Studies on Thermodynamics and No-hair Theorem in Black hole Spacetime

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DOCTOR OF PHILOSOPHY

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CERTIFICATE

Certified that the work presented in this thesis is a bonafide work done by Mr.P.I. Kuriakose, under my guidance in the Department of Physics, Cochin University of Science and Technology and that this work has not been included in any other thesis submitted previously for the award of any degree.

Kochi
September, 2008

Dr. V. C. Kuriakose
(Supervising Guide)

DECLARATION

I hereby declare that the work presented in this thesis is based on the original work done by me under the guidance of Dr. V. C. Kuriakose, Professor (Rtd.), Department of Physics, Cochin University of Science and Technology and has not been included in any other thesis submitted previously for the award of any degree.

Kochi
Sepember, 2008

P.I. Kuriakose

Acknowledgements

When I was a school boy one day I went for a ride on the giant wheel at an exhibition site. As the wheel gained speed I became almost breathless since I felt like floating in the space. It was the first instance that made me realize about the effect of gravity on a human body. The intriguing and omnipresent gravity really fascinated me from that time onwards. When I grew up and became a post graduate student, popular books on *General theory of relativity* and *Theory of curved spacetime* guided me to the exotic world of gravity.

A scientific problem is like a hard piece of log since both are difficult to crack. To cleave the log, a sharp axe with a strong and broad base is necessary. Likewise, to crack a scientific problem, a team work is a must. So this is the right time to render my sincere gratitude to those who have extended their valuable support, however small may be, in fulfilling my thesis.

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P.I. Kuriakose

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Preface

It is therefore possible that the greatest luminous bodies in the Universe are on this very account invisible.
Pierre-Simon Laplace, 1795.

Gravity never eludes us and it is synonymous with a black hole. Black hole may be defined as a region of spacetime, enclosed by a closed one-way membrane created by the spacetime curvature, into which material particles and light can enter but cannot come out. The curved spacetime somehow contrives to create an enclosure with no exit. It is called *event horizon.* The event horizon is not a solid surface, and does not obstruct or slow down matter or radiation that is traveling towards the region within the event horizon. Perhaps, it is because of its intriguing name that so many people are enticed into working on the physics of the black holes. The study of it is an amusing topic and a lot of contemplating brains have been drawing into its fascinating aura since its inception. The idea of black hole was conceived in 1795 by Pierre-Simon Laplace. He thought of a star 250 times bigger than the sun which would hold back all light rays and thereby being invisible. Laplace computed the radius of a star whose escape velocity is equal to speed of light and found to be equal to $\frac{2GM}{c^2}$, where M is the mass of star.

It appears to be inevitable that black holes are formed as a result of gravitational collapse of stars. Black holes, as currently understood, are described by Einstein's general theory of relativity, which he developed in 1915. This theory predicts that when a large enough amount of mass is present in a sufficiently small region of space, all paths through space are warped inwards towards the centre of the volume, preventing all matter and radiation within it from escaping. General relativity describes a black hole as a region of empty space with a pointlike singularity at the centre and an event horizon at the outer edge.

The mathematician Karl Schwarzschild went for an exact solution of Einstein's famous field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$. At the time of inception of his solution, no one identified, to what kind of object the solution was referring and only later on the scientific community came to know that it was indeed a star which holds back everything including light. In 1963, Roy Kerr found solution to Einstein's field equation describing spinning star and later on named as Kerr black hole. In 1964, the world witnessed the first evidence of a black hole and named as Cygnus X-1 and only after long ten years had the scientific community agreed that what they had witnessed was really a black hole.

When Wheeler coined the name black hole in 1967, there was no solid evidence to prove its presence, since black hole theory tells us that there are only three secrets a black hole divulges: its mass, its angular momentum and its electric charge. Almost all galaxies harbour black holes. Hidden deep in the hearts of most of the galaxies lurk gigantic black holes, each brooding in anticipation of an unsuspecting star that may stray into their ambit of terminal attraction and having captured one, they shred and swallow it, growing larger in size. Our neighboring galaxy *Andromeda* is said to have a black hole of mass ten million solar mass. Another galaxy $M87$ has a black hole of mass three billion solar mass. Despite its interior being invisible, a black hole may reveal its presence through an interaction with matter that lies in orbit outside its event horizon. Alternatively, one may observe gas (from a nearby star, for instance) that has been drawn into the black hole. The gas spirals inward, heating up to very high temperatures and emitting large amounts of radiation that can be detected from earthbound and earth-orbiting telescopes.

Black hole can be said to be a testing ground for various disciplines such as thermodynamics, quantum field theory, quantum gravity, to name a few. The mystery of a black hole is so tempting that every one will be drawn into its mystic aura of singularity. The singularity that generally happens only in mathematics is physically exhibited in a black hole eventhough hidden behind the horizon. At the origin, there is a real singularity where spacetime curvature becomes infinite and Einstein's equation breaks down.

Ever since my school days I was fascinated and bewildered by the hugeness and complexity of this Universe. Later on, I came to know about the black holes, which made me more curious about nature, since it is assumed to be the door or exit to a new world. This thesis is an attempt to give attention to the tempting call of black hole, to lie on its lap and to hear some of the mysteries of the universe which will be told by it.

This thesis presents a study of thermodynamics and no-hair theorem in black hole spacetime. In **Chapter 1**, first we give the evolution of a black hole, concept of event horizon, how to detect a black hole, etc. Then we describe the spacetime structure, its symmetry, Killing horizon, etc. We have then explained how the spacetime of a black hole naturally exhibits temperature by using the Unruh effect. The similarity between black hole physics and ordinary laws of thermodynamics have been explained thereafter. We then discuss Hawking effect, information loss paradox, area theorem and generalized second law. The famous Wheeler's no-hair theorem of black holes which stood against the test of time has been examined subsequently. Sequel to that we briefly describe the validity of the state equation of thermal radiation near the horizon. We then give an idea about a *self gravitating radiation systems* and the Bekenstein upper bound on entropy.

In **Chapter 2**, we discuss thermodynamical aspects and back reaction in a black hole. The cornerstone of the relationship between gravitation, thermodynamics and quantum theory is the black hole mechanics, where it appears that certain laws of black hole mechanics are, in fact, simply the ordinary laws of thermodynamics applied to a system containing a black hole. The fields other than gravity perturbs the metric of a black hole and the perturbed metric in turn change many of the physical properties of the black hole, like entropy and effective potential of the spacetime. The back reaction problem is then to solve the semiclassical Einstein's equation $G^{\mu\nu} = 8\pi[T^{\mu\nu} + \tau^{\mu\nu}(\Psi)]$, where, $\tau^{\mu\nu}(\Psi)$ represents the quantum source. The quantum fluctuation in the metric, $\triangle g_{\mu\nu}$, gives the measure of back reaction. The back reaction can be measured indirectly by noting the entropy change. In this chapter, the back reaction is determined by solving the Einstein's field equation and by solving thermodynamical equations for an *extremal anti-de Sitter-Schwarzschild black hole.*

In **Chapter 3**, we discuss the back reaction in a *Schwarzschild de Sitter black hole* dipped in a massless quantum field. Here we have solved the Einstein's semi-classical field equation to calculate the entropy change which is a measure of back reaction. We assume that the black hole is situated inside a highly reflecting cavity having many physical properties such as entropy, surface tension, thermodynamic potential, etc. Inside the cavity the quantum field and the Hawking radiation are in thermal equilibrium. When a metric is perturbed by a scalar field, the effective potential of the spacetime around the black hole will be modified. We have investigated the effective potential of the spacetime with and with out back reaction. The Hamilton-Jacobi approach has been employed in calculating the effective potential. We have found that the perturbed spacetime modifies the stable and unstable orbits of massive and massless particles. The change in effective potential will then be a measure of back reaction. Knowing the effective potential, we can determine the positions of stable and unstable orbits. The results are in agreement with standard ones.

General state equations of thermal radiation are not universal laws and hence must have affected by gravity, i.e., equations must have a form different from the asymptotic form, near the horizon of a black hole. But there are laws which are universally true such as generalized second law and upper bound on the entropy. How the equations of radiation are modified near the horizon of a Reissner-Nordström black hole have been discussed in **Chapter 4**. We have introduced a gedanken experiment to verify the conservation of *gen-*

eralized second law (GSL). Since the GSL is a universal law, it must be conserved in all situations. The conservation is realized only by validating the equations of radiation near the horizon. We have shown that the GSL is violated when the asymptotic equations of radiation are employed in the calculations, but with modified equations of radiation, the GSL is conserved. The upper bound on the entropy of thermal radiation has been verified and found to be similar to the upper bound proposed by Bekenstein.

In **Chapter 5**, we have discussed various gravitating spacetimes, such as, Euclidean, Rindler, Schwarzschild and have shown that how the temperature implicitly generate at the horizons. Subsequently the temperature of a scalar field in the vicinity of a Rindler like spacetime and the trajectory of a test particle in that spacetime have been determined. We then discuss the solution to the scalar field equation near the Rindler spacetime. Subsequent to that, we have explored the possible temperature of the scalar field near the horizon. We have discussed the thermodynamics and entropy of *self gravitating radiation systems* ($SGRS$) thereafter. The best example for an ($SGRS$) is a collapsing star. We then discuss the transit of a scalar field across the horizon as if it is collapsed and calculate the entropy of the scalar field and the entropy bound.

Black holes have no-hair is referred to the theory that there are only three parameters that can be measured by an outside observer relating to a black hole: mass, electric charge and angular momentum. We discuss the evidence of weak scalar hair in an AdS black hole (BTZ – *Bananas* – *Teitelboim* – *Zanelli*) and in Reissner-Nordström black hole in **Chapter 6**. We have derived the scalar field solutions in both cases and have showed the connection between the mass of hairy black hole and non-hairy black hole. Whether the mass of a hairy black hole would blow up or not is a serious question that one needs to examine in the investigation of hair in a black hole. We have studied the stability of a black hole with hair for 1^{st} and 2^{nd} order perturbations. The hair of a black hole will be stable only if the scalar solution is stable against perturbations.

Strong interpretation of scalar hair is always a challenge to the physicists because getting a non-trivial solution and a proper metric simultaneously is always cumbersome. We discuss the evidence of strong hair in a static (3+1) black hole in **Chapter 7**. A strong hair demands non-trivial solution as well as a proper metric with a new conserved quantity. A proper metric is proposed with a radius and temperature and entropy. We have calculated the temperatures of different black holes by the Hamilton-Jacobi method. We have also calculated the entropy of the black hole dressed with a massive scalar field and that of a naked black hole.

In **Chapter 8**, we present the various results and conclusions of this thesis. The scope of the present work and the future plans are also discussed in this chapter.

Part of the results of the thesis have been published in journals and presented in conferences.

In refereed journals

1. P. I. Kuriakose, V. C. Kuriakose , Back reaction in static Einstein spaces-change of entropy, (*Gen. Rel. Grav* **36**, 2433 (2004)).

2. P. I. Kuriakose, V. C. Kuriakose, Back reaction in Schwarzschild-de Sitter space time with a massless quantum field, (*Mod. Phys. Lett. A* **21**, 169 (2006)).

3. P. I. Kuriakose, V. C. Kuriakose, Scalar hair for an AdS black hole, (*Mod. Phys. Lett. A***21**, 2893 (2006)).

4. P. I. Kuriakose, V. C. Kuriakose, Scalar hair for a static black hole, (under revision - *Class. Quan. Grav*), arXiv:0805.4554 (2008).

5. P. I. Kuriakose, V. C. Kuriakose, Generalized second law and entropy bound for a Reissner-Nordström black hole, (to be communicated), arXiv:0806.2192 (2008).

6. P. I. Kuriakose, V. C. Kuriakose, Thermodynamics and entropy of a self gravitating radiation system (to be communicated).

In conferances

1. P. I. Kuriakose and V. C. Kuriakose, "Back reaction in an extremal Reissner-Nordström black hole", XXII IAGRG, IUCAA, Pune, 2002.

2. P.I.Kuriakose and V.C.Kuriakose, "State equations of radiation in the extremal Reissner-Nordström black hole", ICGC, Kochi, Kerala, 2004.

3. P.I.Kuriakose and V.C.Kuriakose, "Scalar hair in a Static spacetime",ICGC, IUCAA, Pune, 2007.

List of other Publications

1. P. I. Kuriakose, V. C. Kuriakose, "Extremal Reissner-Nordström black hole in thermal equilibrium: The back-reaction-Change of entropy", (*Gen. Rel. Grav* **35**, 863 (2003)).

Chapter 1

Introduction and thesis outline

Spacetime grip mass, telling it how to move;
And mass grips spacetime, telling it how to curve
John Archibald Wheeler.

We believe that the Universe began with a mighty explosion referred to as a Big Bang which occurred about 15 billion years ago. A few minutes after the Universe was born, it was assumed to be filled almost entirely with hydrogen. In course of time, blobs of gas formed in this hydrogen atmosphere, which then began to shrink under the influence of its own gravity. As the shrinking continued, a stage then came when the core of the gas became so hot as to trigger a nuclear reaction. That was the birth of a star. In a star under equilibrium, the outward thermal pressure of the nuclear reaction balances the inward gravity. This equilibrium continues until almost all the hydrogen is used for the nuclear reaction. As the star runs out of hydrogen, the gas pressure starts coming down. Gravity now gains the upper hand and the star starts shrinking again. The core keeps on shrinking and becomes hot and a stage comes when a new thermonuclear cycle starts operating, this time involving helium. Each burning cycle involves several steps, essentially leading to the conversion of light elements into slightly heavier ones. When the element feeding a particular fusion reaction is nearly exhausted, the burning ceases and the core of the star begins to shrink under gravity. The collapse is stopped when the next cycle of thermonuclear reaction gets triggered. This process goes on repeatedly till the core becomes iron.

But all the stars may not start off from the hydrogen cycle and go through all the nuclear burn cycles ending up finally with an iron core. It all depends on the initial mass and the composition of star. The important fact is that all stars at some stage, for some reason or other may quit the thermonuclear process before it reaches the end point (iron core). If the initial mass of star is $\leq 1.4M_{\odot}$, after exhaustion of the fuel, the shrinking of the star continues until a new pressure called electron degeneracy pressure arrests it. Hence such stars do not shrink endlessly to disappear into a point but the shrinking stops much earlier to become a **White dwarf**. The limit $1.4M_{\odot}$ is called Chandrasekhar limit.

If the initial mass of the star is more than $1.4M_{\odot}$, the electron degeneracy pressure is no longer sufficient to win over the gravity. So the collapse continues. As the protons and electrons come closer to become neutrons, resulting in the out ward pressure called neutron degeneracy pressure, the gravitational collapse is arrested, thus creating a **Neutron star**.

As the thermonuclear fuel in a massive star ($> 5M_\odot$) is exhausted, the contraction of the star can't be arrested either at the white dwarf stage or at the stage of neutron star. This situation triggers a gravitational collapse which will make the star close in on itself. Such a star then destines towards its ultimate fate, i.e., a **black hole** in the universe. Black hole is the inevitable outcome of Einstein's general theory of relativity which says that matter warps spacetime. When a large enough amount of mass is present in a sufficiently small region of spacetime, all paths through the spacetime are warped towards the centre of that volume, preventing all matter and radiation within it from escaping. Thus a black hole is a region of spacetime in which the gravitational field is so powerful that nothing, not even light, can escape its pull after having fallen past its event horizon (outer edge of black hole). Thus black hole may be referred to a surface called event horizon which encloses a space including the singularity at the centre. In a spherically symmetric gravitational collapse all matter fall through a fictitious spherical surface called event horizon whose radius depends on the features of black hole. On the other hand, a black hole exerts the same force on something far away from it as any other object of the same mass would. For example, if our Sun were crushed until it was about 2 km in size, it would become a black hole, but the Earth would remain in its same orbit. The term black hole comes from the fact that the hole's interior is invisible to an external observer, since everything is hidden behind the horizon.

Black holes manifest themselves in many different ways such as swallow everything that comes near by, emit thermal radiation, scatter waves, etc. There must be a plenty of physical phenomena going on around the black hole that differ from other celestial bodies. General relativity says that mass deforms the structure of spacetime. Light cones have a slope ± 1 far from a star but their slope tends to $\pm\infty$ as they approach the star. This means that they become more vertical: the cone *closes up* (Fig. 1.1). As the cone closes up, its velocity decreases and finally becomes zero at a point. The surface upon which such points lie is defined as the event horizon. There arises a question that how does the speed of light change against the concept of special theory of relativity?. The answer is that gravitational field changes the geometry of spacetime and the speed of light is fundamentally tied to the nature of the spacetime geometry the light is passing through.

According to general theory of relativity, gravity manifests as the bending and stretching of spacetime, caused by matter, energy and pressure. Light rays follow geodesics through this bent, stretched or compressed spacetime. The warping of spacetime wrap the paths of the light rays. Relative to an observer at rest far away from a black hole, space is compressed (contracted) and time is stretched out (dilated) near the event horizon, i.e., each unit of space is shorter and each unit of time is longer near the horizon. The collapse of a star is not a quick process, since infinite time would be elapsed before completing the collapse as far a distant observer is concerned. The collapse takes place across the event horizon which will hide the black hole from becoming naked.

1.0.1 Event horizon

Popular accounts commonly try to explain the black hole phenomenon by using the concept of escape velocity, the speed needed for a body starting at the surface of a massive object to completely clear the object's gravitational field. It follows from Newton's law of gravity that a sufficiently dense object's escape velocity can be equal to or even exceed the speed of light depending upon the mass and radius of the object. Thus event horizon may also be defined as the surface on which the escape velocity is equal to the speed of light. Event horizon is characterized by three properties. First, it is a *static limit*, i.e., no one can remain static on the event horizon because of the immense gravitational pull of black hole. As we cross the event horizon, time becomes spacelike and space becomes timelike. Since time can only flow forward and singularity lies in the future, falling inward to the singularity is inevitable.

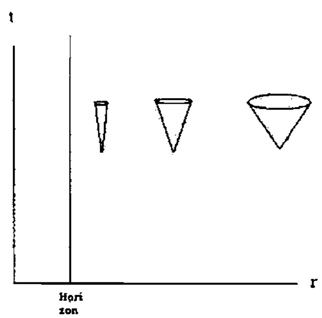

Figure 1.1: Light cones drawn in black hole spacetime close up near the horizon.

Second, it is an *infinite redshift surface*, i.e., the wavelength of the radiation received by a distant observer is greater than the original wavelength of the source near the horizon. As the source is placed nearer and nearer to the black hole surface, the redshift keeps increasing, tending to an infinite value in the limit, eventually making the emitted radiation not to be observed at all. This is the essence of the invisibility of a black hole. In strong gravitational field the clock runs slow and on the horizon time stands still.

Third, it is a *one-way membrane*, i.e., matter can fall into it but cannot come out. Spacetime curvature in the vicinity of a black hole manifests as tidal force. In a freely falling frame, we can get rid of gravity, but we are still stuck with tidal forces, which depends inversely on the cube of distance. Since surface like event horizon in spacetime is tangential to a light cone, it cannot be recrossed again, i.e., it acts as a *one-way membrane.*

The event horizon is analogous to a light wavefront, i.e., like a geometric surface traveling with the speed of light under the action of gravity. As we have seen, gravity can slow down the propagation of electromagnetic waves and hence the wavefront. If the gravitational field is increased steadily, as we move towards the gravitating source, there comes a critical point where gravity can hold this geometric surface fixed in space. We may cross it in one direction and go in, but can never re-cross it and come out. In short, the black hole is nothing but a light wave front, shorn of its electromagnetism but retaining its geometric properties, held in position by gravity and frozen in spacetime. The event horizon, which is a sphere, is represented by a circle drawn out in time. So it looks like a cylinder with time as its axis (Fig. 1.2). If light is emitted here, the inward ray crosses the event horizon and travels ultimately to hit the singularity and the outgoing ray gets stuck at one point, i.e., light never comes out.

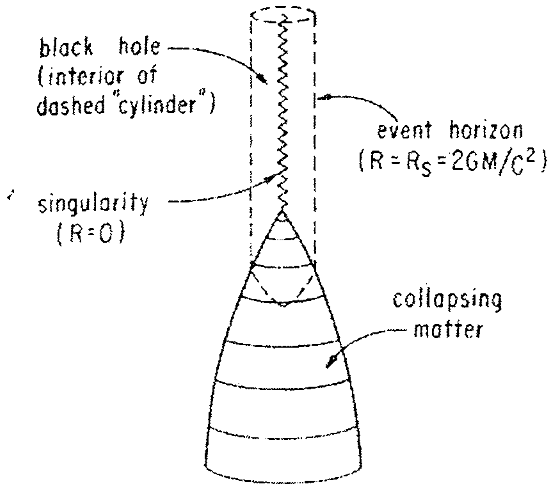

Figure 1.2: A (1+1) dimensional black hole showing the collapse and singularity.

What the distant observer sees is the surface of the star appearing progressively redder and fainter, inching towards the horizon slower and slower, but never reaching it. As an adventurous astronomer falls towards the event horizon, since the gravitational force acting on his feet is greater than on his head, he himself stretches out of proportion. According to general relativity, a black hole's mass is entirely compressed into a region with zero volume, which means its density and gravitational pull are infinite, and so is the curvature of spacetime that it causes. These infinite values cause most physical equations, including those of general relativity, to stop working at the centre of a black hole. So physicists call the zero-volume, infinitely dense region that represents the black hole, a singularity (Fig. 1.3). The singularity in a non-rotating black hole is a point, in other words, it has zero length, width and height. The singularity of a rotating

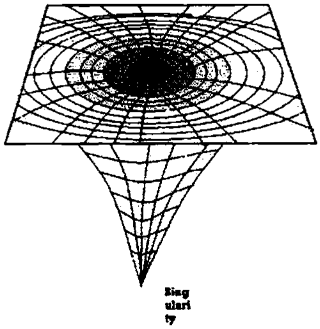

Figure 1.3: Warping of spacetime in the neighbourhood of a black hole and pinching of spacetime at the singularity.

black hole is smeared out to form a ring shape lying in the plane of rotation. The ring still has no thickness and hence no volume.

1.0.2 Detection of black holes

Classical gravity says that black hole is an object with temperature absolute zero so that nothing comes out of it. This makes the black hole inaccessible to the outer world. A black hole may be perceived by tracking the movement of a group of stars that orbit with the company of a black hole. Suppose a star moves as if there is an invisible partner to it so that they move about a common centre of mass. This invisible partner could be a black hole. The spectrum of the visible star may then be investigated. The spectrum oscillates about a mean value, i.e., swings between red and blue shift. From doppler formula we can find the velocity of rotation and period of revolution of the visible star. The mass of visible star can be deduced from the brightness. Considering the equation of motion of two stars (one is invisible black hole) about their common centre of mass and feeding the parameters of visible star, we will be able to get the mass of the invisible star. If the mass thus obtained is greater than five solar mass, it could be a black hole. The mass of the first detected black hole (**Cygnus X-1**) is seven times the solar mass (Fig. 1.4).

The black holes interact with matter that lie in the orbit out side the event horizon. The matter like gas, spirals inward, heating up to very high temperatures and emitting large amounts of radiation that can be detected from earthbound and earth-orbiting telescopes[1]. Perturbation in the black hole spacetime can be evaluated by adding

Figure 1.4: Photo courtesy Nasa/cxc; X-ray image of Cygnus X-1 taken from orbiting Chandra X-ray observatory.

relevant terms to the metric function and feed them into Einstein's field equation and get the solution that governs the behaviour of the perturbations. One important problem that was handled by the perturbation theory was the stability of the black hole. If the perturbation dies or oscillates, then the black hole is stable. If the perturbation grows with time and blows up, then the black hole is unstable. Perturbation formalism revealed the existence of *quasi-normal modes* of the black hole vibrations, which carry the imprint of the black hole. The quasi-normal modes generally appear during the formation of a black hole by the gravitational collapse and when two black holes coalesce. These quasi-normal modes show up in the process of gravitational wave scattering. It is like pelting somebody in the dark and identify the location by noting the direction from where the screaming sound comes. A part of the wave packet directed to the black hole is scattered off the black hole and we can observe the out-coming wave form. As the black hole is disturbed, it vibrates, generating a decaying wave at a characteristic frequency. It has come to be known as the quasi-normal mode of the black hole. The quasi-normal mode by itself reveals the existence of black hole and frequency gives the information on the black hole parameter, namely the mass.

Gravitational radiation is yet another tool to detect a black hole. Radiations are ripples in the fabric of spacetime. A binary system, **Eagle** (in the constellation **Aquila**) demonstrated the existence of gravitational radiation. These binary stars revolve in close orbits with break-neck speed. The gravitational field at such a close separation is quite high. The system sends out gravitational radiation by shredding its own energy, associated with diminishing radius of orbit. When the binary stars are far apart, the wave is essentially a regular sine wave. The frequency increases slowly at first as the orbit of the black hole shrinks due to emission of gravitational waves. As the two black holes come close to each other gravitational waves are emitted with increasing amplitude and frequency. This wave pattern is called a *chirp*, which carries the unmistakable signature of black hole.

1.0.3 Types of black holes

The simplest possible black hole is the one that has only mass. These black holes are often referred to as Schwarzschild black holes [2] after Karl Schwarzschild who discovered this solution in 1916. It was the first non-trivial exact solution to the Einstein equations to be discovered and according to Birkhoff's theorem, the only vacuum solution that is spherically symmetric. In general relativity, Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be stationary and asymptotically flat. Hence the popular notion of a black hole *sucking in everything* in its surroundings is therefore incorrect; the external gravitational field, far from the event horizon, is essentially like that of ordinary massive bodies.

More general black hole solutions were discovered later with more features for the black holes. The Reissner-Nordström solution [3] describes a black hole with electric charge, while the Kerr solution yields [4] a rotating black hole. The most generally known stationary black hole solution having both charge and angular momentum is the Kerr-Newman metric [5]. All these general solutions share the property that they converge to the Schwarzschild solution at distances that are large compared to the ratio of charge and angular momentum to mass (in natural units).

How the spacetime around a black hole behaves is of utmost importance, because of its symmetric properties. The solution of the Einstein's field equation, in spherical polar coordinates, to the exterior part of black hole is singular at the horizon and hence needs to be modified, since there seems no physical pathology at the horizon.

1.1 Spacetime structure

1.1.1 Metric of a black hole

All physical phenomena, like the geodesic which defines the spacetime structure, gets modified near the black hole. In the Newtonian gravity circular orbits of material bodies around a heavy gravitating mass, like the sun, can exist at all radius. For a Schwarzschild black hole the inner most stable orbit is at $6GM/c^2$. Between $6GM/c^2$ and $3GM/c^2$, the orbits are unstable. The orbit of radius $3GM/c^2$ is the geodesic of light so that light moves in a circle. The vacuum solution to a static spherically symmetric black hole in spherical polar coordinates is given as [2]

$$ds^2 = (1 - \tfrac{2M}{r})dt^2 - (1 - \tfrac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.1)$$

This solution is singular at $r = 2M$, i.e., on the event horizon. Since the curvature at the horizon is finite, proportional to M/r^6, the singularity at the horizon is unwarranted. So this singularity is not a physical one but only an outcome of a wrong coordinate selection. To remove this singularity, Eddington-Finkelstein (EF) coordinates, named after Arthur Stanley Eddington and David Finkelstein, were introduced [6, 7]. It is a pair of coordinate systems for a Schwarzschild geometry which is adapted to radial null geodesics (i.e. the worldlines of photons moving directly towards or away from the central mass). The transformation of the type

$$\begin{aligned} v &= t + r + 2m\log((r/2M) - 1) \\ u &= t - r - 2m\log((r/2M) - 1), \end{aligned} \tag{1.2}$$

would change the usual Schwarzschild metric into a metric in the ingoing and outgoing Eddington-Finkelstein coordinates [6, 7] as

$$\begin{aligned} ds^2 &= (1 - \tfrac{r_s}{r})dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ ds^2 &= (1 - \tfrac{r_s}{r})du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{1.3}$$

In both these coordinates the metric is explicitly non-singular at the Schwarzschild radius, r_s. If $r_s = 0$, the metric represents just a flat spacetime, then $4\pi r^2$ is the area of sphere of symmetry. For the outgoing radial light rays, $ds^2 = (1 - \frac{r_s}{r})dv^2 - 2dvdr = 0$. Hence it satisfies, $\frac{dr}{dv} = \frac{1}{2}(1 - r_s/r)$. For $r = r_s$, $\frac{dr}{dv}$ vanishes, so the out going light rays remain static at the horizon, i.e., the out going spherical wave front has a constant area of $4\pi r_s^2$. This is called event horizon. So the event horizon is like a light wave front of radius r_s, but frozen in the spacetime. For $r < r_s$ the out going light rays are dragged inward to decreasing r and eventually reach $r = 0$, i.e., singularity (Fig. 1.5). The singularity is disconnected from the exterior if $r_s > 0$, i.e., if the mass M is positive. Now $r_s > 0$ implies that there is mass hidden behind the horizon, which is the black hole. When $r_s < 0$, the metric function will be, $(1 + r_s/r)$. Since, $1 + r_s/r \neq 0$, there is no horizon and at $r = 0$, there is a singularity which is naked. So, when $M > 0$, we get a black hole and when $M < 0$, singularity becomes naked [8, 9, 10]. But Cosmic Censorship says that singularity can't be naked, it must be hidden behind the horizon. The first two well behaved coordinate systems were introduced by

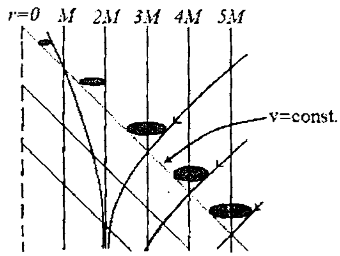

Figure 1.5: Diagram of the Positive mass (EF) spacetime, suppressing the angular coordinate with constant r surfaces vertical and constant v surfaces at 45°

Eddington and Finkelstein. Motivated by these systems, Kruskel and Szekeres [11, 12] independently introduced a coordinate system known as *Kruskel- Szekeres coordinates* for the Schwarzschild black hole (SBH). They use a dimensionless radial coordinate u and a dimensionless time coordinate v related to r and t by

$$\begin{aligned} u &= (r/2M - 1)^{1/2} e^{r/4M} \cosh(t/4M) \\ v &= (r/2M - 1)^{1/2} e^{r/4M} \sinh(t/4M), \end{aligned} \tag{1.4}$$

for region, $r > 2M$, and

$$\begin{aligned} u &= (1 - r/2M)^{1/2} e^{r/4M} \sinh(t/4M) \\ v &= (1 - r/2M)^{1/2} e^{r/4M} \cosh(t/4M), \end{aligned} \tag{1.5}$$

for region, $r < 2M$. The metric of SBH in the *Kruskel- Szekeres coordinates* [11, 12] is given as

$$ds^2 = (32M^3)e^{-r/2M}(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.6)$$

In this metric, the singularity at the horizon is not present, making the system well behaved. It is often useful, in visualizing the structure of a spacetime, to introduce coordinates that attribute finite coordinate values to infinity. We can transform the *Kruskel-Szekeres coordinates* into new coordinates ψ, ξ, θ, ϕ by introducing

$$\begin{aligned} v + u &= \tan \tfrac{1}{2}(\psi + \xi) \\ v - u &= \tan \tfrac{1}{2}(\psi - \xi). \end{aligned} \quad (1.7)$$

The metric of the SBH in the new system is

$$ds^2 = (\tfrac{32M^3}{r})\frac{e^{-r/2M}(-d\psi^2+d\xi^2)}{4\cos^2\frac{1}{2}(\psi+\xi)\cos^2\frac{1}{2}(\psi-\xi)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.8)$$

The resulting coordinate diagram depicts clearly the connections between the horizons, the singularities and the various regions of infinity. Penrose had developed [13] a powerful mathematical technique for studying asymptotic properties of spacetime near infinity. The key to his technique is a *conformal transformation* of spacetime, which brings *infinity* into a finite radius and converts asymptotic calculations into calculations at finite points. These are the various infinities proposed by him.

