1r}\int_{t_1}^{t_2} x^{r-1} f(x)dx - b_{0r}(r-2)\int_{t_1}^{t_2} x^{r-3} f(x)dx. \tag{5.32}$$

Differentiating (5.32) with respect to t_i; $i = 1,2$ and simplifying, we get

$$\frac{f'(t)}{f(t)} = \frac{(A_0 + A_1 t + A_2 t^2)}{(B_0 + B_1 t + B_2 t^2)}. \tag{5.33}$$

From (5.33) it follows that the distribution of X belongs to generalized Pearson family with $A_0 = (a_{0r} - (r-1)b_{1r})$, $A_1 = (a_{1r} - rb_{2r})$, $A_2 = 1$ and $B_i = b_{ir}$; $i = 0,1,2$.

Similarly we can prove the case $a_2 = 0$ as that of the case $a_2 \neq 0$.

Conversely assume (1.32). Multiplying both sides of it by x^{r-2} and on integrating over the limits t_1 to t_2 we get

$$(b_0 + b_1 t_2 + b_2 t_2^2)t_2^2 f(t_2) - (b_0 + b_1 t_1 + b_2 t_1^2)t_1^2 f(t_1) - ((r-1)b_1 + a_0)\int_{t_1}^{t_2} x^{r-2} f(x)dx$$

$$-(rb_2 + a_1)\int_{a}^{t} x^{r-1} f(x)dx - b_0(r-2)\int_{t_1}^{t_2} x^{r-3} f(x)dx - a_2\int_{t_1}^{t_2} x^r f(x)dx = 0. \tag{5.34}$$

Multiplying both sides of (5.34) by $(R(t_1) - R(t_2))^{-1}$ and using (5.30), we obtain the required form. Substitute $a_2 = 0$ in (5.34) and following the similar steps we get (5.31) for $a_2 = 0$.

Remarks 5.1: 1) When $t_1 = 0$, Theorem 5.8 reduces to the Theorem 2.8 given in Chapter 2.

2) When $a_2 = 0$, this theorem reduces to that for the Pearson family of distributions (1.30).

5.4 Weighted models

Now we study the application of these uncertainty measures in the context of weighted distributions. We examine the functional relationships of the GFR functions and the uncertainty measures in the context of weighted distributions and prove some useful characterizations arising out of it. For the weighted rv X_w, the functional relationship connecting the GFR functions are

$$F^w(t_2) - F^w(t_1) = P(t_1 < X_w < t_2) = \frac{E\left[w(X)\middle|t_1 < X < t_2\right]}{\mu_w}\left(F(t_2) - F(t_1)\right) \quad (5.35)$$

and

$$h_i^w(t_1,t_2) = \frac{w(t_i)h_i(t_1,t_2)}{E\left[w(X)\middle|t_1 < X < t_2\right]}; \; i = 1,2 \quad (5.36)$$

where $h_i^w(t_1,t_2) = \frac{f^w(t_i)}{\left(F^w(t_2) - F^w(t_1)\right)}$; $i = 1,2$ and $(t_1,t_2) \in D$.

Remark 5.2: When $w(t) = t$, (5.35) and (5.36) reduces to the forms given in Sankaran and Sunoj (2004).

Next few theorems prove the relationship connecting the ratio of the distribution functions of weighted and original models and the GFR functions.

Theorem 5.9: If $\alpha(t_1,t_2) = \left(\frac{F^w(t_2) - F^w(t_1)}{F(t_2) - F(t_1)}\right)$ for $(t_1,t_2) \in D$, then $\alpha(t_1,t_2)$ determines $F(t)$.

Proof: Differentiating $\alpha(t_1,t_2)$ with respect to t_1 and t_2 respectively, we obtain

$$\frac{\partial}{\partial t_1}\alpha(t_1,t_2)=h_1(t_1,t_2)\left(\alpha(t_1,t_2)-\frac{w(t_1)}{\mu_w}\right) \tag{5.37}$$

and

$$\frac{\partial}{\partial t_2}\alpha(t_1,t_2)=h_2(t_1,t_2)\left(\frac{w(t_2)}{\mu_w}-\alpha(t_1,t_2)\right). \tag{5.38}$$

(see Navarro et al. (2001) and Sunoj and Maya (2006). From (5.37) and (5.38), we get $h_i(t_1,t_2)$; $i=1,2$ as

$$h_1(t_1,t_2)=\frac{\frac{\partial}{\partial t_1}\alpha(t_1,t_2)}{\left(\alpha(t_1,t_2)-\frac{w(t_1)}{\mu_w}\right)} \tag{5.39}$$

$$h_2(t_1,t_2)=\frac{\frac{\partial}{\partial t_2}\alpha(t_1,t_2)}{\left(\frac{w(t_2)}{\mu_w}-\alpha(t_1,t_2)\right)}. \tag{5.40}$$

Now (5.39), (5.40) and the Theorem 4.1 in Navarro and Ruiz (1996) (see Chapter 1), implies the required result.

Theorem 5.10: Under length-biased sampling, for a non-negative rv X with pdf $f(t)$ and df $F(t)$, the ratio

$$\frac{F^L(t_2)-F^L(t_1)}{F(t_2)-F(t_1)}=1+t_1(1+Ct_1)h_1(t_1,t_2)-t_2(1+Ct_2)h_2(t_1,t_2), \tag{5.41}$$

where $\left(F^L(t_2)-F^L(t_1)\right)$ is the df corresponding to the length-biased model, holds for all $(t_1,t_2)\in D$, if and only if X follows Pareto II (2.17), exponential (2.18) and beta (2.19) according as $C>0$, $C=0$ and $C<0$.

Proof: From (5.35), under the weight function $w(t)=t$, we have

$$\frac{F^L(t_2)-F^L(t_1)}{F(t_2)-F(t_1)}=\frac{m(t_1,t_2)}{\mu}=\frac{1}{\mu\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2}xf(x)dx. \tag{5.42}$$

Comparing (5.41) and (5.42), we have

$$\frac{1}{\mu}\int_{t_1}^{t_2}xf(x)dx=\left(R(t_1)-R(t_2)\right)+t_1(1+Ct_1)f(t_1)-t_2(1+Ct_2)f(t_2). \tag{5.43}$$

On differentiating (5.43) with respect to t_i; $i=1,2$, we get

$$\frac{f'(t_i)}{f(t_i)}=\frac{(1+2\mu C)}{\mu(1+Ct_i)}. \tag{5.44}$$

Integrating (5.44), we obtain the densities of Pareto II, exponential and beta distributions according as $C>0, C=0$ and $C<0$.

Conversely, substituting (2.17) in the definition of conditional moment function $m(t_1,t_2)$ and division by μ, yields the required form for the Pareto II distribution. The case is similar to that of exponential and beta distributions.

