A STUDY OF ELEMENTARY OPERATORS

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DECLARATION

I, Sindhu G. Nair hereby declare that this thesis entitled "**A Study of Elementary Operators**" is based on the original research work carried out by me under the guidance of Dr.M.N.Narayanan Namboodiri, Department of Mathematics, Cochin University of Science and Technology, Cochin-22 and no part of this work has previously formed the basis of the award of any degree, diploma, associateship, fellowship or any other title or recognition.

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CERTIFICATE

*This is to certify that the thesis entitled "**A Study of Elementary Operators**" submitted to the Cochin University of Science and Technology by Sindhu G. Nair for the award of degree of Doctor of Philosophy in the Faculty of Science is a bonafide record of studies done by her under my supervision. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.*

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CONTENTS

1 CHAPTER

INTRODUCTION

The study of elementary operators, which is evolved from the theory of matrix equations, was originated by Stephanos and Sylvester. But a symmetric study was begun in the late 50's by Lumer and Rosenblum. They emphasized the spectral properties of these operators and their applications to systems of operator equations. The study of these operators was developed in two branches, Spectral properties and Structural properties. The survey articles of R.E.Curto and L.A.Fialkov [17] give a very good picture of these two aspects.

Let X be a Complex Banach Space and $B(X)$ denote the set of all bounded linear operators on X. A bounded linear map Φ on $B(X)$ is called an elementary operator if there exists operators $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ in $B(X)$ such that

$$\Phi(T) = \sum_{i=1}^{n} A_i T B_i, \forall T \in B(X).$$

These operators are studied by researchers in connection with invariant subspace problem, multi variable spectral theory, structure theory of operator algebras, Riccati equations, Soliton equations etc.,. For A and B in $B(X)$, the so called generalized derivation τ_{AB} on B(X) defined by $\tau_{AB}(X) = AX - XB$ and the 2 sided multiplication operator M_{AB} given by $M_{AB}(X) = AXB$ are operators belonging to this class.

One important aspect of elementary operators is its compactness. K.Vala in his paper [26] proved that the elementary operator Φ given by $\Phi(X) = AXB$ is compact iff A and B are compact operators. Later on C.K.Fong and A.R.Sourour [12] proved that an elementary operator Φ on $B(X)$ is compact iff it has a representation $\Phi(T) = \sum_{i=1}^{n} A_i T B_i$, where each A_i and B_i is compact. Here we study some structural properties of a family of elementary operators based on Anselone's theory of collectively compact operators [1].

1.1 Summary of the thesis

The thesis is divided into four chapters including the introductory chapter I and an Appendix. In Chapter II, we study collective compactness and total boundedness of a family of elementary operators motivated by the work of Fong and Sourour [12] and P.M.Anselone [1]. Anselone developed the intimate connection between collective compactness and total boundedness of a family of operators in $B(X)$. This chapter is divided into 2 sections. In section 1, it is proved that, under some conditions, a family $(\Phi_\alpha)_{\alpha\in I}$ of elementary operators defined by $\Phi_\alpha(T) = \sum_{i=1}^{n} A_i^\alpha T B_i^\alpha$ form a collectively compact set implies $(A_i^\alpha)_{\alpha\in I}$ is collectively compact for each i. Examples are provided to show that the converse need not be true and this is not the case of the second coefficients $(B_i^\alpha)_{\alpha\in I}$. In section 2, we consider the intimate connection between total boundedness and collective compactness of a family of compact elementary operators and we give a necessary and sufficient condition for the total boundedness of a family of compact elementary operators. Also we give sufficient conditions for a family of elementary operators to be totally bounded and collectively compact.

In the 3^{rd} chapter we give some applications of the results in Chapter II to operator equations involving integral operator coefficients. This chapter is divided in to two sections. In section 1, we use Anselone's theory to approximate an operator equation by a collective compact sequence. In section 2, Rice's theory [14] is used to approximate elementary operators with integral operator coefficients.

The 4^{th} and final chapter is divided into 2 sections. In section 1, we introduce the concept of locally elementary operators. It is proved that in the case of a finite dimensional space every locally elementary operator is elementary. Also it is proved that every locally elementary operator is the strong limit of a sequence of elementary operators and we give a sequence of locally elementary operators which are not elementary. In addition to that we give a sufficient condition for a bounded linear operator on a Hilbert space to be diagonal and a theorem showing the existence of such a sequence of locally elementary operators. In section 2, the concept of Random elementary operators is introduced and we discuss it briefly. Also we give some examples of random elementary operators.

In the Appendix of this thesis, we furnish some problems for future work.

1.2 Basic Definitions and Theorems

The definitions and theorems that are quoted in the subsequent chapters are given here.

Let X be a complex Banach space and $B(X)$ denote the set of all bounded linear operators on X.

1.2.1 Collective Compactness [1]

A subset $\mathcal{K}$ of $B(X)$ is said to be collectively compact if the set $\{K(x), K \in \mathcal{K}, x \in X, \| x \| \leq 1\}$ is relatively compact. A sequence of operators in $B(X)$ is collectively compact whenever the corresponding set is collectively compact. For example

Let $X = l^2$ Define $K_n : l^2 \to l^2$ by

$K_n(x) = < x, e_n > e_1$

Let $\mathcal{K} = (K_n)$.

Since $\{K_n(x)/x \in X, \| x \| \leq 1\}$ is bounded and dim $\mathcal{K}\mathcal{X} = 1$, $\mathcal{K}$ is collectively compact.

1.2.2 Definition

A subset E of X is said to be totally bounded if for every $\epsilon > 0$, there are $x_1, x_2, \ldots, x_n$ in X such that $E \subset \bigcup_{i=1}^{n} U(x_i, \epsilon)$. Every totally bounded set is bounded. The converse is true only when X is finite dimensional.

1.2.3 Definition

Let X^* denote the normed dual of X. For $K \in B(X)$, the adjoint of K is the unique operator $K^* \in B(X^*)$ defined by

$$(K^*f)(x) = f(K(x)) \forall f \in X^*, x \in X$$

when X is a Hilbert space, it is customary for K^* to denote the usual Hilbert space adjoint of K given by

$$< K(x), y >=< x, K^*(y) > \forall x, y \in X$$

Since $\| K^* \|=\| K \| \forall K \in B(X)$, a subset $\mathcal{K} \subset B(X)$ is totally bounded iff $\mathcal{K}^* = \{K^*/K \in \mathcal{K}\}$ is totally bounded.

1.2.4 Theorem [1]

Let $\mathcal{K}$ be a set of compact operators in $B(X)$. Then $\mathcal{K}$ is totally bounded iff both $\mathcal{K}$ and $\mathcal{K}^*$ are collectively compact.

1.2.5 Theorem [1]

Let $\mathcal{K} \subset B(X)$ be collectively compact. Then each of the following sets is collectively compact.

1. $\{\lambda K/\lambda \in \Delta, K \in \mathcal{K}\}$ for any bounded scalar set Δ.

2. $\{KM/K \in \mathcal{K}, M \in \mathcal{M}\}$ for any bounded set $\mathcal{M} \subset B(X)$.

3. $\{NK/N \in \mathcal{N}, K \in \mathcal{K}\}$ for any relatively compact set $\mathcal{N} \subset B(X)$.

4. The norm closure $\bar{\mathcal{K}}$ of $\mathcal{K}$.

5. $\{\sum_{i=1}^{n} \lambda_i K_i : K_i \in \mathcal{K}, \sum_{i=1}^{n} \mid \lambda_i \mid \leq b\}$ for any $b < \infty$ and $n \leq \infty$.

1.2.6 Trace Class Operator

Let H be a Hilbert space and $B(H)$, $\mathcal{K}$(H) respectively be the set of all bounded linear operators and compact operators. For each ω in $\mathcal{K}$ $(H)^*$, the dual of $\mathcal{K}$(H), there exists a t_ω in $B(H)$ defined by

$$\omega(T) = trace(Tt_\omega), Tin\mathcal{K}(H).$$

$\{t_\omega : \omega \in \mathcal{K}(H)^*\}$ *is called the set of all trace class operators.*

1.2.7 Definition [22]

Let A be an $m \times n$ matrix.A generalized inverse of A is a matrix A^- of order $n \times m$ such that $AA^-A = A$.

1.2.8 Theorem [22]

A general solution of a consistent non homogeneous equation AX = Y is

$$X = A^-Y + (I - A^-A)WY \quad \textit{where } W \textit{ is an arbitrary } n \times m \textit{ matrix.}$$

1.2.9 Definition [25]

Let A and B be C^*-algebras and let $M_n(A)$ be the set of all matrices of order n with entries in A. For each linear map $\Phi : A \to B$, we define a linear map $\Phi_n : M_n(A) \to M_n(B)$ by $\Phi[a_{ij}] = [\Phi(a_{ij})]$. If Φ_n is positive, then Φ is said to be n-positive. If Φ is n-positive for all n, then Φ is said to be completely positive.

1.2.10 Definition [9]

Let H be a complex Hilbert space and $\mathcal{L}_s$(H) be thc set of all bounded self adjoint linear operators on H. A positive linear functional Φ from $\mathcal{L}_s$(H)$\to R$ is normal if $A_n \to A$ implies $\Phi(A_n) \to \Phi(A)$.

1.2.11 Definition [9]

Let H be a Hilbert space and let $\mathcal{V}$ be a von Newmann algebra in $B(H)$. A normal conditional expectation of $B(H)$ on to $\mathcal{V}$ is a linear map ϵ of $B(H)$ on to $\mathcal{V}$ such that

(1) $\epsilon(X^*) = \epsilon(X)^*$ for all $X \in B(H)$.

(2) $\epsilon(X) = X$ iff $X \in \mathcal{V}$

(3) If $X \geq 0$ then $\epsilon(X) \geq 0$.