$I^+ \equiv$ future timelike infinity.

$I^- \equiv$ past timelike infinity.

$I^0 \equiv$ spacelike infinity

$\jmath^+ \equiv$ future null infinity

$\jmath^- \equiv$ past null infinity

The Schwarzschild spacetime depicted in $(\psi, \xi, \theta, \phi)$ is shown in Fig. (1.6). When a gravitational collapse is spherically symmetric, the

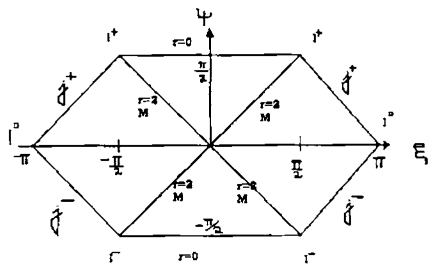

Figure 1.6: In the diagram with ψ, ξ coordinates, the infinities are brought to finite distances. Each of the asymptotically flat regions has its own set of infinities $I^+, I^-, I^0, \jmath^+, \jmath^-$.

spacetime around the resulting black hole possess certain symmetric properties. The best mathematical tool to describe the symmetry of the spacetime is a Killing vector.

1.1.2 Spacetime symmetry

The covariant approach to the unraveling of black hole geometry is through the spacetime symmetries or Killing vector fields. Let the metric function $g_{\mu\nu}$ relative to some coordinate basis, be independent of t and ϕ, then $\frac{\partial g_{\mu\nu}}{\partial t} = 0$ and $\frac{\partial g_{\mu\nu}}{\partial \phi} = 0$. This implies that the spacetime is static and spherically symmetric. Now translate an arbitrary curve ς through an infinitesimal displacement, $\epsilon \frac{\partial}{\partial \phi}$, to form a new curve ς'. Since $\frac{\partial g_{\mu\nu}}{\partial \phi} = 0$, the curves ς and ς' have the same length. Thus the geometry of the black hole spacetime is left unchanged by a translation of all points through $\epsilon\frac{\partial}{\partial\phi}$. The vector, $\xi = \frac{\partial}{\partial\phi}$ provides an infinitesimal description of these length preserving translations. This length preserving geometrical operator is called a Killing vector. In the case of time translation, the Killing vector is $\frac{\partial}{\partial t}$. The symmetry operation is well depicted in Fig. (1.7). The surface on which ξ^a becomes null ($\xi^a\xi_a = 0$) is itself

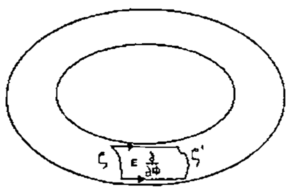

Figure 1.7: A doughnut manifold with a symmetry described by a Killing vector.

a null surface, equivalently a one-way surface or an event horizon. When the geometry of a black hole spacetime is invariant under transitions, $t \to t + \Delta t$ and $\phi \to \phi + \Delta\phi$, the coordinates t and ϕ are cyclic, then E and L are conserved in such a spacetime, where E and L represent energy and angular momentum of a test particle. When the gravitational field is constant the metric function is independent of time, i.e., spacetime displays the property of time and space symmetry.

1.1.3 Killing horizon

A Killing horizon is a null hypersurface on which there is a null Killing vector field. Associated to a Killing horizon there is a geometrical quantity known as surface gravity, κ. In order to discuss the laws of black hole mechanics, we must introduce the notions of stationary, static and axisymmetric black holes as well as the notion of a Killing horizon. If an asymptotically flat spacetime (M, g_{ab}) contains a black hole B, then B is said to be stationary if there exists a one-parameter group of isometries on (M, g_{ab}) generated by a Killing field t^a which is unit timelike at infinity. The black hole is said to be static if it is stationary and if, in addition, t^a is a hypersurface orthogonal to the Killing horizon.

In a wide variety of cases of interest, the event horizon H of a stationary black hole must be a Killing horizon. Carter [14] states that for a static black hole the static Killing field t^a must be normal to the horizon, whereas for a stationary-axisymmetric black hole with the $t - \phi$ orthogonality property there exists a Killing field ξ^a of the form

$$\xi^a = t^a + \Omega \phi^a, \tag{1.9}$$

which is normal to the event horizon and Ω is called the angular velocity of the horizon. Hawking proved [15, 16] that in vacuum the event horizon of any stationary black hole must be a Killing horizon. Consequently, if t^a fails to be normal to the horizon, then there must exist an additional Killing field ξ^a which is normal to the horizon, i.e., a stationary black hole must be non-rotating.

Now, let K be any Killing horizon (not necessarily required to be the event horizon H of a black hole), with normal Killing field

ξ^a. Since $\nabla^a(\xi^a\xi_a)$ also is normal to K , these vectors must be proportional at every point on K. Hence, there exists a function, κ on K, known as the surface gravity of K, which is defined by the equation

$$\nabla^a(\xi^a\xi_a) = -2\kappa\xi^a. \tag{1.10}$$

It can be shown that [17]

$$\kappa = lim(V_a), \tag{1.11}$$

where a is the acceleration of the orbits of ξ^a in the region of K where they are time like, $V \equiv (-\xi^a\xi_a)^{1/2}$ is the red shift factor of ξ^a. Note that the surface gravity of a black hole is defined only when it is *in equilibrium*, i.e., stationary, so that its event horizon is a Killing horizon. There is no notion of the surface gravity of a general, non-stationary black hole, although the definition of surface gravity can be extended to isolated horizons.

1.1.4 Negative curvature

We know that a heavy gravitating bodies like a black hole would warp the spacetime around it. The spacetime is a curved Riemannian manifold globally and Minkowskian locally. The potential gradient pulls free particles towards the gravitating source as the space curvature acts in unison with a potential gradient. Consider the motion of a particle in a 2-space metric given by $ds^2 = (1-2M/r)^{-1}dr^2 + r^2d\phi^2$, which has a negative curvature $-M/r^3$. It can be embedded into the 3-Euclidean space by writing $z^2 = 8M(r-2M)$, which is a parabola and would generate a paraboloid of revolution (Fig. 1.8). Clearly it has a negative curvature which would tend free particle to roll down towards the centre and thus work in unison with the potential gradient [18]. Spacetime under gravity have shown remarkable

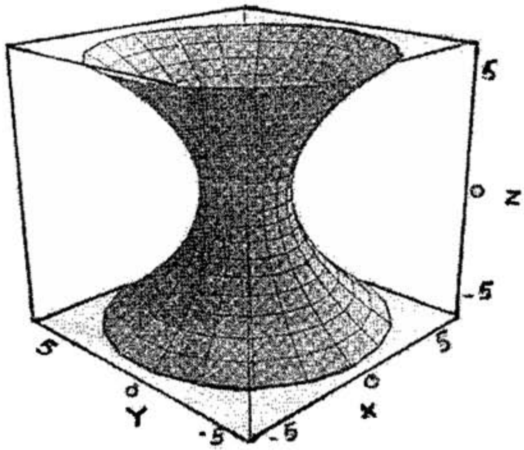

Figure 1.8: Figure shows the paraboloid of revolution for parabola $z^2 = 8r - 16$, where $r^2 = x^2 + y^2$

properties due to quantum effects, giving rise to epoch making discoveries such as Hawking effect and Unruh effect. It can be shown that temperature is implicitly present in the spacetime of a black hole.

1.2 Black hole as a thermodynamic system

1.2.1 Hawking effect

A quantum field in the black hole spacetime back ground will have vacuum fluctuations that permeates all of the spacetime. Hence, there is always something going on, even in the empty space around a black hole. In 1974, Stephen Hawking showed that black holes are not entirely black but emit thermal radiations [19] with a characteristic temperature. He got this result by applying quantum field theory in a static black hole background. The result of his calculations is that a black hole should emit particles with a characteristic temperature distribution. This effect has become known as *Hawking effect*. Since Hawking's result, many others have verified this effect through various methods [20].

Spontaneous emission from a rotating black hole can be visualized as a pair production of real and virtual photons in the ergoregion (Fig. 1.9). The classical field is said to have no temperature, but a quantum field, because of its inherent fluctuations, give rise to pairs of virtual and real particles. The negative energy photons fall across the event horizon and the positive energy photons escape to infinity. The temperature of photons as they reach infinity is $\frac{\hbar}{8\pi M}$ (for SBH), where M is the mass of black hole. Hawking showed that the photons have the spectrum characteristic of a black body with a temperature $T = \frac{\hbar}{8\pi kM}$. Thus Hawking effect has provided a remarkable unification of gravity and thermodynamics. From the expression of the temperature of black hole, it can be seen that large black holes are very cold and emit very little radiation. A stellar black hole of 10 solar masses, for example, would have a Hawking temperature of several nanokelvin, much less than the 2.7K produced by the Cosmic Microwave Background. Micro black holes on the other hand could be quite bright producing high energy gamma rays. Due to low Hawking temperature of stellar black holes, Hawking radiation has never been observed at any of the black hole candidates.

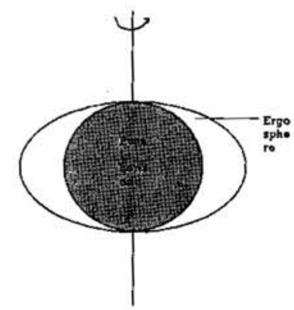

Figure 1.9: The ergosphere in a rotating black hole. In the space between horizon and the ergosphere particle pairs are formed due to quantum phenomenon.

1.2.2 Unruh effect

Unruh effect says that the vacuum in the Minkowski space appears to be in a thermal state at temperature $\frac{\hbar a}{2\pi}$ [21], when viewed by an observer with acceleration $'a'$. Consider a static observer sitting at a fixed radius r out side the horizon R_s. The acceleration due to gravity a (or the surface gravity κ) there is very large and the associated time scale is $1/a$ (periodicity is $2\pi/a$), which is very small compared to R_s. The curvature of spacetime is very small on this time scale, so we expect the vacuum fluctuations of quantum field on this spacetime to have the usual flat spacetime form, provided the quantum field is in a state which is regular near the horizon. Under this condition, the observer will experience the Unruh effect. The ratio of the temperatures measured by static observers at two different radii is $\frac{T_2}{T_1} = \frac{\chi_1}{\chi_2}$, where χ is the norm of the time translation Killing field. At infinity $\chi_2 = \chi_\infty = 1$. So we have an out going thermal flux in the rest frame of the black hole at Hawking temperature [22, 23]. Then

$$T_2 = T_\infty = T_1\chi_1 = \frac{\hbar a}{2\pi}\chi_1 = \frac{\hbar\kappa}{2\pi}. \tag{1.12}$$

with $a\chi_1 = \kappa$. Thus temperature of Unruh radiation [21] near the horizon is $\frac{\hbar a}{2\pi}$ and the temperature of Hawking radiation at infinity is $\frac{\hbar\kappa}{2\pi}$. For Schwarzschild black hole $\kappa = \frac{1}{4M}$. Hence, $T_\infty = T_{bh} = \frac{\hbar}{8\pi M}$. Thus it can be seen that at the heart of the Hawking effect is the Unruh effect. The surface gravity κ or the acceleration due to gravity at the event horizon of the black hole can be determined from the metric by the relation, $\frac{\partial_r g_{00}}{2\sqrt{-g_{00}g_{11}}}|_{r=r_h}$. Thus gravity is very naturally ferreted out of the spacetime of a black hole and the gravity manifests in the curvature. For a SBH, $\kappa = \frac{GM}{r^2}|_{r=r_H}$. But, $\frac{GM}{r^2}$ is nothing but the usual expression for acceleration due to gravity. A black hole with a proper metric will have a surface gravity and hence temperature.

1.2.3 Classical black hole thermodynamics

Classically, black holes are perfect absorbers but do not emit anything; their physical temperature is absolute zero. However, in quantum theory, black holes emit Hawking radiation with a perfect thermal spectrum. This allows a consistent interpretation of the laws of black hole mechanics as physically corresponding to the ordinary laws of thermodynamics. The classical laws of black hole mechanics together with the formula for the temperature of Hawking radiation allows one to identify a quantity associated with black holes as playing the mathematical role of entropy.

In comparing the laws of black hole mechanics in general relativity with the laws of thermodynamics, it should be first noted that the black hole uniqueness theorems [24] establish that stationary black holes, i.e., black holes in equilibrium, are characterized by a small number of parameters, analogous to the state parameters of ordinary thermodynamics. In the corresponding laws, the role of energy E is played by the mass M of the black hole; the role of temperature T is played by a constant times the surface gravity κ of the black hole; and the role of entropy S is played by a constant times the area A of the black hole. The fact that E and M represent the same physical quantity provides a strong hint that the mathematical analogy between the laws of black hole mechanics and the laws of thermodynamics might be of physical significance. However, as temperature of black hole is zero in general relativity, the physical relationships between κ and T; S and A were difficult to evolve [25].

1.2.4 Area theorem

As a classical object with zero temperature it was assumed that black holes had zero entropy; if so, the second law of thermodynamics would be violated by an entropy-laden material entering the black hole, resulting in a decrease of the total entropy of the universe. Therefore, Jacob Bekenstein [26] proposed that a black hole should have an entropy and that it should be proportional to its horizon area. Since black holes do not classically emit radiation, the thermodynamic viewpoint seems to be simply an analogy, not a physical reality. As we shall now see, this situation changes dramatically when quantum effects are taken into account.

Stephen Hawking showed that the total area of the event horizons of any collection of classical black holes can never decrease, even if they collide and swallow each other or merge, i.e., $\frac{dA}{dt} \geq 0$. For a Schwarzschild black hole the area of horizon is

$$A = 16\pi M^2 \tag{1.13}$$
$$dM = \frac{1}{32\pi M} dA = \frac{1}{8\pi M} d(\frac{A}{4}).$$

Since dM is the change in the hole's total energy E and since $1/(8\pi M)$ is the black hole temperature, we can write Eq. (1.13) in the form $dE = TdS$ with

$$S = A/4. \tag{1.14}$$

Since, by the area theorem, the quantity S in Eq. (1.14) can never decrease. So in the **area theorem** and in Eq. (1.13), we can find the first and second laws of thermodynamics as they apply to black holes. That is, a black hole behaves in every respect as a thermodynamic black body with temperature $\frac{1}{8\pi M}$ and entropy $A/4$. This analogy had been noticed as soon as the area theorem was discovered, but at that time it was thought to be a futile exercise since black hole was assumed to have no temperature. But the Hawking effect completed the missing link.

The above universal result can be extended to apply to cosmological horizons such as de Sitter space. It was later suggested that black holes are maximum-entropy objects, meaning that the maximum possible entropy of a region of space is the entropy of the largest black hole that can be fitted into it. The area increase law (Fig. 1.10) in black holes implies that the total entropy of black holes never decreases. Let S_1 and S_2 are the entropies of two black holes respec-

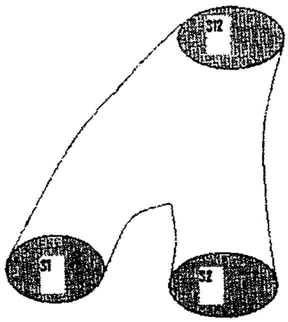

Figure 1.10: When two black holes merge the total entropy would be greater than the individual entropies.

tively and when they coalesce, the new entropy, say, $S_{12} \geq S_1 + S_2$. It seems that this resemblance is a superficial one, since the area law is a theorem in geometry whereas the second law of thermodynamics is understood to have a statistical origin. This resemblance together with the idea that information is irretrievably lost when a body falls into a black hole led Bekenstein to propose [26, 27] that a suitable multiple of the area of the event horizon should be interpreted as its entropy, and that a generalized second law (GSL) should hold.

1.2.5 Generalized second law

Total entropy of all matter in the universe can never decrease. Similarly surface area of event horizon of a black hole never decreases. But when matter falls into the black hole and disappear, the entropy of the universe decreases, which is contrary to the 2^{nd} law of thermodynamics. As Hawking radiation comes out of the black hole, its surface area reduces, which is against area theorem. In order to avoid such an unpleasant situation, a new entropy was proposed as $S' = S + \frac{1}{4}A$, where S is the total entropy of all matter and radiation in the universe excluding all the black holes and A is the total area of all the black holes in the universe. Eventhough S and A may individually change, S' never decreases. When matter is swallowed, S is decreased, but at the same time A is increased. Again, emission of thermal radiation reduces surface area of the horizon but the entropy of universe increases. Thus neither the 2^{nd} law of thermodynamics nor the black hole area theorem are satisfied individually, but it appears that we have a new law, the *generalized second law* as proposed by Bekenstein, of thermodynamics in which $\Delta S' \geq 0$.

At nearly the same time as Bekenstein proposed a relationship between the area theorem and the second law of thermodynamics, Bardeen, Carter, and Hawking [28] provided the general proof of the laws of black hole mechanics which are direct mathematical analogs of the zeroth and first laws of thermodynamics. These laws of black hole mechanics generally apply only to stationary black holes.

1.2.6 Four laws of black hole thermodynamics

The laws of black hole mechanics resemble the ordinary laws of thermodynamics. The laws of black hole physics are

$Zero^{th}$ law

The surface gravity κ of a black hole is constant everywhere on the surface of the event horizon.

1^{st} law

When black hole switches from stationary state to another, its mass changes by $\delta m = \Omega\delta J + \Phi\delta Q + \Theta\delta S + \delta q$, where δJ is the change in angular momentum, δQ is the change in electric charge, δS is the change in entropy, δq is the change in matter distribution.

$2^{nd} law$

In any process, the area of black hole and hence its entropy S does not decrease, i.e., $\delta S \geq 0$.

$3^{rd} law$

The temperature of a black hole cannot be reduced to absolute zero by a finite number of operations.

The close mathematical analogy of the zeroth, first, and second laws of thermodynamics to corresponding laws of classical black hole mechanics is broken by the Planck-Nernst form of the third law of thermodynamics, which states that $S \rightarrow 0$ (or a universal constant) as $T \rightarrow 0$. The analog of this law fails in black hole mechanics, although analogs of alternative formulations of the third law do appear to hold for black holes [29], since there exist extremal black holes (i.e., black holes with $\kappa = 0$) with finite A. However, there is good reason to believe that the Planck-Nernst theorem should not be viewed as a fundamental law of thermodynamics [30] but rather as a property of the density of states near the ground state in the thermodynamic limit, which happens to be valid for commonly studied materials. Indeed, examples can be given of ordinary quantum systems that violate the Planck-Nernst form of the third law in a manner very much similar to the violations of the analog of this law that occur for black holes [31].

1.2.7 Information and naked singularity

The information from a black hole is available only when the black hole area remains unchanged and in less efficient processes the area always increases. The *future event horizon* of an asymptotically flat black hole spacetime is defined as the boundary of the past of future null infinity, i.e., the boundary of the set of points that can communicate with the remote regions of the spacetime to the future. Hawking proved that if $R_{ab}k^a k^b \geq 0$, and if there is no naked singularity (i.e., Cosmic censorship holds), the cross sectional area of future horizon can not be decreasing anywhere [32, 33, 34], where, k^a is the tangent vector to the geodesic and R_{ab} is the Ricci tensor. In Fig. (1.11), it is shown that the circumference of the horizon cross-section increases with the elapse of time. That is, getting information from behind the horizon is difficult if the area of horizon increases. Hawking showed

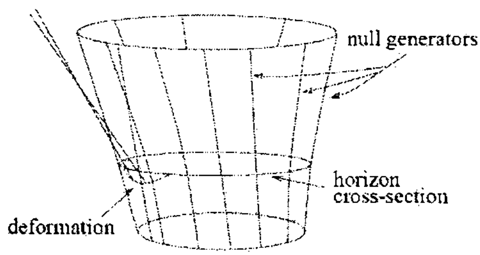

Figure 1.11: Portion of an event horizon with some converging generators that reach a crossing point. The generators of the boundary of the future of the deformation also reach a crossing point. The impossibility of this crossing point is used in proving the area theorem.

that the convergence of the horizon generators does imply existence of naked singularity. The basic idea is to deform the horizon cross-section outward a bit from the point where the null generators are assumed to be converging and to consider the boundary of the future of the part of the deformed cross-section that lies outside the horizon. If the deformation is sufficiently small, all of the generators of this boundary are initially converging and therefore reach a crossing point. The crossing of generators implies that the horizon area shrinks and hence the singularity would become naked, i.e., information could be retrieved from such a black hole. Thus unraveling of naked singularity leads to the retrieval of information that have lost when the matter fell through the horizon.

1.2.8 Membrane paradigm

We can regard the entropy S_{bh} of a black hole as Boltzmann's constant k_B times the logarithm of the total number N_{bh} of quantum mechanically distinct ways that the black hole could have been made, that is

$$S_{bh} = k_B \log N_{bh}. \tag{1.15}$$

It was shown that the entropy of a baryon system with spherical symmetry has only entropy 10^{20} times less than a black hole of same mass [35, 36]. This suggests that the gravitational collapse provides huge entropy production. The knowledge about the reason behind the entropy production in the gravitational collapse would lead us to the ultimate theory, i.e., quantum gravity. A black hole in a box at fixed energy would have a short wavelength cut off at the box but, it has no long wavelength cut off at the box [37]. The reason is that the horizon is an infinite redshift surface.

Black hole entropy is also said to be a measure of the information hidden in correlation between the degree of freedom on either side of the horizon. For instance, although the full state of a quantum field may be pure, the reduced density metrix ρ_{ext} will be mixed. The associated entropy, $S_{entanglement} = -Tr\rho_{ext} \log \rho_{ext}$, should perhaps thus be part of the black hole entropy. This is called an entanglement entropy or geometric entropy. $S_{entanglement}$ is identical to the thermal entropy of the quantum field out side the horizon [38, 39].

1.3 Semi-classical back reaction program

When there is a field other than gravity in the black hole spacetime it would perturb the metric of the black hole. This quantum field which is characterized by a renormalized stress-energy tensor will be in thermal equilibrium with the Hawking radiation. The heat bath around the black hole could be composed of the quanta of the field in the black hole geometry. The expectation value of the renormalized stress-energy tensor in an appropriate vacuum state is regarded as the source in the Einstein semiclassical field equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and this equation will be solved self consistently for the metric. This is called the back reaction program. Such tensors are constructed by using renormalization techniques on the real Euclidean section of the black hole geometry with its Euclidean time coordinate identified with period β_0 so as to eliminate singularity at $r = r_h$. The procedure of back reaction analysis is given here. The interaction of gravity with other fields can be described at three different levels.

1. Classical gravitational field (g) plus other classical fields (f) obey classical equations.

2. In a full quantum description of both g and f, by means of a wave function $\Psi(g, f)$, which obeys the Wheeler-DeWitt equation.

3. g is a perturbed metric such that

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \Psi_{\mu\nu}, \tag{1.16}$$

where, $\bar{g}_{\mu\nu}$ is a classical field and $\Psi_{\mu\nu}$ represents the effect of quantum fluctuation of the metric. The quantum field f is quantized in the field $\bar{g}$ and is described by some wave function $X(\bar{g}, f)$. According to York [40, 41, 42], the back reaction program is as follows.

1. Let a curved back ground spacetime with metric $\bar{g}_{\mu\nu}$ be Ricci flat, so that

$$\bar{R}_{\mu\nu} = \bar{G}_{\mu\nu} = 0, \tag{1.17}$$

2. Let there be an external non-gravitational field Φ on the back ground spacetime and Φ is in a vacuum state, i.e.,

$$\begin{gathered} <\Phi>= 0. \\ <\Phi^2>\neq 0, \end{gathered} \tag{1.18}$$

for quantum fluctuations of Φ. The expectation value of a renormalized symmetric stress-energy tensor $<T_{\mu\nu}>^{ren}$ of Φ satisfies

$$\bar{\nabla}_\mu <T^{\mu\nu}>^{ren}= 0, \tag{1.19}$$

where $\bar{\nabla}_\mu$ represents the covariant derivative w.r.t. $\bar{g}_{\mu\nu}$

3. In Eq. (1.16), $\Psi_{\mu\nu}$ represents the effect of quantum fluctuation of the metric, so that

$$\begin{gathered} <\Psi>= 0 \\ <\Psi^2>\neq 0. \end{gathered} \tag{1.20}$$

The effective stress-energy tensor $\tau_{\mu\nu}$ of Ψ satisfies

$$\bar{\nabla}^{\mu} < \tau_{\mu\nu} >^{ren} (\Psi) = 0. \tag{1.21}$$

4. The back reaction problem is then to solve the Einstein equation

$$G_{\mu\nu}(g) = 8\pi[\tau_{\mu\nu}(\Psi) + T_{\mu\nu}(\Phi)], \tag{1.22}$$

for a classical metric

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \triangle g_{\mu\nu}. \tag{1.23}$$

5. To ignore $\tau^{\mu\nu}$, consider an ideal massless perfectly reflecting spherical wall of area $4\pi r_0^2$ that encloses the black hole so that the microcanonical boundary condition envisages the total effective energy at r_0

$$m(r_0) = M + E_{rad}(r_0), \tag{1.24}$$

where E_{rad} is the energy of radiation.
6. Now the equilibrium temperature distribution is given as

$$T_{loc}(r)|g_{tt}(r)|^{1/2} = T_{\infty} = \tfrac{1}{2\pi}\kappa\hbar, \tag{1.25}$$

where the surface gravity at the event horizon is

$$\kappa = \tfrac{1}{4M}[1 + \tfrac{\hbar}{M^2}(\tfrac{K_0+12}{3840\pi}], \tag{1.26}$$

and K_0 is a constant. The back reaction defines the extend to which the physical properties of a black hole changes. It can be calculated by solving the Einstein' equation and the same will be manifested in terms of, like change in entropy of black hole and of change in effective potential of the black hole spacetime.

In **chapters 2 and 3**, we discuss the semi-classical back reaction in an extremal anti-de Sitter-Schwarzschild black hole dipped in spin-2 quantum field and de Sitter-Schwarzschild black hole immersed in a massless quantum field respectively.

1.4 State equation of thermal radiation

The state equations of thermal radiation in the asymptotic limit are given as $\rho = \alpha T_r^4; s = \frac{4}{3}\alpha T_r^3$. Are they universally valid?. Unruh and Wald said that one could not go near the horizon of a black hole because of the immense buoyancy pressure of the Hawking radiation near the horizon [43, 44]. So we can't test the validity of equation of thermal radiation near the horizon. But the infinite value of Hawking radiation pressure at the horizon is unwarranted, since there is no physical pathology at the horizon. So the best possible way of presenting the Hawking radiation near the horizon is by means of the thermal stress-energy tensor [45, 46, 47], which is finite at the horizon.

Hence it will be possible for the matter in a box to be emptied on to the horizon and conduct a gedanken experiment. Since the gravity near the horizon is intense, the asymptotic state equations of thermal radiation may not be valid there. So the probable answer to the above question is, **no**. Since black hole is a testing ground for extreme cases, this equation must be validated near the horizon of a black hole to make it universally acceptable. After all, the GSL must be valid in any situation [48]. So one should conduct a gedanken experiment, with the hope of conserving the GSL, to test the universality of the equation of radiation. The general equations of thermal radiation which are true in any situation have been determined for different black holes [49]. The validity of equation of radiation can also be performed by verifying the upper bound on entropy as suggested by Bekenstein [50]. In **chapter 4**, we discuss the validity of the equation of thermal radiation in the spacetime of Reissner-Nordström black hole and the upper bound on entropy. We have derived modified expressions which are valid everywhere.

1.5 Thermodynamics of self gravitating radiation system

Self gravitating radiation system ($SGRS$) sees a good example in the gravitational collapse of a star to become a black hole. When the collapse is symmetric, the entropy of the $SGRS$ of mass M (as the only parameter) confined to a spherical box of radius R, is $4\pi M^2$. This is exactly the entropy of the black hole into which the $SGRS$ of mass M collapses. This entropy can be taken as the upper bound on the entropy of $SGRS$ of mass M.

The Rindler frame can mimic gravity, since it is accelerating. This frame has many properties of a black hole except the mass. It has horizon, surface gravity, temperature, etc. Hence entropy calculations and entropy bound may be examined in the Rindler frame also. In **chapter 5**, we have discussed the thermodynamics of spacetimes such as that of Schwarzschild, Rindler, etc. The gravitational collapse of scalar field, its entropy and upper bound in a Rindler spacetime have been also analyzed in this chapter.

