INCOME MODELING USING QUANTILE FUNCTIONS

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CERTIFICATE

Certified that the thesis entitled "INCOME MODELING USING QUANTILE FUNCTIONS" is a bonafide record of work done by Smt. Haritha N. Haridas under our guidance in the Department of Statistics, Cochin University of Science and Technology, Cochin and that no part of it has been included anywhere previously for the award of any degree or title.

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This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

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CHAPTER I

INTRODUCTION

Ever since the publication of the book 'Cours d' economie Politique by Vilfredo Pareto in 1897, the study of income distributions has continued to be a fertile area of research with improvements and implications of the Pareto model itself as the central theme and other viable alternatives to replace the Paretian logic with in a much broader framework that could explain the generating mechanism behind income data. What makes the study still interesting and challenging is the intricated random behaviour of incomes of individuals due to multiplicity of causes of changes such as those pertaining to structure of the populations, sources of income, behaviour of the economy, duration of stay of individuals in particular interval, to mention a few. While it is difficult to accommodate all the contributory causes, let alone account adequately for each, a model obtained under simplifying assumptions may ultimately render a reasonable fit to data, but only at the risk of discarding vital conclusions on the data due to the over-simplification already effected to achieve a pleasing model. A reasonable agreement with a probability distribution that could be conceived as representing size distribution of incomes in different countries, regions or strata of society over a period of time is still eluding research, keeping people busy with excavation of newer models along with methods of validation. We have made a comprehensive account of all important milestones in this journey covering an array of distributions when a review of literature on income distributions is taken up in the next chapter.

The creative work of Pareto demonstrated the usefulness of his distribution, not only in finding a suitable fit for the upper tail of income distributions in many instances, but also in other directions. In the first place it led to the consideration of fat tailed distributions either obtained as modifications or generalizations of the classical Pareto law and discussion of other heavy tail distributions in connection with income and other economic phenomena. Even when other models than Pareto were thought of, it became almost customary, especially in earlier works, to compare the behaviour of the tail of the new distribution with that of Pareto, beyond a certain value of the variable. Besides the empirical validity, the shapes of income and wealth distributions are unaltered to changes in the different measurement units such as family, personal or household. It also gave the impetus to developing various measures of income inequality during the pre-second world war period. The post-war period saw a re-surgence of the Pareto law, mainly from the characterizations obtained from those of the exponential law by applying monotone transformations. The dullness property of incomes, relation between true and reported incomes, residual incomes, truncation invariant Gini index etc. are such examples of identifying Pareto law as the only income model satisfying these properties.

Some unanswered questions while persisting with the Pareto law were, an inadequate justification for what economic phenomenon causes Pareto incomes especially in the upper tail, the determination of the point beyond which it holds and finally model that could apply for the entire range of incomes. It appears that the answers to these questions came in the introduction of flexible families of distributions with focus on obtaining good fit to income data from various sources. In the process most of the standard continuous distributions like exponential, Lomax, gamma, beta, Weibull, lognormal and some of their generalizations were proposed with means of estimating their parameters and various measures of income inequality. There were many attempts at explaining the data generating mechanism also, like the law of proportional effect leading to the lognormal distribution, income power models and diffusion models leading to Champernowne distribution etc., but most of them were not found holding universally. Such a scenario has thrown up opportunities to search for alternative methodologies to unify the existing procedures of data analysis.

The study of probability distributions as models in applied problems can be accomplished in two different ways; one by specifying the distribution function (or equivalently the probability density function) and other through the quantile distribution function. In modeling size distribution of incomes to our knowledge, all except one paper have adopted the first approach and proposed distribution functions. A modest beginning to the latter law of thought was made by Tarsitano (2004) when he used a general version of the Tukey Lambda family of quantile distributions to model income data. Different schools of thought prevail in forming the attitude towards analysis of statistical data, broadly classified as confirmatory and exporatory analyses. The former lays stress on what could be actually established from the data in a probabilistic approach while the latter is the willingness to look for what we expect the data to provide as well as exploration of things that are not explicit. It is argued that the second form of analysis is more complete and the quantile method is best suited for the same. Quantiles provide descriptive statistics in the data and are easily amenable to analytic treatment besides giving meaningful graphical representations. The treatment using quantile functions and distribution functions are theoretically equivalent as one implies the other. Several versions of quantile distributions have been introduced since the nineteen sixties as replacements for distribution functions, though it is always possible to transform one form into the other, especially for the standard distributions. An interesting feature of these quantile distributions is that they are families of distributions that are flexible enough to take different shapes as the data situation demands. They provide empirical models in the sense that, without much information of the physical properties of the data generating mechanism, one will be able to locate a member of the family that fits the data. In cases where some of physical properties are known or hypothesized, they are easily convertible in terms of quantiles so that the appropriate distribution can still be arrived at.

Tarsitano (2004) has used a four parameter quantile distribution function in his novel approach to modeling income distribution. Comparatively simple methods were adopted by him in estimation of the parameters of the model and measures of income inequality. The present study is a continuation of this direction of approach intended to supplement and strengthen the existing results.

A close analysis of the Tarsitano model reveals that there is a substantial limitation to the parameter space that compels the analyst to verify whether there exist models corresponding to the set of solutions obtained. It would therefore be advantageous to effect modifications in the model toward off this problem. A modified lambda family available in literature though slightly complex in form, satisfying our requirement is therefore a plausible candidate as the quantile distribution function. Another general problem common to all quantile functions is that there may exist more than one set of parameter values capable of generating a proper distribution. Often one may have to appeal to some property of the empirical distribution that is consistent with the parameter values in the ultimate choice. This difficulty can be avoided to a large extent if the estimation procedure itself is designed in such a way that the basic characteristics derived from the observations match with those in the theoretical model. The present work explores the possibilities of this aspect while inferring the parameters. Achievement of a good fit to the data does not necessarily mean that we have a meaningful model, unless it is shown that the distribution results from physical characteristics present in the observations. This means that we should be able to verify the presence of some characteristics associated with the income distribution with those in the data. Measures of income inequalities widely used for the purpose of characterizing income distributions are the basic tools employed for such verifications. The income gap ratios for the poor and affluent along with the truncated Gini indices have been proposed in earlier literature as characteristics suitable for differentiating income distributions. The problem of characterizing income distributions by the functional form of these quantities is still open. The present thesis is an attempt to resolve the above mentioned problems and the work in this connection is organized into five chapters with the following contents.

After this introduction, Chapter II provides a review on existing income distributions in literature. The properties of modified lambda family are discussed in Chapter III along with the justifications for using it as an income model. A new estimation procedure involving quantile measures of location, dispersion, skewness and kurtosis is introduced to estimate the parameters of modified lambda family in Chapter IV. Using this procedure, modified lambda family is fitted to a real income data. A simulation study is also conducted in the fourth chapter to compare the new estimation procedure with two other estimation techniques in the existing literature. Finally, in Chapter V, income distributions are characterized using the functional forms of income gap ratio and truncated Gini coefficient.

CHAPTER II[1]
REVIEW OF LITERATURE

2.1 Introduction

There are two general approaches to the study of income distributions, one motivated by the distribution of income among factors of production initiated by Ricardo (1817) and the other analyzing the distribution of total income of a given population receiving incomes making use of probability distribution functions. The latter stream of thought introduced by Pareto (1895) has generated considerable interest among statisticians and economists to produce a large body of literature on new models based on empirical as well as theoretical justifications. The dynamic nature of income distributions always leaves scope for refinements to existing models and theories to make this a fertile area of research. The present chapter undertakes a survey of the important results in this respect so as to provide the background material for further study and research.

The review material in this chapter is categorized into three sections. Section 2.2 discusses the probability distributions used as size distribution of incomes along with their justifications and inferential procedures that is of interest to distribution theory and practitioners of income modeling. A major theme in income analysis is how the distribution of income units is made on the basis of income size and the degree of income inequality. The literature on these concepts is surveyed in Section 2.3. The use of distribution functions in income modeling can be indirectly accomplished through quantile functions, which is a major tool employed in the present study. Accordingly we present some basic results in this area in Section 2.4.

[1] Part of the work in this chapter is being published in Nair & Haritha (2005).

2.2 Size distribution of incomes

One stream of research in income distribution models attempts to describe stochastic mechanisms that generate income distributions via certain properties. This class includes the work of Gibrat (1931), Champernowne (1953, 1973), Ericson (1945), Rutherford (1955), Wold and Whittle (1957), Lydall (1959) etc. For details of these models we refer to Arnold (1983). The present study being focused on the distribution function approach, we confine our attention largely to parametric families of distributions that are proposed for modeling income data.

The model most frequently applied in the literature to fit the distribution of personal income is the Pareto model. It is generally accepted for high income groups. Pareto (1895,1896) observed a decreasing linear relationship between the logarithm of income and the logarithm of the number N_x of income receivers with income greater than x, viz.

$$\log N_x = A - \alpha \log x, \quad x \geq \sigma.$$

Normalizing by the number of income receivers $N = N_\sigma$,

$$\frac{N_x}{N} = 1 - F(x) = \left(\frac{x}{\sigma}\right)^{-\alpha}, x \geq \sigma > 0 \tag{2.1}$$

which is the classical Pareto distribution, where $\alpha > 0$ and $\sigma > 0$ are respectively the shape (also measuring the heaviness of the right tail) and scale parameters. Champernowne (1953) demonstrated that under certain assumptions the stationary income distribution of an appropriately defined Markov process will approximate the Pareto distribution irrespective of the initial distribution. Mandelbrot (1961, 1964), Wold and Whittle (1957), Lydall (1959) and Sphilberg (1977) arrived at Pareto distribution from true models in different situations. Introducing an additional location parameter Pareto (1896) suggested Pareto(II) distribution with distribution function

$$F(x) = 1 - \left[1 + \left(\frac{x-\mu}{\sigma}\right)\right]^{-\alpha}, x > \mu; \alpha > 0; \sigma > 0. \tag{2.2}$$

The shape parameter α is sometimes interpreted as an index of inequality. The maximum likelihood estimates of Pareto I are given by

$$\hat{\sigma} = X_{1:n} \quad \text{and} \quad \hat{\alpha} = \left[\frac{1}{n} \sum_{i=1}^{n} \log \frac{X_i}{X_{1:n}} \right]^{-1}.$$

Quandt (1966) proposed the moment estimates (by equating the sample minimum and the sample mean to their corresponding expectations) as

$$\hat{\alpha}_M = \left(n\bar{X} - X_{1:n}\right) / n\left(\bar{X} - X_{1:n}\right)$$

and

$$\hat{\sigma}_M = \left(n \hat{\alpha}_M - 1 \right) X_{1:n} \Big/ n \hat{\alpha}_M$$

and the quantile estimates (selecting two probability levels p_1 and p_2 $[1 - p_i = P(X > x_i), i = 1,2]$ and substituting the corresponding sample quantiles $X_{[np_i]:n}$ for x_1, x_2 in $1 - p_i = \left(\frac{x_i}{\sigma} \right)^{-\alpha}, i = 1,2$) as

$$\hat{\alpha}_Q = \frac{\log\left[(1 - p_1)/(1 - p_2)\right]}{\log\left[X_{[np_2]:n} / X_{[np_1]:n} \right]}$$

and

$$\hat{\sigma}_Q = X_{[np_1]:n} (1 - p_1)^{\frac{1}{\hat{\alpha}_Q}}.$$

Several authors have considered the problem of deriving Bayes estimates for the parameters of the Pareto type I distribution. [Malik (1970), Zellner (1971), Lwin (1972), Rao Tummala (1977) and Sinha and Howlader (1980)]. Silcock (1954) and Harris (1968) proposed maximum likelihood estimates, Harris (1968) (when μ=0) and Arnold and Laguna (1977) used the method of moments, while Moore and Harter (1967, 1969), Kulldorff and Vannman (1973) and Vannman (1976) suggested estimates based on order statistics for the parameters of Pareto(II) distribution. The properties of the above distributions, interpretations for the parameters and all the above estimation techniques are discussed in detail in Arnold (1983). Classical regression type estimators and several recent developments, notably in connection with UMVU estimation are discussed in Kleiber and Kotz (2003).

The pioneering study marking initial use of the lognormal distribution as an economic size distribution was Gibrat's thesis of 1931. Assuming X(0) to be the initial income to which a random number N(t) of independent increments z_is are added so that at the end of a period of time the income X(t) of the individual becomes

$$\log X(t) = \log X(0) + \sum_{i=1}^{N(t)} z_i$$

Using central limit theorem $\log X(t)$ is asymptotically normal and hence $X(t)$ has an approximate lognormal distribution for large values of t. The probability density function of the lognormal distribution is given by

$$f(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(\log x - \mu)^2\right\}, x > 0, \sigma, \mu > 0 \quad (2.3)$$

This distribution as well as methods of estimation of parameters are discussed in detail in Aitchison and Brown (1957), Crow and Shimizu (1988), Johnson, Kotz and Balakrishnan (1994) and Kleiber and Kotz (2003). The maximum likelihood estimates are given by

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(\log x_i - \log \bar{x}).$$

Quensel (1944) found that the lognormal curve is the better approximation in the lower range of incomes. The Gibrat's law of proportional effect has not found acceptance as a generating mechanism of income distributions.

As an alternative to Pareto I which does not fit the entire range of income size adequately Champernowne (1937, 1952) suggested the distribution of log income Y=log X, termed income power and assumed that it has a density function of the form

$$f(y) = \frac{n}{Cosh\left[\alpha(y - y_0)\right] + \lambda}, -\infty < y < \infty, \alpha, \lambda, y_0, n > 0 \quad (2.4)$$

and fitted successfully income data pertaining to Bohemian and United Kingdom income data sets. A stochastic model leading to the

Champernowne distribution as the equilibrium distribution was briefly discussed by Ord (1975). This distribution has Paretian tails at both extremities, which was the initial attraction of the model. As far as estimation is concerned, Champernowne (1952) considered methods starting with some form of average income and Harrison (1974) suggested a minimum distance estimator, determining parameters simultaneously using an iterative generalized least squares approach.

Simon (1955,1958), Metcalf (1969) and Thurow (1970) proposed the Yule, displaced lognormal and beta distributions, with improved fit than Pareto and lognormal. But the interpretation of parameters is more difficult in these cases.

Salem and Mount (1974) approximated the distribution of personal income by a two parameter gamma density function with probability density function

$$f(x)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x},0<x<\infty,\alpha,\lambda>0. \tag{2.5}$$

They showed that the gamma density provides an alternative model that fits the income data by Amoroso (1925). Salem and Mount (1974) used maximum likelihood method for the estimation of parameters.

In Economics, Weibull distribution specified by

$$f(x)=\frac{a}{\beta}\left(\frac{x}{\beta}\right)^{a-1}e^{-\left(\frac{x}{\beta}\right)^{a}},x>0,a,\beta>0 \tag{2.6}$$

is probably less prominent, but D'Addario (1974) noticed its potential for income data. Bartels (1977), McDonald (1984), Atoda, Suruga and Tachibanaki (1988), Brachmann, Stich and Trede (1996) etc. fit Weibull distribution to incomes of various countries. Parameter estimation (maximum likelihood method) for the Weibull distribution is discussed in Cohen and Whitten (1988), Johnson, Kotz and Balakrishnan (1994) and Kleiber and Kotz (2003). The maximum likelihood estimators satisfy the equations

$$\hat{a} = \left[\left[\sum_{i=1}^{n} x_i^{\hat{a}} \log x_i\right]\left[\sum_{i=1}^{n} x_i^{\hat{a}}\right]^{-1} - \frac{1}{n}\sum_{i=1}^{n} \log x_i\right]$$

and

$$\hat{\beta} = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{a}}\right]^{\frac{1}{\hat{a}}}$$

for a random sample $(x_1,...,x_n)$ from (2.6).

Singh and Maddala (1976) derived the model with distribution function

$$F(x) = 1 - \left[1 + \left(\frac{x}{b}\right)^a\right]^{-q}, x > 0, a, b, q > 0 \tag{2.7}$$

which includes the Pareto and Weibull distributions as special cases. The distribution has been fitted to United States income data and has been found to fit remarkably well. Singh and Maddala (1976), McDonald and Ransom (1979), Dagum (1983), McDonald (1984), Majumder and Chakravarty (1990) fits this distribution to various income data and found to outperform almost all the other distributions so far discussed. Singh and Maddala (1976) estimated parameters by using a regression method minimizing

$$\sum_{i=1}^{n}\left\{\log[1 - F(x_i)] + q\log\left[1 + \left(\frac{x_i}{b}\right)^a\right]\right\}^2.$$

See also Stoppa (1995) for a regression method utilizing the elasticity $\frac{d\log F(x)}{d\log x}$ of the distribution. The likelihood equations are difficult to solve and therefore special methods to solve the equations are proposed by Mielke and Johnson (1974), Wingo (1983) and Watkins (1999). Another alternative is to employ the maximum product of spacings (MPS) estimation (Shah and Gokhale (1993)) which maximizes

$$H = \frac{1}{n+1}\sum_{i=1}^{n} \log\{F(x_i, \theta) - F(x_{i-1}, \theta)\}, \; i = 1, 2, ..., n+1,$$

with $x_0 = -\infty$ and $x_{n+1} = \infty$.

Dagum (1977) introduced the distribution with the distribution function

$$F(x)=\left[1+\left(\frac{b}{x}\right)^{a}\right]^{-p},x>0,a>0,b\geq 0,p\geq o \qquad (2.8)$$

and two further generalizations [Dagum type II and III, Dagum (1977,1980)] as models for the size distribution of personal income. The Dagum type II distribution has the cumulative distribution function

$$F(x)=\alpha+(1-\alpha)\left[1+\left(\frac{x}{b}\right)^{-a}\right]^{-p},x\geq 0 \; a,b,p>0\,,\alpha\in(0,1)\,. \qquad (2.9)$$

The type II distribution was proposed as a model for income distributions with null and negative incomes. The cumulative distribution function of Dagum type III distribution is given by

$$F(x)=\alpha+(1-\alpha)\left[1+\left(\frac{x}{b}\right)^{-a}\right]^{-p},a,b,p>0\,,\alpha<0\,,x_0<x<\infty$$

$x_0>0$, where $x_0=\left\{b\left[\left(1-\frac{1}{a}\right)^{\frac{1}{p}}-1\right]\right\}^{-\frac{1}{a}}$ is determined such that $F(x)\geq 0$.

(2.10)

The introduction of the model was justified on the basis of empirical observation that the income elasticity of the cumulative distribution function of income is a decreasing and bounded function of F. Kleiber (1996) showed that the Dagum and the Singh-Maddala income distributions are closely related through a reparametrization of the reciprocal random variable. That is,

$$X\sim SM(a,b,q)\Leftrightarrow\frac{1}{X}\sim D(a,\frac{1}{b},q)\,.$$

He exploited this relationship to derive Lorenz ordering results for the Dagum distributions from known results for the Singh-Maddala family. Further he explained why the Dagum distribution almost necessarily gives the better fit. Lukasiewicz and Orlowski (2004) showed that Dagum's model well describes distributions of incomes both in American and Polish households. As a further theoretical support Fattorini and Lemmi (1979) derived Dagum distribution as the equilibrium distribution of a continuous time stochastic process under certain assumptions on its infinitesimal mean and variance. Dagum (1977) discussed five methods for estimating the model parameters and recommended a nonlinear least squares method by minimizing

$$\sum_{i=1}^{n}\left\{F_n(x_i)-\left[1+\left(\frac{x_i}{b}\right)^{-a}\right]^{-p}\right\}.$$

A further regression type estimator utilizing the income elasticity of the cumulative distribution function was considered by Stoppa (1995).

The Fisk (1961a, b) distribution with cumulative distribution function

$$F(x)=\left[1+\left(\frac{x}{b}\right)^{-a}\right]^{-1}, x>0\,, a,b>0 \tag{2.11}$$

is a Singh-Maddala distribution with q=1, a Dagum distribution with p=1 and a special case of three parameter Champernowne distribution. We refer to the Pareto III distribution in Arnold (1983) for a variant of this model.

McDonald (1984) proposed a generalization of the beta distribution of the second kind, denoted GB(a,b,p,q) represented by the probability density function

$$f(x)=\frac{ax^{ap-1}}{b^{ap}B(p,q)\left[1+\left(\frac{x}{b}\right)^{a}\right]^{p+q}}, x\geq 0, a,b,p,q>0 \tag{2.12}$$

that could subsume the majority of the models suggested in literature. This distribution in (2.12) covers as special cases the Singh Maddala distribution (p=1), Dagum distribution (q=1), the beta distribution of the second kind (a=1), the Fisk distribution (p=q=1), Pareto type II distribution (a=p=1). See McDonald and Xu (1995) for further details. It is to be noted that the family is closely related to the type III and XII of the Burr family (Tadikammala (1980)). The generalization offered by Majumder and Chakravarty (1990) of the Singh-Maddala and the Dagum models is not reviewed separately it being only a reparametrized version of (2.12) (McDonald and Mantrala (1995)). Much work has not been done in estimating the parameters of (2.12) except Venter (1983) who considered ML estimation. Recently, Brazauskas (2002) obtained the Fisher information matrix of the GB2 distributions. For details one can refer Kleiber and Kotz (2003). Specializing (2.12) for a=1 one arrives at the beta distribution of the second kind with probability density function

$$f(x)=\frac{x^{p-1}}{b^{p}B(p,q)\left[1+\frac{x}{b}\right]^{p+q}},x>0 \qquad (2.13)$$

which appeared as a realistic model for Finnish income data (Vartia and Vartia (1980)).

Almost along the same logic the beta distribution of first kind used for modeling income data by Thurow (1970), was also extended to general form by McDonald (1984) with density function

$$f(x)=\frac{ax^{ap-1}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}}{b^{ap}B(p,q)},0\le x\le b \qquad (2.14)$$

where all the four parameters a, b, p, q are positive. Here b is a scale and a, p, q are shape parameters.

Esteban (1986) has shown that the (inverse) generalized gamma distribution with probability density function

$$f(x)=\frac{a}{\beta^{ap}\Gamma(p)}x^{ap-1}e^{-\left(\frac{x}{\beta}\right)^{a}},x>0;\beta,p>0;a<0 \qquad (2.15)$$

can be used as an income distribution.