(4) If $X_1, X_2 \in \mathcal{V}$ and $Y \in B(H)$ then $\epsilon(X_1 Y X_2) = X_1 \epsilon(Y) X_2$.

(5) If $X_n \uparrow X$, then $\epsilon(X_n) \uparrow \epsilon(X)$.

Example :- Let P(.) be a projection-valued measure on Z and define

$\mathcal{V}$=$\{A \in B(H)$:$AP_n = P_nA$, for all $n \in Z\}$.

Then $\mathcal{V}$ is a von Newmann algebra and the map ϵ on B(H) defined by $\epsilon(A) = \sum_{n \in Z} P_nAP_n$ is a normal conditional expectation of B(H) on to $\mathcal{V}$.

1.2.12 Definition [21]

Let $(\Omega, \mathcal{B})$ be a measurable space and X a metric space. A function $g : \Omega \to X$ is called a generalized random variable if for any $B \in \mathcal{B}_X$, the σ-algebra generated by closed subsets of X, $g^{-1}(B)$ belongs to the σ-algebra $\mathcal{B}$.

1.2.13 Definition [21]

Let $(\Omega, \mathcal{B})$ be a measurable space, Γ an arbitrary set and X a metric space. A mapping $T : \Omega \times \Gamma \to X$ is called a random operator if for each $\gamma \in \Gamma$, $T(., \gamma)$ is an X-valued generalized random variable.

1.3 Notations that are frequently used

X	– Complex Banach space.
$B(X)$	– Set of all bounded linear operators on X.
Φ	– Elementary operator.
H	– Complex Hilbert space.

2 CHAPTER

COLLECTIVELY COMPACT & TOTALLY BOUNDED FAMILY OF ELEMENTARY OPERATORS

This chapter deals with the collective compactness and total boundedness of a family of elementary operators. The chapter is divided into 2 sections. In section 1, collective compactness is investigated. Second one deals with total boundedness aspect.

2.1 Collective Compactness

In this section we apply Anselone's theory of collective compactness to a family of elementary operators.

2.1.1 Definition

Let X be a complex Banach space and $B(X)$ denote the set of all bounded linear operators on X. A linear mapping Φ on B(X) is called an elementary operator if $\exists A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ in B(X) such that

$\Phi(T) = \sum_{i=1}^{n} A_i T B_i$, $T \in B(X)$.

First we recall the following lemma

2.1.2 Lemma [12]

Let $B_1, \ldots, B_n$ be in B(X) which are linearly independent and let B be in $B(X)$. Then B is not in the linear span of $B_1, \ldots, B_n$ iff there are finitely many vectors $x_1, \ldots, x_r$ in X and equally many linear functionals $f_1, \ldots, f_r$ in X^* such that

$\sum_{i=1}^{r} f_i(B_j(x_i)) = 0$, j=1,2,...,n. and

$\sum_{i=1}^{r} f_i(B(x_i)) = 1$.

In order to cope with the complex situation that arise while dealing with a family of such finite collections we define property #.

2.1.3 Property #.

Let $\{B_1^\alpha, B_2^\alpha, \ldots, B_n^\alpha\}_{\alpha \in I}$ (I an index set) be a family in $B(X)$ such that for each $\alpha \in I$, $\{B_1^\alpha, B_2^\alpha, \ldots, B_n^\alpha\}$ is linearly independent. Then the above collection is said to have property # if there are finite dimensional subspaces Y_k of X and Y_k^* of X^* and there are vectors $x_{1k}^\alpha, x_{2k}^\alpha, \ldots, x_{r_\alpha k}^\alpha$ in Y_k, functionals $f_{1k}^\alpha, f_{2k}^\alpha, \ldots, f_{r_\alpha k}^\alpha$ in Y_k^* which are uniformly bounded such that

$\sum_{i=1}^{r_\alpha k} f_{ik}^\alpha(B_j^\alpha(x_{ik}^\alpha)) = 0,\ j \neq k, j = 1, 2, \ldots, n$ and

$\sum_{i=1}^{r_\alpha k} f_{ik}^\alpha(B_k^\alpha(x_{ik}^\alpha)) = 1$, k=1,2,...,n.

Here we give some examples of family of functions having property #.

Example 1:-

Let $H = l^2$, the Hilbert space of all square summable sequences of real or complex numbers and $\{e_1, e_2, \ldots\}$ be the standard orthonormal basis in l^2. Define (P_n) on H by,

$$P_n(x_1, x_2, \ldots) = (x_1, \tfrac{x_2}{2}, \tfrac{x_3}{3}, \ldots, \tfrac{x_n}{n}, 0, 0, \ldots).$$

The sequence (P_n) has property #.

Example 2:-

Let $\{L_i(s,t), 1 = 1,2,\ldots,n\}$ be nonnegative real, continous functions on $[a,b] \times [a,b]$ such that for each i, there is a subinterval $[a_i,b_i] \subseteq [a,b]$ such that

$$\begin{aligned} L_k(s,t) &= 0 \text{ for every } (s,t) \in [a_i,b_i] \times [a_i,b_i], k \neq i \\ &\neq 0 \text{ for } k = i, (s,t) \in [a_i,b_i] \times [a_i,b_i] \end{aligned}$$

Also assume that $[a_i,b_i] \cap [a_j,b_j] = \phi, i \neq j$.

Let $f \in C[a,b]$. For $\omega_{nj} > 0$ and t_{nj} in $[a,b]$, $j = 1,2,\ldots,n$, let the numerical quadrature formula satisfies

$$\lim_{n\to\infty} \sum_{j=1}^{n} \omega_{nj} f(t_{nj}) = \int_a^b f(t)dt$$

Also for $i = 1,2,\ldots,m$

$$L_i^{(n)}(f)(s) = \sum_{j=1}^{n} \omega_{nj} L_i(s,t_{nj}) f(t_{nj}), \qquad f \in C[a,b]$$

This sequence $(L_i^{(n)})$ has property #.

Proof.

For each i let f_i be a nonnegative continuous real function on $[a,b]$ such that

$$\begin{aligned} f_i(t) &\neq 0 \text{ for every } t \in (a_i,b_i) \\ &= 0 \text{ otherwise.} \end{aligned}$$

Also let,

$$\phi_{f_i}(g) = \int_a^b f_i(t)g(t)dt, \text{ g in } C[a,b]$$

Then ϕ_{f_i} is in the dual $C[a,b]^*$ of $C[a,b]$.

Now consider the m-dimensional subspaces X and Y generated by $\{f_i/i = 1,2,\ldots,m\}$ and $\{\phi_{f_i}/i = 1,2,\ldots,m\}$ respectively. Then,

$$\phi_{f_i}(L_k^{(n)}(f_i)) = \int_a^b \sum_{j=1}^n \omega_{nj} L_k(s,t_{nj})(f_i(t_{nj})f(s)ds$$

$$= \sum_{j=1}^n \omega_{nj} f_i(t_{nj}) \int_a^b L_k(s,t_{nj})f_i(s)ds$$

$$= 0 \text{ for } k \neq i$$

$$= \sum_{j=1}^n \omega_{nj} f_i(t_{nj}) \int_a^b L_i(s,t_{nj})f_i(s)ds, \; k = i$$

Let $\alpha_n = \sum_{j=1}^n \omega_{nj} f_i(t_{nj}) \int_a^b L_i(s,t_{nj})f_i(s)ds$

By our choice, (α_n) is a bounded sequence of positive real numbers such that $\alpha_n > \epsilon > 0$ for some $\epsilon > 0$. Now put $g_{in} = \frac{f_i}{\alpha_n}$. Thus $\| g_{in} \| < \frac{\|f_i\|}{\epsilon}$ for every n, and is in X^*.

Thus $\phi_{f_i}(L_k^{(n)}(g_{in})) = 0$ for $k \neq i$

$= 1$ for $k = i, i = 1,2,\ldots,m.$

Hence by definition $(L_i^{(n)})$ has property $\#$. $\square$

Now we prove one of the main theorems of this chapter.

2.1.4 Theorem.

The collection $(\Phi_\alpha)_{\alpha\in I}$ of elementary operators on $B(X)$ where $\Phi_\alpha(T) = \sum_{i=1}^{n} A_i^\alpha T B_i^\alpha, \alpha \in I, T \in B(X)$ is collectively compact implies that $(A_i^\alpha)_{\alpha\in I}$ is collectively compact for each i provided $(B_i^\alpha)_{\alpha\in I}$ has property #.

Proof.

We prove the result for i=1. The other cases can be proved similarly. Let Y_1 and Y_1^* be finite dimensional subspaces of X and X^* respectively as in property #. The number r_{α_1} given in # can be assumed to be smaller than maximum of dim Y_1 and dim Y_1^* with out any difficulty. Let $\{x_1, x_2, \dots, x_{N_1}\}$ and $\{f_1, f_2, \dots, f_{N_2}\}$be basis for Y_1 and Y_1^* respectively.

Since $(\Phi_\alpha)_{\alpha\in I}$is collectively compact for each bounded set U in B(X), $\bigcup_{\alpha\in I} \Phi_\alpha(U)$ is relatively compact. Now,

$\sum_{k=1}^{r_\alpha 1} f_{k1}^\alpha(B_j^\alpha(x_{k1}^\alpha)) = \sum_{k=1}^{r_\alpha 1} \sum_{j=1}^{N_2} \delta_{\alpha kj} f_j(B_j^\alpha(x_{k1}^\alpha))$

$$= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \beta_{\alpha ij} f_j(B_j^\alpha(x_i))$$

where $\beta_{\alpha ij}=\sum_{k=1}^{r_{\alpha_1}} \delta_{\alpha kj}\theta_{\alpha ki}$ for suitable scalars $\delta_{\alpha kj}$ and $\theta_{\alpha ki}$.