1.6 No hair theorem

Physicist John Wheeler [51] once made a famous remark, ***black holes have no hair.*** This referred to the theory that there are only three parameters that can be applied by an outside observer relating to a black hole: mass, electric charge and angular momentum. This is because, in all solutions of the Einstein-Maxwell Equations in general relativity, all other information about a black hole are hidden from observation by the event horizon. This means the no-hair theorem is essentially a restatement of the Cosmic Censorship Hypothesis. Its original form, *black holes have no-hair*, held that a black hole can be *dressed* only by electromagnetic field which are associated with a Gauss-like law.

In astrophysics, the no-hair theorem states that black holes are completely characterized only by the three above said externally observable parameters. All other information about the matter which formed a black hole or falling into it, disappear behind the black-hole event horizon and are therefore permanently inaccessible to external observers. The statement that black holes have no-hair means, there are no features other than mass, charge and angular momentum that distinguish one black hole from another.

If we construct two black holes with the same masses, electrical charges, and angular momenta, the first black hole being made out of ordinary matter and the second one out of anti-matter, they would be completely indistinguishable. None of the pseudo-charges (baryonic, leptonic, etc.) is conserved in the black holes. No-hair theorem may also be defined as the fact that black holes will emit the same radiation regardless of what goes into the black hole. It should be noted however that, not all theoreticians believe that the *no-hair* holds completely, since some of the information lost at the event horizon would be recovered during the process of evaporation.

1.6.1 Information loss paradox

Classically, the laws of physics are the same run forward or in reverse, i.e., if the position and velocity of every particle in the universe were measured, we could work backward to discover the history of the universe arbitrarily far in the past. In quantum mechanics, this corresponds to a vital property called unitarity which has to do with the conservation of probability [52]. Black holes, however, might violate this rule, i.e., if we throw a pure quantum state into a black hole, we will get a mixed state. This runs counter to the rules of quantum mechanics and is known as the black hole information loss paradox. An open question in fundamental physics is the so-called information loss paradox, or black hole unitarity paradox.

As seen from outside, information is never actually destroyed as the matter falling into the black hole takes an infinite time to reach and cross the event horizon. Hence, collapse is an infinitely long process. For a radially outgoing light, dt/dr approaches infinity near the horizon. This is a time dilation effect. Any message sent via light signal from near the event horizon to an observer far from the black hole will be stretched out. The closer the emitter of the light signal is to the event horizon, the more stretched out the message will appear to the far away observer. As the frequency of the light signal decreases or redshifts, the ability of the radiation to store information (information per unit time) decreases, which will refrain us from getting information from the black hole.

At a quantum level, is the quantum state of the Hawking radiation uniquely determined by the history of what has fallen into the black hole?; or is the history of what has fallen into the black hole uniquely determined by the quantum state of the black hole and the radiation? This is what determinism and unitarity would require. For a long time, Stephen Hawking had opposed such ideas, holding to his original position that the Hawking radiation is entirely thermal and therefore entirely random, containing none of the information held in material the hole has swallowed in the past; this information he reasoned had been lost.

However, in 2004 he presented a new argument, reversing his previous position [53]. On this new calculation, the loss of entropy (and hence information) associated with the Hawking radiation is difficult to conceive, until the black hole completes its evaporation; until then it is impossible to relate in a 1:1 way the information in the Hawking radiation to the initial state of the system. Once the black hole evaporates completely, then such an identification can be made, and unitarity is preserved. By the time Hawking completed his calculation, it was already very clear from the AdS/CFT correspondence that black holes decay in a unitary way.

1.6.2 Hair?

With the developments in particle physics, solutions for black holes with various *hairs* have been found. Among them are black holes dressed with Yang-Mills, Proca-type Yang-Mills, and Skyrme fields in various combination with Higgs fields [54, 55, 56]. The first black hole solution demonstrating the failure of the no-hair conjecture was obtained by Gibbons [57] in 1982 within Einstein-Maxwell (EM)-

dilaton theory.

The fact that the Gibbons solution carries no dilatonic charge makes it asymptotically indistinguishable from a Reissner-Nordström black hole with the same mass and electric charge. However, since the latter is not a consistent solution of the EM-dilaton equations, one might expect that, within a given matter model, the stationary black hole solutions are still characterized by a set of global charges (generalized no-hair conjecture). In fact, the Gibbons black hole supports the generalized no-hair conjecture; its uniqueness within EM-dilaton theory was established by Masood-ul-Alam in 1992 [58]. However, neither the original nor the generalized no-hair conjecture are correct. For instance, the latter fails to be valid within Einstein-Yang-Mills (EYM) theory: According to the generalized version, any static solution of the EYM equations should either coincide with the Schwarzschild metric or have some non-vanishing Yang-Mills charges [59, 60]. Since these solutions are asymptotically indistinguishable from the Schwarzschild solution, and since the latter is a particular solution of the EYM equations, the non-Abelian black holes violate the generalized no-hair conjecture. As the non-Abelian black holes are not stable [61, 62], one might adopt the view that they do not present actual threats to the generalized no-hair conjecture. However, during the last years, various authors have found stable black holes which are not characterized by a set of asymptotic flux integrals: For instance, there exist stable black hole solutions with hair to the static, spherically symmetric Einstein-Skyrme equations [63, 64] and to the EYM equations coupled to a Higgs triplet [65]. Hence, the restriction of the generalized no-hair conjecture to stable configurations is not correct either.

1.6.3 Weak and strong interpretation

The interpretations of hair are of two types: ***Weak and Strong.*** A weak interpretation of hair means the existence of a non-trivial solution for a field other than electromagnetic field which obeys gauss-like law so that in the expression for the field there must have conserved quantities other than *mass, charge and angular momentum.* Fig. (1.12) shows the profile of a scalar field for an AdS Schwarzschild black hole with a double well potential. The condition that the mass of hairy black hole, which is greater than the mass of non-hairy black hole, should not blow up is a major criterion for the existence of hair. The field equation must not fade out under perturbation so as to ascertain the stability of hair. Situations of preservation of *no-hair conjecture* [51, 66, 67, 68, 69] and its violations [70, 71] had been reported earlier many times. A possible black hole solution is given as

$$\Phi = Ar^{-3/2}\cos(\frac{\sqrt{4\beta-9}}{2}) \tag{1.27}$$

where β is a constant. The strong interpretation says about the need of a non-trivial solution of the field as well as a proper metric that has the trace of a new conserved quantity. A proper metric means a metric with a well defined horizon that would hide the singularity and that holds temperature [72]. The major challenge that we face in the investigation of strong hair is that when the solution is non-trivial, the singularity would become naked, i.e., the metric would not have a proper horizon and the metric would be proper only for a trivial solution [73, 74]. Weak interpretation of scalar hair in a BTZ and RN black holes are discussed in **chapter 6** and in **chapter 7**,

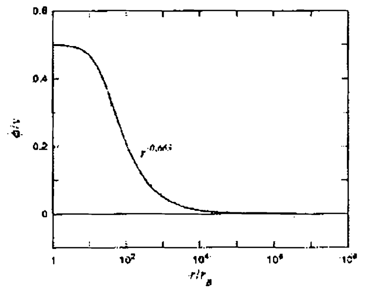

Figure 1.12: The configuration of the scalar field Φ in a symmetric double well potential.

a strong interpretation of scalar hair in a static (3+1) black hole has been described. In this chapter, a non-trivial solution and a proper metric have been obtained.

Chapter 2

Thermodynamics of static Einstein spaces-Back reaction

2.1 Introduction

If it isn't a black hole, it really has to be something exotic !
S. W. Hawking.

Classically, black holes are perfect absorbers but do not emit anything; their physical temperature is absolute zero. The situation that black holes swallow everything that comes near by would adversely affect the validity of second law of thermodynamics, since the entropy of universe would decrease by means of absorption. If this amassing of matter continues, the mass of black hole and hence its surface area increases, which leads to the area theorem $\frac{dA}{dt} \geq 0$.

In 1974, Stephen Hawking [19] showed that black holes are not entirely black but emit thermal radiation. The continuous emission of radiation will be resulted in the complete vaporization of black holes, which is contrary to the area theorem. But the area theorem and second law of thermodynamics are basic laws and hence must be protected. So Bekenstein [21] proposed that the black hole should possess an entropy and it must be a function of the area of the black hole horizon. The temperature and entropy of a black hole are given as $\frac{\hbar}{8\pi M}$ (for SBH) and $\frac{A}{4}$ respectively, where M is the mass of the black hole and A is the area of the horizon.

After the advent of Hawking's discovery that a black hole in empty space radiates energy with a thermal spectrum[19], it has been believed that a black hole can exist in thermal equilibrium with a heat bath possessing a characteristic temperature distribution. The heat bath around the black hole could be composed of the quanta of the field in the black hole geometry. The gravitational effect of the heat bath is characterized by its gravitationally induced renormalized stress-energy tensor [75].

Had no equilibrium been established between black hole and the thermal field, black hole would have evaporated. To save the black hole from such an unpleasant situation, York [40] considered the Schwarzschild black hole and proposed a cavity at the outer event horizon to contain the field which results in a back reaction. This cavity has some physical properties such as; surface tension, charge and temperature. In this model two event horizons are proposed; one at $r = M$, where M is the mass of black hole alone (naked black hole) and the other at $r = m$, where m is the sum of the mass of black hole and of radiation surrounding it (dressed black hole). The introduction of a cavity of radius r_0 as an effective boundary at a finite distance is very important since the back reaction problem does not have definite solution unless boundary conditions are specified. The spacetime geometry inside the cavity has two parameters, such as, the mass M_{bh} of the black hole and the radius r_0 of the cavity containing the thermal radiation.

The theory of quantum fields in curved spacetime reached a high level of development ever since the Hawking's celebrated result that black holes radiate energy with a thermal spectrum. The rigorous description of the quantum behaviour of a black hole requires the back reaction of the emitted radiation on the back ground spacetime. Temperature of a black hole differs from the Hawking's temperature if the back reaction contribution in the form of an energy density term is taken into consideration [76]. Hawking's formula of temperature of black hole predicts an ever-increasing temperature, resulting in the complete evaporation of the hole. But we can check the complete evaporation by properly taking the back reaction into account. The remnant mass still be much greater than Planck mass.

The expectation value of the renormalized stress-energy tensor in an appropriate vacuum state is regarded as the source in the Einstein semi-classical field equation and this equation is solved self consistently for the metric [77]. This is the back reaction program. Such tensors are constructed by using renormalization techniques on the real Euclidean section of the black hole geometry with its Euclidean time coordinate identified with period β_0 so as to eliminate singularity at $r = r_h$.

Huang, Liu, Xu and Zhao proposed a thermodynamical approach to tackle the back reaction program [78, 79]. They had used this approach to solve the back reaction problem of Schwarzschild black hole and the results agree with the York's one which was obtained by solving the Einstein's equation [40]. Lin-Xin Li made use of the thermodynamical approach to solve the back reaction of Kerr black hole [80]. Using the functional formalism, a generalized Einstein equation in the form of a Langevin equation for the description of the back reaction of quantum field and their fluctuations on the dynamics of curved spacetime had already been investigated [81].

A static spacetime is one which has a property that

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor of the spacetime and Λ is the Cosmological constant. The outline of this chapter is as follows. In section 2.2, back reaction program is presented. Solution to back reaction is discussed in section 2.3. In section 2.4, a thermodynamical approach to back reaction is presented. We give conclusion in section 2.5.

2.2 Back reaction program

Back reaction manifests itself in the change in the metric of a black hole. When metric changes, the radius of event horizon changes, thereby changing the surface area and hence entropy. In the present context we concentrate on a static black hole, like an anti-de Sitter-Schwarzschild black hole, its asymptotic region being not far away from the horizon. The spacetime would become Lorentzian at $(\frac{6M}{\Lambda})^{1/3}$. Let the anti-de Sitter-Schwarzschild black hole be placed inside an axisymmetric spherical cavity which is being filled with Hawking's radiation. The thermodynamical system composed of a black hole and radiation in *curved space* can be treated as a thermodynamic system composed of a black hole with surface (dressed black hole) and radiation in *flat space.* The surface tension of surface must be negative in order to balance the radiation pressure. Eventhough thermal radiation differs from Hawking radiation [45], its influence can be absorbed into the properties of the surface at the outer event horizon. Suppose a quantum spin-2 field is present around the black hole. Now the thermal radiations getting reflected by the surface is balanced by the renormalized stress energy tensor of a spin-2 field in the static spacetime of the black hole. The total effective mass energy inside a cavity of radius r_0 for static observer is [40]

$$m(r) = M + R_{rad}(r_0) = M[1 + \epsilon(\mu + \varrho)], \tag{2.2}$$

where ϵ is a constant equal to $\frac{h}{2\pi M^2}$ and μ and ϱ are two position dependent variables. In the above equation M represents the mass of naked black hole, E_{rad} the mass of radiation surrounding the black hole. The interaction between the black hole and spin-2 field in thermal equilibrium is incorporated in the surface term $M\epsilon(\mu + \varrho)$. The back reaction program is then to solve the Einstein's equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle, \tag{2.3}$$

where $\langle T_{\mu\nu}\rangle$ is the renormalized stress-energy tensor of a spin-2 field in Einstein static space and $G_{\mu\nu}$ is the Einstein's tensor.

2.3 Solution of back reaction program

A Schwarzschild black hole in a spacetime with negative cosmological constant is known as the anti-de Sitter-Schwarzschild black hole. The metric to the static and spherically symmetric anti-de Sitter-Schwarzschild black hole is given as

$$\begin{aligned} ds^2 &= -(1 - \tfrac{2m(r_0)}{r} + \tfrac{\Lambda}{3}r^2)e^{2\epsilon\varrho(r)}dt^2 \\ &+(1 - \tfrac{2m(r_0)}{r} + \tfrac{\Lambda}{3}r^2)^{-1}dr^2 + r^2 d\Omega^2, \end{aligned} \tag{2.4}$$

where r_0 is the radius of cavity. Then the linear perturbations to the metric result from expanding the metric function in ϵ as

$$e^{2\epsilon\varrho(r)} = 1 + 2\epsilon\varrho(r). \tag{2.5}$$

Here we impose a condition that $9m^2\Lambda = 1$. Generally in the extremal case, the temperature of the black hole is zero. Even though in the extremal case, the temperature of the horizon is still not zero for an anti-de Sitter situation. At the horizon, $1 - \frac{2m}{r} + \frac{r^2}{27m^2} = 0$. Solving the equation, the radius of the event horizon is obtained as $r_h \approx 1.8m$. Since r is a function of mass only, we can write $r_h = \alpha m(r_0)$ where α is a constant. When the region inside the cavity is considered, $r_h = \alpha m(r)$. So that the metric is modified as

$$ds^2 = -[1 - w]e^{2\epsilon\varrho(r)}dt^2 + [1 - w]^{-1} + r^2 d\Omega^2, \tag{2.6}$$

where, $w = \frac{\alpha m(r)}{r}$ and $\alpha m(r)$ the radius of event horizon. As the metric is affected by the quantum field, the radius of the horizon is modified as

$$r_h = \alpha M[1 + \epsilon(\mu + \varrho)], \tag{2.7}$$

as the radius of the outer horizon. Now the area of the horizon may be written as $A = 4\pi\alpha^2 M^2[1 + 2\epsilon(\mu + \varrho)]$. Differentiating A w.r.t. M and putting $T_{bh} = \frac{1}{2\pi\alpha^2 M}$ and $dM = T_{bh} dS_0$ (S_0 the entropy of naked black hole) we get

$$dS_0 = \tfrac{1}{4} dA[1 - 2\epsilon(\mu + \varrho)]. \tag{2.8}$$

On integrating Eq. (2.8), we get

$$S_0 = \tfrac{1}{4} A[1 - 2\epsilon(\mu + \varrho)] + c. \tag{2.9}$$

As the boundary condition, we have, $S_0 = 0$, when $A = 0$. So $c = 0$. Then, Eq. (2.9) is simplified as

$$S = S_0 + \tfrac{1}{2}\epsilon(\mu + \varrho)A = S_0 + S_s, \tag{2.10}$$

where, $S = \frac{1}{4}A$ is the entropy of dressed black hole, A is the area of the outer horizon or the cavity, S_s is the correction to the entropy or the surface entropy due to back reaction.

We have the Einstein's equation as

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.11}$$

where, $R_{\mu\nu}$ is the *Ricci* curvature tensor and R is the scalar curvature and $T\mu\nu$ is the stress energy tensor of a spin-2 quantum field in the black hole spacetime which acts as a source term in the semi-classical

Einstein's equations. The Ricci tensor is given as

$$R_{\mu\nu} = \frac{-\partial\Gamma^{i}_{\mu\nu}}{\partial x^{i}} + \frac{\partial^{2}(\ln\sqrt{-g})}{\partial x^{\mu}\partial x^{\nu}} + \Gamma^{m}_{\mu n}\Gamma^{n}_{\nu m} - \frac{\partial}{\partial x^{n}}[\ln\sqrt{-g}]\Gamma^{n}_{\mu\nu}, \quad (2.12)$$

where, $\Gamma^{n}_{\mu\nu}$ is the Christoffel symbol, which can be given as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \quad (2.13)$$

The Ricci scalar is given as

$$R = R^{0}_{0} + R^{1}_{1} + R^{2}_{2} + R^{3}_{3}. \quad (2.14)$$

The stress energy tensor of spin-2 quantum field in the static Einstein's spacetime is [46, 82]

$$T^{\nu}_{\mu} = a_{s}T^{(tr)\nu}_{\mu} + b_{s}T'^{\nu}_{\mu} + c_{s}T''^{\nu}_{\mu}. \quad (2.15)$$

All three tensors are finite at the horizon ($r_h = \alpha m$). The first tensor $T^{(tr)\nu}_{\mu}$ is the only one with nonzero trace at the horizon:

$$T^{(tr)\nu}_{\mu} = 48\kappa^{4}_{h}w^{6}(\delta^{\nu}_{\mu} + 3\Pi^{\nu}_{\mu} - 6\Psi^{\nu}_{\mu}), \quad (2.16)$$

where,

$$\kappa_{h} = \frac{1}{2}\frac{\partial_{r}g_{tt}}{\sqrt{-g_{tt}g_{rr}}}\,|_{r=r_{H}} = \frac{1}{2\alpha^{2}M}, \quad (2.17)$$

is the surface gravity. Defining the constant tensors as,

$$\Pi^{\nu}_{\mu} = \delta^{\nu}_{\mu} - 4\delta^{0}_{\mu}\delta^{\nu}_{0}.$$

$$\Psi^{\nu}_{\mu} = \delta^{1}_{\mu}\delta^{\nu}_{1} - \delta^{0}_{\mu}\delta^{\nu}_{0}, \quad (2.18)$$

where, δ^{ν}_{μ} is the usual Kronecker delta function. At the horizon ($w = 1$), only the first tensor in Eq. (2.15) is of interest and the coefficient $a_s = a_2 = \frac{212}{2880\pi^2}$ [82]. The tensor components are then given as

$$a_2 T^0_0 = -\tfrac{212*96}{2880\pi^2}\kappa_h^4; a_2 T^1_1 = -\tfrac{212*96}{2880\pi^2}\kappa_h^4$$
$$a_2 T^2_2 = \tfrac{212*192}{2880\pi^2}\kappa_h^4; a_2 T^3_3 = \tfrac{212*192}{2880\pi^2}\kappa_h^4. \tag{2.19}$$

The components of the *Ricci* curvature are

$$R_{00} = \tfrac{3}{2}w'(1-w)e^{2\epsilon\varrho(r)}\epsilon\varrho' - (1-w)^2 e^{2\epsilon\varrho(r)}\epsilon^2\varrho'^2$$
$$+\tfrac{1}{2}w''(1-w)e^{2\epsilon\varrho(r)} - \tfrac{2}{r}(1-w)^2 e^{2\epsilon\varrho(r)}\epsilon\varrho' + \tfrac{w'}{r}(1-w)e^{2\epsilon\varrho(r)}$$
$$R_{11} = \epsilon^2\varrho'^2 - \tfrac{1}{2}w'(1-w)^{-1}\epsilon\varrho' - \tfrac{1}{2}w''(1-w)^{-1} + \tfrac{w'}{r}(1-w)^{-1}$$
$$R_{22} = -\tfrac{w}{r^2} - \tfrac{w'}{r} + \tfrac{(1-w)}{r}\epsilon\varrho'$$
$$R_{33} = (1-w)\sin^2\theta - rw'\sin^2\theta - \sin^2\theta + r(1-w)\sin^2\theta\epsilon\varrho'. \tag{2.20}$$

The mixed Ricci tensors are given as

$$R^0_0 = -\frac{3}{2}w'\epsilon\varrho' + (1-w)\epsilon^2\varrho'^2 - \frac{1}{2}w'' + \frac{2}{r}(1-w)\epsilon\varrho' - \frac{w'}{r}$$
$$R^1_1 = (1-w)\epsilon^2\varrho'^2 - \frac{1}{2}w'\epsilon\varrho' - \frac{1}{2}w'' + \frac{w'}{r}$$
$$R^2_2 = -\frac{w}{r^4} - \frac{w'}{r^3} + \frac{1-w}{r^3}\epsilon\varrho'$$
$$R^3_3 = \frac{1-w}{r^2} - \frac{w'}{r} - \frac{1}{r^2} + \frac{1-w}{r}\epsilon\varrho'. \tag{2.21}$$

The *Ricci* scalar is

$$R = 2(1-w)\epsilon^2\varrho'^2 + \tfrac{3(1-w)}{r}\epsilon\varrho' - 2w'\epsilon\varrho' + \tfrac{1-w}{r^3}\epsilon\varrho'$$
$$-\tfrac{w'}{r} - \tfrac{w}{r^4} - \tfrac{w'}{r^3} - \tfrac{1}{r^2} - w'' + \tfrac{(1-w)}{r^2}. \quad (2.22)$$

Components of Einstein's mixed tensors at the horizon ($w = 1$)

$$G^0_0 = -\tfrac{1}{2}w'\epsilon\varrho' - \tfrac{1}{4}w'' - \tfrac{w'}{2r} + \tfrac{1}{2r^4} + \tfrac{1}{2r^2} + \tfrac{w'}{2r^3} = 8\pi T^0_0$$

$$G^1_1 = \tfrac{1}{2}w'\epsilon\varrho' - \tfrac{1}{4}w'' + \tfrac{3w'}{2r} + \tfrac{1}{2r^4} + \tfrac{1}{2r^2} + \tfrac{w'}{2r^3} = 8\pi T^1_1$$

$$G^2_2 = w'\epsilon\varrho' - \tfrac{w'}{2r^3} - \tfrac{1}{2r^4} + \tfrac{1}{4}w'' + \tfrac{w'}{2r} + \tfrac{1}{2r^2} = 8\pi T^2_2 \quad (2.23)$$

$$G^3_3 = w'\epsilon\varrho' + \tfrac{3w'}{2r} + \tfrac{1}{4}w'' - \tfrac{1}{2r^2} + \tfrac{1}{2r^4} + \tfrac{w'}{2r^3} = 8\pi T^3_3,$$

where, $w' = \frac{\alpha}{r}\frac{dm}{dr} - \frac{\alpha m}{r^2}$ and $w'' = -\frac{2\alpha}{r^2}\frac{dm}{dr} + \frac{2\alpha m}{r^3}$. From Eq. (2.23)

$$-\tfrac{1}{2}w'' + \tfrac{w'}{r} + \tfrac{1}{r^4} + \tfrac{1}{r^2} + \tfrac{w'}{r^3} = 8\pi[T^0_0 + T^1_1]. \quad (2.24)$$

We have, $m = M[1 + \epsilon(\mu + \varrho)]$. Then, $\frac{dm}{dr} = \epsilon M(\frac{d\mu}{dr} + \frac{d\varrho}{dr})$ and at $w = 1$, we have from Eq. (2.24)

$$\epsilon(\tfrac{d\mu}{dr} + \tfrac{d\varrho}{dr}) = \tfrac{1}{M}[k_1 - 2k_2 r - \tfrac{k_3}{r^2}] \; . \quad (2.25)$$

On integrating Eq. (2.25), we get

$$\epsilon(\mu + \varrho) = \tfrac{1}{M}[k_1 r - k_2 r^2 + \tfrac{k_3}{r}] + c. \quad (2.26)$$

With the boundary condition that $\epsilon(\mu+\varrho)=0$ at $r=M$, we have

$$\epsilon(\mu+\varrho)=\frac{1}{M}[k_1 r-k_2 r^2+\frac{k_3}{r}]-k_1+k_2 M-\frac{k_3}{M^2}. \quad (2.27)$$

We have taken an extremal case, $9m^2\Lambda=1$ and $T=\frac{1}{2\pi\alpha^2 M}$. Then at $r=\alpha m$, we have

$$\epsilon(\mu+\varrho)\simeq(\frac{c_1}{3\Lambda^{1/2}}-\frac{c_2}{9\Lambda}+3c_3\Lambda^{1/2})T+\frac{c_4}{T}-c_5T^2, \quad (2.28)$$

where c_1, c_2, c_3, c_4, c_5 are constants. Substituting Eq. (2.28) in Eq. (2.10), we get

$$S=S_0+\frac{1}{2}[\epsilon(\mu+\varrho)]A\simeq(\frac{c_1}{3\Lambda^{1/2}}-\frac{c_2}{9\Lambda}+3c_3\Lambda^{1/2})T+\frac{c_4}{T}-c_5T^2]A, \quad (2.29)$$

where, $\frac{1}{2}[\epsilon(\mu+\varrho)]A\simeq(\frac{c_1}{3\Lambda^{1/2}}-\frac{c_2}{9\Lambda}+3c_3\Lambda^{1/2})T+\frac{c_4}{T}-c_5T^2]A$ is called the surface entropy, which is a measure of back reaction. A black hole immersed in a quantum field and Hawking radiation kept inside a cavity is called a dressed black hole. Eq. (2.29), reveals that the back reaction depends on temperature, since the surface entropy, which is a measure of back reaction, is a function of temperature. Thus the entropy of a dressed black hole depends on temperature also. An isolated black hole is called a naked black hole, whose entropy is a function of area of event horizon only. So a black hole may be assumed to be a thermodynamic system (micro canonical) taken inside a cavity with a boundary, in which the thermal radiation and quantum field do exist in thermal equilibrium. The entropy of the field and radiation inside the cavity does not reside in itself, but, get transferred to the surface and hence the name surface entropy.

2.4 Thermodynamic approach

Black hole is assumed to be kept inside a cavity of radius r_0. The surface possess a negative surface tension σ. The negative surface tension is proposed to balance the thermal pressure. Semi-classical quantum gravity plays a key role in the investigation of thermodynamics of black hole [83]. Now the thermodynamic potential of the black hole with surface is given as [84]

$$\Phi = \Phi_0 + \Phi_s$$
$$\Phi_s = \sigma A, \tag{2.30}$$

where, Φ_0 and Φ_s are respectively the thermodynamic potentials of the naked black hole and that of surface. The thermodynamic relation reads as

$$d\Phi = -SdT + \sigma dA, \tag{2.31}$$

where, S is the total entropy of the black hole. The above equation can be expanded as

$$d\Phi_0 = -S_0 dT,$$
$$d\Phi_s = -S_s dT + \sigma dA. \tag{2.32}$$

Then

$$S_s = -(\tfrac{\partial \Phi_s}{\partial T})_A = -(\tfrac{\partial \sigma}{\partial T})_A A. \tag{2.33}$$

Let, V be the volume of radiation surrounding the black hole, V_{bh} the volume of the inner event horizon and V_{ext} the volume of external cavity. Then, let the radiation undergoes a change in volume adiabatically, so that

$$\delta V = \delta V_{bh} + \delta V_{ext}. \tag{2.34}$$

By the first law of thermodynamics

$$\delta(M + M_R) = -P\delta V_{ext} = T(\delta S_0 + \delta S_R) + \sigma dA - P\delta V, \tag{2.35}$$

where, M the mass of naked black hole, M_R the mass of radiation, S_R the entropy of radiation. Here we assume the condition that, $(\delta S_0 + \delta S_R) = 0$. The reason is that the change in the naked black hole entropy is compensated by the change in the entropy of radiation. Therefore

$$P(\delta V - \delta V_{ext}) = \sigma dA$$
$$P\delta V_{bh} = \sigma dA. \tag{2.36}$$

Then

$$\sigma = P(\tfrac{\delta V_{bh}}{\delta A}) = \tfrac{1}{3}aT^4(\tfrac{\delta V_{bh}}{\delta A}). \tag{2.37}$$

The term, $\frac{1}{3}aT^4$ is the thermal pressure. Surface tension σ always try to minimize the surface area of the cavity, hence $\frac{\delta V_{bh}}{\delta A} < 0$, i.e., the back reaction makes the black hole more symmetric. So, as volume increases area reduces. Fig. (2.1), shows how the area becomes minimum when cavity is symmetric. Considering the condition that the back reaction ceases to occur as $T \to 0$ and dimensionally, $\frac{\delta V_{bh}}{\delta A}$

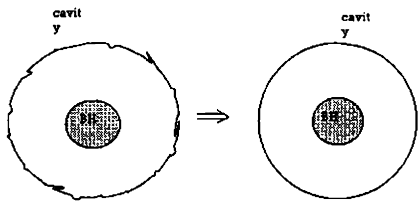

Figure 2.1: When the cavity becomes symmetric, even though the surface area decreases, volume increases.

is length, a particular form is proposed to $\frac{\delta V_{bh}}{\delta A}$ [85]. Let

$$\begin{aligned} \frac{\delta V_{bh}}{\delta A} &= -\alpha\frac{m}{T^2} - \gamma\frac{m^3}{T^4}\log T + \frac{\beta}{T} \\ &= -\frac{\alpha}{3\Lambda^{1/2}}\frac{1}{T^2} - \frac{\gamma}{27\Lambda^{3/2}}\frac{1}{T^4} + \frac{\beta}{T}. \end{aligned} \tag{2.38}$$

since, $9m^2\Lambda = 1$ and α, β, γ are dimensionless constants. Then from Eq. (2.37)

$$\sigma = -\frac{a\alpha}{9\Lambda^{1/2}}T^2 - \frac{a\gamma}{81\Lambda^{3/2}}\log T + \frac{\alpha\beta}{3}T^3, \tag{2.39}$$

and the surface entropy is given as

$$S_s = -(\frac{\partial\sigma}{\partial T})_A A = [\frac{2a\alpha}{9\Lambda^{1/2}}T + \frac{a\gamma}{81\Lambda^{3/2}}\frac{1}{T} - \alpha\beta T^2]A. \tag{2.40}$$

The above equation for surface entropy is in agreement with what obtained in the Eq. (2.29) by the metric approach. In Eq. (2.40), the surface entropy may be assumed to be a linear function of temperature, the other non-linear terms may be neglected. The surface entropy may be then written as

$$S_s = \alpha T A. \tag{2.41}$$

By keeping the average radius constant, a symmetric surface encloses more volume with minimum surface area. So with more pressure inside the cavity, the cavity becomes more symmetric. Hence the pressure is inversely proportional to area of the cavity. So we have

$$P S_s = \alpha T \ . \tag{2.42}$$

This is just like the gas equation. The graphical presentation of Eq. (2.42) is shown in Fig. (2.2). Each point on the surface in Fig. (2.2), gives the values of P, S_s, T at which the black hole system remains in equilibrium.