Similar attempts at finding a more flexible family led McDonald and Xu (1995) to the five parameter distribution termed as the generalized beta (GB), with probability distribution function

$$f(x)=\frac{|a|x^{ap-1}\left(1-(1-c)\left(\frac{x}{b}\right)^{a}\right)^{q-1}}{b^{ap}B(p,q)\left(1+c\left(\frac{x}{b}\right)^{a}\right)^{p+q}},$$

$$\text{for } 0<x^{a}<b^{a}; 0\le c\le 1; b,p,q>0, \qquad (2.16)$$

and fitted this model to the 1985 US family incomes but selected the subfamily represented by (2.12).

Ripsy Bandarian et al. (2002) took ten special cases GB1, GB2,Beta 1, Beta 2, Generalized Gamma, Singh Maddala, Dagum, Lognormal, Gamma and Weibull and fitted them to 82 data sets comprising of 23 countries at different time periods. The conclusion of this empirical comparison was that the Weibull, Dagum and GB2 gave better fits and the number of shape parameters is a significant factor in improving the fit.

Reed and Jorgensen (2004) suggested a new parametric model named Double Pareto-Lognormal distribution (dPlN) for modeling size distributions. This distribution which has four parameters arises as that of the state of a geometric brownian motion (GBM) with lognormally distributed initial state, after an exponentially distributed length of time (or equivalently as the distribution of the killed state of such a GBM with constant killing rate). As an income distribution the explanation revolves around the assumption that an individual's earnings follows GBM and that the population of individuals is approximately growing at a fixed rate. Starting incomes are assumed to be lognormally distributed and evolving as GBM. The assumption of a growing population implies that the time that an individual has been earning is approximately exponentially distributed, and thus that current earnings or income follow close to that of a GBM killed with a constant killing rate. The probability density function of dPlN distribution in terms of the cumulative distribution function and complementary cumulative distribution function ϕ and ϕ^{c} of N(0,1) can be given by

$$f(x)=\frac{\alpha\beta}{\alpha+\beta}\left[A(\alpha,\upsilon,\tau)x^{-\alpha-1}\phi\left(\frac{\log x-\upsilon-\alpha\tau^2}{\tau}\right)+x^{\beta-1}A(-\beta,\upsilon,\tau)\phi^c\left(\frac{\log x-\upsilon+\beta\tau^2}{\tau}\right)\right]$$

$$\text{where } A(\theta,\upsilon,\tau)=\exp\left(\theta\upsilon+\frac{\alpha^2\tau^2}{2}\right). \qquad (2.17)$$

Examples of the fit of the dPIN (with plots) to various income distributions are presented in Reed (2003). The estimation techniques such as method of moments and maximum likelihood method have been discussed in Reed and Jorgensen (2004).

The book by Kleiber and Kotz (2003) investigated parametric statistical distributions of economic size phenomena of various types and listed Pareto, Lognormal, Gamma-type and Beta type distributions. They also discuss the properties, characterizations, inequality measures and estimation of parameters of the above distributions.

2.3 Measures of income inequality

In general measures of income inequality can be classified as intra distribution measures and inter distribution measures. The former is restricted to a single population of income receivers and the latter depicts the inequality between population of income receivers. Dalton (1920) introduced the following desiderata to be satisfied by intra distribution measures.

(1) Principle of transfers:- transferring of income from higher income receiver to lower one should reduce the measure.

(2) Principle of proportionate addition to incomes (scale independence):- proportionate addition or subtraction to all incomes should leave the measure unaffected.

(3) Principle of proportional addition to persons:- the measure should be invariant to proportionate additions to the population of income receivers.

(4) Principle of symmetry:- invariance of the measure to any permutation of income among the income receivers.

(5) Principle of normalization:- the range of the measure should be in the interval [0, 1] with zero(one) for perfect equality(inequality).

(6) Principle of operationality:- the intra distribution inequality measure should provide a unique, straight forward and unambiguous estimate of the income inequality by all researchers using the same observed or fitted income distribution, independently of their subjective inequality aversion.

The most widely used measure of inequality is the Lorenz Curve, (Lorenz (1905)) defined for finite populations as a function $L(u)$ on $[0,1]$ such that, for fixed u, $L(u)$ represents the proportion of the total income in the population accounted for by the $100u$% poorest individuals in the population. Gastwirth (1971) gave a rigorous definition by defining the corresponding inverse distribution function by $F^{-1}(y) = \sup\{x : F(x) \le y\}, o < y < 1$ and defining the Lorenz Curve by the equation

$$L(u) = \frac{\int_0^u F^{-1}(y)dy}{\int_0^1 F^{-1}(y)dy}, 0 < u < 1. \tag{2.18}$$

The significance of the Lorenz Curve lies in two derived measures of inequality viz., the Gini index [Gini (1914)] and Pietra index [Pietra (1932)] defined respectively as

$$G = 2\int_0^1 [u - L(u)]du \tag{2.19}$$

and

$$P = \frac{E|X - E(X)|}{2E(X)}. \tag{2.20}$$

Two other important income inequality measures are Atkinson (1970) measures

$$A_\varepsilon = 1 - \frac{1}{E(X)}\left\{\int_0^\infty x^{1-\varepsilon} dF(x)\right\}^{\frac{1}{1-\varepsilon}} \quad (2.21)$$

where $\varepsilon > 0$ is a sensitivity parameter giving more weight to the small incomes as it increases, and the generalized entropy measures [Cowell and Kuga (1981)]

$$GE_\theta = \frac{1}{\theta(\theta-1)}\int_0^\infty\left[\left[\frac{x}{E(X)}\right]^\theta - 1\right]dF(x)$$

As θ tends 0 and 1 we have

$$T_1 = GE_1 = \int_0^\infty \frac{x}{E(X)}\log\left[\frac{x}{E(X)}\right]dF(x) \quad (2.22)$$

and

$$T_2 = GE_0 = \int_0^\infty \log\left[\frac{E(X)}{x}\right]dF(x). \quad (2.23)$$

The latter two measures are known as the Theil coefficients (Theil (1967)). Some usual measures of dispersion suitably scaled have also been proposed in literature. They are

Absolute Mean Deviation:

$$\tau_1(X) = \int_0^\infty |x - E(X)|\, dF(x) \quad (2.24)$$

Relative Mean Deviation:

$$\tau_2(X) = \frac{1}{E(X)}\int_0^\infty |x - E(X)|\, dF(x) \quad (2.25)$$

Absolute Standard Deviation:

$$\tau_3(X) = \sqrt{\int_0^\infty (x - E(X))^2\, dF(x)} \quad (2.26)$$

Coefficient of Variation:

$$\tau_4(X) = \frac{\tau_3(X)}{E(X)} \quad (2.27)$$

Gini's Mean Difference:

$$\tau_5(X) = E|X_1 - X_2| \tag{2.28}$$

In the above notations the Gini index becomes

$$\frac{\tau_5(X)}{2E(X)}.$$

Ord et al. (1978) discussed a general class of inequality measures given by

$$\tau_g(X) = E[g(X)/E(X)]$$

where g is a convex function on $(0,\infty)$. In particular, $g(x) = \left(x^{\gamma+1} - 1\right)/\gamma(\gamma+1)$ and $\gamma = -2, -1, 0$ and 1 respectively can be associated with the ratio of the arithmetic to the harmonic mean, the ratio of the arithmetic to the geometric mean, the Theil index and the Herfindhal index (the squared coefficient of variation).

Ord et al. (1981) also suggested the entropy measures given by

$$e_\gamma(X) = \frac{1}{\gamma}\int_0^\infty f(x)\left[1 - f^\gamma(x)\right]dx, -1 < \gamma < \infty \tag{2.29}$$

to serve as inequality measures.

Frigyes (1965) proposed three measures which have direct economic interpretation (other inequality measures including gini index lack this property) and are given by

$$u = \frac{m}{m_1}, v = \frac{m_2}{m_1}, w = \frac{m_2}{m} \tag{2.30}$$

where $m = E(X)$, $m_1 = E\left(X \middle| X < m\right)$, $m_2 = E\left(X \middle| X \geq m\right)$. The measure v may be regarded as a measure of inequality for the entire income distribution, while u and w indicate the inequalities of the two respective parts of the distribution below and above the mean. The properties and applications of these measures have been discussed by Elteto amd Frigyes (1968).

Hart (1975) introduced the concept of moment distribution and explained some of the common measures in terms of moments of this distribution. The r^{th} moment distribution of a non-negative random variable X (with distribution function F) is given by

$$F_r(x) = \int_0^x t^r dF(t) \Big/ E\left(X^r\right) \tag{2.31}$$

The Lorenz curve can be described as the set of points in the unit square with coordinates $\left(F(x), F_1(x)\right)$ where x ranges from 0 to ∞. The function $L(u)$ may be defined implicitly by

$$u = \int_0^x dF(\xi)$$

and

$$L(u) = \frac{1}{E(X)} \int_0^x \xi dF(\xi) \tag{2.32}$$

Gini index can be related to the moment distribution as

$$G = 1 - 2\int_0^\infty F_1(x) dF(x) \tag{2.33}$$

Further expressions of other inequality measures in terms of moment distributions are given in Arnold (1983).

A class of linear measures of income inequality due to Mehran (1976) is

$$I = \frac{1}{\mu} \int_0^1 \left[F^{-1}(p) - \mu\right] W(p) dp \tag{2.34}$$

where $W(p)$ is a score function chosen independently of the shape of F with $\int_0^1 W(p) dp = 0$. Each score function defines a particular linear inequality measure.

The Gini index was generalized by Kakwani (1980), Donaldson and Weymark (1980, 1983) and Yitzhaki (1983) by applying different weight functions for the area under the Lorenz curve.

$$G_n = 1 - n(n-1)\int_0^1 L(u)(1-u)^{n-2}\,du\,, \quad n > 1\,. \tag{2.35}$$

Muliere and Scarsini (1989) observed that

$$G_n = 1 - \frac{E(X_{1:n})}{E(X)}$$

Zenga (1984) used the first moment distribution and the quantiles of the size distribution to define a new measure

$$Z(u) = 1 - \frac{F^{-1}(u)}{F_{(1)}^{-1}(u)}, 0 < u < 1 \tag{2.36}$$

$\{[u, Z(u)] | u \in (0,1)\}$ is often referred to as the Zenga concentration curve.

Most of these measures have been subjected to study for the Pareto distribution (see Arnold (1983)) and individual measures calculated for certain distributions. However, for analytic discussion of the various indices pertaining to a particular model or for comparing their relative behaviour in different distributions it is necessary to have an account of the measures for all the income models discussed in this chapter. In the absence of such a comprehensive list of measures of inequality we have presented in Table 2.1 some important measures relating to various distributions for reference.

Table 2.1:

Income Inequality measures

Distribution	Coefficient of Variation	Lorenz Curve	Gini Index
Pareto I	$\left(\alpha^2-2\alpha\right)^{-\frac{1}{2}}$	$1-\left(1-u\right)^{1-\frac{1}{\alpha}}$	$\frac{1}{2\alpha-1}$
Pareto II($\mu=0$)	$\alpha^{\frac{1}{2}}\left(\alpha-2\right)^{-\frac{1}{2}}$	$\alpha\left[1-\left(1-u\right)^{1-\frac{1}{\alpha}}\right]-\left(\alpha-1\right)u$	$\frac{\alpha}{2\alpha-1}$
Lognormal	$\sqrt{e^{\sigma^2}-1}$	$\phi\left[\phi^{-1}\left(u\right)-\sigma^2\right]$ (1)	$2\phi\left(\frac{\sigma}{\sqrt{2}}\right)-1$
Gamma	$\frac{1}{\sqrt{\alpha}}$	(2)	$\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha+1\right)\sqrt{\pi}}$
Weibull	$\frac{\Gamma\left(1+\frac{1}{a}\right)}{\left\{\Gamma\left(1+\frac{2}{a}\right)-\left[\Gamma\left(1+\frac{1}{a}\right)\right]^2\right\}^{\frac{1}{2}}}$	$\frac{\gamma\left(\frac{1}{a}+1,-\ln\left(1-p\right)\right)}{\Gamma\left(\frac{1}{a}+1\right)}$ (3)	$1-2^{-\frac{1}{a}}$
Singh Maddala	$\sqrt{\frac{\Gamma\left(q\right)\Gamma\left(1+\frac{2}{a}\right)\Gamma\left(q-\frac{2}{a}\right)}{\Gamma^2\left(1+\frac{1}{a}\right)\Gamma^2\left(q-\frac{1}{a}\right)}-1}$	(4) $I_{\left[1-\left(1-u\right)^{\frac{1}{q}}\right]}\left(1+\frac{1}{a},q-\frac{1}{a}\right)$	$1-\frac{\Gamma\left(q\right)\Gamma\left(2q-\frac{1}{a}\right)}{\Gamma\left(q-\frac{1}{a}\right)\Gamma\left(2q\right)}$
Dagum	$\sqrt{\frac{\Gamma\left(p\right)\Gamma\left(p+\frac{2}{a}\right)\Gamma\left(1-\frac{2}{a}\right)}{\Gamma^2\left(p+\frac{1}{a}\right)\Gamma^2\left(1-\frac{1}{a}\right)}-1}$	$1-I_{\left(u^{-1/p}-1\right)}\left(1-\frac{1}{a},p+\frac{1}{a}\right)$	$\frac{\Gamma\left(p\right)\Gamma\left(2p+\frac{1}{a}\right)}{\Gamma\left(2p\right)\Gamma\left(p+\frac{1}{a}\right)}-1$
Generalized Beta of second kind	$\sqrt{\frac{B\left(p,q\right)B\left(p+\frac{2}{a},q-\frac{2}{a}\right)}{B^2\left(p+\frac{1}{a},q-\frac{1}{a}\right)}-1}$	(5)	(6)
Generalized Gamma	$\sqrt{\frac{\Gamma\left(p\right)\Gamma\left(p+\frac{2}{a}\right)}{\Gamma^2\left(p+\frac{1}{a}\right)}-1}$	(7)	(8)

Contd..

Distribution	Generalized Gini Coefficient	Pietra Index
Pareto I	$\frac{n-1}{n\alpha-1}$	$\frac{(\alpha-1)^{-1}}{\alpha^{\alpha}}$
Pareto II($\mu=0$)	$\frac{\alpha(n-1)}{n\alpha-1}$	$\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$
Lognormal	(9)	$2\phi\left(\frac{\sigma}{2}\right)-1$
Gamma	(10)	$\left(\frac{\alpha}{e}\right)^{\alpha}\frac{1}{\Gamma(\alpha+1)}$
Weibull	$1-n^{-\frac{1}{a}}$	(11)
Singh Maddala	$\frac{\Gamma\left(nq-\frac{1}{a}\right)\Gamma(q)}{\Gamma(nq)\Gamma\left(q-\frac{1}{a}\right)}$	(12)
Dagum	$\frac{\Gamma(p)}{\Gamma\left(p+\frac{1}{a}\right)}\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\frac{\Gamma\left(jp+\frac{1}{a}\right)}{\Gamma(jp)}$	(13)
Generalized Beta of second kind	(14)	(15) $F(\mu)-F_{(1)}(\mu)$
Generalized Gamma	(16)	(17)

Contd..

Distribution	Generalized Entropy Measure	Atkinson's Measure
Pareto I	$\frac{1}{\theta(\theta-1)}\left\{\frac{(\alpha-1)^{\theta}\alpha\sigma^{\alpha\theta}}{(\alpha-\theta)}-1\right\}$	$1-\frac{\alpha-1}{\alpha\sigma}\left[\frac{\alpha\sigma^{1-\varepsilon}}{\alpha+\varepsilon-1}\right]^{1/1-\varepsilon}$
Pareto II($\mu=o$)	$\frac{1}{\theta(\theta-1)}\left\{(\alpha-1)^{\theta}\alpha B(\theta+1,\alpha-\theta)-1\right\}$	$1-\frac{\alpha(\alpha-1)}{\sigma^{\varepsilon}}B(2-\varepsilon,\alpha+\varepsilon-1)$
Lognormal	$\frac{e^{\frac{\sigma^2}{2}\left(\theta^2+\theta+1\right)}}{\theta(\theta-1)}$	$1-e^{-\frac{\sigma^2\varepsilon}{2}}$
Gamma	$\frac{1}{\theta(\theta-1)}\left[\frac{\Gamma(\theta+\alpha)}{\Gamma(\alpha)\alpha^{\theta}}-1\right]$	$1-\frac{1}{\alpha}\left[\frac{\Gamma(\alpha-\varepsilon+1)}{\Gamma(\alpha)}\right]^{1/1-\varepsilon}$
Weibull	$\frac{1}{\theta(\theta-1)}\left[\frac{\Gamma\left(\frac{\theta}{a}+1\right)}{\left(\Gamma\left(\frac{1}{a}+1\right)\right)^{\theta}}-1\right]$	$1-\frac{1}{\Gamma\left(1+\frac{1}{a}\right)}\left[\Gamma\left(\frac{1-\varepsilon}{a}+1\right)\right]^{1/1-\varepsilon}$
Singh Maddala	$\frac{1}{\theta(\theta-1)}\left[\frac{1}{q^{\theta-1}}\frac{B\left(\frac{\theta}{a}+1,q-\frac{\theta}{a}\right)}{\left(B\left(\frac{1}{a}+1,q-\frac{1}{a}\right)\right)^{\theta}}-1\right]$	$1-\frac{b}{B\left(\frac{1}{a}+1,q-\frac{1}{a}\right)}\left[B\left(\frac{1-\varepsilon}{a}+1,q-\frac{1-\varepsilon}{a}\right)\right]^{1/1-\varepsilon}$
Dagum	$\frac{1}{\theta(\theta-1)}\left[\frac{1}{p^{\theta-1}}\frac{B\left(1-\frac{\theta}{a},p+\frac{\theta}{a}\right)}{\left(B\left(1-\frac{1}{a},p+\frac{1}{a}\right)\right)^{\theta}}-1\right]$	$1-\frac{1}{pB\left(1-\frac{1}{a},p+\frac{1}{a}\right)}\left[pB\left(\frac{\varepsilon-1}{a}+1,p-\frac{\varepsilon-1}{a}\right)\right]^{1/1-\varepsilon}$
Generalized Beta of second kind	$\frac{1}{\theta(\theta-1)}\left\{\left[B(p,q)\right]^{\theta-1}\frac{B\left(\frac{\theta}{a}+p,q-\frac{\theta}{a}\right)}{\left(B\left(\frac{1}{a}+p,q-\frac{1}{a}\right)\right)^{\theta}}-1\right\}$	$1-\frac{B(p,q)}{B\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\left[\frac{B\left(\frac{1-\varepsilon}{a}+p,q+1-\frac{1-\varepsilon}{a}\right)}{B(p,q)}\right]^{1/1-\varepsilon}$
Generalized Gamma	$\frac{1}{\theta(\theta-1)}\left\{\left[\Gamma(p)\right]^{\theta-1}\frac{\Gamma\left(\frac{\theta}{a}+p\right)}{\left[\Gamma\left(\frac{1}{a}+p\right)\right]^{\theta}}-1\right\}$	$1-\frac{\Gamma(p)}{\beta\Gamma\left(p+\frac{1}{a}\right)}\left[\frac{\Gamma\left(p+\frac{1}{a}(1-\varepsilon)\right)}{\Gamma(p)}\right]^{1/1-\varepsilon}$

Expressions used in Table 2.1

(1) ϕ denotes the cumulative distribution function of the standard normal distribution.

(2) $\{[u, L(u)]\} = \{[F(x:\lambda,\alpha), F(x:\lambda,\alpha+1)] | x \in (0,\infty)\}$

(3) $\gamma(\alpha,x) = \int_0^x e^{-t} t^{\alpha-1}$

(4) $I_z(p,q) = \frac{1}{B(p,q)} \int_0^z \frac{t^{p-1}}{(1+t)^{p+q}} dt, z > 0$

(5) $\{[u, L(u)]\} = \left\{\left[F(x:a,b,p,q), F\left(x:a,b,p+\frac{k}{a},q-\frac{k}{a}\right)\right] | x \in (0,\infty)\right\}$

(6) $\frac{2B\left(2p+\frac{1}{a},2q-\frac{1}{a}\right)}{pB(p,q)B\left(p+\frac{1}{a},q-\frac{1}{a}\right)} \times$

$$\left\{\frac{1}{p}\,{}_3F_2\left[1,p+q,2p+\frac{1}{a};p+1,2(p+q);1\right] - \frac{1}{p+\frac{1}{a}}\,{}_3F_2\left[1,p+q,2p+\frac{1}{a};p+\frac{1}{a}+1,2(p+q);1\right]\right\}$$

(7) $\{[u, L(u)]\} = \left\{\left[F(x:a,\beta,p), F\left(x:a,\beta,p+\frac{1}{a}\right)\right] | x \in (0,\infty)\right\}$

(8) $\frac{1}{2^{2p+\frac{1}{a}} B\left(p,p+\frac{1}{a}\right)} \times \left\{\frac{1}{p}\,{}_2F_1\left(1,2p+\frac{1}{a};p+1;\frac{1}{2}\right) - \frac{1}{p+\frac{1}{a}}\,{}_2F_1\left(1,2p+\frac{1}{a};p+1+\frac{1}{a};\frac{1}{2}\right)\right\}$

(9) $1 - \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1-\phi(v+\sigma)]^{n-1} e^{-\frac{v^2}{2}} dv$

(10) $1-\frac{n\lambda^{n\alpha+1}}{\alpha(\Gamma(\alpha))^{n}}\int_{0}^{\infty}\Gamma^{n-1}(\alpha,\lambda;x)x^{\alpha}e^{-\lambda x}dx$

(11) $\beta\left[\Gamma\left(1+\frac{1}{a},\Gamma^{a}\left(1+\frac{1}{a}\right)\right)-\gamma\left(1+\frac{1}{a},\Gamma^{a}\left(1+\frac{1}{a}\right)\right)+\Gamma\left(1+\frac{1}{a}\right)\left(\gamma\left(1,\Gamma^{a}\left(1+\frac{1}{a}\right)\right)-\Gamma\left(1,\Gamma^{a}\left(1+\frac{1}{a}\right)\right)\right)\right]$

where $\gamma(\alpha,x)=\int_{0}^{x}u^{\alpha-1}e^{-u}du$ and $\Gamma(\alpha,x)=\int_{x}^{\infty}u^{\alpha-1}e^{-u}du$.