Now by theorem 2.4 in [13], there exist $c > 0$ such that

$$\| x_{k1}^\alpha \| = \| \sum_{i=1}^{N_1} \theta_{\alpha ki} x_i \| > c \sum_{i=1}^{N_1} | \theta_{\alpha ki} |$$

Then $\sum_{i=1}^{N_1} \mid \theta_{\alpha ki} \mid < \frac{\lambda}{c}$ where λ is a bound for $\{\| x_{k1}^{\alpha} \|,\ \alpha \in I, k = 1, \ldots, r_{\alpha 1}\}$

$$= \lambda_1(\text{say}).$$

Similarly $\exists \lambda_2 > 0$ such that $\sum_{j=1}^{N_2} \mid \delta\alpha kj \mid < \lambda_2$.

Therefore $|\beta_{\alpha ij}| < \lambda_1 \cdot \lambda_2 . \max\{N_1, N_2\}$, since $r_{\alpha 1} \leq \max\{N_1, N_2\}$.

Hence the set B$=\{\beta_{\alpha ij}/\alpha \in I\ i = 1, \ldots, N_1\ ,\ j = 1, \ldots, N_2\}$ is bounded.

Since $(\Phi_\alpha)_{\alpha \in I}$ is collectively compact and B is bounded $(B\Phi_\alpha)_{\alpha \in I}$is collectively compact by Theorem 1.2.5. Hence $\bigcup_{\alpha \in I}\{\beta_{\alpha ij}\Phi_\alpha(f_j \otimes x)(x_i)/\|x\| \leq 1\}$is relatively compact. Thus $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\bigcup_{\alpha \in I}\{\beta_{\alpha ij}\Phi_\alpha(f_j \otimes x)(x_i)/\|x\| \leq 1\}$ is relatively compact.

Therefore $\bigcup_{\alpha \in I}\{\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\beta_{\alpha ij}\Phi_\alpha(f_j \otimes x)(x_i)/\|x\| \leq 1\}$ is relatively compact.

Now $\Phi_\alpha(f_j \otimes x)(x_i) = \sum_{k=1}^{n} A_k^\alpha(f_j \otimes x)B_k^\alpha(x_i)$

$$= \sum_{k=1}^{n} f_j(B_k^\alpha(x_i))A_k^\alpha(x).$$

Therefore $\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\beta_{\alpha ij}\Phi_\alpha(f_j \otimes x)(x_i)$

$$= \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\beta_{\alpha ij}\sum_{k=1}^{n} f_j(B_k^\alpha(x_i))A_k^\alpha(x)$$

$$= \sum_{k=1}^{n}[\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\beta_{\alpha ij}f_j(B_k^\alpha(x_i))]A_k^\alpha(x)$$

$$= \sum_{k=1}^{n}[\sum_{m=1}^{r\alpha 1} f_{m_1}^{\alpha}(B_j^{\alpha}(x_{m_1}^{\alpha}))]A_k^{\alpha}(x)$$

$$= A_1^{\alpha}(x) \text{ by the definition of } \{f_{k_1}^{\alpha}\}_{k=1,2,\ldots,r_{\alpha 1}} \text{ and } \{x_{k_1}^{\alpha}\}_{k=1,2,\ldots,r_{\alpha 1}}$$

Therefore $\{A_1^{\alpha}(x)/\|x\| \leq 1\}$ is relatively compact.Hence $(A_1^{\alpha})_{\alpha\in I}$ is collectively compact. □

2.1.5 Remarks

As in the case of compact elementary operators, the analogous collective compactness of $(B_k^{\alpha})_{\alpha\in I}$ does not remain valid. The following example shows this.

2.1.5 (1) Example

Let H=l^2, the Hilbert space of all square summable sequences of real or complex numbers and $\{e_1, e_2, \ldots\}$ be the standard orthonormal basis in l^2.

Put $K_n(x) = \langle x, e_n\rangle e_1, x \in l^2$. For any bounded set U in $l^2, \{K_n(U)\}_{n\in N}$ is bounded and since $\dim\{K_n(H)\} = 1$, $(K_n)_{n\in N}$ is collectively compact. Now $K_n^{\star}(x) = \langle x, e_1\rangle e_n$, $x \in l^2$. Since $\|K_n^{\star}(e_1) - K_m^{\star}(e_1)\| = \sqrt{2}$, whenever $n \neq m$, $(K_n^{\star}(e_1))_{n\in N}$ does not have a convergent subsequence. Therefore $(K_n^{\star})$ is not collectively compact. Now for each n, let $\Phi_n(T) = K_n T \tilde{K_n}$ where $\tilde{K_n} = K_1 + K_n^{\star}$, n=1,2,3,....

$(\tilde{K_n})$ satisfies property #.

Now,

$$\begin{aligned}\Phi_n(T)(x) &= K_n T \tilde{K_n}(x)\\ &= [\langle T(e_1), e_n\rangle + \langle T(e_n), e_n\rangle] K_1(x)\\ &= \lambda_n(T) K_1(x)\end{aligned}$$

Therefore (Φ_n) is collectively compact.

But $(\tilde{K_n})$ is not collectively compact.

2.1.6 Remarks

Here we provide two more examples. This will show that the collective compactness of the coefficient operators need not imply the collective compactness of the associated elementary operators. This is an interesting aspect when we look at the corresponding result for a single elementary operator.

2.1.6 (1) Example.

Let H=l^2. Consider the operator defined as in Example 2.1.5(1). We know that (K_n) is collectively compact.Define (P_n) and P on H as

$$P_n(x_1, x_2, \ldots) = (x_1, \tfrac{x_2}{2}, \tfrac{x_3}{3}, \ldots, \tfrac{x_n}{n}, 0, 0, \ldots)$$
$$P(x_1, x_2, \ldots) = (x_1, \tfrac{x_2}{2}, \tfrac{x_3}{3}, \ldots, \tfrac{x_n}{n}, \tfrac{x_{n+1}}{n+1}, \ldots)$$

Since $\bigcup_n \{P_n(x)/x \in U\} \subseteq \{P(x)/x \in U\}$ and P is compact, (P_n) is collectively compact.

Put $\Phi_n(T) = P_n T K_n$, T in $B(H)$.

Since $\|\Phi_n(I) - \Phi_m(I)\| \geq 1, (\Phi_n)$ is not collectively compact.

2.1.6 (2) Example.

For x in l^2, let $B_n(x) = x_1e_1 + x_ne_2$, $x = (x_1, x_2, \ldots) \in l^2$.

(B_n) satisfies property # and (B_n) is collectively compact.

Let $\Phi_n(T) = K_nTB_n, T \in B(l^2)$.

Since $\|\Phi_n(B_n^\star)(e_n) - \Phi_m(B_m^\star)(e_n)\| = 1$ for $m \neq n$,

$\|\Phi_n(B_n^\star) - \Phi_m(B_m^\star)\| \geq 1$.

Therefore (Φ_n) is not collectively compact.

We conclude this section with an interpretation of property # so as to reduce any kind of obsecurity or artificiality inherent in the very definition of it, through the following proposition.

2.1.7 Proposition

Let H be a Hilbert space and $(B^\alpha)_{\alpha\in I}$ be in $B(H)$. This collection $(B^\alpha)_{\alpha\in I}$ has property # iff there is a finite dimensional subspace Y of H and a real number $\theta > 0$ such that $\eta_\alpha = \max_{x,y\in S_Y} |< B_\alpha x, y >| \geq \theta$, $\forall\ \alpha \in\ I$ where $S_Y = \{x \in Y / \parallel x \parallel \leq 1\}$.

Proof.

Assume that the family $(B_\alpha)_{\alpha\in I}$ has property #. Then there exists finite dimensional subspaces Y_1 , $Y_2 \in H$ and vectors $x_1^\alpha, x_2^\alpha, \ldots, x_{r_\alpha}^\alpha$ in Y_1 and $y_1^\alpha, y_2^\alpha, \ldots, y_{r_\alpha}^\alpha$ in Y_2 which are uniformly bounded such that

$$\textstyle\sum_{i=1}^{r_\alpha} < B_\alpha x_i^\alpha, y_i^\alpha > = 1$$

Let $Y = span\{Y_1, Y_2\}$ and $N = dimY$. Clearly $r_\alpha \leq N$. Also,

$$1 = \left| \sum_{i=1}^{r_\alpha} < B_\alpha x_i^\alpha, y_i^\alpha > \right|$$

$$\leq \sum_{i=1}^{r_\alpha} |< B_\alpha x_i^\alpha, y_i^\alpha >|$$

$\leq kN\eta_\alpha$ since $x_\alpha^{i's}$ and $y_\alpha^{i's}$ are uniformly bounded.

Therefore $\quad \eta_\alpha \geq \frac{1}{kN} = \theta.$

Conversely let $\exists$ a finite dimensional subspace Y of H such that $\eta_\alpha \geq \theta$, $\forall \alpha \in I$. Since Y is finite dimensional, $\exists$ some $\alpha \in I$ such that

$$\eta_\alpha =< B_\alpha(x_\alpha), y_\alpha >, \qquad x_\alpha, y_\alpha \in S_Y. \text{ Then,}$$

$< B_\alpha(x_\alpha), \frac{y_\alpha}{\eta_\alpha} > \; = 1$, where $\| \frac{y_\alpha}{\eta_\alpha} \| \leq \frac{1}{|\eta_\alpha|} \leq \frac{1}{\theta}$.

Hence $(B_\alpha)_{\alpha \in I}$ has property $\#$.

□

2.1.8 Remarks

The general case, when there is a family $\{B_K^\alpha / K = 1, 2, \ldots, n\}_{\alpha \in I}$, can be reduced to the above Case by considering matrices

$$\begin{pmatrix} B_1^\alpha & & & \\ & B_2^\alpha & 0 & \\ & & \ddots & \\ 0 & & & B_n^\alpha \end{pmatrix}$$

acting on $H \oplus H \oplus \ldots \oplus H(ncopies)$ and can have a similar geometric interpretation. However this property is very crucial to the results we obtained.