Theorem 5.11: The ratio of the relation

$$\frac{F^L(t_2)-F^L(t_1)}{F(t_2)-F(t_1)}=1-g^*(t_2)h_2(t_1,t_2)+g^*(t_1)h_1(t_1,t_2), \tag{5.45}$$

where $g^*(t_i)=\frac{g(t_i)}{\mu}$; $i=1,2$, holds for all $(t_1,t_2)\in D$ for the family (1.31).

Proof: Integrating (2.24) over the limits t_1 to t_2 and by dividing $\left(F(t_2)-F(t_1)\right)$, we get

$$g(t_2)h_2(t_1,t_2)-g(t_1)h_1(t_1,t_2)=\mu-m(t_1,t_2). \tag{5.46}$$

Equations (5.42) and (5.46) together imply the required result. The converse part is obtained by direct calculation.

Corollary 5.1: When $g(t)=b_0+b_1t+b_2t^2$ in Theorem 5.11, reduces to that of Pearson family of distributions (1.30).

Theorem 5.12: Let X_w be a weighted rv associated to X and $w(t)=t$, then the ratio of the relationship

$$\frac{F^L(t_2)-F^L(t_1)}{F(t_2)-F(t_1)} = K\left[1+t_1h_1(t_1,t_2)-t_2h_2(t_1,t_2)+n_a^1(t_1,t_2)\right], \tag{5.47}$$

where $K=\mu c(\theta)$, a constant and $n_a^1(t_1,t_2)=E\left[X\frac{a'(X)}{a(X)}|t_1<X<t_2\right]$ holds for all $(t_1,t_2)\in D$ for the class of distributions (Sankaran and Gupta (2005)) defined by

$$f(t,\theta)=\begin{cases} a(t)(c(\theta))\exp(-c(\theta)t); a<t<b \\ 0; otherwise \end{cases}. \tag{5.48}$$

Proof: Assuming (5.47) and using (5.42), we have

$$\int_{t_1}^{t_2} xf(x)dx = K\left[\left(R(t_1)-R(t_2)\right)+t_1f(t_1)-t_2f(t_2)-\int_{t_1}^{t_2} x\frac{a'(x)}{a(x)}f(x)dx\right] \tag{5.49}$$

differentiating (5.49) with respect to t_i; $i=1,2$ and on simplification, we get (5.48). Substitution of (5.49) in (5.42) and on simplification, yields the converse part of the theorem.

In view of the form-invariance property for the generalized Pearson family of distributions (1.32), the analogous statement for Theorem (5.8) in the context of size-biased model is immediate. This is stated in the following theorem.

Theorem 5.13: The df of a non-negative rv X belongs to generalized Pearson family (1.32) under size-biased sampling, if and only if its r^{th} order conditional moments satisfies a recurrence relation of the form

$$m^r(t_1,t_2)=(q_{1r}+q_{2r}t_2)t_2^{r-1}h_2(t_1,t_2)-(q_{1r}+q_{2r}t_1)t_1^{r-1}h_1(t_1,t_2)$$
$$-p_{0r}m^{r-2}(t_1,t_2)-p_{1r\alpha}m^{r-1}(t_1,t_2),$$

where $q_{ir}=\frac{q_i}{p_2}$; $i=1,2$, $p_{0r}=\frac{\left[p_0+\left(r-(\alpha+1)\right)q_1\right]}{p_2}$ and $p_{1r}=\frac{\left(p_1+(r-\alpha)q_2\right)}{p_2}$, provided $p_2\neq 0$

and

$$m^r(t_1,t_2)=(q_{1r}+q_{2r}t_2)t_2^r h_2(t_1,t_2)-(q_{1r}+q_{2r}t_1)t_1^r h_1(t_1,t_2)-p_{0r\alpha}m_{r-1}(t_1,t_2), \quad (5.50)$$

where $q_{ir}=\dfrac{q_i}{\left((r+1-\alpha)q_2+p_1\right)}$; $i=1,2$ and $p_{0r}=\dfrac{\left(p_0+(r-\alpha)q_1\right)}{\left((r+1-\alpha)q_2+p_1\right)}$, provided $\left((r+1-\alpha)q_2+p_1\right)\neq 0$ and $p_2=0$.

Proof: The proof is similar to that of the Theorem 5.8.

Remarks 5.3: 1) When $t_1=0$, Theorem 5.13 reduces to the Theorem 2.9 given in the second Chapter.

2) When $p_2=0$, this Theorem 5.13 reduces to that of the Pearson family of distributions (2.48).

Now we consider the geometric vitality function for the weighted models. The geometric vitality function corresponding to weighted model is denoted as $\log G^w(t_1,t_2)$ and it is given by

$$\log G^w(t_1,t_2)=E\left(\log X_w \middle| t_1<X_w<t_2\right),\ (t_1,t_2)\in D$$

$$=\frac{1}{\left(F^w(t_1)-F^w(t_2)\right)}\int_{t_1}^{t_2}(\log x_w)f^w(x)dx. \quad (5.51)$$

(5.51) can be written as

$$\log G^w(t_1,t_2)=\frac{1}{m_w(t_1,t_2)\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2}w(x)f(x)\log x\,dx, \quad (5.52)$$

where $m_w(t_1,t_2)=E\left[w(X)\middle|t_1<X<t_2\right]$.

Corollary 5.2: When $w(t)=t$, (5.52) reduces to the geometric vitality function of a length-biased model and it is denoted as $\log G^L(t_1,t_2)$. Substituting this and applying integration by parts, we obtain

$$\log G^L(t_1,t_2)=\frac{1}{m(t_1,t_2)\left(R(t_1)-R(t_2)\right)}\Big[t_1R(t_1)\log t_1-t_2R(t_2)\log t_2 .$$

$$\left.+\int_{t_1}^{t_2}R(x)dx+\int_{t_1}^{t_2}R(x)\log x dx\right]. \qquad (5.53)$$

Theorem 5.14: For a non-negative rv X , the relation

$$\lambda m(t_1,t_2)\log G^L(t_1,t_2)-\log G(t_1,t_2)=1+t_1h_1(t_1,t_2)\log t_1-t_2h_2(t_1,t_2)\log t_2 \qquad (5.54)$$

holds for all $(t_1,t_2)\in D$ if and only if X follows an exponential distribution (2.18).