(12) $F_{SM}(\mu;a,b,q)-F_{GB2}\left(\mu;a,b,1+\frac{1}{a},q-\frac{1}{a}\right)$ where $F_{SM}(.)$ and $F_{GB2}(.)$ are the distribution functions of SinghMaddala and Generalised Beta of second kind respectively.

(13) $I_{\left(\frac{k}{b}\right)^{-a}}\left(-\frac{1}{a}+1,p+\frac{1}{a}\right)+\left(1+\left(\frac{k}{b}\right)^{-a}\right)^{-p}-1$

(14) $1-\frac{n}{B\left(p+\frac{1}{a},q-\frac{1}{a}\right)}\int_{0}^{\infty}\left[1-I_{u}(p,q)\right]^{n-1}\frac{u^{p+\frac{1}{\alpha}-1}}{(1+u)^{p+q}}du$

(15) $F(.)$ and $F_{(1)}(.)$ are the distribution function and the first moment distribution of Generalised Beta of second kind respectively.

(16) $1-\frac{n}{(\Gamma(p))^{n-1}\Gamma\left(p+\frac{1}{a}\right)}\int_{0}^{\infty}\Gamma^{n-1}(p,u)e^{-u}u^{p+\frac{1}{a}-1}du$

(17) $\frac{\beta}{\Gamma(p)}\left[\Gamma\left(p+\frac{1}{a},\left(\frac{k}{\beta}\right)^{a}\right)-\gamma\left(p+\frac{1}{a},\left(\frac{k}{\beta}\right)^{a}\right)\right]-\frac{k}{\Gamma(p)}\left[\Gamma\left(p,\left(\frac{k}{\beta}\right)^{a}\right)-\gamma\left(p,\left(\frac{k}{\beta}\right)^{a}\right)\right]$

2.4 Tukey Lambda Distributions

In the previous sections we discussed various forms of the distribution functions that could represent data on incomes. An alternative way of describing a continuous distribution is to use the quantile functions defined as

$$Q(u) = F^{-1}(u) = \inf(x | F(x) \ge u),\ 0 < u < 1.$$

Since $F(x) \ge u$ if and only if $Q(u) \le x$, the knowledge of the form of $Q(u)$ is equivalent to the knowledge of the functional form of $F(x)$. Taking account of this aspect, Hastings et al. (1947) introduced the one parameter symmetric lambda distribution given by the quantile function

$$Q(u) = \frac{u^{\lambda} - (1-u)^{\lambda}}{\lambda} \tag{2.37}$$

where u follows a uniform distribution in $(0,1)$. Subsequently several symmetric and asymmetric forms with increased number of parameters of $Q(u)$ were introduced as alternative models from a variety of applications. Of these the four parametric version introduced by Ramberg and Schmeiser (1974) is specified by

$$Q(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2},\ 0 < u < 1, \tag{2.38}$$

where λ_1 is the location parameter, λ_2 the scale parameter and λ_3 and λ_4 determine the skewness and kurtosis of the distribution respectively. Model (2.38) is widely used in literature because of its ability to give distributions of different shapes. The density function is

$$f(x) = \frac{1}{Q'(u)}$$

$$= \frac{\lambda_2}{\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}} \tag{2.39}$$

Notice that (2.39) specifies a valid distribution if and only if $\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}$ has the same sign (positive or negative) for all u in $[0,1]$, as long as λ_2 takes that sign also. The four regions of parameter values where the GLD is a legitimate probability distribution are given in Table 2.2.

Table 2.2 :

Parameter space of GLD

Region	Value of			LowerBound	UpperBound
	λ_2	λ_3	λ_4		
1	$\lambda_2 < 0$	$\lambda_3 < -1$	$\lambda_4 > 1$	$-\infty$	$\lambda_1 + 1/\lambda_2$
2	$\lambda_2 < 0$	$\lambda_3 > 1$	$\lambda_4 < -1$	$\lambda_1 - 1/\lambda_2$	∞
3	$\lambda_2 > 0$	$\lambda_3 > 0$	$\lambda_4 > 0$	$\lambda_1 - 1/\lambda_2$	$\lambda_1 + 1/\lambda_2$
	$\lambda_2 > 0$	$\lambda_3 = 0$	$\lambda_4 > 0$	λ_1	$\lambda_1 + 1/\lambda_2$
	$\lambda_2 > 0$	$\lambda_3 > 0$	$\lambda_4 = 0$	$\lambda_1 - 1/\lambda_2$	λ_1
4	$\lambda_2 < 0$	$\lambda_3 < 0$	$\lambda_4 < 0$	$-\infty$	∞
	$\lambda_2 < 0$	$\lambda_3 = 0$	$\lambda_4 < 0$	λ_1	∞
	$\lambda_2 < 0$	$\lambda_3 < 0$	$\lambda_4 = 0$	$-\infty$	λ_1

While any λ_3, λ_4 values in these regions produce proper statistical distributions, the regions do not include all the λ_3, λ_4 values that do so. Karian, Dudewicz and Mc Donald (1996) report that sections of the regions excluded by Ramberg and Schmeiser (1974) can also produce proper statistical distributions. They therefore added two new regions, adjoining regions 1 and 2.

For $\lambda_1 = 0$, the k^{th} moment of (2.38) is given by

$$E\left(X^k\right) = \lambda_2^{-k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i B\left(\lambda_3(k-i)+1, \lambda_4 i+1\right) \qquad (2.40)$$

when it exists, where $B(\alpha,\gamma)$ denotes the beta f unction evaluated at α,γ. Ramberg et al.(1979), Ozturk and Dale (1985), King and MacGillivray (1999), Karian and Dudewicz (1999), King and MacGillivray (2006) discussed some estimation procedures for (2.38). Mac Gillivray (1982) and Groeneveld (1986) discuss some properties of this family.

Tarsitano (2004) proposed (2.38) as a flexible and adaptable model to fit the distribution of incomes. He discussed the methods of least squares, quantiles and moments to fit the model to grouped income data. Some well known income inequality measures of the distribution are also evaluated in that paper.

CHAPTER III
MODIFIED LAMBDA FAMILY

3.1 Introduction

In the previous chapter we have seen that a wide variety of statistical distributions have been considered as possible models for incomes. Some of these models are derived by postulating the data generating mechanism through stochastic processes or by specifying the physical characteristics governing the income distribution through concepts in economics. In situations where none of these approaches satisfactorily represent the complexities in the observations a statistical model is selected for its ability to fit the given data and then it is used to analyse the income characteristics as a best approximation. Recently in statistical practice, the use of the quantile function (in the place of the distribution function) in data analysis and inference is finding greater acceptance due to its simplicity and flexibility. As pointed out in the previous chapter, Tarsitano (2004) proposed the generalized lambda distribution introduced by Ramberg and Schmeiser (1974), as a flexible and adaptable model to fit the distribution of incomes. The density function for generalized lambda family cannot be expressed in closed form. However, it can be expressed in terms of its quantile function. But the model proposed by Tarsitano (2004) is not valid in the entire parametric space. In order to avoid this problem in the present study, we consider another four- parameter generalized lambda distribution proposed by Freimer et al. (1988) for modeling income. In Section 3.2 we present a general theory of quantile function, including its properties, relative advantages over the distribution function approach and the various characteristics of the distribution such as location, dispersion, skewness, kurtosis and shape, in terms of quantile functions. This will form the essential background material for further discussions in the present thesis. Following this, in Section 3.3, the modified lambda family of Freimer et al. (1988) is studied with respect to the quantile function approach, by describing several new properties of the family that were not illuminated in the basic paper of the authors. The focus in our study being the use of the distribution to income modeling in Section 3.4 the conventional measures of income inequality are reframed in terms of quantile measures to facilitate their subsequent analysis.

3.2 Quantile functions

Historically, the idea of quantiles seems to have originated in the work "Statistics by inter comparison : with remarks on the Law of Frequency of Error" by Francis Galton published in Philosophy Magazine in 1875, although the term quantiles was first introduced only by Kendall (1940). Tukey's (1970) work on exploratory data analysis (as against prevailing confirmatory analysis) and Parzen's (1979) paper stimulated the development of quantile functions as an essential tool instead of the distribution function in statistical analysis.

For a general distribution function which is continuous from the right

$$Q(p) = F^{-1}(p) = \inf\{x : F(x) \geq p\} \tag{3.1}$$

is defined as the quantile function, which has the fundamental property that for every $0 \leq p \leq 1$, $F(x) \geq p$, iff $Q(p) \leq x$, where $-\infty < x < \infty$. When the functions $Q(p)$ and $F(x)$ are continuous and increasing in the respective arguments, $Q(p) = F^{-1}(p)$ and $F(x) = Q^{-1}(x)$. The derivative of $Q(p)$, is called the quantile density function denoted by $q(p)$. Thus

$$q(p) = \frac{dQ(p)}{dp} \tag{3.2}$$

is non-negative for $0 \leq p \leq 1$. Various properties of $Q(p)$ that makes it useful in modeling and analysis of statistical data include

(i) $-Q(1-p)$ is the reflection of $Q(p)$ on the line $x = 0$.

(ii) If $Q_i(p)$'s are quantile functions $\sum_{i=1}^{n} Q_i(p)$ is also a quantile function.

(iii) $Q_i(p) > 0$, $i = 1,2$ implies $Q_1(p)Q_2(p)$ also represent a distribution quantile function.

(iv) $Q(p) = \alpha Q_1(p) + (1-\alpha)Q_2 p$, $0 \le \alpha \le 1$,

the mixture of two quantile functions, lies between the two distributions with quantile functions $Q_1(p)$ and $Q_2(p)$.

(v) If $Q_1(p)$ has zero median and unit interquartile range (or some other measures of location and scale) then

$Q_2(p) = \mu + \sigma Q_1(p)$ has location μ and scale σ.

(vi) If X has quantile function $Q(p)$, $(X)^{-1}$ has quantile function $(Q(1-p))^{-1}$.

(vii) For a non-decreasing function $H(p)$, $0 \le p \le 1$, with $H(0) = 0$ and $H(1) = 1$, $Q(H(p))$ is a quantile function in the same range of $Q(p)$.

It is clear from (i) through (vii) that many of the properties of $Q(p)$ are not shared by the distribution function $F(x)$ which brings the advantages of using the quantile functions and its flexibility, especially in modeling problems. Just like the distribution function, but with lesser effort (dispensing with the expected values that require integration of functions with respect to the density function) the characteristics of the distribution such as location, dispersion etc. can be worked out directly from the $Q(p)$ function. We briefly discuss the quantile measures associated with the distribution.

Generally, the distribution parameters of the position are the

Median, $M = Q(0.5)$, (3.3)

lower quartile $Q_1 = Q(0.25)$ and the upper quartile $Q_3 = Q(0.75)$.Sometimes along with M, Q_1 and Q_2, we may use $Q(0)$ and $Q(1)$ to get a feel of the spread of the distribution. As a measure of dispersion we have the interquartile range

$$IQR = Q_3 - Q_1 \qquad (3.4)$$

and for skewness, the Galton coefficient of skewness

$$S = (Q_1 + Q_3 - 2M)/Q_3 - Q_1 \qquad (3.5)$$

Notice that S is independent of position and scale and lies in (-1,1), with $S = 0$ indicating a symmetric distribution and a large positive S is indicative of a long right tail. All these three measures M, IQR and S are available in classical literature on descriptive statistics and many analysts have favoured the mean, variance and Pearson coefficient of skewness β_1 in their places in new of the developments that took place after the discovery of the Pearson family of distributions that are uniquely characterized by the four-tuple (mean, variance, β_1, β_2) or by the first four moments. The non-robustness of these measures, susceptibility to outliers, instability of corresponding sample characteristics while matching with population values, all have made model building through them far from universally acceptable. While the relative advantages of median and interquartile range are well documented in literature, the role of β_1 as a measure of skewness is also subject to scrutiny. The relative position that mean should be greater than median for a positively skewed distribution which is basic to the concept of skewness is not satisfied by on β_1 values, $\beta_1 = 0$ holds for asymmetric distributions, unusually abrupt changes in β_1 for relatively small changes in the parameter values etc have motivated several proposals for alternative measures of skewness and re-affirmation of the utility of S. Pearson's second β_2 as a measure of kurtosis is also not free of criticism. For a standardized variable Z, the relationship

$$E(Z^4) = V(Z^2) + 1$$

would mean that the interpretation of β_2 depends on the concentration of probability near the central tendency as well as at the tails of the distribution.

Moors (1988) have proposed a new quantile measure of kurtosis that takes into account the above two aspects viz

$$T = \left[(e_7 - e_5) + (e_3 - e_1)\right] / IQR$$
$$= \left[Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)\right] / IQR \qquad (3.6)$$

so that e's are the octiles of the distribution. In addition to these standard quantile measures one can also use the shape index

$$s(p) = \frac{Q(1-p) - Q(p)}{IQR} \ , \quad 0 \le p \le 0.5 \qquad (3.7)$$

and the skewness ratio

$$S(p) = \frac{Q(1-p) - M}{M - Q(p)} \qquad (3.8)$$

in making judgements on the shape and skewness.

Sometimes a median based measure of dispersion, that is often a more robust estimator than the variance, called median absolute deviation defined as

$$MAD = Median(|X - M|)$$

is used. Further the correlation between two random variables X and Y can be proposed by extending the definition of MAD in the form (Falk (1997))

$$\xi(X,Y) = \frac{Median\left[(X - M_X)(Y - M_Y)\right]}{MAD(X)MAD(Y)}$$

Although the emphasis made so far is on descriptive measures based on quantiles, the evaluation of moments to facilitate the mean, variance, β_1 and β_2 is equally relevant and straightforward. In fact, $F(x) = p$ implies $f(x)dx = dp$ and therefore

$$E(X^r) = \int_0^1 \left[Q(p)\right]^r dp. \qquad (3.9)$$

More generally, for any non-decreasing transformation $T(X)$ of X

$$E\left[T(X)\right] = \int_0^1 T(Q(p))dp \qquad (3.10)$$

enables one to look at expected values of most desirable functions. As pointed our earlier higher order moments when used in inference and model building provide statistics with huge variability, especially in situations of multi-parameter distributions. The contributory factor to such instabilities is the use of higher powers of X, which suggests that employing linear functions can solve the problem to some extent. Accordingly linear functions of order statistics are considered for the purpose of describing distributional characteristics. Denoting by $X_{r:n}$ the rth order statistic from a sample of size n, the first four moments (called L-moments) in samples of 1,2,3 and 4 is defined as (Gilchrist (2000))

$$\rho_1 = E(X_{1:1}) \tag{3.11}$$

$$\rho_2 = E\tfrac{1}{2}(X_{2:2} - X_{1:2}) \tag{3.12}$$

$$\rho_3 = E\tfrac{1}{3}(X_{3:3} - 2X_{2:3} + X_{1:3}) \tag{3.13}$$

and

$$\rho_4 = E\tfrac{1}{4}(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}). \tag{3.14}$$

This leads to formation of L - Coefficient of Variation

$$CV = \rho_2/\rho_1 \tag{3.15}$$

Skewness $S = \rho_3/\rho_2$ (3.16)

Kurtosis $K = \rho_4/\rho_2 \quad , \quad \frac{5S^2 - 1}{4} < K < 1$ (3.17)

Using the relationship between the quantile density function $f_{(r)}(p)$ of $X_{(r)}$ with that of X,

$$EX_{r:n}{}^{k} = \frac{n!}{(r-1)!(n-r)!}\int_0^1 Q^k(p)\,p^{r-1}(1-p)^{n-r}\,dp$$

$$\rho_2 = \int_0^1 Q(p)(2p-1)\,dp \tag{3.18}$$

$$\rho_3 = \int_0^1 Q(p)\left(6p^2 - 6p + 1\right)dp \tag{3.19}$$

$$\rho_4 = \int_0^1 Q(p)\left(20p^3 - 30p^2 + 12p - 1\right)dp \tag{3.20}$$

Another type of moments used in the context of quantile functions is probability-weighted moments (PWM) given by

$$w_{t,r,s} = E\left[Q^t p^r (1-p)^s\right] \tag{3.21}$$

where generally t is taken to be unity to avoid inclusion of powers that may lead to complications in the manner explained for X^r. Two interesting special cases are then $t=1$, $s=0$ and $t=1$ and $r=0$, giving $w_{r,0}$ and $w_{0,s}$, satisfying

$$w_{r,0} = \sum_{s=0}^{r} \binom{r}{s} (-1)^s w_{0,s} \text{ and } w_{0,s} = \sum_{r=0}^{s} \binom{s}{r} (-1)^r w_{r,o} .$$

3.3 Modified Lambda Family

A disturbing feature of the Tarsitano model (2.38) for incomes is that it is not always a quantile function and provides a proper density function of X only for certain regions of the parameter space viz $(\lambda_3 \geq 0, \lambda_4 \geq 0)$, $(\lambda_3 \leq 0, \lambda_4 \leq 0)$, $(\lambda_3 > 1, \lambda_4 < -1)$ and $(\lambda_3 < -1, \lambda_4 > 1)$. Further no positive moments exist for the last two regions (Ramberg et al. (1979)). Motivated by these limitations which adversely affect the fitting process and to utilize the advantages of the quantile function approach, we consider the modified lambda family (MLF) as an income model in the present work introduced by Friemer et al. (1988) as an alternative to the Ramberg and Schmeiser (1974) lambda distribution discussed in Chapter II.

3.3.1 Quantile Function

The quantile function of MLF is given by

$$Q(p) = \lambda_1 + \frac{1}{\lambda_2}\left[\frac{p^{\lambda_3}-1}{\lambda_3} - \frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right], \lambda_1, \lambda_2, \lambda_3, \lambda_4 \text{ real.} \tag{3.22}$$

Here λ_1 is a location parameter, λ_2 is a scale parameter and λ_3 and λ_4 determine the shape of the distributions in the family. This parameterization is well defined for the values of the shape parameters (λ_3,λ_4) over the entire two dimensional plane in a continuous manner. Freimer et al. (1988) considers the canonical form of the MLF (3.22) obtained by setting $\lambda_1=0$ and $\lambda_2=1$ while discussing the properties of the family. Since λ_1 and λ_2 are also essential parameters in income modeling we look at the four parameter version and present the properties of the latter in the present chapter. Thus the range of the random variable X in modification of the results in Freimer et al. (1988) becomes

$$\left(\lambda_1-\frac{1}{\lambda_2\lambda_3},\lambda_1+\frac{1}{\lambda_2\lambda_4}\right) \text{ if } \lambda_3,\lambda_4\geq 0;$$

$$\left(\lambda_1-\frac{1}{\lambda_2\lambda_3},\infty\right) \text{if } \lambda_3>0,\lambda_4\leq 0;$$

$$\left(-\infty,\lambda_1+\frac{1}{\lambda_2\lambda_4}\right) \text{ if } \lambda_3\leq 0,\lambda_4>0$$

and

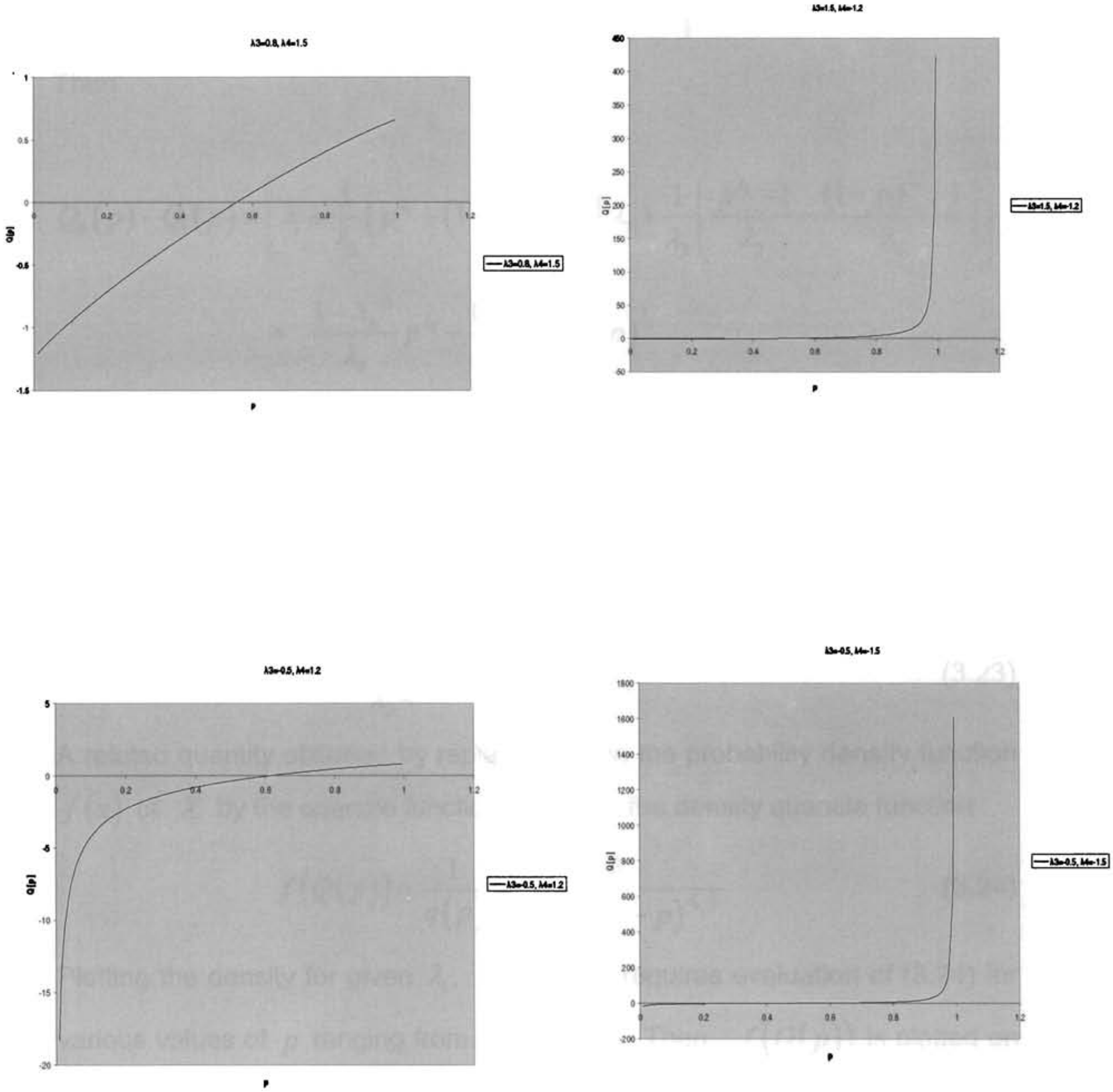
$(-\infty,\infty)$ if both λ_3 and λ_4 are ≤ 0.