We conclude this section with the following simple theorem.

2.1.9 Theorem

Let X be a normed space and Y be a dense subspace of X. Let (K_n) be a collectively compact sequence in $B(Y)$. Let $\tilde{K_n}$ be the extension of K_n to X. Then $(\tilde{K_n})$ is collectively compact.

Proof.

Since (K_n) is collectively compact, each K_n is compact and by a theorem in [13], each $\tilde{K_n}$ is compact.

Let $U = \{x \in X/ \parallel x \parallel \leq 1\}$. It is enough to show that $\cup_n\{\tilde{K_n}(U)\}$ is relatively compact.

Let $k > 1$ be any fixed real number.

Consider $\cup_n\{K_n(x)/x \in Y, \parallel x \parallel \leq k\}$.

Since (K_n) is collectively compact, the above set is relatively compact.

Let $x \in U$. Since Y is dense in X, we can select a sequence (x_n) in Y, $\parallel x_n \parallel \leq 1$ $\forall n$ (by passing to a subsequence if necessary) such that $x_n \to x$. Then

$$\cup_n\{\tilde{K_n}(x)/x \in U\} \subseteq \text{closure of } \cup_n\{K_n(x)x \in Y, \parallel x \parallel \leq k\}.$$

Therefore $\cup_n\{\tilde{K_n}(x)/x \in U\}$ is relatively compact. Hence $(\tilde{K_n})$ is collectively compact. □

2.2 Total Boundedness.

The intimate connection between total boundedness and collective compactness is well known [1]. Here we take up this in the context of elementary operators. First we show that total boundedness of the coefficient operators implies the total boundedness of the associated family of elemenary operators.

2.2.1 Theorem.

Let $(A_i^\alpha)_{\alpha\in I}$ and $(B_i^\alpha)_{\alpha\in I}$ be totally bounded families for i=1,2,..., n. Then the associated family $(\Phi_\alpha)_{\alpha\in I}$ of elemenary operators on $B(X)$ is totally bounded.

Proof.

For each i=1,2,..., n, let k_i and h_i be positive real numbers such that
$\|A_i^\alpha\| \leq k_i$ and $\|B_i^\alpha\| \leq h_i$. Given $\epsilon > 0$, let $\epsilon_i = \frac{\epsilon}{2nk_i}$.
Since (B_i^α) is totally bounded for each i, it has a finite ϵ net say $\{B_i^{\beta_1}, B_i^{\beta_2}, \ldots, B_i^{\beta_m}\}$.

Let $k_i' = \max\{\|B_i^{\beta_j}\|$, j=1,2,..., m$\}$ and

$\delta_i = \frac{\epsilon}{2nk_i'}$, i=1,2,..., n.

Since $(A_i^\alpha)_{\alpha\in I}$ is totally bounded, let $\{A_i^{\alpha_1}, A_i^{\alpha_2}, \ldots, A_i^{\alpha_e}\}$ be a finite δ_i net for i=1,2,..., n.

Put $\Phi_{jk}(T) = \sum_{i=1}^n A_i^{\alpha_j} T B_i^{\beta_k}$, T$\in B(X)$.

Then $\{\Phi_{jk}/^{j=1,2,\ldots,m}_{k=1,2,\ldots,e}\}$ is a finite ϵ net for $\{\Phi_\alpha/\alpha \in I\}$.

Hence $(\Phi_\alpha)_{\alpha\in I}$ is totally bounded. $\square$

The following proposition reveals the connection between total boundedness of coefficient operators and collective compactness of the associated family of elementary operators.

2.2.2 Proposition.

Let $(A_i^\alpha)_{\alpha\in I}$ and $(B_i^\alpha)_{\alpha\in I}$ be collectively compact families of operators in $B(X)$ for i=1,2,..., n and $(\Phi_\alpha)_{\alpha\in I}$ the associated family of elementary operators on $B(X)$. If $(B_i^\alpha)_{\alpha\in I}$ is totally bounded for each i, then (Φ_α) is collectively compact.

Proof.

Let U=$\{T \in B(X)/\|T\| \leq 1\}$.

It is enough to show that$\bigcup_{\alpha\in I}\{\Phi_\alpha(U)\}$ and $\bigcup_{\alpha\in I}\{\Phi_\alpha(U)^*\}$ are collectively compact by theorem 5.5 in [1]. Since $\{TB_i^\alpha/\|T\| \leq 1\}$ is bounded, $\bigcup_{\alpha\in I}\{A_i^\alpha TB_i^\alpha/\|T\| \leq 1\}$ is collectively compact (Proposition 4.2 in [1]). Similarly $\bigcup_{\alpha\in I}\{B_i^{\alpha^*}T^*A_i^{\alpha^*}/\|T\| \leq 1\}$ is collectively compact.

Therefore $\bigcup_{\alpha\in I}\{A_i^\alpha TB_i^\alpha/\|T\| \leq 1\}$ ic relatively compact. Hence $(\Phi_\alpha)_{\alpha\in I}$ is collectively compact. $\square$

In the light of the above theorem, it would be interesting to look at Example 2.1.6(2). Observe that the coefficient operators (B_n) in this example, fails to be totally bounded since $\|B_n - B_m\| \geq 1$, $\forall m \neq n$. We conclude this section by characterizing certain classes of totally bounded families of elementary operators in terms of its coefficient operators.

2.2.3 Theorem.

Let $(A_i^\alpha)_{\alpha \in I}$, $(B_i^\alpha)_{\alpha \in I}$, i=1,2,..., n be 2n families of compact operators on a Hilbert space H having property # . Then the associated family $(\Phi_\alpha)_{\alpha \in I}$ of elementary operators on B(H) is totally bounded iff (A_i^α) and (B_i^α) are totally bounded for i=1,2,..., n.

Proof.

Let $\mathcal{K}$(H) and $\tau(H)$ denote the class of all compact operators and the class of all trace class operators on H respectively. For ω in $\mathcal{K}\ (H)^*$, let $t_\omega \in \tau(H)$ be defined by $\omega(T) = trace(Tt_\omega)$, $T \in \mathcal{K}$(H). It is well known that the map Π defined by $\Pi(\omega) = t_\omega$ is an isometric * isomorphism of $\mathcal{K}\ (H)^*$ onto τ(H).
Let $\tilde{\Phi_\alpha} = \Phi_\alpha / \mathcal{K}$(H), $\alpha \in I$.
If (Φ_α) is totally bounded, so is $(\tilde{\Phi_\alpha})$.
Now for ω in $\mathcal{K}\ (H)^*$.

$$\begin{aligned}\tilde{\Phi_\alpha}^*(\omega)(T) &= \omega(\tilde{\Phi_\alpha}(T)) \\ &= \omega(\textstyle\sum_{i=1}^n A_i^\alpha T B_i^\alpha) \\ &= trace((\textstyle\sum_{i=1}^n A_i^\alpha T B_i^\alpha)(t_\omega))\end{aligned}$$

$= trace((\sum_{i=1}^{n} B_i^\alpha t_\omega A_i^\alpha)(T)), T \in \mathcal{K}(\mathrm{H})$ since $trace(Tt_\omega) = trace(t_\omega T)$

$= \omega(T)$

$= \sum_{i=1}^{n} B_i^\alpha \omega A_i^\alpha(T)$

Therefore

$$\Pi(\tilde{\Phi_\alpha})^* \Pi^{-1}(\omega) = \sum_{i=1}^{n} B_i^\alpha t_\omega A_i^\alpha \tag{1}$$

Notice that Theorem 2.1.3 holds if we replace $B(H)$ by $\mathcal{K}(H)$ when H is a Hilbert space.

Hence (1) shows that $(B_k^\alpha)_{\alpha\in I}$ is collectively compact. Since (Φ_α) is totally bounded, the following family $(\psi_\alpha)_{\alpha\in I}$ of elementary operators namely $\psi_\alpha(T) = \sum_{i=1}^{n} B_i^{\alpha^*} T A_i^{\alpha^*}$, $T \in B(H)$ is totally bounded and hence collectively compact. Since $(B_i^{\alpha^*})_{\alpha\in I}$ and $(A_i^{\alpha^*})_{\alpha\in I}$ have property #, it follows that $(B_i^{\alpha^*})_{\alpha\in I}$ is collectively compact. Therefore $(B_i^\alpha)_{\alpha\in I}$ is totally bounded. Similarly $(A_i^\alpha)_{\alpha\in I}$ is also totally bounded.

Converse follows from theorem 2.2.1. □

3 CHAPTER

APPLICATIONS

This chapter is divided into two sections. In section 1 we apply the results of chapter II to operator equations of the form $\sum_{i=1}^{n} A_i X B_i - X = Y$ which are important in the numerical stability analysis of differential equations. Here we give some applications of our observations to operator equations involving integral operators using Anselone's theory. In section 2 Rice theory of approximation of functions [14] is applied to collective compact family of elementary operators.

3.1 Applications to Operator equations with integral operator coefficients.

Let (Φ_n) be a sequence of compact elementary operators on $B(X)$ which converges to a compact elementary operator Φ on $B(X)$, point wise in the norm of $B(X)$. For a known T_0 in $B(X)$,consider the following operator equations:

(1) $\Phi(T) - T = T_0$

(2) $\Phi_n(T) - T = T_0$, $\qquad n = 1, 2, \ldots$

Here the problem of solving equation (1) approximately using the solutions of equation (2) and estimating the error involved is considered.

The theory of collectively compact families of bounded linear operators on a complex Banach space have already been developed and applied very successfully to integral operator equations by Anselone [1]. Here we supply the additional work needed when Anselone's theory is applied to the elementary operator setup. First we recall some of Anselone's theorems.