Proof: Suppose that the relation (5.54) holds, then by definition,

$$\frac{\lambda}{\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2}xf(x)\log x dx-\frac{1}{\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2}f(x)\log x dx$$

$$=1+t_1\frac{f(t_1)}{\left(R(t_1)-R(t_2)\right)}\log t_1-t_2\frac{f(t_2)}{\left(R(t_1)-R(t_2)\right)}\log t_2 . \qquad (5.55)$$

Multiply both sides of (5.55) by $\left(R(t_1)-R(t_2)\right)$ and on differentiation with respect to t_i; $i=1,2$, yields the required result. Converse part can be proved by direct calculation. The measure of uncertainty for the weighted model is denoted as $H^w(t_1,t_2)$ and it is defined by

$$H^w(t_1,t_2)=-\int_{t_1}^{t_2}\frac{f^w(x)}{\left(R^w(t_1)-R^w(t_2)\right)}\log\left(\frac{f^w(x)}{\left(R^w(t_1)-R^w(t_2)\right)}\right)dx$$

$$=\frac{-1}{m_w(t_1,t_2)\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2}w(x)f(x)\log\left(\frac{w(x)f(x)}{m_w(t_1,t_2)\left(R(t_1)-R(t_2)\right)}\right)dx \qquad (5.56)$$

and the corresponding conditional measure of uncertainty is

$$M^w(t_1,t_2)=-E\left(\log f^w(X)\middle|t_1<X_w<t_2\right)$$

$$=\frac{-1}{m_w(t_1,t_2)\left(R(t_1)-R(t_2)\right)}\int_{t_1}^{t_2}w(x)f(x)\log\left(\frac{w(x)f(x)}{E\left(w(X)\right)}\right)dx . \qquad (5.57)$$

CHAPTER SIX

LOWER PARTIAL MOMENTS*

6.1 Introduction

Various measurements of risk have been considered as important tools for decision making problems where some risk exist. If we consider an individual investors risk perception in a financial decision making context, there are different modes of information presentation are available. In a recent survey, it is found that corporate managers mostly concerned about downside risk which is a measure of distance between a risky situation and the corresponding risk free situation. There are several classes of downside risks of interest in finance. They generally involve the tail of the relevant distribution of returns below some specific target return. Bawa (1975) introduced lower partial moment (LPM) as a measure of downside risk in financial economics. Consider an individual with a given portfolio that generates a random return X and the individual have a target return t. Then the risk associated when X falls short of t leads to the natural definition of downside risk, an uncertainty associated with the shortfalls below the target return. This uncertainty is measured using LPM, and in the case of continuous distribution, for a positive integer, the r^{th} order LPM of X is defined as

$$l_r(t) = E\left[(X-t)^-\right]^r ; r = 0,1,2,\dots,\ t > 0 \qquad (6.1)$$

where

$$(X-t)^- = \begin{cases} (t-X); & X < t \\ 0; & X \geq t \end{cases}.$$

* Some of the results in this Chapter have been communicated to an International Journal.

When the pdf associated with X is $f(t)$ and t is the target rate of returns, then (6.1) can be written as

$$l_r(t)=\int_a^t (t-x)^r f(x)dx. \tag{6.2}$$

Some of the most frequently used risk measures are special cases of LPMs. For example, when the weighing coefficient $r=0$, the probability of loss equals the 0^{th} order LPM $l_0(t)$ and for $r=1$, it is the expected loss $l_1(t)$. Here the target value t is considered as a threshold point separating gains and losses. A survey of literature in this area is available in Bawa (1975), Price et al. (1982), Harlow (1991), Eftekhari (1998) and Lien and Tse (2001).

However, when X represents the lifetime/repair time of a component/system, $l_r(t)$ can be related with various reliability measures viz. reversed hazard rate, reversed mean residual life (expected inactivity time) etc. Further, LPMs can also be used for model identification in the same way as the truncated moments are employed.

6.2 Properties

By virtue of the relationship (6.2), we have

$$l_r(t)=r\int_a^t (t-x)^{r-1} F(x)dx. \tag{6.3}$$

Differentiating (6.2) with respect to t, successively we get

$$\frac{d}{dt}l_r(t)=r\int_a^t (t-x)^{r-1} f(x)dx,$$

and

$$\frac{d^2}{dt^2}l_r(t)=r(r-1)\int_a^t (t-x)^{r-2} f(x)dx.$$

Proceeding similarly r times, we obtain

$$\frac{d^r}{dt^r}l_r(t)=r!F(t), \text{ where } F(t)=\int_a^t f(x)dx$$

or

$$F(t) = \frac{l_r^{(r)}(t)}{r!}, \tag{6.4}$$

where $l_r^{(r)}(t)$ is the r^{th} derivative of $l_r(t)$ with respect to t. Thus from (6.4), $l_r(t)$ determines the df uniquely.

Theorem 6.1: Let X be a non-negative rv having an absolutely continuous df $F(t)$ such that $\lim_{t\to a} tf(t) = 0$ and $l_r(t)$ be defined as in (6.1). Then the ratio of consecutive lower partial moments, is of the form

$$\frac{l_r(t)}{l_{r-1}(t)} = Ct, \tag{6.5}$$

where $0 < C < 1$ is a constant characterizes power distribution with df (4.5).

Proof: Suppose that the relation (6.5) holds, by using (6.2), we have

$$\int_a^t (t-x)^r f(x)dx = Ct\int_a^t (t-x)^{r-1} f(x)dx. \tag{6.6}$$

Use $t = (t - x + x)$ in (6.6) and on simplification, we get

$$(1-C)\int_a^t (t-x)^r f(x)dx = C\int_a^t x(t-x)^{r-1} f(x)dx. \tag{6.7}$$

Differentiating (6.7) r times and using (6.4), we obtain

$$(1-C)F(t) = Ctf(t) \tag{6.8}$$

or

$$\lambda(t) = \frac{(1-C)}{Ct}. \tag{6.9}$$

From the uniqueness property of $\lambda(t)$, (6.9) corresponds to the power distribution with df (4.5).

Conversely assuming (4.5) and by substituting for $f(t)$ in (6.2) and on simplification we get (6.5) and it is obtained from Table 6.4.

Theorem 6.2: For a rv X considered in Theorem 6.1, and for any $r>0$, the r^{th} order LPM satisfies a relation of the form

$$l_r(t)+Cl_{r-1}(t)=t^r \tag{6.10}$$

where $C>0$ for all $t>0$ if and only if X follows exponential distribution with df (2.18).

Proof: Assume that the relation (6.10) holds, using (6.2), we get

$$\int_a^t (t-x)^r f(x)dx + C\int_a^t (t-x)^{r-1} f(x)dx = t^r . \tag{6.11}$$

Differentiating (6.11) r times with respect to t and using (6.4), we get

$$\frac{f(t)}{1-F(t)}=\frac{r}{C}. \tag{6.12}$$

From the uniqueness property of hazard rate, (6.12) corresponds to the exponential distribution. The converse part is obtained by direct calculation and it is given in Table 6.4.