We discuss below only those properties that pertain to the four parameter model and refer to Freimer et al. (1988) for other characteristics that are invariant with respect to λ_1 and λ_2. Figure 3.1 illustrates the shapes of quantile function for certain values of the parameters.

Some comments regarding the interpretation of the model (3.22) and its comparison with the Ramberg and Schmeiser family, not stated in earlier papers on the subject, seems to be in order. In the first place the component p^{λ_3} is the quantile function corresponding to the power distribution $F(x)=x^{\frac{1}{\lambda_3}}$, $0<x<1$ and the factor $(1-p)^{\lambda_4}$ to the quantile function of the Pareto distribution $F(x)=1-x^{\frac{1}{\lambda_4}}$, $x>1$. The two distributions have special relevance as models of income. Further (3.22) can be thought of

Figure 3.1

Quantile Functions



as generated to be the sum of two quantile functions (with appropriate location and scale changes) according the result stated in (ii) of Section 3.2. To compare the Ramberg-Schmeiser (1974) and Friemer et al. (1988) families, we assume without loss of generality that the two have the same set of parameters with $Q_1(p)$ and $Q_2(p)$ as their respective quantile functions. Then

$$Q_2(p)-Q_1(p)=\left[\lambda_1+\frac{1}{\lambda_2}\left(p^{\lambda_3}-(1-p)^{\lambda_4}\right)\right]-\left[\lambda_1+\frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}-1}{\lambda_3}-\frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right)\right]$$

$$=\frac{1-\lambda_3^{-1}}{\lambda_2}p^{\lambda_3}-\frac{1-\lambda_4^{-1}}{\lambda_2}(1-p)^{\lambda_4}$$

so that the changes between the two is basically in scale and in the shape.

3.3.2 Density Quantile Function

The quantile density function is obtained from (3.2) as

$$q(p)=\frac{1}{\lambda_2}\left[p^{\lambda_3-1}+(1-p)^{\lambda_4-1}\right] \tag{3.23}$$

A related quantity obtained by replacing x, in the probability density function $f(x)$ of X by the quantile function $Q(p)$ is the density quantile function

$$f(Q(p))=\frac{1}{q(p)}=\frac{\lambda_2}{p^{\lambda_3-1}+(1-p)^{\lambda_4-1}} \tag{3.24}$$

Plotting the density for given λ_1, λ_2, λ_3, λ_4 requires evaluation of (3.24) for various values of p ranging from zero to one. Then $f(Q(p))$ is plotted on the Y-axis versus $Q(p)$ on the X-axis. Eventhough λ_1 does not explicitly appear in (3.24), the density is a function of λ_1 since it is defined in terms of $Q(p)$, which depends upon λ_1, as can be seen from (3.22). The ordinates at the extremes are given by

$$f(Q(0))=f(Q(1))=\lambda_2 \qquad \text{when } \lambda_3>1 \text{ and } \lambda_4>1$$

$$=0 \qquad \text{when } \lambda_3<1 \text{ and } \lambda_4<1$$

$$f(Q(0))=0 \text{ and } f(Q(1))=\lambda_2 \quad \text{when } \lambda_3<1 \text{ and } \lambda_4>1$$

$$f(Q(0))=\lambda_2 \text{ and } f(Q(1))=0 \quad \text{when } \lambda_3>1 \text{ and } \lambda_4<1.$$

A detailed study of density shape classification of MLF is given in Freimer et al. (1988). MLF contains unimodal ($\lambda_3,\lambda_4<1$ or $\lambda_3,\lambda_4>2$), U-shaped (λ_3 and λ_4 lies in $[1,2]$), J-shaped ($\lambda_3>2$ and λ_4 in $[1,2]$) and monotone ($\lambda_3>1$ and $\lambda_4<1$) pdf's. We have illustrated in Figure 3.2 the shapes of density functions for different values of the shape parameters.

3.3.3 Characteristics of MLF

Defining the quartiles Q_i by $P(X<Q_i)\le\frac{i}{4}$ and $P(X>Q_i)\le 1-\frac{i}{4}$, $i=1,2,3$, the Median of MLF is given by

$$Me=Q_2=\lambda_1+\frac{1}{\lambda_2}\left[\frac{0.5^{\lambda_3}-1}{\lambda_3}-\frac{0.5^{\lambda_4}-1}{\lambda_4}\right] \tag{3.25}$$

When $\lambda_3=\lambda_4$, $Me=\lambda_1$.

Inter quartile range is given by

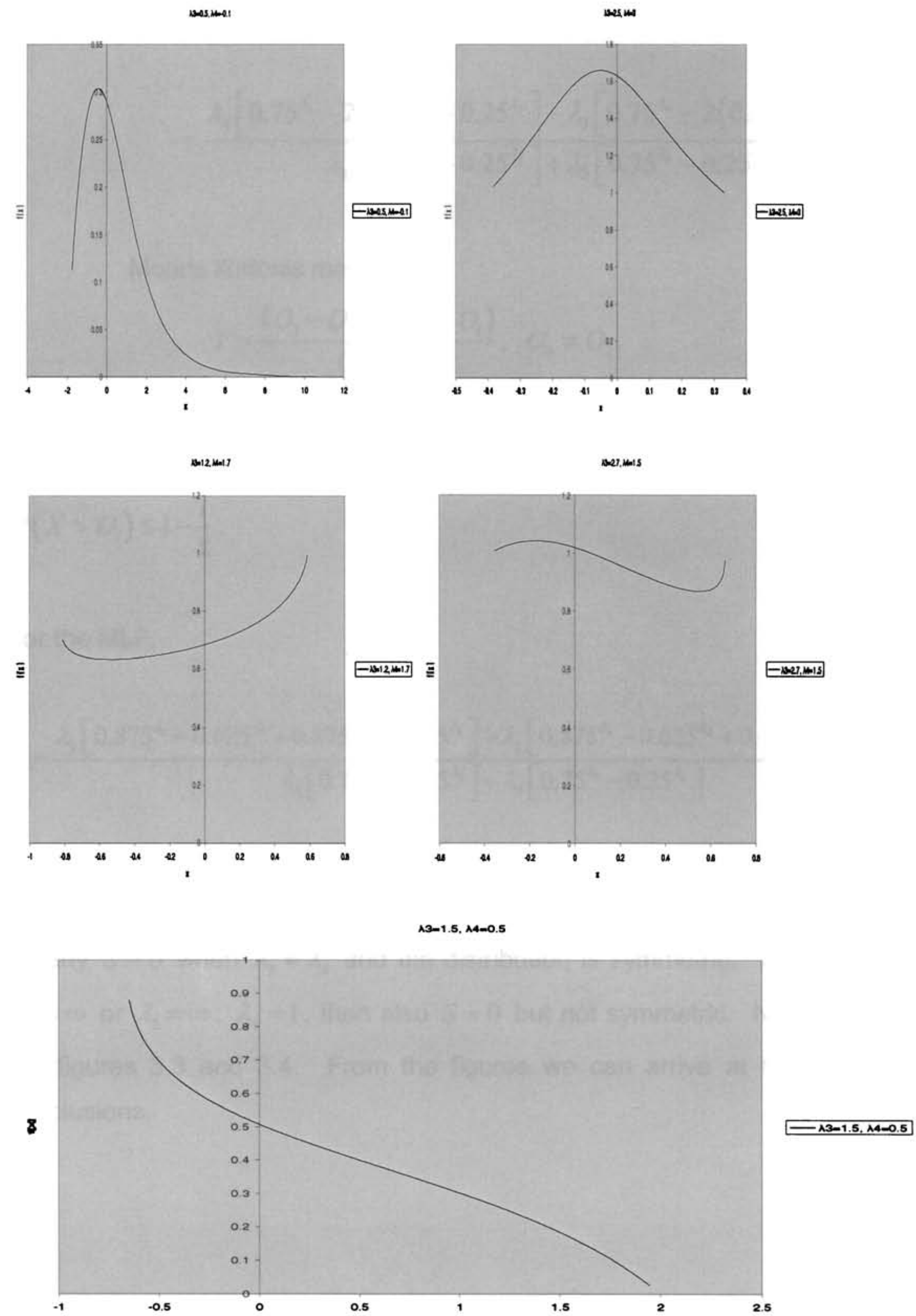
$$IQR=Q_3-Q_1=\frac{1}{\lambda_2}\left[\frac{0.75^{\lambda_3}-0.25^{\lambda_3}}{\lambda_3}+\frac{0.75^{\lambda_4}-0.25^{\lambda_4}}{\lambda_4}\right] \tag{3.26}$$

The quartile deviation now becomes

$$R=\frac{IQR}{2}=\frac{1}{2\lambda_2}\left[\frac{0.75^{\lambda_3}-0.25^{\lambda_3}}{\lambda_3}+\frac{0.75^{\lambda_4}-0.25^{\lambda_4}}{\lambda_4}\right] \tag{3.27}$$

Figure 3.2

Density Function



Galton's measure of skewness defined in terms of quartiles is given by

$$S = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}$$

$$= \frac{\lambda_4\left[0.75^{\lambda_3} - 2\left(0.5^{\lambda_3}\right) + 0.25^{\lambda_3}\right] - \lambda_3\left[0.75^{\lambda_4} - 2\left(0.5^{\lambda_4}\right) + 0.25^{\lambda_4}\right]}{\lambda_4\left[0.75^{\lambda_3} - 0.25^{\lambda_3}\right] + \lambda_3\left[0.75^{\lambda_4} - 0.25^{\lambda_4}\right]}$$

(3.28)

Moor's kurtosis measure is

$$T = \frac{(O_7 - O_5) + (O_3 - O_1)}{O_6 - O_2}, \quad O_6 \neq O_2$$

where O_i, $i = 1, \ldots, 7$ are the octiles defined by $P(X < O_i) \leq \frac{i}{8}$ and $P(X > O_i) \leq 1 - \frac{i}{8}$.

For the MLF,

$$T = \frac{\lambda_4\left[0.875^{\lambda_3} - 0.625^{\lambda_3} + 0.375^{\lambda_3} - 0.125^{\lambda_3}\right] + \lambda_3\left[0.875^{\lambda_4} - 0.625^{\lambda_4} + 0.375^{\lambda_4} - 0.125^{\lambda_4}\right]}{\lambda_4\left[0.75^{\lambda_3} - 0.25^{\lambda_3}\right] + \lambda_3\left[0.75^{\lambda_4} - 0.25^{\lambda_4}\right]}$$

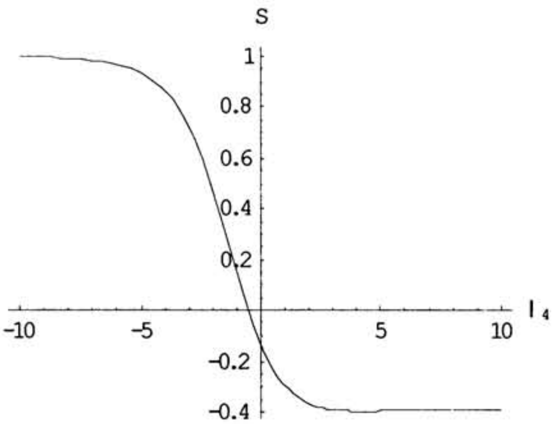
(3.29)

Clearly $S = 0$ when $\lambda_3 = \lambda_4$ and the distribution is symmetric. When $\lambda_3 = 1$, $\lambda_4 = \infty$ or $\lambda_3 = \infty$, $\lambda_4 = 1$, then also $S = 0$ but not symmetric. Now consider the figures 3.3 and 3.4. From the figures we can arrive at the following conclusions.

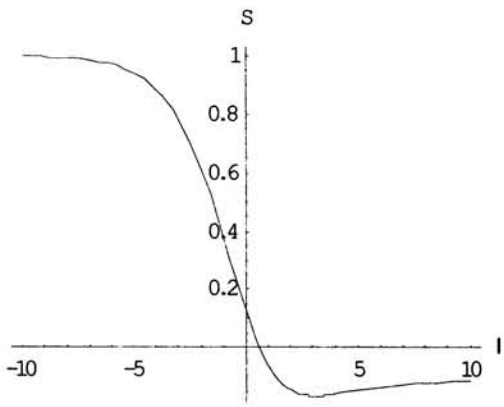
Figure 3.3

Skewness (λ_3 fixed)

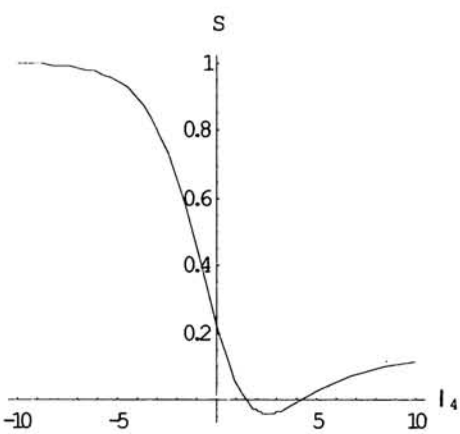
λ_3=-0.5



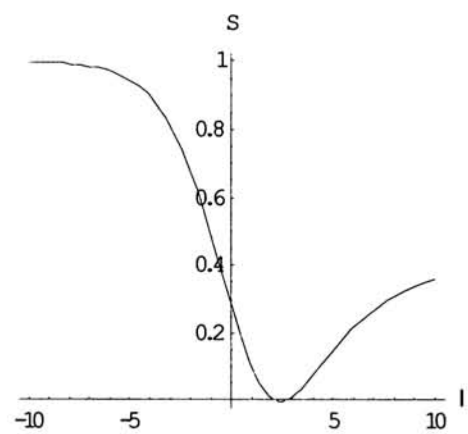
λ_3=0.6



λ_3=1.5



λ_3=2.7



λ_3=3.5

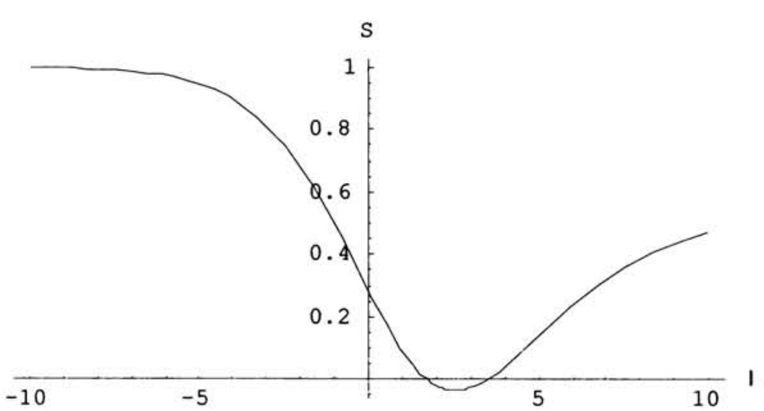
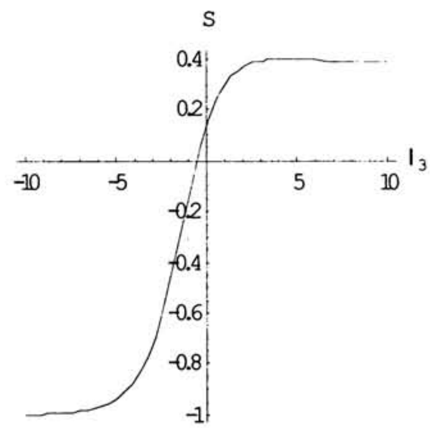


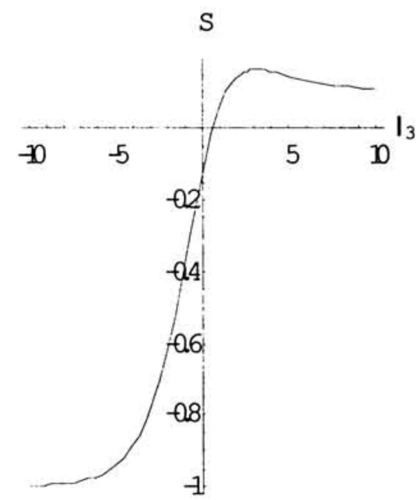
Figure 3.4

Skewness (λ_4 fixed)

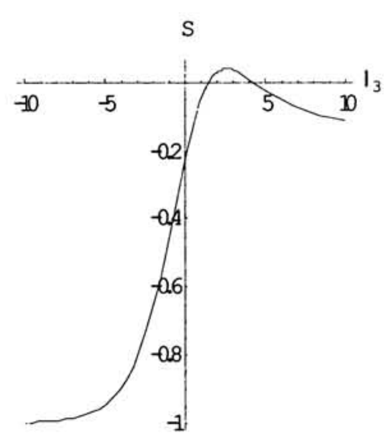
λ_4=-0.5



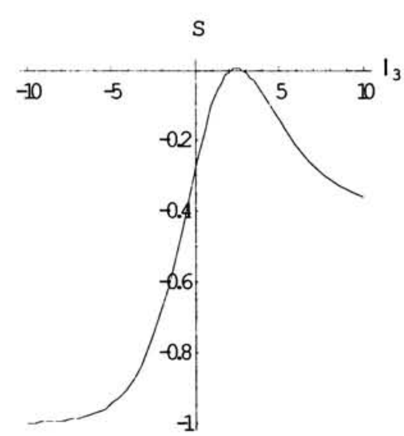
λ_4=0.6



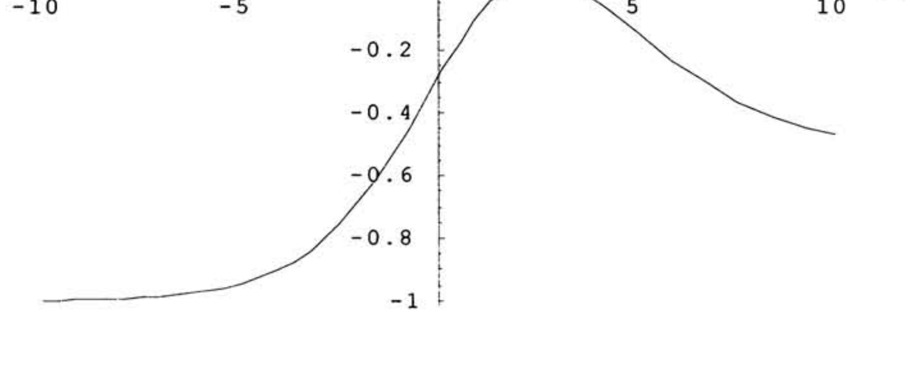
λ_4=1.5



λ_4=2.7



λ_4=3.5



Case (i) $\lambda_3 < 0$

$S > 0$ when $\lambda_3 > \lambda_4$

$S < 0$ when $\lambda_3 < \lambda_4$

S decreases monotonically with respect to λ_4.

Case (ii) $0 < \lambda_3 < 1$

Here also $S > 0$ when $\lambda_3 > \lambda_4$

$S < 0$ when $\lambda_3 < \lambda_4$

But S decreases to a negative value and then slightly increases but attains a constant negative value itself.

Case (iii) $1 < \lambda_3 \leq 2$

$S = 0$ for two values of λ_4, one exactly at $\lambda_4 = \lambda_3$ and another point which is greater than λ_3, say λ_0. $S > 0$ when $\lambda_4 < \lambda_3$ and $\lambda_4 > \lambda_0$ and $S < 0$ in the interval (λ_3, λ_0).

Case (iv) $2 < \lambda_3 < 3$

In this region, $S \geq 0$ for every value of λ_4 except for some values in $2 < \lambda_4 < 3$.

Case (v) $\lambda_3 \geq 3$

Here also $S = 0$ for two values of λ_4, one at a point which is less than λ_3, say λ_0 and another at $\lambda_4 = \lambda_3$. Here also $S > 0$ when $\lambda_4 < \lambda_0$ and $\lambda_4 > \lambda_3$ and $S < 0$ in the interval (λ_0, λ_3).

Similarly we can observe the variation of S with λ_3 by fixing the value of λ_4.

Case (i) $\lambda_4 < 0$

S increases monotonically with λ_3.

$S < 0$ for $\lambda_3 < \lambda_4$

$S > 0$ for $\lambda_3 > \lambda_4$

Case (ii) $0 < \lambda_4 < 1$

$S > 0$ for $\lambda_3 > \lambda_4$

$S < 0$ for $\lambda_3 < \lambda_4$

But S monotonically increases to a positive value and then slightly decreases and attains a constant positive value.

Case (iii) $1 < \lambda_4 < 2$

$S = 0$ at two points, one at $\lambda_3 = \lambda_4$ and another at a point greater than λ_4, say λ_0.

$S < 0$ when $\lambda_3 < \lambda_4$ and $\lambda_3 > \lambda_0$

$S > 0$ in the interval (λ_4, λ_0).

Case (iv) $2 < \lambda_4 < 3$

$S < 0$ for all values of λ_3 except for some values in $2 < \lambda_3 < 3$.

Case (v) $\lambda_4 > 3$

$S = 0$ at two points of λ_3, one at a point less than λ_4, say λ_0 and another at $\lambda_3 = \lambda_4$. $S < 0$ when $\lambda_3 < \lambda_0$ and $\lambda_3 > \lambda_4$. $S > 0$ in the interval (λ_0, λ_4).

Now the three dimensional view of skewness by taking λ_3 along the X-axis and λ_4 along the Y-axis is given in figure 3.5.

Moor's kurtosis measure $T = 1$ when $\lambda_3 = \lambda_4 = 1$ and $\lambda_3 = \lambda_4 = 2$. The three dimensional view of kurtosis is given in figure 3.6.