3.1.1 Theorem [1]

Let X be a complex Banach space and let K,K_n be in $B(X)$ such that

(1) $\lim_{n\to\infty} K_n(x) = K(x)$ for every x in X

(2) (K_n) is collectively compact, and

(3) K is compact.

Whenever $(I - K_n)^{-1}$ exists, define

$\Delta^n = \| (I - K_n)^{-1} \| \quad \| (K_n - K)K \|$

For a particular n assume that $(I - K_n)^{-1}$exists and $\Delta^n < 1$.

Then $(I - K)^{-1}$exists,

$$\begin{aligned} \| (I-K)^{-1} \| &\leq \tfrac{1+\|(I-K_n)^{-1}\|\|K\|}{1-\Delta^n} \text{ and} \\ \| x_n - x \| &\leq \tfrac{\|(I-K_n)^{-1}\|\|K_n(y)-K(y)\|+\Delta^n\|x_n\|}{1-\Delta^n} \end{aligned}$$

where $x = (I-K)^{-1}(y)$ and $x_n = (I-K_n)^{-1}(y)$

Moreover,$(I - K_n)^{-1}$ exists for all n sufficiently large, $\Delta_n \to 0$ as $n \to \infty$, the estimates for $\| (I-K)^{-1} \|$ are bounded uniformly with respect to n and the estimates for $\| x_n - x \|$ tends to zero as $n \to \infty$.

3.1.2 Theorem [1]

Let X be a complex Banach space and let K,(K_n) be in $B(X)$ such that $K_n \to K$ point wise and (K_n) is collectively compact. Then $\| (K_n - K)K \| \to 0$, $\| (K_n - K)K_n \| \to 0$ as $n \to \infty$.

In order to apply the above theory, the following simple observation would be essential.

3.1.3 Theorem.

Let $\Phi_n(T) = \sum_{i=1}^{m} A_i^n T B_i^n$, A_i^n, B_i^n, T are in $B(X)$, be elementary operators on $B(X)$ such that

(1) $\{A_i^n / i = 1, 2, \ldots, m\}$, $\{B_i^n / i = 1, 2, \ldots, m\}$ are linearly independent.

(2) $\lim_{n\to\infty} A_i^n(x) = A_i(x)$, $\lim_{n\to\infty} B_i^n(x) = B_i(x)$,$x \in X$, $i = 1, 2, \ldots, m$.

(3) (A_i^n) and (B_i^n) are collectively compact for each n.

If (B_i^n) is totally bounded for each n, then

(a) (Φ_n) is collectively compact

(b) $(\Phi_n)(T)$ converges to $\Phi(T)$ in $B(X)$ for each T in $B(X)$.

(c) $\| (\Phi n - \Phi)\Phi \| \to 0$ as $n \to \infty$, where

$\Phi(T) = \sum_{i=1}^{n} A_i T B_i$, T in $B(X)$.

Proof.

(a) is a simple consequence of proposition 2.2.2 and (c) follows from theorem 3.1.2.

It is enough to prove (b). For each n,

$$\begin{aligned}\Phi_n(T) &= \sum_{i=1}^m A_i^n T B_i^n \\ &= \sum_{i=1}^m (A_i^n - A_i + A_i) T (B_i^n - B_i + B_i) \\ &= \sum_{i=1}^m (A_i^n - A_i) T (B_i^n - B_i) + \sum_{i=1}^m (A_i^n - A_i) T B_i + \\ &\qquad \sum_{i=1}^m A_i T (B_i^n - B_i) + \sum_{i=1}^m A_i T B_i\end{aligned}$$

Since (B_i^n) is totally bounded and by a theorem 1.8 in [1], the right side tends to $\sum_{i=1}^m A_i T B_i$, for each T in $B(X)$. □

3.1.4 Remarks

The above theorem reveals that the problem can be tackled with extra conditions on the coefficient operators. Now we show that the extra condition is not very hard to achieve for integral operator coefficients with positive definite continuous kernels.

Let K(s,t) be nonnegative continuous functions on $[a, b] \times [a, b]$ and let K be the corresponding integral operators on C[a,b], the space of continuous real or complex functions on [a,b] with supremum norm:

$$K(f)(s) = \int_a^b K(s,t) f(t) dt, \text{ f in C[a,b]}$$

For $\omega_{nj} > 0$ and t_{nj} in [a,b], $j = 1, \ldots, n$, let the numerical quadrature formula satisfies

$\lim_{n\to\infty} \sum_{j=1}^n \omega_{nj} f(t_{nj}) = \int_a^b f(t)dt$, f in C[a,b]. Now let,

$$K_n(f)(s) = \sum_{j=1}^{n} \omega_{nj} K(s, t_{nj}) f(t_{nj}), \text{ f in C[a,b]}$$

It is well known that (K_n) is collectively compact and $\lim_{n\to\infty} K_n(f) = K(f)$, for all f in C[a,b].

3.1.5 Proposition.

Let $K(s,t)$ be a nonnegative real valued continuous function on $[a,b] \times [a,b]$ and let $[a_n, b_n] \subset [a,b]$, n=1,2,... be such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Let $K_n(s,t)$ be real valued nonnegative continuous function on $[a,b] \times [a,b]$ such that

$$\begin{aligned} K_n(s,t) &= K(s,t) \ , (s,t) \in [a,b] \times [a,b] - [a_n, b_n] \times [a_n, b_n] \\ &= 0 \qquad\quad , (s,t) \in [a_{n_0}, b_{n_0}] \times [a_{n_0}, b_{n_0}] \end{aligned}$$

for some subinterval $[a_{n_0}, b_{n_0}] \subseteq [a_n, b_n]$. If K and K_n are the corresponding integral operators then $\| K_n - K \| \to 0$ as $n \to \infty$. Consequently (K_n) is totally bounded.

Proof.

Simple measure theoretic argument leads to the proof. □

3.1.6 Remarks

The above proposition gives us a totally bounded sequence of integral operators. Now put,

$\Phi_n(T) = L_n T K_n$ and

$\Phi(T) = LTK$ where (L_n) be as in the remark 3.1.4 and (K_n) be as in proposition 3.1.5. Then the numerical solvability of the corresponding operator equations reduces to the question whether (Φ_n) is collectively compact or not. We answer this by the following proposition.

3.1.7 Proposition

Let K,K_n be as in theorem 3.1.5 and let (L_n) be a sequence of collectively compact integral operators on C[a,b]. Then the elementary operators (Φ_n) is collectively compact where

$\Phi_n(T) = L_n T K_n$, T in $B(C[a,b])$.

3.1.8 Remarks

We conclude this section with the following remark. The observations made in this section reveals the scope of approximating the solutions and estimating the error involved numerically for operator equations involving elementary operators whose coefficients are integral operators with positive definite continuous kernels.

3.2 Application of approximation of functions

First of all we give some basic definitions that are used in this section.

3.2.1 Least-square approximation problem [14]

Let f be a continuous function on [0,1] and let L(A,x)=$\sum_{i=1}^{n} a_i\phi_i(x)$ be the linear approximating function where $\phi_i(x)$ are n linearly independent functions and $A = (a_1, \ldots, a_n)$ in E_n, the n dimensional Euclidean space.

The problem is to determine A^* so that

$L_2(f - L) = [\int_0^1 [f(x) - L(A,x)]^2 dx]^{\frac{1}{2}}$ is a minimum.

3.2.2 Functions orthogonal on finite point sets [14]

Least-square approximation to a function f(x) defined on a finite point set, $X = \{x_i/i = 1, 2, \ldots, m\}$ requires that one determine A^* so that

$L_2(f - L) = [\sum_{i=1}^{m} [f(x_i) - L(A, x_i)]^2]^{\frac{1}{2}}$ is a minimum.

In particular, we say that a system $\{\phi_i(x)/i = 1, 2, \ldots, n\}$ is an orthogonal system on X with weights $\{w_i/i = 1, 2, \ldots, m; \quad w_i > 0\}$ if

$\sum_{i=1}^{m} \phi_j(x_i\phi_k(x_i)w_i = 0, \quad j \neq k$. The system is said to be orthonormal if, in addition to the above, we have

$\sum_{i=1}^{m} [\phi_j(x_i)]^2 w_i = 1.$

3.2.3 Approximation on an interval as the limit of approximation on a finite point set [14]

If we approximate f(x) with the L_2-norm on a very large number of points in [0,1], we would expect the approximation obtained to be close to the best L_2 approximation on [0,1]. Let

$X = \{x_i / i = 1, 2, \ldots\}$ be dense in [0,1] and set

$X_m = \{x_i \in X; \qquad i = 1, 2, \ldots, m\}.$

For each x_i in X_m define

$\delta_m(x_i) = \min\{\mid x_i - x_j \mid / \ x_j < x_i, x_j \in X_m\}.$

Let $\{\phi_i(x)/i = 1, 2, \ldots, n\}$ be a system of linearly independent functions on [0,1] and for each m, let $L(A_m, x)$ be the function of the form

$L(A, x) = \sum_{i=1}^{n} a_i \phi_i(x)$ which minimizes

$\sum_{x \in X_m} w(x)[f(x) - L(A, x)]^2 \delta_m(x) \qquad (1)$

Also let $L(A^*, x)$ be the best weighted L_2 approximation to f(x) on [0,1] with the continuous weight function w(x). Then we have

Theorem A.

Let f(x) be continuous on [0,1], and let $L(A_m, x)$ minimize (1). If

$$\lim_{m \to \infty} \max_{x \in X_m} \delta_m(x) = 0,$$

then

$$\lim_{m \to \infty} L(A_m, x) = L(A^*, x)$$

3.2.4 Remarks

We apply this theory to elementary operators with integral operator coefficients.