Theorem 6.3: For a rv X considered in Theorem 6.1, a relation connecting the r^{th} order LPM

$$l_r(t)=C(t-a)^{r+1} \tag{6.13}$$

where $C>0$ is a constant is satisfied for all $t>0$ if and only if X follows uniform distribution with pdf

$$f(t)=\frac{1}{(b-a)};\ a<t<b. \tag{6.14}$$

Proof: Assume (6.13) holds, using the similar steps as in Theorem 6.2, we get (6.14). The converse part is directly obtained from Table 6.4.

Theorem 6.4: For a non-negative rv X with an absolutely continuous df with $\lim_{t \to a} tf(t) = 0$, a relation of the form

$$l_r(t) + r(C_1 t + C_2) l_{r-1}(t) = (C_1 + 1)t^r \tag{6.15}$$

where $C_i > 0$; $i = 1,2$ are constants holds for all $t > 0$ if and only if X follows generalized Pareto distribution with df

$$F(t) = 1 - \left(\frac{q}{pt + q} \right)^{\frac{1}{p}+1} ; \; t > 0, p > -1, q > 0 . \tag{6.16}$$

Proof: The 'if' part of the theorem can be obtained from Table 6.4. To prove the 'only if' part, assume (6.15) holds. Using (6.2) and on simplification, we obtain

$$\int_a^t (t-x)^r f(x)dx + r(C_1 t + C_2) \int_a^t (t-x)^{r-1} f(x)dx = (1 + C_1)t^r . \tag{6.17}$$

Putting $t = (t - x + x)$ in (6.17) and on simplification, we get

$$(1 + C_1) \int_a^t (t-x)^r f(x)dx + \int_a^t (C_1 x + C_2)(t-x)^{r-1} f(x)dx = (1 + C_1)t^r . \tag{6.18}$$

Differentiating (6.18) r times with respect to t using (6.4), and on further simplification

$$r(1 + C_1)F(t) + (C_1 t + C_2) f(t) = r(1 + C_1) \tag{6.19}$$

which implies that

$$\frac{f(t)}{1 - F(t)} = \frac{r(1 + C_1)}{(C_1 t + C_2)} . \tag{6.20}$$

Now from the uniqueness property of hazard rate, (6.20) yields the required result.

6.3 Recurrence relationships

In this section, we identify some of the recurrence relationships between various LPM's for certain important families of distributions.

Theorem 6.5: Assume $l_{g(r-1)}(t)=\int_a^t (t-x)^{r-1} f(x)g(x)dx$, then the pdf of a random variable X belongs to general family of distributions (1.31) if and only if its r^{th} order LPM satisfies a recurrence relation of the form

$$K(t-a)^r + r l_{g(r-1)}(t) = (\mu - t) l_r(t) + l_{r+1}(t), \tag{6.21}$$

where K is a constant such that $K=-f(a)g(a)$.

Proof: Assume that the relation (6.21) holds. By using (6.2) and on simplification, (6.21) becomes

$$K(t-a)^r + r\int_a^t (t-x)^r f(x)g(x)dx = \mu\int_a^t (t-x)^r f(x)dx - \int_a^t x(t-x)^r f(x)dx. \tag{6.22}$$

Differentiating (6.22) $(r+1)$ times using (6.4) and on simplification, we get (1.31).

Conversely assuming (1.31) and multiplying both sides of (1.31) by $(t-x)^r$ and on integration using the assumption given in the theorem, we get (6.21).

Table 6.1 provides some of the important members of the family (1.31) and identifies each of its recurrence relationships using (6.21).

Corollary 6.1: When $g(t)=b_0+b_1t+b_2t^2$ with $b_2 \neq -\frac{1}{2}$, (1.31) reduces to the Pearson family (1.30). Substitute for $g(t)$ in (6.21) and on integration we get the recurrence relation for Pearson family given by

$$K(t-a)^r + r\left[b_0+b_1t+b_2t^2\right] l_{r-1}(t)$$
$$-\left[r(b_1+2b_2t)+(\mu-t)\right] l_r(t) + \left[rb_2-1\right] l_{r+1}(t) = 0, \tag{6.23}$$

where $K=-(b_0+b_1t+b_2t^2)f(a)$.

Table 6.1: Recurrence relationships connecting some members of general family

Distribution	$f(t)$	$g(t)$	$l_{r+1}(t)$
Beta	$\frac{1}{B(a,b)}t^{(a-1)}(1-t)^{(b-1)}$; $0<t<1,\ a>0,\ b>0$	$\frac{t(1-t)}{(a+b)}$	$\left[r+a+b\right]^{-1}\left[\left((2r+a+b)t-r-\mu(a+b)\right) l_r(t)+rt(1-t)l_{r-1}(t)\right]$
Gamma	$\frac{m^p}{\Gamma p}\exp(-mt)t^{(p-1)}$; $t>0,\ m>0,\ p>0$	mt	$rmtl_{r-1}(t)+(t-rm-\mu)l_r(t)$
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\exp-\left(\frac{t-\mu}{\sigma}\right)^2$; $-\infty<t<\infty, \sigma>0$, $-\infty<\mu<\infty$	σ^2	$r\sigma^2 l_{r-1}(t)+(t-\mu)l_r(t)$
Maxwell	$4\lambda^{-3/2}\pi^{-1/2}t^2\exp\left(-\frac{t^2}{\lambda^2}\right)$; $t>0, \lambda>0$	$\frac{\lambda^2}{2}\left(1+\frac{\lambda^2}{t^2}\right)$	$2tl_r(t)-\frac{1}{2}\left[\lambda^2(r+3)-2t^2\right]l_{r-1}(t) +\frac{r}{2}\lambda^2 t l_{r-2}(t)$

Theorem 6.6: Assume $\lim_{t\to a} tf(t)=0$, then the distribution of a rv X belongs to generalized Pearson family of distributions (1.32) if and only if its r^{th} order LPM satisfies a recurrence relation of the form

$$l_r(t)=C(t-a)^{r-2}+[(c_1+2c_2t)+rd_2]l_{r-1}(t)$$
$$-[(r-1)(d_1+2d_2t)+(c_0+c_1t+c_2t^2)]l_{r-2}(t)+(r-2)(d_0+d_1t+d_2t^2)l_{r-3}(t)$$

where $C=\frac{-1}{a_2}\left((b_0+b_1a-b_2a^2)f(a)\right)$, $c_i=\frac{a_i}{a_2}$ and $d_i=\frac{b_i}{a_2}$; $i=0,1,2$ are real constants provided $a_2\neq 0$, and when $a_2=0$,

$$l_r(t)=[r(d_1+2d_2t)+(c_0+c_1t)]l_{r-1}(t)-(r-1)(d_0+d_1t+d_2t^2)l_{r-2}(t)-C(t-a)^{r-1} \quad (6.24)$$

Where $c_i = \frac{a_i}{\left(a_1 + (r+1)b_2\right)}$; $i = 0,1$, $d_j = \frac{b_j}{\left(a_1 + (r+1)b_2\right)}$; $j = 1,2$ and

$C = \frac{-1}{\left(a_1 + (r+1)b_2\right)}\left((b_0 + b_1 a - b_2 a^2) f(a)\right)$ provided $\left(a_1 + (r+1)b_2\right) \neq 0$.