Figure 3.5

Three dimensional view of skewness

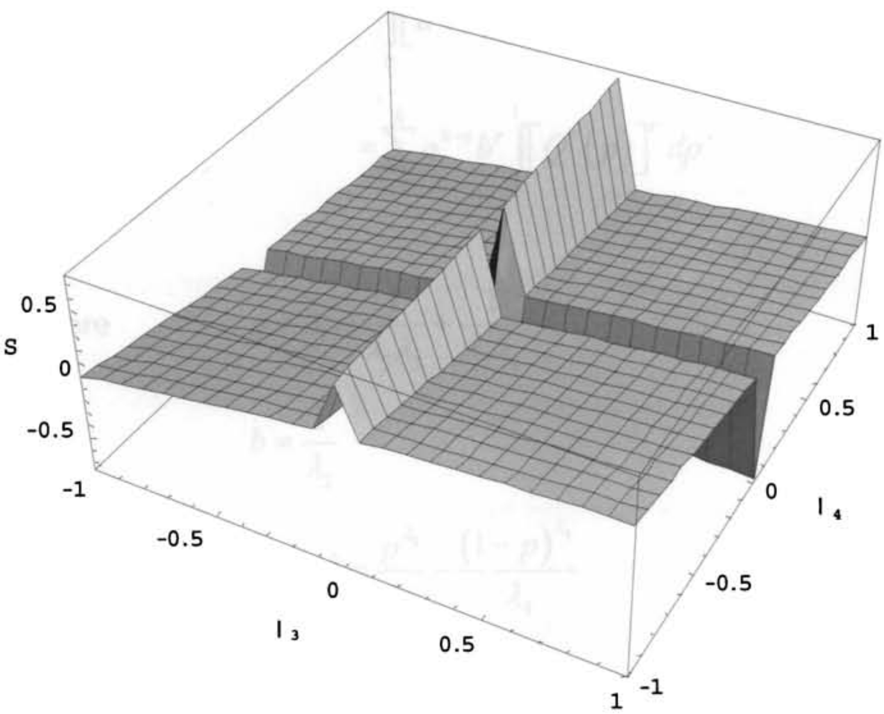
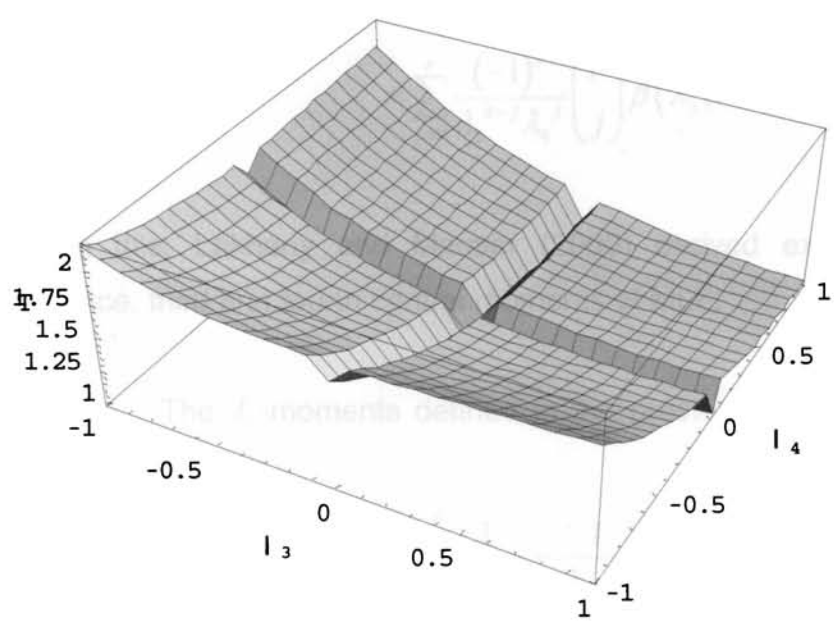


Figure 3.6

Three dimensional view of kurtosis



Now the k^{th} raw moment of MLF is

$$E(X^k) = \int_0^1 [Q(p)]^k \, dp$$

$$= \int_0^1 [a + bQ^*(p)]^k \, dp$$

$$= \sum_{r=0}^{k} a^{k-r} b^r \int_0^1 [Q^*(p)]^r \, dp \qquad (3.30)$$

where $a = \lambda_1 - \dfrac{1}{\lambda_2 \lambda_3} + \dfrac{1}{\lambda_2 \lambda_4}$

$$b = \frac{1}{\lambda_2}$$

$$Q^*(p) = \frac{p^{\lambda_3}}{\lambda_3} - \frac{(1-p)^{\lambda_4}}{\lambda_4}$$

and

$$\int_0^1 [Q^*(p)]^r \, dp = \int_0^1 \sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{p^{\lambda_3(r-j)} (1-p)^{\lambda_4 j}}{\lambda_3^{r-j} \lambda_4^{j}} \, dp$$

$$= \sum_{j=0}^{r} \frac{(-1)^j}{\lambda_3^{k-j} \lambda_4^{j}} \binom{r}{j} \beta\left(\lambda_3(r-j)+1, \lambda_4 j + 1\right) \qquad (3.31)$$

Using this, Lakhany and Mauser (2000) derived expressions of mean, variance, third and fourth central moments of MLF.

The L-moments defined in the previous section are obtained for MLF as follows.

$$\rho_1 = \lambda_1 + \frac{1}{\lambda_2} \left(\frac{1}{\lambda_4 + 1} - \frac{1}{\lambda_3 + 1} \right) \qquad (3.32)$$

$$\rho_2 = \frac{1}{\lambda_2}\left(\frac{1}{2+3\lambda_3+\lambda_3^2} + \frac{1}{1+\lambda_4} - \frac{1}{2+\lambda_4}\right) \tag{3.33}$$

$$\rho_3 = \frac{1}{\lambda_2}\left(\frac{\lambda_3 - 1}{6+11\lambda_3+6\lambda_3^2+\lambda_3^3} + \frac{1}{1+\lambda_4} - \frac{3}{2+\lambda_4} + \frac{2}{3+\lambda_4}\right) \tag{3.34}$$

$$\rho_4 = \frac{1}{\lambda_2}\left(\frac{1}{1+\lambda_3} - \frac{6}{2+\lambda_3} + \frac{10}{3+\lambda_3} - \frac{5}{4+\lambda_3} + \frac{1}{1+\lambda_4} - \frac{6}{2+\lambda_4} + \frac{10}{3+\lambda_4} - \frac{5}{4+\lambda_4}\right) \tag{3.35}$$

Using these we get the L -coefficient of variation, skewness and kurtosis of MLF as $\frac{\rho_2}{\rho_1}$, $\frac{\rho_3}{\rho_2}$ and $\frac{\rho_4}{\rho_2}$ respectively.

Now two special cases of probability weighted moments defined in (3.21), when $t=1$, $s=0$ and $t=1$, $r=0$ are obtained for MLF as follows.

$$w_{r,0} = \frac{\lambda_1}{r+1} - \frac{1}{\lambda_2(r+1)(\lambda_3+r+1)} + \frac{1}{\lambda_2\lambda_4}\left[\frac{1}{r+1} - \beta(r+1, \lambda_4+1)\right] \tag{3.36}$$

and

$$w_{0,s} = \frac{\lambda_1}{s+1} + \frac{1}{\lambda_2(s+1)(\lambda_4+s+1)} + \frac{1}{\lambda_2\lambda_3}\left(\beta(\lambda_3+1, s+1) - \frac{1}{s+1}\right) \tag{3.37}$$

3.3.4 Modified lambda family as a model of income

The primary objective of present study being modeling income using the quantile function approach in which the modified lambda family is the basic tool, it is necessary to examine how far that family is appropriate in such a context. Various properties of the family derived in the previous section serve as back ground materials for application, provided that there is sufficient justification for the lambda family to represent income data. We have provided in Chapter II a review of probability distributions used in literature that could serve as models of income. These include the Pareto type I, Pareto type II, exponential, lognormal, gamma, Weibull, Singh-Maddala, Dagum type I, II and III, Fisk, generalized beta etc. with each model justified in terms of its appropriateness in particular situations, with no model enjoying universal acceptance over time and space. In the present section we study the modified lambda family vis-à-vis its relationship with the above distributions either as a particular case or as a limiting case or as an approximation.

(i) In the quantile function of MLF at (3.22)

$$\lim_{\lambda_3 \to \infty} \lambda_1 + \frac{1}{\lambda_2}\left(\frac{p^{\lambda_3}-1}{\lambda_3} - \frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right) = \lambda_1 - \frac{1}{\lambda_2}\left(\frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right) + \frac{1}{\lambda_2}\lim_{\lambda_3 \to \infty}\left(\frac{p^{\lambda_3}-1}{\lambda_3}\right)$$

$$= \lambda_1 + \frac{1}{\lambda_2\lambda_4} - \frac{1}{\lambda_2\lambda_4}(1-p)^{\lambda_4}$$

so that

$$\lambda_2\lambda_4\left(x_p - \lambda_1 - \frac{1}{\lambda_2\lambda_4}\right) = -(1-p)^{\lambda_4}$$

or

$$1-F(x) = -(\lambda_2\lambda_4)^{\frac{1}{\lambda_4}}\left(x_p - \lambda_1 - \frac{1}{\lambda_2\lambda_4}\right)^{\frac{1}{\lambda_4}}$$

Setting $\lambda_1 = \sigma$, $\lambda_2 = \alpha\sigma^{-1}$ and $\lambda_4 = -\alpha^{-1}$, we get the Pareto type I distribution (2.1). Since $p < 1$, the convergence of p^{λ_3}/λ_3 to zero in the above limit for a desired degree of accuracy is attained for a moderate value of $\lambda_3 > 1$. Notice that in the above case, $\lambda_4 < 0$ and $x_p \geq \lambda_1 > 0$ are necessary conditions for the MLF to fit a Pareto I data. In this case, the quantile estimates proposed by Quandt (1966) becomes useful.

(ii) As before taking limits as $\lambda_3 \to \infty$

$$\lambda_2\lambda_4\left(x_p - \lambda_1 - \frac{1}{\lambda_2\lambda_4}\right) = -(1-p)^{\lambda_4}$$

which shows that the MLF reduces to the Pareto type II distribution with moderate $\lambda_3 > 1$, $\lambda_4 = -\frac{1}{\alpha} < 0$ and $\lambda_2 = \frac{\alpha}{\sigma}$. Notice in this case that in (2.2) the parameter $\mu = \lambda_1$.

(iii) The Weibull distribution (2.6) has the quantile function

$$Q(p) = \beta\left[-\log(1-p)\right]^{\frac{1}{a}}$$

so that, it is not a member of the MLF but can be approximated through the relationship

$$1 - \lambda_2\lambda_4 x = (1-p)^{\lambda_4}.$$

(iv) The Singh-Maddala (1976) model in equation (2.7) is governed by the equation

$$\left(\frac{x}{b}\right)^a = (1-p)^{-\frac{1}{q}} - 1$$

which means that the transformed variable $Y = \left(\frac{X}{b}\right)^a$ has MLF with parameters $\lambda_1 = 0$, $\lambda_2 = q$, $\lambda_4 = -\frac{1}{q}$ as $\lambda_3 \to \infty$.

(v) Arguing in the same way as above, the Dagum distribution provides

$$\left(\frac{b}{x}\right)^a = p^{-1/c} - 1.$$

Thus the transformation $Y = \left(\frac{b}{X}\right)^a$ leads to MLF with $\lambda_1 = 0$, $\lambda_2 = -c$, $\lambda_3 = -\frac{1}{c}$ and $\lambda_4 \to \infty$.

(vi) On similar lines the Fisk distribution (2.11) can be expressed as

$$p^{-1} = 1 + \left(\frac{x}{b}\right)^{-a}$$

or

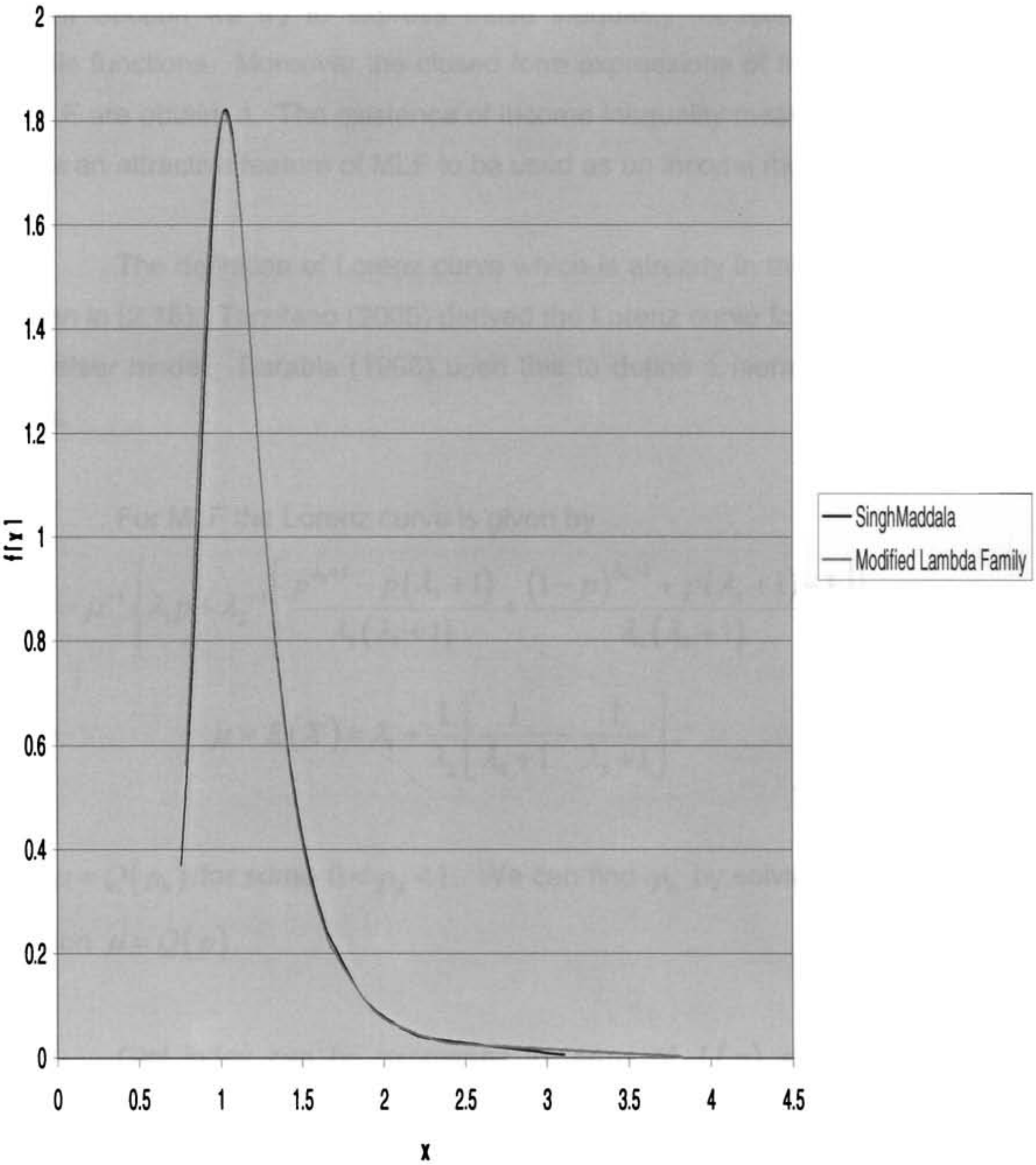
$$x = b\left(\frac{1-p}{p}\right)^{-\frac{1}{a}} = bp^{\frac{1}{a}}(1-p)^{-\frac{1}{a}} \tag{3.38}$$

The right side of equation (3.38) represents the product of quantile functions that represent the power distribution $F(x) = \left(\frac{x}{b}\right)^{a}$ with scale parameter b and shape parameter a and the Pareto distribution $F(x) = 1 - x^{-a},\ x > 1$ with shape parameter a. Both the component distributions are in MLF.

In addition to the above as $\lambda_3 \to 0$ or $\lambda_4 \to 0$ we have the exponential model, as both λ_3 and $\lambda_4 \to 0$ the logistic model and as λ_3, λ_4 both tends to unity or two or when $\lambda_3 \to \infty$ and λ_4 tends to unity or when λ_3 tends to unity and λ_4 tends to infinity the resulting distributions are uniform. Also when $\lambda_3 = \lambda_4 = 0.1349$ MLF describes the normal with a maximum error of 0.001. By equating the quantile measures of location, dispersion, skewness and kurtosis of the other distributions to those of the MLF we can get reasonably good approximations. For e.g. the Singh-Maddala distribution with $a = 10$, $b = 1$, $q = 0.5$ is a close approximation to the MLF with $\lambda_1 = 1.09246$, $\lambda_2 = 7.01713$, $\lambda_3 = 0.310545$ and $\lambda_4 = -0.36819$ (see Fig. 3.7). Thus the MLF appears to be a flexible family that could accommodate many of the income models through a judicious choice of the parameter values in a practical situation.

Figure 3.7

Modified Lambda Family approximation to Singh Maddala Distribution



3.5 Income Inequality Measures

A detailed review of income inequality measures which are common in the analysis of income data has been done in Chapter II. In the present Section we try to express those inequality measures in terms of quantile functions. Moreover the closed form expressions of those measures for MLF are obtained. The existence of income inequality measures in closed form is an attractive feature of MLF to be used as an income model.

The definition of Lorenz curve which is already in the quantile form is given in (2.18). Tarsitano (2005) derived the Lorenz curve for Ramberg and Schmeiser model. Sarabia (1996) used this to define a hierarchy of Lorenz curves.

For MLF the Lorenz curve is given by

$$L(p)=\mu^{-1}\left\{\lambda_1 p+\lambda_2^{-1}\left[\frac{p^{\lambda_3+1}-p(\lambda_3+1)}{\lambda_3(\lambda_3+1)}+\frac{(1-p)^{\lambda_4+1}+p(\lambda_4+1)-1}{\lambda_4(\lambda_4+1)}\right]\right\} \tag{3.39}$$

where $$\mu=E(X)=\lambda_1+\frac{1}{\lambda_2}\left[\frac{1}{\lambda_4+1}-\frac{1}{\lambda_3+1}\right].$$

Also $\mu=Q(p_0)$ for some $0<p_0<1$. We can find p_0 by solving for p in the equation $\mu=Q(p)$.

Gini index can be expressed in terms of $L(p)$ and is given in (2.19). For MLF it is given by

$$G=\mu^{-1}\lambda_2^{-1}\left\{\frac{1}{(\lambda_3+1)(\lambda_3+2)}+\frac{1}{(\lambda_4+1)(\lambda_4+2)}\right\} \tag{3.40}$$

The absolute mean deviation (2.24) can be given in quantile form as

$$\tau_1 = \int_0^1 \left| Q(u) - Q(p_0) \right| du$$

and for MLF τ_1 is given by

$$\tau_1 = 2\left\{ p_0 Q(p_0) - \left[\lambda_1 p_0 + \frac{p_0^{\lambda_3+1}}{\lambda_2\lambda_3(1+\lambda_3)} + \frac{(1-p_0)^{\lambda_4+1} - 1}{\lambda_2\lambda_4(1+\lambda_4)} + \frac{p_0}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3} \right) \right] \right\}$$

(3.41)

The relative mean deviation and Pietra index are given respectively by $\tau_2 = \frac{\tau_1}{\mu}$ and $P = \frac{\tau_1}{2\mu}$. Now the coefficient of variation (2.27) of MLF is obtained as

$$CV = \frac{\lambda_1\lambda_2 + \frac{1}{1+\lambda_4} - \frac{1}{1+\lambda_3}}{\sqrt{\frac{1}{\lambda_3^2(1+2\lambda_3) - \frac{2\beta(1+\lambda_3, 1+\lambda_4)}{\lambda_3\lambda_4} + \frac{1}{\lambda_4^2(1+2\lambda_4)} - \left(\frac{1}{\lambda_3(1+\lambda_3)} - \frac{1}{\lambda_4(1+\lambda_4)} \right)^2}}}$$

(3.42)

The three measures proposed by Frigyes given in (2.30) can be translated into quantile forms as

$$u = \frac{p_0 Q(p_0)}{\int_0^{p_0} Q(p)\,dp}, \qquad v = \frac{p_0}{1-p_0} \frac{\int_{p_0}^{1} Q(p)\,dp}{\int_0^{p_0} Q(p)\,dp}$$

and

$$w = \frac{\int_{p_0}^{1} Q(p)\,dp}{(1-p_0) Q(p_0)}.$$

For MLF,

$$u=\frac{p_0^{1+\lambda_3}}{\lambda_2\lambda_3(1+\lambda_3)}+\frac{(1-p_0)^{\lambda_4}-1}{\lambda_2\lambda_4(1+\lambda_4)}+p_0\left(\lambda_1+\frac{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}-\frac{(1-p_0)^{\lambda_4}}{\lambda_4+\lambda_4^2}}{\lambda_2}\right)$$

(3.43)

$$v=\frac{p_0\left[(1-p_0)\lambda_1+\frac{p_0-1}{\lambda_2\lambda_3}+\frac{1-p_0^{\lambda_3+1}}{\lambda_2\lambda_3+\lambda_2\lambda_3^2}+\frac{1-p_0}{\lambda_2\lambda_4}-\frac{(1-p_0)^{1+\lambda_4}}{\lambda_2\lambda_4+\lambda_2\lambda_4^2}\right]}{(1-p_0)\left\{\frac{p_0^{\lambda_3+1}}{\lambda_2\lambda_3(1+\lambda_3)}+\frac{(1-p_0)^{\lambda_4}-1}{\lambda_2\lambda_4(\lambda_4+1)}+p_0\left[\lambda_1+\frac{\frac{1}{\lambda_4}-\frac{1}{\lambda_3}-\frac{(1-p_0)^{\lambda_4}}{\lambda_4+\lambda_4^2}}{\lambda_2}\right]\right\}}$$

(3.44)

$$w=\frac{(1-p_0)\lambda_1-\frac{1-p_0}{\lambda_2\lambda_3}+\frac{1-p_0^{\lambda_3+1}}{\lambda_2\lambda_3+\lambda_2\lambda_3^2}+\frac{(1-p_0)}{\lambda_2\lambda_4}-\frac{(1-p_0)^{\lambda_4+1}}{\lambda_2\lambda_4+\lambda_2\lambda_4^2}}{(1-p_0)\left[\lambda_1+\frac{\frac{p_0^{\lambda_3}-1}{\lambda_3}-\frac{(1-p_0)^{\lambda_4}-1}{\lambda_4}}{\lambda_2}\right]}$$

(3.45)

The quantile form of Atkinson measures given in (2.21) is

$$A_\varepsilon=1-\frac{\left\{\int_0^1 (Q(p))^{1-\varepsilon}dp\right\}^{\frac{1}{1-\varepsilon}}}{\int_0^1 Q(p)dp}$$

and that of generalized entropy measures (2.22) and (2.23) respectively are

$$T_1 = \int_0^1 \frac{Q(p)}{Q(p_0)} \log \frac{Q(p)}{Q(p_0)} dp \text{ and } T_2 = \int_0^1 \log \frac{Q(p_0)}{Q(p)} dp .$$

The quantile form of the entropy measure (2.29) suggested by Ord et al. (1981) is

$$e_\gamma(X) = \frac{1}{\gamma} \int_0^1 \left[1 - q^{-\gamma}(p)\right] dp, \quad -1 < \gamma < \infty .$$

The expressions of the above four measures do not exist in closed form for MLF.