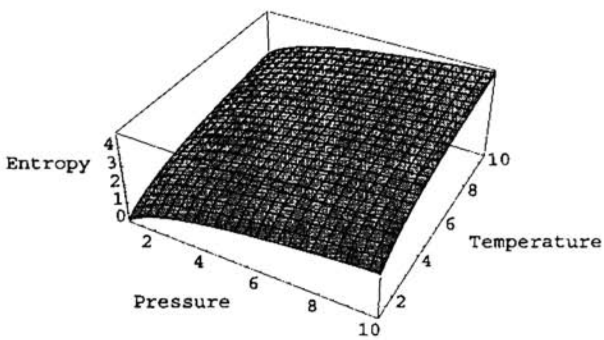

Figure 2.2: The surface of thermal equilibrium inside the cavity. Each point on the surface gives S_s, P, T at which equilibrium exists.

2.5 Conclusion

An anti-de Sitter-Schwarzschild black hole surrounded by a spin-2 quantum field in thermal equilibrium is held captive by a cavity of radius r_0. Two horizons, one at $r_1 = M$ and second at $r_2 = M[1 + \epsilon(\mu + \varrho)]$ were proposed so that the spin-2 field and thermal field coexist in equilibrium in between the two horizons. The change of entropy of a black hole which is a measure of back reaction by the York and thermodynamical approaches gave identical results. The entropy of a dressed black hole (sum of entropy of naked black hole and surface entropy) is found to depend on temperature of black hole and the area of the event horizon. Generally the entropy of a black hole is $\frac{1}{4}A$, where A is the horizon's area, but the surface entropy which is a measure of back reaction depends on the temperature also. The surface entropy or back reaction is due to the quantum field other than gravitational field that is present in the vicinity of the black hole. The cavity, which has many physical properties such as thermodynamic potential and surface tension holds the quantum field and the Hawking radiation so as to keep the black hole from evaporation into nothing.

Chapter 3

Back reaction in a static black hole with a massless quantum field

3.1 Introduction

A black hole - a tremendous creation. Its physics defies imagination. Time and space it can bend. Wow! I can't comprehend the gravity of this situation,
a layman.

Since a complete and self consistent quantum gravity theory still eludes us, the evaporation of black holes can be studied only by using a semi-classical approach. One can assume that the expectation value of the stress-energy tensor operator of the quantized fields in a suitably chosen state, acts as a source for the dynamical evolution of the back ground spacetime. This theory would correspond to an appropriate limit of a self consistent quantum theory of gravity [86].

In this chapter we make use of the stress-energy tensor given in Eq. (2.15) in solving the Einstein's semi-classical field equation [45, 46]. Since black hole acts as thermodynamic system, a stable equilibrium may be achieved by putting the black hole in a cavity [78, 79, 80]. There will be stable and unstable orbits in the spacetime of a black hole. The stability of an orbit is decided by the nature of effective potential of the spacetime. In a perturbed spacetime, the effective potential will be altered, which results in the change of the stable and unstable orbits of massive and massless particles [82, 87, 88]. The order-$\hbar$ fluctuation of gauge fields would act as a repulsive gravity and its strength increases as the higher order in the back reaction are included [89].

Recently it is reported that the cosmological constant is positive [90]. This has led to a renewed interest in de Sitter space [91, 92, 93] and hectic research is going on in the field of de Sitter space. Thermodynamics of de Sitter black hole [94, 95, 96] and its surface gravity for different event horizons had already been evaluated [97, 98]. For a given observer somewhere in space, the black hole radiation is unidirectional, but radiation from the cosmological horizon would arrive to him from all directions. So, the back reaction problem in the de Sitter spacetime has aroused special interest, since the result may bring us close to the knowledge about the stability of a black hole.

In this chapter, back reaction at the cosmic horizon of a Schwarzschild-de Sitter black hole in thermal equilibrium with conformal massless quantum field is investigated. Two methods have been proposed to measure the back reaction: one by solving Einstein's semi-classical field equation and the result is in agreement with that of the standard results, second by evaluating the perturbed potential in the presence of a quantum field using the Hamilton-Jacobi method and thereby suggesting stable and unstable orbits for massive and massless particles. The outline of the chapter.3 is as follows. In Sec 3.2, the entropy change is calculated from the change in the metric. In Sec 3.3, the theory of back reaction program is discussed by solving the Einstein's field equation. In Sec 3.4, the nature of the perturbed spacetime geometry is presented by examining the effective potential and orbits of massive and massless particles by making use of Hamilton-Jacobi method. In Sec 3.5, we give the conclusion explaining the differences in the results of chapter.2 and chapter.3. Back reaction at the black hole event horizon can also be investigated using similar methods.

3.2 Entropy change

Suppose a massless quantum field is present in the vicinity of a black hole which is conformally coupled to the spacetime geometry. Back reaction can be measured by measuring the entropy change as given Eq. (2.10). Again the perturbed metric conformal with the quantum field changes the effective potential of the black hole spacetime. The extent of change in the potential gives a measure of back reaction. The presence of field would change the radii of the stable and unstable orbits of massive and massless particles.

For a given arrangement of a black hole and quantum field in thermal equilibrium, the sum of the masses of the naked black hole M and radiation E_{rad} is a constant. Then the metric of the static and spherically symmetric Schwarzschild-de Sitter black hole of mass

$m(r_0)$ in vacuum is given as

$$ds^2 = -(1 - \tfrac{2m(r_0)}{r} - \tfrac{\Lambda}{3}r^2)dt^2 + (1 - \tfrac{2m(r_0)}{r} - \tfrac{\Lambda}{3}r^2)^{-1}dr^2 + r^2 d\Omega^2, \tag{3.1}$$

where, r_0 is the radius of cavity. The horizon surface (here outer horizon is at r_0) equation of de Sitter black hole is as follows

$$1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 = 0. \tag{3.2}$$

The above equation is a unitary cubic equation. When $9m^2\Lambda < 1$ is satisfied, it has three different real roots: $r_- = -2\sqrt{\frac{1}{\Lambda}}\cos(\frac{\varphi}{3} - \frac{\pi}{3})$; $r_h = -2\sqrt{\frac{1}{\Lambda}}\cos(\frac{\varphi}{3} + \frac{\pi}{3})$; $r_c = 2\sqrt{\frac{1}{\Lambda}}\cos(\frac{\varphi}{3})$, where

$$\cos\frac{\varphi}{3} = \frac{1}{2}[(\sqrt{9m^2\Lambda - 1} - 3m\sqrt{\Lambda})^{1/3} - (\sqrt{9m^2\Lambda - 1} + 3m\sqrt{\Lambda})^{1/3}]. \tag{3.3}$$

Now r_- is a negative root with out a physical meaning; $r_h$is the smaller positive root, which is the radius of the black hole event horizon; r_c is the larger positive root corresponding to the cosmic horizon. Between the inner horizon (horizon of naked black hole)and outer horizon (horizon of dressed black hole or cavity) the mass m is a variable so that, $m(r) = M[1 + \epsilon\mu(r)]$, in which ϵ is the linearity constant equal to $\frac{h}{2\pi M^2}$ and M represents the mass of naked black hole. The perturbed metric is given as

$$ds^2 = -[1 - \tfrac{2m(r)}{r} - \tfrac{\Lambda}{3}r^2]e^{2\epsilon\rho(r)}dt^2 + [1 - \tfrac{2m(r)}{r} - \tfrac{\Lambda}{3}r^2]^{-1}dr^2 + r^2 d\Omega^2. \tag{3.4}$$

In Eq. (3.4), the linear perturbation term can be given by the Eq. (2.5). The radius of the cosmic horizon is given as

$$r_c = \frac{1}{\sqrt{\Lambda}}[(\sqrt{9m^2\Lambda - 1} - 3m\sqrt{\Lambda})^{1/3} - (\sqrt{9m^2\Lambda - 1} + 3m\sqrt{\Lambda})^{1/3}]. \tag{3.5}$$

The cosmological radius in terms of Λ alone is [96]

$$r_c = \sqrt{\frac{1}{4\Lambda}}. \tag{3.6}$$

The area of the cosmic horizon is given by

$$A_c = \tfrac{4\pi}{\Lambda}[(\sqrt{9m^2\Lambda - 1} - 3m\sqrt{\Lambda})^{2/3} + (\sqrt{9m^2\Lambda - 1} + 3m\sqrt{\Lambda})^{2/3}$$

$$-2(\sqrt{9m^2\Lambda - 1} - 3m\sqrt{\Lambda})^{1/3}(\sqrt{9m^2\Lambda - 1} + 3m\sqrt{\Lambda})^{1/3}]. \tag{3.7}$$

We know that, $0 < 9M^2\Lambda < 1$, then by applying the linearity condition we get $A_c \simeq 24\pi m^2 \simeq 24\pi M^2(1+\epsilon\mu)^2$. As the area A_c of a black hole is a function of mass M, then

$$\frac{dA_c}{dM} = 48\pi M(1 + 2\epsilon\mu). \tag{3.8}$$

The temperature range of cosmic horizon is, $\frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}} < T_c < \frac{1}{2\pi}\sqrt{\Lambda}$ [96]. As an average, $T_c \simeq \frac{1}{12\pi M}$ (with the assumption $24M^2\Lambda = 1$). But, $dM = T_c dS_0$. Hence from Eq. (2.10), we have

$$S_0 = \tfrac{1}{4}A_c - S_s. \tag{3.9}$$

In the above equation, $\frac{1}{4}A_c = S$ is the entropy had the entire mass m (including naked mass M and radiation mass E_{rad}) been concentrated as a black hole and S_0 is the entropy of the naked black hole and S_s is the entropy of the surface. Therefore the entropy of the dressed black hole

$$S = S_0 + S_S = S_0 + \frac{1}{2}\epsilon\mu A_c. \tag{3.10}$$

In the above equation, S_s is a measure of back reaction since it results from the change in metric. From Eq. (3.10) we can see that the entropy of a dressed black hole of mass m is greater than the entropy of a naked black hole of mass M by an amount equal to the entropy of the surface.

3.3 Theory of back reaction

Suppose there is an external non-gravitational free field ψ with a quadratic lagrangian on a curved background spacetime with metric g_{ab} and ψ is in vacuum state $\langle\psi\rangle = 0$, appropriate to the back ground. Quantum fluctuations of ψ give rise to $\langle\psi^2\rangle \neq 0$ and we can obtain the expectation value of a renormalized symmetric stress-energy tensor $\langle T^{ab}\rangle$. In accordance with current methods [75], we assume that the background spacetime is Ricci flat ($R_{ab} = 0$) and T^{ab} satisfies the background conservation law, $\nabla_b T^{ab} = 0$. The back-reaction program is then to solve the semi-classical Einstein equation $G_{\mu\nu} = 8\pi[T_{ab} + \tau_{ab}]$, where τ_{ab} is the stress-energy tensor due to quantum fluctuation of the metric. In our present work, we ignore τ_{ab}, by putting the black hole in a cavity and treating them as a micro-canonical ensemble.

When the Hawking radiation is fully thermal, the thermal pressure is $\frac{1}{3}\alpha T_{loc}^4$, where $T_{loc} = \frac{T_{bh}}{\sqrt{-g_{00}}}$. But as proposed by Page [45] and Frolov. [46], the best method to present a pressure is the stress-energy tensor of radiation. The stress-energy tensor of radiation is a function of black hole temperature T_c(temperature of cosmic horizon), which is the temperature as measured by an observer at flat space time and also finite. The back reaction involves solving of Einstein's field equation, $G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$. The stress energy tensor of the massless quantum field in the static Einstein spacetime is given by the Eq. (2.15) as, $a_s T_\mu^{(tr)\nu} + b_s T_\mu^{\prime\nu} + c_s T_\mu^{\prime\prime\nu}$ [46, 82]. The tensors in Eq. (2.15) are finite at the horizon. But only the first tensor $T_\mu^{(tr)\nu}$ is with non zero trace:

$$T_\mu^\mu = a_s \frac{48M^2}{r^6}, \tag{3.11}$$

where $a_s = \frac{[h(0)+\frac{7}{8}h(\frac{1}{2})-\frac{13}{2}2h(1)]}{(2880\pi^2)}$; $b_s = \frac{[2h(0)+\frac{7}{4}h(\frac{1}{2})+2h(1)]}{(2880\pi^2)}$; $c_s = \frac{[-\frac{4}{35}h(0)+92h(0)]}{(2880\pi^2)}$ [46]. Here $h(s)$ is the number of helicity states for the spin s. The second tensor $T_\mu^{\prime\nu}$ is the one that contributes at far distances, and the third is finite at the horizon and vanishes asymptotically at infinity as r^{-3}. The third tensor is important only for the region close to horizon. The components of the tensor at the horizon $w = 1$ [82]

$$\begin{aligned} &T_0^0 = -(96a_s + 171b_s)\kappa_c^4;\ T_1^1 = -(96a_s - 169b_s + 8c_s)\kappa_c^4; \\ &T_2^2 = (192a_s + b_s + 4c_s)\kappa_c^4;\ T_3^3 = (192a_s + b_s + 4c_s)\kappa_c^4, \end{aligned} \tag{3.12}$$

where κ_c is the surface gravity at the cosmic horizon. We have

$$\kappa_c = \frac{\partial_r g_{tt}}{2\sqrt{-g_{tt}g_{rr}}}\,|_{r=r_C}$$
$$= \frac{M}{r^2} - \frac{\Lambda}{3}r\,|_{r=r_C} \simeq \frac{1}{6M}. \quad (3.13)$$

So $\frac{1}{12\pi M}$ is the approximate temperature of cosmic horizon. The components of *Ricci* tensor are (with $w = \frac{2m}{r} + \frac{\Lambda r^2}{3}$)

$$R_{00} = \frac{3}{2}w'(1-w)e^{2\epsilon\rho}\epsilon\rho' - (1-w)^2e^{2\epsilon\rho}\epsilon^2\rho'^2 +$$
$$\frac{1}{2}w''(1-w)e^{2\epsilon\rho} - \frac{2}{r}(1-w)^2e^{2\epsilon\rho}\epsilon\rho' + \frac{w'}{r}(1-w)e^{2\epsilon\rho}. \quad (3.14)$$

$$R_{11} = \epsilon^2\rho'^2 - \frac{1}{2}w'(1-w)^{-1}\epsilon\rho'$$
$$-\frac{1}{2}w''(1-w)^{-1} + \frac{w'}{r}(1-w)^{-1}. \quad (3.15)$$

$$R_{22} = -\frac{w}{r^2} - \frac{w'}{r} + \frac{(1-w)}{r}\epsilon\rho'. \quad (3.16)$$

$$R_{33} = (1-w)\sin^2\theta - rw'\sin^2\theta - \sin^2\theta + r(1-w)\sin^2\theta\epsilon\rho'. \quad (3.17)$$

The mixed Ricci tensors are given as

$$R_0^0 = -\frac{3}{2}w'\epsilon\rho' + (1-w)\epsilon^2\rho'^2 -$$
$$\frac{1}{2}w'' + \frac{2}{r}(1-w)\epsilon\rho' - \frac{w'}{r}$$
$$R_1^1 = \epsilon^2\rho'^2(1-w) - \frac{1}{2}w'\epsilon\rho' - \frac{1}{2}w'' + \frac{w'}{r}$$
$$R_2^2 = -\frac{w}{r^4} - \frac{w'}{r^3} + \frac{(1-w)}{r^3}\epsilon\rho'$$

$$R_3^3 = \frac{(1-w)}{r^2} - \frac{w'}{r} - \frac{1}{r^2} + \frac{(1-w)}{r}\epsilon\rho'. \tag{3.18}$$

The *Ricci* scalar is given as

$$\begin{aligned} R &= 2(1-w)\epsilon^2\rho'^2 + \tfrac{3(1-w)}{r}\epsilon\rho' - \tfrac{w'}{r} - \tfrac{w}{r^4} \\ &-\tfrac{1}{r^2} - w'' - 2w'\epsilon\rho' + \tfrac{(1-w)}{r^2} - \tfrac{w'}{r^3} + \tfrac{(1-w)}{r^3}\epsilon\rho'. \end{aligned} \tag{3.19}$$

Components of Einstein's mixed tensors are

$$\begin{aligned} G_0^0 &= -\tfrac{1}{2}w'\epsilon\rho' + \tfrac{1}{2r}(1-w)\epsilon\rho' + \tfrac{1}{2}w'' \\ &-\tfrac{w'}{2r} + \tfrac{w}{2r^4} + \tfrac{1}{2r^2} + \tfrac{w'}{2r^3} - \tfrac{1-w}{2r^3}\epsilon\rho' = 8\pi T_0^0 - \Lambda. \end{aligned} \tag{3.20}$$

$$\begin{aligned} G_1^1 &= \tfrac{1}{2}w'\epsilon\rho' + \tfrac{3}{2r}(1-w)\epsilon\rho' + \tfrac{3}{2}\tfrac{w'}{r} + \tfrac{w}{2r^4} \\ &+\tfrac{1}{2r^2} + \tfrac{w'}{2r^3} - \tfrac{1-w}{2r^3}\epsilon\rho' - \tfrac{1-w}{2r^2} = 8\pi T_1^1 - \Lambda. \end{aligned} \tag{3.21}$$

$$\begin{aligned} G_2^2 &= \tfrac{1-w}{2r^3}\epsilon\rho' - \tfrac{w}{2r^4} - \tfrac{w'}{2r^3} + \tfrac{1}{2r^2} - \tfrac{1-w}{2r^2} + \tfrac{w'}{2r} \\ &-\tfrac{3(1-w)}{2r}\epsilon\rho' - (1-w)\epsilon^2\rho'^2 + w'\epsilon\rho' + \tfrac{w''}{2} = 8\pi T_2^2 - \Lambda. \end{aligned} \tag{3.22}$$

$$\begin{aligned} G_3^3 &= \tfrac{w'}{2r^3} - \tfrac{1-w}{2r^3}\epsilon\rho' + w'\epsilon\rho' + \tfrac{1-w}{2r^2} - \tfrac{1-w}{2r}\epsilon\rho' \\ &-(1-w)\epsilon^2\rho'^2 - \tfrac{w'}{2r} - \tfrac{1}{2r^2} + \tfrac{w}{2r^4} + \tfrac{w''}{2} = 8\pi T_3^3 - \Lambda, \end{aligned} \tag{3.23}$$

where, $w = \frac{2m}{r} + \frac{\Lambda r^2}{3}$; $w' = \frac{2}{r}\frac{dm}{dr} - \frac{2m}{r^2} + \frac{2\Lambda r}{3}$ and $w'' = -\frac{4}{r^2}\frac{dm}{dr} + \frac{4m}{r^3} + \frac{2\Lambda}{3}$. From Eq. (3.21) and Eq. (3.22) and with the linearity condition that $\epsilon^2 \to 0$, we have at $w = 1$ (at the horizon)

$$-\frac{1}{2}w'' + \frac{w'}{r} + \frac{1}{r^4} + \frac{w'}{r^3} = 8\pi[T_1^1 - T_2^2] = 8\pi(-288a_s + 168b_s - 12c_s)\kappa_c^4. \tag{3.24}$$

Substituting a_s, b_s, c_s, w' , w'' in Eq. (3.24), we have

$$\frac{dm}{dr} - 2\frac{m}{r} = (k\kappa_c^4 - \frac{2\Lambda}{15})r^2, \tag{3.25}$$

where k is a constant. Solution to Eq. (3.25) is

$$m = (k\kappa_c^4 - \frac{2\Lambda}{15})r^3 + c, \tag{3.26}$$

where c is a constant. From the relation $m = M(1 + \epsilon\mu)$, we have $\frac{dm}{dr} = \epsilon M \frac{d\mu}{dr}$, then Eq. (3.26) is modified as

$$\epsilon\mu = \frac{(k\kappa_c^4 - \frac{2\Lambda}{15})r^3 + c}{M} - 1. \tag{3.27}$$

Applying the boundary condition, $\epsilon\mu = 0$ at the cosmic horizon, we have at the cavity $r = r_0$,

$$\epsilon\mu = \frac{(k\kappa_c^4 - \frac{2\Lambda}{15})}{M}r_0^3, \tag{3.28}$$

where r_0 is the radius of the cavity. The above result is in agreement with the York's approach [40]. Substituting Eq. (3.28) in Eq. (3.9) we get

$$S_0 = \frac{1}{4}A_c - \frac{1}{2}\{\frac{(k\kappa_c^4 - \frac{2\Lambda}{15})}{M}r_0^3\}A_c. \tag{3.29}$$

Eq. (3.29) shows that the entropy of a naked black hole is less than $\frac{1}{4}A_c$. The term S_s is the surface entropy, which is a measure of back reaction. Thus we may conclude that the cavity ensures the entropy of the cosmic horizon as $\frac{1}{4}A_c$. The expression for the surface entropy depends on the area A_c of the event horizon. This difference between the entropy of dressed and naked black holes would appear as the surface term. The Eq. (3.29) reveals the stability of the black hole. Continuous emission of Hawking radiation would finish off a black hole or would reduce the entropy thereby destabilizing the black hole. Eq. (3.29) says that any decrease in the entropy of the black hole would appear as the entropy of the surface thereby keeping the black hole stable.

In the same way, from Eq. (3.22) and Eq. (3.23) we can find the expression for $\epsilon\rho$ which is the term corresponding to the quantum field. We have

$$\epsilon\frac{d\rho}{dr} = \frac{w'}{(1-w)} - \frac{1}{r} + \frac{1}{(1-w)r} - \frac{1}{r^3(1-w)} + \frac{w'}{(1-w)r^2}. \quad (3.30)$$

Since cosmic horizon r_c is large, terms such as $\frac{1}{(1-w)r}$, $\frac{1}{r^2(1-w)}$, $\frac{w'}{(1-w)r^3}$ can be neglected. Then we get

$$\epsilon\rho = -\log(1-w) + \frac{1}{r^2}. \quad (3.31)$$

The expression for $\epsilon\rho$ can be substituted in the metric equation to find the limit of perturbation. In the coming section, the effective potential is evaluated in the presence of quantum field.

3.4 Effective potential

The effective potential of the test particles moving in static and spherically symmetric back ground geometry is determined based on the Hamilton-Jacobi approach. The metric of the Schwarzschild-de Sitter black hole in thermal equilibrium with a massless quantum field is

$$ds^2 = -[1-w]e^{2\epsilon\rho(r)}dt^2 + [1-w]^{-1}dr^2 + r^2 d\Omega^2. \tag{3.32}$$

The trajectory of the particle of mass m_0 moving in this back ground can be obtained by making use of Hamilton-Jacobi equation

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m_0^2 = 0, \tag{3.33}$$

where S denotes the action. Expanding this equation in the black hole geometry we get in the equatorial plane($\theta = \pi/2$),

$$-[1-w]^{-1}e^{-2\epsilon\rho(r)}(\frac{\partial S}{\partial t})^2 + [1-w](\frac{\partial S}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial S}{\partial \phi})^2 + 1 = 0. \tag{3.34}$$

Now to determine the $H-J$ function S, we assume that

$$S(t,r,\phi) = -\widetilde{E}(\tau) + A(r) + \widetilde{L}\phi, \tag{3.35}$$

where $\widetilde{E}$ and $\widetilde{L}$ are the constant energy and angular momentum of the test particle. Then substituting the Eq. (3.35) in the $H-J$ equation (Eq. (3.34)) we get

$$-[1-w]^{-1}e^{-2\epsilon\rho(r)}\widetilde{E}^2 + [1-w](\frac{\partial A}{\partial r})^2 + \frac{\widetilde{L}^2}{r^2} + 1 = 0. \tag{3.36}$$

From the above equation

$$\frac{\partial A}{\partial r} = \left(\frac{\widetilde{E}^2}{(1-w)^2 e^{2\epsilon\rho}} - \frac{(\widetilde{L}^2 + r^2)}{(1-w) r^2} \right)^{1/2} \tag{3.37}$$

In a time translational symmetry $\widetilde{E}$ is a constant, so $\frac{\partial S}{\partial \widetilde{E}} = \delta$ (a constant). The $H - J$ function is obtained as

$$S(t, r, \phi) = -\widetilde{E}t + \int^r \left(\frac{\widetilde{E}^2}{(1-w)^2 e^{2\epsilon\rho}} - \frac{(\widetilde{L}^2 + r^2)}{(1-w) r^2} \right)^{1/2} + \widetilde{L}\phi, \tag{3.38}$$

and therefore,

$$\frac{\partial S}{\partial \widetilde{E}} = \delta = -t + \int^r \left(\frac{\widetilde{E}^2}{(1-w)^2 e^{2\epsilon\rho}} - \frac{(\widetilde{L}^2 + r^2)}{(1-w) r^2} \right)^{-1/2} \frac{\widetilde{E}}{(1-w)^2 e^{2\epsilon\rho}}. \tag{3.39}$$

The radial velocity of the test particle is given by

$$\frac{dr(t)}{dt} = (1-w) e^{\epsilon\rho} \frac{1}{\widetilde{E}} [\widetilde{E}^2 - (1-w) e^{2\epsilon\rho} \{ \frac{\widetilde{L}^2}{r^2} + 1 \}]^{1/2}. \tag{3.40}$$

This equation governs the radii of allowed orbits of test particles in the black hole space time geometry. Then the function

$$\widetilde{V}^2(r) = (1-w) e^{2\epsilon\rho} \{ \frac{\widetilde{L}^2}{r^2} + 1 \}, \tag{3.41}$$

plays the role of the effective potential with the condition that $\widetilde{E}^2 > V(r)$. Substituting Eq. (2. 2) in Eq.(3.41) we get

$$\widetilde{V}^2(r) = [1 - \frac{2M(1+\epsilon\mu)}{r} - \frac{\Lambda}{3} r^2] e^{2\epsilon\rho} \left(\frac{\widetilde{L}^2}{r^2} + 1 \right). \tag{3.42}$$

But $\epsilon\mu = \frac{1}{M}(kk_c^4 - \frac{2\Lambda}{15})r^3$. Substituting this in the above equation we get the potential for particle as

$$\widetilde{V}^2(r) = [1 - \frac{2M}{r} - (2kk_c^4 + \frac{9\Lambda}{15})r^2](1 + 2\epsilon\rho)\left(\frac{\widetilde{L}^2}{r^2} + 1\right). \quad (3.43)$$

When photon is considered, the effective potential is

$$V^2(r) = [1 - \frac{2M}{r} - (2kk_c^4 + \frac{9\Lambda}{15})r^2](1 + 2\epsilon\rho)\left(\frac{L^2}{r^2}\right). \quad (3.44)$$

In the unperturbed situation the expressions for the above potentials are $\widetilde{V}^2(r) = [1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2]\left(\frac{\widetilde{L}^2}{r^2} + 1\right)$, $V^2(r) = [1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2]\frac{L^2}{r^2}$. In Fig. (3.1), the unperturbed potentials of quantum particle and photon are shown. The graph of particle (bold line) shows a maximum at $r = 3.2M$ and crosses zero and enters a negative region at $r = 11.5M$. In the graph for photon (narrow line) the maximum is at $r = 3M$ and crosses zero at $r = 11.5M$. In Fig.(3.2), perturbed potentials of a particle (bold line) and a photon (narrow line) are shown. The maximum potentials for the particle and photon are at $r = 2.95M$ and $r = 2.9M$ respectively and cross zero at $r = 6.75M$. The presence of a quantum field has resulted in the increase of attraction. With the condition that $54M^2\Lambda = 1$ and $\widetilde{L}^2 = 81M^2$, in the unperturbed situation, the radius of the unstable circular orbit for particle and photon are respectively at

$$r \simeq 3.2M,$$
$$r = 3M. \quad (3.45)$$

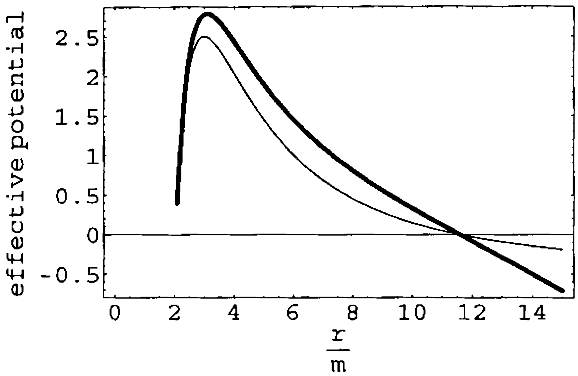

Figure 3.1: Variation of effective potential in the absence of back-reaction.

When a quantum field is present around the black hole, the mass M of naked black hole is to be replaced by mass m of dressed black hole. In this case, the particle and photon will have the unstable circular orbits at

$$r = 2.95M, \qquad r = 2.9M. \tag{3.46}$$

respectively. Both the graphs show that there are no stable orbits.