Proof: When $a_2 \neq 0$: Assume that the distribution of X belongs to the generalized Pearson family, multiply both sides of (1.32) by $(t-x)^{r-2}$ and on integration over the limits a to t we get (6.24).

Conversely assume (6.24), by using (6.2) and substituting for c_i and d_i; $i = 0,1,2$ we obtain

$$C(t-a)^{r-2} - \int_a^t (b_1 + 2b_2 x)(t-x)^{r-2} f(x)dx - \int_a^t (a_0 + a_1 x + a_2 x^2)(t-x)^{r-2} f(x)dx$$

$$+(r-2)\int_a^t (b_0 + b_1 x + b_2 x^2)(t-x)^{r-3} f(x)dx. \qquad (6.25)$$

Differentiating (6.25) r times and using (6.4) we get (1.32).

Case II: When $a_2 = 0$: By putting $a_2 = 0$ in (1.32) and then following similar steps to that of case I, we get the required result.

Some members of the generalized Pearson family and their corresponding values of the constants involved in the theorem 6.6 are given in Tables 6.2 and 6.3.

Table 6.2: Some members of generalized Pearson family and the values of the constants involved in the Theorem 6.6 (when $a_2 \neq 0$)

Members and Distribution	c_0	c_1	c_2	d_0	d_1	d_2
Inverse Gaussian $\left(\frac{\lambda}{2\pi t^3}\right)^{1/2} \exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\right)$; $t, \lambda, \mu > 0$	$-\mu^2$	$\frac{3\mu^2}{\lambda}$	1	0	0	$\frac{-2\mu^2}{\lambda}$

Maxwell $4\left(\frac{\lambda^3}{\pi}\right)^{1/2} t^2 \exp\left(-\lambda t^2\right)$; $t, \lambda > 0$	$\frac{-1}{\lambda}$	0	1	0	$\frac{-1}{2\lambda}$	0
Rayleigh $2\lambda t \exp\left(-\lambda t^2\right)$; $t, \lambda > 0$	$\frac{-1}{2\lambda}$	0	1	0	$\frac{-1}{2\lambda}$	0

Table 6.3: Some members of generalized Pearson family and the values of the constants involved in the Theorem 6.6 (when $a_2 = 0$)

Members and Distribution	c_0	c_1	d_0	d_1	d_2
Gamma $\frac{m^p}{\Gamma p} \exp(-mt) t^{p-1}$; $t > 0, m, p > 0$	$\frac{(1-p)}{m}$	1	0	$\frac{-1}{m}$	0
Pareto I $ck^c t^{-(c+1)}$; $t > k$, $c, k > 0$	0	$\frac{(c+1)}{(c-r)}$	0	0	$\frac{1}{(r-c)}$
Normal $\frac{1}{\sqrt{2\pi}\sigma} \exp -1/2\left(\frac{t-\mu}{\sigma}\right)^2$; $-\infty < t, \mu < \infty, \sigma > 0$	$-\mu$	1	$-\sigma^2$	0	0
Beta $\frac{d}{R}\left(1-\frac{t}{R}\right)^{d-1}$; $0 < t < R, d > 1$	0	$\frac{(d-1)}{(d+r)}$	0	$\frac{-R}{(d+r)}$	$\frac{1}{(d+r)}$
Exponential $\lambda \exp(-\lambda t)$; $t, \lambda > 0$	0	1	0	$\frac{1}{\lambda}$	0

Theorem 6.7: The distribution of X belongs to exponential family with pdf (1.28) if and only if its r^{th} order LPMs satisfy the recurrence relation

$$l_{r+1}(t) = \left(t + D'(\theta)\right) l_r(t) - \frac{d}{d\theta} l_r(t), \tag{6.26}$$

where $D'(\theta)$ is the derivative of $D(\theta)$ with respect to θ.

Proof: From the definition of LPMs

$$l_r(t) = \int_0^t (t-x)^r \exp\left(\theta x + C(x) + D(\theta)\right) dx. \tag{6.27}$$

Differentiating (6.27) with respect to θ, we get (6.26).

Conversely assume (6.26), then by using (6.2) and on simplification, we get

$$\frac{d}{d\theta} l_r(t) = \int_0^t (t-x)^r \left(x + D'(\theta)\right) f(x) dx. \tag{6.28}$$

Differentiating (6.28) $(r+1)$ times with respect to t and on simplification we get (1.28).

A list of various distributions with its pdf and the corresponding recurrence relationships using LPM are given in Table 6.4.

In the next section, we examine the various properties of LPM's in the context of weighted distributions.

6.4 Weighted models

In this case, the r^{th} order LPM corresponding to the weighted distribution at a point t is denoted as $l_r^w(t)$ and it is defined as

$$l_r^w(t) = E\left((X_w - t)^-\right)^r ; \; r = 0,1,2,\ldots, \; t > 0. \tag{6.29}$$

By using (1.1), (6.29) can be written as

$$l_r^w(t) = \frac{1}{\mu_w} \int_a^t (t-x)^r w(x) f(x) dx. \quad (6.30)$$

Now we find the relations connecting LPMs of some specific models

6.4.1 Length-biased models

The r^{th} order LPM corresponding to the length-biased model (1.2) is denoted as $l_r^L(t)$ and it is given by

$$l_r^L(t) = \frac{1}{\mu} \int_a^t (t-x)^r x f(x) dx. \quad (6.31)$$

Substituting $x = (t-(t-x))$, (6.31) becomes

$$l_r^L(t) = \frac{1}{\mu} \left(t l_r(t) - l_{r+1}(t) \right). \quad (6.32)$$

The following theorem characterizes power distribution using the relation connecting the r^{th} order LPMs of original and length-biased models.

Theorem 6.8: Let X be a non-negative rv with an absolutely continuous df $F(t)$ and $l_r(t)$ is defined as in (6.1). Assume $\lim_{t \to a} t f(t) = 0$, then the ratio of the r^{th} order LPMs of original and length-biased model satisfies a relation of the form

$$\frac{l_r^L(t)}{l_r(t)} = Ct \quad (6.33)$$

where $C > 0$ is a constant is satisfied for all $t > 0$ if and only if X follows power distribution with distribution function (4.5).