Now generalized Gini index (2.35) and Zenga curve (2.36) are in quantile forms itself.

For MLF, Generalized Gini index,

$$G_n = 1 - n(n-1)\mu^{-1} \left\{ \lambda_1 \beta(2, n-1) + \frac{1}{\lambda_2 \lambda_3 (\lambda_3 + 1)} \beta(n-1, \lambda_3 + 2) - \frac{\lambda_3 + 1}{\lambda_2 \lambda_3 (\lambda_3 + 1)} \beta(2, n-1) \right.$$
$$\left. - \frac{1}{\lambda_2 \lambda_4 (\lambda_4 + 1)(n + \lambda_4)} + \frac{\lambda_4 + 1}{\lambda_2 \lambda_4 (\lambda_4 + 1)} \beta(2, n-1) \right\}$$

(3.46)

(2.36) is the same as

$$Z(p) = 1 - \frac{Q(p) Q(p_0)}{\int_0^p Q(u) du}$$

For MLF,

$$Z(p) = 1 - \frac{\left(\dfrac{\lambda_1 + \dfrac{p^{\lambda_3} - 1}{\lambda_3} - \dfrac{(1-p)^{\lambda_4} - 1}{\lambda_4}}{\lambda_2} \right) \left(\lambda_1 + \dfrac{\dfrac{p_0^{\lambda_3} - 1}{\lambda_3} - \dfrac{(1-p_0)^{\lambda_4} - 1}{\lambda_4}}{\lambda_2} \right)}{p\lambda_1 - \dfrac{p}{\lambda_2 \lambda_3} + \dfrac{p^{\lambda_3 + 1}}{\lambda_2 \lambda_3 (\lambda_3 + 1)} + \dfrac{p}{\lambda_2 \lambda_4} + \dfrac{(1-p)^{\lambda_4 + 1} - 1}{\lambda_2 \lambda_4 (\lambda_4 + 1)}}$$

(3.47)

CHAPTER IV
ESTIMATION AND FITTING

4.1 Introduction

In the previous chapter we have introduced the modified lambda family with the objective of considering it as a plausible model of income distribution. Supplementing the theoretical justifications given earlier for using MLF as an income model due to its versatility it is essential to establish its empirical validity by showing that members of the family fits income data. Towards this endeavour in the present chapter our attempt is to devise procedures for estimating the parameters of the MLF, establish some theoretical results that supports the use of such estimators and finally show by goodness of fit procedures that the distribution describes the data adequately. Since our aim is to substantiate the relevance of MLF as an income model, a deeper analysis of the proposed estimation procedure vis-à-vis other competing methods is not attempted. However, a short review of the existing procedures for estimating the parameters of the MLF has been conducted in Section 2, for the sake of completion. The new estimation procedure based on comparing selected characteristics in the population and in the sample is presented in Section 3. In Section 4, MLF is fitted to a real income data along with the assessment of the goodness of fit through Chisquare criterion. Finally in Section 5, the accuracy of the estimates arrived at by the proposed procedure is compared with those of the method of moments and percentiles through a simulation study.

4.2 Review of estimation techniques

Due to the mathematical form of the quantile function and the extent of the parameter space induced by the four parameters, the likely correlation between the estimates and time consuming operational problems renders the question of obtaining appropriate parameters for the MLF often a challenging task. There are many general methods of estimations prescribed for the Ramberg and Schmeiser (1974) generalized lambda distribution mentioned in equation (2.38), but the formulas therein do not apply themselves to MLF. Hence our discussion of the present section confine only to those specifically confined to the MLF.

King and Mac Gillivray (1999) proposed the starship method which consists in

(a) transforming the data on X to $F(X)$

(b) calculating the value of the Kolmogorov distance or Anderson-Darling distance for the values of $F(x)$ and the uniform distribution over $(0,1)$

(c) choosing λ - values that minimizes the distance. In a discussion of the method, the authors point out that it is of 'numerically intensive nature' requiring computer power to fit the data and analytical results are not available for the expected value and standard errors of the estimates.

Tarsitano (2005) considered the quantile function

$$Q(p) = \lambda_1 + \lambda_2 p^{\lambda_4} - \lambda_3 (1-p)^{\lambda_5}$$

which contains five parameters, λ_1 for location, λ_2 and λ_3 representing scale and λ_4 and λ_5 describing the shape. The model according to the author contains MLF and therefore his general conclusions about the estimates remain valid for the latter. Various methods of estimation discussed are percentile method that matches five selected sample percentiles with corresponding theoretical percentiles, method of moments, matching probability weighted moments $E\left[Q(p_i)\right]$, $i = 0,1,2,3,4$ with the sample counterparts

$$t_0 = \frac{1}{n}\sum_{j=1}^{n} C_j \,, \quad t_i = \sum_{j=i+1}^{n} C_j \prod_{r=1}^{j} (j-r) / \prod_{r=0}^{i} (n-r) \,, \ i = 1,2,3,4$$

where C's midpoints of class intervals, minimum distance method using Cramer – Von Mises statistic, that minimizes

$$D = \sum (p_i - F(X_i))^2$$

where F is the estimated p value that would generate the observation X_i, maximum likelihood estimates obtained by minimizing the negative log-likelihood

$$L = -\sum_{i=1}^{k} n_i \left(F(X_i) - F(X_{i-1}) \right)$$

and the Pseudo least square approach based on

$$X_i = E(X_i) + \varepsilon_i .$$

The simulation study for comparing the different methods for 36 configurations revealed that the minimum distance and probability weighted moments approaches gave the 'worst' results. Further, the method of maximum likelihood had given results 'slightly better than these obtained by minimum distance, but not by an amount of any practical importance besides both being computationally demanding. The percentile, moment and Pseudo least squares were reported to give desirable results.

King and Mac Gillivray (2006) introduced the notion of spread functions

$$S_F(p) = F^{-1}(p) - F^{-1}(1-p)$$

in defining shape functionals

$$r(p) = \frac{F^{-1}(p) + F^{-1}(1-p) - 2m}{S(p)}$$

and

$$\eta(p,q) = F^{-1}(p) + F^{-1}(1-p) - \frac{\left[F^{-1}(q) + F^{-1}(1-q)\right]}{S(q)}, \ \tfrac{1}{2} < q < p < 1$$

by which estimates that minimize the distance between sample and population values of the functionals were proposed. For the MLF short tails were found to be problematic in the estimation procedure. Due to the various limitations pointed out in starship method, maximum likelihood, minimum distance and shape functionals, consideration will be given in the present study to computationally simple and reasonably accurate methods using percentiles and moments. First we present a new procedure for estimation by matching selected characteristics in the population and in the sample.

4.3 New Estimation Procedure

The method proposed in the present section resembles that of the classical methods of selected points, with the difference that the points chosen here is derived by matching the basic characteristics of the distribution viz. location, dispersion, skewness and kurtosis with those in the sample. The choice of the characteristics ensures that the parameter values determined there from corresponds to the true values that provide the same location, scale and shapes with a reasonable degree of accuracy. The accuracy results empirically from the criterion for optimization and theoretically from the asymptotic properties established in the sequel. The measures of location, dispersion, skewness and kurtosis involved in the new estimation procedure are the quantile based measures viz. Median, Quartile deviation, Galton's coefficient of skewness and Moor's kurtosis measure respectively. The expressions relating to these measures were obtained in equations (3.25),(3.27),(3.28) and (3.29) in the previous chapter. Hence the method of estimation of the parameters λ_i, $i=1,2,3,4$ in the model (3.22) is by solving for the λ_i's from the equations obtained by setting (3.25), (3.27), (3,28) and (3.29) respectively equal to the corresponding measures in the sample. To accomplish this, we define the pth quantile corresponding to a random sample $(X_1, X_2, ..., X_n)$ of observations on X as the pth quantile ξ_p of the sample distribution function $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x)$ where $I(.)$ is the indicator function. Denoting the sample median, quartile deviation, measures of skewness and kurtosis by m, r, s and t based on ξ_p the problem is to solve the simultaneous equations

$$\lambda_1 + \frac{1}{\lambda_2}\left[\frac{0.5^{\lambda_3} - 1}{\lambda_3} - \frac{0.5^{\lambda_4} - 1}{\lambda_4}\right] = m \tag{4.1}$$

$$\frac{1}{2\lambda_2}\left[\frac{0.75^{\lambda_3}-0.25^{\lambda_3}}{\lambda_3}+\frac{0.75^{\lambda_4}-0.25^{\lambda_4}}{\lambda_4}\right]=r \tag{4.2}$$

$$\frac{\lambda_4\left[0.75^{\lambda_3}-2\times0.5^{\lambda_3}+0.25^{\lambda_3}\right]-\lambda_3\left[0.75^{\lambda_4}-2\times0.5^{\lambda_4}+0.25^{\lambda_4}\right]}{\lambda_4\left[0.75^{\lambda_3}-0.25^{\lambda_3}\right]+\lambda_3\left[0.75^{\lambda_4}-0.25^{\lambda_4}\right]}=s \tag{4.3}$$

$$\frac{\lambda_4\left[0.875^{\lambda_3}-0.625^{\lambda_3}+0.375^{\lambda_3}-0.125^{\lambda_3}\right]+\lambda_3\left[0.875^{\lambda_4}-0.625^{\lambda_4}+0.375^{\lambda_4}-0.125^{\lambda_4}\right]}{\lambda_4\left[0.75^{\lambda_3}-0.25^{\lambda_3}\right]+\lambda_3\left[0.75^{\lambda_4}-0.25^{\lambda_4}\right]}=t \tag{4.4}$$

It may be noted that the above equations are non-linear and therefore, ends up with more than one quadruple of $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ values that satisfy them. The solutions being unrestricted, some set of solutions may not be within the range prescribed for the $\lambda's$ in Chapter 3, so that a proper probability distribution will not result. Secondly when more than one set arise as solution with admissible range, there is the question of some criterion that distinguishes the best solution. A solution to the first problem is to discard λ values that do not fall within the parameter space. Answering how a choice be made when multiple admissible solutions occur can be made with the aid of an optimality criterion. One simple way of devising such a criterion is to ensure that the difference between the estimated values and the sample values of the measures of location, dispersion , skewness and kurtosis are within a preassigned small value. Since there are four such differences the criterion is prescribed as

$$e=\max\left(|M-m|,|R-r|,|S-s|,|T-t|\right)<\varepsilon$$

for some positive ε , sufficiently small.

The computation of the solutions are carried using the command 'FindRoot' in Mathematica, which requires a set of initial values for the $\lambda's$ to initiate the solution. While different initial values in some cases, may give different solutions, the e criterion is invoked to find the best among them. Thus empirically the method proposed leads to a unique solution within the parameter space that nearly reproduces population characteristics that matches these found in the sample.

Our next step is to show that the procedure can also be justified from a theoretical stand point. Consider a sample of size n from a one-dimensional distribution of the continuous type with distribution function $F(x)$ and density function $f(x)$. Let ξ_p denote the pth quantile, $0 < p < 1$ and suppose that in some neighbourhood of ξ_p, $f(x)$ is continuous and has a continuous derivative $f'(x)$. Then it is known that (Serfling, R. J. (1980)) the pth sample quantile z_p is asymptotically normal $\left(\xi_p, \frac{1}{f(\xi_p)}\sqrt{\frac{pq}{n}}\right)$. Further as a special case the median of the sample m is a strongly consistent estimator of the population median $M = \xi_{\frac{1}{2}}$ and further $m = z_{\frac{1}{2}}$ is asymptotically normal $N\left(M, \frac{1}{4}n\left[f'(M)\right]^2\right)$. Then m belongs to the class of CAN estimators. In the same manner, $r = \frac{1}{2}(z_{0.75} - z_{0.25})$ is also a CAN estimator with distribution

$$\mathrm{N}\left(R, \frac{1}{64n}\left(\frac{3}{f^2(Q(0.75))} - \frac{2}{f(Q(0.25))f(Q(0.75))} + \frac{3}{f^2(Q(0.25))}\right)\right)$$

for large values of n. The results in Serfling (1980) concerning the functions of quantiles can be adapted suitably to the result that (s,t) is consistent for (S,T) and $\sqrt{n}(s-S, t-T)^*$ has asymptotic bivariate normal distribution with mean $(0,0)^*$ and dispersion matrix $\phi'(c)A\left[\phi'(c)\right]^*$ where

$$A = (\sigma_{ij}),\ \sigma_{ij} = \frac{i(8-j)}{64 f(E_i) f(E_j)},\ i \le j \qquad \phi(c) = (S,T)^*$$

and $*$ denotes the transpose. The expected values of $m-M$, $r-R$, $s-S$ and $t-T$ being zero for large samples, our estimating equations $m=M$, $r=R$, $s=S$ and $t=T$ provide λ values that agrees with the above expected values with small variations implied by the consistency of the estimators.

4.4 Fitting MLF to income data

Having set the background material for inference, the next important stage in model building is to test the model against the observations for adequacy. For the purpose, we consider the income data from Arnold (1983); referred to as 'Texas counties data' consisting of 157 observations. Each observation represents the total personal income accruing to the population of one of the 254 counties in Texas in 1969. The 157 included in the present data set represent all the Texas counties in which total personal income exceeds $20,000,000.

From the observations, the sample characteristics required for our estimation procedure are computed as

$$m = 46.3,\ r = 37.25,\ s = 0.5651,\ t = 2.4362$$

Substitution in equations (4.1) through (4.4) and following the ε criteria in the computational process gave the following admissible solutions (using Mathematica)

$$\hat{\lambda}_1 = 27.3207,\ \hat{\lambda}_2 = 0.0441223,\ \hat{\lambda}_3 = 3.86057,\ \hat{\lambda}_4 = -1.19399$$

Of the various possible values of the $\lambda's$ for different initial values above estimates are optimum according the ε criteria that gave the maximum error of $\varepsilon = 0.0282$. In order to assess the appropriateness of the MLF with the above parameter values for the given observations, a frequency distribution was formed by classifying the data into 10 intervals (the class interval was taken unequal for larger incomes to accommodate a reasonable frequency) and the corresponding expected frequencies using the above estimates of the parameters are exhibited in Table 4.1.

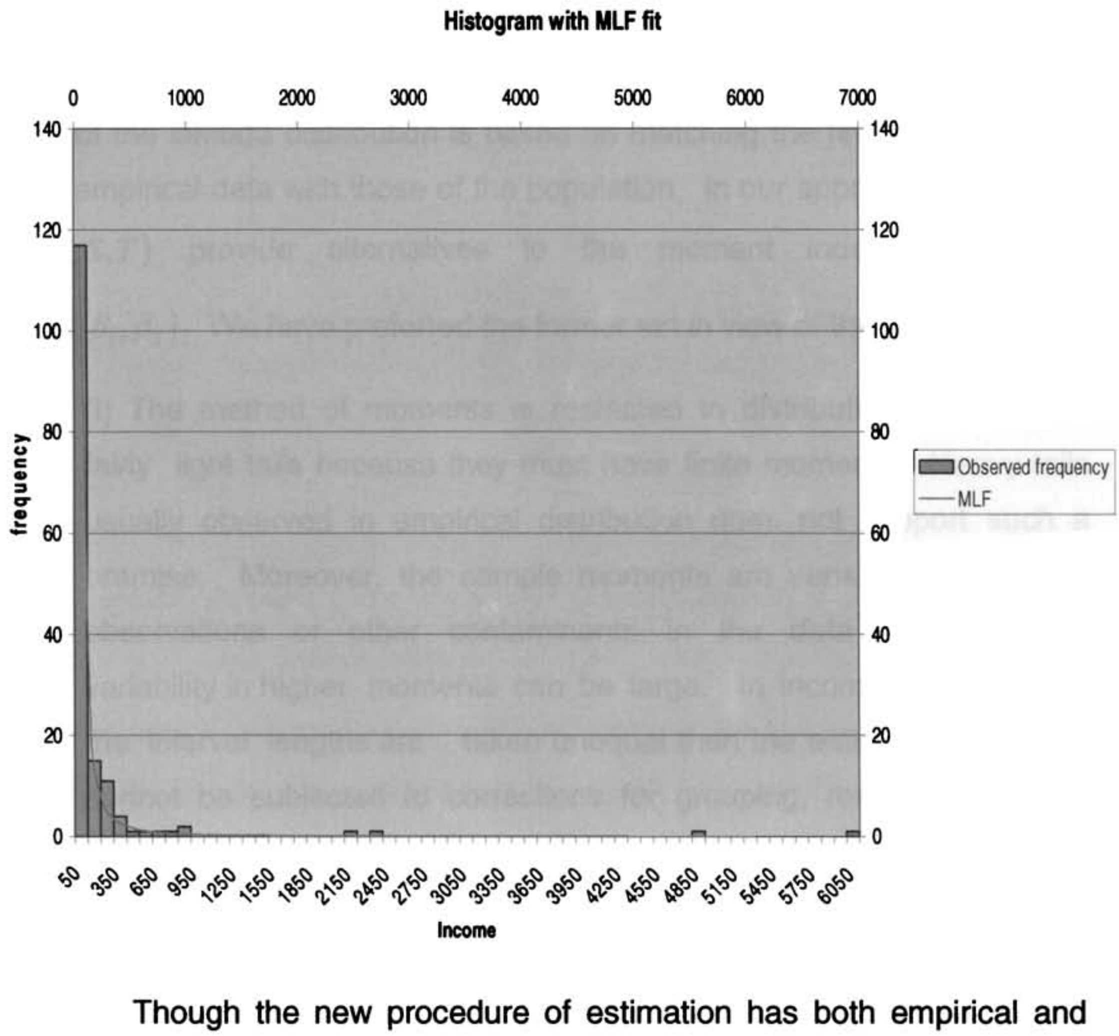
Table 4.1

Lambda distribution fit for Texas Counties Data

Class intervals	Observed frequency	Expected frequency
<30	39	41.8895
30-40	27	25.9491
40-50	13	15.7135
50-60	13	10.6738
60-70	6	7.78014
70-80	11	5.94857
80-100	8	8.53515
100-200	15	18.1278
200-500	16	12.121
>500	9	10.2614

The chi-square goodness of fit test provides $\chi^2 = 7.884$ that do not reject the hypothesis that observations follow the MLF. The histogram of the data and the frequency curve from the expected frequencies for the various class intervals are shown in Fig 4.1. Thus it is clear that MLF could be used as a model of incomes and that our method of estimation provides estimates of reasonable accuracy.

Figure 4.1



Though the new procedure of estimation has both empirical and theoretical support, it is of interest to know how it fares in comparison with some of the standard methods. For reasons noted in Section 2 of the current chapter where review of the different approaches were taken up, for comparison purposes we choose the method of percentiles and method of moments.

Since our expressions for the sample statistics are non-linear in the parameters analytic derivations of the standard errors of the $\lambda's$ are difficult to obtain. Hence a quick assessment of the sampling variations in the estimates for the given data is not possible. Therefore we have conducted a simulation study to assess the standard errors of the estimates for comparison with other methods. These are presented in the next Section.

4.5 Comparison with the Methods of Moments and Percentiles

The most popular approach for estimating parameters in various forms of the lambda distribution is based on matching the first four moments of the empirical data with those of the population. In our approach, measures (M, R, S, T) provide alternatives to the moment induced quantities $(\mu, \sigma^2, \beta_1, \beta_2)$. We have preferred the former set in view of the following.

(i) The method of moments is restricted to distributions possessing fairly light tails because they must have finite moments. Heavy tails usually observed in empirical distribution does not support such a premise. Moreover, the sample moments are sensitive to extreme observations or other contaminants in the data and sampling variability in higher moments can be large. In income data usually the interval lengths are taken unequal then the estimated moments cannot be subjected to corrections for grouping, resulting in highly biased estimates.

(ii) Existence of moments requires restrictions on the parameter space that are not always satisfied by the solution. This is not a necessary condition for using M, R, S and T.

(iii) The quantities M, R, S and T can be found graphically and further has the advantage of being operative without the necessity of knowing every measurement.

(iv) S and T are invariant under location and scale and R is location invariant.

(v) The numerical values (S,T) show the same pattern of behaviour as (β_1,β_2), except for the difference in size of the numerical values. Thus from a theoretical stand point the method proposed in the present study has several advantages over the method of moments.

The percentile method to fit MLF to a given data consists in equating four suitably sample quantiles to their MLF counterparts and solving the resulting equations for λ_1, λ_2, λ_3 and λ_4. The four sample statistics are defined by

$$\hat{\rho}_1 = \hat{Q}(0.5)$$

$$\hat{\rho}_2 = \hat{Q}(0.9) - \hat{Q}(0.1)$$

$$\hat{\rho}_3 = \frac{\hat{Q}(0.5) - \hat{Q}(0.1)}{\hat{Q}(0.9) - \hat{Q}(0.5)}$$

$$\hat{\rho}_4 = \frac{\hat{Q}(0.75) - \hat{Q}(0.25)}{\hat{\rho}_2}$$

These sample statistics have the following interpretations. $\hat{\rho}_1$ is the sample median; $\hat{\rho}_2$ is the inter-decile range, i.e. , the range between the 10^{th} percentile and 90^{th} percentile; $\hat{\rho}_3$ is the left-right tail-weight ratio, a measure of relative tail weights of the left tail to the right tail (distance from median to the 10^{th} percentile in the numerator and distance from 90^{th} percentile to the median in the denominator); and $\hat{\rho}_4$ is the tail weight factor or the ratio of the inter-quartile range to the inter-decile range, which is ≤ 1 and measures how great tail weight is (values close to 1.00 indicate the distribution is not spread out greatly in its tails, while values close to 0 indicate the distribution has long tails).