3.5 Conclusion

Hawking's evaporation eventually finishes off a black hole, but we know that black hole is stable. To have such an equilibrium, there must be a pair of balancing forces. A cavity is proposed to contain the black hole, so that Hawking radiation and quantum field exist in equilibrium by means of surface tension of the cavity. In this work, a Schwarzschild-de Sitter black hole with mass M and

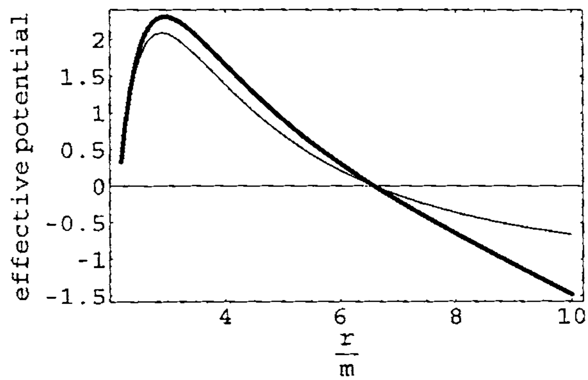

Figure 3.2: Variation of effective potential in the presence of back-reaction.

temperature $T_c = \frac{1}{12\pi M}$ surrounded by a massless quantum field in thermal equilibrium is held captive by a cavity of radius r_0. The quantum field and thermal field coexist in equilibrium in between the horizon and cavity. A cavity, enclosing the black hole and field, should be there to ensure the exact entropy. It is determined that the entropy of the dressed black hole is $\frac{1}{4}A_c$. The solution of $\epsilon\mu$ is in agreement with that of York. The change of entropy of a black hole is a measure of back reaction. Again the change in the gravitational potential of the black hole space time shows the effect of back reaction. The maximum potentials for particle and photon are at $r = 2.95M$ and $r = 2.9M$. Potentials cross zero at $r = 6.75M$ in the presence of an external quantum field. The particles and photons are easily captured if they possess enough energy to reach $r = 2.95M$ and $r = 2.9M$, otherwise they would have scattered away.

The surface entropy in anti-de Sitter-Schwarzschild black hole depends on temperature and area of horizon whereas in de Sitter-Schwarzschild case surface entropy is a function of horizon area only.

Chapter 4

Generalized second law and entropy bound in a black hole

4.1 Introduction

Things fall apart; the centre cannot hold and more anarchy is loosed upon the world.
W. B. Keats, The Second coming.

The studies on black holes during the last 30 years have brought to light strong hints of a deep and fundamental relationship among gravitation, thermodynamics and quantum theory. The cornerstone of this relationship is the black hole thermodynamics, where it appears that laws of black hole mechanics are, in fact, simply the ordinary laws of thermodynamics, at least in a theoretical perspective. There is a huge increase, of the order of 10^{20}, in entropy of a star during its gravitational collapse to become a black hole. It is presumably associated with the gravitational microstates of the black hole through the number of ways in which a black hole of a given mass 'm' and area 'A' can be formed.

It has been proposed that black hole entropy can be due to quantum entanglement between the interior and exterior states of the black hole and also that entanglement entropy S is equal to quantum corrections of Bekenstein-Hawking entropy, S_{bh}[99, 100, 101, 102]. In these arguments, the black hole entropy is related to the entanglement entropy in the QFT in the same spacetime [103, 104]. Even now calculations of black hole entropy, GSL and the conditions to conserve GSL are live subjects in black hole physics. Appropriate vacuum state for the interior of a box with reflecting walls being lowered towards a Schwarzschild black hole is the Boulware state, which warrants finite but high value for the stress-tensor [105]. In a gedanken experiment, Jenson [106] showed that GSL is valid when matter with negative gravitating energy is added to a near-extremal U(1)-charged static black hole in Einstein-Maxwell theory.

The analogy between laws of black hole mechanics and classical statistical mechanics will break down once the GSL is violated. Unruh and Wald [43, 44] analyzed this situation by performing a gedanken experiment. They considered a thought experiment of lowering a box containing matter toward a black hole taking into account of the effect of "acceleration radiation"(the effective radiation a stationary observer near a black hole would observe) [21]. The resulting change in entropy of the black hole ΔS_{bh} in the round-trip process was shown to be greater than S_s, the original entropy of the contents of the box. The existence of Hawking radiation [19] preserves the validity of the GSL because the thermal radiation is the state of matter and radiation which maximizes entropy at fixed energy and volume[44].

In order to make GSL valid, Bekenstein [50, 107] proposed a conjecture: There exists a universal upper bound on entropy S for an arbitrary system of effective radius R and energy E, which can be expressed in Planck units ($c = \hbar = G = k_B = 1$) as $\frac{S}{E} \leq 2\pi R$.

If the Hawking radiation were fully thermal, the radiation pressure at the horizon would be infinitely high, since $P(r) = \frac{1}{3}\alpha \frac{T_{bh}^4}{\chi(r)}$, where $\chi(r)$ is the red shift factor. This situation refrain us from bringing the box in the gedanken experiment to the event horizon. The concept that the thermal pressure of Hawking radiation assumes infinite value at the horizon is unwarranted as no physical pathology is believed to exist at the horizon. To overcome this problem, many have [45, 46, 47, 75] proposed stress-energy calculations and obtained a finite value for the temporal and radial components of stress-energy tensor at the horizon.

The information loss paradox in a black hole can be resolved by treating the Hawking radiation as not exactly thermal [108] and this concept will be used in this chapter. This implies that the pressure of Hawking radiation will have only a finite value at the horizon eventhough Boulware vacuum inside the box suggests high value of stress energy tensor to explain the thermal radiation. Hence the box containing matter can be brought to the horizon. The state equations of radiation in asymptotic limit are given as

$$\rho = \alpha T_r^4; s = \frac{4}{3}\alpha T_r^3, \tag{4.1}$$

where, ρ is the energy density, s is the entropy density, T_r is the temperature of radiation and α is a constant. The asymptotic state equations of radiation when applied in the calculations of gedanken experiment, it is obtained that the GSL is violated. As the sanctity of GSL cannot be questioned, the state equations of radiation (Eq. (4.1)) need to be modified.

In a gedanken experiment, a box filled with radiation is lowered on to the horizon. For an inertial observer (freely falling), out side the box, he sees a Hartle-Hawking vacuum. This vacuum has a finite positive energy density value at the horizon. But he sees a vacuum inside the box with a negative energy density which blows up at the horizon. This vacuum is called the Boulware vacuum. The interior of the box which is initially empty will acquire a negative energy density through the lowering process. This negative energy density is an outcome of Boulware vacuum. So in evaluating the buoyancy on the box by the Hawking radiation, the pressures due to the stress-energies of Boulware vacuum and Hartle-Hawking vacuum must be taken into consideration.

The knowledge that the Hawking radiation near the horizon is not fully thermal, leads us to the conjecture that the gravitational field near the horizon can influence the equations of state of radiation. The state equations of radiation near the Schwarzschild black hole were earlier studied [109]. In this chapter we discuss the acceptability of general state equations of radiation near the horizon of a Reissner-Nordström (RN) black hole. The spacetime around the RN black hole is static and spherically symmetric so that the Ricci scalar R is zero but $R_{ab} \neq 0$. But in a Schwarzschild black hole both R and R_{ab} equal to zero. The scheme of the chapter is as follows. In Sec.

4.2, we describe the violation of GSL where ordinary equations of radiation are used. In Sec. 4.3, new equations of radiation and the upper bound are given . In Sec. 4.4, we give the conclusion.

4.2 Violation of GSL?

In a gedanken experiment, a box filled with matter or radiation is brought from infinity to the horizon and the bottom lid is opened so that the contents are released to the black hole. The box is then filled with Hawking radiation and is lifted back to infinity. In this process, we can determine the gain of entropy of the black hole as matter is swallowed and the loss of entropy as the Hawking radiation is lost. So, whether the loss of entropy is greater than the gain of entropy is the bone of contention in the study of GSL.

A RN black hole with mass M and charge Q is situated inside a spherical cavity with radius r_0 greater than r_h, negligible mass and perfect reflectability. Let us imagine that the black hole and Hawking radiation be in thermal equilibrium in the cavity. We fill a rectangular box of volume aA (a the height and A the cross section area of the box) with thermal radiation of temperature T_r at infinity. Now lower the box (Fig.4.1)adiabatically through a hole on the cavity to the horizon, release the contents, then slowly raise the box back to infinity. In general $T_r \gg T_{bh}$. The increase in the energy of the black hole in the above process is [43, 44]

$$\varepsilon = E_r - W_\infty, \tag{4.2}$$

where, W_∞ is the work delivered to infinity and E_r is the rest energy

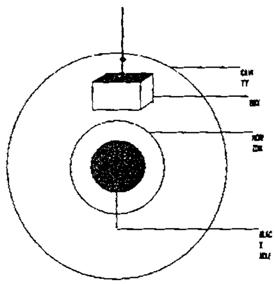

Figure 4.1: Gedankenexperiment: Black hole is kept inside a cavity and a box filled with radiation is brought to the horizon.

of radiation in the box. This increase in the energy of the black hole manifests as the increase in the entropy. We have

$$E_r = \alpha a A T_r^4$$
$$W_\infty = W_1 - W_2, \tag{4.3}$$

where, E_r is the energy of radiation in the box, W_1 is the work delivered to infinity on account of the weight of box and radiation and W_2 the work delivered to the black hole on account of the buoyancy force of Hawking radiation. The entropy of radiation inside the box is

$$S_r = \frac{4}{3}\alpha a A T_r^3. \tag{4.4}$$

Since the process of lowering and raising the box is adiabatic, S_r remains constant, since no heat exchange takes place in an adiabatic process .

4.2.1 Calculation of W_1

W_1 is the energy delivered to infinity as the box is dropped on to the horizon under the action of the gravitational force of the black hole and may be given as

$$W_1 = E_r - E;$$
$$E = A \int_l^{l+a} \rho(x)\chi(x)dx. \tag{4.5}$$

When the box is brought to the horizon, the bottom lid of the box is opened so that the radiation in the box will be in contact with the Hawking radiation. Then, E is the energy of the radiation inside the box after it has attained the thermal equilibrium with the Hawking radiation near the horizon, l is the distance from the horizon to the bottom of box, $\chi(x)$ is the red shift factor and x is the proper distance from the horizon to the box. Under thermodynamic equilibrium between acceleration radiation and radiation inside the box at a height l, the temperature of radiation becomes $T_0(l)$. Then we have

$$\rho(x) = \alpha T_{loc}^4$$
$$\chi(x) = [1 - \frac{2M}{r(x)} + \frac{Q^2}{r^2(x)}]^{1/2}, \tag{4.6}$$

where T_{loc} is the temperature of acceleration radiation locally. T_{loc} is related to the equilibrium temperature $T_0(l)$ as [110]

$$T_{loc} = \frac{T_0(l)}{\chi(x)}. \tag{4.7}$$

On the horizon, $T_0(l=0) = T_{bh}$. For $l \ll r_h$ and writing $r = r_h + x$, $\chi(x)$ in Eq. (4.6) may be modified as

$$\chi(x) = \frac{2^{1/2}(M^2-Q^2)^{1/4}}{r_h} x^{1/2}. \tag{4.8}$$

$\chi(x)$ will be zero at $x = 0$, i.e., on the horizon. Eq. (4.8) is a reasonably good approximation of the metric function inside the cavity in which the black hole is situated. On substituting Eqs. (4.6, 4.8) in the expression for E in Eq. (4.5), we get

$$E = \frac{\alpha A r_h^3}{\sqrt{2l}} \frac{T_0^4}{(M^2-Q^2)^{3/4}}. \tag{4.9}$$

Similarly, the entropy of the contents of the box is

$$\begin{aligned} S_r &= \frac{4}{3}\alpha T_0^3 A \int_i^{l+a} \frac{dx}{\chi(x)^3} \\ &= \frac{4\alpha A r_h^3}{3\sqrt{2l}} \frac{T_0^3}{(M^2-Q^2)^{3/4}}. \end{aligned} \tag{4.10}$$

But in an adiabatic process entropy never changes. So

$$S_r = \tfrac{4}{3}\alpha a A T_r^3. \tag{4.11}$$

From (4.10) and (4.11), the equilibrium temperature is obtained as

$$T_0 = (2l)^{1/6} a^{1/3} (M^2 - Q^2)^{1/4} \frac{T_r}{r_h}. \tag{4.12}$$

From Eq. (4.9), energy of the radiation after attaining thermal equilibrium is obtained as

$$E = (2l)^{1/6} \alpha a^{4/3} A (M^2 - Q^2)^{1/4} \frac{T_r^4}{r_h}. \tag{4.13}$$

But, $E_r = aA\alpha T_r^4$, which is the energy of thermal radiation in the box. So

$$E = \frac{(2l)^{1/6}a^{1/3}(M^2 - Q^2)^{1/4}}{r_h}E_r. \tag{4.14}$$

Now work done to infinity on account of gravity is

$$W_1 = E_r - E = E_r - \frac{(2l)^{1/6}a^{1/3}(M^2 - Q^2)^{1/4}}{r_h}E_r. \tag{4.15}$$

The work W_1 depends on the distance l from the horizon to the bottom of the box.

4.2.2 Calculation of W_2

The work done on the black hole on account of the buoyancy of Hawking radiation is [43, 44]

$$W_2 = A\int_l^{l+a} P(x)\chi(x)dx, \tag{4.16}$$

where, $P(x)$ is the pressure of Hawking radiation. If the Hawking radiation is fully thermal, then

$$P(x) = \frac{1}{3}\alpha T_{loc}^4 = \frac{1}{3}\alpha\frac{T_{bh}^4}{\chi^4(x)}. \tag{4.17}$$

So, at the horizon, $P(x) \to \infty$. This makes $W_2 \to \infty$, which means the box cannot be dropped on to the horizon. If the pressure is finite, the box can be brought to the horizon. In classical gravity, the geometry is treated classically while matter fields are quantized. In examining the semiclassical perturbations of the RN metric caused by the vacuum energy of the quantized scalar fields, we can treat the background electromagnetic field as a classical field. The right hand side of the semiclassical Einstein equations will then contain both classical and quantum stress-energy contributions

$$G^{\mu}_{\nu} = 8\pi[T^{\mu}_{\nu} + \langle T^{\mu}_{\nu}\rangle]. \tag{4.18}$$

T^{μ}_{ν} represents the classical stress-energy tensor of scalar field and $\langle T^{\mu}_{\nu}\rangle$ is its quantum counterpart. Now consider the situation where the black hole is in thermal equilibrium with the quantized field, so that the perturbed geometry continues to be static and spherically symmetric. To first order in $\epsilon = \frac{\hbar}{M^2}$, the general form of the perturbed RN metric may be written as:

$$ds^2 = -[1 + 2\epsilon\rho(r)]f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2, \tag{4.19}$$

where, $f = (1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2})$ and $[1 + 2\epsilon\rho(r)]$ represents the perturbation due to the scalar field. In order to save the black hole from extinction due to evaporation, the black hole is assumed to be placed inside a massless reflecting spherical shell. Inside the shell the quantum field and Hawking radiation are in thermodynamic equilibrium and hence the black hole mass function $m(r)$ contains classical mass and the quantum first-order perturbation. So [111]

$$m(r) = M[1 + \epsilon\mu(r)]. \tag{4.20}$$

This equation explains the back reaction. The metric perturbation functions, $\rho(r)$ and $\mu(r)$ are determined by solving the semi-classical Einstein's equation expanded to first order in ϵ [111].

$$\frac{d\mu}{dr} = -\frac{4\pi r^2}{M\epsilon}\langle T^t_t\rangle,$$

$$\frac{d\rho}{dr} = \frac{4\pi r}{\epsilon} f^{-1}[\langle T_r^r \rangle - \langle T_t^t \rangle]. \tag{4.21}$$

The right hand side of Eq. (4.21) is divergent on the horizon unless $[\langle T_r^r \rangle - \langle T_t^t \rangle]$ vanishes there. Not only that both $\langle T_r^r \rangle$ and $\langle T_t^t \rangle$ must be finite at the horizon. The expectation value of stress-energy tensor of a quantized massive scalar field in the RN spacetime is given as [111]

$$\langle T_r^r \rangle \mid_{r_h} = \langle T_t^t \rangle \mid_{r_h} \cong \frac{6\pi^3 \epsilon}{105 \bar{m}^2} T_{bh}^4, \tag{4.22}$$

where, $\bar{m}$ the mass of scalar field and $\epsilon = \frac{\hbar}{M^2}$. Hence, a probable form of the Hawking pressure at the horizon is given by Eq. (4.22) multiplied by a constant α.

But, it has been shown that [105], a box which is static on the horizon suffers the pressure of acceleration radiation and it induces a Boulware state inside box. In particular, the Hartle-Hawking state has been used in the computation of the renormalized stress-energy tensor in Eq. (4.22) and it is valid only for a freely falling observer near the horizon, not for the stationary (that is, accelerating) box. Therefore the effect of acceleration radiation also needs to be taken into consideration in the calculation of work W_2.

Let the vacuum state inside the box is Boulware ($B - state$) and that outside is Hartle-Hawking ($H - H$). Considering the contributions of radiation pressures on the top and bottom of the box, the pressure on each side will be the difference of the pressure of $H - H$ vacuum on outside and pressure of $B - vacuum$ on the inside [105].

So we get

$$P_{net} = P_{HH} - P_B \tag{4.23}$$

The pressure inside the box due to $B - vacuum$ is given as [112]

$$T_t^t(B) = kT_{bh}^4(\frac{r_+}{r})^6[\frac{A_t^t}{(1-\frac{r_\pm}{r})^2} + B_t^t] \tag{4.24}$$

where A_t^t and B_t^t are finite tensors and $r = (r_+ + l)$, where l is the proper distance of the box from the horizon. Hence Eq. (4.24) is modified as

$$T_t^t(B) = kT_{bh}^4(\frac{r_+}{r_+ + l})^6[A_t^t\frac{(r_+ + l)^2}{l^2} + B_t^t] \tag{4.25}$$

So the net force to be applied to bring the box to the horizon is

$$P_{net} = \alpha\frac{6\pi^3\epsilon}{105\bar{m}^2}T_{bh}^4 + \alpha kT_{bh}^4[\frac{r_+^6}{(r_+ + l_T)^6}(A_t^t\frac{(r_+ + l_T)^2}{l_T^2} + B_t^t) - \frac{r_+^6}{(r_+ + l_B)^6}(A_t^t\frac{(r_+ + l_B)^2}{l_B^2} + B_t^t)] \tag{4.26}$$

where, l_T and l_B are the proper distances of the top and bottom of the box from the horizon. By using Eq. (4.16), we now get

$$W_2 = \frac{8(M^2 - Q^2)^{1/4}\pi^3\epsilon a^{3/2}A}{105\sqrt{2}\bar{m}^2 r_H}\alpha T_{bh}^4 + \frac{4(M^2 - Q^2)^{1/4}a^{3/2}A\alpha kT_{bh}^4}{3\sqrt{2}r_h}[\frac{r_+^6}{(r_+ + l_T)^6} (A_t^t\frac{(r_+ + l_T)^2}{l_T^2} + B_t^t) - \frac{r_+^6}{(r_+ + l_B)^6}(A_t^t\frac{(r_+ + l_B)^2}{l_B^2} + B_t^t)]. \tag{4.27}$$

The increase in the energy of the black hole in the gedanken experiment is obtained from Eqs. (4.2,4.3,4.15,4.27) as

$$\varepsilon = E_r - W_\infty = E_r - W_1 + W_2,$$

$$\varepsilon = \frac{(2l)^{1/6}a^{1/3}(M^2-Q^2)^{1/4}}{r_h}E_r + \frac{8(M^2-Q^2)^{1/4}\pi^3\epsilon a^{3/2}A}{105\sqrt{2}\bar{m}^2 r_h}\alpha T_{bh}^4$$
$$+\frac{4(M^2-Q^2)^{1/4}a^{3/2}A\alpha k T_{bh}^4}{3\sqrt{2}r_h}[\frac{r_+^6}{(r_++l_T)^6}(A_t^t\frac{(r_++l_T)^2}{l_T^2}+B_t^t) -$$
$$\frac{r_+^6}{(r_++l_B)^6}(A_t^t\frac{(r_++l_B)^2}{l_B^2}+B_t^t)] \quad (4.28)$$

On the horizon, $l \simeq 0$, for the H_H vacuum, but $l_B \neq 0$ for the B vacuum because of the thickness of the box. Hence

$$\varepsilon = \frac{8(M^2-Q^2)^{1/4}\pi^3\epsilon a^{3/2}A}{105\sqrt{2}\bar{m}^2 r_h}\alpha T_{bh}^4 +$$
$$\frac{4(M^2-Q^2)^{1/4}a^{3/2}A\alpha k T_{bh}^4}{3\sqrt{2}r_h}[\frac{r_+^6}{(r_++l_T)^6}(A_t^t\frac{(r_++l_T)^2}{l_T^2}+B_t^t) -$$
$$\frac{r_+^6}{(r_++l_B)^6}(A_t^t\frac{(r_++l_B)^2}{l_B^2}+B_t^t)] \quad (4.29)$$

The increase of the black hole entropy in the gedankenexperiment may be given as,

$$\Delta S_{bh} = \frac{8(M^2-Q^2)^{1/4}\pi^3\epsilon a^{3/2}A\alpha T_{bh}^3}{105\sqrt{2}\bar{m}^2 r_h} +$$
$$\frac{4(M^2-Q^2)^{1/4}a^{3/2}A\alpha k T_{bh}^3}{3\sqrt{2}r_H}[\frac{r_+^6}{(r_++l_T)^6}(A_t^t\frac{(r_++l_T)^2}{l_T^2}+B_t^t) -$$
$$\frac{r_+^6}{(r_++l_B)^6}(A_t^t\frac{(r_++l_B)^2}{l_B^2}+B_t^t)] \quad (4.30)$$

Since, $T_{bh} \ll T_r$ we find, $\Delta S_{bh} \ll S_r$. This is a violation of GSL, but GSL is more or less a universal law, hence must be conserved. In the above calculations, we took, $\rho = \alpha T_{loc}^4$ and $s = \frac{4}{3}\alpha T_{loc}^3$, which are not true, near the horizon. These equations don't prevail, unless the Hawking radiation is fully thermal. So we would expect a modified state equations of radiation under gravity.

4.3 State equations of radiation

As Hawking radiation is not fully thermal, the buoyancy would be finite and hence the box can be brought to the horizon. Since the gravity is very strong near the horizon, the equations of radiation near the horizon would be affected. By the first law of thermodynamics, we have

$$d(\rho V) = T_{loc} ds - p dV, \tag{4.31}$$

where, P is the pressure of thermal radiation, ρ is the energy density, s is the entropy density and V is the volume of the box. The local temperature of Unruh radiation is T_{loc}. The above equation yields [43]

$$\begin{aligned} \rho + p &= sT_{loc}, \\ dp &= sdT_{loc}. \end{aligned} \tag{4.32}$$

For a static spacetime, the hydrostatic equilibrium equation, derived from $\nabla^a T_{ab} = 0$, for a perfect fluid stress-energy tensor [43] is

$$\begin{aligned} \nabla_a p &= (\rho + p)[\frac{\zeta^b}{\chi}]\nabla_b[\frac{\zeta_a}{\chi}] \\ &= -(\rho + p)\frac{1}{\chi}\nabla_a \chi, \end{aligned} \tag{4.33}$$

where, ζ^a is a static Killing vector field. Since the Hawking radiation satisfies the hydrostatic equilibrium, from Eq. (4.31), we have

$$\frac{d[\chi(x)p]}{dx} = -\rho(x)\frac{d\chi(x)}{dx}, \tag{4.34}$$

where, $\chi(x)$, is the metric function close to the horizon. In the flat space situation, the relation connecting ρ and s is given as

$$s_r = \frac{4}{3}\frac{1}{T_r}\rho_r. \tag{4.35}$$

The term $\frac{4}{3}\frac{1}{T_r}$ is the proportionality term connecting ρ and s, which can be expressed as $C(\infty)$. This term is not a constant, but a parameter that depends on the distance from horizon and may be expressed as $C(l)$, where l is the distance from the horizon to the bottom of the box. In the spacetime of black hole, red shift factor also must be taken into account. Therefore, we may propose that [109]

$$s = C(l)\rho(x)\chi(x). \tag{4.36}$$

This relation will converge to the flat space situation when there is no gravity. Substituting Eq. (4.36) in Eq. (4.32), we get

$$\begin{aligned} \rho(x) + p = C(l)\rho(x)\chi(x)T_{loc} = C(l)\rho(x)T_0; \\ p = \rho(x)[C(l)T_0 - 1]. \end{aligned} \tag{4.37}$$

From Eqs. (4.34) and (4.37), we get the expressions of radiation in the context of RN black hole as

$$\rho(x) = \rho_0\chi^{\frac{-CT_0}{(CT_0-1)}}$$

$$s(x) = C(l)\rho_0 \chi^{\frac{-1}{(CT_0-1)}}. \tag{4.38}$$

Eq. (4.38) represents the modified state equations of radiation and are more realistic in explaining the physical situation near the horizon. If ρ_0 is the energy density in the asymptotic limit and in the asymptotic limit $\chi(\infty) = 1$, then

$$\begin{aligned} \rho(\infty) &= \rho_0 \\ s(\infty) = C(\infty)\rho_0 &= \frac{4}{3}\frac{1}{T_r}\rho_0. \end{aligned} \tag{4.39}$$

Eq. (4.38) converges to flat spacetime equations (Eq. (4.1)), as $\chi(\infty) \rightarrow 1$. The state equation of radiation in the context of Schwarzschild black hole had been utilized in calculating the entropy of self-gravitating radiation systems [113]. As we approach the horizon, $\chi \rightarrow 0$, hence the energy density increases but never becomes infinity because of the thickness of the box. From Eq. (4.38) and $S_r = \frac{4}{3}\alpha T_r^3 aA$, it can be shown that

$$C(l)\chi^{\frac{-1}{(CT_0-1)}} = \frac{4}{3}\frac{1}{T_r}. \tag{4.40}$$

The *R.H.S* of Eq. (4.40) is a constant. As $l \rightarrow 0$, both χ and $\frac{1}{CT_0-1} \rightarrow 0$. $C(l)$ increases as we approach the horizon and on the horizon, $C(l \rightarrow 0) = \frac{4}{3}\frac{1}{T_{bh}}$.

4.3.1 Generalized second law

In calculating the entropy change of black hole, we have earlier considered the flat spacetime equations of radiation. Now we will evaluate W_1 with the new equations of radiation. We have

$$\begin{aligned} W_1 &= E_r - E; \\ E &= A\int_l^{l+a} \rho(x)dx \\ &= A\rho_0 \int_l^{l+a} \chi^{-\xi}dx, \end{aligned} \qquad (4.41)$$

where, $\xi = \frac{CT_0}{(CT_0-1)}$. By substituting Eq. (4.8) and Eq. (4.38) in Eq. (4.41), we get

$$E = \frac{1}{(2-\xi)}\frac{2aA\rho_0}{[4(M^2-Q^2)]^{\xi/4}}[r_H/\sqrt{a}]^{\xi}. \qquad (4.42)$$

The entropy may be calculated as

$$\begin{aligned} S_r = A\int_l^{l+a} sdx &= A\int_l^{l+a} C(l)\rho(x)\chi(x)dx \\ &= AC(l)\rho_0 \int_l^{l+a} \chi^{1-\xi}dx. \end{aligned} \qquad (4.43)$$

Eq. (4.43) is evaluated using Eq. (4.8) near the horizon as ($l \ll r_H$)

$$S_r = \frac{1}{(3-\xi)}\frac{2aAC(l)\rho_0}{[4(M^2-Q^2)]^{\frac{\xi-1}{4}}}[r_h/\sqrt{a}]^{\xi-1}. \qquad (4.44)$$

But entropy can also be written as, $S_r = \frac{4}{3}aA\alpha T_r^3$. Equating this equation with Eq. (4.44) and evaluating for ρ_0, we get

$$\rho_0 = \frac{(3-\xi)[4(M^2-Q^2)]^{\frac{\xi-1}{4}}}{\frac{3}{2}C(l)(\frac{r_h}{\sqrt{a}})^{\xi-1}}\alpha T_r^3. \qquad (4.45)$$

In the asymptotic limit, $C(\infty) = \frac{4}{3}\frac{1}{T_r}$. We can calculate the asymptotic value of energy density ρ_0 by using the relation, $\xi(\infty) =$

$\frac{C(\infty)T_0}{C(\infty)T_0-1} \simeq 0$, considering the fact that $T_r \gg T_0$. Substituting $C(\infty)$and $\xi(\infty)$ in Eq. (4.45), we get

$$\rho_0 = \frac{3}{2}\frac{r_h/\sqrt{a}}{[4(M^2-Q^2)]^{1/4}}\alpha T_r^4. \quad (4.46)$$

But $\frac{r_H/\sqrt{a}}{[4(M^2-Q^2)]^{1/4}}$ is a dimensionless constant and it may be absorbed in $\frac{3}{2}$. Now substitute ρ_0 in Eq. (4.42)

$$E = \frac{3}{2}\frac{\alpha T_r^4}{(2-\xi)}\frac{2aA}{[4(M^2-Q^2)]^{(\xi+1)/4}}[r_h/\sqrt{a}]^{\xi+1}. \quad (4.47)$$

As $l \to 0$, $\xi(l \to 0) = \frac{4/(3T_{bh})T_0}{(4/(3T_{bh})T_0-1)} \simeq 1$. Energy of radiation near the horizon is obtained from Eq. (4.47) as

$$E = 3\frac{[r_h/\sqrt{a}]^2}{[4(M^2-Q^2)]^{1/2}}aA\alpha T_r^4. \quad (4.48)$$

The term $\frac{[r_h/\sqrt{a}]^2}{[4(M^2-Q^2)]^{1/2}}$ is a dimensionless constant. Had we taken the asymptotic expressions in calculating the energy of radiation near the horizon, the value would have been approximately zero. Now in Eq. (4.28)

$$\begin{aligned}\varepsilon = \frac{3[r_h/\sqrt{a}]^2 aA\alpha T_r^4}{[4(M^2-Q^2)]^{1/2}} + \frac{8(M^2-Q^2)^{1/4}\pi^3\epsilon a^{3/2}A\alpha T_{bh}^4}{105\sqrt{2}\bar{m}^2 r_h} \\ + \frac{4(M^2-Q^2)^{1/4}a^{3/2}A\alpha k T_{bh}^4}{3\sqrt{2}r_h}[\frac{r_+^6}{(r_++l_T)^6} \\ (A_t^t\frac{(r_++l_T)^2}{l_T^2}+B_t^t) - \frac{r_+^6}{(r_++l_B)^6}(A_t^t\frac{(r_++l_B)^2}{l_B^2}+B_t^t)]. \quad (4.49)\end{aligned}$$

The entropy change of the black hole in the round trip process is

$$\Delta S_{bh} = \frac{3[r_h/\sqrt{a}]^2 aA\alpha T_r^4}{[4(M^2-Q^2)]^{1/2}T_{bh}} + \frac{8(M^2-Q^2)^{1/4}\pi^3\epsilon a^{3/2}AT_{bh}}{105\sqrt{2}\bar{m}^2 r_h}T_{bh}$$
$$+\frac{4(M^2-Q^2)^{3/2}a^{3/2}A\alpha kT_{bh}^3}{3\sqrt{2}r_h}[\frac{r_+^6}{(r_++l_T)^6}$$
$$(A_t^t\frac{(r_++l_T)^2}{l_T^2}+B_t^t)-\frac{r_+^6}{(r_++l_B)^6}(A_t^t\frac{(r_++l_B)^2}{l_B^2}+B_t^t)] \quad (4.50)$$

The entropy of thermal radiation is $\frac{4}{3}\alpha T_r^3$. Eq. (4.50) says that the increase in the entropy of the black hole is greater than $\frac{4}{3}T_r^3$. So the GSL is conserved.