Proof: Suppose that the relation (6.33) holds. Using (6.32) and (6.2) and on simplification, we get

$$\left(\frac{1}{\mu} - C - 1 \right) \int_a^t (t-x)^{r+1} + \left(\frac{1}{\mu} - C \right) \int_a^t x (t-x)^r = 0. \quad (6.34)$$

Differentiating (6.37) $(r+1)$ times and on simplification using (6.4), we obtain

$$\lambda(t)=\frac{\left(\frac{1}{\mu}-C-1\right)}{\left(C-\frac{1}{\mu}\right)t}. \tag{6.35}$$

Now using the uniqueness property of reversed hazard rate, (6.35) implies the required result.

The converse part is obtained from Table 6.4.

Theorem 6.9: For a rv X defined in Theorem 6.8, and $\lim_{t\to a} tf(t)=0$, then the r^{th} order LPMs of original and length-biased model satisfies a relation of the form

$$(C_1 t + C_2) l_r(t) - l_r^L(t) = C_1 t^{r+1} \tag{6.36}$$

where $C_i(>0)$; $i=1,2$ are constants, holds for all $t>0$ if and only if X follows generalized Pareto distribution with distribution function (6.16).

Proof: Assuming (6.36), from (6.2) and (6.31), we have

$$C_1\int_a^t (t-x)^{r+1} f(x)dx + \int_a^t (C_3 x + C_2)(t-x)^r f(x)dx = C_1 t^{r+1} \tag{6.37}$$

where $C_3=\left(C_1-\frac{1}{\mu}\right)$, differentiating (6.37) $(r+1)$ times with respect to t and on simplification using (6.4), we get $\frac{f(t)}{1-F(t)}=\frac{C_1(r+1)}{(C_3 t + C_2)}$, which is the hazard rate of generalized Pareto distribution. From the uniqueness property of hazard rate, the remaining part of the theorem can be proved.

Conversely assume that X is specified by generalized Pareto distribution. Substituting (6.16) in (6.32) and using (6.2) and on simplification, we get (6.36) with $C_1 = \frac{(p+1)}{q(1-pr)}$ and $C_2 = \frac{(r+1)}{(1-pr)}$.

The following table provides LPM of some of the distributions in original and length biased case.

Table 6.4: $l_r(t)$ and $l_r^L(t)$ of certain probability distributions

Distribution	$f(t)$	$l_r(t)$	$l_r^L(t)$
Exponential	$\lambda \exp(-\lambda t)$; $\lambda > 0, t > 0$	$t^r - \frac{r}{\lambda} l_{r-1}(t)$	$(r+1+\lambda t) l_r(t) - \lambda t^{r+1}$
Pareto I	$ck^c t^{-(c+1)}$; $t > k$, $k, c > 0$	$\frac{c}{(c-r)}\left[(t-k)^r - \frac{rt}{c} l_{r-1}(t)\right]$	$\frac{(c-1)}{k(c-r-1)}\left[t l_r(t) - (t-k)^{(r+1)}\right]$
Power function	$\frac{c}{b^c} t^{(c-1)}$; $t > 0$, $b, c > 0$	$\frac{rt}{(r+c)} l_{r-1}(t)$	$\frac{(c+1)t}{b(r+c+1)} l_r(t)$
Beta	$Rd(1-Rt)^{d-1}$; $t > 0, R, d > 0$	$\frac{d}{(d+r)}\left[t^r - \frac{r}{Rd}(1-Rt) l_{r-1}(t)\right]$	$\frac{Rd(d+1)}{(d+r+1)}\left[\frac{(r+1+Rdt)}{Rd} l_r(t) - t^{r+1}\right]$
Pareto II	$pq(1+pt)^{-q-1}$; $t > 0, p, q > 0$	$\frac{q}{(q-r)}\left[t^r - \frac{r}{pq}(1+pt) l_{r-1}(t)\right]$	$\frac{pq(q-1)}{(q-r-1)}\left[\frac{(r+1+pqt)}{pq} l_r(t) - t^{r+1}\right]$
Uniform	$\frac{1}{(b-a)}$; $a < t < b$	$\frac{(t-a)^{r+1}}{(b-a)(r+1)}$	$\frac{2[t+(r+1)a]}{(b^2-a^2)(r+1)(r+2)}(t-a)^{r+1}$
Generalized Pareto	$q^{\frac{1}{p}+1}(p+1)\left(\frac{1}{(pt+q)}\right)^{\frac{1}{p}+2}$ $t > 0, p > -1, q > 0$	$\frac{1}{[1+p(1-r)]}\left[(p+1)t^r - r(pt+q) l_{r-1}(t)\right]$	$\frac{(p+1)}{q(1-pr)}\left[t l_r(t) - t^{r+1}\right] + \frac{(r+1) l_r(t)}{(1-pr)}$

6.4.2 Equilibrium Models

The equilibrium distribution arises naturally in renewal theory (see Cox (1962)). It is the distribution of the backward or forward recurrence time in the limiting case. A formal definition of the equilibrium distribution is as follows. Let X be a rv admitting an absolutely continuous distribution function $F(t)$ with respect to Lebesgue measure in the support of the set of non-negative real numbers. Associated with X, a rv X_E is defined with pdf

$$f_E(t) = \frac{R(t)}{\mu}; \quad t > 0 \tag{6.38}$$

The form of the equilibrium distribution (6.38) can also be obtained as a particular case of weighted distribution (1.1) with weight function $w(t) = \frac{R(t)}{f(t)}$. Then the r^{th} order LPM corresponding to the equilibrium model is denoted as $l_r^E(t)$ and it is defined as

$$l_r^E(t) = \frac{1}{\mu(r+1)}\left(t^{r+1} - l_{r+1}(t)\right); \ t > 0. \tag{6.39}$$

Theorem 6.10: Let X be a non-negative rv with an absolute continuous df $F(t)$, then a relationship $l_r(t) = l_r^E(t)$ is satisfied for all $t > 0$ if and only if X follows an exponential distribution (2.18).

Proof: Assume $l_r(t) = l_r^E(t)$, by using (6.38) and (6.2) we get

$$\frac{1}{\mu(r+1)}\left(t^{r+1} - \int_0^t (t-x)^{r+1} f(x)dx\right) = \int_0^t (t-x)^r f(x)dx. \tag{6.40}$$

Differentiating (6.40) $(r+1)$ times with respect to t and on simplification using (6.4) we obtain

$$\frac{f(t)}{1-F(t)} = \frac{1}{\mu}. \tag{6.45}$$

From the uniqueness property of hazard rate, (6.41) corresponds to exponential distribution. The converse part is obtained from Table 6.5.

Theorem 6.11: For a non-negative rv X having an absolute continuous df $F(t)$ and assume $\lim_{t\to 0} tf(t)=0$, then the r^{th} order LPM of original and equilibrium model satisfies a relation of the form

$$l_r^E(t)+Btl_r(t)=At^{r+1} \tag{6.42}$$

where $A>0$ and $B>0$ are constants, holds for all $t>0$ if and only if X follows power distribution with df (4.5).