For MLF, these measures are obtained as

$$\rho_1 = \lambda_1 + \frac{1}{\lambda_2}\left[\frac{0.5^{\lambda_3}-1}{\lambda_3} - \frac{0.5^{\lambda_4}-1}{\lambda_4}\right]$$

$$\rho_2 = \frac{1}{\lambda_2}\left[\frac{0.9^{\lambda_3}-0.1^{\lambda_3}}{\lambda_3} + \frac{0.9^{\lambda_4}-0.1^{\lambda_4}}{\lambda_4}\right]$$

$$\rho_3 = \frac{\lambda_4\left[0.5^{\lambda_3}-0.1^{\lambda_3}\right]+\lambda_3\left[0.9^{\lambda_4}-0.5^{\lambda_4}\right]}{\lambda_4\left[0.9^{\lambda_3}-0.5^{\lambda_3}\right]+\lambda_3\left[0.5^{\lambda_4}-0.1^{\lambda_4}\right]}$$

$$\rho_4 = \frac{\lambda_4\left[0.75^{\lambda_3}-0.25^{\lambda_4}\right]+\lambda_3\left[0.75^{\lambda_4}-0.25^{\lambda_4}\right]}{\lambda_4\left[0.9^{\lambda_3}-0.1^{\lambda_3}\right]+\lambda_3\left[0.9^{\lambda_4}-0.1^{\lambda_4}\right]}$$

Now solving the equations $\rho_1 = \hat{\rho}_1$, $\rho_2 = \hat{\rho}_2$, $\rho_3 = \hat{\rho}_3$ and $\rho_4 = \hat{\rho}_4$ using mathematica we get the estimates $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$ and $\hat{\lambda}_4$. The computational aspects discussed for the new method are valid for the percentile method as the four equations are nonlinear.

To assess the performance of the above three competing methods we have conducted a simulation study by generating samples of size 33 (to accommodate the quantiles) from MLF with parameters $\lambda_1 = 13.7$, $\lambda_2 = 0.2$, $\lambda_3 = 0.4$ and $\lambda_4 = 0.01$. The parameters were then estimated using the three methods. The same procedure was repeated for samples of size 66. The bias and S.E are presented in Table 4.2.

Based on the simulation studies carried over several samples have revealed the following features associated with the various methods.

The absolute bias in the new method tend to decrease with increasing sample size for the estimates of all the four parameters, though the reduction in bias is not considerable. The estimates are thus more or less stable. This is confirmed by almost the same type of behaviour seen in the case of the mean square error as well.

Table 4.2 :

Bias and Mean Square Error

Estimation Method	Parameters	Sample Size	Absolute Bias	Mean Square Error
New Method	λ_1	$n = 33$	0.0407	0.163
		$n = 66$	0.03942	0.0927
	λ_2	$n = 33$	0.00088	0.000697
		$n = 66$	0.00084	0.000682
	λ_3	$n = 33$	0.0608	0.0116
		$n = 66$	0.0564	0.01365
	λ_4	$n = 33$	0.0612	0.0346
		$n = 66$	0.0563	0.0322
Method of Percentiles	λ_1	$n = 33$	0.0459	0.1522
		$n = 66$	0.0622	0.1543
	λ_2	$n = 33$	0.0007	0.0008
		$n = 66$	0.0099	0.00377
	λ_3	$n = 33$	0.0571	0.0169
		$n = 66$	0.01499	0.0184
	λ_4	$n = 33$	0.0622	0.0397
		$n = 66$	0.0165	0.03796
Method of Moments	λ_1	$n = 33$	0.23695	0.1894
		$n = 66$	0.1854	0.1451
	λ_2	$n = 33$	0.0314	0.0063
		$n = 66$	0.0296	0.0044
	λ_3	$n = 33$	0.2422	0.1459
		$n = 66$	0.2074	0.1088
	λ_4	$n = 33$	0.1234	0.062
		$n = 66$	0.1186	0.0597

The percentile method shows larger bias for λ_1, fluctuating bias for λ_2 and decreasing bias for λ_3 and λ_4. Generally the numerical value of the bias is seen larger than that of the new method. The mean square errors are lower for the new method, showing that there is more concentration of the estimates about the location measure..

Of the three methods, the method of moments fares the worst having produced considerably larger bias and mean square errors than the other two methods. Over all the impression gathered from the simulation study is that our method compares favourably and at times better than the method of percentiles and moments.

CHAPTER V

IDENTIFICATION OF MODELS BY INCOME CHARACTERISTICS

5.1 Introduction

The present study so far focussed attention on the approach to modeling income data by using the quantile functions as an alternative to the distribution function traditionally employed in most of the situations. A particular quantile function proposed in Chapter III capable of generating the flexible family of distributions named as the modified lambda family, was seen to represent a potential income model in the sense of rendering a good fit to income data. A serious limitation to this approach was the lack of a stochastic mechanism that account for the distribution. Alternatively in the absence of stochastic arguments, one may also think of inherent characteristics of income data that may give rise to a unique distribution so that the data generating mechanism can be spelt out through the concerned characteristic. In other words the model appropriate to a given population of incomes can be based on a characterization satisfying the particular nature of an income characteristic suitable to that population. The objective of this chapter is to build up a theoretical frame work for this purpose. This needs well accepted choice of income characteristics that can distinguish various distributions and amenable to analytic treatment. We have selected the concept of income gap ratio and the truncated form of Gini index for characterizing income distributions. Results are obtained for both distribution functions and quantile functions, by starting with the former and then making deductions to the latter case. The rest of the chapter consists of five more sections. In Section 2 we introduce the definitions of the Income Gap Ratio and Truncated Gini Index. We show in Section 3 that $\alpha(t)$ uniquely determines the distribution $F(x)$ and that the power distribution is the only continuous distribution for which $\alpha(t)$ and $G(t)$ are independent of the truncation point t. Further it is proved that $(1+G(t))\mu(t)$ can characterize the income distribution. Similar results concerning the affluent are proved in Section 4. The monotonic behaviour of the income gap ratio can be used as a criterion to classify income distributions that help the choice of the distribution as model of income. In Section 5 some results in this connection are presented. In the last Section almost all the results in the above sections are converted into the context of quantile functions and the income gap ratio and truncated gini index of the MLF have been evaluated.

5.2 Income gap ratio and truncated Gini index

Most of the indexes of poverty or affluence associated with income data are generally based on the proportion of people belonging to that category along with their income distribution through the income gap ratio and some measure of income inequality like the Gini index truncated at the appropriate level of income. Sen (1976), Takayama (1979) and Sen (1986, 1988) deal with such indexes and their properties. Since the income gap ratio and truncated Gini coefficient have a vital role in the definition of an index, it is important to investigate their relationships with the basic income distribution. In situations where these quantities are estimated from the observations without knowing the form of the distribution of incomes, (e.g. non-parametric estimation of income gap ratio and Gini coefficient) one basic question is whether the values of these functions at different levels of income enable the determination of the income distribution of the population. Theoretically the problem looks at the derivation of the distribution function of incomes based on the functional form of the income gap ratio and the truncated Gini coefficient. The present chapter focuses attention on this problem and some related issues like classification of income distributions on the basis of the behaviour of income gap ratio. Analogous results for quantile function of incomes have also been discussed in this chapter. We first present the basic definitions of the income gap ratio and truncated Gini index using the distribution function approach.

Let X be a non-negative random variable representing the income of a community of individuals with absolutely continuous distribution function

$F(x)$, survival function $\bar{F}(x)=1-F(x)$ and density function $f(x)$. Assuming the poverty line $X=t$, the proportion of poor people is $F(t)$ and their income distribution becomes that of the random variable $(X|X\leq t)$ viz.

$$
\begin{aligned}
{}_tF(x) &= \frac{F(x)}{F(t)} \qquad x\leq t \\
&= 1 \qquad x>t. \qquad (5.1)
\end{aligned}
$$

The income gap ratio of the poor people is defined as

$$
\begin{aligned}
\alpha(t) &= 1-E\left(\frac{X}{t}|X\leq t\right) \\
&= 1-\frac{\int_0^t yf(y)dy}{tF(t)} \qquad (5.2)
\end{aligned}
$$

Using the standard definition of the Gini coefficient in (2.33) the truncated version relating to those below the poverty line is

$$
\begin{aligned}
G(t) &= 1-2[\mu(t)]^{-1}\int_0^t y_t\bar{F}(y)_t f(y)dy \\
&= 1-2[\mu(t)]^{-1}\int_0^t y\left(1-\frac{F(y)}{F(t)}\right)\frac{f(y)}{F(t)}dy \qquad (5.3)
\end{aligned}
$$

where

$$
\mu(t)=\frac{1}{F(t)}\int_0^t yf(y)dy=E(X|X\leq t) \qquad (5.4)
$$

is the average income below the poverty line.

Analogously with reference to an affluence line $X=t^*$ the incomes for the affluent $\{X|X>t^*\}$ has distribution specified by

$$
F_{t^*}(x)=0 \qquad x\leq t^*
$$

$$= \frac{F(x) - F(t^*)}{1 - F(t^*)} \qquad x > t^* \tag{5.5}$$

The income gap ratio of the affluent is then

$$\alpha^*(t^*) = 1 - \frac{t^*}{E(X|X \geq t^*)}$$

$$= 1 - \frac{t^* \bar{F}(t^*)}{\int_{t^*}^{\infty} y f(y) dy} \tag{5.6}$$

and the corresponding truncated Gini coefficient becomes,

$$G^*(t^*) = 1 - 2\left[\mu^*(t^*)\right]^{-1} \int_{t^*}^{\infty} y \frac{\bar{F}(y)}{\bar{F}(t^*)} \frac{f(y)}{\bar{F}(t^*)} dy \tag{5.7}$$

where

$$\mu^*(t^*) = \frac{1}{\bar{F}(t^*)} \int_{t^*}^{\infty} y f(y) dy = E(X|X > t^*). \tag{5.8}$$

These definitions will be employed in the next section to develop characterization of F in terms of $\alpha(t)$, $\alpha^*(t^*)$, $G(t)$ and $G^*(t^*)$.

5.3 Characterization of income distributions

First we establish a one-to-one correspondence between income gap ratio and the base line income distribution.

Theorem 5.1:

If X has an absolutely continuous distribution function over $(0,\infty)$ with finite mean and income gap ratio $\alpha(t)$ which is differentiable, then

$$F(t) = \exp\left[-\int_{t}^{\infty} \frac{1 - \alpha(y) - y\alpha'(y)}{y\alpha(y)} dy \right], t > 0. \tag{5.9}$$

Proof: From the definition (5.2),

$$(1-\alpha(t))tF(t)=\int_0^t yf(y)\,dy$$

Differentiating with respect to t and re-arranging terms

$$\frac{f(t)}{F(t)}=\frac{1-\alpha(t)}{t\alpha(t)}-\frac{\alpha'(t)}{\alpha(t)}$$

Integrating from t to ∞,

$$\left[\ln F(t)\right]_t^\infty=\int_t^\infty\frac{1-\alpha(y)-y\alpha'(y)}{y\alpha(y)}dy$$

which leads to (5.9).

This theorem shows that using the functional form of $\alpha(t)$ one can determine the income distribution . Usually income gap ratios computed at several points of income are available directly from the income data without making assumptions about the income distribution. Empirically it is possible to draw conclusion about the form of the income gap ratio by plotting $\alpha(t)$ against t. We now establish some sample functional forms of $\alpha(t)$ that characterize income distributions.

Theorem 5.2:

The only continuous distribution for which the income gap ratio is a constant is the power distribution

$$F(x)=\left(\frac{x}{c}\right)^{\alpha},0<x<c,c,\alpha>0. \tag{5.10}$$

Proof: Suppose X follows the power distribution (5.10). Then from definition (5.2),

$$\alpha(t)=\frac{1}{\alpha+1}\text{, which is a constant.}$$

Conversely, let $\alpha(t)=k$, a constant less than unity. Then from (5.9),

$$F(x)=\left(\frac{x}{c}\right)^{\frac{1-k}{k}}$$

which is the power distribution with

$$\alpha = \frac{1-k}{k} > 0.$$

Remark:

The above result can be used to ascertain the changes in the income distribution (e.g. number of individuals whose income has to be raised to the next level) inorder to have a designated reduction in the income gap ratio.

An analogous result for the power distribution exists regarding the truncated Gini coefficient relating to the poor.

Theorem 5.3:

If X has absolutely continuous distribution function $F(x)$ satisfying $E(XF(X)) < \infty$, then the truncated Gini coefficient $G(t)$ is independent of t if and only if X has power distribution (5.10).

Proof: From (5.3) and (5.4),

$$G(t) = 1 - 2[\mu(t)]^{-1}\left[\int_0^t y\frac{f(y)}{F(t)}dy - \int_0^t \frac{yF(y)f(y)}{F^2(t)}dy\right]$$

$$= 1 - 2 + 2[\mu(t)]^{-1}\int_0^t \frac{yF(y)f(y)}{F^2(t)}dy$$

which gives

$$\frac{1}{2}\mu(t)[1+G(t)] = \int_0^t \frac{yF(y)f(y)}{F^2(t)}dy \qquad (5.11)$$

When X follows power distribution (5.10), from (5.11)

$$\frac{1}{2}\mu(t)[1+G(t)] = \frac{1}{\left(t/c\right)^{2\alpha}}\int_0^t y\left(\frac{y}{c}\right)^{\alpha} \alpha c^{-\alpha} y^{\alpha-1} dy$$

$$= \frac{\alpha t}{2\alpha+1}$$

and further

$$\mu(t)=\frac{\alpha}{(\alpha+1)t}$$

so that $G(t)=\frac{1}{2\alpha+1}$, a constant. This proves the if part.

To see that the only if part holds we assume $G(t)=k$ and write (5.11) as

$$\frac{1}{2}F(t)(1+G(t))\int_0^t yf(y)dy=\int_0^t yF(y)f(y)dy$$

Differentiating the last equation twice

$$(1+k)tf(t)+\frac{1}{2}(1+k)F(t)=tf(t)+F(t)$$

or

$$\frac{f(t)}{F(t)}=\frac{(1-k)}{2kt}$$

Integrating from t to c,

$$\left[\ln F(t)\right]_t^c=\int_t^c\frac{1-k}{2ky}dy$$

$$=\ln\left(\frac{c}{t}\right)^{\frac{1-k}{2k}}$$

or

$$F(t)=\left(\frac{t}{c}\right)^{\frac{1-k}{2k}}$$

which is the power distribution (5.10) with $\alpha=\frac{(1-k)}{2k}$.

The problem of obtaining a general inversion formula for finding $F(t)$ in terms of $G(t)$ was found to be difficult in view of the presence of $\mu(t)$ in (5.11). However, if one sets $g(t)=\mu(t)\left[1+G(t)\right]$, we can express $F(t)$ in terms of $g(t)$.

From (5.11)

$$\frac{1}{2}\left[F(t)\right]^2 g(t) = \int_0^t yF(y)f(y)dy$$

Differentiating with respect to t, we obtain

$$f(t)g(t) + \frac{1}{2}F(t)g'(t) = tf(t)$$

or

$$\frac{f(t)}{F(t)} = \frac{g'(t)}{2\left[t - g(t)\right]}$$

Integrating from t to ∞,

$$\left[\ln F(t)\right]_t^\infty = \int_t^\infty \frac{g'(y)}{2\left[y - g(y)\right]}dy$$

or

$$F(t) = \exp\left[-\frac{1}{2}\int_t^\infty \frac{g'(y)}{y - g(y)}dy\right], \quad t > 0. \tag{5.12}$$

The practical utility of (5.12) is the identification of $F(x)$ through the functional form of $g(x)$. Expressions of $g(x)$ for some income distributions are evaluated below, to indicate its usefulness in a practical situation.

(i) Pareto distribution:

$$f(x) = \alpha\sigma^\alpha x^{-\alpha-1}, \; x > \sigma > 0, \; \alpha > 0$$

$$g(x) = \left[1 + G(x)\right]\mu(x)$$

$$= 2\int_\sigma^x \frac{yF(y)f(y)dy}{F^2(x)}$$

$$= \frac{2}{\left[1 - \left(\frac{x}{\sigma}\right)^{-\alpha}\right]^2}\int_\sigma^x y\left[1 - \left(\frac{y}{\sigma}\right)^{-\alpha}\right]\alpha\sigma^\alpha y^{-\alpha-1}dy$$

$$=\frac{2\alpha\sigma}{\left[1-\left(\frac{x}{\sigma}\right)^{-\alpha}\right]^2}\left\{\frac{1-\left(\frac{x}{\sigma}\right)^{-\alpha+1}}{\alpha-1}-\frac{1-\left(\frac{x}{\sigma}\right)^{-2\alpha+1}}{2\alpha-1}\right\}.$$

(ii) Exponential:

$$F(x)=1-e^{-\lambda x},\ x>0,\ \lambda>0$$

$$g(x)=2\int_0^x\frac{y\left[1-e^{-\lambda y}\right]\lambda e^{-\lambda y}}{\left[1-e^{-\lambda x}\right]^2}dy$$

$$=\frac{2}{\left[1-e^{-\lambda x}\right]^2}\left[\frac{e^{-2\lambda x}}{2}\left(x+\frac{1}{2\lambda}\right)-e^{-\lambda x}\left(x+\frac{1}{\lambda}\right)+\frac{3}{4\lambda}\right].$$

(iii) Power:

$$F(x)=\left(\frac{x}{c}\right)^\alpha$$

$$g(x)=\frac{2}{\left[\left(\frac{x}{c}\right)^\alpha\right]^2}\int_0^x y\left[\frac{y}{c}\right]^\alpha c^{-\alpha}\alpha y^{\alpha-1}dy$$

$$=\frac{2\alpha x}{(2\alpha+1)}$$

(iv) Dagum:

$$F(x)=\left[1+\left(\frac{b}{x}\right)^a\right]^{-p},\ x>0$$

$$g(x)=\frac{2}{\left[1+\left(\frac{b}{x}\right)^a\right]^{-2p}}\int_0^x y\left[1+\left(\frac{b}{y}\right)^a\right]^{-p}apb^a y^{-a-1}\left[1+\left(\frac{b}{y}\right)^a\right]^{-p-1}dy$$

$$=2ap\left[1+\left(\frac{b}{x}\right)^a\right]^{2p}\int_0^x\left(\frac{y}{b}\right)^{-a}\left[1+\left(\frac{y}{b}\right)^{-a}\right]^{-2p-1}dy$$

$$= 2\left[1+\left(\frac{b}{x}\right)^{a}\right]^{2p} pbB_{\left(\frac{x}{b}\right)^{a}}\left(2p+a^{-1},1-a^{-1}\right)$$

where $B_x(p,q)=\int_0^x t^{p-1}(1+t)^{-(p+q)}\,dt$.

5.4 Measures of affluence

The income gap ratio and Gini coefficient for the affluent hold analogous properties as for the poor.

Theorem 5.4:

If X has absolutely continuous distribution function over $(0,\infty)$ with finite mean and income gap ratio $\alpha^*(t^*)$, then

$$\bar{F}(t^*) = \exp\left[-\int_0^{t^*}\frac{1-\alpha^*(y)+y\alpha^{*\prime}(y)}{\alpha^*(y)\left(1-\alpha^*(y)\right)y}dy\right] \qquad (5.13)$$

Proof: From (5.6),

$$\left[1-\alpha^*(t^*)\right]\int_{t^*}^{\infty} yf(y)dy = t^*\bar{F}(t^*)$$

Differentiating with respect to t^* and simplifying

$$\frac{f(t^*)}{\bar{F}(t^*)} = \frac{1-\alpha^*(t^*)+t^*\alpha^{*\prime}(t^*)}{\alpha^*(t^*)\left(1-\alpha^*(t^*)\right)t^*}$$

Integrating from 0 to t^*, we get (5.13).

Remarks:

1. The only continuous distribution over the set of positive reals for which $\alpha^*(t^*)$ is a constant is the Pareto distribution. This is easily

verified by noting that, $\alpha^*(t^*) = \frac{1}{\alpha}$ (in the form given above) and substituting in (5.13) gives the desired form for $F(x)$.

2. The generalized Pareto family with

$$\bar{F}(x) = \left(\frac{b}{ax+b}\right)^{\frac{1}{a}+1}, x > 0, b > 0, a > -1 \qquad (5.14)$$

is characterized by an income gap ratio in the bilinear form

$$\alpha^*(t^*) = \frac{at^* + b}{(a+1)t^* + b}$$

Note that (5.14) contains the exponential distribution as $a \to 0$, the Pareto II distribution of the form

$$\bar{F}(x) = \alpha^c (x+\alpha)^{-c}, \; x, \alpha > 0$$

with $a = (c-1)^{-1}$, $b = \alpha(c-1)^{-1}$ when $0 < a < 1$ in (5.14)

and the beta

$$\bar{F}(x) = \left(1 - \frac{x}{R}\right)^d, 0 < x < R, d > 0,$$

$a = -(d+1)^{-1}$, $b = R(d+1)^{-1}$ when $-1 < a < 0$ in (5.14).

3. Ord et. al (1983) have used the gamma entropy measure

$$e_\gamma(t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)}\left[1 - \frac{f^\gamma(x)}{\bar{F}^\gamma(t)}\right] dx \Big/ \gamma \qquad (5.15)$$

as a measure of inequality and shows that (5.15) is truncation invariant (independent of t) if and only if X has exponential distribution. We now discuss the relationship of the measure $e_\gamma(t)$ with the income gap ratio.