4.3.2 Upper bound on S/E

The upper bound on the entropy was identified as a necessary condition to conserve the GSL. Hence it is desirable to look into the verification of the upper bound on S/E. We have from Eqs. (4.42, 4.44)

$$\frac{S}{E} = \frac{(2-\xi)}{(3-\xi)}C(l)[4(M^2-Q^2)]^{1/4}\frac{\sqrt{a}}{r_h}. \quad (4.51)$$

RN Black hole temperature is given as

$$T_{bh} = \frac{\sqrt{M^2-Q^2}}{2\pi r_h^2}, \quad (4.52)$$

where, M is the mass, Q is the charge and r_h is the horizon radius of the black hole. From Eq. (4.51)

$$\frac{3-\xi}{2-\xi} = \frac{E}{S}C(l)T_{bh}\frac{4\pi\sqrt{a}r_h}{\sqrt{2}(M^2-Q^2)^{1/4}}. \quad (4.53)$$

Near the horizon, $C(l \to 0) = \frac{4}{3}\frac{1}{T_{bh}}$. Eq. (4.53) is modified with the situation $\frac{3-\xi}{2-\xi} > 1$, as

$$1 < \frac{E}{S}\frac{4}{3}\frac{4\pi\sqrt{a}r_h}{\sqrt{2}(M^2-Q^2)^{1/4}}$$
$$\frac{S}{E} < \frac{4}{3}\frac{4\pi\sqrt{a}r_h}{\sqrt{2}(M^2-Q^2)^{1/4}}. \quad (4.54)$$

Dimensionally, this formula is of the Bekenstein form [107, 50]. The Bekenstein upper bound is $S/E \leq 2\pi R$. Dimensionally, $\frac{r_h}{\sqrt{M^2-Q^2}} = L^{1/2} \equiv a^{1/2}$. Hence Eq. (4.54) may be written as $S/E \leq 2\pi a$, where a is the dimension of the box.

4.4 Conclusion

Generalized second law must be valid in all situations. When evaluating the GSL, if the asymptotic state equations of radiation are considered, the GSL will be violated. Since the Hawking radiation is not fully thermal, the gedanken experiment could be conducted close to the horizon, as the buoyancy force of Hawking radiation is finite at the horizon. The gravity is so strong near the horizon that the state equations of radiation must have been affected by it. Here we have obtained the state equation of radiation near the horizon of a Reissner-Nordström black hole and found that the GSL is conserved. In the asymptotic limit, the equations converge to the usual expressions αT_r^4 and $\frac{4}{3}\alpha T_r^3$. The parameter $C(l)$ connecting the entropy and energy density is $\frac{4}{3}\frac{1}{T_r}$ in the asymptotic limit and $\frac{4}{3}\frac{1}{T_{bh}}$ near the horizon.

In the above calculation, the upper bound on S/E is analogous to the one given by Bekenstein. The upper bound on S/E is a necessary condition to have the conservation of GSL. The above procedure has a slight disadvantage that the Eq. (4.38), doesn't give the exact value of ρ and s on the horizon because of the wrong coordinate. The correct equation will be obtained only in the absence of coordinate singularity and will be initiated somewhere else.

Chapter 5

Thermodynamics and entropy of self gravitating radiation systems (SGRS)

5.1 Introduction

Symmetry, as wide or as narrow as you may define it, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.
Hermann Weyl, Symmetry.

The theory of classical gravity warps up regions of spacetime in the vicinity of a black hole so as to produce surfaces that act as one-way membranes called event horizons. The event horizon, which is said to be the outer edge of a black hole has a radius that depends on the three parameters of the black hole such as mass, charge and angular momentum. Cosmic censorship says that the singularity of a black hole should not be naked to an outside observer, i.e., the horizon should hide the singularity. The event horizon actually restricts us from knowing the whereabouts of what had gone into the black hole. This constraint compels us to believe that there must have some fundamental relationship between gravity, horizon and thermodynamics, since thermal radiation carries no information because of its random character. The presence of event horizon or one way membrane is not necessarily an outcome of gravitational collapse since horizons can exist even in Minkowski spacetime for an accelerated observer. Rindler frame describes a uniformly accelerated frame of reference. Rindler spacetime has a non-compact surface as the event horizon, which is coordinate dependent. [75, 114].

Temperature is implicit in the Rindler spacetime and this leads to the study of QFT in this spacetime. The Rindler metric is symmetric under time reversal and there exists a natural definition of a time symmetric vacuum state. We can associate an *entropy* with such spacetime, since notion of temperature already exists. Having a notion of temperature, there are two ways of defining the entropy: (1) The partition function $Z(\beta)$ of the canonical ensemble of systems with temperature β^{-1}, is related to the entropy S and energy E by $Z(\beta) \propto e^{S-\beta E}$. (2) In thermodynamics, we have $dS = dE/T(E)$. So the situation that entropy can be determined by the above two methods led to the unification of thermodynamics with mechanics.

In the case of time symmetric vacuum state like Rindler spacetime, there will be no change of entropy dS and the thermodynamic method cannot be used to define the entropy. But, we can construct a canonical ensemble of a class of spacetimes with fixed value for β

and evaluate the partition function $Z(\beta)$. Knowing $Z(\beta)$, we can calculate S. For the Rindler spacetime with a horizon, the partition function has the form $Z \propto e^{S-\beta E}$, where the entropy per unit transverse area turns to be 1/4 while the energy is zero. Mathematically there is no distinction between the horizons which arise in the Schwarzschild, de Sitter, and Rindler spacetimes. Rindler and de Sitter spacetimes are natural choices which exhibit temperature and entropy [115].

In relativistic physics, Rindler coordinate chart is an important coordinate chart representing part of flat spacetime, also called the Minkowski vacuum. In special relativity, a uniformly accelerating particle undergoes hyperbolic motion. For each such particle, a Rindler frame can be chosen in which it is at rest. The interest in Rindler spacetime lies in its similar geometrical structure with the Schwarzschild black hole (SBH) near the horizon and it can mimic the gravitational collapse. Unruh effect is at the heart of the Hawking effect and the Hawking radiation in the SBH resembles with the Unruh radiation in the Rindler spacetime [21, 116, 117].

The entropy of $SGRS$ with *mass* as the only parameter is shown to be the entropy of the black hole into which the $SGRS$ of mass M would collapse [113]. The motivation for the present work is to know the temperature of horizon, by measuring the temperature of scalar field in thermal equilibrium with horizon and to study the collapse of $SGRS$ in the Rindler frame and hence to calculate the entropy and the entropy bound. The $SGRS$ may be assumed to be a scalar field in the Rindler spacetime so that it will transit through the horizon as if it is collapsed. In Sec. 5.2, the thermodynamics of Euclidean, Rindler and Schwarzschild spacetimes are investigated.

The temperature of scalar field near the horizon of Rindler horizon is also calculated in this section. In Sec. 5.3, the structure of Rindler spacetime and how scalar field would behave are given. In Sec. 5.4, the entropy of $SGRS$ and entropy bound are calculated. In Sec. 5.5, we give the conclusion.

5.2 Thermodynamics of different spacetimes

5.2.1 Euclidean spacetime

The close link between gravity and thermodynamics can be shown mathematically based on the relationship between temperature and the Euclidean extension of spacetime. The mean value of a dynamical variable $f(q)$ in quantum statistical mechanics can be expressed in the form

$$< f >= \frac{1}{Z}\sum_{E}\int \phi_E^*(q)f(q)\phi_E(q)e^{-\beta E}dq, \tag{5.1}$$

where $\phi_E(q)$ is the stationary state eigen function of Hamiltonian with $H\phi_E(q) = E\phi_E(q)$, $\beta = \frac{1}{T}$ is the inverse temperature and $Z(\beta)$ is the partition function. This expression calculates the mean value $< E|f|E >$ in a given energy state and then averages over a Boltzmann distribution of energy states with the weightage $\frac{1}{Z}e^{-\beta E}$. The quantum mechanical kernel giving the probability amplitude for the system to go from the state q at time $t = 0$ to the state q' at time t is given by

$$K(q', t; q, 0) = \sum_{E} \phi_E^*(q')\phi_E(q)e^{-itE}. \tag{5.2}$$

Comparing Eqs. (5.1, 5.2), the thermal average in Eq. (5.1) can be obtained as

$$< f >= \frac{1}{Z} \int dq K(q', -i\beta; q, 0) f(q), \tag{5.3}$$

in which the following have been done: (1) The time coordinate has been analytically continued to imaginary values with $it = \tau$. (2) The system is assumed to exhibit periodicity in the imaginary time τ with period β in the sense that the state variable q has the same values at $\tau = 0$ and at $\tau = \beta$. This can be extended to quantum field theory with q denoting the field.

Spacetime with horizon possesses natural analytic continuation from Minkowski signature to the Euclidean signature with $t \to \tau = it$. If the metric is periodic in τ, then a temperature associates naturally with such a spacetime. This is the basis of the thermodynamics of a spacetime under gravity.

5.2.2 Rindler spacetime

It is possible to introduce coordinate charts in Minkowski spacetime such that regions are separated by horizons, an example being the coordinate system used by a uniformly accelerated frame (Rindler frame) which has a non-compact horizon. The natural coordinate system (t, x, y, z) used by an observer moving with a uniform acceleration g along the x- axis is related to the inertial frame (T, X, Y, Z) through the relation [118]

$$\begin{gathered} gT = \sqrt{1 + 2gx} \sinh(gt), \\ (1 + gX) = \sqrt{1 + 2gx} \cosh(gt); Y = y; Z = z. \end{gathered} \tag{5.4}$$

The metric in the accelerated frame will be given as

$$ds^2 = f(x)dt^2 - \frac{dx^2}{f(x)} - dy^2 - dz^2, \tag{5.5}$$

where $f(x) = (1 + 2gx)$. This metric has a horizon at $x = -\frac{1}{2g}$ with the surface gravity $\kappa = g$ and temperature $T = \frac{g}{2\pi}$. Near $x = x_H$, $f(x)$ can be expanded in a Taylor series and obtain $f(x) = B(x - x_H)$ where $B = f'(x_H)$. The surface gravity on the horizon, $\kappa = \frac{1}{2}|f'(x_H)| = \frac{1}{2}|B|$. Near the horizon, $f(x) = 2\kappa(x - x_H)$. By introducing the transformation, $\xi = [2\kappa^{-1}(x - x_H)]^{1/2}$, the metric near the horizon becomes

$$ds^2 = \xi^2\kappa^2 dt^2 - d\xi^2 - dy^2 - dz^2. \tag{5.6}$$

Wick rotation $t \to \tau = it$ leads to

$$-ds^2 = \xi^2 d(\kappa\tau)^2 + d\xi^2 + dy^2 + dz^2, \tag{5.7}$$

which is essentially the metric in the polar coordinates in the $\tau - \xi$ plane. For this metric to be well defined near the origin, $\kappa\tau$ should behave like an angular coordinate θ with periodicity 2π. Now the periodicity in τ is $2\pi/\kappa$. So the temperature of the Rindler spacetime can be written as

$$T = \frac{\kappa}{2\pi} = \frac{g}{2\pi}. \tag{5.8}$$

This implies that an accelerating frame in a Minkowski space exhibits temperature naturally. This is called Unruh effect.

5.2.3 Schwarzschild metric near the horizon

The metric of Schwarzschild black hole near the horizon can be written with the assumptions, $r = r_h + x$ and $dr = dx$ and $1 - \frac{2m}{r} = 1 - \frac{r_h}{r_h+x} \simeq \frac{x}{r_h}$, as

$$ds^2 = -\frac{x}{r_h}dt^2 + \frac{r_h}{x}dx^2 + r_h^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.9}$$

The term, $r_h^2(d\theta^2 + \sin^2\theta d\phi^2)$ can be shown to be $dx^2 + dy^2 + dz^2$ and dx^2 can be neglected in comparison with $\frac{r_h}{x}dx^2$. We now propose a transformation of the type

$$\rho = 2\sqrt{r_h}x^{1/2}. \tag{5.10}$$

So, we get, $d\rho = dx\sqrt{\frac{r_h}{x}}$. The metric is then modified as

$$ds^2 = -\frac{\rho^2}{4r_h^2}dt^2 + d\rho^2 + dy^2 + dz^2. \tag{5.11}$$

To convert the frame into Euclidean, a Wick rotation, $t = -i\tau$, is applied, so that

$$-ds^2 = \rho^2 d(\frac{\tau}{2r_h})^2 + d\rho^2 + dy^2 + dz^2. \tag{5.12}$$

For this metric to be well-behaved, $\frac{\tau}{2r_h}$ should behave as if it is an angular coordinate with periodicity 2π. So periodicity in τ is given as (comparing Eq. (5.7) and Eq. (5.12))

$$\begin{aligned} \beta = 1/T_{bh} &= \frac{2\pi}{1/(2r_h)} = 4\pi r_h; \\ T_{bh} &= \frac{1}{4\pi r_h}. \end{aligned} \tag{5.13}$$

In the case of Schwarzschild black hole, $T_{bh} = \frac{1}{8\pi M}$. In the Rindler spacetime, $T_{bh} = \frac{1}{4\pi/2g} = \frac{g}{2\pi}$. So temperature is implicit in the horizon.

5.3 Scalar field in Rindler frame

The static Rindler metric in two dimensions can be obtained from Eq. (5.5) as

$$ds^2 = \frac{1}{1+2gx}dx^2 + dy^2. \tag{5.14}$$

Now, introduce the principle of calculus of variation to find the trajectory of the particle so that the function which has the stationary value can be given as

$$f = \sqrt{\frac{1}{1+2gx} + y'^2}, \tag{5.15}$$

In Eq. (5.15), $y' = dy/dx$. We now apply the Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$, to find the expression for trajectory. The equation of motion of a particle in the Rindler spacetime is obtained as

$$y'' + \frac{g}{1+2gx}y' = 0 \tag{5.16}$$

The solution to Eq. (5.16) is

$$y(x) = \frac{-e^{\frac{-gx}{1+2gx}}(1+2gx)C(1)}{g} + C(2), \tag{5.17}$$

where, $C(1)$ and $C(2)$ are constants. Fig. (5.1) shows the trajectory of a particle in the static two dimensional Rindler space. The graph

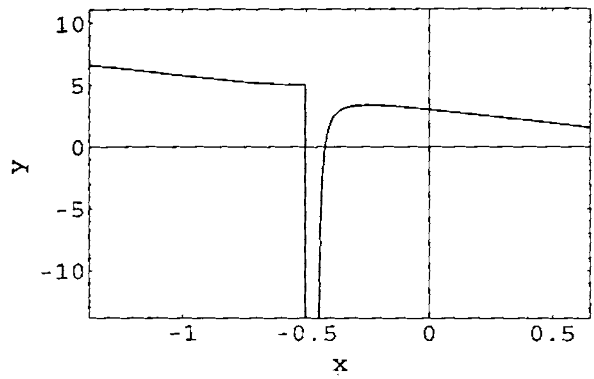

Figure 5.1: Trajectory of a particle in a static Rindler space with $C(1) = 2$, $C(2) = 5$ and $g = 1$.

clearly shows that the horizon is situated at $x_h = -\frac{1}{2g} = -\frac{1}{2}$, with $g = 1$. There is no physical singularity in the Rindler spacetime, since at $x = 0$, the metric does not blow up. The only singularity is the coordinate singularity that appears at the horizon.

5.3.1 Scalar field solution

Consider a massless scalar field equation in a Rindler spacetime, so that its field equation is given as

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu]\Phi = 0. \qquad (5.18)$$

Introducing the metric $ds^2 = -\frac{\rho^2}{4r_h^2}dt^2 + d\rho^2 + dy^2 + dz^2$ and $\sqrt{-g} = \frac{\rho}{2r_h}$ in the above scalar field equation, the spatial equations of scalar field are

$$\frac{\partial^2\Phi_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi_\rho}{\partial\rho} - [k_1^2 + k_2^2 - \frac{4r_h^2\omega^2}{\rho^2}]\Phi_\rho \qquad (5.19)$$

$$\frac{\partial^2 \Phi_y}{\partial y^2} + k_1^2 \Phi_y = 0; \frac{\partial^2 \Phi_z}{\partial z^2} + k_2^2 \Phi_z = 0. \tag{5.20}$$

Put $k_1^2 + k_2^2 = k^2$, k can be called the root and $\pm i 2 r_h \omega$ the order, where ω represents the frequency of the scalar field. Eq. (5.19) in ρ is of Bessel type. The full solution to the scalar field equation is

$$\Phi = \Phi_\rho e^{-i\omega t + i k_1 y + i k_2 z} \tag{5.21}$$

The solution to Bessel equation is $\Phi_\rho = I_{\pm i\omega}(\sqrt{k_1^2 + k_2^2})\rho$ [119]. Suppose the frequency $\omega = \alpha + i\beta$ with α, β real and $\beta > 0$, then $I_{-i\omega}(\sqrt{k_1^2 + k_2^2})\rho$ goes to infinity exponentially as $\rho \to \infty$ and vanishes at $\rho = 0$. On the contrary, $I_{+i\omega}(\sqrt{k_1^2 + k_2^2})\rho$ goes to infinity at $\rho = 0$ and falls to zero exponentially as $\rho \to \infty$ [120]. The boundary of the Bessel equation will consist of $\rho = 0$, the horizon of the Rindler spacetime, and $\rho \to \infty$. We must demand that the scalar field function Φ initially well behaved at the boundaries. For the infinity, we could require the field Φ falling off to zero. So the suitable solution must be

$$\Phi = e^{-i\omega t + i k_1 y + i k_2 z} I_{+i\omega}(\sqrt{k_1^2 + k_2^2})\rho \tag{5.22}$$

Now the temperature of scalar field in thermal equilibrium with the Unruh radiation can be determined as follows. In Eq. (5.19), the term, $-4r_h^2\omega^2 = -(16\pi^2 r_h^2)/T^2$. Making $T \to iT = \bar{T}$, we get the term as, $(16\pi^2 r_h^2)/\bar{T}^2$. Then the periodicity in $\bar{T}$ will be the temperature of scalar field, which is also the temperature of horizon. So the temperature of the horizon is given as

$$T_{bh} = \frac{1}{4\pi r_h}. \tag{5.23}$$

Thus scalar field is in thermal equilibrium with the horizon. The temperatures of the Rindler and Schwarzschild horizons have a common expression for temperature, since by substituting the respective values for r_h in Eq. (5.23) the temperatures of these black holes will be obtained.

5.4 Entropy of self gravitating radiation system (SGRS)

Consider a Rindler frame which accelerates in the x-direction so that a dense cloud of gravitating particles converge on itself due to gravitation, with their centre coinciding with the origin of the Rindler frame. Here we study the transit of the matter through the horizon of the black hole to which the matter system is going to collapse. Because of the acceleration along the x-direction, the dimension of the matter along that direction shrinks. As the configurations of $SGRS$ approach the horizon, the entropy will become infinite by virtue of the asymptotic equation of radiation, such as, $\rho = \alpha T_r^4$ and $s = \frac{4}{3}\alpha T_r^3$. The divergence of entropy can be avoided by using the modified equation of radiation [109].

The entropy of $SGRS$ had been investigated earlier to examine the validity of the entropy upper bound and concluded that it is true only outside the horizon by one radiation wavelength [121]. The total entropy of $SGRS$ with spherical symmetry (Schwarzschild black hole) is expressed as

$$S = 4\pi\alpha \int_0^R \rho^{3/4} r^{5/2} \varepsilon^{-1/2} dr, \tag{5.24}$$

in which the asymptotic value of equation of radiation has been used. In Eq. (5.24) ρ represents the energy density of $SGRS$ and ε is given as $\varepsilon(r) = r - 2m(r) > 0$. Then $\varepsilon(r) > 0$ indicates that each configuration of $SGRS$ is outside the corresponding Schwarzschild radius $2m(r)$. The mass of $SGRS$ is given as $m(r) = \int_0^r 4\pi r^2 \rho dr$. The blowing up ($\varepsilon \to 0$) of entropy of $SGRS$ can be avoided by considering the configuration with proper distance from their own Schwarzschild radius equal to at least one radiation wavelength of magnitude,

$$R(r) = \sqrt{g_{rr}\varepsilon} = \sqrt{r\varepsilon} \geq \lambda(r). \tag{5.25}$$

But such a prerequisite is not necessary to find the entropy of $SGRS$ once we make use of the modified equation of radiation near the horizon [113].

Here, we discuss the collapse of a massive scalar field across the horizon of the black hole to which the matter is going to collapse. The gravitational collapse is then studied in the Rindler frame which acts in unison with the collapse. The mass of a layer of $SGRS$ in the Rindler frame before it takes off is given as $dm = 4\pi\rho r^2 dr$. The matter is then allowed to collapse through the event horizon. Let s be the entropy density which can be given as $s = C(l)\rho\chi$ [109], where $C(l)$ is the term that parameterize the different layers of the scalar field that transit through horizon and χ is the red-shift factor.

The entropy of the $SGRS$ that crosses the event horizon is given as

$$S = 4\pi \int_0^R s\sqrt{g_{11}}r^2 dr = 4\pi \int_0^R C(l)\rho\sqrt{-g_{00}}\sqrt{g_{11}}r^2 dr. \quad (5.26)$$

In the collapse, matter layer by layer, crosses the event horizon and finally converge to the singularity at the origin of the Rindler frame. With $g_{11} = (1 - 2m/r)^{-1}$ and $g_{00} \le (1 - 2M/r)$, we have from Eq. (5.26)

$$S \le 4\pi \int_0^R C(l)\rho r^2 dr. \quad (5.27)$$

As $SGRS$ collapses, any data regarding it will be the information it holds just before crossing the horizon, once it crosses the horizon, information it holds will be lost. So the total entropy of the system must depend on the limit values of $C(l)$ and $\rho(l)$ as $l \to 0$, i.e., near the horizon. $C(l \to 0)$ is given as

$$C(l \to 0) = \frac{4}{3}\frac{1}{T_r}, \quad (5.28)$$

where, T_r is the temperature of the $SGRS$. As the $SGRS$ approaches the horizon, it attains the temperature of horizon (Eq. (5.23)). The temperature of the horizon is $T_{bh} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}$. Hence, $C(l \to 0) = \frac{32}{3}\pi M$. The elementary mass of $SGRS$ that has gone through the horizon may be given as

$$dm(r) = 4\pi r^2 \rho dr = \rho dx dy dz. \quad (5.29)$$

Making a transformation into the Rindler frame ($d\bar{x} = \frac{dx}{\sqrt{1+2gx}}$; $dy =$

$d\bar{y}; dz = d\bar{z}$), we get

$$\begin{aligned} dm(r) &= \rho d\bar{x}\sqrt{1+2gx}d\bar{y}d\bar{z} \\ &= \bar{\rho} d\bar{x} d\bar{y} d\bar{z}, \end{aligned} \tag{5.30}$$

with, $\bar{\rho}$ is the new density in the Rindler frame. Therefore in Eq. (5.27)

$$\begin{aligned} S &\leq \int_0^M \frac{32}{3}\pi m dm \\ &\leq \frac{32}{3}\pi M^2 = 4\pi M^2, \end{aligned} \tag{5.31}$$

where M is the total mass of the $SGRS$ which has collapsed across the horizon. So, the entropy of the spherical mass is same as the entropy of the black hole into which the collapse is occurred. Because of the shrinking along the x-direction, the collapse may not be symmetric and since the mass remains same, the entropy is still proportional to M^2, which is true in the case of a Schwarzschild black hole. But when the entropy is written in terms of area, since the surface is not fully symmetric, it may not be exactly $\frac{1}{4}A$.

5.4.1 Upper bound on S/E

We know that the GSL must be valid in all situations and the upper bound on entropy is a necessary condition to hold the generalized second law. So it is advisable to calculate the upper bound in this case also. The upper bound on entropy is given as

$$S/E = 4\pi M \leq 2\pi R \tag{5.32}$$

where R is the average size of the $SGRS$ which has gone through the horizon of radius r_h. Since $R > r_h$, we get the inequality in Eq. (5.32), which is similar to the Bekenstein upper bound $S/E \leq 2\pi R$, where R is the dimension of the box.

5.5 Conclusion

Unruh effect is the basis of Hawking effect. We can treat a gravitating system as a thermodynamical one if the spacetime of the system possess a periodicity in the imaginary time τ. We have shown that the spacetime close to the event horizon of the Schwarzschild black hole is Rindler like. Using the field theory approach, the temperatures of the scalar field near the horizons of Rindler spacetime and Schwarzschild spacetime have been determined. The field around a black hole is in thermal equilibrium with the horizon. The trajectory of a particle in the static Rindler space has been determined by solving the Euler's equation.

We have investigated the gravitational collapse of $SGRS$ in a Rindler space and found that the entropy of the $SGRS$ is the same as the entropy of the black hole into which it is collapsed. We have also calculated the entropy bound and obtained that it is in unison with the Bekenstein upper bound.

Chapter 6

Scalar hair for an AdS black hole

6.1 Introduction

If the facts don't fit the theory, change the facts.,
Albert Einstein.

The no-hair theorem in general relativity says that in the exterior of black hole the only information available regarding the black hole may be that of its mass, charge and angular momentum. All other informations about the matter which formed a black hole or infalling into it, disappear behind the black hole event horizon and are therefore permanently inaccessible to external observers. The statement that the black hole has no-hair means, there are no features other than mass, charge and angular momentum that distinguish a black hole from another one. If we construct two black holes with the same mass, charge and angular momentum, the first being made out of ordinary matter and the second out of anti-matter, they would be completely indistinguishable.

Scalar hair in a black hole demands a non-trivial solution for the scalar field in the vicinity of a black hole. The profile of the field must be Gaussian type, as if a scalar source is present on the black hole. But, it is found that the scalar field becomes trivial if one demands a regular horizon at a finite distance from the centre [122]. A regular horizon means, a horizon which has a radius and temperature so that it would hide the singularity. A non-trivial solution seemed possible only for a black hole that exhibit naked singularity.

The question of *scalar hair* for a static black hole has been a matter of debate for quite some time. Situations of preservation of *no-hair conjecture* [123, 66, 67, 68, 69] and its violations [70, 71] had been reported earlier many times. Saa [73] deduced a theorem which shows that the static spherically symmetric exterior solution for the gravitational field equations in a wide class of scalar tensor theories will essentially reduce to the well known Schwarzschild solutions if one has to hide the naked singularity at the centre of a black hole by the event horizon [73, 74]. Bocharova and Bekenstein [124, 125] constructed solutions with regular horizon for a scalar field conformally coupled to Einstein's gravity. They are

$$ds^2 = -(1 - \frac{r_0}{r})^2 dt^2 + (1 - \frac{r_0}{r})^{-2} dr^2 + r^2 d\Omega^2 \qquad (6.1)$$

$$\Phi = -\frac{r_0 \alpha^{-1/2}}{r - r_0},$$

where $\alpha = \frac{8\pi G}{6}$. Initially, divergence of field at the horizon was considered as a pathology of the solution. However, further analysis [126] suggested that the divergence of Φ on the horizon might be innocuous. In general, asymptotically flat, static, spherically symmetric nontrivial solutions of scalar field Φ coupled to Einstein's gravity do not possess a regular horizon [127]. As a strong interpretation, Bizon [128] and Weinberg [72] showed that a theory allows a hairy black hole, if there is a need to specify quantities other than the conserved charges defined at asymptotic infinity, in order to characterize completely a stationary black hole solution. Eq. (6.1) is characterized by the Arnowitt-Deser-Misner (ADM) mass $r_0/2$ [129], and scalar charge $Q = 4\pi r_0 \alpha^{-1/2}$. So Eq. (6.1) carries "hair" and violates the "no-hair conjecture".

But the divergence of the scalar field at the horizon is so severe that Eq. (6.1) doesn't satisfy Einstein's equation at the horizon, hence Eq. (6.1) need not be a black hole solution [69]. As a weak interpretation of scalar hair, non-trivial solution in terms of conserved charges was mooted [72]. Using that ideology, scalar hair was reported in asymptotically anti-de Sitter spacetime and asymptotically flat spacetime [70, 71]. There is no regular black hole solution when the scalar field is massless or has a *convex potential* (containing only mass term). Examples of black hole solution, such as, $\Phi \sim r^{-3/2} \cos(\frac{\sqrt{4\beta - 9}}{2} \ln r)$, where β is a constant, in the symmetric and asymmetric double well potential was reported [70]. This unexpected discovery of black hole solutions by considering asymptotically AdS, rather than asymptotically flat spacetime were thoroughly analyzed [69]. Applying the principle of the conservation of the $'r'$ component of the total energy-momentum tensor ($T^{\mu}_{r;\mu} = 0$) in Einstein's equation [130], they showed that there were no nontrivial static and spherically symmetric black hole solutions in the asymptotically AdS with true cosmological constant. The asymptotically AdS region corresponds to one where the effective cosmological constant is

$$\Lambda_{eff} = \Lambda + 8\pi V(\Phi_\infty). \tag{6.2}$$

Eq. (6.2) gives the idea that in the asymptotically flat case we must require V to go to 0 at infinity, while in asymptotically AdS case any non zero value of V at infinity can be absorbed into the effective cosmological constant. So the argument is that only the change in the true cosmological constant makes any sense, not the change in effective cosmological constant.