Proof: Assume that (6.42) holds. Using (6.39), (6.2) and on simplification, we get

$$\left(\frac{1}{\mu(r+1)}-A\right)t^{(r+1)}-\left(\frac{1}{\mu(r+1)}-B\right)\int_a^t (t-x)^{r+1}f(x)dx+B\int_a^t x(t-x)^r f(x)dx=0. \tag{6.43}$$

Differentiating (6.43) $(r+2)$ times with respect to t and using (6.4) and the regularity condition, we obtain

$$\frac{f'(t)}{f(t)}=\frac{B_2}{t}, \tag{6.44}$$

where $B_2=\left(\frac{1}{\mu B}-(r+2)\right)$. Integrating (6.44) with respect to t, yields the required result. Converse part of the theorem is obtained from Table 6.5.

Theorem 6.12: For a non-negative rv X defined in Theorem 6.11, the r^{th} order LPM of original and equilibrium model satisfies a relation of the form

$$(At+B)l_r(t)-l_r^E(t)=At^{(r+1)} \tag{6.45}$$

where A, B (≥ 0) are constants holds for all $t>0$ if and only if X follows generalized Pareto distribution with distribution function (6.16).

Proof: Assume the relation (6.45) holds, from (6.2) and (6.39), we obtain

$$A_1\int_a^t (t-x)^{r+1}f(x)dx+\int_a^t (Ax+B)(t-x)^r f(x)dx=A_1t^{r+1} \tag{6.46}$$

where $A_1 = \left(A + \dfrac{1}{\mu(r+1)} \right)$. Assume $\lim_{t \to 0} t f(t) = 0$ and differentiating (6.46) $(r+1)$ times with respect to t and using (6.4), we get

$$\frac{f(t)}{1-F(t)} = \frac{A_1(r+1)}{(At+B)}. \tag{6.47}$$

From the uniqueness property of hazard rate, (6.47) provides (6.16). The proof of converse part is obtained from Table 6.5.

The r^{th} order LPMs of some probability distributions of original and equilibrium models are listed in Table 6.5.

Table 6.5: $l_r(t)$ and $l_r^E(t)$ of certain probability distributions

Distribution	$f(t)$	$l_r(t)$	$l_r^E(t)$
Exponential	$\lambda \exp(-\lambda t)$; $\lambda > 0, t > 0$	$t^r - \frac{r}{\lambda} l_{r-1}(t)$	$t^r - \frac{r}{\lambda} l_{r-1}(t)$
Pareto I	$ck^c t^{-(c+1)}$; $t > k$, $c, k > 0$	$\frac{c}{(c-r)}\left[(t-k)^r - \frac{rt}{c} l_{r-1}(t)\right]$	$\frac{(c-1)}{kc(r+1)}\left[t^{r+1} - \frac{c}{(c-r-1)}(t-k)^{r+1} - \frac{(r+1)t}{c} l_r(t)\right]$
Power function	$\frac{c}{b^c} t^{(c-1)}$; $t > 0$; $b, c > 0$	$\frac{rt}{(r+c)} l_{r-1}(t)$	$\frac{(c+1)}{bc}\left[\frac{t^{r+1}}{(r+1)} - \frac{t l_r(t)}{(r+c+1)}\right]$
Beta	$Rd(1-Rt)^{d-1}$; $t > 0$, $R, d > 0$	$\frac{d}{(d+r)}\left[t^r - \frac{r}{Rd}(1-Rt) l_{r-1}(t)\right]$	$\frac{(d+1)}{(d+r+1)}\left[Rt^{r+1} + (1+Rt) l_r(t)\right]$
Pareto II	$pq(1+pt)^{-q-1}$; $t > 0, p, q > 0$	$\frac{q}{(q-r)}\left[t^r - \frac{r}{pq}(1+pt) l_{r-1}(t)\right]$	$\frac{(q-1)}{(q-r-1)}\left[(1+pt) l_r(t) - pt^{r+1}\right]$
Uniform	$\frac{1}{(b-a)}$; $a < t < b$	$\frac{(t-a)^{r+1}}{(b-a)(r+1)}$	$\frac{2}{(b+a)(r+1)}\left[t^{r+1} - \frac{(t-a)^{r+2}}{(b-a)(r+2)}\right]$
Generalized Pareto	$q^{\frac{1}{p}+1}(p+1)\left(\frac{1}{(pt+q)}\right)^{\frac{1}{p}+2}$ $t > 0, a > -1, b > 0$	$\frac{1}{[1+p(1-r)]}\left[(p+1)t^r - r(pt+q) l_{r-1}(t)\right]$	$\frac{1}{q(1-pr)}\left[(pt+q) l_r(t) - pt^{r+1}\right]$

6.5 Applications

One of the main applications of LPM is that it can be used to find some poverty indices in the income analysis. Poverty measures are generally a kind of inequality measure that confines attention to a specified bottom slice of the income distribution, *i.e.* they only care for poor people. In measuring the indices of poverty, the most widely used statistic is the proportion of population that falls below the poverty line. The measures of poverty ignore most of the income distribution and often give substantial weight to an individual being or just below the poverty line whereas no weight is given to those slightly above the poverty line. These measurements involve two problems, the identification of the poor and the aggregation of information about the poor (see Sen (1976)). Most studies focus on income distribution as an indicator to identify the poor.

In the present context, suppose X represents the income of a community of individuals and define a minimum income requirement, the poverty line t, such that all individuals i who earn income $x_i < t$ are said to be poor and the rv $(X-t)^-$ takes the value $(t-x)$ and zero for those individuals whose income below or above poverty line respectively. Thus in income analysis, it is a useful measure for studying some poverty measures. Here the zero order LPM $l_0(t)$ gives the proportion of poor people and their income distribution is given as

$$ {}_tF_X(x) = \begin{cases} \dfrac{l_0(x)}{l_0(t)}; X \le t \\ 1; X > t \end{cases} \qquad (6.48) $$

(see Belzuence et al. (1995)), and $l_0^L(t)$ measures the proportion of total income earned by income units having income less than or equal to t. In income studies, an index, which measures how poor the poor are, is the income gap ratio $\alpha(t)$, where the income gap of an individual is $(t-x)$. Another measure useful in income analysis is $\mu(t)$, the average income below the poverty line and they are defined as

$$ \alpha(t) = \frac{l_1(t)}{t l_0(t)} \qquad (6.49) $$

$$ \mu(t) = t - \frac{l_1(t)}{l_0(t)}. \qquad (6.50) $$

From (6.49) and (6.50), it is clear that

$$\mu(t) = t\left(1-\alpha(t)\right). \tag{6.51}$$

Using the above relationships, the following theorems are immediate.