Defining the random variable $Y_{t^*} = \left(\frac{f^\gamma(X)}{t^{*\gamma}\bar{F}^\gamma(t^*)} \middle| X > t^*\right)$ it is easy to see that

$$E(Y_{t^*}) = \frac{1}{t^{*\gamma}}\left(1 - \gamma e_\gamma(t^*)\right)$$

Now the geometric mean of Y_{t^*} is $G(t^*)$, where

$$\log G(t^*) = E\log Y_{t^*} = \frac{\gamma}{\bar{F}(t^*)}\int_{t^*}^{\infty}\left(\log f(x) - \log\bar{F}(t^*) - \log t^*\right)f(x)\,dx$$

$$= \frac{\gamma}{\bar{F}(t^*)}\int_{t^*}^{\infty} f(x)\log f(x)\,dx - \gamma\log\bar{F}(t^*) - \gamma\log t^* \quad (5.16)$$

Again defining $Z_{t^*} = \left(\frac{f(X)}{t^*\bar{F}(x)}\middle| X > t^*\right)$, Z_{t^*} has geometric mean $p(t^*)$ with

$$\log p(t^*) = E\left(\log\frac{f(X)}{t^*\bar{F}(x)}\middle| X > t^*\right)$$

$$= \frac{1}{\bar{F}(t^*)}\int_{t^*}^{\infty}\left(f(x)\log f(x) - f(x)\log\bar{F}(x) - f(x)\log t^*\right)dx$$

$$= \frac{1}{\bar{F}(t^*)}\int_{t^*}^{\infty} f(x)\log f(x)\,dx - \log\bar{F}(t^*) - \log t^* + 1 \quad (5.17)$$

Comparing with (5.16)

$$\log p(t^*) = 1 + \frac{1}{\gamma}\log G(t^*)$$

or

$$p(t^*) = eG^{\frac{1}{\gamma}}(t^*) \quad (5.18)$$

Finally Z_{t^*} has harmonic mean $H(t^*)$ where

$$H^{-1}(t^*) = E\left(\frac{t^*\bar{F}(X)}{f(X)}\middle| X > t^*\right)$$

$$= \frac{t^*}{\bar{F}(t^*)}\int_{t^*}^{\infty}\bar{F}(x)\,dx$$

$$= t^*\left[\frac{1}{\bar{F}(t^*)}\int_{t^*}^{\infty} yf(y)\,dy - t^*\right]$$

$$=t^*\left[\frac{t^*}{1-\alpha^*(t^*)}-t^*\right]$$

or

$$H(t^*)=\frac{1-\alpha^*(t^*)}{t^{*2}\alpha^*(t^*)}$$

Since $p(t^*)\geq H(t^*)$, from (5.18)

$$G(t^*)\geq\frac{\left(1-\alpha^*(t^*)\right)^{\gamma}}{e^{\gamma}t^{*2\gamma}\left(\alpha^*(t^*)\right)^{\gamma}} \tag{5.19}$$

Further $E\left(Y_{t^*}\right)\geq G(t^*)$ gives

$$\frac{1}{t^{*\gamma}}\left(1-\gamma e_{\gamma}(t^*)\right)\geq\frac{\left(1-\alpha^*(t^*)\right)^{\gamma}}{e^{\gamma}t^{*2\gamma}\alpha^{*\gamma}(t^*)}$$

which on simplification provides the following lower bound to the income gap ratio in terms of the entropy measure

$$\alpha^*(t^*)\geq\left[1+et^*\left(1-\gamma e_{\gamma}(t^*)\right)^{\frac{1}{\gamma}}\right]^{-1} \tag{5.20}$$

The Gini coefficient for the affluent is defined in equations (5.7) and (5.8). Ord et. al (1983) have shown that this coefficient is constant among the class of absolutely continuous distributions with positive density almost everywhere in (k,∞), if and only if the distribution is Pareto. This result corresponds to Theorem 5.3. Writing

$$g^*(x)=\left(1-G^*(x)\right)\mu^*(x)$$

one could see that

$$\frac{1}{2}\bar{F}^2(t^*)\left(1-G^*(t^*)\right)\mu^*(t^*)=\int_{t^*}^{\infty}y\bar{F}(y)f(y)dy$$

Proceeding as in the earlier theorems we have the following result .

Theorem 5.5:

If X has absolutely continuous distribution function with $E\left(X\bar{F}(X)\right)<\infty$ with Gini coefficient $G^*\left(t^*\right)$ and average income $\mu^*\left(t^*\right)$ above the affluence line $X=t^*$, then

$$\bar{F}(x)=\exp\left[-\frac{1}{2}\int_0^x \frac{g^{*\prime}(y)}{g^*(y)-y}dy\right] \tag{5.21}$$

where $g^*(y)=\left(1-G^*(y)\right)\mu^*(y)$.

Remarks:

1. The generalized Pareto family (5.14) is characterized by

$$\mu^*\left(t^*\right)\left(1-G^*\left(t^*\right)\right)=\frac{2at^*+2t^*+b}{a+2}=At^*+B \tag{5.22}$$

which is a unification of the result in Sathar, Rajesh and Nair (2003) separately proved for the exponential, Pareto II and beta distributions. They have also proved that the income gap ratio is in constant proportion with the $G^*\left(t^*\right)$ for the above three distributions, the proportionality being $\frac{1}{2}$ for exponential, $>\frac{1}{2}$ for Pareto II and $<\frac{1}{2}$ for the beta. From (5.22) the property $G^*\left(t^*\right)=\frac{1+a}{2+a}\alpha^*\left(t^*\right)$ characterizes the generalized Pareto family.

2. $g^*(y)=cy$, where c is a constant greater than unity characterizes the Pareto distribution.

5.5 Classes of Income Distributions

Based on the monotonic behaviour of the income gap ratio $\alpha^*\left(t^*\right)$ it is possible to classify income distributions. These results are helpful in modeling incomes where the empirical evaluation of the income gap ratios at different values of t^* will give us an idea about the class of distributions from which the appropriate model should be chosen.

Definition: A distribution function $F(x)$ is increasing in income gap ratio for the rich IIR(r) (decreasing in income gap ratio – DIR(r)) if $\alpha^*(x^*)$ is non-decreasing (non-increasing) in x^*.

Belzunce et al. (1998) defines the class of decreasing mean left proportional residual income (DMLPRI) if

$$E\left(\frac{X}{t}|X>t\right)=\frac{1}{t\bar{F}(t)}\int_t^{\infty}xf(x)dx$$

is decreasing in t. Since this criterion is equivalent to DIR(r) all results proved there are true for DIR(r) also, and accordingly we establish some new implications of the DIR(r) class which can supplement the existing results on DMLPRI.

1. The classes of income distributions based on monotonicity of $\alpha^*(t^*)$ are well defined, as the exponential model is DIR(r), the beta discussed in Section 5.4 is IIR(r) while the Pareto distribution is both DIR(r) and IIR(r) with $\alpha^*(t^*)$, a constant.
2. A sufficient condition for $F(x)$ to be DIR(r) is that either of the following conditions hold.

   (a) $f(x)$ is log-concave (b) $F(x)$ has increasing failure rate.

Proof: If $g(x)$ is a monotonic (increasing or decreasing) function on (a,b) with either $g(a)=0$ or $g(b)=0$, then if $g'(x)$ is log-concave then $g(x)$ is also log-concave on (a,b). We use this result repeatedly for different functions in the proof.

To prove (a)Assume that $f(x)$ is log-concave. Then by definition of log-concavity $\frac{f'(x)}{f(x)}$ is decreasing, and since $\frac{f'(x)}{f(x)} = \frac{\bar{F}''(x)}{\bar{F}'(x)}$, $\frac{\bar{F}''(x)}{\bar{F}'(x)}$ and hence $\frac{\bar{F}'(x)}{\bar{F}(x)}$ are decreasing. Defining $H(x) = \int_x^\infty \bar{F}(t)\,dt$, $H'(x) = -\bar{F}(x)$ and $H''(x) = -\bar{F}'(x)$. Thus $\frac{H''(x)}{H'(x)}$ and hence $\frac{H'(x)}{H(x)}$ are decreasing functions. Thus we find that

$$\frac{1-\alpha^*(t^*)}{\alpha^*(t^*)} = -\frac{t^* H'(t^*)}{H(t^*)}$$

is increasing and this implies $\alpha^*(t^*)$ is decreasing or $F(x)$ is DIR(r).

To prove (b) we note that whenever the failure rate $h(x) = \frac{f(x)}{\bar{F}(x)} = -\frac{\bar{F}'(x)}{\bar{F}(x)}$ is increasing $\frac{\bar{F}'(x)}{\bar{F}(x)}$ is decreasing. The rest of the proof follows from that of part (a).

Remarks:

1. Part (a) gives a simple criterion to distinguish income distributions with decreasing income gap ratio. For $IIR(r)$ models the words increasing and log-concave in (a) and (b) are to be replaced by decreasing and log-convex. A classification of some distributions used to model incomes according to the above criteria is given below.
   Note: log-concavity properties are preserved under linear transformations so that scale and location parameters can be introduced without affecting their classifications. Also, the classifications hold for truncated versions of the above distributions.

2. Belzunce et. al (1998) defines the class of increasing proportional failure rate ($IPFR$) distributions in which $xh(x)$ is an increasing function and shows that $IPFR \Rightarrow DMLPRI$. When the class has increasing

failure rate $xh(x)$ is also increasing so that $IFR \Rightarrow IPFR$, but the converse need not be as seen from the case of the distribution

$$f(x)=\frac{1}{2}x^{-1/2}e^{-x^{1/2}},\ x>0.$$

Therefore resulting from (b) above we can write the implications

$$IFR \Rightarrow IPFR \Rightarrow DMLPRI \Leftrightarrow DIR(r).$$

3. A necessary and sufficient condition that $F(x)$ is $DIR(r)$ ($IIR(r)$) is that $$\alpha^*(t^*)<(>)\frac{1}{t^*h(t^*)}.$$

Proof: From the definition in equation (5.6)

$$\left(1-\alpha^*(t^*)\right)\int_{t^*}^{\infty} yf(y)dy = t^*\bar{F}(t^*)$$

Differentiating with respect to t^* and simplifying,

$$t^*\alpha^*(t^*)f(t^*)-\frac{t^*\bar{F}(t^*)}{1-\alpha^*(t^*)}\alpha'(t^*)=\bar{F}(t^*)$$

or

$$\alpha^{*\prime}(t^*)=\frac{1-\alpha^*(t^*)}{t^*}\left(\alpha^*(t^*)h(t^*)t^*-1\right)$$

For $IIR(r)$ distribution, $\alpha^{*\prime}>0$ and hence

$$\alpha^*(t^*)>\frac{1}{t^*h(t^*)}.$$

An analogous discussion holds in the case of income gap ratio for the poor.

Definition: A distribution function $F(x)$ is increasing in income gap ratio for the poor- $IIR(p)$ (decreasing $-DIR(p)$) if $\alpha(x)$ is non-decreasing (non-increasing) in x. This class is the same as that discussed by Belzunce et al.

(1998) in the name of decreasing mean right proportional residual income ($DMRPRI$). They provide an exhaustive discussion of the properties of this class. We observe further that

(i) a necessary and sufficient condition for $F(x)$ to be $IIR(p)$ is that $\alpha < (1+t\lambda(t))^{-1}$, where $\lambda(t)=\frac{f(t)}{F(t)}$, the reversed failure rate of X.

(ii) $\alpha(t)$ is increasing or $F(x)$ is $IIR(p)$ if and only if $\int_0^t F(t)dt$ is log-concave.

5.6 Quantile Forms of Income Gap Ratio and Truncated Gini Coefficient

Since the major theme in the present work is the modeling of income data using the lambda distribution, the transformation of the expressions of inequality measures discussed in the previous sections of this chapter into quantile forms is relevant.

Let p be the proportion of the poor people of a population. Then by the transformation $u=F(x),\ 0<u<1$ or $x=Q(u)$ where $Q(u)=F^{-1}(u)$ in the equation (5.2) we get the income gap ratio of poor in terms of quantile functions and is given by

$$\alpha(p)=1-\frac{\int_0^p Q(u)du}{pQ(p)} \tag{5.23}$$

Similarly, the income gap ratio of the rich is given by

$$\alpha^*(p^*)=1-\frac{(1-p^*)Q(p^*)}{\int_{p^*}^1 Q(u)du} \tag{5.24}$$

where $(1-p^*)$ is the proportion of rich people of the population.

Now the truncated Gini coefficient for poor and rich are given respectively by

$$G(p) = \frac{2[\mu(p)]^{-1}}{p^2} \int_0^p uQ(u)\,du - 1 \tag{5.25}$$

where $\mu(p) = \frac{1}{p}\int_0^p Q(u)\,du$

and

$$G^*(p^*) = 1 - \frac{2[\mu^*(p^*)]^{-1}}{(1-p^*)^2} \int_{p^*}^1 (1-u)Q(u)\,du \tag{5.26}$$

where $\mu^*(p^*) = \frac{1}{1-p^*}\int_{p^*}^1 Q(u)\,du$.

Theorem 5.6:

The quantile function $Q(p)$ can be uniquely determined by the income gap ratio of the poor, $\alpha(p)$ as

$$Q(p) = \frac{\mu}{p[1-\alpha(p)]} \exp\left\{-\int_p^1 \frac{1}{u[1-\alpha(u)]} du\right\} \tag{5.27}$$

Proof: From definition (5.23)

$$[1-\alpha(p)]pQ(p) = \int_0^p Q(u)\,du$$

$$\frac{Q(p)}{\int_0^p Q(u)\,du} = \frac{1}{p[1-\alpha(p)]}$$

Integrating from p to 1,

$$\left[\ln \int_0^p Q(u)\,du\right]_p^1 = \int_p^1 \frac{1}{u[1-\alpha(u)]} du$$

$$\ln \mu - \ln \int_0^p Q(u)\,du = \int_p^1 \frac{1}{u\left[1-\alpha(u)\right]}\,du$$

$$\ln \frac{\int_0^p Q(u)\,du}{\mu} = -\int_p^1 \frac{1}{u\left[1-\alpha(u)\right]}\,du$$

$$\int_0^p Q(u)\,du = \mu \exp\left\{-\int_p^1 \frac{1}{u\left[1-\alpha(u)\right]}\,du\right\}$$

Now differentiating with respect to p, we get 5.27.

Theorem 5.7:

The quantile function can be uniquely determined by the income gap ratio of the rich, $\alpha^*(p^*)$ as

$$Q^*(p^*) = \mu\left[\frac{1-\alpha^*(p^*)}{1-p^*}\right]\exp\left\{-\int_0^{p^*} \frac{1-\alpha^*(u)}{1-u}\,du\right\} \qquad (5.28)$$

Proof: From definition 5.24,

$$\left[1-\alpha^*(p^*)\right]\int_{p^*}^1 Q(u)\,du = (1-p^*)Q(p^*)$$

$$\frac{-Q(p^*)}{\int_{p^*}^1 Q(u)\,du} = \frac{\alpha^*(p^*)-1}{1-p^*}$$

Integrating from 0 to p^*,

$$\left[\ln \int_{p^*}^1 Q(u)\,du\right]_0^{p^*} = \int_0^{p^*} \frac{\alpha^*(u)-1}{1-u}\,du$$

$$\ln \frac{\int_{p^*}^1 Q(u)\,du}{\mu} = \int_0^{p^*} \frac{\alpha^*(u)-1}{1-u}\,du$$

$$\int_{p^*}^{1} Q(u)\,du = \mu \exp\left\{-\int_0^{p^*} \frac{1-\alpha^*(u)}{1-u}du\right\}$$

Now differentiating with respect to p^*, we get 5.28.

Now (5.25) can be written as

$$\left[1+G(p)\right]\mu(p) = \frac{2}{p^2}\int_0^p uQ(u)\,du \tag{5.29}$$

Let $$g(p) = \left[1+G(p)\right]\mu(p)$$

Thus from (5.29),

$$p^2 g(p) = 2\int_0^p uQ(u)\,du$$

Differentiating with respect to p,

$$2pg(p) + p^2 g'(p) = 2pQ(p)$$

Dividing by $2p$,

$$Q(p) = g(p) + \frac{pg'(p)}{2} \tag{5.30}$$

Thus it is possible to determine the quantile function from $g(p)$. As an illustration we have found below the expression of $g(p)$ that characterizes $Q(p)$ for some important distributions.

(i) Pareto:

$$Q(p) = \sigma(1-p)^{-\frac{1}{\alpha}}$$

$$g(p) = \frac{2}{p^2}\int_0^p uQ(u)\,du$$

$$= \frac{2}{p^2}\int_0^p u.\sigma(1-u)^{-1/\alpha}\,du$$

$$=\frac{2\sigma\alpha}{p^2(\alpha-1)(2\alpha-1)}\left\{\alpha-(1-p)^{-\frac{1}{\alpha}+2}\right\}-\frac{2\sigma\alpha(1-p)^{-\frac{1}{\alpha}+1}}{p(\alpha-1)}$$

(ii) Exponential:

$$Q(p)=-\frac{1}{\lambda}\ln(1-p)$$

$$g(p)=\frac{2}{p^2}\int_0^p u\left(-\frac{1}{\lambda}\ln(1-u)\right)du$$

$$=\frac{1}{\lambda}\left[\frac{1}{p}+\frac{1}{2}-\ln(1-p)+\frac{\ln(p-1)}{p^2}\right]$$

(iii) Power:

$$Q(p)=cp^{\frac{1}{\alpha}}$$

$$g(p)=\frac{2}{p^2}\int_0^p ucu^{\frac{1}{\alpha}}du$$

$$=\frac{2c\alpha p^{\frac{1}{\alpha}}}{2\alpha+1}$$

(iv) Dagum:

$$Q(p)=b\left[p^{-\frac{1}{\beta}}-1\right]^{-\frac{1}{a}}$$

$$g(p)=\frac{2}{p^2}\int_0^p u.b\left[u^{-\frac{1}{\beta}}-1\right]du$$

$$=\frac{2b\beta}{p^2}\int_0^{p^{\frac{1}{\beta}}} v^{\frac{1}{a}+2\beta-1}(1-v)^{-\frac{1}{a}}dv, \quad \text{when } v=u^{-\frac{1}{\beta}}$$

$$=\frac{2b\beta}{p^2}B_{p^{\frac{1}{\beta}}}\left(\frac{1}{a}+2\beta,1-\frac{1}{a}\right).$$

Similarly $Q(p^*)$ can be uniquely determined by the function

$$g^*(p^*)=\left[1-G^*(p^*)\right]\mu^*(p^*)$$

which can be proved as follows.

From (5.26)

$$g^*(p^*) = \frac{2}{(1-p^*)^2} \int_{p^*}^{1} (1-u) Q(u)\, du$$

Differentiating with respect to p^*, we obtain

$$Q(p^*) = g^*(p^*) - \frac{(1-p^*) g^{*\prime}(p^*)}{2} \tag{5.31}$$

The characterization results obtained in the previous sections can also be proved using quantile function approach.

Now for MLF,

$$\alpha(p) = \frac{\frac{1}{\lambda_2}\left[\frac{p^{\lambda_3+1}}{\lambda_3+1} + \frac{1-(1-p)^{\lambda_4}(1+\lambda_4 p)}{\lambda_4(\lambda_4+1)}\right]}{\lambda_1 p + \frac{p}{\lambda_2}\left[\frac{p^{\lambda_3}-1}{\lambda_3} - \frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right]} \tag{5.32}$$

$$\alpha^*(p^*) = \frac{\frac{1}{\lambda_2}\left[\frac{p^{*\lambda_3}}{\lambda_3+1} + \frac{(1-p^*)^{\lambda_4+1}}{\lambda_4+1} - \frac{p^{*\lambda_3}}{\lambda_3} + \frac{1}{\lambda_3(\lambda_3+1)}\right]}{(1-p^*)\left[\lambda_1 + \frac{1}{\lambda_2}\left(\frac{p^{*\lambda_3}-1}{\lambda_3} - \frac{(1-p^*)^{\lambda_4}-1}{\lambda_4}\right)\right]} \tag{5.33}$$

$$G(p) = \frac{\frac{1}{\lambda_2}\left[\frac{p^{\lambda_3}}{(\lambda_3+2)(\lambda_3+1)} + \frac{1+(1-p)^{\lambda_4+1}}{p\lambda_4(\lambda_4+1)} + \frac{2\left[(1-p)^{\lambda_4+2}-1\right]}{p^2\lambda_4(\lambda_4+1)(\lambda_4+2)} - \frac{1}{\lambda_4}\right]}{\lambda_1 + \frac{1}{\lambda_2}\left[\frac{p^{\lambda_3}}{\lambda_3(\lambda_3+1)} + \frac{(1-p)^{\lambda_4+1}-1}{p\lambda_4(\lambda_4+1)} + \frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right]} \tag{5.34}$$

$$G^*(p^*)=\frac{\dfrac{1}{\lambda_2}\left[\dfrac{1-p^{*\lambda_3+1}}{(\lambda_3+1)(\lambda_3+2)}-\dfrac{p^*}{\lambda_3(\lambda_3+1)}+\dfrac{p^{*\lambda_3+2}}{\lambda_3(\lambda_3+1)}+\dfrac{(1-p^*)^{\lambda_4+2}}{(\lambda_4+2)(\lambda_4+1)}\right]}{(1-p^*)^2\left[\lambda_1+\dfrac{1}{\lambda_2}\left[\dfrac{1}{\lambda_4}-\dfrac{1}{\lambda_3}+\dfrac{1-p^{*\lambda_3+1}}{\lambda_3(\lambda_3+1)(1-p^*)}-\dfrac{(1-p^*)^{\lambda_4}}{\lambda_4(\lambda_4+1)}\right]\right]} \quad (5.35)$$

In conclusion, we have shown that the modified lambda family has the potential to be used as a model of income, because of its flexibility to assume different distributional shapes. In view of the quantile functions involved in the distribution, it is easier to generate random numbers than many of the competing parametric models. Since analysis of income data usually involves a large number of observations, the asymptotic properties seem to apply with a good amount of accuracy. Further there is closed form expressions for many of the measures of income inequality, making them easier to compute. We have also presented a few theoretical results that help the identification of the distribution of income given the income gap ratio or the truncated Gini coefficient at different values of the poverty or affluence limit. The poverty and affluence measures being directly expressed in terms of the α's and G's, the results established here have relevance in that context also.

In the present study MLF is fitted to one income data remarkably well. It is essential to verify the goodness of fit of the family to the income datas of various countries in different time periods. This model can also be used to project the income distributions of future period. These problems are expected to be presented in a subsequent work.

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