In this chapter, our aim is to find whether the scalar hair would occur in the asymptotically AdS space time with regular horizon and we took (2+1) non-rotating Bananas-Teitelboim-Zanelli (BTZ) black hole as a model [47, 131] which shares many of the features of its (3+1) dimensional counterparts and also in the Reissner-Nordström (RN) black hole. In contrast with (3+1) dimensional general relativity, the (2+1) dimensional model has only finite physical degrees of freedom. As a result, questions about quantum gravity can be explored in considerable detail [132, 133]. we report non-trivial black hole solution showing no divergence at the horizon and asymptotically falling to the vacuum value. The scheme of the chapter is as follows. In Sec. 6.2, non-trivial scalar hair solution is obtained for the BTZ and RN black holes by solving the scalar field equations. Mass of hairy black hole is also discussed in this section. In Sec. 6.3, the complete solution to scalar field equation is obtained. In Sec. 6.4, stability analysis is described by applying the theory of perturbation. In Sec. 6.5, we give the conclusion.

6.2 Solution with a minimal coupling

A non-trivial radial solution of a scalar field, whose source is a massive double well scalar potential, in the vicinity of the BTZ black hole will be discussed here. We will restrict our consideration to the minimally coupled case. Consider the action

$$S = \int d^3x\sqrt{-g}[\frac{1}{16\pi}(R-2\Lambda) - \frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - V(\Phi)], \qquad (6.3)$$

where Φ is the real scalar field and $V(\Phi)$ is the potential. The metric of BTZ black hole is

$$ds^2 = -fe^{-2\delta}dt^2 + f^{-1}dr^2 + r^2d\phi^2. \qquad (6.4)$$

Here we propose a regular horizon so that it hides the naked singularity. For the BTZ, $f = -m + \frac{r^2}{l^2}$, where $\Lambda = -\frac{1}{l^2}$, the cosmological constant. Let the functions m and δ depend only on r. The Lagrangian density for the action of Eq. (6.3) can be obtained from

$$\text{Ł} = \sqrt{-g}[\frac{1}{16\pi}(R-2\Lambda) - \frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - V(\Phi)]. \qquad (6.5)$$

In Eq. (6.5), Φ is a radial function. Using the Euler's equation

$$\frac{d}{dr}(\frac{\partial \text{Ł}}{\partial\Phi'}) - \frac{\partial \text{Ł}}{\partial\Phi} = 0, \qquad (6.6)$$

where $\Phi' = \frac{\partial\Phi}{\partial r}$. We get the scalar field equation from Eq. (6.6) and δ' by varying the action as

$$\begin{aligned}[re^{-\delta}f\Phi']' &= e^{-\delta}r\frac{dV(\Phi)}{d\Phi}\\ \delta' &= -2\pi\Phi'^2. \end{aligned} \qquad (6.7)$$

Eq. (6.7) governs the scalar field in the black hole spacetime and perturbation. Whether a non-trivial solution results from Eq. (6.7) will be the present investigation. Multiplying both sides of Eq. (6.7) by $(\Phi - \Phi_0)$, where Φ_0 is the vacuum value or the asymptotic value of Φ, we get:

$$(\Phi - \Phi_0)[re^{-\delta} f\Phi']' = (\Phi - \Phi_0)e^{-\delta} r\frac{dV}{d\Phi}$$

$$\frac{d}{dr}[(\Phi - \Phi_0)(re^{-\delta} f\Phi')] = (\Phi - \Phi_0)e^{-\delta} r\frac{dV(\Phi)}{d\Phi} + e^{-\delta} r f\Phi'^2. \quad (6.8)$$

Integrating both sides of Eq. (6.8), from r_h to r, we have

$$[(\Phi - \Phi_0)(re^{-\delta} f\Phi')]^r_{r_h} = \int_{r_h}^{r} dre^{-\delta} r[(\Phi - \Phi_0)\frac{dV}{d\Phi} + f\Phi'^2]. \quad (6.9)$$

In the asymptotic limit, $\Phi \to \Phi_0$ and $\Phi' \longrightarrow 0$. At $r = r_h$, $f = 0$. So the left hand side of Eq. (6.9) vanishes. Therefore

$$[(\Phi - \Phi_0)\frac{dV(\Phi)}{d\Phi} + f\Phi'^2] = 0. \quad (6.10)$$

We know that f is positive outside the horizon of the AdS black hole. If $V(\Phi)$ represents a convex potential (mass term only), then $\frac{dV(\Phi)}{d\Phi} > 0$. So Eq. (6.10) is zero only when $\Phi = \Phi_0$ and $\Phi' = 0$. That is, the magnitude of field remains constant throughout the space. But such a solution is trivial. Since we need a non-trivial solution, a double well potential is proposed to act in the Eq. (6.10). The double well potential has a mass term and a self interaction term with an asymptotic value. The potential function against Φ is shown in Fig. (6.1). The double well potential function is given as

$$V(\Phi) = -\frac{1}{2}\mu^2(\Phi - \Phi_0)^2 + \frac{1}{4}\lambda^2(\Phi - \Phi_0)^4 + \frac{1}{4}\mu^4/\lambda^2, \qquad (6.11)$$

where, μ is the mass and λ is the self interaction coefficient of scalar field. In the double well potential case, $\frac{dV(\Phi)}{d\Phi} < 0$ in the limit Φ

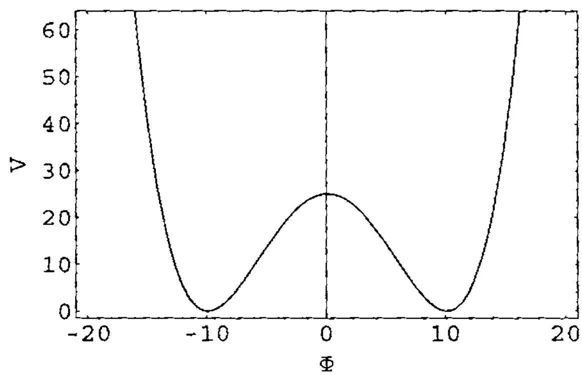

Figure 6.1: Double well potential against field variable Φ, with $\mu = 1, \lambda = 0.1, \Phi_0 = 0.1$

away from Φ_0, so Eq. (6.10) can become zero even if $\Phi \neq \Phi_0$ and $\Phi' \neq 0$. On substituting Eq. (6.11) and $f = \frac{r^2}{l^2} - m$ in Eq. (6.10) and putting, $\Phi - \Phi_0 = x; d\Phi = dx; \frac{\mu}{\lambda} = a$, we get

$$\frac{dr}{\sqrt{r^2 - l^2 m}} = \frac{dx}{\lambda l x \sqrt{a^2 - x^2}}. \qquad (6.12)$$

Integrating Eq. (6.12) and rearranging we get

$$\Phi = \frac{\mu}{\lambda} \sec h[-\mu l arc \cos h(\frac{r}{r_h})] + \Phi_0, \qquad (6.13)$$

with $r_h = l\sqrt{m}$. The presence of $' - \mu l'$ inside the bracket will restrict us from getting a well defined solution. So we put, $\mu = -\frac{1}{l}$, thus defining a negative cosmological constant for the spacetime. So $\Phi = \frac{\mu}{\lambda} \sec h[arc \cos h(\frac{r}{r_h})] + \Phi_0$. At the horizon, $\Phi = \frac{\mu}{\lambda} + \Phi_0$, which is finite and in the asymptotic limit, $\Phi = \Phi_0$. The profile of Eq. (6.13) is shown in Fig. (6.2), which brings the characteristic of scalar field. The bold line is for BTZ black hole and thin line is for RN black hole (will be shown in next sub-section). In the figure, we put $\mu = 1, \lambda = 0.1, \Phi_0 = 0.1$. It can be shown that the curve drops at the rate $1/r^2$, i.e., the field depends inversely on r. We can

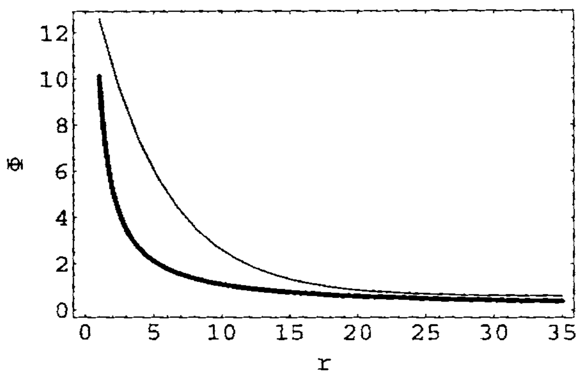

Figure 6.2: Variation of field variable against r, with $\mu = 1, \lambda = 0.1, \Phi_0 = 0.1$.

now conclude that scalar hair is possible with a potential function of double well type (Fig. (6.1)). The potential is maximum at $\Phi = \Phi_0$ and is zero at $\Phi = \mu/\lambda + \Phi_0$. The effective cosmological constant is

$$\Lambda_{eff} = \Lambda + 2\pi\mu^4/\lambda^2 = \Lambda + \Lambda_{add}. \tag{6.14}$$

Eq. (6.14) reveals that the cosmological constant has an origin in the scalar field. The term Λ_{add} may be assumed to be a trace of scalar field.

6.2.1 Scalar hair in Reissner-Nordström black hole

In the expression, $[(\Phi - \Phi_0)\frac{dV(\Phi)}{d\Phi} + f\Phi'^2] = 0$, substitute, $f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}$. Now solve for Φ. We get

$$\Phi = 4\frac{\mu}{\lambda}\frac{e^{-\mu\sqrt{Q^2-2mr+r^2}}}{(-m+\sqrt{Q^2-2mr+r^2})^{\mu m}} + \Phi_0. \tag{6.15}$$

At the horizon, $\sqrt{Q^2 - 2mr + r^2} = 0$. Since, μm is negligibly small, the field at the horizon is

$$\Phi = 4\frac{\mu}{\lambda} + \Phi_0. \tag{6.16}$$

In the asymptotic limit, $\Phi \to \Phi_0$. From Eqs. (6.13) and (6.15), we can see that the scalar field solutions are expressed in terms of the prevailing parameters of the black hole. That shows that the hair is weak. To have a strong hair, the solutions must have conserved quantities other than mass, charge and angular momentum.

6.2.2 Mass of hairy black hole

Mass of hairy black hole must be in general greater than the mass of non-hairy black hole. Now let us compare $m(r_h)$, the mass of the non-trivial static black hole of radius r_h, with $M(r_h)$, the mass of the corresponding naked black hole of the same radius. We have

$$M(r_h) = \frac{r_h^2}{l^2}. \tag{6.17}$$

The mass of the non-trivial black hole is [69]

$$m(r_h) = M(r_h) + 2\pi\int_{r_h}^{r}[V(\Phi) - V(\Phi_\infty) + (1/2)f\Phi'^2]dr. \tag{6.18}$$

In the above equation, $V(\Phi)$ is zero at the horizon and $V(\Phi_\infty) = \frac{1}{4}\mu^4/\lambda^2$. We have, $f = m(-1 + \frac{r^2}{r_h^2})$; $\Phi' = -\frac{\mu}{\lambda}\frac{r_h}{r^2}$ and $\Phi = \frac{\mu}{\lambda}\frac{r_h}{r}$ (as can be shown from Eq. (6.13)). We get, $2\pi r[V(\Phi) - V(\Phi_\infty)] = -\frac{\pi\mu^4 r_h^2}{\lambda^2 r} + \frac{\pi\mu^4 r_h^4}{2\lambda^2 r^3}$ and $\pi r f \Phi'^2 = \frac{\pi\mu^2 r_h^2}{\lambda^2 l^2}(-\frac{r_h^2}{r^3} + \frac{1}{r})$. On integrating Eq. (6.18) we get

$$m(r_h) = M(r_h) - \frac{\pi\mu^4 r_h^2}{\lambda^2}\log r/r_h - \frac{\pi\mu^4 r_h^4}{4\lambda^2}(\frac{1}{r^2} - \frac{1}{r_h^2}) + \frac{\pi\mu^2 r_h^4}{2\lambda^2 l^2}(\frac{1}{r^2} - \frac{1}{r_h^2}) + \frac{\pi\mu^2 r_h^2}{\lambda^2 l^2}\log r/r_h, \quad (6.19)$$

with $m(r_h)$ is the mass of black hole with scalar hair and $M(r_h)$ the mass with out scalar hair. From the above assumption, $\mu = -\frac{1}{l}$ we get

$$m(r_h) = M(r_h) + \frac{\pi\mu^4 r_h^4}{4\lambda^2}(\frac{1}{r^2} - \frac{1}{r_h^2}). \quad (6.20)$$

At the horizon, $m(r_h) = M(r_h)$. As the distance from centre in-

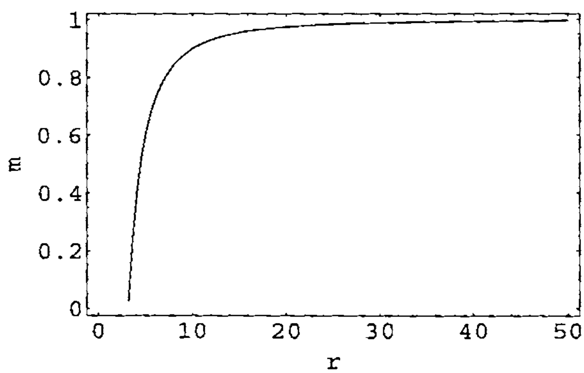

Figure 6.3: Variation of mass of hairy black hole up on non-hairy black hole against r.

creases, $m(r_h)$ increases and at infinity, $m(r_h) = M(r_h) - \frac{\pi\mu^4 r_h^2}{4\lambda^2}$.

From Eq. (6.14), $\Lambda_{eff} = \Lambda + 2\pi\mu^4/\lambda^2 = \Lambda + \Lambda_{add}$, where, $\Lambda_{add} = -\frac{1}{l^2_{add}}$. Therefore, $m(r_h) = M(r_h) + \frac{r_h^2}{8l^2_{add}}$. Since $\frac{1}{l^2}$ is positive, $m(r_h) > M(r_h)$. This is an essential condition to have scalar hair [69, 130]. It is seen that $m(r_h)$ is a function of r. The function $m(r_h)$ in principle diverges, but under condition $\mu = -\frac{1}{l}$, the mass never blows up. The profile of a mass function for non-trivial Schwarzschild black hole was given earlier [130]. The mass function of nontrivial BTZ black hole is depicted in Fig. (6.3).

6.3 Solution to scalar field equation

The solution to scalar field equation under the action of gravity and scalar potential will be now calculated. Defining [131, 134]

$$dr_* = e^\delta \frac{dr}{f}, \tag{6.21}$$

The metric of BTZ in the tortoise co-ordinate is obtained as

$$ds^2 = -fe^{-2\delta}dt^2 + fe^{-2\delta}dr_*^2 + r(r_*)^2 d\phi^2, \tag{6.22}$$

with

$$\sqrt{-g} = fe^{-2\delta}r(r_*). \tag{6.23}$$

The field equation of massive scalar field under the action of a scalar potential coupled to gravity is

$$[\Box + \xi R]\Phi = \frac{dV(\Phi)}{d\Phi}. \tag{6.24}$$

The mass term and interaction term of the scalar field have been included in the potential $V(\Phi)$. The scalar curvature, $R = -\frac{6}{l^2}$

[47, 135]. For minimal coupling, Eq. (6.24) is modified as

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu]\Phi = \frac{dV(\Phi)}{d\Phi}. \tag{6.25}$$

On expanding Eq. (6.25) we get field equation as

$$\begin{aligned}&-\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{r(r_*)}\frac{\partial\Phi}{\partial r(r_*)} + \frac{\partial^2\Phi}{\partial r(r_*)^2} + \frac{fe^{-2\delta}}{r(r_*)^2}\frac{\partial^2\Phi}{\partial\phi^2}\\ &-fe^{-2\delta}\{-\mu^2(\Phi-\Phi_0)+\lambda^2(\Phi-\Phi_0)^3\} = 0.\end{aligned} \tag{6.26}$$

For separating the variables, express $\Phi = T(t)R_k^p(r(r_*))Y_m(\phi)$. The radial field equation is

$$\begin{aligned}&\frac{d^2R_k^p}{d^2r(r_*)^2} + \frac{1}{r(r_*)}\frac{dR_k^p}{dr(r_*)} + [k^2 + \frac{fe^{-2\delta}}{r(r_*)^2}\beta^2\\ &-fe^{-2\delta}\{-\mu^2(\Phi-\Phi_0)\tfrac{1}{\Phi} + \lambda^2(\Phi-\Phi_0)^3\tfrac{1}{\Phi}\}]R_k^p = 0,\end{aligned} \tag{6.27}$$

where $-\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = k^2$ and $\frac{1}{Y}[\frac{\partial^2 Y}{\partial\phi^2}] = \beta^2$.

The wave function $\Phi = \frac{\mu}{\lambda}\frac{r_h}{r}$ (as can be shown from Eq. (6.13)) and $(\Phi-\Phi_0)/\Phi \simeq 1, (\Phi-\Phi_0)^3/\Phi \simeq \Phi^2$. So the Eq. (6.27) can be modified as

$$\frac{d^2R_k^p}{d^2r(r_*)^2} + \frac{1}{r(r_*)}\frac{dR_k^p}{dr(r_*)} + [k^2 + \frac{fe^{-2\delta}\beta^2}{r(r_*)^2} + fe^{-2\delta}\mu^2 - \frac{fe^{-2\delta}\mu^2r_h^2}{r(r_*)^2}]R_k^p = 0. \tag{6.28}$$

In Eq. (6.28), R_k^p represents the radial field equation. The effective potential of scalar field from Eq. (6.28) is

$$V_{eff} = -\frac{fe^{-2\delta}\beta^2}{r(r_*)^2} - fe^{-2\delta}\mu^2 + \frac{fe^{-2\delta}\mu^2r_h^2}{r(r_*)^2}. \tag{6.29}$$

At the horizon, $f = 0$, hence, $V_{eff} = 0$. From Eq. (6.13), $\Phi' = -\frac{\mu}{\lambda}\frac{r_h}{r^2}$. Substituting the value of Φ' in Eq. (6.7), we get

$$\delta = \frac{\pi\mu^2 r_h^2}{\lambda^2 r^2}. \tag{6.30}$$

As we go away from the horizon, $f \sim \frac{r^2}{l^2}$. Substituting the value of f and $e^{-2\delta}$ in Eq. (6.29), we get

$$V_{eff} = \frac{-\beta^2}{l^2} + \frac{2\pi\mu^2\beta^2 m}{\lambda^2 r^2} + \frac{2\pi\mu^4 m}{\lambda^2} + \mu^4 - \frac{2\pi\mu^4 m}{\lambda^2 r^2}. \tag{6.31}$$

In the above equation, terms with μ^4 have been eliminated, since they are negligibly small. The role of μ and λ of the scalar field, eventhough evident in the calculation of V_{eff}, may be omitted . Then, $V_{eff} = \frac{-\beta^2}{l^2} + \frac{2\pi\mu^2\beta^2 m}{\lambda^2 r^2} \simeq \frac{\alpha^2}{r^2}$. From Eq. (6.31) it can be shown that for values of $r > r_h$, V_{eff} very soon rises to a positive value and then falls and finally reaches the asymptotic value. So for majority parts of the spacetime, V_{eff} is inversely proportional to r^2. In the deSitter spacetime the effective potential is zero both at the horizon and in the asymptotic limit. The V_{eff} rises only slowly as we go away from the horizon, reaches a maximum value and then falls to zero in the asymptotic limit[136]. As we approach the horizon, $f \to 0$ and hence Eq. (6.28) reduces to

$$\frac{d^2 R_k^p}{d^2 r(r_*)^2} + \frac{1}{r(r_*)}\frac{dR_k^p}{dr(r_*)} + k^2 R_k^p = 0. \tag{6.32}$$

This is a Bessel equation with root k and order *zero.* As we go away from horizon, the potential function drops approximately to the order $1/r^2$. So Eq. (6.28) away from the horizon is

$$\frac{d^2R_k^\alpha}{d^2r(r_*)^2} + \frac{1}{r(r_*)}\frac{dR_k^\alpha}{dr(r_*)} + [k^2 - \frac{\alpha^2}{r^2}]R_k^\alpha = 0. \tag{6.33}$$

This is a Bessel equation of root k and order α. So through out the spacetime, the radial field can be represented by a Bessel function. The above two equations have solutions which are normalizable.

$$\int_0^\infty R_k^\alpha(r)R_{k'}^{*\alpha}(r)r(r_*)dr(r_*) = \delta(k-k'). \tag{6.34}$$

Solution to Eq. (6.32) can be represented as

$$R_k^0(r(r_*)) = \sum_{n=0}^\infty[(-1)^n(\tfrac{kr}{2})^{2n}\tfrac{1}{\Gamma(n+1)n!}]. \tag{6.35}$$

Solution to Eq. (6.33) is given as

$$R_k^\alpha(r(r_*)) = \sum_{n=0}^\infty[(-1)^n(\tfrac{kr}{2})^{\alpha+2n}\tfrac{1}{\Gamma(\alpha+n+1)n!}]. \tag{6.36}$$

where Γ is the usual Gamma function. Eq. (6.35) and Eq. (6.36) represent the full solution of scalar field which may be normalized and hence can be quantized.

6.4 Stability analysis

In Sec 6.2 we have found a scalar hair for non-rotating BTZ and RN black holes. A mere non-trivial solution is not enough to show that there is definite hair, since, no-hair conjecture still holds if the solution soon falls out. So we will now consider the stability of the scalar field solution. The first order perturbed equation for the scalar field is obtained from the Eq. (6.28) as [69]

$$\ddot{\Phi} = -\hat{A}\Phi = [\frac{d^2}{dr(r_*)^2} + \frac{1}{r(r_*)}\frac{d}{dr(r_*)} - V_{eff}]\Phi, \tag{6.37}$$

where $dr_* = e^{\delta}\frac{dr}{f}$ and $\hat{A} = -[\frac{d^2}{dr(r_*)^2} + \frac{1}{r(r_*)}\frac{d}{dr(r_*)} - V_{eff}]$ is a Hermitian operator. If Φ is a vector of the Hilbert space, then the inner product in the context of BTZ black hole is given as [137]

$$\langle\Phi_1, \Phi_2\rangle = \int_0^{2\pi}\int_0^{\infty} r(r_*)\Phi_1\Phi_2 d\phi dr(r_*). \tag{6.38}$$

If, $\langle\Phi, \hat{A}\Phi\rangle > 0$, then the state function Φ which represents the scalar field around the black hole is stable. The solution to scalar field equation, $[\Box + \xi R]\Phi = \frac{dV(\Phi)}{d\Phi}$, can be expressed as, $\Phi(t, r, \phi) = \Phi(r)e^{ikt}Y_m(\phi)$, where $\Phi(r)$ is the radial part of the solution which is a Bessel function as given by Eq. (6.36). Using the radial part of the scalar field solution and substituting the operator $\hat{A}$ in Eq. (6.38), we get

$$\langle\Phi, \widehat{A}\Phi\rangle = -2\pi \int_0^{\infty} dr(r_*) R_k^{\alpha}[rR_k^{\alpha''} - R_k^{\alpha'} - rV_{eff}R_k^{\alpha}]. \tag{6.39}$$

If R_k^{α} is positive definite, then $R_k^{\alpha''}$ is also positive definite but $R_k^{\alpha'}$ is negative definite. Then, $rR_k^{\alpha''}$ is negative definite and '$-rV_{eff}R_k^{\alpha}$' is positive definite in the range 0 to ∞. So the net term inside the bracket of Eq. (6.39) is negative definite. Again, if R_k^{α} is negative definite, the net term inside the bracket is positive definite. In both cases, the expression $\langle\Phi, \widehat{A}\Phi\rangle \geq 0$. That means, the scalar field equation is stable under the first order perturbation and hence the configuration is stable. This shows the existence of a stable hair.

As a second order perturbation, we have

$$\frac{d^2 R_k^\alpha}{dr(r_*)^2} + (k^2 - V_{eff}) R_k^\alpha = 0 \tag{6.40}$$

with smooth real potentials independent of k and of short range. If k^2 is negative, the perturbations diverge exponentially with time and then the solution is unstable [67]. The wave function R_k^α must approach zero at the horizon when the eigenmode is negative [67]. But in our case the eigenmode is positive and hence the wave function is not zero at the horizon (Eqs. (6.13) and (6.16)). As a result solution is stable against radial perturbations and the scalar hair does not fall out.

6.5 Conclusion.

There is a general feeling that anything that is added to a black hole will not induce any trace of it on the black hole except mass, angular momentum and charge and hence *no-hair conjecture.* As a strong interpretation, in the presence of a scalar field, black hole should possess a trace different from mass, angular momentum and charge inorder to have the notion of a scalar hair. The scalar potential at infinity acts as an added cosmological constant. That added cosmological constant can be treated as a signature of the scalar field, and hence *scalar hair.* But since, only the effective cosmological constant is affected and not the true cosmological constant, the above argument did not draw much attention. As a weak interpretation, a non-trivial solution in terms of the existing conserved quantities is enough to show that there is hair. Saa [73] and Banerjee [74] argued that a regular horizon is possible only when the scalar solution is trivial and when the solution is non-trivial, the horizon will be a surface of singularity. Torii [67] argued that the hair falls out easily, since for every non-trivial solution the eigenmode is negative.

In our case a non-trivial scalar black hole solution is obtained with a double well potential as the source, for BTZ black hole and RN black hole with regular horizons. The horizon is not singular and it hides singularity. Since the eigenmode is positive, the scalar field is finite at the horizon and falls to a minimum value in the asymptotic limit. The mass of black hole with hair is greater than that with out hair with a condition $\mu = -\frac{1}{l}$. All these conclusions show that scalar hair solution is possible for the BTZ and RN black holes.

Chapter 7

Scalar hair for a static (3+1) black hole

7.1 Introduction

I never thought that others would take them so much more seriously than I did,
Albert Einstein.

No-hair conjecture [123] demands the non existence of any information other than mass, charge and angular momentum of a black hole. In order to prove the no-hair conjecture, no-hair theorems had been established by coupling the classical fields with the Einstein gravity [138]. It had been shown that the scalar field would be trivial if one demands a regular horizon at a finite distance from the centre of the black hole and also that stationary black hole solutions are hairless in a variety of cases, coupling different classical fields to gravity [122, 139, 140, 141]. In **Chapter 6**, we have noted that the co-existence of non-trivial solution and a proper metric is difficult to evolve.

It is widely believed that black holes with scalar hair generally exist only when the scalar potential has negative region [70, 71, 72, 142]. There is a common belief that there are no static asymptotically flat and asymptotically AdS black holes with spherical scalar hair, if the scalar field theory coupled to gravity, satisfies the Positive Energy Theorem [143]. A charged de Sitter black hole in the Einstein-Maxwell-Scalar-Λ system possesses only unstable solutions [68]. But an unexpected development of scalar hair in AdS black hole with minimal [70] as well as nonminimal [144] coupling of scalar field, demanded a heuristic study of scalar hair [69].

As a strong interpretation, black hole has hair if there is a need to specify quantities other than the conserved charges defined at asymptotic infinity in order to characterize completely a stationary black hole solution [72, 128]. Efforts were done to reveal strong hair [124, 125] and they came up with a scalar solution conformally coupled to Einstein's gravity through a metric for extremal case. Eventhough innocuous, the solution has a divergence at the horizon. It is given as

$$\Phi = \frac{-r_0 \alpha^{-1/2}}{r - r_0}, \tag{7.1}$$

with $\alpha = \frac{8\pi G}{6}$. In Eq. (7.1), Φ blows up at the horizon, which is against the principle that Φ shall be finite everywhere [66]. In another attempt, a four dimensional solution of the Einstein equation with a positive cosmological constant coupled to a massless self interacting conformal scalar field was put forwarded [135]. The scalar solution in that case is

$$\Phi(r) = \sqrt{3/4\pi}\frac{\sqrt{G}M}{r - GM}. \tag{7.2}$$

But Eq. (7.2) does not give any information other than mass of black hole and the cosmological constant. Hence strong interpretation of scalar hair is not guaranteed. The motivation of the present work is to know whether a strong interpretation of the scalar hair can be possible in a static (3+1) black hole, which requires a nontrivial solution of scalar field in the vicinity of black hole with regular horizon.

In this chapter, we report a nontrivial black hole solution of a massive but self interacting scalar field showing no divergence at the horizon and asymptotically falling to the vacuum value. The proposed metric shows trace of scalar charge.

Whether a non-trivial scalar solution and a metric with a horizon are mutually compatible or not has been the objective of scalar hair investigations. Many contend that only when the solution is trivial that a metric with a horizon is established and for a non-trivial solution, the singularity will become naked. The criterion of scalar hair is the co-existence of non-trivial solution and a proper metric (having horizon and temperature) that holds the trace of scalar field.

The scheme of the chapter is as follows. In Sec. 7.2, non-trivial scalar hair solution is obtained for a (3+1) static black hole. In Sec. 7.3, the metric of the hairy black hole is obtained by solving the scalar stress-energy tensor. Entropy and mass of hairy black hole are also discussed in this section. Sec. 7.4, discusses thermodynamics of black hole. The conclusion is given in section 7.5.