Theorem 6.13: Let X be a non-negative rv representing the income of a community of individuals and have an absolute continuous df $F(t)$, then the average income below poverty line satisfies a relation of the form

$$\mu(t) = \mu[1-t(1+Ct)\lambda(t)] \tag{6.52}$$

where C is a constant and $\lambda(t) = \dfrac{f(t)}{F(t)}$, if and only if X follows Pareto II, exponential and beta distributions with distribution functions (2.17), (2.18) and (2.19) respectively according as $C > 0$, $C = 0$ and $C < 0$.

Proof: Assume that the relation (6.52) holds. Using (6.50), (6.2) and on simplification, we get

$$\int_0^t xf(x)dx = \mu\left[F(t) - t(1+Ct)f(t)\right] \tag{6.53}$$

On differentiating (6.53) with respect to t using the assumption that $\lim_{t\to 0} tf(t) = 0$, we obtain

$$\frac{f'(t)}{f(t)} = \frac{(1+2\mu C)}{\mu(1+Ct)}. \tag{6.54}$$

Integrating (6.54) with respect to t yields the distributions Pareto II, exponential and beta according to $C > 0$, $C = 0$ and $C < 0$ respectively.

The converse part is obtained by direct calculation.

Remark 6.1: Even if the three distributions in Theorem 6.15 satisfy the relation (6.52), the inequality $\mu(t) \le t$ holds for only Pareto II and exponential distributions. For beta distribution, the inequality fails as it is not useful for modeling poverty data.

Corollary 6.2: The income gap ratio for the poor people satisfies the relationship

$$\alpha(t) = 1 - \frac{\mu}{t}\left(1 - t(1+kt)\lambda(t)\right) \tag{6.55}$$

for all $t > 0$ if and only if X follows Pareto II, exponential and Beta distributions respectively according as $k > 0, k = 0$ and $k < 0$.

Theorem 6.14: For a rv X defined in Theorem 6.13 and if $\lim_{t\to 0} g(t)f(t) = 0$, then the income gap ratio $\alpha(t)$ satisfies a relation of the form

$$\alpha(t) = 1 - \frac{1}{t}\left(\mu - g(t)\lambda(t)\right) \tag{6.56}$$

for all $t > 0$ if and only if the distribution belongs to the general family (1.31).

Proof: Assume the relation (6.56) holds, then using (6.49) and (6.2) and on simplification we get

$$\int_0^t (t-x)f(x)dx = tF(t) - \mu F(t) + g(t)f(t). \tag{6.57}$$

Differentiating (6.57) with respect to t and on simplification, we obtain the required result.

To prove the converse, assuming (1.31), and integrating (2.24) using the assumption and on simplification, we get

$$f(t)g(t) = (\mu - t)l_0(t) + l_1(t). \tag{6.58}$$

Dividing each term of (6.58) by $tl_0(t)$, we get (6.56).

Corollary 6.3: For the family given in (1.31) the average income below poverty line is

$$\mu(t) = \mu - g(t)\lambda(t).$$

Remark 6.2: Based on the properties of poverty measures, it has been known for some time that a close formal ties between risk and inequality exists. The income inequality arises in situations where not all people in a society earn the same income in a given period. Similarly, a distribution of returns is called risky if there are events where portfolio values are different. Besides, the resemblance of poverty and downside risk is also striking as both have their focus on the lower part of the distribution, concentrating on income of the poor and the bad outcomes respectively. The poverty line t in income studies divides the poor from the non poor corresponds to the critical line t that divides critical events with portfolio values less than or equal to t from the uncritical events with portfolio values greater than t. Thus the result that we obtained in poverty studies is also useful in downside risk studies where X represents the random return of a portfolio.

6.6 Bivariate lower partial moments

An extension to the univariate lower partial moment is quite straightforward. Let $\underline{X}=(X_1,X_2)$ be a bivariate random vector in the support of $(a_1,b_1)\times(a_2,b_2)$, $b_i>a_i$; $i=1,2$, where (a_i,b_i) is an interval in the real line with an absolutely continuous distribution function $F(t_1,t_2)$ with respect to Lebesgue measure. Assume $E\left(X_1^r X_2^s\right)<\infty$. Then the $(r,s)^{\text{th}}$ lower partial moment of $\underline{X}$ about $\underline{t}=(t_1,t_2)$ is

$$l_{r,s}(t_1,t_2)=E\left[(t_1-X_1)^r(t_2-X_2)^s I(X_1<t_1,X_2<t_2)\right]$$

$$=\int_{a_1}^{t_1}\int_{a_2}^{t_2}(t_1-x_1)^r(t_2-x_2)^s f(x_1,x_2)dx_1dx_2\ . \tag{6.59}$$

where $I(X_1<t_1,X_2<t_2)$ is an indicator function in the bivariate setup. Further, $l_{r,s}(t_1,t_2)$ uniquely determines the distribution function through the relation

$$F(t_1,t_2)=\frac{1}{r!s!}\frac{d^{r+s}}{dt_1^r dt_2^s}l_{r,s}(t_1,t_2)\,. \tag{6.60}$$

Table 6.6 provides the mathematical relationships among the LPM's of the bivariate weighted distribution for some important weight functions.

Table 6.6: Weighted BLPM for various weight functions

$w(t_1,t_2)$	Lower partial moment
t_1	$l_{r,s}^{w1}(t_1,t_2)=t_1 l_{r,s}(t_1,t_2)-l_{r+1,s}(t_1,t_2)$
t_2	$l_{r,s}^{w2}(t_1,t_2)=t_2 l_{r,s}(t_1,t_2)-l_{r,s+1}(t_1,t_2)$
$t_1 t_2$	$l_{r,s}^{w3}(t_1,t_2)=l_{r+1,s+1}(t_1,t_2)-t_2 l_{r+1,s}(t_1,t_2)-t_1 l_{r,s+1}(t_1,t_2)+t_1 t_2 l_{r,s}(t_1,t_2)$
t_1+t_2	$l_{r,s}^{w4}(t_1,t_2)=(t_1+t_2)l_{r,s}(t_1,t_2)-l_{r+1,s}(t_1,t_2)-l_{r,s+1}(t_1,t_2)$

6.7 Future study

The present study gave emphasis on characterizing continuous probability distributions and its weighted versions in univariate set up. Therefore a possible work in this direction is to study the properties of weighted distributions for truncated random variables in discrete set up. The problem of extending the measures into higher dimensions as well as its weighted versions is yet to be examined. As the present study focused attention to length-biased models, the problem of studying the properties of weighted models with various other weight functions and their functional relationships is yet to be examined. These works are proposed to be undertaken in the future study.

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