Let K(s,t) be real valued continuous function on $[0, 1] \times [0, 1]$ and let K be the corresponding integral operator on C[0,1]. Keeping $s \in [0, 1]$ fixed, let $K^s(t)$ be a real valued function on [0,1]. Using the theory 3.2.3

let $L(A^s_m, t) = \sum_{i=1}^{n} a_i^s \phi_i(t)$ be the function as in theorem A. Each $a_i^{;s}$ is a continuous function on [0,1]. Again applying the theory let

$L(B^i_m, s) = \sum_{j=1}^{n} b_{ij}\phi_j(s)$ be the approximating function as in theorem A. Let $K_m = \sum_i \sum_j b_{ij}\phi_i(t)\phi_j(s)$ and $\tilde{K} = \lim_{m\to\infty} K_m$. We can regard $\tilde{K}$ as a best approximation of K(s,t) on $[0,1]\times[0,1]$.

Now we give the following theorem

3.2.5 Theorem

Let X=C[0,1] be the Banach space of all real or complex valued continuous functions on [0,1] and let $\{K_1, K_2, \ldots, K_n\}$, $\{L_1, L_2, \ldots, L_n\}$ be linearly independent sets of integral operators on C[0,1], with continuous kernels $\{K_1(s,t), K_2(s,t), \ldots, K_n(s,t)\}$ and $\{L_1(s,t), L_2(s,t), \ldots, L_n(s,t)\}$. Let $\Phi(T) = \sum_{i=1}^{n} K_i T L_i$. For each positive integer m let $K_1^m, K_2^m, \ldots, K_n^m$ and $L_1^m, L_2^m, \ldots, L_n^m$, m=1,2,... be as above (Remark). Let $\tilde{K}_i$ and $\tilde{L}_i$ be the integral operators corresponding to the kernels $\lim_{m\to\infty} K_i^m$ and $\lim_{m\to\infty} L_i^m$, i=1,2,...,n. If

$\tilde{\Phi}(T) = \sum_{i=1}^{n} \tilde{K}_i T \tilde{L}_i$ and

$\tilde{\Phi}_m(T) = \sum_{i=1}^{n} K_i^m + S_i^m T L_i^m$, m=1,2,...

Where S_i^m's are collectively compact operators on X such that $S_i^m(x) \to 0$ as $m \to \infty$, $i = 1, 2, \ldots, n$. Then

$(\tilde{\Phi}_m)$ converges to $(\tilde{\Phi})$ in the collective compact sense.

Proof.

For each $T \in B(C[0,1])$,

$$\| \tilde{\Phi}_m(T) - \tilde{\Phi}(T) \| = \| \sum_{i=1}^{n} (K_i^m + S_i^m) T L_i^m - \tilde{K}_i T \tilde{L}_i \|$$

$$\leq \sum_{i=1}^{n} \| (K_i^m + S_i^m - \tilde{K}_i) T L_i^m \| + \sum_{i=1}^{m} \| \tilde{K}_i T (L_i^m - \tilde{L}_i) \|$$

$\rightarrow 0$ as $m \rightarrow \infty$ by Proposition 1.8 in [1] and

since $L_i^m \rightarrow \tilde{L}_i$ uniformly.

Therefore $\tilde{\Phi}_m(T) \rightarrow \tilde{\Phi}(T)$ $\forall T \in B(C[0,1])$. $(\tilde{\Phi}_m)$ is collectively compact by Proposition 2.2.2. Also by Theorem 3.1.2 $\tilde{\Phi}_m \rightarrow \tilde{\Phi}$ in the collective compact sense. □

3.2.6 Remarks

Even if some perturbation affects total boundedness, it need not affect collective compactness. For example a possible class of perturbations may be, take $X = l^2$

Define, $K_m : X \rightarrow X$ by

$K_m(x) = x_m e_1$.

Then (K_m) is collectively compact but not totally bounded.

4 CHAPTER

LOCALLY AND RANDOM ELEMENTARY OPERATORS

In this chapter we introduce the concept of locally elementary operators and Random elementary operators. Also we provide examples of locally elementary operators which are not elementary.

4.1 Locally Elementary Operators

Let us define a locally elementary operator

4.1.1 Definition

Let H be a separable Hilbert space, $\{e_1, e_2, \ldots\}$ be an orthonormal basis for H and $B(H)$ denote the set of all bounded linear operators on H. A bounded linear operator Φ:$B(H) \longrightarrow B(H)$ is called locally elementary if for each n , $\exists A_1^{e_n}, A_2^{e_n}, \ldots, A_{m_n}^{e_n}$ and $B_1^{e_n}, B_2^{e_n}, \ldots, B_{m_n}^{e_n}$ in $B(H)$ such that

$$\Phi(T)(e_n) = \sum_{k=1}^{m_n} A_k^{e_n} T B_k^{e_n}(e_n), \forall T \in B(H).$$

4.1.2 Theorem

On a finite dimensional Hilbert space, every locally elementary operator is an elementary operator.

Proof.

Let H be a finite dimensional Hilbert space and $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for H.

Let $\Phi: B(H) \longrightarrow B(H)$ be locally elementary.

Let $x \in H$, Then $x = \sum_{i=1}^{n} \alpha_i e_i$, $\alpha_i \in K$.

Now for all $T \in B(H)$ and for all $x \in H$

$$\Phi(T)(x) = \Phi(T)(\textstyle\sum_{i=1}^{n} \alpha_i e_i)$$

$$= \textstyle\sum_{i=1}^{n} \alpha_i \Phi(T)(e_i) \text{ since } \Phi(T) \text{ is linear}$$

$$= \textstyle\sum_{i=1}^{n} \alpha_i \sum_{k=1}^{m_i} A_k^{e_i} T B_k^{e_i}(e_i)$$

$$= \textstyle\sum_{i=1}^{n} \sum_{k=1}^{m_i} A_k^{e_i} T B_k^{e_i} P_i(x) \text{ where } P_i \text{ is the orthogonal projection of H onto span}\{e_i\}.$$

So $\qquad \Phi(T) = \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_k^{e_i} T B_k^{e_i} P_i$

Hence Φ is elementary. □

4.1.3 Theorem

Every locally elementary operator is the strong limit of a sequence of elementary operators.

Proof.

Let H be a separable Hilbert space and (e_n) be an orthonormal basis for H.

Let $\Phi: B(H) \longrightarrow B(H)$ be locally elementary.

Let $x \in H$, Then $x = \sum_{i=1}^{\infty} \alpha_i e_i$, $\alpha_i \in K$.

Now for all $T \in B(H)$, let

$$\Phi(T)(x) = \Phi(T)(\textstyle\sum_{i=1}^{\infty} \alpha_i e_i)$$

$$= \textstyle\sum_{i=1}^{\infty} \Phi(T)(\alpha_i e_i) \text{ since } \Phi(T) \text{ is continous}$$

$$= \textstyle\sum_{i=1}^{\infty} \alpha_i \Phi(T)(e_i)$$

$$= \lim_{n \longrightarrow \infty} \textstyle\sum_{i=1}^{n} \sum_{k=1}^{m_i} A_k^{e_i} T B_k^{e_i} P_i(x) \ \forall x \in H \text{ and } \forall T \in B(H).$$

Therefore,

$$\Phi(T) = \lim_{n \longrightarrow \infty} \Phi_n(T) \text{ where } \Phi_n(T) = \textstyle\sum_{i=1}^{n} \sum_{k=1}^{m_i} A_k^{e_i} T B_k^{e_i} P_i$$

□

In order to give an example of a locally elementary operator which is not elementary, we need the following results. The next Lemma gives a sufficient condition for a bounded linear operator on H to be diagonal.

4.1.4 Lemma

Let H be a separable Hilbert Space and (e_n) be an orthonormal basis for H. Let $A : H \longrightarrow H$ be a bounded linear operator such that $A(e_{i_1} + e_{i_2} + \ldots + e_{i_n})$ is a linear span of $e_{i_1}, e_{i_2}, \ldots, e_{i_n}$ $\forall\, i_1, i_2, \ldots, i_n \in N$, $i_j \neq i_k$, for some n. Then A is diagonal.

Proof.

The proof is by induction on n.

First we prove the result for n=1,2.

Let n=1, Then $A(e_i) = \lambda_i e_i \Rightarrow [A] = diag(\lambda_1, \lambda_2, \ldots)$.

Now let n=2 and assume

$$A(e_i + e_j) = \lambda_{ij}^i e_i + \lambda_{ij}^j e_j \;\; \forall i, j \in N, \quad i \neq j$$

Then $\quad A(e_i - e_k) = A(e_i + e_j) - A(e_j + e_k)$

$$= \lambda_{ij}^i e_i + (\lambda_{ij}^j - \lambda_{jk}^j) e_j - \lambda_{jk}^k e_k \tag{1}$$

Also $\quad A(e_i - e_k) = \lambda_{il}^i e_i + (\lambda_{il}^l - \lambda_{lk}^l) e_l - \lambda_{lk}^k e_k \quad (2)$

Equating these two we have

$$\lambda_{ij}^i = \lambda_{il}^i \qquad j, l \in N, \quad j, l \neq i$$

Also $\quad A(e_i + e_k) = \lambda^i_{ik} e_i + \lambda^k_{ik} e_k \qquad (3)$

From (1) and (3), we have

$$A(e_i) \quad = (\tfrac{\lambda^i_{ij}+\lambda^i_{ik}}{2})e_i + (\tfrac{\lambda^j_{ij}-\lambda^j_{jk}}{2})e_j + (\tfrac{\lambda^k_{ik}-\lambda^k_{jk}}{2})e_k$$

$$= \lambda^i_{ij} e_i \qquad \text{since } \lambda^i_{ij} = \lambda^i_{il} \text{ j,l} \in N,\ \text{j,l} \neq i.$$

Therefore A is diagonal. Hence the result is true for n=2.