7.2 Solution with a conformal coupling

A non-trivial radial solution of a scalar field, whose source is a scalar double well potential, in the vicinity of a static (3+1) black hole will be discussed in this section. We will restrict our consideration to the conformally coupled case. Consider the action

$$I = \int d^4x\sqrt{-g}[\frac{R}{2k} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi \\ -\frac{1}{2}\xi R(\Phi - \Phi_0)^2 - V(\Phi - \Phi_0)], \tag{7.3}$$

where Φ is a massive, self interacting and conformally coupled scalar field. In the static case, Φ represents a radial field. For a (3+1) case, $\xi = \frac{1}{6}$. A double well potential of the type in Eq. (6.11) and Fig. (6.1) has been considered in Eq. (7.3). In Fig. (6.1), V has global minima at $\Phi = \pm\frac{\mu}{\delta}$ and a local maximum at $\Phi = \Phi_0$. The scalar field equation is given by

$$\Box(\Phi - \Phi_0) - \xi R(\Phi - \Phi_0) + \mu^2(\Phi - \Phi_0) - \delta^2(\Phi - \Phi_0)^3 = 0, \tag{7.4}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplace-Beltrami operator and R represents the Ricci scalar. For the present situation, we do not consider cosmological constant. Hence $R = 0$. The stress-energy tensor of scalar field under gravity can be given by the relation

$$T_{\mu\nu} = \nabla_\mu\Phi\nabla_\nu\Phi - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi +$$

$$\tfrac{1}{6}[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}](\Phi - \Phi_0)^2 \tag{7.5}$$

$$+\tfrac{1}{2}g_{\mu\nu}\mu^2(\Phi - \Phi_0)^2 - \tfrac{1}{4}g_{\mu\nu}\delta^2(\Phi - \Phi_0)^4 - \tfrac{1}{4}g_{\mu\nu}\tfrac{\mu^4}{\delta^2},$$

with

$$\Box(\Phi - \Phi_0)^2 = 2(\Phi - \Phi_0)\Box\Phi + 2\nabla_\mu\Phi\nabla_\nu\Phi,$$
$$\nabla_\mu\nabla_\nu(\Phi - \Phi_0)^2 = 2(\Phi - \Phi_0)\nabla_\mu\nabla_\nu\Phi + 2\nabla_\mu\Phi\nabla_\nu\Phi. \tag{7.6}$$

Here ∇_μ represents a covariant derivative in the metric $g_{\mu\nu}$. For a static and spherically symmetric space time, the $t-t$ component of scalar stress energy tensor is given as

$$T_0^0 = -\tfrac{1}{2}g^{11}(\nabla_1\Phi)^2 + \tfrac{1}{6}G_0^0(\Phi - \Phi_0)^2$$
$$+\tfrac{1}{6}\mu^2(\Phi - \Phi_0)^2 - \tfrac{1}{12}\delta^2(\Phi - \Phi_0)^4 - \tfrac{1}{12}\tfrac{\mu^4}{\delta^2}, \tag{7.7}$$

and the $r-r$ component of stress energy tensor is given as

$$T_1^1 = \tfrac{1}{6}g^{11}(\nabla_1\Phi)^2 - \tfrac{1}{3}g^{11}(\Phi - \Phi_0)\nabla_1^2\Phi + \tfrac{1}{3}(\nabla_1\Phi)^2 + \tfrac{1}{6}G_1^1(\Phi - \Phi_0)^2$$
$$+\tfrac{1}{6}\mu^2(\Phi - \Phi_0)^2 - \tfrac{1}{12}\delta^2(\Phi - \Phi_0)^4 - \tfrac{1}{12}\tfrac{\mu^4}{\delta^2}. \tag{7.8}$$

The metric of a static (3+1) black hole may be given as

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2. \tag{7.9}$$

In the above metric, λ is a function of r only and let $\nu = \lambda(r) + f(t)$. Here, $f(t)$ is an arbitrary function of t. There is no loss of generality in setting $f(t) = 0$, since it can be absorbed in the definition of t, i.e., by replacing $e^{f(t)}dt$ by dt. With this redefinition of the time coordinate, $\nu = -\lambda$. Then

$$G_0^0 = G_1^1 = \frac{1}{r^2}\frac{d}{dr}[r(1-e^{2\nu})]. \tag{7.10}$$

Applying Eq. (7.10) in Einstein's equation $G_\nu^\mu - \kappa T_\nu^\mu = 0$, we get

$$T_0^0 - T_1^1 = 0. \tag{7.11}$$

The concept of scalar hair is applicable only to a static black hole, since only in that case that we will be able to solve the field equation. From Eq. (7.11), we get

$$-2g^{11}(\nabla_1\Phi)^2 - (\nabla_1\Phi)^2 + g^{11}(\Phi-\Phi_0)\nabla_1^2\Phi = 0. \tag{7.12}$$

Eq. (7.12) is a covariant differential equation. In Eq. (7.12), properties such as mass and self interaction terms of scalar field do not come explicitly. In Eq. (7.12), $\nabla_1^2\Phi$ can be written in the ordinary derivative as

$$\nabla_1^2\Phi = \partial_1^2\Phi - \Gamma_{11}^i\partial_i\Phi, \tag{7.13}$$

where Γ is the usual Christoffel symbol and i runs from $0 \to 3$. In the above case, all the Christoffel symbols except Γ_{11}^1 are zeroes and $\Gamma_{11}^1 = \lambda' = -\nu'$. Now, Eq. (7.12) gets modified as,

$$-2g^{11}(\partial_1\Phi)^2 - (\partial_1\Phi)^2 + g^{11}(\Phi-\Phi_0)[\partial_1^2\Phi + \partial_1\nu\partial_1\Phi] = 0. \tag{7.14}$$

In quest of scalar hair, the general principle is to get a non-trivial solution which is compatible with a proper metric that hides singularity. So we propose a solution of the form

$$\Phi(r) = \frac{a}{r} + \Phi_0, \tag{7.15}$$

where Φ_0 is the asymptotic value of scalar field and $'a'$ is a constant which may be derived from the mass of black hole and scalar field. Non-trivial solution of scalar field have been proposed by many people earlier but never extended it to the concept of scalar hair [135, 145]. By substituting Eq. (7.15) in Eq. (7.14), we get

$$-\partial_1\Phi + g^{11}(\Phi - \Phi_0)\partial_1\nu = 0. \tag{7.16}$$

From Eq. (7.16), we can determine the metric function which will be given in the Sec.7.3. The profile of Eq. (7.15) shows that the field has a finite value Φ_h at the horizon and then falls to the asymptotic value Φ_0. The variation of Φ against r is shown in Fig. (7.1). The field has the highest magnitude at the horizon and falls to the asymptotic value Φ_0. The profile of the scalar field shows that a trace of scalar field is hidden behind the event horizon. If we get a proper metric (a metric with horizon and temperature) in addition to the solution, then that shows the existence of a strong hair.

7.3 Metric of a static (3+1) black hole

The form of metric, which is compatible with the scalar solution, will be now determined. Since $g^{11} = -e^{2\nu}$, Eq. (7.16) becomes

$$\partial_1\Phi + e^{2\nu}(\Phi - \Phi_0)\partial_1\nu = 0. \tag{7.17}$$

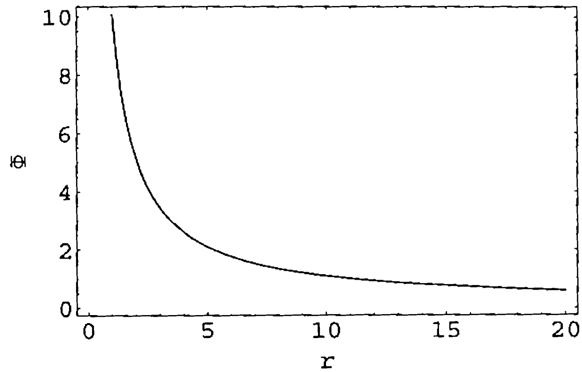

Figure 7.1: Variation of scalar field against r with $a = 1$ and $\Phi_0 = 0.1$.

By introducing a transformation of the type [146]

$$\nu = \frac{1}{2}\log(1+f), \tag{7.18}$$

we get from Eq. (7.17)

$$\frac{d\Phi}{(\Phi - \Phi_0)} = -\frac{1}{2}df, \tag{7.19}$$

where f is a radial function. Now introduce a gauge transformation of the type, $\Phi - \Phi_0 = \bar{\Phi}$. Then

$$\begin{aligned} d\Phi &= d\bar{\Phi}, \\ \frac{d\bar{\Phi}}{\bar{\Phi}} &= -\frac{1}{2}df. \end{aligned} \tag{7.20}$$

Integrating Eq. (7.20), we get

$$\log\bar{\Phi} = -\frac{f}{2} + C_0. \tag{7.21}$$

In the asymptotic limit, $\nu = 0$ and hence $f = 0$. Putting a new value, say $\bar{\Phi}_0$, as the asymptotic value of $\bar{\Phi}$ in the new scale, we can obtain

$$\log(\frac{\bar{\Phi}}{\bar{\Phi}_0}) = -\frac{f}{2}. \qquad (7.22)$$

Since the transformation is only of a gauge type, we get that, $\frac{\bar{\Phi}}{\bar{\Phi}_0} = \frac{\Phi}{\Phi_0}$. Thus from Eqs. (7.18), (7.22), we find

$$e^{2\nu} = 1 - 2\log(\frac{\Phi}{\Phi_0}). \qquad (7.23)$$

Eq. (7.23) represents the metric function which is compatible with the scalar solution, $\Phi(r) = \frac{a}{r} + \Phi_0$. In the asymptotic limit the metric function,

$$e^{2\nu} = 1, \qquad (7.24)$$

coincides with those of Schwarzschild and RN black holes. Denoting the field at the horizon as $\Phi_h = \Phi_0 e^{1/2}$, the radius of the horizon can be obtained as:

$$r_h = \frac{a}{\Phi_0(e^{1/2} - 1)}. \qquad (7.25)$$

In Eq. (7.23), $\log(\frac{\Phi}{\Phi_0})$ can be expanded as a series if $\frac{\Phi}{\Phi_0} \leq 2$ and it is true in the present case. Therefore,

$$\log(\frac{\Phi}{\Phi_0}) = (\frac{\Phi}{\Phi_0} - 1) - \frac{1}{2}(\frac{\Phi}{\Phi_0} - 1)^2 + \frac{1}{3}(\frac{\Phi}{\Phi_0} - 1)^3.... \qquad (7.26)$$

The metric function then may be written as a series as,

$$e^{2\nu} = 1 - 2(\frac{\Phi}{\Phi_0} - 1) + (\frac{\Phi}{\Phi_0} - 1)^2 - \frac{2}{3}(\frac{\Phi}{\Phi_0} - 1)^3.... \qquad (7.27)$$

In Eq. (7.27), let $(\frac{\Phi}{\Phi_0} - 1) = \frac{a}{\Phi_0 r} = \frac{b}{r}$. The metric function gets modified as,

$$e^{2\nu} = 1 - \frac{2b}{r} + \frac{b^2}{r^2} - \frac{2}{3}\frac{b^3}{r^3}\cdots \tag{7.28}$$

The above mentioned metric is in unison with a recent work[147]. Eq. (7.28) represents a composite metric which would converge to different metrics by truncating appropriate terms.

7.3.1 Study of metric

Eq. (7.27) may be written in a concise form as,

$$e^{2\nu} = 1 - 2[\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{1}{n}(\frac{\Phi}{\Phi_0} - 1)^n]. \tag{7.29}$$

When $n = 1$, we have the Schwarzschild like black hole:

$$e^{2\nu} = 1 - 2(\frac{\Phi}{\Phi_0} - 1) = 1 - \frac{2b}{r}. \tag{7.30}$$

In this case, $\frac{\Phi_h}{\Phi_0} = \frac{3}{2}$. The radius of horizon is $r_h = 2\frac{a}{\Phi_0} = 2b$.

When $n = 2$, we have the extremal case similar to extremal RN black hole. The metric function is reduced to,

$$\begin{aligned} e^{2\nu} &= 1 - 2(\tfrac{\Phi}{\Phi_0} - 1) + (\tfrac{\Phi}{\Phi_0} - 1)^2 = [1 - (\tfrac{\Phi}{\Phi_0} - 1)]^2 \\ &= (1 - \tfrac{b}{r})^2. \end{aligned} \tag{7.31}$$

In the extremal case, $\frac{\Phi_h}{\Phi_0} = 2$, which gives the maximum value of $\frac{\Phi_h}{\Phi_0}$. The radius of horizon is $r_h = \frac{a}{\Phi_0} = b$. When $n = 3$, $\frac{\Phi_h}{\Phi_0} = \frac{13}{8}$ and $r_h = \frac{5}{3}\frac{a}{\Phi_0} = \frac{5}{3}b$. When $n = 4$, $\frac{\Phi_h}{\Phi_0} = \frac{17}{10}$ and $r_h = \frac{2}{3}\frac{a}{\Phi_0} = \frac{2}{3}b$. It is obtained that the horizon's radius increases and decreases with

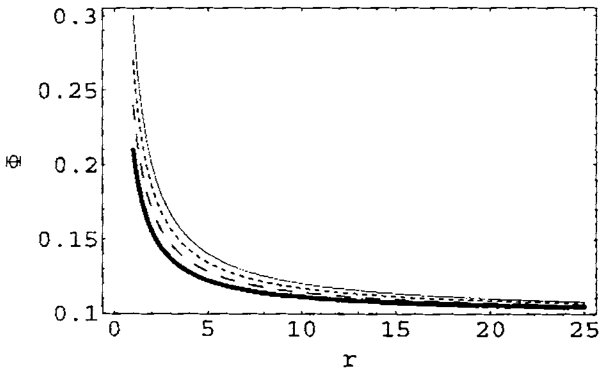

Figure 7.2: Variations of scalar field Φ against r for different black holes, with, $\Phi_0 = 0.1$.

diminishing magnitude as n increases. We can thus introduce more types of black holes by putting $n = 5, 6....$ But as the series progress, the series very quickly diminishes. The variation of scalar field Φ against r for different black holes are shown in Fig. (7.2). The thick line represents the case with $n = 1$. The normal line graph represents the case with $n = 2$. The dashed line represents the case with $n = 3$ and the dotted line represents the case with $n = 4$. It may be thought that the signature $'a'$ in the scalar solution is derived from the mass of the black hole. But,the signature of scalar field in the metric is due to the asymptotic value of scalar tensor T_0^0. The asymptotic value of T_0^0 is given as (Eq. (7.7)),

$$T_0^0 = \frac{1}{6}G_0^0(\infty)\Phi_0^2 - \frac{1}{12}\frac{\mu^4}{\delta^2}. \tag{7.32}$$

As G_0^0 vanishes in the asymptotic limit we get,

$$T_0^0 = -\frac{1}{12}\frac{\mu^4}{\delta^2}. \tag{7.33}$$

The trace term corresponding to scalar field may be written as,

$$\eta(r) = -\int_{r_h}^{r_0} 4\pi r^2 T_0^0 dr. \tag{7.34}$$

Substituting Eq. (7.33) in Eq. (7.34), we get,

$$\eta = \frac{\pi}{9}\frac{\mu^4}{\delta^2}(r_0^3 - r_h^3). \tag{7.35}$$

η of Eq. (7.35) has its contribution in the making of signature of scalar field in the metric.

7.3.2 Stability of field

There exist a possibility that a transformation of $'r'$ coordinate may eliminate the scalar field, making it unstable. The technique to ascertain the stability of the field is to see whether the field has a finite value in a different coordinate at the horizon. With $n = 1$, the metric would become, $e^{2\nu} = 1 - 2b/r$. So if we define

$$dr_* = \frac{dr}{\sqrt{1 - 2b/r}}, \tag{7.36}$$

we get

$$r_* = \frac{\sqrt{r}(r - 2b) + 2b(\sqrt{2b - r})arctan[\frac{\sqrt{r}}{\sqrt{2b-r}}]}{\sqrt{r - 2b}} + C_1, \tag{7.37}$$

where C_1 is a constant integration. At the horizon, $r_* = C_1$. So the field at the horizon in the new coordinate is

$$\Phi = \frac{a}{C_1} + \Phi_0, \tag{7.38}$$

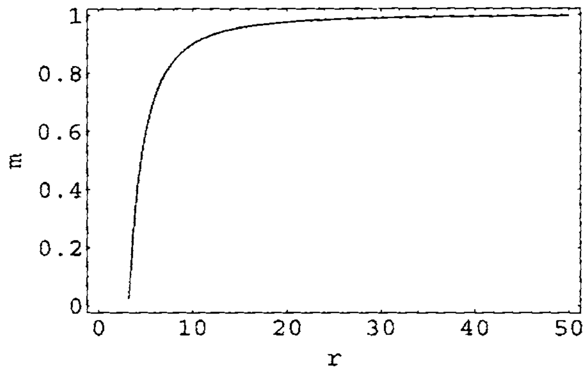

Figure 7.3: Variation of mass of hairy black hole against r, with, $a = 1$ and $\Phi_0 = 0.1$.

which is finite. Hence the field can not be eliminated at the horizon by a coordinate transformation, i.e., the field is stable. Stability of field shows that the hair does not fall out.

7.3.3 Mass of hairy black hole

Mass of a hairy black hole, in general, must be greater than a non-hairy black hole. The mass (m_{rh}) of a hairy black hole is related to the mass (M_{rh}) of a non-hairy black hole [69] through the relation,

$$m_{r_h} = M_{r_h} + 4\pi r^2 \int_{r_h}^{r} r[V(\Phi) - V(\Phi_\infty) + \frac{1}{2} e^{2\nu} \Phi'^2] dr. \quad (7.39)$$

In Eq. (7.39), $V(\Phi) = 0$ at the horizon and $V(\Phi_\infty) = \frac{1}{4}\frac{\mu^4}{\delta^2}$. As the distance from the center of black hole increases, $m(r_h)$ increases, but never blows up since it attains a steady value as r increases. The variation of $m(r_h)$ against r is shown in the Fig. (7.3).

7.3.4 Entropy

The black hole with a cavity is called a dressed black hole and without it is called a naked black hole. A black hole dressed with a scalar field can be stable only inside a cavity, whose entropy may be the sum of entropy of naked black hole and that of surface. A dressed black hole results in the phenomenon called back reaction and the surface entropy is the result of back reaction. Black hole entropy is nothing but the Noether charge, i.e., $\frac{\kappa}{2\pi}S = \int_\Sigma Q$, where $\int_\Sigma$ is the surface integral and Q is the Noether charge[148]. The entropy of a stationary black hole can be expressed as $2\pi \oint Q$ over any cross-section of the horizon [149]. The entropy of a black hole can be evaluated as a local quantity on the horizon using two dimensional gravity [150]. Surface gravity of the black hole is given as,

$$\kappa = \frac{\partial_r g_{tt}}{2\sqrt{-g_{tt}g_{rr}}} |_{r=r_h} . \tag{7.40}$$

With $g_{tt} = e^{2\nu} = 1 - 2\log(\frac{\Phi}{\Phi_0})$ and $\Phi(r) = \frac{a}{r} + \Phi_0$, we get,

$$\kappa = \frac{a}{ar_h + \Phi_0(r_h)^2}. \tag{7.41}$$

Substituting for $r_h = \frac{a}{\Phi_0(e^{1/2}-1)}$ in Eq. (7.41), we find the temperature of black hole as,

$$T_{bh} = \frac{\Phi_0(e^{1/2}-1)^2}{2\pi a e^{1/2}}. \tag{7.42}$$

The area of the horizon,

$$A_h = 4\pi r_h^2 = 4\pi \frac{a^2}{\Phi_0^2(e^{1/2}-1)^2}, \tag{7.43}$$

exclusively depends on the parameter 'a'. Following the standard procedure, we can determine the entropy of black hole in the presence of scalar field as

$$dA_h = \frac{4}{e^{1/2}\Phi_0} dS. \tag{7.44}$$

On integrating Eq. (7.44),

$$S = \Phi_0 e^{1/2} \frac{A_h}{4} + c = \Phi_h \frac{A_h}{4} + c, \tag{7.45}$$

where c is a constant and S the entropy of black hole in the presence of scalar field. By applying the boundary condition that when $\Phi_h = 0$, $S = S_0$, the entropy of naked black hole. Then $c = S_0$. But by Hawking's theory, $S_0 = \frac{A_h}{4}$. So the total entropy is written as,

$$S = \Phi_h \frac{A_h}{4} + \frac{A_h}{4} = S_s + S_0. \tag{7.46}$$

Eq. (7.46) clearly indicates that the scalar field contributes to the entropy of a black hole.

7.4 Thermodynamics

When the Hawking radiation is fully thermal, the thermal pressure is $\frac{1}{3}\alpha T_{loc}^4$, where $T_{loc} = \frac{T_{bh}}{\sqrt{-g_{00}}}$. A dressed black hole (with e.m field, dilation field etc) possess a black hole temp, $T_{bh} \leq \frac{\hbar}{\sqrt{4\pi A_h}}$ [151]. The stress-energy tensor of the radiation is a function of black hole temperature T_{bh}. The hairy black hole is a thermodynamical system with a modified temperature.

The effective potential of the test particles moving in static and spherically symmetric background geometry is determined by the

Hamilton-Jacobi approach. By Hamilton-Jacobi equation,

$$g^{k\lambda}\partial_k S\partial_\lambda S + \mu^2 = 0, \tag{7.47}$$

where μ is the mass of scalar field and,

$$S(t, r, \phi) = -Et + S(r) + L\phi. \tag{7.48}$$

E and L are the constant energy and angular momentum of the test particle. Substituting Eq. (7.48) in Eq. (7.47) and simplifying we get the action as

$$S(r) = \mp \int [E^2 - (\frac{L^2}{r^2} + \mu^2)f]^{1/2}\frac{1}{f}dr, \tag{7.49}$$

where $f = e^{2\nu}$. From the expression for the action, we can measure the temperature of black hole by the following method.

7.4.1 Temperature of different black holes

(a). In Eq. (7.29), with $n = 1$, $e^{2\nu} = 1 - 2(\frac{\Phi}{\Phi_0} - 1) = 1 - \frac{2b}{r}$. This is SBH like. As $r \to r_h$, $S(r)$ is modified as,

$$S(r) = \mp \int \frac{r_h E dr}{r - r_h} = \mp\beta \log(r - r_h). \tag{7.50}$$

with $\beta = Er_h$. Assuming that the scalar field gets reflected at the horizon, the scalar field in the neighborhood of the horizon can be written as [152],

$$\Phi(r) = e^{-i\beta \log(r - r_h)} + Re^{i\beta \log(r - r_h)}, \tag{7.51}$$

with R as the coefficient of reflection. The coefficient of reflection R and the probability of reflection by horizon P may be given as,

$$R = e^{-2\pi\beta}; P = \mid R \mid^2 = e^{-4\pi\beta}. \tag{7.52}$$

Using the thermodynamical relation,

$$P = e^{-E/T_{bh}}, \tag{7.53}$$

we get,

$$E/T_{bh} = 4\pi\beta, \tag{7.54}$$

which gives the black hole temperature as,

$$T_{bh} = \frac{1}{4\pi r_h}. \tag{7.55}$$

This is the temperature of Schwarzschild like black hole. Eq. (7.55) reveals that the temperature of a black hole is a matter related to the radius of horizon. Since, radius of horizon is a function of parameters, such as mass, charge (vector as well as scalar) and angular momentum, it can be seen that black hole temperature depends on these parameters.

(b). In Eq. (7.29), with $n = 2$, $e^{2\nu} = [1 - (\frac{\Phi}{\Phi_0} - 1)]^2 = [1 - \frac{b}{r}]^2$. This is extremal case in which no black hole temperature is observed.

(c). In Eq. (7.29), with $n = 3$, $e^{2\nu} = 1 - 2(\frac{\Phi}{\Phi_0} - 1) + (\frac{\Phi}{\Phi_0} - 1)^2 - \frac{2}{3}(\frac{\Phi}{\Phi_0} - 1)^3 = 1 - \frac{2b}{r} + \frac{b^2}{r^2} - \frac{2}{3}\frac{b^3}{r^3}$. The action $S(r)$ as $r \to r_h$ is given as,

$$S(r) = \mp \int \frac{E r_h^3 dr}{(r - r_h)(r_c - r_h)(r_h - r_0)}, \tag{7.56}$$

with, $\beta = \frac{Er_h^3}{(r_c - r_h)(r_h - r_0)}$ and $(r_c + r_h + r_0) = -2b; (r_h r_c + r_c r_0 + r_b r_0) = -b^2; r_h r_c r_0 = -\frac{2}{3}b^2$. Now,

$$S(r) = \mp\beta \log(r - r_h). \tag{7.57}$$

By proceeding as in the previous case, the temperature of the black hole can be shown to be,

$$T_{bh} = \frac{(r_c - r_h)(r_h - r_0)}{4\pi r_h^3}. \tag{7.58}$$

Thus we see that the hairy black hole acts as a thermodynamical system with proper horizon and temperature.

7.5 conclusion

There is an argument that a regular horizon is possible only when the scalar solution is trivial [73, 74]. So when the solution is non-trivial, the event horizon will be a surface of singularity and hence it will not represent a black hole. A black hole exists only when the event horizon hides the naked singularity.

As a weak interpretation of scalar hair, a non-trivial solution of scalar field in terms of the existing conserved quantities is enough to show that there is hair [70, 71]. Whether a horizon naturally occurs, even when the solution is non-trivial, will be the primary objective of strong interpretation of scalar hair. Thus as a strong interpretation, in the presence of a scalar field, a black hole would have a signature different from mass, angular momentum and vector charge.

We have shown that a non-trivial scalar black hole solution for a massive self interacting conformal scalar field would be obtained in the case of static (3+1) black hole. The metric proposes a horizon and temperature for the black hole. The horizon and surface temperature ensure a true black hole. The metric element has a term other than the existing conserved quantities. In our case, only a particular pair of scalar field and metric are found to be mutually compatible. We have also shown that the scalar field never vanishes in a transformed coordinate, making it stable. The hair does not fall out if the field is stable.

In the proposed metric, only one parameter, i.e., b has appeared. This may invite some criticisms against the strong interpretation of scalar hair. But in the standard extremal case, same parameter does the job of mass and vector charge. Another argument against b is that it may have been originated from the mass of black hole itself. But Eq. (7.35) clearly indicates that the origin of the parameter b is from scalar field and hence we may conclude that scalar field can depict its signature in the proposed metric.

Chapter 8

Results and conclusion

There are grounds for cautious optimism that we may now be near the end of the search for the ultimate laws of nature,
Stephen W. Hawking.

8.1 Results

I believe that I have given a vivid profile of my research work in the concluded chapters. Now let me summarize the results.

1. The most important question on black hole evaporation that one has to deal is the back reaction problem, i.e., the calculation of the effects of the emitted quantum radiation on the spacetime geometry of the black hole. Semi-classical Einstein field equations have been solved in **Chapter 2**, in the context of an extremal anti-de Sitter-Schwarzschild black hole surrounded by a spin-2 quantum field in thermal equilibrium with the Hawking radiation. The change

of entropy of a black hole, which is a measure of back reaction, have been measured using the method of York and thermodynamical approach. The surface entropy (or entropy correction) of a dressed black hole is found to depend also on temperature of black hole.

2. In **Chapter 3**, back reaction in a Schwarzschild-de Sitter black hole surrounded by a massless quantum field in thermal equilibrium, has been investigated. The change of entropy of the black hole has been determined by solving the Einstein's equation. Again the change in the effective potential of the black hole spacetime bathed in a quantum field shows the effect of back reaction. We have showed that there are no stable orbit in this spacetime and the unstable orbits situate at $r = 2.95M$ and $r = 2.9M$ respectively for a particle and a photon in the presence of back reaction.

3. The gravity is so strong near the horizon that the asymptotic expression of state equations of radiation will be affected by it. In **Chapter 4**, we have obtained the modified state equations of radiation near the horizon of a Reissner-Nordström black hole and found that both the GSL and Bekenstein's upper bound on entropy are conserved in a gedanken experiment. In the asymptotic limit, the new equations converge to the asymptotic expressions.

4. Temperature is implicitly present in a spacetime which is periodic in the imaginary time τ. We have shown in **Chapter 5**, that the spacetime close to the event horizon of the Schwarzschild black hole is Rindler like. Using the field theory approach, the temperature of the scalar field near the horizon of Rindler spacetime and Schwarzschild spacetime have been determined. The trajectory of a particle in a static Rindler space has been determined by solving the Euler's equation. We have calculated the entropy of an $SGRS$

and is found to be proportional to the square of mass. We have also calculated the entropy bound and obtained that it is similar to the Bekenstein upper bound

5. A weak hair has been observed in **Chapter 6** for BTZ and RN black holes with regular horizon. The horizon is not singular and it hides singularity. The scalar field is stable against the 1^{st} and 2^{nd} order perturbations. The mass of black hole with hair is greater than that with out hair with a condition $\mu = -\frac{1}{l}$. All these show that there is scalar hair for the BTZ and RN black holes.

6. Whether a horizon naturally occurs, even when the solution is nontrivial, will be the primary objective of strong interpretation of scalar hair. We have shown in **Chapter 7**, that a non-trivial scalar black hole solution, for a massive self interacting conformal scalar field, would be obtained in the case of static (3+1) black hole. The metric proposes horizon and temperature for the black hole. The horizon and surface temperature ensure a true black hole. The metric element has a term other than the existing conserved quantities such as mass, vector charge and angular momentum. Not only that the scalar field is not carried away by a coordinate transformation, implies its stability.

8.2 Future prospects

A complete description of black hole evaporation and back reaction needs a comprehensive theory of quantum gravity. Since quantum gravity still eludes us, we have to be content with semi-classical methods to solve the back reaction program, making it insufficient. So quantization of spacetime is still a dream for the physicists. The black hole spacetime may be an ideal one to get quantized, since it holds the unique property of singularity. It indicates a situation where physical parameters of a theory become untenable by attaining infinite values. The singularity in GR not only marks the end of GR but also of everything else. No theory can survive in the absence of proper spacetime back ground. So with the advent of a quantum gravity we could venture deep into the black hole evaporation and back reaction.

The *information loss paradox* is related to the fact that whether the Hawking radiation is fully thermal or not. If it is fully thermal, information will be lost in the gravitational collapse. If the correlations between the inside and outside of the black hole are not restored during the evaporation process, then by the time that the black hole has evaporated completely, an initial pure state will have evolved to a mixed state, i.e., *information* will be lost. So we need a thorough investigation regarding this. The statement that the *Black holes have no-hair* is a bone of contention. Weak interpretation of hair is not so rare to occur. But strong interpretation requires a new conserved quantity in the metric and proper horizon that hides the singularity. What we have obtained in **chapter.7**, is only a mutually compatible scalar solution and a metric. Our aim is to evolve an independent solution and a metric that contains new information.

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