Assume the result is true for n=m-1 and let n=m, ie For any choice of distinct integers $i_1, i_2, \ldots, i_m$

$$A(e_{i_1} + e_{i_2} + \ldots + e_{i_{m-1}}) = \textstyle\sum_{k=1}^{m} \lambda^{i_k}_{i_1 i_2 \ldots i_m} e_{i_k}.$$

Now

$$(m-1)A(e_{i_1} + e_{i_2} + \ldots + e_{i_{m-1}}) + A(e_{j_1} + e_{j_2} + \ldots + e_{j_{m-1}})$$

$$= \textstyle\sum_{k=1}^{m-1} A(e_{i_1} + e_{i_2} + \ldots + e_{i_{m-1}} + e_{j_k})$$

$$= \textstyle\sum_{k=1}^{m-1} [\sum_{l=1}^{m-1} \lambda^{i_l}_{i_1 i_2 \ldots i_{m-1} j_k} e_{i_l} + \lambda^{j_k}_{i_1 i_2 \ldots i_{m-1} j_k} e_{j_k}]$$

and $\quad A(e_{i_1} + e_{i_2} + \ldots + e_{i_{m-1}}) - A(e_{j_1} + e_{j_2} + \ldots + e_{j_{m-1}})$

$$= A(e_{i_1} + e_{i_2} + \ldots + e_{i_{m-1}} + e_{i_m})$$
$$-A(e_{j_1} + e_{j_2} + \ldots + e_{j_{m-1}} + e_{i_m})$$

$$= \sum_{k=1}^{m-1} \lambda^{i_k}_{i_1 i_2 \ldots i_{m-1} i_m} e_{i_k}$$
$$+(\lambda^{i_m}_{i_1 i_2 \ldots i_{m-1} i_m} - \lambda^{i_m}_{j_1 j_2 \ldots j_{m-1} i_m}) e_{i_m}$$
$$- \sum_{k=1}^{m-1} \lambda^{j_k}_{j_1 j_2 \ldots j_{m-1} i_m} e_{j_k}$$

From theses two, we have

$$mA(e_{j_1} + e_{j_2} + \ldots + e_{j_{m-1}}) = \sum_{k=1}^{m-1} [\lambda^{j_k}_{i_1 i_2 \ldots i_{m-1} j_k}$$
$$+(m-1)\lambda^{j_k}_{j_1 j_2 \ldots j_{m-1} i_m}] e_{j_k}$$

Since $\lambda^{i_k}_{i_1 i_2 \ldots i_{m-1} j_k} = \lambda^{i_k}_{i_1 i_2 \ldots i_{m-1} i_m}$ $\quad \forall k, m \in N$

$i_k, i_m \neq i_1, i_2, \ldots i_{m-1}$

and $\lambda^{i_m}_{i_1 i_2 \ldots i_{m-1} i_m} = \lambda^{i_m}_{j_1 j_2 \ldots j_{m-1} i_m}$ by considering

$A(\sum_{k=1}^{m-1} e_{i_k} - \sum_{k=1}^{m-1} e_{j_k})$ in two different ways

ie $A(e_{i_1} + e_{i_2} + \ldots + e_{i_{m-1}})$ is a linear span of $e_{i_1}, e_{i_2}, \ldots, e_{i_{m-1}}$. Since the result is true for n=m-1, A is diagonal. Hence the lemma. $\square$

4.1.5 Theorem

Let H be separable Hilbert space and $\{e_1, e_2, \ldots\}$ be an orthonormal basis for H. Let Φ be a bounded linear operator on $B(H)$ with the following properties.

(1). $\Phi(P_k^m) = P_k^m$ where P_k^m is the orthogonal projection of H on to span $\{e_{((k-1)m+1)}, \ldots, e_{km}\}$ and $\Phi(I) = I$.

(2). Φ maps every operator in $B(H)$ to some diagonal operator.

Then Φ can not be elementary.

Proof.

Let m be a fixed positive integer and let $\Phi(P_k^m) = P_k^m$ for all k=1,2,... and $\Phi(I) = I$. If possible assume Φ is elementary. Then there exists $A_1, A_2, \ldots, A_N$ and $B_1, B_2, \ldots, B_N$ such that $\Phi(T) = \sum_{i=1}^{N} A_i T B_i$. Take $T = P_{n_1}^m + P_{n_2}^m + \ldots + P_{n_N}^m$, where $\{P_{n_1}^m, P_{n_2}^m, \ldots, P_{n_N}^m\}$ are a set of N arbitrary projections and let $\{e_{n_i}^1, \ldots, e_{n_i}^m\}$ spans range of $P_{n_i}^m$, i=1,2,...,N.

$$\begin{bmatrix} < B_1(e_{n_1}^1), e_{n_1}^1 > & \ldots & < B_N(e_{n_1}^1), e_{n_N}^m > \\ \vdots & & \\ < B_1(e_{n_N}^{m-1}), e_{n_1}^1 > & \ldots & < B_N(e_{n_N}^{m-1}), e_{n_N}^{m-1} > \\ < B_1(e_{n_N}^m), e_{n_1}^1 > & \ldots & < B_N(e_{n_N}^m), e_{n_N}^m > \end{bmatrix} \begin{bmatrix} A_1(e_{n_1}^1) \\ A_1(e_{n_1}^2) \\ \vdots \\ A_N(e_{n_N}^m) \end{bmatrix} = \begin{bmatrix} e_{n_1}^1 \\ e_{n_1}^2 \\ \vdots \\ e_{n_N}^m \end{bmatrix}$$

From this we have

$$\left[< B_i(e_{n_j}^r), e_{n_k}^s >\right]_{mN \times mN^2} \begin{bmatrix} < A_1(e_{n_1}^1), x > \\ < A_1(e_{n_1}^2), x > \\ \vdots \\ < A_N(e_{n_N}^m), x > \end{bmatrix}_{mN^2 \times 1} = \begin{bmatrix} < e_{n_1}^1, x > \\ < e_{n_1}^2, x > \\ \vdots \\ < e_{n_N}^m, x > \end{bmatrix}_{mN \times 1} \quad \forall x \in H$$

This is of the form AX = Y

Applying generalized inverse [22], we have

$X = A^{-}Y + (I - A^{-}A)W_xY$ where A^{-} is any generalized inverse of A, which is an $mN^2 \times mN$ matrix and W_x an $mN^2 \times mN$ matrix.

Taking $x = e_n$ where $e_n \neq e^k_{n_j}$ k=1,2,...,m , $j = 1, 2, \ldots, N$

We have $< A_i(e^k_{n_j}), e_n > = 0 \; \forall \; k = 1, 2, \ldots, m$

$$j = 1, 2, \ldots, N$$

Therefore $A_i(e^k_{n_j})$ is a linear combination of $e^1_{n_1}, e^2_{n_1}, \ldots, e^m_{n_N}$ for $j = 1, 2, \ldots, N$, k=1,2,...,m. Therefore by Lemma 4.1.4, A_i is diagonal for i=1,2,..., N.

The $(i, j)^{th}$ element of the matrix $\sum_{k=1}^{N} A_kTB_k$ is

$\sum_{k=1}^{N} a^k_i t_{ij} b^k_j$ where $A_i = diag(a^i_1, a^i_2, a^i_3, \ldots)$

$$B_i = diag(b^i_1, b^i_2, b^i_3, \ldots)$$

Therefore $\Phi(T) = \sum_{k=1}^{N} A_kTB_k$ and $\Phi(I) = I$ only if

$$\sum_{k=1}^{N} a^k_i b^k_j = 1 \qquad i = j$$
$$= 0 \qquad i \neq j.$$

Consider any (n+1) vectors say $B_1, B_2, \ldots, B_{N+1}$ where

$B_i = (b^1_i, b^2_i, \ldots, b^N_i), i = 1, 2, \ldots, N + 1.$

Since these vectors are linearly independent, $\exists$ constants $\alpha_1, \alpha_2, \ldots, \alpha_{N+1}$ not all zero such that $\sum_{k=1}^{N+1} \alpha_k B_k = 0.$

Taking inner product with $(a^1_i, a^2_i, \ldots, a^N_i)$ for $i = 1, 2, \ldots, N + 1,$

we get $\alpha_i = 0, i = 1, 2, \ldots, N+1$, a contradiction.

This is because of the wrong assumption that

$$\Phi(T) = \sum_{k=1}^{N} A_k T B_k$$

Therefore Φ is not elementary. $\square$

When m=1, we get the following corollary.

4.1.6 Corollary

Let Φ be a bounded linear operator on $B(H)$ with the following properties.

(1). $\Phi(P_i)$'s are mutually orthogonal projections and $\Phi(I) = I$ where P_i's are the projections of H on to span $\{e_i\}$.

(2). Φ maps every operator in $B(H)$ to some diagonal operator.

Then Φ can not be elementary.

Proof.

Case 1:

$\Phi(P_i) = P_i$ for all i=1,2,3,... and $\Phi(I) = I$.

The proof follows by putting m=1 in the previous theorem.

Case 2:

$\Phi(P_i)$'s are mutually orthogonal projections for all $i = 1, 2, \ldots$.

Since $\sum \Phi(P_i)$=I, There exists a partial isometry $U : H \to H$ such that $U^*\Phi(P_i)U = P_i$.

Let $\tilde{\Phi}(T) = U^*\Phi(T)U = \sum_{j=1}^{n} U^*A_jTB_jU$.

Then for each i, $\tilde{\Phi}(P_i) = P_i$ and $\tilde{\Phi}(I) = I$.

By case 1, U^*A_j is diagonal for $j = 1, 2, \ldots, n$.

With out loss of generality, assume that A_j's and B_j's are hermitian or skew-hermitian.

Then U^*A_j is diagonal implies A_jU is diagonal $\forall j = 1, 2, \ldots, n$.

Now consider the operator $\Delta(T) = \Phi(UTU^*)$.

$\Delta(T)$ is diagonal for all $T \in B(H)$, $\Delta(I) = I$ and $\Delta(P_i) = P_i$ for all i=1,2,....

By case 1, Δ can not be elementary, a contradiction. Therefore, Φ can not be elementary. □

Now we give a sequence of locally elementary operators which are not elementary.

4.1.7 Theorem

Let H be separable Hilbert space and $\{e_1, e_2, \ldots\}$ be an orthonormal basis for H and let P_k^m be as in the above theorem.

Define Φ_m:$B(H) \to B(H)$ by

$\Phi_m(T) = \sum_{k=1}^{\infty} P_k^m T P_k^m$

Then Φ_m is locally elementary, but not elementary.

Proof.

For each $n \in \{m(k-1)+1, \ldots, mk\}$, k=1,2,...,

$\Phi_m(T)(e_n) = P_k^m T P_k^m(e_n)$. Therefore Φ_m is locally elementary.

But by theorem 4.1.5, Φ_m is not elementary. □

4.1.8 Remarks

Here few more properties of the random elementary operator Φ_n, n=1,2,... introduced in Theorem 4.1.7, are presented. Most of the properties are either direct consequences of some well known results or can be verified very easily.
(1) Φ_n is completely positive and continuous with respect to σ-weak topology of operators. Hence it is a normal completely positive map on $B(H)$ for each n. The theory of normal completely positive maps can be found in Quantum theory of open systems by E.B.Davies [9].

(2) Φ_n maps each operator T to block diagonal operators which are band limited. It can be seen that $\Phi_n(T) \to T$ (strongly) as $n \to \infty$. If T is compact then $\Phi_n(T) \to T$ uniformly as $n \to \infty$.

(3) Let $\mathcal{A}_n = \{\Phi_n(T)/T \in B(H)\}$. Then $\mathcal{A}$ is a von Newmann sub algebra of B(H), containing identity. Now $\Phi_n \circ \Phi_n = \Phi_n$, ie., Φ_n's are idempotent operators and $\Phi_n(X) = X$ for all X in $\mathcal{A}_n$. Hence each Φ_n is a normal conditional expectation on the von Newmann algebra B(H) on to $\mathcal{A}_n$. The theory of non commutative conditional expectations can be seen in E.B.Davies's book [9]. Hence the collection $\{\Phi_n : n \in N\}$ forms a one parameter family of conditional expectations on B(H). Certainly this will not be a semi group.

4.2 Random Elementary Operators.

In this section, we give the definition of a Random elementary operator and some properties of these operators.

4.2.1 Definition.

Let $(\Omega, \mathcal{B})$ be a measurable space and H a Hilbert space. A Random operator Φ from $\Omega \times B(H)$ to $B(H)$ is called a random elementary operator if for each $\omega \in \Omega$, the operator $\Phi(\omega, .)$ from $B(H)$ to $B(H)$ is an elementary operator.

First, we give an example showing that the operator Φ is random elementary need not imply the coefficient operators to be random.

Example.

Let $(\Omega, \mathcal{B})$ be a measurable space and H a Hilbert space. Let E be a non-measurable subset of Ω and $A, B \in B(H)$.

Define $\psi : \Omega \times H \to H$ by

$$\psi(\omega, x) = \chi_E(\omega) A(x) \text{ and}$$

$\Theta : \Omega \times H \to H$ by

$$\Theta(\omega, x) = \chi_{E^c}(\omega) B(x)$$

Ψ and Θ are not random.

But $\Phi : \Omega \times B(H) \to B(H)$ defined by

$$\Phi(\omega, T) = \psi_\omega T \Theta_\omega$$
$$\equiv 0 \text{ is random.}$$

4.2.2 Definition

Let E_i's are pairwise disjoint measurable subsets of Ω and $A_i^{'s}$ are functions from Γ to X, where Γ is a metric space. Then the random operator Φ defined by $\Phi(\omega, x) = \sum_{i=1}^{\infty} \chi_{E_i}(\omega) A_i(x)$ is called a countably valued random operator.

4.2.3 Theorem

Let Ψ and Θ defined by $\Psi(\omega, x) = \sum_{i=1}^{\infty} \chi_{E_i}(\omega) A_i(x)$ and $\Theta(\omega, x) = \sum_{i=1}^{\infty} \chi_{F_j}(\omega) B_j(x)$ be two countably valued random operators, where A_i,B_i $\in B(H)$, H a separable Hilbert space. Then the operator $\Phi : \Omega \times B(H) \to B(H)$ defined by $\Phi(\omega, T) = \Psi_\omega T \Theta_\omega$ is random elementary.

Proof.

$$\Phi(\omega, T)(x) = \sum_j \sum_i \chi_{E_i \cap F_j}(\omega) A_i T B_j(x)$$

ie $$\Phi(\omega, T) = \sum_j \sum_i \chi_{E_i \cap F_j}(\omega) A_i T B_j$$

Now let $T \in B(H)$ and B a closed set in $B(H)$

$$\{\omega \in \Omega / \phi(\omega, T) \in B\} = \bigcup_{i,j} \{\omega \in \Omega / A_i T B_j \in B\} \cup \{\omega \in \Omega / 0 \in B\}$$

which is measurable. Therefore Φ is random.

Generalizing, we have if $\Psi_1, \Psi_2, \ldots, \Psi_n$ and $\Theta_1, \Theta_2, \ldots, \Theta_n$ are countably valued random variables on $\Omega \times H \to H$ where H a separable Hilbert space, the operator $\Phi(\omega, T) = \sum_{i=1}^{n} \Psi_{\omega}^{i} T \Theta_{\omega}^{i}$ is random elementary. □

4.2.4 Lemma

Let H be a separable Hilbert space and f a non-negative measurable function on Ω. Then the operator $\Psi : \Omega \times H \to H$ defined by $\Psi(\omega, x) = f(\omega)A(x)$ where $A \in B(H)$ is random.

Proof.

Since H is separable, every open set in H is a countable union of open balls. Let B an open ball centered at the origin of radius r. Let $x \in H$ is fixed. Then if $A(x) \neq 0$, $\{\omega \in \Omega / f(\omega)A(x) \in B\}$ is measurable.
Suppose $A(x) \neq 0$.

$\{\omega \in \Omega / f(\omega)A(x) \in B\} = \{\omega \in \Omega / \| f(\omega)A(x) \| < r\}$

$= \{\omega \in \Omega / \mid f(\omega) \mid < \frac{r}{\|A(x)\|}\}$ which is measurable.

Now let B is centered at x_0 of radius r

$\{\omega \in \Omega / f(\omega)A(x) \in B\} = \{\omega \in \Omega / \mid f(\omega) \mid < \frac{r + \|x_0\|}{\|A(x)\|}\}$
which is measurable.

Now for any B open, $B = \cup_i B_i$ where $B_i^{'s}$ are open balls.

$\{\omega \in \Omega / f(\omega)A(x) \in B\} = \cup_i \{\omega \in \Omega / f(\omega)A(x) \in B_i\}$ is measurable.

Therefore Θ is random. □

4.2.5 Theorem

Let Ψ and Θ be such that

$\Psi(\omega,x)=\sum_{i=1}^{\infty} f_i(\omega)A_i(x)$ and $\Theta(\omega,x)=\sum_{j=1}^{\infty} g_j(\omega)B_j(x)$ where $f_i^{'s}$ and $g_j^{'s}$ are such that $f_if_j=0$ and $g_ig_j=0$ for $i\neq j$. Then the operator Φ defined by $\Phi(\omega,T)=\Psi_\omega T\Theta_\omega$ is random elementary.

4.2.6 Lemma

Let (Ω,B) be a measurable space. Let $\Psi_1,\Psi_2,\ldots,\Psi_n$ and $\Theta_1,\Theta_2,\ldots,\Theta_n$ are random matrices on Ω. Then the operator Φ defined by $\Phi(\omega,x)=\sum_{i=1}^{n}\Psi_\omega^i T\theta_\omega^i$ is random elementary.

4.2.7 Remarks

The spectral properties of elementary operators where studied by R.E.Curto [23] and he proved that the spectrum of the elementary operator is the product of Taylor spectrums of the coefficient operators. Also it is proved that the eigen values and eigen vectors of random matrices are random variables, so that, the eigen values of a random elementary operator acting on $B(H)$, when H is finite dimensional, are random variables.

APPENDIX

The problems under consideration for future work are

(1) Let H be a separable Hilbert space and B(H) be the set of all bounded linear operators on H. Consider the operator Φ given by $\Phi(T) = \sum_{i=1}^{n} A_i T B_i$ where the coefficients $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are two sets of commuting n-tuples of operators in B(H). Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for H and let P_m be the orthogonal projection of H onto span$\{e_1, e_2, \ldots, e_m\}$. For $T \in B(H)$ let $T^m = P_m T P_m$.
Now consider the operator $\Phi_m(T^m) = \sum_{i=1}^{m} A_i^m T^m B_i^m$

Question : Can one approximate spectrum of Φ using the spectrum of Φ_m as $n \to \infty$?. These Φ_m's can be regarded as elementary operators on $m \times m$ matrices. But the trouble is that even if A_i's and B_i's are commuting n-tuples, their truncations A_i^m's and B_i^m's need not commute. So one may have to assume that A_i^m's and B_i^m's are also commuting. The Riccati operator $\Phi(T) = AT - TB$ has this property.

(2) Can one get an estimate of $\| \Phi - \tilde{\Phi} \|$ in theorem 3.2.5 ?. Also can one obtain a measure of dependence of $\tilde{\Phi}$ on the distribution of the points of the dense set X in [0,1] ?.

(3) The usual spectral approximation problem of T using $\Phi_m(T)$ where Φ_m's are the locally elementary operators provided in theorem 4.1.7.

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